Combinatorial Enumeration of Weighted Catalan Numbers

by

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Abstract

This thesis is devoted to the divisibility property of weighted Catalan and Motzkin numbers and its applications. In Chapter 1, the definitions and properties of weighted Catalan and Motzkin numbers are introduced. Chapter 2 studies Wilf conjecture on the complementary Bell number, the alternating sum of the Stirling number of the second kind. Congruence properties of the complementary Bell numbers are found by weighted Motkin paths, and Wilf conjecture is partially proved. In Chapter 3, Konvalinka conjecture is proved. It is a conjecture on the largest power of two dividing weighted Catalan number, when the weight function is a polynomial. As a corollary, we provide another proof of Postnikov and Sagan of weighted Catalan numbers, and we also generalize Konvalinka conjecture for a general weight function.

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Chapter 1

Introduction

1.1 Background

In combinatorics, the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

has been studied for a long time and has involved various problems, especially in enumerative combinatorics. Stanley[18] has listed more than 170 combinatorial examples of Catalan numbers. For more information, see [2], [4], [6], [7], [8], [15], and [17].
C_n is the number of Dyck paths with length $2n$, a combinatorial interpretation of Catalan numbers. A Dyck path consists of two steps $(1,1)$ and $(1,-1)$ which does not pass below the x-axis. If a step $(1,0)$ is also allowed, the number of paths from $(0,0)$ to $(n,0)$ is M_n, the nth Motzkin number. Motzkin numbers also have been studied in many papers including [4], [5], [6], [7], [8], [10], [15], and [18]. Donaghey and Shapiro[5] provided 14 combinatorial examples of Motzkin numbers. The first Catalan numbers(Sloane[16]'s $A000108$) and Motzkin numbers(Sloane[16]'s $A001006$) are

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \quad C_5 = 42, \quad C_6 = 132$$

$$M_0 = 1, \quad M_1 = 1, \quad M_2 = 2, \quad M_3 = 4, \quad M_4 = 9, \quad M_5 = 21, \quad M_6 = 51$$

The number of Motzkin paths with length n that are composed of $2i$ $(1,1)$ and $(1,-1)$ steps is $\binom{n}{2i}C_i$. Therefore, the relation between Catalan and Motzkin number is

$$M_n = \sum_{0\leq i\leq \frac{n}{2}} \binom{n}{2i} C_i$$

and arithmetic properties of Catalan and Motzkin numbers are studied in [2], [3], [4], and [6].

Weighted Catalan and Motzkin numbers are generalized Catalan and Motzkin numbers by giving weights on Dyck and Motzkin paths. A lot of combinatorial numbers including Bell, Euler, and Stirling numbers can be expressed by weighted Catalan numbers with the corresponding weight functions, and congruences properties are found in [11], [13], and [14].

In this paper, divisibility properties of weighted Catalan and Motzkin numbers are found and proved. The complementary Bell numbers are also investigated by using the properties of weighted paths.

1.2 Weighted Catalan numbers

A Dyck path P with length $2n$ is a path from $(0,0)$ to $(2n,0)$ consisting of steps $(1,1)$(a rise step) and $(1,-1)$(a fall step) that lies above the x-axis. It can be expressed by $p_0, p_1, \cdots, p_{2n}$, a sequence of points in $(N\cup\{0\})\times(N\cup\{0\})$ where

(1) $p_0 = (0,0)$, $p_{2n} = (2n,0)$

(2) $p_{i+1} - p_i = (1,1)$ or $(1,-1)$

$b(x)$(respectively, $d(x)$) is the given weight function from $N\cup\{0\}$ to Z. The weight of a rise step from (x,y) to $(x+1,y+1)$ is $b(y)$(respectively, the weight of a fall step from $(x,y+1)$ to $(x+1,y)$ is $d(y)$). For a Dyck path P, its weight(i.e. $w(P)$) is defined as the product of the weights of rise and fall steps. See Figure 1-1 for an example.

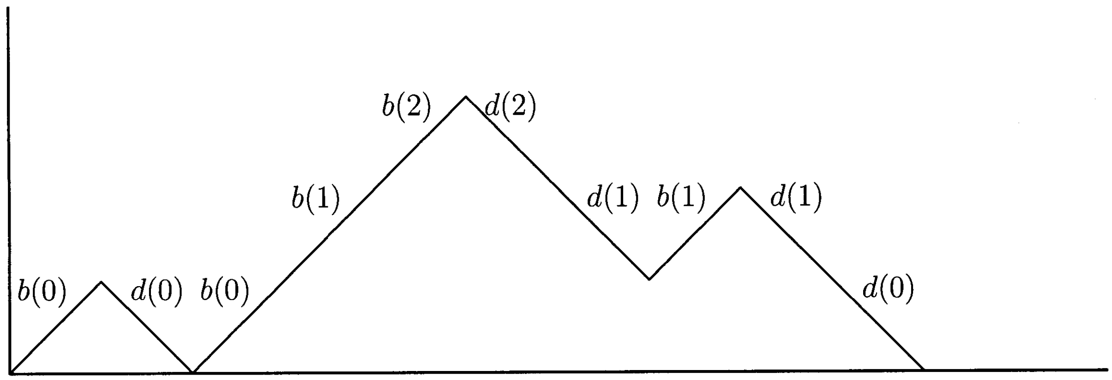

Figure 1-1: A Dyck path with weight $b(0)^2b(1)^2b(2)d(0)^2d(1)^2d(2)$

The corresponding nth weighted Catalan number $C_n^{b,d}$ is given by

$$C_n^{b,d} = \sum w(P) \tag{1.1}$$

where the sum is over all Dyck paths from $(0,0)$ to $(2n,0)$. Since $C_n^{b,d} = C_n^{bd,1}$, $d(x)$ is not concerned in most cases(assume that $d(x) = 1$) unless it is mentioned. The first five numbers are

$$\begin{aligned}
C_0^{b,1} &= 1 \\
C_1^{b,1} &= b(0) \\
C_2^{b,1} &= b(0)^2 + b(0)b(1) \\
C_3^{b,1} &= b(0)^3 + 2b(0)^2b(1) + b(0)b(1)^2 + b(0)b(1)b(2) \\
C_4^{b,1} &= b(0)^4 + 3b(0)^3b(1) + 3b(0)^2b(1)^2 + b(0)b(1)^3 + 2b(0)^2b(1)b(2) \\
&+ 2b(0)b(1)^2b(2) + b(0)b(1)b(2)^2 + b(0)b(1)b(2)b(3)
\end{aligned}$$

From $[8, Chapter\ 5]$, the generating function of $C_n^{b,d}$ is

$$\sum_{n\geq 0} C_n^{b,d}x^n = \cfrac{1}{1 - \cfrac{b(0)d(0)x}{1 - \cfrac{b(1)d(1)x}{1 - \cfrac{b(2)d(2)x}{\cdots}}}} \tag{1.2}$$

$C_n^{1,1}$ is the nth Catalan number C_n. For $b(x) = q^x$, $C_n^{b,1}$ is the q-Catalan number $C_n(q)$, and the corresponding (1.2) is the Ramanujan continued fraction. One property of

$C_n(q)$ is $w(P) = q^{area(P)}$ for a Dyck path P, where $area(P)$ is the area between P and the x-axis.

$C_n^{b,1}$ has another combinatorial interpretation generalizing C_n as the number of binary trees with n nodes. A binary tree is a rooted tree in which every vertex has at most two children, a left or a right child. Each node of a binary tree has the weight $b(i)$, where i is the number of left edges from the root. The weight of a binary tree T(i.e. $w(T)$) is defined as the product of the weights of nodes. It can be checked by the depth-first search that

$$C_n^{b,1} = \sum w(T) \tag{1.3}$$

where the sum is over all binary trees with n nodes. Postnikov and Sagan[14] combinatorially found the power of two dividing weighted Catalan numbers by group actions on binary trees. Konvalinka[11] defined a generalized q-analogue weighted Catalan number with m-ary trees and found similar results.

In Chapter 3, Konvalinka conjecture, a conjecture related to the power of two dividing weighted Catalan numbers, is proved and divisibility properties are studied.

1.3 Weighted Motzkin numbers

Similar to a Dyck path, a Motzkin path Q with length n is a path from $(0,0)$ to $(n,0)$ consisting of steps $(1,1)$(a rise step), $(1,0)$(a level step), and $(1,-1)$(a fall step) that lies above the x-axis. It can also be expressed by $p_0, p_1, \cdots, p_n$, a sequence of points in $(N \cup \{0\}) \times (N \cup \{0\})$ where

(1) $p_0 = (0,0)$, $p_n = (n,0)$

(2) $p_{i+1} - p_i = (1,1)$, $(1,0)$, or $(1,-1)$

For the given weight function $b(x)$(respectively, $c(x)$ and $d(x)$) from $N \cup \{0\}$ to Z, the weight of a rise step from (x,y) to $(x+1,y+1)$ is $b(y)$(respectively, the weight of a level step from (x,y) to $(x+1,y)$ is $c(y)$ and the weight of a fall step from $(x,y+1)$ to $(x+1,y)$ is $d(y)$). The weight of a Motzkin path Q(i.e. $w(Q)$) is defined as the product of the weight of steps. See Figure 1-2 for an example.

The corresponding nth weighted Motzkin number $M_n^{b,c,d}$ is given by

$$M_n^{b,c,d} = \sum w(Q) \tag{1.4}$$

where the sum is over all Motzkin paths from $(0,0)$ to $(n,0)$. Since $M_n^{b,c,d} = M_n^{bd,c,1}$, $d(x)$ is not concerned in most cases(assume that $d(x) = 1$) unless it is mentioned. The first five numbers are

$$\begin{aligned}
M_0^{b,c,1} &= 1 \\
M_1^{b,c,1} &= c(0) \\
M_2^{b,c,1} &= b(0) + c(0)^2 \\
M_3^{b,c,1} &= 2b(0)c(0) + b(0)c(1) + c(0)^3 \\
M_4^{b,c,1} &= b(0)^2 + b(0)b(1) + 3b(0)c(0)^2 + 2b(0)c(0)c(1) + b(0)c(1)^2 + c(0)^4
\end{aligned}$$

From [8, *Chapter* 5], the generating function of $M_n^{b,c,d}$ is

$$\sum_{n\geq 0} M_n^{b,c,d} x^n = \cfrac{1}{1 - c(0)x - \cfrac{b(0)d(0)x^2}{1-c(1)x-\cfrac{b(1)d(1)x^2}{1-c(2)x-\cfrac{b(2)d(2)x^2}{\cdots}}}}$$

$M_n^{1,1,1}$ is the nth Motzkin number. For $b(x) = 1$ and $c(x) = 2$, $M_n^{b,c,1}$ is the $(n+1)$th Catalan number(i.e. C_{n+1}). For $b(x) = 2$ and $c(x) = 3$, $M_n^{b,c,1}$ is the $(n+1)$th little Schroeder number or super Catalan number(i.e. s_{n+1}). For more information, see

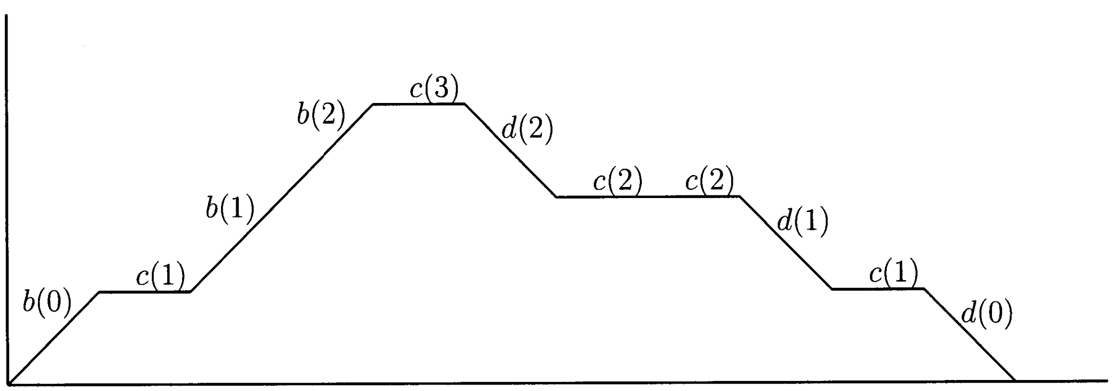

Figure 1-2: A Motzkin path with weight $b(0)b(1)b(2)c(1)^2c(2)^2c(3)d(0)d(1)d(2)$.

