Mixed Volumes of Hypersimplices, Root Systems and Shifted Young Tableaux

by

Dorian Croitoru

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Author . . . .
Department of Mathematics
July 21, 2010

Certified by . . . .
Alexander Postnikov
Associate Professor of Applied Mathematics
Thesis Supervisor

Accepted by . . . .
Bjorn Poonen
Chairman, Department Committee on Graduate Students

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Abstract

This thesis consists of two parts. In the first part, we start by investigating the classical *permutohedra* as Minkowski sums of the *hypersimplices*. Their volumes can be expressed as polynomials whose coefficients – the *mixed Eulerian numbers* – are given by the mixed volumes of the hypersimplices. We build upon results of Postnikov and derive various recursive and combinatorial formulas for the mixed Eulerian numbers. We generalize these results to arbitrary root systems Φ, and obtain cyclic, recursive and combinatorial formulas for the volumes of the *weight polytopes* (Φ-analogues of permutohedra) as well as the *mixed Φ-Eulerian numbers.* These formulas involve Cartan matrices and weighted paths in Dynkin diagrams, and thus enable us to extend the theory of mixed Eulerian numbers to arbitrary matrices whose principal minors are invertible.

The second part deals with the study of certain patterns in standard Young tableaux of *shifted* shapes. For the staircase shape, Postnikov found a bijection between vectors formed by the diagonal entries of these tableaux and lattice points of the (standard) *associahedron.* Using similar techniques, we generalize this result to arbitrary shifted shapes.

Thesis Supervisor: Alexander Postnikov
Title: Associate Professor of Applied Mathematics

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Contents

Chapter 1

Introduction

My main research interests lie in geometric and algebraic combinatorics and in particular in the geometry of convex polytopes. Convex polytopes lie at the crux of combinatorics. Studying their classical invariants, such as their volumes, numbers of lattice points, f-vectors and Erharht polynomials has been a central problem, not only in its own interest, but more importantly because of the vast connections of these invariants with other areas of mathematics such as algebraic geometry and representation theory. The first problem investigated here is stuying invariants of a certain class of polytopes called *permutohedra.* The classical *permutohedron* $P(x_1, \ldots, x_{n+1})$ is defined as the convex hull of the $(n+1)!$ points obtained by permuting the coordinates of $(x_1, ..., x_{n+1})$. Permutohedra and more importantly, their various generalizations, have been studied sistematically by Postnikov and others. Special cases of permutohedra include many interesting polytopes such as *graphical zonotopes* and *graph associahedra* in graph theory, as *moment polytopes* (and in particular *matroid polytopes*) in algebraic geometry, and as *alcoved polytopes* and *weight polytopes* in representation theory. It is a general principle that the volumes, numbers of lattice points and other invariants of these polytopes should have alternative descriptions in terms of other objects (both combinatorial and non combinatorial) such as trees, Young tableaux, degrees of toric varieties, Weyl group elements with special properties, and so on. For example, $P(n-1, \ldots, 1, 0)$ is the *zonotope* corresponding to the complete graph K_{n+1}, and it is known that $n! Vol\,(P(n-1, ..., 1, 0))$ is the number of spanning trees of the complete graph K_{n+1}. A more interesting example is the *hypersimplex* $\Delta_{n+1,k} = P\left(1^k, 0^{n+1-k}\right)$ (1^k means k ones), which are matroid polytopes, the moment polytopes corresponding to a toric varieties X_p for generic points p in the Grassmanian, and also appear as weight polytopes of the fundamental representations of $\mathfrak{gl}_n$. In this work I have investigated volumes of permutohedra by computing the mixed volumes of hypersimplices, called the *mixed Eulerian numbers.* These numbers include many classical combinatorial numbers such as the Catalan numbers, binomial coefficients and Eulerian numbers. I have found various recursive, combinatorial and cyclic formulas that enable one to compute the mixed Eulerian numbers easily. All of these results generalize to arbitrary root systems, and some results even to arbitrary positive definite matrices. The second problem that I have explored in this thesis is studying patterns in Young tableaux of *shifted* shapes. Generalizing a result of Postnikov, I have shown that diagonal vectors of these tableaux are in bijection with lattice points of a certain polytope, which can be represented as a Minkowski sum of coordinate simplices (depending on the Young shape). These polytopes are generalized permutohedra, and are often combinatorially equivalent to the *associahedron.*

The thesis is organized as follows. Chapter 2 is devoted to Permutohedra, Mixed Eulerian numbers and their generalizations. In Section 2.1, we give a brief overview on permutohedra and their volumes. In Section 2.2, we introduce the (classical) mixed Eulerian numbers, motivate their study and discuss known results about them. Section 2.3 generalizes the setup to any affine root system Φ. Next, in Section 2.4 we derive recursive formulas for the mixed Eulerian numbers, and in particular show that they are positive integers for any root system. We illustrate these results in Section 2.5, as well as provide new simple proofs for known results on the mixed Eulerian numbers. In Section 2.6 we prove a cyclic relation between volumes of weight polytopes associated to root subsystems of an extended affine root system. In Section 2.7 we use the dependence of the mixed Eulerian numbers solely on the Cartan matrix A_Φ of the root system to generalize the theory to arbitrary positive definite matrices. We specialize some of these general results to A_Φ in Section 2.8 and obtain an alternate characterization of the mixed Eulerian numbers in terms of weighted paths in Dynkin diagrams. Chapter 3 is about Shifted Young Tableaux. Section 3.1 reviews basic definitions. In Section 3.2 we use a result of Baryshnikov and Romik to derive a generating function for the diagonal vectors of shifted Young tableaux. We use this later in Section 3.3 to establish a one-to-one correspondence between diagonal vectors of shifted λ-tableaux (λ is the shape of the tableaux) and lattice points of a certain polytope $\mathbf{P}_\lambda$. This polytope is a Minkowski sum of simplices in $\mathbb{R}^n$ and its combinatorial structure only depends on the length of the partition λ. In particular, if the length of λ is n, $\mathbf{P}_\lambda$ turns out to be combinatorially equivalent to the *associahedron* Ass_n. In Section 3.4 we describe the vertices of $\mathbf{P}_\lambda$ in terms of certain binary trees, and give a simple construction of the corresponding "extremal" λ-shifted tableaux.

Chapter 2

Permutohedra, Mixed Eulerian Numbers and beyond

2.1 Permutohedra

The classical *permutohedron* $P(x_1, \ldots, x_{n+1})$ is defined as the convex hull of the $(n+1)!$ points obtained by permuting the coordinates of the point $(x_1, ..., x_{n+1})$. According to G. Ziegler, permutohedra appeared for the first time in the work of Schoute in 1911 ([20]), though the term permutohedron was only coined much later. For generic $x_1, ..., x_{n+1}$, $P(x_1, ..., x_{n+1})$ is n-dimensional, lying in the hyperplane $t_1 + ... + t_{n+1} = x_1 + ... + x_{n+1}$. For a polytope $P \subseteq \mathbb{R}^{n+1}$ included in a hyperplane $t_1 + ... + t_{n+1} = c$, define its volume as the usual n-dimensional volume of the projection of P onto $t_{n+1} = 0$. Permutohedra as well as their various generalizations have been studied extensively in [16, 17]. In particular their combinatorial structure has been described in terms of certain posets – *building posets* – which will also appear in Chapter 3. Special cases of permutohedra appear as *graphical zonotopes* and *graph associahedra* in graph theory, as *moment polytopes* in algebraic geometry, as *alcoved polytopes* arising from affine Coxeter arrangements (see [13]), and as *weight polytopes* of fundamental representations of Lie groups. Of particular interest are invariants associated to them such as their volumes, numbers of lattice points or their Ehrhart polynomials. For example, $P(n, n-1, n-2, \ldots, 0)$ is the *graphical zonotope* corresponding to the complete graph K_{n+1}:

$$P(n, n-1, n-2, \ldots, 0) = \sum_{1 \leq i < j \leq n+1} [e_i, e_j]$$

In other words, it is the Minkowski sum of all the line segments $[e_i, e_j]$ between $e_i$and e_j, where e_i denotes the ith standard basis vector in $\mathbb{R}^{n+1}$. As such, a basic result of zonotope theory tells us that $n! Vol\,(P(n, n-1, n-2, ..., 0))$ is the number of spanning trees of K_{n+1} (see [22, Ex.4.32]), namely $(n+1)^{n-1}$, whereas its number of lattice points is the number of spanning forests of K_{n+1}.

An important example of a permutohedron is the *hypersimplex*

$$\Delta_{n+1,k} = P\left(1^k, 0^{n+1-k}\right)$$

(a^b means a sequence of b a's), which is the intersection of the unit hypercube with the hyperplane $t_1 + \cdots + t_{n+1} = k$. This is the *matroid polytope* corresponding to the uniform matroid of rank k consisting of all the $k-$subsets of $[n+1]$. It appears in algebraic geometry as the moment polytope of the toric variety $X_p = \overline{\mathbb{T}p}$ of a point $p \in Gr_{k,n+1}$ whose Plucker coordinates are all non-zero (the action of the torus $\mathbb{T} = (\mathbb{C}^*)^{n+1}$on $Gr_{k,n+1}$ is given by $(t_1, ..., t_{n+1}) \cdot (x_1, ..., x_{n+1}) = (t_1x_1, ..., t_{n+1}x_{n+1})$). As such, it is known that the normalized volume of $\Delta_{n+1,k}$ is the degree of X_p as a subvariety of $\mathbb{CP}^{\binom{n+1}{k}-1}$(see [9, 8]). On the other hand, it is well-known that

$$Vol(\Delta_{n+1,k}) = \frac{1}{n!}A_{n,k},$$

where $A_{n,k}$ is the number of permutations of size n with $k-1$ descents (i.e. $A_{n,k}$'s are the *Eulerian numbers*). This is a famous old result, dating back to Euler. However, the first published proof seems to be due to Laplace ([14]). A simple half-page proof constructing a triangulation of $\Delta_{n+1,k}$ into $A_{n,k}$ unit simplices was found by Stanley ([24]).

More recently, Postnikov has computed the volume of $P(x_1, \ldots, x_{n+1})$ explicitly as a homogeneous polynomial of degree n in $x_1, \ldots, x_{n+1}$. To state the theorem, recall that there is a natural bijection between sequences of nonnegative integers $c_1, \ldots, c_{n+1}$ such that $c_1 + \cdots + c_{n+1} = n$, and lattice paths in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) with "up" or "right" steps: The path L corresponding to $(c_1, ..., c_n)$ has c_i vertical steps along the line $x = i-1$. Let $I_{c_1 \ldots c_{n+1}} \subseteq [n]$ be the set of indices i such that both the $(2i-1)^{\text{th}}$ and $2i^{\text{th}}$ steps of L are below the $x = y$ axis (see Figure 2.1), and $D_n(I_{c_1 \ldots c_n})$ be the number of permutations in S_{n+1} with descent set $I_{c_1 \ldots c_n}$.

Theorem 2.1.1. *[16, Theorem 3.2] The volume of $P(x_1, \ldots, x_{n+1})$ is given by*

$$Vol(P(x_1, \ldots, x_{n+1})) = \sum_{c_1 + \ldots + c_{n+1} = n, c_i \geq 0} (-1)^{|I_{c_1 \ldots c_{n+1}}|} D_n(I_{c_1 \ldots c_{n+1}}) \frac{x_1^{c_1}}{c_1!} \cdots \frac{x_{n+1}^{c_{n+1}}}{c_{n+1}!}.$$

Example 2.1.2. The path in Figure 2.1 corresponds to the composition $(2,1,0,0,2,0,0,2)$. We have $I_{21002002} = \{4,6,7\}$ and there are $\binom{7}{3} \cdot 3 + \binom{7}{2} \cdot 2 = 189$ permutations in S_8 with descents in positions 4,6,7. Hence, by Theorem 1, the coefficient of $x_1^2 x_2 x_5^2 x_8^2$ in $Vol\left(P(x_1, ..., x_8)\right)$ is $-\frac{189}{8}$.

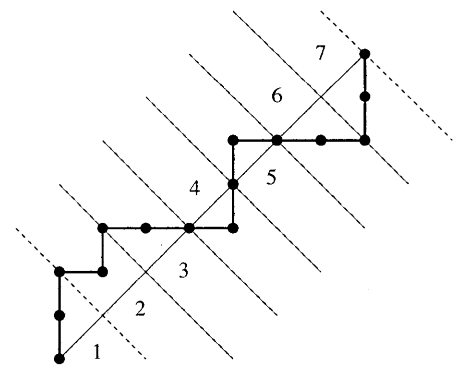


Figure 2.1.1: Lattice path corresponding to $(c_1, \ldots, c_8) = (2, 1, 0, 0, 2, 0, 0, 2)$

2.2 Classical Mixed Eulerian Numbers

Theorem 2.1.1 is a strong result, however it is not presented in a way that can be naturally generalized to other root systems. Let us instead introduce the new variables $u_1 = x_1 - x_2, u_2 = x_2 - x_3, \ldots, u_n = x_n - x_{n+1}$. We have the following Minkowski sum decomposition:

$$\begin{aligned} P(x_1, ..., x_{n+1}) &= P(u_1 + ... + u_n + x_{n+1}, ..., u_n + x_{n+1}, x_{n+1}) \\ &= u_1\Delta_{n+1,1} + u_2\Delta_{n+1,2} + \cdots + u_n\Delta_{n+1,n} + x_{n+1}(1, \ldots, 1) \end{aligned}$$

Since the Minkowski sum of Q and a point v is just Q translated by v, we may ignore the term $x_{n+1}(1, ..., 1)$ when taking volumes of both sides in above to obtain

$$\begin{aligned} Vol\,(P(x_1, ..., x_{n+1})) &= Vol\,(u_1\Delta_{n+1,1} + u_2\Delta_{n+1,2} + \cdots + u_n\Delta_{n+1,n}) \\ &= \sum_{(i_1,\ldots,i_n)\in[n]^n} Vol(\Delta_{n+1,i_1}, \ldots, \Delta_{n+1,i_n}) u_{i_1} \cdots u_{i_n}, \end{aligned} \qquad (2.2.1)$$

where $Vol\,(\Delta_{n+1,i_1}, \ldots, \Delta_{n+1,i_n})$ is the *mixed volume* of the polytopes $\Delta_{n+1,i_1}, \Delta_{n+1,i_2}, \ldots, \Delta_{n+1,i_n}$. The (Brunn-Minkowski) theory of mixed volumes of polytopes is one of the cornerstones of classical convexity theory, and was pioneered by Minkowski in [15]; the last equality in (2.2.1) is essentially a restatement of Minkowski's main theorem in the case of the hypersimplices. Mixed volumes of integer polytopes have important connections to algebraic geometry. For example, by a famous theorem of Bernstein, they count common zeroes of generic polynomials whose Newton polytopes are the given polytopes (see [2]). Computing mixed volumes of integer polytopes is very difficult in general, but for standard coordinate simplices Postnikov has managed to find combinatorial formulas by using Bernstein's result and ingenious linear algebra techniques ([16]). An excellent treatment of the Brunn-Minkowski theory is contained in [19]; see also [4] for formulas and inequalities involving mixed volumes. Since the mixed volume of n polytopes does not depend on their listed order, we may combine similar terms in (2.2.1) to obtain

$$Vol\,(P(x_1,\ldots,x_{n+1})) = \sum_{c_1+\ldots+c_n=n, c_i\geq 0} A_{c_1\ldots c_n}\frac{u_1^{c_1}}{c_1!}\cdots\frac{u_n^{c_n}}{c_n!}, \tag{2.2.2}$$

where $A_{c_1\ldots c_n} = n!Vol\left(\Delta_{n+1,1}^{c_1},\ldots,\Delta_{n+1,n}^{c_n}\right)$, and $\Delta_{n+1,i}^{c_i}$ denotes c_i copies of $\Delta_{n+1,i}$. The coefficients $A_{c_1\ldots c_n}$ are called the *(classical) mixed Eulerian numbers.* They are positive integers because hypersimplices are integer polytopes of full dimension (see [8]).