Sloane[16]'s $A001003$.

The power of two that divides the complementary Bell numbers is analyzed in Chapter 2 by using the property of weighted Motzkin numbers. Wilf conjecture, a conjecture of the complementary Bell numbers, is partially proved.

Chapter 2

Wilf conjecture

2.1 Background

$S(n,k)$ is the Stirling number of the second kind(i.e. the number of partitioning $[n]$ into k nonempty subsets). The nth Bell number is $B_n = \sum_{k=0}^{n} S(n,k)$, the number of partitioning $[n]$. These numbers appear in several combinatorial problems.
The complementary Bell numbers(or the Uppuluri-Carpenter numbers) are $f(n) = \sum_{k=0}^{n}(-1)^k S(n,k)$. The first $f(n)$(Sloane[16]'s $A000587$) for $n = 0,1,2,3,4,\cdots$ is

$$1, -1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180, -17731, -50533, \cdots$$

$f(n)$ is the difference between the number of even partitions and odd partitions, and it is related to p-adic numbers and multiplicative partition functions. The generating function of $f(n)$ is

$$\sum_{n\geq 0} f(n)x^n = e^{1-e^x}$$

Wilf[9] conjectured that $f(n)$ is nonzero for all $n > 2$.

Conjecture 1 [*9*] *$f(n) \neq 0$ for all $n > 2$.*

Yang[21] proved that the number of zeros smaller than x is at most $x^{\frac{2}{3}}$ with the sum estimates. Murty and Sumer[12] approached the conjecture by the congruences of

$f(n)$, and Wannemacker, Laffey, and Osburn[20] proved that $f(n) \neq 0$ for all $n \not\equiv 2$, $2944838 (mod\ 3145728)$. In this chapter, the main result is the following.

Theorem 2 [*1, Theorem* 2] *There is at most one $n > 2$ satisfying $f(n) = 0$.*

Alexander[1] proved the theorem with the umbral calculus, but weighted Motzkin numbers are used to prove Theorem 2 in this paper. Section 2.2 deals with congruence properties of $f(n)$ by using the properties of weighted Motzkin numbers. Theorem 2 is finally proved in Section 2.3.

2.2 Congruence properties of $f(n)$

$f(n)$ is expressed by weighted Motzkin numbers and investigated by the properties of weighted Motzkin paths. The theorems and lemmas in this section are used to prove Theorem 2.

Flajolet[7] found the direct relationship between $f(n)$ and weighted Motzkin numbers.

Theorem 3 [*7, Theorem* 2]

$$\sum_{k=0}^{n} S(n,k)u^k = M_n^{b'',c''}$$

where $b''(x) = u(x+1)$ and $c''(x) = u + x$.

Flajolet[7] proved the above theorem by using Path diagrams. A bijection was constructed between set partitions and weighted Motzkin paths by generalizing Francon-Viennot decomposition in [8]. B_n(Bell numbers) and I_n(the number of involutions on $[n]$) also can be expressed by weighted Motzkin numbers with Theorem 3. In particular,

$$f(n) = M_n^{b',c'}$$

for $b'(x) = -x-1$ and $c'(x) = x-1$. Slightly changing the weight functions,

$$f(n) = M_n^{b,c,d}$$

for

$$\begin{cases} b(x) = (-x-1)/2, & c(x) = x-1, \quad \text{and } d(x) = 2 \quad \text{if } x \text{ is odd} \\ b(x) = -x-1, & c(x) = x-1, \quad \text{and } d(x) = 1 \quad \text{if } x \text{ is even} \end{cases}$$

From now on, the above weight functions $b(x)$, $c(x)$, and $d(x)$ are used. If $W_{n,k}$ is the sum of weighted paths from $(0,0)$ to (n,k) that lies above the x-axis,

$$W_{n+1,k+1} = b(k)W_{n,k} + c(k+1)W_{n,k+1} + d(k+1)W_{n,k+2}$$

for $n, k \geq 0$ and $W_{n+1,0} = c(0)W_{n,0} + d(0)W_{n,1}$. If A_r is the following $(r+1) \times (r+1)$ matrix,

$$\mathbf{A_r} := \begin{pmatrix} c(0) & d(0) & 0 & \dots & \dots \\ b(0) & c(1) & d(1) & \dots & \dots \\ \vdots & \vdots & \ddots & \dots & \dots \\ \vdots & \vdots & b(r-2) & c(r-1) & d(r-1) \\ \vdots & \vdots & 0 & b(r-1) & c(r) \end{pmatrix}$$

$(W_{n+k,0}, W_{n+k,1}, \cdots, W_{n+k,r}) \equiv A_r^k(W_{n,0}, W_{n,1}, \cdots, W_{n,r})$ $(mod\ b(0)b(1)\cdots b(r))$ because $W_{n,l} \equiv 0(mod\ b(0)b(1)\cdots b(r))$ for $l \geq r+1$.
Since $b(4k-1) = -2k \equiv 0(mod\ 2)$ and $d(4k-1) \equiv 0(mod\ 2)$ for $k \geq 1$,

$$\mathbf{A_{4k-1}} \equiv \begin{pmatrix} A & \mathbf{0} & \dots & \dots & \dots \\ \mathbf{0} & A & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \dots & \dots \\ \vdots & \vdots & \vdots & A & \mathbf{0} \\ \vdots & \vdots & \vdots & \mathbf{0} & A \end{pmatrix} (mod\ 2)$$

where

$$\mathbf{A} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Therefore,

$$A_{4k-1}^{6} \equiv I(mod\ 2) \tag{2.1}$$

because $A^6 \equiv I(mod\ 2)$, where I is an identity matrix.

The next lemma deals with congruences of $W_{n,k}$ for n. S is the shift operator(i.e. $S(W_{n,k}) = W_{n+1,k}$ and $S(f(n)) = f(n+1)$), and $(\sum_{i=0}^{t} a_i S^i)(W_{n,k})$ means $\sum_{i=0}^{t} a_i S^i(W_{n,k}) = \sum_{i=0}^{t} a_i W_{n+i,k}$.

Lemma 4

$$(E-1)^r(W_{n,k}) \equiv 0(mod\ 2^r)$$

for $r \geq 1$, *where* $E = S^{6(2t-1)}$ *and* $t \in N$.

Proof For a given r, $(E-1)^r(W_{n,k}) \equiv 0(mod\ 2^r)$ for $k > 4r-1$ because $W_{n,k} \equiv 0(mod\ 2^r)$ for $k > 4r-1$ from $b(0)b(1)\cdots b(4r-1) \equiv 0(mod\ 2^r)$. Therefore, it can be assumed that $S = A_{4r-1}$.

It is proved by mathematical induction on r. For $r = 1$, $A_3^6 \equiv I(mod\ 2)$ from (2.1). It is assumed that the statement is true for $r = m(m \geq 1)$, and it is proved for $r = m+1$. Using the inductive assumption for $r = m$(i.e. $(E-1)^m(W_{n,k}) \equiv 0(mod\ 2^m)$) and $(W_{n+s,0}, \cdots, W_{n+s,4m+3}) \equiv A_{4m+3}^{s}(W_{n,0}, \cdots, W_{n,4m+3})(mod\ 2^{m+1})$,

$$\begin{aligned} &\left(\frac{(E-1)^m W_{n+s,0}}{2^m}, \cdots, \frac{(E-1)^m W_{n+s,4m+3}}{2^m}\right) \\ &\equiv A_{4m+3}^{s}\left(\frac{(E-1)^m W_{n,0}}{2^m}, \cdots, \frac{(E-1)^m W_{n,4m+3}}{2^m}\right)(mod\ 2) \end{aligned}$$

Therefore, $\frac{(E-1)^m E(W_{n,k})}{2^m} \equiv \frac{(E-1)^m (W_{n,k})}{2^m}(mod\ 2)$ and $(E-1)^{m+1}(W_{n,k}) \equiv 0(mod\ 2^{m+1})$ for $k \leq 4m+3$. The proof is done. ■

We remark that $g(E) \equiv 0(mod\ 2^r)$ for $r > 0$ if $(x-1)^r$ divides $g(x) \in Z[x]$.

Corollary 5

$$(E^{2^k} - 1)^2(f(n)) \equiv 0(mod\ 2^{2k+2})$$

for $k \geq 0$.

Proof It is true for $k = 0$ from Lemma 4($r = 2$). For $k \geq 1$, $E^{2^k} - 1 = (E-1)(E+1)(E^2+1)\cdots(E^{2^{k-1}}+1)$ and $E^{2^s}+1 = (E-1)g_s(E)+2$ for $s \geq 0$, where $g_s(E) = E^{2^s-1} + \cdots + E + 1$. If we expand $(E^{2^k}-1)^2$, each term is divisible by $2^m(E-1)^{2k+2-m}$ for some $0 \leq m \leq 2k+2$, and $2^m(E-1)^{2k+2-m}(f(n)) \equiv 0(mod\ 2^{2k+2})$ from Lemma 4. ∎

Similar to Corollary 5, it can be proved that $(E^{2^k}-1)(f(n)) \equiv 0(mod\ 2^{k+1})$. It implies that

$$f(n + 3 \times 2^{k+1}) \equiv f(n)(mod\ 2^{k+1})\ for\ all\ n \tag{2.2}$$

2.3 Proof of Theorem 2

In this section, *Theorem* 2 is proved. It is proved by mathematical induction and Corollary 5.

Proof From Corollary 5,

$$(E^{2^{k+1}} - 1)(f(n)) \equiv 2(E^{2^k} - 1)(f(n))(mod\ 2^{2k+2}) \tag{2.3}$$

for $k \geq 0$. It implies that $f(n + 3 \times 2^{k+2}(2t-1)) - f(n) \equiv 2(f(n + 3 \times 2^{k+1}(2t-1)) - f(n))(mod\ 2^{2k+2})$ for $k \geq 0$ and $t \geq 1$.

It is shown that

$$f(n) \not\equiv 0(mod\ 2^{k+2})\ for\ n \not\equiv 2, a_k(mod\ 3 \times 2^k) \tag{2.4}$$

where $k \geq 5$, $a_k \equiv 38(mod\ 3 \times 2^5)$, and $0 \leq a_k < 3 \times 2^k$. The statement is true for $k = 5$ and $a_5 = 38$ because Wannemacker, Laffey, and Osburn[20] showed that $f(n) \not\equiv 0(mod\ 2^7)$ for $n \not\equiv 2, 38(mod\ 3 \times 2^5)$. It is assumed that the statement is true for $r = m(m \geq 5)$, and it is proved for $r = m+1$. From the inductive assumption for $r = m$, $f(n) \not\equiv 0(mod\ 2^{m+3})$ for $n \not\equiv 2, 2+3\times 2^m, a_m, a_m + 3 \times 2^m (mod\ 3 \times 2^{m+1})$. If there exist $a \geq 0$ and $b \geq 0$ such that $f(2+3\times 2^m + 3\times 2^{m+1}a) \equiv f(2+3\times 2^{m+1}b) \equiv 0(mod\ 2^{m+3})$, let

$$\begin{aligned} A &= 2 + 3 \times 2^m + 3 \times 2^{m+1}a \\ B &= 2 + 3 \times 2^{m+1}b \\ C &= \frac{A+B}{2} = 2 + 3 \times 2^{m-1} + 3 \times 2^m(a+b) \end{aligned}$$

(if $a < b$, change a into $a + 4b$ by using (2.2)). From (2.3),

$$f(A) - f(B) \equiv 2(f(C) - f(B))(mod\ 2^{m+3}) \tag{2.5}$$

by taking $n = 2 + 3 \times 2^{m+1}b$, $t = a - b + 1$ and $k = m - 2$(i.e. $n = 2 + 3 \times 2^{m+1}b$, $n + 3 \times 2^{k+3} \times (2t-1) = 2 + 3 \times 2^m + 3 \times 2^{m+1}a$, $n + 3 \times 2^{k+2} \times (2t-1) = 2 + 3 \times 2^{m-1} + 3 \times 2^m(a+b)$, and $2k+2 = 2m-2 \geq m+3$ for $m \geq 5$). Therefore, $f(C) \equiv 0(mod\ 2^{m+2})$, where $C \equiv 2 + 3 \times 2^{m-1} \not\equiv 2, a_k(mod\ 3 \times 2^m)$, and it contradicts the inductive assumption for $r = m$. From $f(2) = 0$, $f(n) \not\equiv 0(mod\ 2^{m+3})$ for $n \equiv 2 + 3 \cdot 2^m(mod\ 3 \times 2^{m+1})$.

Similarly, $f(n) \not\equiv 0(mod\ 2^{m+3})$ for $n \equiv a_m(mod\ 3\times 2^{m+1})$ or $n \equiv a_m + 3\times 2^m(mod\ 3 \times 2^{m+1})$. The proof of (2.4) is completed, and $a_{m+1} = a_m$ or $a_m + 3 \times 2^m$.

If there exist x and y such that $f(x) = f(y) = 0$ and $x \neq y > 2$, we can find some k such that $0 \leq x, y < 3 \times 2^k$. Therefore, $x = y = a_k$ from (2.4) and it contradicts $x \neq y$. The proof is done. ∎

a_5	38	a_{13}	$20294 = a_{12} + 3 \times 2^{12}$
a_6	$134 = a_5 + 3 \times 2^5$	a_{14}	$44870 = a_{13} + 3 \times 2^{13}$
a_7	$326 = a_6 + 3 \times 2^6$	a_{15}	$94022 = a_{14} + 3 \times 2^{14}$
a_8	326	a_{16}	$192326 = a_{15} + 3 \times 2^{15}$
a_9	326	a_{17}	192326
a_{10}	$1862 = a_9 + 3 \times 2^9$	a_{18}	$585542 = a_{17} + 3 \times 2^{17}$
a_{11}	1862	a_{19}	$1371974 = a_{18} + 3 \times 2^{18}$
a_{12}	$8006 = a_{11} + 3 \times 2^{11}$	a_{20}	$2944838 = a_{19} + 3 \times 2^{19}$

Table 2.1: a_i for $5 \leq i \leq 20$

2.4 Remark

From (2.4), it can be checked that Wilf conjecture is true if and only if a_i's are increasing. If a_i's are increasing and there exists $n > 2$ such that $f(n) = 0$, we can find some k such that $a_k > n$. Then, $n \not\equiv a_k (mod\ 3 \times 2^k)$ and Wilf conjecture is true from (2.4). Table 2.1 shows a_i for $5 \leq i \leq 20$.

Subbarao and Verma asked whether $f(n)$ takes some value infinitely or not in *Problem* 5.7 and 5.8 in [19]. The following theorem partially answers the question by using Table 2.2 and $f(n+96) \equiv f(n)(mod\ 128)$ in Corollary 5.

Theorem 6 *[10] $f(n)$ takes some value except $-2(mod\ 128)$ at most 3 times.*

Proof Similar to the proof of Corollary 5, it is proved that

$$f(n + 3(2t-1)2^m) \not\equiv f(n)(mod\ 2^{m+3}) \tag{2.6}$$

for $t \geq 1$, $m \geq 5$, and $n \geq 1$ except the case $f(n) \equiv -2(mod\ 128)$ because $f(n + 3(2t-1)2^4) \not\equiv f(n)(mod\ 128)$.

If there exist $a > b$ such that $a \equiv b(mod\ 96)$ and $f(a) = f(b)$, it contradicts (2.6). The proof is done. ■

$f(n) (mod\, 128)$	$n (mod\, 96)$
0	2, 38
1	3, 4
5	36, 87, 88
15	21, 49
17	48, 52
32	61, 74, 96
35	19, 33
39	25, 54
51	67, 85
53	40, 82
55	73, 93
57	22, 72
61	12, 64
64	50, 86
75	18, 43
96	14, 26
99	37, 42
109	16, 58
119	6, 7
126	5, 17, 29, 41, 53, 65, 77, 89
127	1, 31

Table 2.2: f(n)(mod 128)

Chapter 3

Konvalinka conjecture

3.1 Background

Alter and Kubota[2] arithmetically determined the largest power of p dividing C_n, where p is a prime number. For $p > 3$, if $n+1 = \sum_{i=0}^{k} n_i p^i$ where $0 \leq n_i < p$ for $0 \leq i \leq k$ and $n_k > 0$,

$$\xi_p(C_n) = \#\{i | \sum_{j=0}^{i} n_j p^j \geq \frac{p^{i+1}+1}{2}\}$$

where $\xi_p(m)$ is the largest power of p dividing m[2, *Theorem* 7]. For $p = 3$, if $n+1 = \sum_{i=j}^{k} n_i 3^i$ where $0 \leq n_i < 3$ for $j \leq i \leq k$ and $n_j, n_k > 0$, it will be proved that $\xi_3(C_n) = \#\{j < i \leq k | n_i = 2\}$ in Theorem 21.

$\xi_2(C_n)$ is $s(n+1)-1$ where $s(m)$ is the sum of digits in the binary expansion of m, and Deutsch and Sagan[4, *Theorem* 2.1] found a combinatorial proof with the standard binary tree interpretation. G_n is the group of automorphisms of a complete binary tree with height n, a binary tree which has all possible descendants at height n. In [4, *Lemma* 2.3], for an orbit O of G_n acting on T_n where T_n is a set of binary trees with n nodes,

$$\xi_2(|O|) \geq s(n+1) - 1 \tag{3.1}$$

with equality for $2(s(n+1)-1)!!$ orbits, where $(2m)!! = (2m-1)(2m-3)\cdots 1$. Since T_n is partitioned by orbits and $2(s(n+1)-1)!!$ is odd, $\xi_2(C_n) = \xi_2(|T_n|) = s(n+1)-1$. Arithmetic properties of weighted Catalan numbers have been investigated by [2], [4], [11], [13], and [14]. Postnikov and Sagan[14] found sufficient conditions for $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ by giving weights on (3.1).

Theorem 7 *[14, Theorem 2.1] If $b(0) \equiv 1(mod\ 2)$ and $\Delta^n b(x) \equiv 0(mod\ 2^{n+1})$ for all $n \geq 1$ and $x \geq 0$,*

$$\xi_2(C_n^{b,1}) = \xi_2(C_n) = s(n+1) - 1$$

for all $n \geq 0$, where $\Delta b(x) = b(x+1) - b(x)$.

For a polynomial $b(x)$, Konvalinka[11] conjectured equivalent conditions for $\xi_2(C_n^{b,1}) = \xi_2(C_n)$. It was checked for some $b(x)$ and $n \leq 250$ with a computer program $C++$ in [11]. It is interesting that the first four $b(i)$ determines whether $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ or not.

Conjecture 8 *[11, Conjecture] For $b(x) \in Z[X]$, $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ for all $n \geq 0$ if and only if $b(0) \equiv 1(mod\ 2)$, $b(1) \equiv b(0)(mod\ 4)$, and $b(3) \equiv b(2)(mod\ 4)$.*

The forward direction of Konvalinka conjecture is not trivial. But, it is easy to show the backward direction by using $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ for $0 \leq n \leq 4$ as follow.

$$\begin{aligned}
C_0^{b,1} &= 1 \equiv 1(mod\ 2) \\
C_1^{b,1} &= b(0) \equiv 1(mod\ 2) \\
C_2^{b,1} &= b(0)^2 + b(0)b(1) \equiv 2(mod\ 4) \\
C_3^{b,1} &= b(0)^3 + 2b(0)^2b(1) + b(0)b(1)^2 + b(0)b(1)b(2) \equiv 1(mod\ 2) \\
C_4^{b,1} &= b(0)^4 + 3b(0)^3b(1) + 3b(0)^2b(1)^2 + b(0)b(1)^3 + 2b(0)^2b(1)b(2) \\
&+ 2b(0)b(1)^2b(2) + b(0)b(1)b(2)^2 + b(0)b(1)b(2)b(3) \equiv 2(mod\ 4)
\end{aligned}$$

For a generalized version and its proof, see Proposition 11 in [11].

Proof of the backward direction $b(0) \equiv 1(mod\ 2)$ because $C_1^{b,1} \equiv 1(mod\ 2)$. Since $C_2^{b,1} \equiv 2(mod\ 4)$, $b(0)(b(0)+b(1)) \equiv 2(mod\ 4)$ and $b(0)+b(1) \equiv 2(mod\ 4)$. Therefore, $b(1) \equiv b(0)(mod\ 4)$. Furthermore,

$$\begin{aligned} C_4^{b,1} &= b(0)^4 + 3b(0)^3b(1) + 3b(0)^2b(1)^2 + b(0)b(1)^3 + 2b(0)^2b(1)b(2) \\ &+ 2b(0)b(1)^2b(2) + b(0)b(1)b(2)^2 + b(0)b(1)b(2)b(3) \\ &\equiv 1+3+3+1+2b(1)b(2)+2b(0)b(2)+1+b(2)b(3) \\ &\equiv 1+b(2)b(3) \\ &\equiv 2(mod\ 4) \end{aligned}$$

because from $b(1) \equiv b(0) \equiv 1$ or $3(mod\ 4)$ implies $b(0)^2 \equiv b(0)b(1) \equiv b(1)^2 \equiv 1(mod\ 4)$ and $b(2) \equiv b(0) \equiv 1(mod\ 2)$ implies $b(2)^2 \equiv 1(mod\ 4)$. $b(2)b(3) \equiv 1(mod\ 4)$ means $b(3) \equiv b(2)(mod\ 4)$. ■

The next corollary gives another version of Konvalinka conjecture, Conjecture 8. The following are conditions for the coefficients of a polynomial $b(x)$, and both conditions are used in our proof.

Corollary 9 *If $b(x) = b_0 + b_1x + b_2x^2 + \cdots + b_lx^l \in Z[X]$, the conditions of $b(x)$ in Conjecture 8 are equivalent to the following conditions.*

1. $b_0 \equiv 1(mod\ 2)$
2. $b_1 + b_2 + \cdots + b_l \equiv 0(mod\ 4)$
3. $b_3 + b_5 + b_7 + \cdots \equiv 0(mod\ 2)$

Proof $b(0) = b_0 \equiv 1(mod\ 2)$ is equivalent to condition 1 and $b(1) - b(0) = b_1 + b_2 + \cdots + b_l \equiv 0(mod\ 4)$ is equivalent to condition 2. Furthermore,

$$\begin{aligned} b(3) - b(2) &\equiv b(-1) - b(2) \\ &\equiv (b_0 - b_1 + b_2 - b_3 + b_4 - \cdots) - (b_0 + 2b_1) \\ &\equiv (b_1 + b_2 + \cdots + b_l) - 2(b_3 + b_5 + b_7 + \cdots)(mod\ 4) \end{aligned}$$

implies that $b(3) \equiv b(2)(mod\ 4)$ is equivalent to condition 3 under condition 1 and 2. Therefore, the conditions that $b(0) \equiv 1(mod\ 2)$, $b(1) \equiv b(0)(mod\ 4)$, and $b(3) \equiv b(2)(mod\ 4)$ are equivalent to condition 1, 2, and 3. ■

In Section 3.2, $\xi_2(C_n) = s(n+1) - 1$ is proved by finding the recurrence relation of Catalan numbers in Theorem 10. Its coefficients have the simple largest power of two, and the proof is completed by mathematical induction on n.

In Section 3.3, Konvalinka conjecture is divided into two theorems. Theorem 11 shows $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ if $b(0) \equiv 1(mod\ 2)$ and $b(0) \equiv b(1) \equiv b(2) \equiv b(3)(mod\ 4)$, and Theorem 14 shows $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ if $b(0) \equiv 1(mod\ 2)$ and $b(0) \equiv b(1) \equiv -b(2) \equiv -b(3)(mod\ 4)$. The proof of Theorem 11 is similar to that of Theorem 10 by finding the recurrence relations of weighted Catalan numbers in Section 3.3 and Section 3.4 with two lemmas. Lemma 12 and Lemma 13 provide the largest power of two dividing the coefficients of the recurrence relations.

In Section 3.5, Theorem 14 is similarly proved by Lemma 15 and Lemma 16. The lemmas are more complicated than the previous lemmas but they are similar.

In Section 3.6, Theorem 20 provides the conditions of $\xi_2(C_n^{b,1}) = \xi_2(C_n)$ for a general function $b(x)$. $\xi_3(C_n)$ is arithmetically investigated in Theorem 21.

3.2 The power of two in Catalan numbers

$\xi_2(C_n) = s(n+1) - 1$ is proved by Dyck path interpretations. The recurrence relations of $W_{n,k}$, the number of Dyck paths from $(0,0)$ to (n,k), are represented by a matrix and its characteristic polynomial provides the recurrence relations of Catalan numbers. Its coefficients are binomial coefficients, and we know that $\xi_2(\binom{n}{k})$ is the number of carries when we add k and $n-k$ in the base 2. $\xi_2(C_n) = s(n+1) - 1$ is finally proved.