Example 2.2.1. The kth hypersimplex $\Delta_{n+1,k}$ is the Newton polytope of the k-th elementary symmetric polynomial in $x_1,...,x_{n+1}$. Bernstein's theorem says that A_{120} equals the number of distinct solutions in $\mathbb{CP}^3$ of the system

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0 \\ b_1x_1x_2 + b_2x_2x_3 + b_3x_3x_4 + b_4x_1x_3 + b_5x_2x_4 + b_6x_3x_4 = 0 \\ c_1x_1x_2 + c_2x_2x_3 + c_3x_3x_4 + c_4x_1x_3 + c_5x_2x_4 + c_6x_3x_4 = 0 \end{cases}$$

where a_i, b_j, c_k are generic.

It is natural to ask whether these coefficients have a simple combinatorial interpretation. This is one of the main problems that will be explored here. The question becomes more relevant given the following list of known results.

Theorem 2.2.2. *[16, 6, 24] The mixed Eulerian numbers have the folowing properties:*

(1) $A_{c_1\ldots c_n} = A_{c_n\ldots c_1}$.

(2) For $1 \leq k \leq n$, $A_{0^{k-1},n,0^{n-k}} = A_{n,k}$ *(the usual Eulerian number). Here and below* 0^l *denotes a sequence of* l *zeros.*

(3) $A_{k,0\ldots0,n-k} = \binom{n}{k}$.

(4) For $1 \leq k \leq n$, $i = 0,\ldots,n$ *the number* $A_{0^{k-1},n-i,i,0^{n-k-1}}$ *is the number of permutations* $w \in S_{n+1}$ *with* k *descents and* $w(n+1) = i+1$.

(5) If $c_1+\cdots+c_i \geq i$ *for* $i = 1,\ldots,n$ *then* $A_{c_1\ldots c_n} = 1^{c_1}2^{c_2}\ldots n^{c_n}$. *There are* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *such sequences* $(c_1,\ldots,c_n)$.

(6) Let $\sim$ *be the equivalence relation on sequences* $(c_1,\ldots,c_n)$ *given by* $(c_1,\ldots,c_n) \sim (d_1,\ldots,d_n)$ *if and only if* $(c_1,\ldots,c_n,0)$ *is a cyclic shift of* $(d_1,\ldots,d_n,0)$. *Then the sum of mixed Eulerian numbers in each equivalence class is* $n!$, *and the number of equivalence classes is* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(7) $\sum_{c_1\ldots c_n}\frac{1}{c_1!\cdots c_n!}A_{c_1c_2\ldots c_n} = (n+1)^{n-1}$.

Some comments about the above theorem are in place. Part (1) of the theorem follows because the volume-preserving isometry of $\mathbb{R}^{n+1}$ given by

$$(x_1, ..., x_n) \mapsto (1 - x_1, ..., 1 - x_n)$$

maps $\Delta_{n+1,i}$ to $\Delta_{n+1,n+1-i}$, $i = 1, \ldots, n$. Part (2) is essentially a restatement of the classical formula $n! \cdot Vol(\Delta_{n+1,k}) = A_{n,k}$. Part (4) was originally proved by Ehrenborg, Readdy and Steingrimsson in [6]. The idea behind their proof is that $u_k\Delta_{n+1,k}+u_{k+1}\Delta_{n+1,k+1}$ (the volume of which is $\sum_{i=0}^{n} A_{0^{k-1},n-i,i,0^{n-k-1}} u_k^{n-i} u_{k+1}^i$) turns out to be a slice of another cube, and so can more or less be handled directly. However it's not clear how one could generalize their method to compute other mixed Eulerian numbers (for example of form $A_{0\ldots0,a,b,c,0\ldots0}$), because the Minkowski sum of even three (rescaled) hypersimplices is, in general, a complicated polytope. Part (6) is an interesting result which was conjectured by Stanley and proved by Postnikov in [16, Theorem 16.4]. This claim has a simple geometric explanation in terms of alcoves of the affine Weyl group in type A. In some sense, it comes from symmetries of the extended Dynkin diagram of type A, which allow one to express the volume of the fundamental alcove (which is easy to compute) as a sum of volumes of n permutohedra. We will generalize this result in Section 2.6. It is known that there are C_n equivalence classes $\sim$ and each of them contains exactly one *Catalan sequence* $(c_1, \ldots, c_n)$ such that $c_1 + \cdots + c_i \geq i, \forall i = 1 \ldots n$. The right side in (7) is exactly the volume of the permutohedron $P(n, n-1, \ldots, 0)$, which as we have seen, is the number $(n+1)^{n-1}$ of spanning trees of K_{n+1}.

Problem 2.2.3. It is known that there are C_n equivalence classes $\sim$ and each of them contains exactly one *Catalan sequence* $(c_1, \ldots, c_n)$ such that $c_1 + \cdots + c_i \geq i, \ i = 1 \ldots n$. For each Catalan sequence $(c_1, \ldots, c_n)$, can we find some statistic on permutations of S_n (or S_{n+1}) whose distribution gives the mixed Eulerian numbers in the $\sim$-equivalence class of $(c_1, \ldots, c_n)$? For example, part (4) of 2.2.2 implies that for $i = 1, \ldots, n-1$ the mixed Eulerian numbers of the sequences in the $\sim$-class of $(n - i, i, 0^{n-2})$ are given by counting permutations $w \in S_{n+1}, w(n+1) = i+1$ with a fixed number of descents.

Rather than handle the (classical) mixed Eulerian numbers directly, we generalize the setup to other root systems, and then discuss particular cases.

2.3 Root Systems, Hypersimplices and Mixed Eulerian Numbers

Let Φ be a reduced root system of rank n spanning a real vector space V. Fix a choice of simple roots $\alpha_1, \ldots, \alpha_n$ in Φ. The roots are ordered in accordance with the standard labelling of the Dynkin diagram of Φ (see [12], p.58 for example). Let Λ be the associated weight lattice, W_Φ - the Weyl group of Φ, and $(\cdot,\cdot)$ - the correponding W_Φ-invariant inner product on V. Let $\lambda_1,\ldots,\lambda_n \in V$ be the fundamental dominant weights of Φ; they form the dual basis of the simple coroots $\frac{2\alpha_i}{(\alpha_i,\alpha_i)}$. For $x \in V$, we can define the *weight polytope* $P_\Phi(x)$ as the convex hull in V of the orbit $W_\Phi x$. Write $x = u_1\lambda_1 + \cdots + u_n\lambda_n,\ u_i \in \mathbb{R}$. Note that the coefficients u_i are given by

$$u_i = \left(x, \frac{2\alpha_i}{(\alpha_i,\alpha_i)}\right) \tag{2.3.1}$$

We have the following Minkowski sum decomposition

$$P_\Phi(x) = u_1 P_\Phi(\lambda_1) + \cdots + u_n P_\Phi(\lambda_n).$$

The polytopes $P_\Phi(\lambda_1),\ldots,P_\Phi(\lambda_n)$ are called the Φ-*hypersimplices.* Taking volumes of both sides we obtain:

$$Vol\,(P_\Phi(x)) = \sum_{c_1,\ldots,c_n\geq 0, c_1+\cdots+c_n=n} A^\Phi_{c_1\ldots c_n} \frac{u_1^{c_1}}{c_1!}\cdots\frac{u_n^{c_n}}{c_n!}, \tag{2.3.2}$$

where

$$A^\Phi_{c_1\ldots c_n} = n!\cdot Vol\,(P_\Phi(\lambda_1)^{c_1},\ldots,P_\Phi(\lambda_n)^{c_n})$$

is the mixed volume of c_1 copies of $P_\Phi(\lambda_1)$, c_2 copies of $P_\Phi(\lambda_2)$,..., c_n copies of $P_\Phi(\lambda_n)$. We define

$$V_\Phi(u_1,...,u_n) = \frac{1}{|\det(\alpha_1\ ...\ \alpha_n)|} Vol\,(P_\Phi(x)) = \frac{1}{|\det(\alpha_1\ ...\ \alpha_n)|} Vol\,(P_\Phi(u_1\lambda_1+...+u_n\lambda_n))$$

Thus, V_Φ is a homogeneous polynomial of degree $n = \text{rank}\Phi$. It is the volume of $P_\Phi(x)$, if the volume form were normalized so that the box spanned by $\alpha_1,...,\alpha_n$ had unit volume. The coefficients $A^\Phi_{c_1\ldots c_n}$ are called the *mixed Φ-Eulerian numbers.* As before, they are positive because the Φ-hypersimplices have full dimension.

The main purpose of this Chapter is to study the mixed Φ-Eulerian numbers. Postnikov has expressed these coefficients as complicated sums over weighted doubly-labelled trees [16]. Our purpose is to find simpler formulas, or combinatorial interpretations involving paths in the Dynkin diagrams, or the geometry of the root system Φ.

Example 2.3.1. $\Phi = \mathrm{A}_n$. Recall that the standard realization of A_n is given by $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, ..., \alpha_n = e_n - e_{n+1}$ inside the hyperplane $V = \{(x_1,...,x_{n+1}) | x_1 + ... + x_{n+1} = 0\}$. If P is the box generated by $\alpha_1,...,\alpha_n$ then the volume of the image of P under the projection $(x_1,...,x_{n+1}) \to (x_1,...,x_n)$ is

$$\det\begin{pmatrix} 1 & -1 & \cdots & & \\ & \ddots & \ddots & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} = 1$$

hence Vol on V agrees with the volume form defined in Section 2.1. The Weyl group is S_{n+1} which acts on $V \subset \mathbb{R}^{n+1}$ by permuting the coordinates. The fundamental dominant weights are $\lambda_i = (1^i, 0^{n+1-i}) - \frac{i}{n+1}(1,1,...,1)$, hence the Φ-hypersimplices are $P_{W_{A_n}}(\lambda_i) = P_{n+1}(1^i, 0^{n+1-i}) - \frac{i}{n+1}(1,...,1) = \Delta_{n+1,i} - \frac{i}{n+1}(1,...,1)$, i.e. the classical hypersimplices translated into V. Therefore, for $\Phi = \mathrm{A}_n$, $A^{\Phi}_{c_1 \ldots c_n}$ are just the classical mixed Eulerian numbers.

Lemma 2.3.2. *If Φ is the direct sum of two root systems Φ_1 and Φ_2, spanned by $\alpha_1, ..., \alpha_m$ and $\alpha_{m+1}, ..., \alpha_n$ respectively, then*

$$V_\Phi(u_1, ..., u_n) = V_{\Phi_1}(u_1, ..., u_m)\, V_{\Phi_2}(u_{m+1}, ..., u_n)$$

Proof. This follows because $P_\Phi(x)$ is the direct product of $P_{\Phi_1}(x_1)$ with $P_{\Phi_2}(x_2)$, where x_i is the projection of x onto the subspace of V spanned by Φ_i. □

2.4 Recursive Formulas For Mixed Eulerian Numbers

In this section we derive recursive formulas for the mixed Eulerian numbers. One of the main tools that we employ is the following result due to Postnikov.

Proposition 2.4.1. *[16, Proposition 18.6] For $i = 1, \ldots, n$, we have*

$$\frac{\partial}{\partial u_i} V_\Phi(u_1, \ldots, u_n) = \sum_{j=1}^{n} \frac{|W_\Phi|(\lambda_i, \lambda_j)}{|W_{\Phi_j}|(\alpha_j, \lambda_j)} V_{\Phi_j}(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n),$$

where $\Phi_j = \Phi - \{j\}$ is the root subsystem of Φ with node α_j removed.

Recall that the Cartan matrix associated to Φ is given by $A_\Phi = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}\right)_{1 \le i,j \le n} = (\alpha_1 \ldots \alpha_n)^T \left(\frac{2\alpha_1}{(\alpha_1, \alpha_1)} \;\cdots\; \frac{2\alpha_n}{(\alpha_n, \alpha_n)} \right)$. Since $\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$, it follows that

$$A_\Phi^{-1} = (\lambda_1 \ldots \lambda_n)^T (\lambda_1 \ldots \lambda_n) \begin{pmatrix} \frac{2}{(\alpha_1,\alpha_1)} & & \\ & \ddots & \\ & & \frac{2}{(\alpha_n,\alpha_n)} \end{pmatrix} = \left(\frac{2(\lambda_i, \lambda_j)}{(\alpha_j, \alpha_j)} \right)_{i,j}.$$

Now, letting $\overline{V}_\Phi = \frac{1}{|W_\Phi|} V_\Phi$, 2.3.2 can be rewritten as follows:

$$\begin{pmatrix} \partial/\partial u_1(\overline{V}_\Phi) \\ \vdots \\ \partial/\partial u_n(\overline{V}_\Phi) \end{pmatrix} = A_\Phi^{-1} \begin{pmatrix} \overline{V}_{\Phi_1}(u_2, ..., u_n) \\ \vdots \\ \overline{V}_{\Phi_n}(u_1, ..., u_{n-1}) \end{pmatrix},$$

or

$$A_\Phi \begin{pmatrix} \partial/\partial u_1(\overline{V}_\Phi) \\ \vdots \\ \partial/\partial u_n(\overline{V}_\Phi) \end{pmatrix} = \begin{pmatrix} \overline{V}_{\Phi_1}(u_2, ..., u_n) \\ \vdots \\ \overline{V}_{\Phi_n}(u_1, ..., u_{n-1}) \end{pmatrix}. \tag{2.4.1}$$

The polynomials $\overline{V}_\Phi$ are completely (over)determined by the recurrence (2.4.1) and the initial condition $\overline{V}_{A_1}(u_1) = \frac{1}{2}u_1$. Tables of polynomials $V_\Phi(u_1, ..., u_n)$ for irreducible root systems of rank up to 4 can be found in Appendix A. The table is missing V_{C_4}, but this polynomial is easily readable from V_{B_4} as the following observation shows.

Lemma 2.4.2. *For any n,*

$$V_{B_n}(u_1, ..., u_n) = 2V_{C_n}\left(u_1, ..., u_{n-1}, \frac{1}{2}u_n\right)$$

Proof. This follows since the root systems B_n and C_n are dual to each other. Indeed, assume $\alpha_1, ..., \alpha_n$ is a set of simple roots for B_n in some space V, (corresponding to the nodes of the Dynkin diagram). Then $\alpha_1, ..., \alpha_{n-1}, 2\alpha_n$ can be taken as a set of simple roots for a root system of type C_n in V. Let $x \in V$. Since the Weyl groups of B_n and C_n are the same (as subgroups of $GL(V)$), we have $P_{B_n}(x) = P_{C_n}(x)$. By definition, the volume of $P_{B_n}(x)$ is

$$|\det(\alpha_1 \, ... \, \alpha_n)| V_{B_n}(u_1, ..., u_n)$$

where $u_i = \left(x, \frac{2\alpha_i}{(\alpha_i,\alpha_i)}\right)$. Similarly, the volume of $P_{C_n}(x)$ is

$$|\det(\alpha_1 \, ... \, \alpha_{n-1} \; 2\alpha_n)| V_{C_n}(u_1, ..., u_{n-1}, v_n)$$

where now $v_n = \left(x, \frac{2(2\alpha_n)}{(2\alpha_n,2\alpha_n)}\right) = \frac{1}{2}u_n$. Equating the two volumes implies the result. □

Theorem 2.4.3. *For any Φ, the mixed Φ-Eulerian numbers are positive integers.*

Proof. We have seen that $A^\Phi_{c_1...c_n} > 0$. We show that $A^\Phi_{c_1...c_n} \in \mathbb{Z}$, by using the recursion (2.4.1). This recursion implies that for each i, $\frac{\partial}{\partial u_i}V_\Phi$ is a linear combination of the polynomials $V_{\Phi_1}(u_2, ..., u_n), ..., V_{\Phi_n}(u_1, ..., u_{n-1})$ with coefficients of the form $\frac{1}{\det A_\Phi} \cdot \frac{|W_\Phi|}{|W_{\Phi_j}|}$. It is straightforward to show that these numbers are always integers by using explicit tables for $\det A_\Phi$ and $|W_\Phi|$ (see [12, p.66,68]). For example, suppose $\Phi = E_7$. Then

$$\Phi_j \in D_6, A_6, A_1 \oplus A_5, A_2 \oplus A_1 \oplus A_3, A_4 \oplus A_2, D_5 \oplus A_1, E_6$$

hence

$$\det A_\Phi |W_{\Phi_j}| \in 2\cdot 2^5 6!, 2\cdot 7!, 2\cdot 2!6!, 2\cdot 3!2!4!, 2\cdot 5!3!, 2\cdot 2^4 5!2!, 2\cdot 2^7 3^4 5$$

which always divides $|W_{\mathrm{E}_7}| = 2^{10}3^4 5\cdot 7$. Thus, $\frac{\partial}{\partial u_i} V_\Phi$ is an integer linear combination of $V_{\Phi_1}, ..., V_{\Phi_n}$. Integrating in u_i, it follows that for $c_i > 0$, the coefficient $A^\Phi_{c_1 \dots c_n}$ of $\frac{u_1^{c_1}\dots u_n^{c_n}}{c_1!\cdots c_n!}$ in V_Φ is an integer linear combination of numbers of the form $A^{\Phi_j}_{d_1\dots d_{n-1}}$. Hence the result follows by induction on the rank n of Φ (we have already seen this result for $\Phi = \mathrm{A}_n$). □

Rather than deal with the entries $(A_\Phi)_{i,j}$ directly, it's more convenient to consider the natural weight function wt on the edges of the Dynkin diagram of Φ, defined as follows: $wt(i \to j)$ is $\frac{1}{2}$ times the number of edges in the diagram from i to j, where undirected edges count both ways. For example, in type G_2, there are 3 directed edges from node 2 to node 1, so we set $wt(1 \to 2) = 1/2$ and $wt(2 \to 1) = 3/2$. In types A, D, E all edges are undirected so $wt(i \to j) = 1/2$ if i and j are connected, and 0 otherwise. In other words, $wt(i \to j) = 2\,(A_\Phi)_{i,j}$. The function wt will also appear later in Section 2.8. Next, given a weak composition $(c_1, ..., c_n)$ of n, we identify it with the weight function $w : i \mapsto c_i$ on the nodes of the Dynkin diagram. We write A^Φ_w for $\frac{1}{|W_\Phi|} A^\Phi_{c_1\dots c_n}$. Let $w_{i\to j}$ denote the labelling which is identical to w except $w_{i\to j}(i) = w(i) - 1, w_{i\to j}(j) = w(j) + 1$.

Theorem 2.4.4. *Fix i such that $w(i) > 0$. The mixed Eulerian numbers satisfy the following recursive formula*

$$A^\Phi_w - \sum_{i,j-\text{connected}} wt(i \to j) A^\Phi_{w_{i\to j}} = \frac{1}{2} A^{\Phi-\{i\}}_{w|_{\Phi-\{i\}}}. \tag{2.4.2}$$

Proof. The result follows by comparing the coefficients of $\frac{u_1^{c_1}}{c_1!}\cdots\frac{u_i^{c_i-1}}{(c_i-1)!}\cdots\frac{u_n^{c_n}}{c_n!}$ in the ith equation of (2.4.1). □

Looking at these recursions, it seems that one could generalize the mixed Eulerian numbers to graphs. However, they overdetermine A^Φ_w, and one can show that the only simple connected graphs which admit such positive coefficients come from the Dynkin diagrams.

Consider the case $\Phi = \mathrm{A}_n$. Then $\Phi - \{i\} = \mathrm{A}_{i-1} \oplus \mathrm{A}_{n-i}$. Let $d_1, ..., d_n \geq 0$ be integers with $d_1 + ... + d_n = n - 1$. Applying Theorem 2.4.4 for w corresponding to the composition $d_1, ..., d_i + 1, ..., d_n$ of n, we obtain

Proposition 2.4.5. *For every nonnegative integers $d_1, ..., d_n$ which sum to $n - 1$, we have*

$$\begin{aligned} &2A_{d_1\dots d_{i-1}, d_i+1, d_{i+1}\dots d_n} - A_{d_1\dots, d_{i-1}+1, d_i, \dots d_n} - A_{d_1\dots d_i, d_{i+1}+1, d_{i+2}\dots d_n} \\ &= \begin{cases} \binom{n+1}{i} A_{d_1\dots d_{i-1}} \cdot A_{d_{i+1}\dots d_n} & \text{if } d_1 + ... + d_{i-1} = i-1,\ d_i = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.4.3}$$

These recurrences (together with $A_1 = 1$) determine all of the mixed Eulerian numbers uniquely. On the other hand, Proposition 2.4.1 gives us directly another set of recurrences which characterize the coefficients $A_{c_1 \dots c_n}$. Following the setup of Example 2.3.1, we have $(\lambda_i, \lambda_j) = i - \frac{ij}{n+1}$ for $i < j$. Therefore

$$\frac{\partial}{\partial u_i} V_{\mathrm{A}_n}(u_1, \dots, u_n) = \sum_{j=1}^{n} \left(\min\{i, j\} - \frac{ij}{n+1} \right) V_{\mathrm{A}_{j-1}}(u_1, \dots, u_{j-1}) V_{\mathrm{A}_{n-j}}(u_{j+1}, \dots, u_n)$$

To find $A_{c_1 \dots c_n}$, choose any i such that $c_i > 0$. Extracting the coefficient of $\frac{u_1^{c_1}}{c_1!} \cdot \ldots \cdot \frac{u_{i-1}^{c_{i-1}}}{c_{i-1}!} \cdot \frac{u_i^{c_i - 1}}{(c_i - 1)!} \cdot \frac{u_{i+1}^{c_{i+1}}}{c_{i+1}!} \cdot \ldots \cdot \frac{u_n^{c_n}}{c_n!}$ on both sides of the last equation, we obtain (formally)

$$\begin{aligned} \frac{1}{(n+1)!} A_{c_1 \dots c_n} &= \sum_{1 \le j < i} (j - \frac{ij}{n+1}) \frac{1}{j!} \cdot \frac{1}{(n-j+1)!} A_{c_1 \dots c_{j-1}} \cdot A_{c_{j+1} \dots c_{i-1}(c_i - 1) \dots c_n} \\ &+ \sum_{i \le j \le n} (i - \frac{ij}{n+1}) \frac{1}{j!} \cdot \frac{1}{(n-j+1)!} A_{c_1 \dots c_{i-1}(c_i - 1) \dots c_{j-1}} \cdot A_{c_{j+1} \dots c_n} \end{aligned}$$

However, remember that $A_{c_1 \dots c_n}$ is only defined when $c_1 + \dots + c_n = n$, so we have to set it equal to 0 otherwise. Therefore in the above sums, the terms for $j < i$ only appear when $c_1 + \dots + c_{j-1} = j - 1$ and $c_j = 0$, and similarly, the terms for $j \ge i$ only appear when $c_j = 0$ and $c_1 + \dots + c_j = j$ (with the exception for the case $i = j$, in which case the restrictions are $c_i = 1, c_1 + \dots + c_i = i$). We call such indices j *good*. Therefore,

$$\begin{aligned} A_{c_1 \dots c_n} &= \sum_{j < i,\ j-good} \frac{j(n+1-i)}{n+1} \binom{n+1}{j} A_{c_1 \dots c_{j-1}} \cdot A_{c_{j+1} \dots c_{i-1}(c_i - 1) \dots c_n} \\ &+ \sum_{j \ge i, j-good} (n+1)(i - \frac{in}{n+1}) A_{c_1 \dots (c_i - 1) \dots c_{j-1}} \cdot A_{c_{j+1} \dots c_n} \end{aligned}$$

or

$$\begin{aligned} A_{c_1 \dots c_n} &= (n-i+1) \sum_{j < i, j-good} \binom{n}{j-1} A_{c_1 \dots c_{j-1}} \cdot A_{c_{j+1} \dots c_{i-1}(c_i - 1) \dots c_n} \qquad (2.4.4) \\ &+ i \sum_{j \ge i, j-good} \binom{n}{j} A_{c_1 \dots (c_i - 1) \dots c_{j-1}} \cdot A_{c_{j+1} \dots c_n}. \end{aligned}$$

In particular, by choosing i to be maximal such that $c_i > 0$ in the last formula, we obtain the following result.

Theorem 2.4.6. *Let $c_1, ..., c_n$ be a composition of n, and suppose $i = \max\{j | c_j > 0\}$. Then the classical mixed Eulerian number $A_{c_1 \ldots c_n}$ is given recursively, by*

$$A_{c_1 \ldots c_n} = (n - i + 1) \sum_{j<n, j-good} \binom{n}{j-1} A_{c_1 \ldots c_{j-1}} \cdot A_{c_{j+1} \ldots c_{i-1}(c_i - 1) \ldots c_n} + i A_{c_1 \ldots (c_i - 1) \ldots c_{n-1}}, \tag{2.4.5}$$

where the last term only appears if $c_n \leq 1$. This recurrence together with $A_1 = 1$ determines all the mixed Eulerian numbers uniquely.

2.5 Examples.

In this section we apply Theorem 2.4.6 to give new proofs of parts (2)-(5) of Theorem 2.2.2.

Example 2.5.1. (*volumes of hypersimplices*) Let $A(n, k) = A_{0 \ldots n \ldots 0}$, where n is in position k. Thus $A(n, k)$ is the normalized volume of the kth hypersimplex in $\mathbb{R}^{n+1}$, i.e. the volume of the slice of the unit cube in $\mathbb{R}^n$ lying inside $k - 1 \leq x_1 + \ldots + x_n \leq k$. We have $A(1, 1) = 1$ and for $n \geq 2$ 2.4.5 becomes $A(n, k) = (n - k + 1) \cdot A(n - 1, k - 1) + kA(n - 1, k)$, where the last term appears unless $k = n$. This is exactly the recurrence characterizing the Eulerian numbers, hence $A(n, k)$ is the number of permutations in S_n with $k - 1$ descents. Thus we recover Euler's famous result.

Example 2.5.2. (*mixed volumes of the opposite hypersimplices*) Let $A(n, k) = A_{k0 \ldots 0(n-k)}$ for $k = 0, 1, ..., n$. We have $A(n, 0) = A(0, n) = 1$. Theorem 2.4.6 implies $A(n, k) = \binom{n}{k} A_{k0 \ldots 0} A_{0 \ldots (n-k-1)} = \binom{n}{k}$ for $0 < k < n - 1$, and $A(n, n - 1) = nA_{(n-1)0 \ldots 0} = n$. Therefore, in all cases we have $A(n, k) = \binom{n}{k}$, as claimed in part (3) of Theorem 2.2.2.

Example 2.5.3. (*mixed volumes of two adjacent hypersimplices*) Let

$$A(n, i, k) = A_{0 \ldots i, n-i, \ldots 0}$$

with $n - i$ in position k. We have $A(n, n, k) = A(n, k - 1), A(n, 0, k) = A(n, k)$ and for $n \geq 2, 0 < i < n$ the recurrence 2.4.5 becomes $A(n, i, k) = (n-k+1)A(n-1, i, k-1) + kA(n-1, i, k)$.

Claim. $A(n, i, k)$ equals the number of permutations $\pi \in S_{n+1}$ with $k - 1$ descents such that $\pi(n + 1) = n - i + 1$.

Proof. For $i = 0$ or $i = n$ the result follows easily from the previous section. It remains to show that the number $B(n, i, k)$ of permutations $\pi \in S_{n+1}$ with $k - 1$ descents and last coordinate $n - i + 1$ satisfies the same recurrence as $A(n, i, k)$. Consider such a permutation π

written as a sequence of numbers. If we remove 1 from π, and then decrease all the remaining digits by 1, we obtain a new permutation $\tau \in S_n$ such that $\tau(n) = n - i$ and τ has $k-1$ or $k-2$ descents. If τ has $k-1$ descents then 1 must have been inserted in a descent position of $\pi - \{1\}$, or in the beginning. If τ has $k-2$ descents, 1 must have been inserted in one of the $n-1-(k-2) = n-k+1$ ascent positions of $\pi - \{1\}$. $\square$

This establishes part (4) of Theorem 2.2.2.