The idea of the proof is also used in the next sections for Konvalinka conjecture.

Theorem 10 [*4, Theorem* 2.1]

$$\xi_2(C_n) = s(n+1) - 1$$

for all $n \geq 0$, where $s(m)$ is the sum of digits in the binary expansion of m.

Proof $W_{n,k}$ is the number of paths from $(0,0)$ to (n,k) that lies above the x-axis. The recurrence relations for $W_{n,k}$ are

$$W_{n+1,k+1} = W_{n,k} + W_{n,k+2}$$

for $n, k \geq 0$ and $W_{n+1,0} = W_{n,1}$ for $n \geq 0$. Therefore, for any x_i and $y_i (i \geq 1)$,

$$A_r(W_{n,0}, \cdots, W_{n,k-1}, W_{n,k}, x_1, \cdots) = (W_{n+1,0}, \cdots, W_{n+1,k-1}, y_1, y_2, \cdots) \quad \text{if } r \geq k$$
$$A_r(W_{n,0}, \cdots, W_{n,n}, 0, 0, \cdots) = (W_{n+1,0}, \cdots, W_{n+1,n}, W_{n+1,n+1}, 0, \cdots) \quad \text{if } r \geq n+1$$

and the first component of $A_r^{2n}(1, 0, \cdots, 0)$ is $W_{2n,0} = C_n$ for $0 \leq n \leq r$ if A_r is the following $(r+1) \times (r+1)$ matrix.

$$\mathbf{A_r} := \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 1 & 0 & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & 1 & 0 & 1 \\ \vdots & \vdots & 0 & 1 & 0 \end{pmatrix}$$

If D_r is the characteristic polynomial of A_r, $D_0 = x$, $D_1 = x^2 - 1$, and

$$D_r = xD_{r-1} - D_{r-2} \quad for\; r \geq 2$$

From the above recurrence relation, it is shown that $D_r = x^{r+1} - d_{(r,1)}x^{r-1} + d_{(r,2)}x^{r-3} - \cdots$ for

$$d_{(r,k)} = \sum 1 \times 1 \times \cdots \times 1$$

where the sum is over all $(c_1, c_2, \cdots, c_k)$ such that $0 \leq c_1 < c_2 - 1 < c_3 - 2 < \cdots < c_k - (k-1) \leq r - k = (r-1) - (k-1)$. So, $d_{(r,k)} = \binom{r-k+1}{k}$.
For $r = 2^t - 2$, $\xi_2(d_{(2^t-2,k)}) = \xi_2(\binom{2^t-1-k}{k}) = s(k)$ for $1 \leq k < 2^{t-1}$ and $t \geq 2$. From the Cayley-Hamilton theorem for $x^{2s-1}D_{2^t-2}$,

$$C_{s+2^{t-1}-1} - d_{(2^t-2,1)}C_{s+2^{t-1}-2} + d_{(2^t-2,2)}C_{s+2^{t-1}-3} - \cdots + (-1)^{2^{t-1}-1}d_{(2^t-2,2^{t-1}-1)}C_s = 0 \quad (3.2)$$

where $1 \leq s \leq 2^{t-1} - 1$ since C_i is the first component of $A^{2i}_{2^t-2}(1, 0, \cdots, 0)$ for $0 \leq i \leq 2^t - 2$.
Similarly, for $r = 2^t - 1$, $\xi_2(d_{(2^t-1,k)}) = \xi_2(\binom{2^t-k}{k}) = s(k) - 1$ for $1 \leq k \leq 2^{t-1}$ and $t \geq 2$. From $x^{2s}D_{2^t-1}$,

$$C_{s+2^{t-1}} - d_{(2^t-1,1)}C_{s+2^{t-1}-1} + d_{(2^t-1,2)}C_{s+2^{t-1}-2} - \cdots + (-1)^{2^{t-1}}d_{(2^t-1,2^{t-1})}C_s = 0 \quad (3.3)$$

where $0 \leq s \leq 2^{t-1} - 1$.
The proof is done by induction on n with (3.2) and (3.3). For $0 \leq n \leq 4$, it is obvious. It is assumed that the statement is true for $n < k (k \geq 5)$, and it is proved for $n = k$.

Case 1 : $k = 2^t - 1$ for some $t \geq 3$(in this case, $s(k+1) - 1 = 0$)
From (3.3)($r = 2^t - 1$, $s = 2^{t-1} - 1$),

$$\begin{aligned} C_{2^t-1} &\equiv (-1)^{2^{t-1}+1}d_{(2^t-1,2^{t-1})}C_{2^{t-1}-1} \\ &\equiv 1 (mod\; 2) \end{aligned}$$

because $\xi_2(C_n) = 0(2^{t-1} - 1 \leq n < 2^t - 1)$ if and only if $n = 2^{t-1} - 1$.

Case 2 : $2^t \leq k < 2^{t+1} - 1$ for some $t \geq 2$

$\xi_2(C_k) = s(k+1) - 1$ from (3.2) when $s = k - 2^t + 1$(in this case, $r = 2^{t+1} - 2$) because $\xi_2(d_{(2^{t+1}-2,i)}C_{k-i}) = s(i) + s(k+1-i) - 1 \geq s(k+1) - 1$ for $1 \leq i \leq 2^t - 1$ with equality for $2^{s(k+1)-1} - 1$ cases and $s(k+1) - 1 \geq 1$ for $2^t \leq k < 2^{t+1} - 1$.

From *Case* 1 and 2, the proof is done. ■

Remark It can be shown that $s(m) + s(n) \geq s(m+n)$, and $s(m) + s(n)$ is equal to $s(m+n)$ if and only if $m_i \neq n_j$ for all $0 \leq i \leq p$ and $0 \leq j \leq q$, where $m = 2^{m_0} + 2^{m_1} + \cdots 2^{m_p}$ and $n = 2^{n_0} + 2^{n_1} + \cdots 2^{n_q}$ for $0 \leq m_0 < m_1 < \cdots < m_p$ and $0 \leq n_0 < n_1 < \cdots < n_q$. Therefore, *Case* 2 in Theorem 10 is easy to check.

3.3 Proof of Konvalinka Conjecture: Part 1

In this section, Konvalinka conjecture is divided into two theorems: Theorem 11 and Theorem 14. Theorem 11(respectively, Theorem 14) shows that

$$\xi_2(C_n^{b,1}) = \xi_2(C_n) = s(n+1) - 1$$

if $b(0) \equiv 1 (mod\ 2)$ and

$$b(0) \equiv b(1) \equiv b(2) \equiv b(3) (mod\ 4)$$

$$(\text{respectively, } b(0) \equiv b(1) \equiv -b(2) \equiv -b(3) (mod\ 4))$$

Theorem 11(the first part of Konvalinka conjecture) is proved by Lemma 12 and Lemma 13 in Section 3.3 and Section 3.4. Similarly, Theorem 14(the second part of Konvalinka conjecture) is proved by Lemma 15 and Lemma 16 in Section 3.5.

Theorem 11 *$b(x)$ is a polynomial. If $b(0) \equiv 1 (mod\ 2)$ and $b(0) \equiv b(1) \equiv b(2) \equiv b(3) (mod\ 4)$,*

$$\xi_2(C_n^{b,1}) = \xi_2(C_n) = s(n+1) - 1$$

for all $n \geq 0$.

Proof Similar to Theorem 10,

$$\mathbf{A_r} := \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ b(0) & 0 & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & b(r-2) & 0 & 1 \\ \vdots & \vdots & 0 & b(r-1) & 0 \end{pmatrix}$$

and $D_0 = x$, $D_1 = x^2 - b(0)$,

$$D_r = xD_{r-1} - b(r-1)D_{r-2} \quad for\ r \geq 2$$

From the above recurrence relation, it is shown that $D_r = x^{r+1} - d_{(r,1)}x^{r-1} + d_{(r,2)}x^{r-3} - \cdots$ where

$$d_{(r,k)} = \sum b(c_1)b(c_2)\cdots b(c_k) \tag{3.4}$$

and the sum is over all $(c_1, c_2, \cdots, c_k)$ such that $0 \leq c_1 < c_2 - 1 < c_3 - 2 < \cdots < c_k - (k-1) \leq r - k = (r-1) - (k-1)$.

It is easy to check that $\xi_2(C_n^{b,1}) = \xi_2(C_n) = s(n+1) - 1$ for $0 \leq n \leq 4$ and $d_{(2^{t+1}-1, 2^t)} = b(0)b(2)\cdots b(2^{t+1}-2) \equiv 1 (mod\ 2)$. If $\xi_2(d_{(2^t-2,k)}) = s(k)$ for $1 \leq k < 2^{t-1}$ and $\xi_2(d_{(2^t-1,k)}) = s(k) - 1$ for $1 \leq k \leq 2^{t-1} (t \geq 2)$, the proof in Theorem 10 can be used in the same way.

Therefore, it is enough to show that $\xi_2(d_{(2^t-2,k)}) = s(k)$ for $1 \leq k < 2^{t-1}$ and $\xi_2(d_{(2^t-1,k)}) = s(k) - 1$ for $1 \leq k \leq 2^{t-1} (t \geq 2)$. It will be shown on Lemma 12.

■

Remark For $b(x+h)$, $d_{(r,k)}$ in (3.4) is generalized.

$$d^h_{(r,k)} = \sum b(c_1+h)b(c_2+h)\cdots b(c_k+h) \tag{3.5}$$

where the sum is over all $(c_1, c_2, \cdots, c_k)$ such that $0 \le c_1 < c_2 - 1 < c_3 - 2 < \cdots < c_k - (k-1) \le r - k = (r-1) - (k-1)$.

The recurrence relations of $d^h_{(2^n-2,i)}$ and $d^h_{(2^n-1,i)}$ are found in Lemma 13, and $\xi_2(d^h_{(2^n-2,i)})$ and $\xi_2(d^h_{(2^n-1,i)})$ are studied in Lemma 12 by mathematical induction on n. The lemmas are proved in the next section.

Lemma 12 *For* $n \ge 1$, *if* $C^h_{(n,i)} = d^h_{(2^n-2,i)}$ *and* $D^h_{(n,i)} = d^h_{(2^n-1,i)}$,

$$\begin{aligned} \xi_2(D^h_{(n,i)}) &= s(i) - 1 \quad \textit{for} \quad 1 \le i \le 2^{n-1} \\ \xi_2(D^h_{(n,i)} - D^{h-1}_{(n,i)}) &\ge s(i) + 1 \quad \textit{for} \quad 0 \le i \le 2^{n-1} \\ \xi_2(C^h_{(n,i)}) &= s(i) \quad \textit{for} \quad 1 \le i < 2^{n-1} \end{aligned}$$

3.4 Lemmas

For $n = 1$ and 2, C^h and D^h are

$$\begin{array}{lll} C^h_{(1,0)} = 1 & & \\ D^h_{(1,0)} = 1 & D^h_{(1,1)} = b(h) & \\ C^h_{(2,0)} = 1 & C^h_{(2,1)} = b(h) + b(h+1) & \\ D^h_{(2,0)} = 1 & D^h_{(2,1)} = b(h) + b(h+1) + b(h+2) & D^h_{(2,2)} = b(h)b(h+2) \end{array}$$

The recurrence relations for C^h and D^h are provided in the next lemma before we prove Lemma 12. $C^h_{(k+1,i)}$ and $D^h_{(k+1,i)}$ are expressed by $C^h_{(k,j)}$ and $D^h_{(k,j)}$, where $0 \le j \le 2^{k-1}$.