Example 2.5.4. Suppose $c_1, ..., c_n$ satisfies $c_1 + ... + c_i \geq i, \forall i = 1, ..., n$. Let $i = \max\{j | c_j > 0\}$. Since there are no *good* indices $j < i$ and $c_n = n - (c_1 + ... + c_{n+1}) \leq 1$, we have $A_{c_1 \dots c_n} = iA_{c_1 \dots c_{i-1}(c_i - 1) \dots c_{n-1}}$. An easy inductive argument implies $A_{c_1 \dots c_n} = i^{c_i}(i-1)^{c_{i-1}} \dots 1^{c_1 - 1} A_1$, hence part (5) of Theorem 2.2.2 follows.

Theorem 2.4.6 gives another characterization of the mixed Eulerian numbers in type A. It's not obvious at all that 2.4.3 and 2.4.5 are equivalent recursions. The set of good indices of a composition has a simple geometrical interpretation. There is a natural bijection between n-tuples $c_1, ..., c_n$ of non-negative integers which sum to n, and plane lattice paths S between $(1,1)$ and $(n+1, n+1)$ with "up" and "right" steps. Fix i such that $c_i > 0$. Let's modify S by moving 1 unit down the part of S which lies to the right of the $x = i$. Call the new path S'. It is easy to see that the set of good indices j is precisely the set of x coordinates of points where S' crosses the diagonal $x = y$.

Example 2.5.5. The lattice path in Figure 2.5.1 corresponds to the composition $(c_1, \dots, c_8) = (1,0,3,0,0,1,3,0)$. For $i = 7$, the set of good indices is $\{2, 5\}$. In this case formula 2.4.5 gives

$$A_{10300130} = 2\left(\binom{8}{1} A_1 A_{300120} + \binom{8}{4} A_{1030} A_{120}\right) + 7A_{1030012}.$$

Using 2.4.6 repeatedly, we find $A_{1030} = 2\left(\binom{4}{1} A_1 A_{20}\right) + 3A_{102} = 2 \cdot 4 \cdot 1 \cdot 1 + 3\binom{3}{1} = 17$ and $A_{1030012} = \binom{7}{1} A_1 A_{30011} + \binom{7}{4} A_{1030} A_{11} = 7 \cdot 1^3 \cdot 4 \cdot 5 + \binom{7}{4} \cdot 17 \cdot 2! = 1330$, and finally $A_{10300130} = 16 \cdot 1^3 \cdot 4 \cdot 5^2 + 140 \cdot 17 \cdot 1 \cdot 2^2 + 7 \cdot 1330 = 20430$.

The mixed Eulerian numbers include the factorials, binomial coefficients, numbers of permutations with various restrictions, numbers of the form $1^{c_1} \dots n^{c_n}$. While finding a simple closed formula for $A_{c_1 \dots c_n}$ is unlikely (there is no such formula already for $A_{0 \dots 0, k, n-k, 0 \dots 0}$), it seems reasonble to try the following

Problem 2.5.6. Find a way to label the n vertical segments of the path S with numbers $1, \dots, n$ with certain order restrictions depending on how S behaves (e.g. how S crosses the diagonal $x = y$), such that the number of labelings is $A_{c_1, \dots, c_n}$.

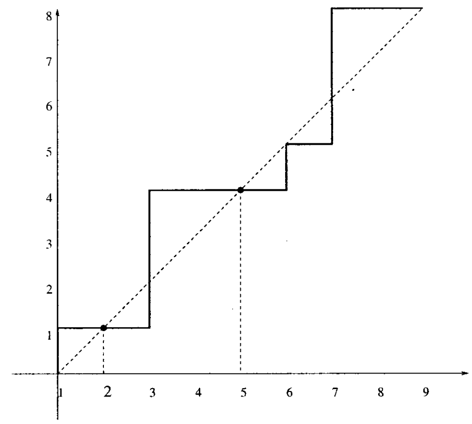


Figure 2.5.1: The lattice path corresponding to the composition $(c_1, \ldots, c_8) = (1, 0, 3, 0, 0, 1, 3, 0)$.

2.6 A cyclic formula for the volumes of weight polytopes

In this section we investigate the geometry of alcoves in the affine Coxeter arrangement of a root system Φ to obtain a generalization of Theorem 2.2.2(6). Our approach is similar to Postnikov's in [16, Proposition 16.6], and uses symmetries of extended Dynkin diagrams. Recalling the setup of Section 2.3, we introduce additional notation. It is well-known that there is a well-defined *highest root* $\alpha_{n+1} = m_1\alpha_1 + \cdots + m_n\alpha_n$, where $m_1, ..., m_n$ are positive integers and $m_1 + ... + m_n$ is the *height* of α_{n+1}. We also let $m_{n+1} = 1$. See [3, Chapter 6] for more on these coefficients.

Theorem 2.6.1. *Let $\overline{\Phi}_i$ denote the root system in V spanned by $\{\alpha_j | j \neq i\}$. For any $u_1, ..., u_{n+1}$ we have*

$$\sum_{i=1}^{n+1} \frac{m_i}{|W_{\overline{\Phi}_i}|} V_{\overline{\Phi}_i}(u_1, ..., \hat{u}_i, ..., u_{n+1}) = \frac{|\det(\alpha_1 \ldots \alpha_n)|^{-2}}{n! m_1 \cdots m_n} \left(\sum_{i=1}^{n+1} \frac{m_i (\alpha_i, \alpha_i)}{2} u_i \right)^n \tag{2.6.1}$$

Proof. Recall that the *affine Coxeter arrangement of* Φ in V consists of all hyperplanes of the form

$$(\alpha_i, x) = k, \ k \in \mathbb{Z}, i = 1, 2, ..., n+1.$$

These hyperplane arrangements as well as polytopes arising from them have been studied (mostly in type A_n) in [18, 13]. These hyperplanes subdivide V into regions called *alcoves.* The reflections in these hyperplanes generate the *affine Weyl group* $\overline{W}_\Phi$, which is the semidirect product of W_Φ(reflections fixing the origin) and $\mathbb{Z}^n$(translations) . The *fundamental* *alcove* A_0^Φ is given by

$$A_0^\Phi = \{y \in V | 0 \leq (\alpha_i, y), \forall i \leq n, (-\alpha_{n+1}, y) \leq 1\}$$

It is a simplex with vertices $v_1, ..., v_{n+1}$ given by $v_{n+1} = 0$ and $v_i = \frac{2}{m_i(\alpha_i,\alpha_i)}\lambda_i$ for $i \leq n$ (the latter points lying on $(-\alpha_{n+1}, y) = 1$). Hence its volume is

$$\begin{aligned} VolA_0^\Phi &= \frac{1}{n!m_1 \cdots m_n} \det \left(\frac{2\lambda_1}{(\alpha_1,\alpha_1)} \quad \cdots \quad \frac{2\lambda_n}{(\alpha_n,\alpha_n)} \right) \\ &= \frac{1}{n!m_1 \cdots m_n} \det \left((\alpha_1 \; ... \; \alpha_n)^{-1} \right)^T = \frac{1}{n!m_1 \cdots m_n \det (\alpha_1 \; ... \; \alpha_n)}. \end{aligned} \tag{2.6.2}$$

On the other hand, consider any point $x = u_1\lambda_1 + ... + u_n\lambda_n$ in the interior of A_0^Φ. The $\overline{W}_\Phi$-orbit of x has a unique point in each alcove. If we look at the elements of $\overline{W}_\Phi x$ closest to v_i (i.e. the ones in the alcoves adjacent to v_i), they are the vertices of the weight polytope P_i (centered at v_i) corresponding to the root system $\overline{\Phi}_i$ in V generated by the roots $\{\alpha_j | j \neq i\}$ (also centered at v_i). The walls passing through v_i subdivide P_i into $|W_{\overline{\Phi}_i}|$ congruent pieces, one of which is $P_i \cap A_0^\Phi$. Figure 2.6.1 illustrates the geometry in type B$_2$: The shaded region is the fundamental alcove, and there are 3 adjacent weight polytopes at x). Therefore,

$$VolA_0^\Phi = \sum_{i=1}^{n+1} \frac{VolP_i}{|W_{\overline{\Phi}_i}|} \tag{2.6.3}$$

Now, how do we compute $Vol(P_i)$ as a polynomial in $u_1, ..., u_n$? Recall that $x = u_1\lambda_1 + ... + u_n\lambda_n$ implies $u_i = \frac{2(x,\alpha_i)}{(\alpha_i,\alpha_i)}$, i.e. we only need x and the simple roots in order to know where to evaluate $V_\Phi(u_1, ..., u_n)$. We let $u_{n+1} = \frac{2}{(\alpha_{n+1},\alpha_{n+1})}((x, \alpha_{n+1}) + 1)$ for convenience, so that

$$\sum_{i=1}^{n+1} \frac{m_i (\alpha_i, \alpha_i)}{2} u_i = 1 \tag{2.6.4}$$

Since P_i is centered at v_i, the coefficients of the linear expansion of x in terms of the fundamental dominant weights in $\overline{\Phi}_i$ are

$$w_j = \frac{2(x - v_i, \alpha_j)}{(\alpha_j, \alpha_j)} = u_j, \forall j \neq i$$

(This follows easily since $(v_i, \alpha_j) = 0$ for $j \neq n+1$ and $(v_i, \alpha_{n+1}) = 1; i \neq j$). The volume of the parallelotope formed by the α_j's ($j \neq i$) is

$$|\det (\alpha_1 \; ...\overline{\alpha}_i... \; \alpha_{n+1})| = m_i |\det (\alpha_1 \; ... \; \alpha_n)|,$$

Therefore,

$$Vol(P_i) = m_i |\det(\alpha_1 \ldots \alpha_n)| V_{\overline{\Phi}_i}(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}).$$

Combining the latter with (2.6.2) and (2.6.3) we arrive at

$$\frac{1}{n! m_1 \cdots m_n} = \sum_{i=1}^{n+1} \frac{m_i}{|W_{\overline{\Phi}_i}|} \det(\alpha_1 \ldots \alpha_n)^2 V_{\overline{\Phi}_i}(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1})$$

The last formula holds for all $u_1, \ldots, u_{i+1}$ satisfying (2.6.4). Since the right side is a homogeneous polynomial of degree n in $u_1, \ldots, u_{n+1}$, we obtain

$$\det(\alpha_1 \ldots \alpha_n)^2 \sum_{i=1}^{n+1} \frac{m_i}{|W_{\overline{\Phi}_i}|} V_{\overline{\Phi}_i}(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}) = \frac{1}{n! m_1 \cdots m_n} \left(\sum_{i=1}^{n+1} \frac{m_i (\alpha_i, \alpha_i)}{2} u_i \right)^n$$

for any $u_1, \ldots, u_{n+1}$. The theorem is proved. □

Equation (2.6.1) can be somewhat simplified, by using the following classical formula for the size of the Weyl group of an *irreducible* root system (see [3, Proposition 7 on p.190]):

$$|W_\Phi| = n! m_1 m_2 \ldots m_n \det A_\Phi$$

We have

$$\begin{aligned} \det A_\Phi &= \det \left[(\alpha_1 \ldots \alpha_n)^T \left(\tfrac{2\alpha_1}{(\alpha_1, \alpha_1)} \ \cdots \ \tfrac{2\alpha_n}{(\alpha_n, \alpha_n)} \right) \right] \\ &= \det(\alpha_1 \ldots \alpha_n)^2 \prod_{i=1}^{n} \frac{2}{(\alpha_i, \alpha_i)} \end{aligned}$$

With these last formulas, equation (2.6.1) becomes

$$\prod_{i=1}^{n} (\alpha_i, \alpha_i) \sum_{i=1}^{n+1} \frac{m_i}{|W_{\overline{\Phi}_i}|} V_{\overline{\Phi}_i}(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}) = \frac{1}{|W_\Phi|} \left(\sum_{i=1}^{n+1} m_i (\alpha_i, \alpha_i) u_i \right)^n,$$

for any irreducible root system Φ. In types A, D, E all roots have the same length, hence the latter simplifies to

$$\sum_{i=1}^{n+1} m_i \overline{V}_{\overline{\Phi} i}(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}) = \frac{1}{|W_\Phi|} \left(\sum_{i=1}^{n+1} m_i u_i \right)^n$$

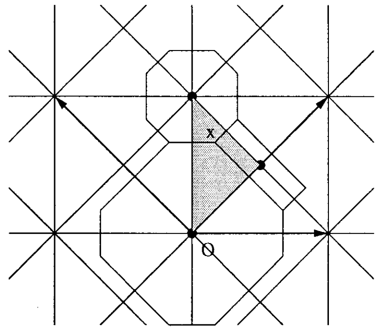


Figure 2.6.1: Alcoves in the affine Coxeter arrangement of type B_2.

2.7 Generalization To Positive Definite Matrices

Let Φ be an affine root system with Cartan matrix A_Φ. Then equation 2.4.1 can be rewritten as

$$A_\Phi \begin{pmatrix} \partial/\partial u_1 V_\Phi \\ \vdots \\ \partial/\partial u_n V_\Phi \end{pmatrix} = \begin{pmatrix} V_{\Phi_1}(u_2, ..., u_n) \\ \vdots \\ V_{\Phi_n}(u_1, .., u_{n-1}) \end{pmatrix}.$$

Since $V_\Phi(u_1, ..., u_n)$ is a homogeneous polynomial of degree $n = \mathrm{rank}(\Phi)$, we obtain

$$nV_\Phi = \sum_{j=1}^{n} u_j \frac{\partial}{\partial u_j} V_\Phi = [u_1 \ ... \ u_n] \, A_\Phi^{-1} \begin{bmatrix} V_{\Phi-\{1\}}(u_2, ..., u_n) \\ \vdots \\ V_{\Phi-\{n\}}(u_1, ..., u_{n-1}) \end{bmatrix}.$$

Since the Cartan matrix of $\Phi - \{j\}$ is just the jth principal minor of A_Φ, the last equation motivates the following construction:

Definition 2.7.1. Let A be a positive definite n by n matrix. We define the homogeneous polynomial $P_A(u_1, ..., u_n)$ by the following recursion:

$$P_A(u_1, ..., u_n) = [u_1 \ ... \ u_n] \, A^{-1} \begin{bmatrix} P_{A_{11}}(u_2, ..., u_n) \\ \vdots \\ P_{A_{nn}}(u_1, ..., u_{n-1}) \end{bmatrix}, \quad P_{[a]}(u) = \frac{u}{a}. \tag{2.7.1}$$

By the above discussion, we have

Theorem 2.7.2. *Let Φ be any root system, and A_Φ its Cartan matrix. Then*

$$P_{A_\Phi}(u_1, ..., u_n) = n! \overline{V}_\Phi (u_1, ..., u_n).$$

We also provide a new direct proof of Theorem 2.7.2.

Proof of Theorem 2.7.2. Consider the point $x = u_1\lambda_1 + ... + u_n\lambda_n$ in the weight lattice of Φ. We may assume without any loss that x is dominant. The Weyl chambers divide the polytope $P_{W_\Phi}(x)$ into $|W_\Phi|$ congruent subpolytopes; denote by P the piece containing x. For $i = 1, 2, ..., n$ let P_i be the projection of x onto the hyperlane orthogonal to α_i, and let Q_i be the projection of x onto the line spanned by λ_i. One can see that the points $x, P_1, ..., P_{i-1}, Q_i, P_{i+1}, ..., P_n$ lie on the hyperplane H_i which passes through x and is orthogonal to λ_i. Thus, these points together with the origin 0 form a pyramid with base $xP_1...P_{i-1}Q_iP_{i+1}...P_n$ and altitude $0Q_i$, and P decomposes into n such pyramids. Now, the projection to the affine hyperplane H_i (considered as having the origin at Q_i), naturally induces a root system structure isomorphic to $\Phi - \{i\}$ on H_i: the simple roots are α_j, and the fundamental dominant weights are just the projections λ'_j of λ_j onto H_i, $j \neq i$. Since projections are linear transformations, $x = u_1\lambda_1 + ... + u_n\lambda_n$ implies $x = x' = u_1\lambda'_1 + ... + u_{i-1}\lambda'_{i-1} + u_{i+1}\lambda'_{i+1} + ... + u_n\lambda'_n$, and one can see that $xP_1...P_{i-1}Q_iP_{i+1}...P_n$ is one of the $|W_{\Phi-\{i\}}|$ congruent pieces of $P_{W_{\Phi-\{i\}}}(x')$, the weight polytope of x' with the respect to the root system $\Phi_i = \Phi - \{i\}$ in H_i. Therefore,

$$Vol\,(xP_1...P_{i-1}Q_iP_{i+1}...P_n) =$$

$$\begin{aligned} &= \frac{1}{|W_{\Phi_i}|} Vol(P_{W_{\Phi_i}}(u_1\lambda'_1 + ... + u_{i-1}\lambda'_{i-1} + u_{i+1}\lambda'_{i+1} + ... + u_n\lambda'_n)) \\ &= V_{\Phi_i}(u_1, ...\hat{u}_i..., u_n)\frac{Vol(\alpha_1, ...\hat{\alpha}_i..., \alpha_n)}{Vol(\alpha_1, ..., \alpha_n)}. \end{aligned}$$

The last quotient is 1 over the length of the projection of α_i onto λ_i (because λ_i is the orthogonal complement of the hyperplane spanned by $\alpha_j, j \neq i$). Thus

$$\frac{Vol(\alpha_1, ...\hat{\alpha}_i..., \alpha_n)}{Vol(\alpha_1, ..., \alpha_n)} = 1/\left\|\frac{(\alpha_i, \lambda_i)}{(\lambda_i, \lambda_i)}\lambda_i\right\| = \frac{2}{(\alpha_i, \alpha_i)}\|\lambda_i\|.$$

Therefore, the volume of the ith pyramid $0xP_1...P_{i-1}Q_iP_{i+1}...P_n$ is

$$\begin{aligned} \frac{1}{n}\|0Q_i\|\, Vol(xP_1...P_{i-1}Q_iP_{i+1}...P_n) &= \frac{1}{n}\left\|\frac{(x, \lambda_i)}{(\lambda_i, \lambda_i)}\lambda_i\right\| \cdot \frac{2}{(\alpha_i, \alpha_i)}\|\lambda_i\|\, V_{\Phi_i}(u_1, ...\hat{u}_i..., u_n) \\ &= \frac{2(x, \lambda_i)}{(\alpha_i, \alpha_i)n} V_{\Phi_i}(u_1, ...\hat{u}_i..., u_n). \end{aligned}$$

Summing the last expression over $i = 1, ..., n$ we obtain the volume of P:

$$V_\Phi(u_1, ..., u_n) = \sum_{i=1}^{n} \frac{2(x, \lambda_i)}{(\alpha_i, \alpha_i)n} V_{\Phi_i}(u_1, ...\hat{u}_i..., u_n),$$

which upon multiplying both sides by $n!$ can be written as

$$\begin{aligned} P_{A_\Phi}(u_1, ..., u_n) &= \sum_{i=1}^{n} \frac{2(x, \lambda_i)}{(\alpha_i, \alpha_i)} P_{A_{\Phi_i}}(u_1, ...\hat{u_i}..., u_n) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_j \frac{2(\lambda_j, \lambda_i)}{(\alpha_i, \alpha_i)} P_{A_{\Phi_i}}(u_1, ...\hat{u_i}..., u_n) \\ &= [u_1 \; ... \; u_n]\, A_\Phi^{-1} \begin{bmatrix} P_{A_{\Phi_1}}(u_2, ..., u_n) \\ \vdots \\ P_{A_{\Phi_n}}(u_1, ..., u_{n-1}) \end{bmatrix}, \text{ as claimed.} \end{aligned}$$

□

We now derive some basic properties of the polynomials $P_A(u_1, ..., u_n)$.

Proposition 2.7.3. *Suppose A is a block diagonal matrix, $A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$ with block sizes $n_1, \ldots, n_k$. Then*

$$P_A(u_1, \ldots, u_n) = \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} P_{A_1}(u_1, \ldots, u_k) \cdots P_{A_k}(u_{n_1+\cdots+n_{k-1}+1}, \ldots, u_{n_1+\cdots+n_k}).$$

Proof. The general case follows from the case $k = 2$ by induction. Henceforth, we assume $A = \begin{pmatrix} B & \\ & C \end{pmatrix}$ where A, B, C are of size $n, m, n-m$ respectively. We induct on n, the case $n = 1$ being trivial. By using the inductive hypothesis and the recurrence formula for the polynomials P_A, we have

$$
\begin{aligned}
P_A &= [u_1 \ldots u_n]\begin{pmatrix} B^{-1} & \\ & C^{-1}\end{pmatrix}\begin{bmatrix} P_{A_{11}}(u_2,\ldots,u_n) \\ \vdots \\ P_{A_{nn}}(u_1,\ldots,u_{n-1})\end{bmatrix} \\
&= [u_1 \ldots u_n]\begin{pmatrix} B^{-1} & \\ & C^{-1}\end{pmatrix}\begin{bmatrix} P_{B_{11}}P_C\binom{n-1}{m-1} \\ \vdots \\ P_{B_{mm}}P_C\binom{n-1}{m-1} \\ \vdots \\ P_B P_{C_{n-m,n-m}}\binom{n-1}{m}\end{bmatrix} \\
&= [u_1 \ldots u_m]\, B^{-1}\begin{bmatrix} P_{B_{11}}(u_2,\ldots,u_m) \\ \vdots \\ P_{B_{mm}}(u_1,\ldots,u_{m-1})\end{bmatrix} P_C(u_{m+1},\ldots,u_n)\cdot\binom{n-1}{m-1} + \\
&+ [u_{m+1} \ldots u_n]\, C^{-1}\begin{bmatrix} P_{C_{11}}(u_{m+2},\ldots,u_n) \\ \vdots \\ P_{C_{n-m,n-m}}(u_{m+1},\ldots,u_{n-1})\end{bmatrix} P_B(u_1,\ldots,u_m)\cdot\binom{n-1}{m} \\
&= P_B(u_1,\ldots,u_m)P_C(u_{m+1},\ldots,u_n)\cdot\left(\binom{n-1}{m-1}+\binom{n-1}{m}\right) \\
&= \binom{n}{m}P_B(u_1,\ldots,u_m)P_C(u_{m+1},\ldots,u_n).
\end{aligned}
$$

□

Corollary 2.7.4. *If* $A = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n\end{pmatrix}$, *then* $P_A = \frac{n!}{d_1 \cdots d_n} u_1 \cdots u_n$.

Proposition 2.7.5. *The coefficient of* $u_1 \cdots u_n$ *in* P_A *is* $\frac{n!}{|A|}$. *In particular, for any root system* Φ, *the mixed volume of the* n Φ*-hypersimplices is* $A^{\Phi}_{1\ldots1} = \frac{|W_\Phi|}{\det A_\Phi}$.

Proof. If we define $Q_A = |A| \cdot P_A$, then 2.7.1 implies that the coefficient of $u_1 \cdots u_n$ in Q_A satisfies the recurrence $[u_1 \cdots u_n]\, Q_A = \sum_{j=1}^{n} [u_1 \cdots \hat{u_j} \cdots u_n] Q_{A_{jj}}$ and $[u_1]Q_{[a]} = 1$. Therefore $[u_1 \cdots u_n]Q_A = n!$. The second part of the Lemma follows since the coefficient of $u_1 \cdots u_n$ in $Vol\,(P_{W_\Phi})$ is $A^{\Phi}_{1\ldots1}$. □

Thus, the determinant of A is encoded by the coefficient of $u_1 \cdots u_n$ in P_A. It would be interesting to find out what other invariants associated to A are represented by coefficients of $P_A(u_1, \ldots, u_n)$.

For an n by n matrix A, and two sequences of numbers $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ we denote by $|A_{a_1 \ldots a_k, b_1 \ldots b_k}|$ the minor of A corresponding to deleting rows $a_1, \ldots, a_k$ and columns $b_1, \ldots, b_k$. We write $|A_{a_1 \ldots a_k}|$ instead of $|A_{a_1 \ldots a_k, a_1 \ldots, a_k}|$.

Theorem 2.7.6. *We have*

$$P_A = \sum_{\pi,\sigma} (-1)^{\binom{n+1}{2}+\sigma+i(\pi)+i(\pi,\sigma)} \frac{|A_{\sigma_1,\pi_1}| \cdot |A_{\pi_1\sigma_2,\pi_1\pi_2}| \cdots |A_{\pi_1\ldots\pi_{n-1}\sigma_n,\pi_1\ldots\pi_n}|}{|A| \cdot |A_{\pi_1}| \cdots |A_{\pi_1\ldots\pi_{n-1}}|} u_{\sigma_1} \cdots u_{\sigma_n} \tag{2.7.2}$$

where the sum is over all permutations $\pi \in S_n$ and sequences $\sigma = (\sigma_1, \ldots, \sigma_n) \in [n]^n$ such that $\sigma_k \neq \pi_1, \ldots, \pi_{k-1}$. Here $i(\pi, \sigma)$ denotes the number of (π, σ)-inversions, i.e. the number of pairs of indices $(i, j), i < j$ such that $\sigma_j < \pi_i$.

Proof. The theorem follows by repeatedly applying the recurrence relation. Indeed, 2.7.1 can be rewritten as

$$P_A = \sum_{i,j=1}^{n} P_{A_i}(u_1 \ldots \hat{u_i} \ldots u_n)(-1)^{i+j} \frac{|A_{j,i}|}{|A|} u_j. \tag{2.7.3}$$

Next, we have $P_{A_i} = \sum_{k,l=1;k.l\neq i}^{n} P_{A_{ik}} \cdot (-1)^{k+l+\epsilon} \frac{|A_{il,ik}|}{|A_i|} u_l$, where ϵ is 0 if $k, l > i$ or $k, l < i$ and 1 otherwise (this ϵ accounts for the fact that we haven't changed the labelling of the rows and columns of A_i). Let's perform one more step: $P_{A_{ik}} = \sum_{r,s=1,r,s\neq i,k}^{n} P_{A_{ikr}} \cdot (-1)^{r+s+\delta} \frac{|A_{iks,ikr}|}{|A_{ik}|} u_s$. Again, δ compesates for the fact that the rows and columns of A_{ik} are still labelled as in A, hence δ is the number of pairs among $(i, r), (k, r), (i, s), (k, s)$ which are in order ((a, b) is *in order* if $a < b$). Now it's easy to see that if we apply 2.7.3 recursively $n - 1$ times, we obtain exactly the right side of 2.7.2 except that the exponent of -1 is $\pi_1 + \cdots + \pi_n + \sigma_1 + \cdots + \sigma_n + \xi + \mu$, where ξ is the number of pairs $(i < j)$ such that $\pi_i < \pi_j$, and μ is the number of pairs $(i < j)$ such that $\pi_i < \sigma_j$. In other words, the exponent of -1 is $1 + 2 + \cdots + n + \sigma_1 + \cdots + \sigma_n + \binom{n}{2} - i(\pi) + \binom{n}{2} - i(\pi, \sigma) \equiv \binom{n+1}{2} + \sigma_1 + \cdots + \sigma_n + i(\pi) + i(\pi, \sigma)$ (*mod* 2). The proof is complete. □

It's interesting to note that similar expressesions to the right side of equation 2.7.2 appear in the famous work in non-commutative algebra on *quasi-determinants* by Gelfand and Retakh ([10]).

Corollary 2.7.7. *The coefficient of u_i^n in $P_A(u_1, \ldots, u_n)$ is*

$$\sum_{\pi \in S_n, \pi_n = i} (-1)^{\binom{n+1}{2}+ni+i(\pi)+\epsilon(\pi)} \frac{|A_{i,\pi_1}| \cdot |A_{\pi_1 i,\pi_1\pi_2}| \cdots |A_{\pi_1\ldots\pi_{n-1}i,\pi_1\ldots\pi_n}|}{|A| \cdot |A_{\pi_1}| \cdots |A_{\pi_1\ldots\pi_{n-1}}|},$$

where $\epsilon(\pi)$ is the number of ordered pairs $(r < s)$ such that $\pi_r > i$.

2.8 Weighted Paths in Dynkin Diagrams

While Theorem 2.7.2 gives an explicit formula for the polynomials $P_A(u_1, ..., u_n)$, it is difficult to compute the coefficients of P_A (or even check whether they are positive). However, in the case of a Cartan matrix $A = A_\Phi$, one can perform a trick which expresses the entries of A_Φ^{-1} in terms of weighted paths in the Dynkin diagram of Φ as follows. We define the weight of each edge of Φ to be $1/2$. For example, in type G_2, there are 3 directed edges from node 2 to node 1, so we set $wt(1 \to 2) = 1/2$ and $wt(2 \to 1) = 3/2$. In types A, D, E all edges are unlabelled so $wt(i \to j) = 1/2$ if i and j are connected, and 0 otherwise. The weight of a (directed) path is defined as the product of weights of its edges.

Theorem 2.8.1. *For any root system Φ, the Φ-mixed Eulerian numbers are given by*

$$A^\Phi_{c_1...c_n} = \frac{|W|c_1!...c_n!}{2^n n!} \sum_{\pi=\pi_1...\pi_n \in S_n} \sum_{P_1,...,P_n} wt(P_1)...wt(P_n) \tag{2.8.1}$$

where the sum is over all (directed) paths $P_i : \pi_i \to \sigma_i$ in the Dynkin diagram such that P_i avoids $\pi_1, ..., \pi_{i-1}$, and such that exactly c_j of these paths end at j.