Lemma 13 *$b(x)$ is a polynomial. If $b(0) \equiv 1(mod\ 2)$ and $b(0) \equiv b(1) \equiv b(2) \equiv b(3)(mod\ 4)$,*

$$\begin{aligned} C^h_{(k+1,i)} &\equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)} + D^h_{(k,i-j)}) - \sum_{j_0} C^h_{(k,j_0)} C^h_{(k,i-j_0)} (mod\ 2^{k+1}) \\ D^h_{(k+1,i)} &\equiv \sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} + b(h-1) \sum_{j''} C^h_{(k,j'')} C^{h+1}_{(k,i-1-j'')} (mod\ 2^{k+1}) \end{aligned}$$

for $k \geq 2$, where the sum is over all $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$, $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$, $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$, and $max\{0, i - 2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

Proof $b(x)$ is $b_0 + b_1 x + b_2 x^2 + \cdots + b_l x^l$. Since $0 \equiv b(2) - b(0) \equiv (b_0 + 2b_1) - b_0 \equiv 2b_1 (mod\ 4)$, $b_1 \equiv 0 (mod\ 2)$. Therefore,

$$\begin{aligned} b(x+2^k) - b(x) &\equiv b_1(x + 2^k - x) + (b_2 \cdot 2 \cdot 2^k + \cdots + b_l \cdot l \cdot 2^k)x \\ &\equiv 2^k b_1 + 2^k(b_3 + b_5 + \cdots)x \\ &\equiv 0(mod\ 2^{k+1}) \end{aligned}$$

for $x \geq 0$ because $(x+2^k)^i - x^i \equiv i2^k x^{i-1} \equiv i2^k x (mod\ 2^{k+1})$ for $i \geq 2$ and $b_3 + b_5 + b_7 + \cdots \equiv 0(mod\ 2)$ from condition 3 in Corollary 9.

Case 1 : The recurrence relation of $C^h_{(k+1,i)}$

$C^h_{(k+1,i)}$ is the sum of products of nonconsecutive i number of $b(l+h)$, where $0 \leq l \leq 2^{k+1} - 3$. If $b(2^k - 2 + h)$ is not used in the product, the sum is

$$\sum_j C^h_{(k,j)} D^{h-1}_{(k,i-j)} (mod\ 2^{k+1}) \tag{3.6}$$

because the first part before $b(2^k - 2 + h)$ with j number of $b(l+h)$ is $C^h_{(k,j)}$ and the second part after $b(2^k - 2 + h)$ with $i - j$ number of $b(l+h)$ is $D^{h+2^k-1}_{(k,i-j)} \equiv D^{h-1}_{(k,i-j)} (mod\ 2^{k+1})$ from $b(x+2^k) \equiv b(x)(mod\ 2^{k+1})$, where $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$.

If $b(2^k-1+h)$ is not used, the sum is

$$\sum_j D^h_{(k,i-j)} C^h_{(k,j)} (mod\ 2^{k+1}) \tag{3.7}$$

because the first part before $b(2^k-1+h)$ with $i-j$ number of $b(l+h)$ is $D^h_{(k,i-j)}$ and the second part after $b(2^k-1+h)$ with j number of $b(l+h)$ is $C^{h+2^k}_{(k,j)} \equiv C^h_{(k,j)}(mod\ 2^{k+1})$ from $b(x+2^k) \equiv b(x)(mod\ 2^{k+1})$, where $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$.
If both $b(2^k-2+h)$ and $b(2^k-1+h)$ are not used, the sum is

$$\sum_{j_0} C^h_{(k,j_0)} C^h_{(k,i-j_0)} (mod\ 2^{k+1}) \tag{3.8}$$

because the first part before $b(2^k-2+h)$ with j_0 number of $b(l+h)$ is $C^h_{(k,j_0)}$ and the second part after $b(2^k-1+h)$ with $i-j_0$ number of $b(l+h)$ is $C^{h+2^k}_{(k,i-j_0)} \equiv C^h_{(k,i-j_0)}(mod\ 2^{k+1})$ from $b(x+2^k) \equiv b(x)(mod\ 2^{k+1})$, where $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$.
From (3.6), (3.7), and (3.8),

$$C^h_{(k+1,i)} \equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)} + D^h_{(k,i-j)}) - \sum_{j_0} C^h_{(k,j_0)} C^h_{(k,i-j_0)} (mod\ 2^{k+1})$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ and $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$.

Case 2 : The recurrence relation of $D^h_{(k+1,i)}$
$D^h_{(k+1,i)}$ is the sum of products of nonconsecutive i number of $b(l+h)$, where $0 \leq l \leq 2^{k+1}-2$. If $b(2^k-1+h)$ is not used in the product, the sum is

$$\sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} (mod\ 2^{k+1}) \tag{3.9}$$

because the first part before $b(2^k-1+h)$ with j' number of $b(l+h)$ is $D^h_{(k,j')}$ and the second part after $b(2^k-1+h)$ with $i-j'$ number of $b(l+h)$ is $D^{h+2^k}_{(k,i-j')} \equiv$

$D^h_{(k,i-j')}(mod\ 2^{k+1})$ from $b(x+2^k) \equiv b(x)(mod\ 2^{k+1})$, where $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$.

If $b(2^k - 1 + h)$ is used in the product, the sum is

$$b(h-1)\sum_{j''} C^h_{(k,j'')} C^{h+1}_{(k,i-1-j'')}(mod\ 2^{k+1}) \tag{3.10}$$

because the first part before $b(2^k - 2 + h)$ with j'' number of $b(l+h)$ is $C^h_{(k,j'')}$ and the second part after $b(2^k + h)$ with $i-1-j''$ number of $b(l+h)$ is $C^{h+2^k+1}_{(k,i-1-j'')} \equiv C^{h+1}_{(k,i-1-j'')}(mod\ 2^{k+1})$ from $b(x+2^k) \equiv b(x)(mod\ 2^{k+1})$, where $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

From (3.9) and (3.10),

$$D^h_{(k+1,i)} \equiv \sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} + b(h-1)\sum_{j''} C^h_{(k,j'')} C^{h+1}_{(k,i-1-j'')}(mod\ 2^{k+1})$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$. ■

The proof of Lemma 12 is completed by mathematical induction with the recurrence relations in Lemma 13.

Proof of Lemma 12 $D^h_{(n,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$, $D^h_{(n,i)} - D^{h-1}_{(n,i)} \equiv 0(mod\ 2^{s(i)+1})$, and $C^h_{(n,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ are shown by mathematical induction on n. For $n = 1$, $D^h_{(1,1)} \equiv b(h) \equiv 1(mod\ 2)$ and $D^h_{(1,0)} - D^{h-1}_{(1,0)} = 1 - 1 = 0$, $D^h_{(1,1)} - D^{h-1}_{(1,1)} \equiv b(h) - b(h-1) \equiv 0(mod\ 4)$. For $n = 2$, $D^h_{(2,1)} \equiv b(h) + b(h+1) + b(h+2) \equiv 1(mod\ 2)$, $D^h_{(2,2)} \equiv b(h)b(h+2) \equiv 1(mod\ 2)$ and $D^h_{(2,0)} - D^{h-1}_{(2,0)} = 1 - 1 = 0$, $D^h_{(2,1)} - D^{h-1}_{(2,1)} \equiv b(h+2) - b(h-1) \equiv 0(mod\ 4)$, $D^h_{(2,2)} - D^{h-1}_{(2,2)} \equiv b(h)b(h+2) - b(h-1)b(h+1) \equiv 0(mod\ 4)$. Similarly, $C^h_{(2,1)} \equiv b(h) + b(h+1) \equiv 2(mod\ 4)$. It is assumed that the statement is true for $n = k(k \geq 2)$, and it is proved for $n = k+1$.

Case 1: $D^h_{(k+1,i)}$

Case 1 deals with $D^h_{(k+1,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \le i \le 2^k$. From Lemma 13,

$$D^h_{(k+1,i)} \equiv \sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} + b(h-1) \sum_{j''} C^h_{(k,j'')} C^{h+1}_{(k,i-1-j'')} (mod\ 2^{k+1}) \qquad (3.11)$$

where the sum is over all $max\{0, i-2^{k-1}\} \le j' \le min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \le j'' \le min\{i-1, 2^{k-1}-1\}$.

For $2 \le i \le 2^k$, $\sum_{j''} C^h_{(k,j'')} C^{h+1}_{(k,i-1-j'')} \equiv 0(mod\ 2^{s(i)})$ can be shown. If $2 \le i \le 2^{k-1}$, the sum is over all $0 \le j'' \le i-1$, and $\xi_2(C^h_{(k,j'')}) + \xi_2(C^{h+1}_{(k,i-1-j'')}) = s(j'') + s(i-1-j'') \ge s(i-1) \ge s(i)-1$ with equality for $2^{s(i-1)}$ cases if $i-1$ is even(in this case, $s(i-1) = s(i)-1$) and 0 cases if $i-1$ is odd(in this case, $s(i-1) > s(i)-1$). If $2^{k-1} < i \le 2^k$, the sum is over all $i-2^{k-1} \le j'' \le 2^{k-1}-1$ and $\xi_2(C^h_{(k,j'')}) + \xi_2(C^{h+1}_{(k,i-1-j'')}) = s(j'') + s(i-1-j'') > s(i-1) \ge s(i)-1$ because $1 \le j'', i-1-j'' < 2^{k-1}$(but $2^{k-1} \le i-1 < 2^k$). Therefore, for $2 \le i \le 2^k$, (3.11) is

$$D^h_{(k+1,i)} \equiv \sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} (mod\ 2^{s(i)}) \qquad (3.12)$$

where the sum is over all $max\{0, i-2^{k-1}\} \le j' \le min\{i, 2^{k-1}\}$.

(1) : $D^h_{(k+1,i)}$ for $i = 1$

$D^h_{(k+1,1)} \equiv 2D^h_{(k,0)} D^h_{(k,1)} + b(h-1) C^h_{(k,0)} C^{h+1}_{(k,0)} \equiv 1(mod\ 2)$. Therefore, $D^h_{(k+1,1)} \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$.

(2) : $D^h_{(k+1,i)}$ for $2 \le i \le 2^{k-1}$

In this case, the sum in (3.12) is over all $0 \le j' \le i$. Therefore,

$$\begin{aligned} D^h_{(k+1,i)} &\equiv \sum_{0 \le j' \le i} D^h_{(k,j')} D^h_{(k,i-j')} \\ &\equiv \sum_{0 < j' < \frac{i}{2}} 2D^h_{(k,j')} D^h_{(k,i-j')} + 1_{\{\text{i is even}\}} D^h_{(k,\frac{i}{2})} D^h_{(k,\frac{i}{2})} \\ &\equiv 2^{s(i)-1}(mod\ 2^{s(i)}) \end{aligned}$$

because $1+(s(j')-1)+(s(i-j')-1) \geq s(i)-1$ for $0 < j' < \frac{i}{2}$ with equality for $2^{s(i)-1}-1$ cases and $2(s(\frac{i}{2})-1) = s(i)+(s(i)-2) \geq s(i)-1$ with equality for one case if $s(i) = 1$.

(3) : $D^h_{(k+1,i)}$ for $2^{k-1} < i < 2^k$

In this case, the sum in (3.12) is over all $i - 2^{k-1} \leq j' \leq 2^{k-1}$. Therefore,

$$\begin{aligned}
D^h_{(k+1,i)} &\equiv \sum_{i-2^{k-1} \leq j' \leq 2^{k-1}} D^h_{(k,j')} D^h_{(k,i-j')} \\
&\equiv \sum_{i-2^{k-1} \leq j' < \frac{i}{2}} 2 D^h_{(k,j')} D^h_{(k,i-j')} + 1_{\{\text{i is even}\}} D^h_{(k,\frac{i}{2})} D^h_{(k,\frac{i}{2})} \\
&\equiv 2^{s(i)-1} (mod\ 2^{s(i)})
\end{aligned}$$

because $1 + (s(j') - 1) + (s(i - j') - 1) \geq s(i) - 1$ for $i - 2^{k-1} \leq j' < \frac{i}{2}$ with equality for one case when $j' = i - 2^{k-1}$ from $1 \leq j', i - j' \leq 2^{k-1}$(but $2^{k-1} < i < 2^k$) and $2(s(\frac{i}{2}) - 1) = s(i) + (s(i) - 2) > s(i) - 1$ from $s(i) > 1$ for $2^{k-1} < i < 2^k$.

(4) : $D^h_{(k+1,i)}$ for $i = 2^k$

$D^h_{(k+1,2^k)} \equiv D^h_{(k,2^{k-1})} D^h_{(k,2^{k-1})} \equiv 1(mod\ 2)$. Therefore, $D^h_{(k+1,2^k)} \equiv 2^{s(2^k)-1}(mod\ 2^{s(2^k)})$.