Proof. Consider the Cartan matrix $A = A_\Phi = \left(\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}\right)_{i,j} = 2(I - B_\Phi)$, where the ij entry of B_Φ is just $wt(i \to j)$. Then $A^{-1} = \frac{1}{2}(I + B_\Phi + B_\Phi^2 + ...)$. It follows that the ij entry of A_Φ^{-1} is $\frac{1}{2}\left(\delta_{ij} + (B_\Phi)_{ij} + (B_\Phi^2)_{ij} + ...\right) = \frac{1}{2}\sum_P wt(P)$, where the sum is over all paths P in Φ starting at i and ending at j (the term $(B_\Phi^k)_{ij}$ is the sum of weights of such paths of length k). More generally, we may consider any principal minor $A_{i_1...i_k} = 2(I - B_{i_1...i_k})$ and by a similar argument we obtain $\left(A^{-1}_{i_1...i_k}\right)_{ij} = \frac{1}{2}\left(\delta_{ij} + (B_{i_1...i_k})_{ij} + (B^2_{i_1...i_k})_{ij} + ...\right) = \frac{1}{2}\sum_Q wt(Q)$, where the sum is over all paths Q in Φ from i to j *avoiding* $i_1, ..., i_k$. Now, recall that $\frac{1}{c_1!...c_n!}A^\Phi_{c_1...c_n}$ is the coefficient of $u_1^{c_1}...u_n^{c_n}$ in $Vol\left(P_{W_\Phi}(u_1\lambda_1 + \cdots + u_n\lambda_n)\right) = \frac{|W_\Phi|}{n!}P_{A_\Phi}(u_1, ..., u_n)$ (cf. Theorem 3). Proceeding as in the proof of 2.7.6, we have

$$\begin{aligned} P_A(u_1, ..., u_n) &= \sum_{\pi_1=1}^{n}\sum_{\sigma_1=1}^{n} P_{A_{\pi_1}}(u_1, ...\hat{u_{\pi_1}}..., u_n)\left(A^{-1}\right)_{\pi_1\sigma_1} u_{\sigma_1} \\ &= \sum_{\pi_1,\sigma_1;\pi_2,\sigma_2\neq\pi_1} P_{A_{\pi_1\pi_2}}(u_1, ...\hat{u_{\pi_1}}\hat{u_{\pi_2}}...u_n)\left(A^{-1}_{\pi_1}\right)_{\pi_2\sigma_2}\left(A^{-1}\right)_{\pi_1\sigma_1} u_{\sigma_2}u_{\sigma_1} \\ &= \cdots \\ &= \sum_{\pi\in S_n,\sigma\in[n]^n} \left(A^{-1}\right)_{\pi_1\sigma_1}\left(A^{-1}_{\pi_1}\right)_{\pi_2\sigma_2}\cdots\left(A^{-1}_{\pi_1...\pi_{n-1}}\right)_{\pi_n\sigma_n} u_{\sigma_1}...u_{\sigma_n}, \end{aligned} \tag{2.8.2}$$

where the sum is over all $\pi \in S_n, \sigma \in [n]^n$ such that $\sigma_i \neq \pi_1, ..., \pi_{i-1}$. By the above discussion,

$$\left(A^{-1}\right)_{\pi_1\sigma_1}\left(A^{-1}_{\pi_1}\right)_{\pi_2\sigma_2}\cdots\left(A^{-1}_{\pi_1...\pi_{n-1}}\right)_{\pi_n\sigma_n} = \sum_{P_1,...,P_{n-1}} \frac{1}{2^n}wt(P_1)...wt(P_{n-1}),$$

where the sum is over all collections of directed paths $P_i : \pi_i \to \sigma_i$ such that P_i avoids $\pi_1, ..., \pi_{i-1}$. We may as well include the weight 1 of the empty path $P_n : \pi_n \to \sigma_n = \pi_n$ (the only path avoiding $\pi_1, ..., \pi_{n-1}$), to each product in the sum. The theorem now easily follows by extracting the coefficients of $u_1^{c_1}...u_n^{c_n}$ on both sides of 2.8.2. □

Remark. By reversing all the paths in 2.8.1, one gets a similar formula for $A^{\Phi}_{c_1...c_n}$ where $P_i : \sigma_i \to \pi_i$ avoids $\pi_1, ...\pi_{i-1}$, and c_j of these paths *start* at j.

Example 2.8.2. Let's illustrate Theorem 2.8.1 by computing the usual mixed-Eulerian number A_{030}. Here, the root system is $\mathbf{A}_3$, all 4 edge weights of the Dynkin diagram are $\frac{1}{2}$. We are interested in triples of paths $P_1 : 2 \to \pi_1, P_2 : 2 \to \pi_2, P_3 : 2 \to \pi_3 = 2$ with P_2 avoiding π_1 (and P_3 is just the empty path which we may ignore). There are 2 possibilities ($\pi_1 = 1, \pi_2 = 3$, and vice versa) which, by symmetry, yield the same contribution to A_{030}. Consider pairs of paths $P_1 : 2 \to 1, P_2 : 2 \to 3$, with P_2 avoiding 1. There is one path P_2 of each odd length k, hence $\sum_{P_2} wt(P_2) = \frac{1}{2} + \frac{1}{8} + ... = \frac{2}{3}$. If a_k is the number of paths P_1 of length k, then a_k satisfies $a_k = 2a_{k-2}, a_1 = 1$. Thus, $a_k = 2^{\frac{k-1}{2}}$ for odd k, and $\sum_{P_1} wt(P_1) = \sum_{k=2n+1 \geq 1} \frac{a_k}{2^k} = \sum_{n \geq 0} \frac{1}{2^{n+1}} = 1$. Therefore, Theorem 2.8.1 implies

$$A_{030} = 2 \cdot \frac{4!0!3!0!}{2^3 3!} \cdot \frac{2}{3} \cdot 1 = 4,$$

which, in accordance with Euler's classical result, is also the number of permutations in S_3 with 2 descents.

Chapter 3

Shifted Young Tableaux

This chapter is based on [5]. In this chapter, we study vectors formed by entries on the diagonal of standard Young tableaux of shifted shapes. We will establish a connection between such vectors and the lattice points of certain generalized permutohedra which are Minkowski sums of coordinate simplices.

3.1 Shifted Young diagrams and tableaux

Definition 3.1.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition into (at most) n parts. The *shifted Young diagram of shape* λ (or just λ*-shifted diagram*) is the set

$$D_\lambda = \left\{(i,j) \in \mathbb{R}^2 | 1 \leq j \leq n, j \leq i \leq n + \lambda_j\right\}.$$

We will think of D_λ as a collection of boxes with $n+1-i+\lambda_i$ boxes in row i, for $i = 1, 2, \ldots, n$ and such that the leftmost box of the i^{th} row is also in the i^{th} column. A *shifted standard Young tableau shape* λ (or just λ*-shifted tableau*) is a bijective map $T : D_\lambda \to \{1, \ldots, |D_\lambda|\}$ which is increasing in the rows and columns, i.e. $T(i,j) < T(i,j+1), T(i,j) < T(i+1,j)$ ($|D_\lambda| = \binom{n+1}{2} + \lambda_1 + \cdots + \lambda_n$ is the number of boxes in D_λ). The *diagonal vector* of such a tableau T is $diag(T) = (T(1,1), T(2,2), \ldots, T(n,n))$.

Figure 3.1.1 shows an example of a shifted standard Young tableau for $n = 4, \lambda = (4,2,1,0)$. Its diagonal vector is $(1,4,7,17)$.

We are interested in describing the possible diagonal vectors of λ-shifted Young tableaux. The problem was solved in the case $\lambda = (0,0,\ldots,0)$ (the empty partition) by A. Postnikov, in [16, Section 15]. Specifically, it was shown that diagonal vectors of the shifted triangular shape $D_\emptyset$ are in bijection with lattice points of the $(n-1)$-dimensional *associahedron* Ass_{n-1}(to be defined in section 2). Moreover, a simple explicit construction was given for the "extreme" diagonal vectors, i.e. the ones corresponding to the vertices of Ass_{n-1}. We use similar

1	2	3	5	8	9	12	13
	4	6	10	11	16		
		7	14	15			
			17				

Figure 3.1.1: A λ-shifted Young tableau of shape $\lambda = (4, 2, 1, 0)$.

techniques to generalize Postnikov's results to arbitrary shifted shapes.

3.2 A generating function for diagonal vectors of shifted tableaux

For a non-negative integer vector $(a_1, ..., a_n)$, let $N_\lambda(a_1, \dots, a_n)$ be the number of standard λ-shifted tableaux T such that $T(i+1, i+1) - T(i,i) - 1 = a_i$ for $i = 1, \dots, n$ and where we set $T(n+1, n+1) = \binom{n+1}{2} + \lambda_1 + \cdots + \lambda_n + 1$.

Theorem 3.2.1. *We have the following identity:*

$$\sum_{a_1,\dots,a_n \geq 0} N_\lambda(a_1, \dots, a_n) \frac{t_1^{a_1}}{a_1!} \cdots \frac{t_n^{a_n}}{a_n!} =$$

$$= \frac{1}{\prod_{i=1}^n (\lambda_i + n - i)!} \cdot \prod_{1 \leq i < j \leq n} (t_i + \cdots + t_{j-1}) \cdot s_\lambda(t_1 + \cdots + t_n, t_2 + \cdots + t_n, \dots, t_n)$$

where s_λ denotes the Schur symmetric polynomial associated to λ.

Proof. Consider a vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ with $x_1 > x_2 > ... > x_n$. Define the polytope

$$P_\lambda(\mathbf{x}) = \{(p_{ij})_{(i,j) \in D_\lambda} \mid 0 \leq p_{ij} \geq p_{i(j+1)},\ p_{ij} \geq p_{(i+1)j},\ p_{ii} = x_i\}.$$

By definition, $P_\lambda(\mathbf{x})$ is exactly the section of the order polytope of shape $D_\lambda$where the values along the main diagonal are $x_{1,\dots}, x_n$. If $\lambda = \emptyset$, this polytope is known as the *Gelfand-Tsetlin polytope*, which has important connections to finite-dimensional representations of $\mathfrak{gl}_n(\mathbb{C})$ (see [11]). Our proof strategy is to compare two different formulas for the volume of $P_\lambda(\mathbf{x})$, one of which is more direct and the other is a summation over standard λ-shifted Young tableaux. On the one hand, by [1, Proposition 12] we have

$$vol(P_\lambda(\mathbf{x})) \quad = \quad \frac{1}{\prod_{i=1}^n (\lambda_i + n - i)!} \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j) \cdot s_\lambda(\mathbf{x}). \tag{3.2.1}$$

Baryshnikov and Romik proved this result directly by an inductive argument (on the number of boxes plus the number of parts of λ) using iterated integrations. On the other hand, there is a natural map ϕ from $P_\lambda(\mathbf{x})$ to the set of standard λ-shifted Young tableaux defined as follows: Let $\mathbf{p} = (p_{ij})_{(i,\, j) \in D_\lambda} \in P_\lambda(\mathbf{x})$ be a point such that $p_{ij} = p_{i'j'} \Leftrightarrow (i,\, j) = (i',\, j')$. Arrange the p_{ij}'s in decreasing order and define the tableau $T = \phi(\mathbf{p})$ by writing k in box $(i,\, j)$ if p_{ij} is the k^{th} element in the above list. By the definition of $P_\lambda(\mathbf{x})$, it is clear that T is a standard λ-shifted Young tableau. Given standard λ-shifted tableau T with diagonal vector $diag(T) = \{d_1, \dots,\, d_n\}$, it is easy to see that $\phi^{-1}(T)$ is isomorphic to the set

$$\{(y_i) \in \mathbb{R}^{|T|} \mid y_1 > y_2 > \cdots > y_{|T|} > 0,\; y_{d_i} = x_i\}$$

which is the direct product of (inflated) simplices

$$\{x_1 = y_1 > y_2 \cdots > y_{d_2 - 1} > x_2\} \times \cdots \times \{x_n = y_{d_n} > y_{d_n+1} \cdots > y_{|T|} > 0\}$$

Therefore,

$$vol(\phi^{-1}(T)) = \frac{(x_1 - x_2)^{a_1}}{a_1!} \cdot \cdots \cdot \frac{(x_{n-1} - x_n)^{a_{n-1}}}{a_{n-1}!} \cdot \frac{x_n^{a_n}}{a_n!}.$$

Summing over all λ-shifted tableaux T, we obtain

$$\begin{aligned} vol(P_\lambda(\mathbf{x})) &= \sum_T vol(\phi^{-1}(T)) \\ &= \sum_{a_1, \dots, a_n \geq 0} N_\lambda(a_1, \dots, a_n) \frac{(x_1 - x_2)^{a_1}}{a_1!} \cdot \cdots \cdot \frac{(x_{n-1} - x_n)^{a_{n-1}}}{a_{n-1}!} \cdot \frac{x_n^{a_n}}{a_n!}. \end{aligned}$$

Comparing the last formula to (3.2.1), and making the substitutions $t_1 = x_1 - x_2, \dots, t_{n-1} = x_{n-1} - x_n, t_n = x_n$ we obtain the identity in the theorem. □

3.3 A bijection between diagonal vectors and lattice points of $\mathbf{P}_\lambda$

In this section we recall the setup from [16, Section 6]. Let $n \in \mathbb{N}$ and let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. For a subset $I \in \{1, 2, \dots, n\}$, let $\Delta_I = Conv\{e_i | i \in I\}$, which is a coordinate simplex of dimension $|I| - 1$. A class of *generalized permutohedra* is given by polytopes in $\mathbb{R}^n$ of the form

$$P_n^y(\{y_I\}) = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} y_I \Delta_I$$

In other words, $P_n^y(\{y_I\})$ is the Minkowski sum of the simplices Δ_I rescaled by $y_I \geq 0$. It is straightforward to see that if $y_I = y_J$, whenever $|I| = |J|$, then $P_n^y(\{y_I\})$ is the classical permutohedron $P_n(z_{[n]}, z_{[n-1]}, ..., z_{\{1\}})$, where

$$z_{[n]} = \sum_{I \subseteq [n]} y_I, z_{[n-1]} = \sum_{I \subseteq [n-1]} y_I, \ldots, z_{\{1\}} = y_{\{1\}}.$$

An extensive study of generalized permutohedra, including their combinatorial structures, volumes, numbers of lattice points was carried by Postnikov and others in [16, 17]. One particular example of a generalized permutohedron, the *associahedron*, in its *Loday realization*, is defined as

$$\mathrm{Ass}_n = \sum_{1 \leq i \leq j \leq n} \Delta_{[i,j]}$$

It is also known as the *Stasheff polytope* [21], and it has generalizations to any Lie type via cluster algebras (see [7], for example).

Proposition 3.3.1. *For any subsets $I_1, \ldots, I_k \subseteq [n]$, and any non-negative integers $a_1, \ldots, a_n$, the coefficient of $t_1^{a_1} \cdots t_n^{a_n}$ in*

$$\prod_{j=1}^{k} \left(\sum_{i \in I_j} t_i \right) \tag{3.3.1}$$

is non-zero if and only if $(a_1, \ldots, a_n)$ is an integer lattice point of the polytope $\sum_{j=1}^k \Delta_{I_j}$.

Proof. It's easy to see that the coefficient of $t_1^{a_1} \cdots t_n^{a_n}$ in (3.3.1) is non-zero if and only if $(a_1, \ldots, a_n)$ can be written as a sum of vertices of the simplices $\Delta_{I_1}, \ldots, \Delta_{I_k}$. By [16, Proposition 14.12], this happens if and only if $(a_1, \ldots, a_n)$ is a lattice point of $\sum_{j=1}^k \Delta_{I_j}$. □

Proposition 3.3.2. *The coefficient of $t_1^{a_1} \cdots t_n^{a_n}$ in $s_\lambda(t_1 + \cdots + t_n, t_2 + \cdots + t_n, \ldots, t_n)$ is non-zero if and only if $(a_1, \ldots, a_n)$ is a lattice point of the polytope $\lambda_1 \Delta_{[1,n]} + \lambda_2 \Delta_{[2,n]} + \cdots + \lambda_n \Delta_{\{n\}}$.*

Proof. Recall that

$$s_\lambda(t_1 + \cdots + t_n, t_2 + \cdots + t_n, \ldots, t_n) \quad = \quad \sum_T (t_1 + \cdots + t_n)^{w_1} \cdots t_n^{w_n}, \tag{3.3.2}$$

where the sum ranges over all *semi-standard* Young tableaux T of shape λ and weight $\mathbf{w} = (w_1, \ldots, w_n)$, i.e. w_i is the number of i's appearing in T (see [23]). Let T be a SSYT of shape λ and weight $\mathbf{w}$. Then $w_1 + \cdots + w_i \leq \lambda_1 + \cdots + \lambda_i$, $\forall i = 1 \ldots n$, because if we consider the boxes containing the numbers $1, 2, \ldots, i$ in T, there can be no more than i of them in the same column. Hence the number of such boxes is at most the size of the first i rows in the Young diagram of λ, which is $\lambda_1 + \cdots + \lambda_i$.

It follows that any monomial $t_1^{a_1}\cdots t_n^{a_n}$ appearing in $(t_1+\cdots+t_n)^{w_1}\cdots t_n^{w_n}$ also appears in $(t_1+\cdots+t_n)^{\lambda_1}\cdots t_n^{\lambda_n}$. On the other hand, $(t_1+\cdots+t_n)^{\lambda_1}\cdots t_n^{\lambda_n}$ does appear in the right side of (3.3.2) as the term corresponding to the tableau T with 1's in the first row, 2's in the second row, etc. Therefore, the coefficient of $t_1^{a_1}\cdots t_n^{a_n}$ in $s_\lambda(t_1+\cdots+t_n, t_2+\cdots+t_n,\ldots,t_n)$ is non-zero if and only if it is non-zero in $(t_1+\cdots+t_n)^{\lambda_1}\cdots t_n^{\lambda_n}$, which by Proposition 3.3.1, is non-zero if and only if $(a_1,\ldots,a_n)$ is a lattice point of $\lambda_1\Delta_{[1,n]}+\lambda_2\Delta_{[2,n]}+\cdots+\lambda_n\Delta_{\{n\}}$. □

We now have all the tools to establish the first main result of this chapter.

Theorem 3.3.3. *The number of (distinct) diagonal vectors of λ-shifted Young tableaux is equal to the number of lattice points of the polytope*

$$\mathbf{P}_\lambda := \sum_{1\le i\le j\le n-1}\Delta_{[i,j]}+\lambda_1\Delta_{[1,n]}+\lambda_2\Delta_{[2,n]}+\cdots+\lambda_n\Delta_{\{n\}}.$$

Proof. By Theorem 3.2.1, and Propositions 3.3.1, 3.3.2 it follows that $N_\lambda(a_1,\ldots,a_n)\neq 0$ if and only if $(a_1,\ldots,a_n)$ is an integer lattice point of the polytope

$$\sum_{1\le i\le j\le n-1}\Delta_{[i,j]}+\lambda_1\Delta_{[1,n]}+\lambda_2\Delta_{[2,n]}+\cdots+\lambda_n\Delta_{\{n\}}.$$

□

In particular, if λ has n parts (i.e. $\lambda_n>0$), we see that $\mathbf{P}_\lambda$ is combinatorially equivalent to Ass_n.

3.4 Vertices of $\mathbf{P}_\lambda$ and extremal Young tableaux

In what follows we describe the vertices of the polytope $\mathbf{P}_\lambda$ by using techniques developed in [16]. Given a generalized permutohedron $P_n^y(\{y_I\})=\sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} y_I\Delta_I$, assume that its *building set* $B=\{I\subseteq[n] \mid y_I>0\}$ satisfies the following conditions:

1. If $I,J\in B$ and $I\cap J\neq\emptyset$, then $I\cup J\in B$.
2. B contains all singletons $\{i\}$, for $i\in[n]$.

A *B-forest* is a rooted forest F on the vertex set $[n]$ such that

1. For any i, $\mathrm{desc}(i,F)\in B$. Here and below, $\mathrm{desc}(i,F)$ denotes the set of *descendants* of i in F (including i).
2. There are no $k\ge 2$ distinct incoparable nodes $i_1,\ldots,i_k$ in F such that

$$\bigcup_{j=1}^{k}\mathrm{desc}(i_j,F)\in B$$

3. $\{\mathrm{desc}(i,F)|i\text{- root of } F\} = \{I \in B | I\text{−maximal}\}$.

We will need the following result of Postnikov:

Proposition 3.4.1. *[16, Proposition 7.9] Vertices of $P_n^y(\{y_I\})$ are in bijection with B-forests. More precisely, the vertex $v_F = (t_1, \ldots, t_n)$ of $P_n^y(\{y_I\})$ associated with a B-forest F is given by $t_i = \sum_{J \in B: i \in J \subseteq \mathrm{desc}(i,F)} y_J$, for $i \in [n]$.*

Remark 3.4.2. It's not hard to see that Proposition 3.4.1 remains essentially true even if we allow the building set B not to contain the singletons $\{i\}$. This is because a term of the form $y_{\{i\}}\Delta_{\{i\}} = u_{\{i\}}e_i$ in a Minkowski sum just translates the other Minkowski summand.

The combinatorial structure of $\mathbf{P}_\lambda$ clearly only depends on its building set, i.e. the number of non-zero parts of the partition λ. Assume λ has k positive parts, so that the building set of $\mathbf{P}_\lambda$ is

$$B_{n,k} = \{[i,j] | 1 \le i \le j \le n-1\} \cup \{[i,n] | 1 \le i \le k\}.$$

We first deal with the case $k = n$. Let T be a plane binary tree on n nodes. For a node v of T, denote by L_v, R_v the left and right branches at v. There is a unique way to label the nodes of T such that for any node v, its label is greater than all labels in L_v and smaller than all labels in R_v. This labelling is called the *binary search labelling* of T.

Proposition 3.4.3. *[16, Proposition 8.1] The $B_{n,n}$-forests are exactly plane binary trees on n nodes with the binary search labeling.*

If $k = 0$, then the building set of $\mathbf{P}_\lambda$ is the same as $B_{n-1,n-1}$ hence $B_{n,0}$-forests are plane binary trees on $n-1$ nodes. The rest of the theory for $k = 0$ is the same as for the case $k = n$, but with n replaced by $n-1$.

Assume now $k \ge 1$. Let T be a B_n-forest. It's easy to see that $\mathrm{desc}(x,T)$ has form $[a,n]$ if and only if the path from the root to x always goes to the right. In this case, $\mathrm{desc}(x,T) = [x - |L_x|, n]$. We want to check when $\mathrm{desc}(x,T) \in B_k$. This will happen if and only if $x - |L_x| \le k$ or $x - |L_x| = n$ (cf. Remark 3.4.2). But $x - |L_x|$ increases as x moves down to the right starting from the root of the tree, and $x - |L_x| = n$ can only happen when $x = n$ and $|L_x| = 0$. It follows that $\{\mathrm{desc}(x,T) | x \in [n]\} \subseteq B_k, \forall x$ if and only if n

$$\{\mathrm{desc}(x,T)|x \in [n]\} \subseteq B_k \Leftrightarrow \begin{cases} n-1-|L_{n-1}| \le k \text{ and } |L_n| = 0 \\ n - |L_n| \le k \text{ and } |L_n| > 0 \end{cases}$$

This argument together with Proposition 3.4.3 implies

Proposition 3.4.4. *Let $k \geq 1$. The B_k-forests are exactly plane binary trees on n nodes with the binary search labeling and such that either $|L_n| \geq \max\{n-k, 1\}$, or $|L_n| = 0$ and $|L_{n-1}| \geq n-1-k$.*

Corollary 3.4.5. *For $1 \leq k \leq n$, the number of vertices of $\mathbf{P}_\lambda$ is*

$$(C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{k-1} C_{n-k}) + (C_0 C_{n-2} + \cdots + C_{k-1} C_{n-1-k}) \tag{3.4.1}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the n^{th} Catalan number, and C_{-1} is taken to be 0.

Proof. By Propositions 3.4.1 and 3.4.4, the number of vertices of $\mathbf{P}_\lambda$ is equal to the number of plane binary trees T on n nodes such that right-most node v in T has a non-empty (left) subtree L_v of size at least $n-k$, or v has no descendants and its parent u has at least $n-k$ descendants. In the first case, if $|L| = i$, then there are C_i ways to choose L and C_{n-1-i} ways to choose the tree $T \backslash L \cup \{v\}$. In the second case, if the size of the left subtree of u is $|L_u| = j$ then there are C_j ways to choose L_u and C_{n-2-j} ways to choose $T \backslash L_u \cup \{u, v\}$. Summing over $i = \max\{1, n-k\}, ..., n-1$ and $j = n-1-k, ..., n-2$ yields the desired formula. $\square$

Remark. There is no difference in the combinatorial structure of $\mathbf{P}_\lambda$ whether λ has n or $n-1$ parts. Indeed, if $\lambda = (\lambda_1, ..., \lambda_n)$ and $\lambda' = (\lambda_1, ..., \lambda_{n-1})$ then $\mathbf{P}_\lambda$ is just the translation of $\mathbf{P}_{\lambda'}$ by $\lambda_n e_n$. In either case, the number of vertices of $\mathbf{P}_\lambda$ is $C_n = C_0 C_{n-1} + ... + C_{n-1} C_0$. On the other extreme, if λ has $k = 0$ parts, then $\mathbf{P}_\lambda = \mathrm{Ass}_{n-1}$ has C_{n-1} vertices.

To describe the vertices of $\mathbf{P}_\lambda$, recall that plane binary trees T on n nodes are in bijective correspondence with the C_n subdivisions of the shifted Young diagram $D_\emptyset$ into n rectangles. This can be defined inductively as follows: Let i be the root of T (in the binary search labeling). Then draw an $(|L_i| + 1) \times (|R_i| + 1)$ rectangle. Then attach the subdivisions corresponding to the binary trees L_i, R_i to the left and, respectively, bottom of the rectangle.

For a subdivision Ξ of $D_\emptyset$ into n rectangles, the i^{th} *rectangle* is the rectangle containing the i^{th} diagonal box of $D_\emptyset$. If T is the binary tree corresponding to Ξ, then the i^{th} rectangle of Ξ has size $(|L_i| + 1) \times (|R_i| + 1)$. In particular, $|L_n| + 1$ is the length of the (bottom-right) vertical strip of the subdivision Ξ.

Example 3.4.6. Figure 3.4.1 depicts a subdivision of the staircase shape $D_\emptyset$ and the corresponding binary tree with the binary search labeling when $n = 4$.

We are finally in a position to prove the second main result of this chapter.

Theorem 3.4.7. *Vertices of $\mathbf{P}_\lambda$ are in bijection with subdivisions of the shifted diagram $D_\emptyset$ into n rectangles such that the bottom-right vertical strip of the subdivision has at least $n-k+1$ boxes. Specifically, let Ξ be such a subdivision. One can obtain a subdivision Ξ^* of $D_{\lambda - \langle 1^k \rangle}$ by merging the rectangles in Ξ with the rows of the Young diagram of $\lambda - \langle 1^k \rangle$ that*

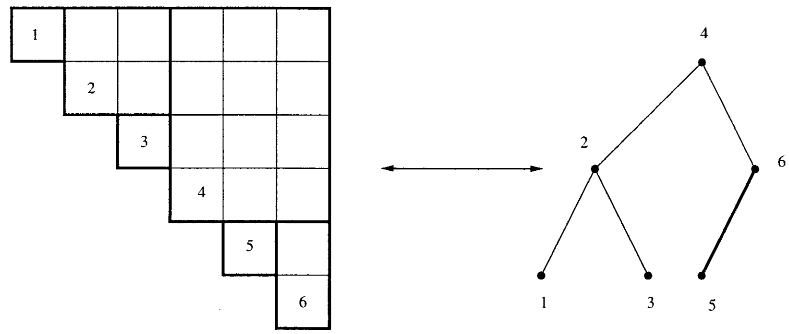


Figure 3.4.1: A subdivision of $D_\emptyset$ and the corresponding labelled binary tree

they border. Then the corresponding vertex of $\mathbf{P}_\lambda$ *is* $v_\Xi = (t_1, \ldots, t_n)$, *where* t_i *is the number of boxes in the* i^{th} *region of* Ξ^*.