Case 2: $D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)}$

Case 2 deals with $D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)} \equiv 0(mod\ 2^{s(i)+1})$ for $0 \leq i \leq 2^k$. For $i = 0$, $D^h_{(k+1,0)} - D^{h-1}_{(k+1,0)} = 1 - 1 = 0$. For $1 \leq i \leq 2^k$, from Lemma 13,

$$\begin{aligned}
&D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)} \\
&\equiv \sum_{j'} (D^h_{(k,j')} D^h_{(k,i-j')} - D^{h-1}_{(k,j')} D^{h-1}_{(k,i-j')}) \\
&+ \sum_{j''} C^h_{(k,j'')} (b(h-1) C^{h+1}_{(k,i-1-j'')} - b(h-2) C^{h-1}_{(k,i-1-j'')}) \\
&\equiv \sum_{j'} (D^h_{(k,j')} + D^{h-1}_{(k,j')})(D^h_{(k,i-j')} - D^{h-1}_{(k,i-j')}) \\
&+ \sum_{j''} C^h_{(k,j'')} (b(h-1) C^{h+1}_{(k,i-1-j'')} - b(h-2) C^{h-1}_{(k,i-1-j'')}) (mod\ 2^{k+1})
\end{aligned}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

By definition in (3.5), $d^{h-1}_{(2^k,i-j'')} = D^h_{(k,i-j'')} + b(h-1)C^{h+1}_{(k,i-1-j'')}$ and $d^{h-1}_{(2^k,i-j'')} = D^{h-1}_{(k,i-j'')} + b(h+2^k-2)C^{h-1}_{(k,i-1-j'')} \equiv D^{h-1}_{(k,i-j'')} + b(h-2)C^{h-1}_{(k,i-1-j'')}(mod\ 2^{k+1})$. So, $b(h-1)C^{h+1}_{(k,i-1-j'')} - b(h-2)C^{h-1}_{(k,i-1-j'')} \equiv D^{h-1}_{(k,i-j'')} - D^h_{(k,i-j'')}(mod\ 2^{k+1})$. Therefore,

$$\begin{aligned} & D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)} \\ & \equiv \sum_{j'}(D^h_{(k,j')} + D^{h-1}_{(k,j')})(D^h_{(k,i-j')} - D^{h-1}_{(k,i-j')}) + \sum_{j''} C^h_{(k,j'')}(D^{h-1}_{(k,i-j")} - D^h_{(k,i-j")}) \\ & \equiv 0(mod\ 2^{s(i)+1}) \end{aligned}$$

because $D^h_{(k,i-j')} - D^{h-1}_{(k,i-j')} \equiv 0(mod\ 2^{s(i-j')+1})$, $D^h_{(k,i-j'')} - D^{h-1}_{(k,i-j'')} \equiv 0(mod\ 2^{s(i-j'')+1})$ and $s(j') + s(i-j') + 1 \geq s(i) + 1$, $s(j'') + s(i-j'') + 1 \geq s(i) + 1$.

Case 3: $C^h_{(k+1,i)}$

Case 3 deals with $C^h_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \leq i < 2^k$. From Lemma 13,

$$C^h_{(k+1,i)} \equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)} + D^h_{(k,i-j)}) - \sum_{j_0} C^h_{(k,j_0)}C^h_{(k,i-j_0)}(mod\ 2^{k+1}) \tag{3.13}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ and $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$.

For $i = 1$, $C^h_{(k+1,1)} = \sum_{0 \leq j \leq 2^{k+1}-3} b(h+j) \equiv 2(mod\ 4)$. Therefore, $C^h_{(k+1,1)} \equiv 2^{s(1)}(mod\ 2^{s(1)+1})$.

For $1 \leq i \leq 2^k - 1$,

$$\sum_{j_0} C^h_{(k,j_0)}C^h_{(k,i-j_0)} \equiv 0(mod\ 2^{s(i)+1}) \tag{3.14}$$

can be shown. If $1 \leq i < 2^{k-1}$, the sum is over all $0 \leq j_0 \leq i$, and $\xi(C^h_{(k,j_0)}) + \xi(C^h_{(k,i-j_0)}) = s(j_0) + s(i-j_0) \geq s(i)$ with equality for $2^{s(i)}$ cases. If $2^{k-1} \leq i \leq 2^k - 1$, the sum is over all $i+1-2^{k-1} \leq j_0 \leq 2^{k-1}-1$, and $\xi(C^h_{(k,j_0)}) + \xi(C^h_{(k,i-j_0)}) = s(j_0) + s(i-j_0) > s(i)$ because $1 \leq j_0, i-j_0 < 2^{k-1}$(but $2^{k-1} \leq i < 2^k$). For

$1 \leq i \leq 2^k - 1$, (3.13) is

$$C^h_{(k+1,i)} \equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)} + D^h_{(k,i-j)})(mod\ 2^{s(i)+1})$$

where the sum is over all $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$. For $i = 0$, it doesn't work because $C^h_{(k+1,0)} = 1 \not\equiv 2 = C^h_{(k,0)}(D^{h-1}_{(k,0)} + D^h_{(k,0)})(mod\ 2)$. Therefore, for $0 \leq i \leq 2^k - 1$,

$$C^h_{(k+1,i)} \equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)} + D^h_{(k,i-j)}) + 1_{\{i=0\}}(mod\ 2^{s(i)+1}) \qquad (3.15)$$

where the sum is over all $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$.
For $1 \leq i \leq 2^k - 1$, (3.15) is

$$C^h_{(k+1,i)} \equiv \sum_j C^h_{(k,j)}(D^h_{(k,i-j)} - D^{h-1}_{(k,i-j)}) + 1_{\{i<2^{k-1}\}}C^h_{(k,i)} + 1_{\{i\geq 2^{k-1}\}}2C^h_{(k,i-2^{k-1})}(mod\ 2^{s(i)+1}) \qquad (3.16)$$

because $\sum_j C^h_{(k,j)} \times 2D^{h-1}_{(k,i-j)} \equiv \sum_{j_0} C^h_{(k,j_0)}C^h_{(k,i-j_0)} + 1_{\{i\leq 2^{k-1}-1\}}C^h_{(k,i)}(2D^{h-1}_{(k,0)} - C^h_{(k,0)}) + 1_{\{i\geq 2^{k-1}\}}C^h_{(k,i-2^{k-1})} \times 2D^{h-1}_{(k,2^{k-1})} \equiv 1_{\{i<2^{k-1}\}}C^h_{(k,i)} + 1_{\{i\geq 2^{k-1}\}}2C^h_{(k,i-2^{k-1})}(mod\ 2^{s(i)+1})$ from (3.14), where the sum is over all $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$. For $i = 0$, it also works because $C^h_{(k+1,0)} = 1 = C^h_{(k,0)}(D^h_{(k,0)} - D^{h-1}_{(k,0)}) + C^h_{(k,0)}$.
From the inductive assumption $D^h_{(k,i-j)} - D^{h-1}_{(k,i-j)} \equiv 0(mod\ 2^{s(i-j)+1})$ and $s(j) + s(i - j) + 1 \geq s(i) + 1$, (3.16) is

$$C^h_{(k+1,i)} \equiv 1_{\{i<2^{k-1}\}}C^h_{(k,i)} + 1_{\{i\geq 2^{k-1}\}}2C^h_{(k,i-2^{k-1})}(mod\ 2^{s(i)+1})$$

$C^h_{(k+1,i)} \equiv C^h_{(k,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ if $1 \leq i < 2^{k-1}$ and $C^h_{(k+1,i)} \equiv 2C^h_{(k,i-2^{k-1})} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ if $2^{k-1} \leq i < 2^k$.

From *Case* 1, 2, and 3, the proof is done. ■

3.5 Proof of Konvalinka Conjecture: Part 2

In this section, Theorem 14, the second part of Konvalinka conjecture, is proved.

Theorem 14 *$b(x)$ is a polynomial. If $b(0) \equiv 1 (mod\ 2)$ and $b(0) \equiv b(1) \equiv -b(2) \equiv -b(3) (mod\ 4)$,*

$$\xi_2(C_n^{b,1}) = \xi_2(C_n) = s(n+1) - 1$$

for all $n \geq 0$.

Proof Same as Theorem 11,

$$\mathbf{A_r} := \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ b(0) & 0 & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & b(r-2) & 0 & 1 \\ \vdots & \vdots & 0 & b(r-1) & 0 \end{pmatrix}$$

and $D_r = x^{r+1} - d_{(r,1)}x^{r-1} + d_{(r,2)}x^{r-3} - \cdots$ for

$$d_{(r,k)} = \sum b(c_1)b(c_2)\cdots b(c_k)$$

where the sum is over all $(c_1, c_2, \cdots, c_k)$ such that $0 \leq c_1 < c_2 - 1 < c_3 - 2 < \cdots < c_k - (k-1) \leq r - k = (r-1) - (k-1)$.

It is easy to check that $\xi_2(C_n^{b,1}) = \xi_2(C_n) = s(n+1) - 1$ for $0 \leq n \leq 4$ and $d_{(2^{t+1}-1,2^t)} = b(0)b(2)\cdots b(2^{t+1}-2) \equiv 1 (mod\ 2)$. If $\xi_2(d_{(2^t-2,k)}) = s(k)$ for $1 \leq k < 2^{t-1}$ and $\xi_2(d_{(2^t-1,k)}) = s(k) - 1$ for $1 \leq k \leq 2^{t-1} (t \geq 2)$, the proof is the same as Theorem 11.

Therefore, it is enough to show that $\xi_2(d_{(2^t-2,k)}) = s(k)$ for $1 \leq k < 2^{t-1}$ and $\xi_2(d_{(2^t-1,k)}) = s(k) - 1$ for $1 \leq k \leq 2^{t-1} (t \geq 2)$. It will be shown on Lemma 16.

■

Similar to Lemma 13, the recurrence relations for C^h and D^h are provided in Lemma 15 before Lemma 16.

Lemma 15 *$b(x)$ is a polynomial. If $b(0) \equiv 1(mod\ 2)$ and $b(0) \equiv b(1) \equiv -b(2) \equiv -b(3)(mod\ 4)$,*

$$\begin{aligned}
C^h_{(k+1,i)} &\equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)} + 2^k(i-j)D^{h-1}_{(k,i-j)}) \\
&+ \sum_j D^h_{(k,i-j)}(C^h_{(k,j)} + 2^k j C^h_{(k,j)}) \\
&- \sum_{j_0} C^h_{(k,j_0)}(C^h_{(k,i-j_0)} + 2^k(i-j_0)C^h_{(k,i-j_0)})(mod\ 2^{k+1}) \\
D^h_{(k+1,i)} &\equiv \sum_{j'} D^h_{(k,j')}(D^h_{(k,i-j')} + 2^k(i-j')D^h_{(k,i-j')}) \\
&+ (b(h-1)+2^k)\sum_{j''} C^h_{(k,j'')}(C^{h+1}_{(k,i-1-j'')} + 2^k(i-1-j'')C^{h+1}_{(k,i-1-j'')})(mod\ 2^{k+1})
\end{aligned}$$

for $k \geq 2$, where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$, $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$, $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$, and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

Proof Similar to Lemma 13, $b(x+2^k) - b(x) \equiv 2^k(mod\ 2^{k+1})$ for $x \geq 0$. If $b(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_l x^l$, $b_1 \equiv 1(mod\ 2)$ since $2 \equiv b(2) - b(0) \equiv (b_0 + 2b_1) - b_0 \equiv 2b_1(mod\ 4)$. Therefore,

$$\begin{aligned}
b(x+2^k) - b(x) &\equiv b_1(x + 2^k - x) + (b_2 \cdot 2 \cdot 2^k + \cdots + b_l \cdot l \cdot 2^k)x \\
&\equiv 2^k b_1 + 2^k(b_3 + b_5 + \cdots)x \\
&\equiv 2^k(mod\ 2^{k+1})
\end{aligned}$$

for $x \geq 0$ because $(x+2^k)^i - x^i \equiv i2^k x^{i-1} \equiv i2^k x (mod\ 2^{k+1})$ for $i \geq 2$ and $b_3 + b_5 + b_7 + \cdots \equiv 0(mod\ 2)$ from condition 3 in Corollary 9.

$$\begin{aligned} C_{(k,i)}^{h+2^k} &\equiv \sum b(c_1+h+2^k)b(c_2+h+2^k)\cdots b(c_i+h+2^k) \\ &\equiv \sum (b(c_1+h)+2^k)(b(c_2+h)+2^k)\cdots(b(c_i+h)+2^k) \\ &\equiv \sum (b(c_1+h)b(c_2+h)\cdots b(c_i+h)+2^k i) \\ &\equiv C_{(k,i)}^h + 2^k i C_{(k,i)}^h (mod\ 2^{k+1}) \end{aligned}$$

where the sum is over all $(c_1, c_2, \cdots, c_i)$ such that $0 \leq c_1 < c_2 - 1 < c_3 - 2 < \cdots < c_i - (i-1) \leq 2^k - 2 - i$. Similarly,

$$\begin{aligned} D_{(k,i)}^{h+2^k} &\equiv \sum b(c_1+h+2^k)b(c_2+h+2^k)\cdots b(c_i+h+2^k) \\ &\equiv \sum (b(c_1+h)+2^k)(b(c_2+h)+2^k)\cdots(b(c_i+h)+2^k) \\ &\equiv \sum (b(c_1+h)b(c_2+h)\cdots b(c_i+h)+2^k i) \\ &\equiv D_{(k,i)}^h + 2^k i D_{(k,i)}^h (mod\ 2^{k+1}) \end{aligned}$$

where the sum is over all $(c_1, c_2, \cdots, c_i)$ such that $0 \leq c_1 < c_2 - 1 < c_3 - 2 < \cdots < c_i - (i-1) \leq 2^k - 1 - i$. Therefore,

$$C_{(k,i)}^{h+2^k} \equiv C_{(k,i)}^h + 2^k i C_{(k,i)}^h (mod\ 2^{k+1}) \tag{3.17}$$

$$D_{(k,i)}^{h+2^k} \equiv D_{(k,i)}^h + 2^k i D_{(k,i)}^h (mod\ 2^{k+1}) \tag{3.18}$$

Case 1 : The recurrence relation of $C_{(k+1,i)}^h$

$C_{(k+1,i)}^h$ is the sum of products of nonconsecutive i number of $b(l+h)$, where $0 \leq l \leq 2^{k+1} - 3$. If $b(2^k - 2 + h)$ is not used in the product, the sum is

$$\sum_j C_{(k,j)}^h (D_{(k,i-j)}^{h-1} + 2^k (i-j) D_{(k,i-j)}^{h-1})(mod\ 2^{k+1}) \tag{3.19}$$

because the first part before $b(2^k-2+h)$ with j number of $b(l+h)$ is $C^h_{(k,j)}$ and the second part after $b(2^k-2+h)$ with $i-j$ number of $b(l+h)$ is $D^{h+2^k-1}_{(k,i-j)} \equiv D^{h-1}_{(k,i-j)}+2^k(i-j)D^{h-1}_{(k,i-j)}(mod\ 2^{k+1})$ from (3.18), where $max\{0,i-2^{k-1}\} \le j \le min\{i,2^{k-1}-1\}$.
If $b(2^k-1+h)$ is not used, the sum is

$$\sum_j D^h_{(k,i-j)}(C^h_{(k,j)}+2^kjC^h_{(k,j)})(mod\ 2^{k+1}) \tag{3.20}$$

because the first part before $b(2^k-1+h)$ with $i-j$ number of $b(l+h)$ is $D^h_{(k,i-j)}$ and the second part after $b(2^k-1+h)$ with j number of $b(l+h)$ is $C^{h+2^k}_{(k,j)} \equiv C^h_{(k,j)}+2^kjC^h_{(k,j)}(mod\ 2^{k+1})$ from (3.17), where $max\{0,i-2^{k-1}\} \le j \le min\{i,2^{k-1}-1\}$.
If both $b(2^k-2+h)$ and $b(2^k-1+h)$ are not used, the sum is

$$\sum_{j_0} C^h_{(k,j_0)}(C^h_{(k,i-j_0)}+2^k(i-j_0)C^h_{(k,i-j_0)})(mod\ 2^{k+1}) \tag{3.21}$$

because the first part before $b(2^k-2+h)$ with j_0 number of $b(l+h)$ is $C^h_{(k,j_0)}$ and the second part after $b(2^k-1+h)$ with $i-j_0$ number of $b(l+h)$ is $C^{h+2^k}_{(k,i-j_0)} \equiv C^h_{(k,i-j_0)}+2^k(i-j_0)C^h_{(k,i-j_0)}(mod\ 2^{k+1})$ from (3.17), where $max\{0,i+1-2^{k-1}\} \le j_0 \le min\{i,2^{k-1}-1\}$.
From (3.19), (3.20), and (3.21),

$$\begin{aligned} C^h_{(k+1,i)} &\equiv \sum_j C^h_{(k,j)}(D^{h-1}_{(k,i-j)}+2^k(i-j)D^{h-1}_{(k,i-j)}) \\ &+ \sum_j D^h_{(k,i-j)}(C^h_{(k,j)}+2^kjC^h_{(k,j)}) \\ &- \sum_{j_0} C^h_{(k,j_0)}(C^h_{(k,i-j_0)}+2^k(i-j_0)C^h_{(k,i-j_0)})(mod\ 2^{k+1}) \end{aligned}$$

where the sum is over all $max\{0,i-2^{k-1}\} \le j \le min\{i,2^{k-1}-1\}$ and $max\{0,i+1-2^{k-1}\} \le j_0 \le min\{i,2^{k-1}-1\}$.

Case 2 : The recurrence relation of $D^h_{(k+1,i)}$

$D^h_{(k+1,i)}$ is the sum of products of nonconsecutive i number of $b(l+h)$, where $0 \le l \le$

$2^{k+1}-2$. If $b(2^k-1+h)$ is not used in the product, the sum is

$$\sum_{j'} D^h_{(k,j')}(D^h_{(k,i-j')} + 2^k(i-j')D^h_{(k,i-j')})(mod\ 2^{k+1}) \tag{3.22}$$

because the first part before $b(2^k-1+h)$ with j' number of $b(l+h)$ is $D^h_{(k,j')}$ and the second part after $b(2^k-1+h)$ with $i-j'$ number of $b(l+h)$ is $D^{h+2^k}_{(k,i-j')} \equiv D^h_{(k,i-j')} + 2^k(i-j')D^h_{(k,i-j')}(mod\ 2^{k+1})$ from (3.18), where $max\{0, i-2^{k-1}\} \le j' \le min\{i, 2^{k-1}\}$.
If $b(2^k-1+h)$ is used in the product,

$$(b(h-1)+2^k)\sum_{j''} C^h_{(k,j'')}(C^{h+1}_{(k,i-1-j'')} + 2^k(i-1-j'')C^{h+1}_{(k,i-1-j'')})(mod\ 2^{k+1}) \tag{3.23}$$

because the first part before $b(2^k-2+h)$ with j'' number of $b(l+h)$ is $C^h_{(k,j'')}$ and the second part after $b(2^k+h)$ with $i-1-j''$ number of $b(l+h)$ is $C^{h+2^k+1}_{(k,i-1-j'')} \equiv C^{h+1}_{(k,i-1-j'')} + 2^k(i-1-j'')C^{h+1}_{(k,i-1-j'')}(mod\ 2^{k+1})$ from (3.17), where $max\{0, i-2^{k-1}\} \le j'' \le min\{i-1, 2^{k-1}-1\}$.
From (3.22) and (3.23),

$$\begin{aligned} D^h_{(k+1,i)} &\equiv \sum_{j'} D^h_{(k,j')}(D^h_{(k,i-j')} + 2^k(i-j')D^h_{(k,i-j')}) \\ &+ (b(h-1)+2^k)\sum_{j''} C^h_{(k,j'')}(C^{h+1}_{(k,i-1-j'')} + 2^k(i-1-j'')C^{h+1}_{(k,i-1-j'')})(mod\ 2^{k+1}) \end{aligned}$$

where the sum is over all $max\{0, i-2^{k-1}\} \le j' \le min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \le j'' \le min\{i-1, 2^{k-1}-1\}$. ■

$\xi_2(C^h_{(n,i)})$ and $\xi_2(D^h_{(n,i)})$ are investigated in Lemma 16 by induction on n. Lemma 16 is similar to Lemma 12 but $\xi_2(D^{2h+1}_{(n,i)} - D^{2h}_{(n,i)})$ and $\xi_2(C^{2h+1}_{(n,i)})$ are different if i is odd.

Lemma 16 *For $n \geq 2$,*

$$\begin{array}{rcll} \xi_2(D^h_{(n,i)}) & = & s(i)-1 & \\ \xi_2(D^{2h}_{(n,i)} - D^{2h-1}_{(n,i)}) & \geq & s(i)+1 & \\ \xi_2(D^{2h+1}_{(n,i)} - D^{2h}_{(n,i)}) & = & s(i) & \text{if } i \text{ is odd} \\ \xi_2(D^{2h+1}_{(n,i)} - D^{2h}_{(n,i)}) & \geq & s(i)+1 & \text{if } i \text{ is even} \end{array}$$

for $1 \leq i \leq 2^{n-1}$ and

$$\begin{array}{rcll} \xi_2(C^{2h}_{(n,i)}) & = & s(i) & \\ \xi_2(C^{2h+1}_{(n,i)}) & = & s(i) & \text{if } i \text{ is even} \\ \xi_2(C^{2h+1}_{(n,i)}) & \geq & s(i)+1 & \text{if } i \text{ is odd} \end{array}$$

for $1 \leq i < 2^{n-1}$.

Proof It is shown by mathematical induction on n. For $n=2$, $D^h_{(2,1)} \equiv b(h) + b(h+1)+b(h+2) \equiv 1(mod\ 2)$, $D^h_{(2,2)} \equiv b(h)b(h+2) \equiv 1(mod\ 2)$ and $D^{2h}_{(2,1)} - D^{2h-1}_{(2,1)} \equiv b(2h+2) - b(2h-1) \equiv 0(mod\ 4)$, $D^{2h+1}_{(2,1)} - D^{2h}_{(2,1)} \equiv b(2h+3) - b(2h) \equiv 2(mod\ 4)$, $D^h_{(2,2)} - D^{h-1}_{(2,2)} \equiv b(h)b(h+2) - b(h-1)b(h+1) \equiv 0(mod\ 4)$. Similarly, $C^{2h}_{(2,1)} \equiv b(2h) + b(2h+1) \equiv 2(mod\ 4)$ and $C^{2h+1}_{(2,1)} \equiv b(2h+1) + b(2h+2) \equiv 0(mod\ 4)$. It is assumed that the statement is true for $n = k(k \geq 2)$, and it is proved for $n = k+1$.

Case 1: $D^h_{(k+1,i)}$

Case 1 deals with $D^h_{(k+1,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^k$. Since $1 \leq s(i) \leq k$ for $1 \leq i \leq 2^k$, the following recurrence relation modulo 2^k from Lemma 15 is enough:

$$D^h_{(k+1,i)} \equiv \sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} + b(h-1) \sum_{j''} C^h_{(k,j'')} C^{h+1}_{(k,i-1-j'')} (mod\ 2^k) \tag{3.24}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

For $i=1$, $D^h_{(k+1,1)} \equiv 2D^h_{(k,0)} D^h_{(k,1)} + b(h-1) C^h_{(k,0)} C^{h+1}_{(k,0)} \equiv 1(mod\ 2)$. Therefore, $D^h_{(k+1,1)} \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$.

For $2 \le i \le 2^k$, $\sum_{j''} C^{2h+1}_{(k,j'')} C^{2h}_{(k,i-1-j'')} \equiv 0 (mod\ 2^{s(i)})$ can be shown. If $2 \le i \le 2^{k-1}$, the sum is over all $0 \le j'' \le i-1$, and $\xi(C^{2h+1}_{(k,j'')}) + \xi(C^{2h}_{(k,i-1-j'')}) \ge s(j'') + s(i-1-j'') \ge s(i-1) \ge s(i) - 1$ with equality for $2^{s(i-1)}$ cases if $i-1$ is even(in this case, j'' and $i-1-j''$ are even) and 0 cases if $i-1$ is odd(in this case, $s(i-1) > s(i) - 1$). If $2^{k-1} < i \le 2^k$, the sum is over all $i - 2^{k-1} \le j'' \le 2^{k-1} - 1$, and $\xi(C^{2h+1}_{(k,j'')}) + \xi(C^{2h}_{(k,i-1-j'')}) \ge s(j'') + s(i-1-j'') > s(i-1) \ge s(i) - 1$ because $1 \le j'', i-1-j'' < 2^{k-1}$(but $2^{k-1} \le i-1 < 2^k$). Therefore, for $2 \le i \le 2^k$, (3.24) is

$$D^h_{(k+1,i)} \equiv \sum_{j'} D^h_{(k,j')} D^h_{(k,i-j')} (mod\ 2^{s(i)}) \tag{3.25}$$

where the sum is over all $max\{0, i - 2^{k-1}\} \le j' \le min\{i, 2^{k-1}\}$.