Proof. The first part of the theorem follows from Proposition 3.4.4 and the discussion preceeding the theorem. To prove the second part, we use Proposition 3.4.1. Recall that the building set of $\mathbf{P}_\lambda$ is $B_k = \{[i,j] | 1 \le i \le j \le n\} \cup \{[i,n] | 1 \le i \le k\}$, and $\mathbf{P}_\lambda = \sum_{[i,j]\in B_k} y_{ij}\Delta_{[i,j]}$ where $y_{ij} = 1$ if $j \neq 1$ and $y_{in} = \lambda_i$. Let T be a B_k-forest, i.e. a binary tree on n nodes with the binary search labeling such that $|L_n| \ge n - k$ (cf. Proposition 3.4.4.) Note that $\mathrm{desc}(i,T) = [i - |L_i|, i + |R_i|]$. Now Proposition 3.4.1 implies that the correponding vertex $v_T = (t_1, \ldots, t_n)$ of $\mathbf{P}_\lambda$ is given by

$$\begin{aligned} t_i &= \sum_{J\in B_k,\ i\in J\subseteq \mathrm{desc}(i,F)} y_J = \sum_{[k,l]\in B_k,\ i-|L_i|\le k\le i\le l\le i+|R_i|} y_{kl} \\ &= (|L_i|+1)\cdot|R_i| + \sum_{k=i-|L_i|}^{i} y_{k(i+|R_i|)}. \end{aligned}$$

If the i^{th} rectangle of Ξ borders the right edge of $D_\emptyset$ (i.e. $n \in \mathrm{desc}(i,\ T)$), then $t_i = (|L_i|+1)\cdot|R_i| + \sum_{k=i-|L_i|}^{i}\lambda_k$. Otherwise, $t_i = (|L_i|+1)\cdot(|R_i|+1)$. In any case, t_i is the number boxes in the i^{th} region of Ξ^*. □

Example 3.4.8. Let $n = 4, \lambda = (4,2,1,0), k = 3$. Figure 3.4.2 shows how a subdivision Ξ of $D_\emptyset$ yields the subdivision Ξ^* of $D_{\lambda-\langle 1^k\rangle} = D_{(3,1,0)}$. The corresponding vertex of $\mathbf{P}_\lambda$ is given by counting boxes in the regions of Ξ^*: $v_{\Xi^*} = (1,10,1,2)$. It follows that there is a (4,2,1,0)-shifted Young tableau T whose diagonal vector is $\mathrm{diag}(T) = (1, 1+1+1, 1+1+1+10+1, 1+1+1+10+1+2) = (1,3,14,16)$.

On the other hand, one can directly construct λ-shifted Young tableaux with diagonal vector $v_{\Xi^*} = (c_1, c_2, \ldots, c_n)$ by using the subdivision Ξ^*. Indeed, we know what the diagonal vector of the tableau $(a_1, \ldots, a_n)$ should be. Consider again the subdivision Ξ^* of $D_{\lambda-\langle 1^k\rangle}$.

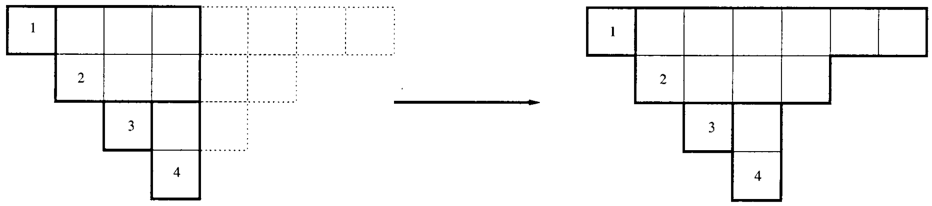


Figure 3.4.2: Constructing the subdivision Ξ^* of $D_{(3,1,0)}$ from a subdivision Ξ of $D_\emptyset$.

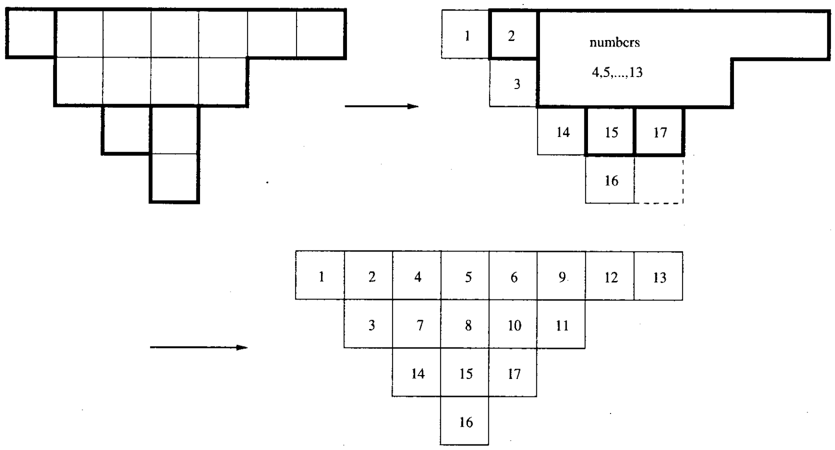


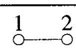
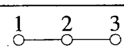
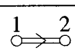
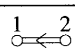
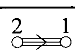
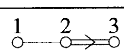
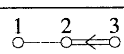
Figure 3.4.3: Constructing shifted Young tableaux from a subdivision Ξ^* of $D_{\lambda-<1^k>} = D_{(3,1,0)}$.

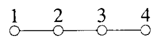
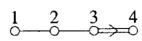
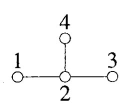
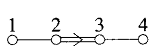
We can extend the diagram $D_{\lambda-\langle 1^k\rangle}$ to D_λ by first adding a box to the left of each row of $D_{\lambda-\langle 1^k\rangle}$, and then, by deleting the last $n-k$ boxes in the n^{th} column of $D_{\lambda-\langle 1^k\rangle}$. Now, we start by putting $a_1, \ldots, a_n$ in the diagonal boxes of D_λ. The remaining part of D_λ is divided into n regions by Ξ^*. Finally, for each $i = 1, \ldots, n$, put the c_i numbers $a_i+1, \ldots, a_{i+1}-1$ in the i^{th} region of Ξ^* in a standard way, i.e. such that entries increase along rows and down columns (as before, we set $a_{n+1} = |D_\lambda| + 1$). In this way we obtain a λ-shifted tableau T such that $\text{diag}(T) = (a_1, \ldots, a_n)$.

Figure 3.4.3 illustrates the above procedure for the subdivision in Example 3.4.8.

Problem 3.4.9. The normalized volume of a polytope is in some sense "dual" to its number of lattice points. Given our results on the lattice points of $\mathbf{P}_\lambda$, it is natural to ask for a combinatorial interpretation of the normalized volume of $\mathbf{P}_\lambda$. Are there any combinatorial objects (vectors of Young tableaux, trees, etc) which would give a triangulation of $\mathbf{P}_\lambda$?

Appendix A

Φ	Diagram	$\lvert W_\Phi \rvert$	$V_\Phi(u_1, ..., u_n)$
A_1		2	u_1
A_2		6	$\frac{u_1^2}{2} + 2u_1u_2 + \frac{u_2^2}{2}$
A_3		24	$\frac{u_1^3}{3!} + 4\frac{u_2^3}{3!} + \frac{u_3^3}{3!} + 2\frac{u_1^2u_2}{2} + 4\frac{u_1u_2^2}{2} + 3\frac{u_1^2u_3}{2}$ $+3\frac{u_1u_3^2}{2} + 4\frac{u_2^2u_3}{2} + 2\frac{u_2u_3^2}{2} + 6u_1u_2u_3$
B_2		8	$4\frac{u_1^2}{2} + 4u_1u_2 + 2\frac{u_2^2}{2}$
C_2		8	$2\frac{u_1^2}{2} + 4u_1u_2 + 4\frac{u_2^2}{2}$
G_2		12	$6\frac{u_1^2}{2} + 12u_1u_2 + 18\frac{u_2^2}{2}$
B_3		48	$8\frac{u_1^3}{3!} + 40\frac{u_2^3}{3!} + 6\frac{u_3^3}{3!} + 16\frac{u_1^2u_2}{2} + 32\frac{u_1u_2^2}{2} + 12\frac{u_1^2u_3}{2}$ $+12\frac{u_1u_3^2}{2} + 24\frac{u_2^2u_3}{2} + 12\frac{u_2u_3^2}{2} + 24u_1u_2u_3$
C_3		48	$4\frac{u_1^3}{3!} + 20\frac{u_2^3}{3!} + 24\frac{u_3^3}{3!} + 8\frac{u_1^2u_2}{2} + 16\frac{u_1u_2^2}{2} + 12\frac{u_1^2u_3}{2}$ $+24\frac{u_1u_3^2}{2} + 24\frac{u_2^2u_3}{2} + 24\frac{u_2u_3^2}{2} + 24u_1u_2u_3$

Φ	Diagram	$\lvert W_\Phi \rvert$	$V_\Phi\,(u_1,\ldots,u_n)$
A_4		120	$\frac{u_1^4}{4!}+11\frac{u_2^4}{4!}+11\frac{u_3^4}{4!}+\frac{u_4^4}{4!}+24u_1u_2u_3u_4$ $+2\frac{u_1^3u_2}{3!}+14\frac{u_3^3u_2}{3!}+3\frac{u_4^3u_2}{3!}+8\frac{u_1u_2^3}{3!}+14\frac{u_3u_2^3}{3!}+17\frac{u_4u_2^3}{3!}$ $+3\frac{u_1^3u_3}{3!}+17\frac{u_3^3u_1}{3!}+4\left(\frac{u_1^3u_4}{3!}+\frac{u_4^3u_1}{3!}\right)+8\frac{u_3^3u_4}{3!}+2\frac{u_4^3u_3}{3!}$ $4\frac{u_1^2u_2^2}{2!2!}+16\frac{u_3^2u_2^2}{2!2!}+9\frac{u_4^2u_2^2}{2!2!}+9\frac{u_1^2u_3^2}{2!2!}+6\frac{u_1^2u_4^2}{2!2!}+4\frac{u_3^2u_4^2}{2!2!}$ $6\left(\frac{u_1^2u_2u_3}{2}+\frac{u_2u_3u_4^2}{2}\right)+16\left(\frac{u_1u_3^2u_4}{2}+\frac{u_1u_2^2u_4}{2}\right)$ $+8\left(\frac{u_1^2u_2u_4}{2}+\frac{u_1u_3u_4^2}{2}\right)+18\left(\frac{u_1u_2u_3^2}{2}+\frac{u_2^2u_3u_4}{2}\right)$ $+12\left(\frac{u_1^2u_3u_4}{2}+\frac{u_1u_2^2u_3}{2}+\frac{u_2u_3^2u_4}{2}+\frac{u_1u_2u_4^2}{2}\right)$
B_4		384	$16\frac{u_1^4}{4!}+192\frac{u_2^4}{4!}+368\frac{u_3^4}{4!}+24\frac{u_4^4}{4!}+192u_1u_2u_3u_4$ $+352\frac{u_3^3u_2}{3!}+128\frac{u_1u_2^3}{3!}+256\frac{u_3u_2^3}{3!}+160\frac{u_4u_2^3}{3!}+336\frac{u_3^3u_1}{3!}$ $+48\left(\frac{u_1^3u_3}{3!}+\frac{u_4^3u_2}{3!}+\frac{u_4^3u_3}{3!}+\frac{u_4^3u_1}{3!}\right)+32\left(\frac{u_1^3u_2}{3!}+\frac{u_1^3u_4}{3!}\right)$ $+192\frac{u_3^3u_4}{3!}+64\frac{u_1^2u_2^2}{2!2!}+320\frac{u_3^2u_2^2}{2!2!}+96\left(\frac{u_4^2u_2^2}{2!2!}+\frac{u_3^2u_4^2}{2!2!}\right)$ $+144\frac{u_1^2u_3^2}{2!2!}+48\frac{u_1^2u_4^2}{2!2!}++128\frac{u_1u_2^2u_4}{2}+288\frac{u_1u_2u_3^2}{2}$ $64\frac{u_1^2u_2u_4}{2}+192\left(\frac{u_1u_2^2u_3}{2}+\frac{u_2^2u_3u_4}{2}+\frac{u_1u_3^2u_4}{2}+\frac{u_2u_3^2u_4}{2}\right)$ $+96\left(\frac{u_1^2u_2u_3}{2}+\frac{u_1^2u_3u_4}{2}+\frac{u_2u_3u_4^2}{2}+\frac{u_1u_2u_4^2}{2}+\frac{u_1u_3u_4^2}{2}\right)$
D_4		192	$8\frac{u_1^4}{4!}+96\frac{u_2^4}{4!}+8\frac{u_3^4}{4!}+8\frac{u_4^4}{4!}+48u_1u_2u_3u_4$ $+16\left(\frac{u_1^3u_2}{3!}+\frac{u_3^3u_2}{3!}+\frac{u_4^3u_2}{3!}\right)+64\left(\frac{u_1u_2^3}{3!}+\frac{u_3u_2^3}{3!}+\frac{u_4u_2^3}{3!}\right)$ $+12\left(\frac{u_1^3u_3}{3!}+\frac{u_3^3u_1}{3!}+\frac{u_1^3u_4}{3!}+\frac{u_4^3u_1}{3!}+\frac{u_3^3u_4}{3!}+\frac{u_4^3u_3}{3!}\right)$ $+32\left(\frac{u_1^2u_2^2}{2!2!}+\frac{u_3^2u_2^2}{2!2!}+\frac{u_4^2u_2^2}{2!2!}\right)+12\left(\frac{u_1^2u_3^2}{2!2!}+\frac{u_1^2u_4^2}{2!2!}+\frac{u_3^2u_4^2}{2!2!}\right)$ $+24\left(\frac{u_1^2u_2u_3}{2}+\frac{u_1^2u_2u_4}{2}+\frac{u_1^2u_3u_4}{2}+\frac{u_1u_2u_3^2}{2}+\frac{u_2u_3^2u_4}{2}\right)$ $+24\left(\frac{u_1u_3^2u_4}{2}+\frac{u_1u_2u_4^2}{2}+\frac{u_2u_3u_4^2}{2}+\frac{u_1u_3u_4^2}{2}\right)$ $+48\left(\frac{u_1u_2^2u_3}{2}+\frac{u_1u_2^2u_4}{2}+\frac{u_2^2u_3u_4}{2}\right)$
F_4		1152	$16u_1^4+232u_2^4+58u_3^4+4u_4^4+1152u_1u_2u_3u_4+$ $128u_1^3u_2+512u_1u_2^3+384u_1^2u_2^2+672u_2^3u_3+336u_2u_3^3$ $+720u_2^2u_3^2+128u_3^3u_4+32u_3u_4^3+96u_3^2u_4^2+96u_1^3u_3$ $+208u_1u_3^3+216\left(u_1^2u_3^2+u_2^2u_4^2\right)+416u_2^3u_4+48u_2u_4^3$ $+64u_1^3u_4+32u_1u_4^3+72u_1^2u_4^2+576\left(u_1^2u_2u_3+u_2u_3^2u_4\right)$ $+768u_1u_2^2u_4+288\left(u_1^2u_3u_4+u_1u_2u_4^2+u_2u_3u_4^2\right)$ $+384\left(u_1^2u_2u_4+u_1u_3^2u_4\right)+192u_1u_3u_4^2$ $+1152u_1u_2^2u_3+864\left(u_2^2u_3u_4+u_1u_2u_3^2\right)$

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