The proof is completed by using the idea of *Case* 1 in Lemma 12 because (3.25) is same as (3.12). Therefore,

$$D^h_{(k+1,i)} \equiv 2^{s(i)-1} (mod\ 2^{s(i)})$$

Case 2: $D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)}$

Case 2 deals with $D^{2h}_{(k+1,i)} - D^{2h-1}_{(k+1,i)}$ and $D^{2h+1}_{(k+1,i)} - D^{2h}_{(k+1,i)}$ for $0 \le i \le 2^k$. From Lemma 15,

$$\begin{aligned} & D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)} \\ & \equiv \sum_{j'} (D^h_{(k,j')} - D^{h-1}_{(k,j')})(D^h_{(k,i-j')} + D^{h-1}_{(k,i-j')}) \\ & + \sum_{j''} C^h_{(k,j'')} (b(h-1) C^{h+1}_{(k,i-1-j'')} - b(h-2) C^{h-1}_{(k,i-1-j'')}) (mod\ 2^{k+1}) \end{aligned}$$

For $i = 1$, $D^{2h}_{(k+1,1)} - D^{2h-1}_{(k+1,1)} \equiv b(2h + 2^k - 2) - b(2h-1) \equiv 0 (mod\ 4)$ and $D^{2h+1}_{(k+1,1)} - D^{2h}_{(k+1,1)} \equiv b(2h + 1 + 2^k - 2) - b(2h) \equiv 2 (mod\ 4)$.

For $i = 2^k$, $D^{2h}_{(k+1,2^k)} - D^{2h-1}_{(k+1,2^k)} \equiv \prod_{0 \le j < 2^k} b(2h+2j) - \prod_{0 \le j < 2^k} b(2h+2j-1) \equiv 0 (mod\ 4)$ and $D^{2h+1}_{(k+1,2^k)} - D^{2h}_{(k+1,2^k)} \equiv \prod_{0 \le j < 2^k} b(2h+2j+1) - \prod_{0 \le j < 2^k} b(2h+2j) \equiv 0 (mod\ 4)$ because $b(2h+2j+1) \equiv b(2h+2j) \equiv -b(2h+2j-1) (mod\ 4)$.

By definition in (3.5), $d^{h-1}_{(2^k,i-j'')} = D^h_{(k,i-j'')} + b(h-1)C^{h+1}_{(k,i-1-j'')}$ and $d^{h-1}_{(2^k,i-j'')} = D^{h-1}_{(k,i-j'')} + b(h+2^k-2)C^{h-1}_{(k,i-1-j'')} \equiv D^{h-1}_{(k,i-j'')} + b(h-2)C^{h-1}_{(k,i-1-j'')}(mod\ 2^{k+1})$ if $i-1-j'' \neq 0$. So, $C^h_{(k,j'')}(b(h-1)C^{h+1}_{(k,i-1-j'')} - b(h-2)C^{h-1}_{(k,i-1-j'')}) \equiv C^h_{(k,j'')}(D^{h-1}_{(k,i-j'')} - D^h_{(k,i-j'')})(mod\ 2^{k+1})$ if $i \geq 2$. Therefore, for $2 \leq i \leq 2^k$,

$$
\begin{aligned}
&D^h_{(k+1,i)} - D^{h-1}_{(k+1,i)} \\
&\equiv \sum_{j'}(D^h_{(k,j')} + D^{h-1}_{(k,j')})(D^h_{(k,i-j')} - D^{h-1}_{(k,i-j')}) + \sum_{j''} C^h_{(k,j'')}(D^{h-1}_{(k,i-j")} - D^h_{(k,i-j")})(mod\ 2^{k+1})
\end{aligned}
$$

(1): $D^{2h}_{(k+1,i)} - D^{2h-1}_{(k+1,i)}$ for $2 \leq i < 2^k$

From $D^{2h}_{(k,i-j')} - D^{2h-1}_{(k,i-j')} \equiv 0(mod\ 2^{s(i-j')+1})$, $D^{2h}_{(k,i-j'')} - D^{2h-1}_{(k,i-j'')} \equiv 0(mod\ 2^{s(i-j'')+1})$ by the inductive assumption and $s(j') + s(i-j') + 1 \geq s(i) + 1$, $s(j'') + s(i-j'') + 1 \geq s(i) + 1$,

$$D^{2h}_{(k+1,i)} - D^{2h-1}_{(k+1,i)} \equiv 0(mod\ 2^{s(i)+1})$$

(2): $D^{2h+1}_{(k+1,i)} - D^{2h}_{(k+1,i)}$ for $2 \leq i < 2^k$

In this case,

$$
\begin{aligned}
&D^{2h+1}_{(k+1,i)} - D^{2h}_{(k+1,i)} \\
&\equiv \sum_{j'}(D^{2h+1}_{(k,j')} + D^{2h}_{(k,j')})(D^{2h+1}_{(k,i-j')} - D^{2h}_{(k,i-j')}) + \sum_{j''} C^{2h+1}_{(k,j'')}(D^{2h}_{(k,i-j'')} - D^{2h+1}_{(k,i-j'')}) \\
&\equiv \sum_{j''}(D^{2h+1}_{(k,j'')} + D^{2h}_{(k,j'')} - C^{2h+1}_{(k,j'')})(D^{2h+1}_{(k,i-j'')} - D^{2h}_{(k,i-j'')}) \\
&+1_{\{j'=min\{i,2^{k-1}\}\}}(D^{2h+1}_{(k,j')} + D^{2h}_{(k,j')})(D^{2h+1}_{(k,i-j')} - D^{2h}_{(k,i-j')})(mod\ 2^{s(i)+1})
\end{aligned}
$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

$D^{2h+1}_{(k,j'')} + D^{2h}_{(k,j'')} \equiv (D^{2h+1}_{(k,j'')} - D^{2h}_{(k,j'')}) + 2D^{2h}_{(k,j'')} \equiv 2^{s(j'')} + 2 \times 2^{s(j'')-1} \equiv 0 \equiv C^{2h+1}_{(k,j'')}(mod\ 2^{s(j'')+1})$ if j'' is odd and $D^{2h+1}_{(k,j'')} + D^{2h}_{(k,j'')} \equiv (D^{2h+1}_{(k,j'')} - D^{2h}_{(k,j'')}) + 2D^{2h}_{(k,j'')} \equiv 0 + 2 \times 2^{s(j'')-1} \equiv$

$2^{s(j'')} \equiv C^{2h+1}_{(k,j'')}(mod\ 2^{s(j'')+1})$ if $j'' > 0$ is even. Therefore,

$$\begin{aligned}
&D^{2h+1}_{(k+1,i)} - D^{2h}_{(k+1,i)}\\
&\equiv 1_{\{i\le 2^{k-1}\}}(D^{2h+1}_{(k,0)} + D^{2h}_{(k,0)} - C^{2h+1}_{(k,0)})(D^{2h+1}_{(k,i)} - D^{2h}_{(k,i)})\\
&+1_{\{j'=min\{i,2^{k-1}\}\}}(D^{2h+1}_{(k,j')} + D^{2h}_{(k,j')})(D^{2h+1}_{(k,i-j')} - D^{2h}_{(k,i-j')})\\
&\equiv 1_{\{i\le 2^{k-1}\}}(D^{2h+1}_{(k,i)} - D^{2h}_{(k,i)})\\
&+1_{\{i>2^{k-1}\}}(D^{2h+1}_{(k,2^{k-1})} + D^{2h}_{(k,2^{k-1})})(D^{2h+1}_{(k,i-2^{k-1})} - D^{2h}_{(k,i-2^{k-1})})\\
&\equiv 1_{\{i\le 2^{k-1}\}}(D^{2h+1}_{(k,i)} - D^{2h}_{(k,i)})\\
&+1_{\{i>2^{k-1}\}}2(D^{2h+1}_{(k,i-2^{k-1})} - D^{2h}_{(k,i-2^{k-1})})(mod\ 2^{s(i)+1})
\end{aligned}$$

because $D^{2h+1}_{(k,2^{k-1})} + D^{2h}_{(k,2^{k-1})} \equiv \prod_{0\le j<2^{k-1}} b(2h+2j+1) + \prod_{0\le j<2^{k-1}} b(2h+2j) \equiv 2\prod_{0\le j<2^{k-1}} b(2h+2j) \equiv 2(mod\ 4)$ from $b(2h+2j+1) \equiv b(2h+2j)(mod\ 4)$ for $0 \le j < 2^{k-1}$.

Finally, $D^{2h+1}_{(k+1,i)} - D^{2h}_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ if i is odd, and $D^{2h+1}_{(k+1,i)} - D^{2h}_{(k+1,i)} \equiv 0(mod\ 2^{s(i)+1})$ if i is even.

Case 3: $C^{2h}_{(k+1,i)}$

Case 3 deals with $C^{2h}_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \le i < 2^k$.

For $i = 1$, $C^{2h}_{(k+1,1)} = \sum_{0\le j\le 2^{k+1}-3} b(2h+j) \equiv 2(mod\ 4)$. Therefore, $C^{2h}_{(k+1,1)} \equiv 2^{s(1)}(mod\ 2^{s(1)+1})$.

For $i = 2^k - 1$, from Lemma 15, $C^{2h}_{(k+1,2^k-1)} \equiv C^{2h}_{(k,2^{k-1}-1)}(D^{2h-1}_{(k,2^{k-1})} + 2^k D^{2h-1}_{(k,2^{k-1})}) + (C^{2h}_{(k,2^{k-1}-1)} + 2^k C^{2h}_{(k,2^{k-1}-1)})D^{2h}_{(k,2^{k-1})} \equiv C^{2h}_{(k,2^{k-1}-1)}(D^{2h-1}_{(k,2^{k-1})} + D^{2h}_{(k,2^{k-1})}) \equiv 2^k(mod\ 2^{k+1})$ because $D^{2h-1}_{(k,2^{k-1})} + D^{2h}_{(k,2^{k-1})} \equiv \prod_{0\le j<2^{k-1}} b(2h+2j-1) + \prod_{0\le j<2^{k-1}} b(2h+2j) \equiv 2\prod_{0\le j<2^{k-1}} b(2h+2j) \equiv 2(mod\ 4)$ from $b(2h+2j) \equiv -b(2h+2j-1)(mod\ 4)$ for $0 \le j < 2^{k-1}$. Therefore, $C^{2h}_{(k+1,2^k-1)} \equiv 2^{s(2^k-1)+1}(mod\ 2^{s(2^k-1)+1})$.

Since $1 \le s(i) < k$ for $2 \le i < 2^k - 1$, the following recurrence relation modulo 2^k from Lemma 15 is enough:

$$C^{2h}_{(k+1,i)} \equiv \sum_j C^{2h}_{(k,j)}(D^{2h-1}_{(k,i-j)} + D^{2h}_{(k,i-j)}) - \sum_{j_0} C^{2h}_{(k,j_0)} C^{2h}_{(k,i-j_0)}(mod\ 2^k) \tag{3.26}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ and $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$.

For $1 \leq i < 2^k - 1$,

$$\sum_{j_0} C^{2h}_{(k,j_0)} C^{2h}_{(k,i-j_0)} \equiv 0 (mod\ 2^{s(i)+1})$$

is shown in (3.14). Therefore, (3.26) is

$$C^{2h}_{(k+1,i)} \equiv \sum_j C^{2h}_{(k,j)} (D^{2h-1}_{(k,i-j)} + D^{2h}_{(k,i-j)}) (mod\ 2^{s(i)+1})$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$. For $i = 0$, it doesn't work because $C^{2h}_{(k+1,0)} = 1 \not\equiv 2 = C^{2h}_{(k,0)}(D^{2h-1}_{(k,0)} + D^{2h}_{(k,0)}) (mod\ 2)$. For $i = 2^k - 1$, it works because $C^{2h}_{(k+1,2^k-1)} \equiv 2^k \equiv C^{2h}_{(k,2^{k-1}-1)}(D^{2h-1}_{(k,2^{k-1})} + D^{2h}_{(k,2^{k-1})}) (mod\ 2^{k+1})$. Therefore, for $0 \leq i \leq 2^k - 1$,

$$C^{2h}_{(k+1,i)} \equiv \sum_j C^{2h}_{(k,j)} (D^{2h-1}_{(k,i-j)} + D^{2h}_{(k,i-j)}) + 1_{\{i=0\}} (mod\ 2^{s(i)+1}) \tag{3.27}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$.

For $2 \leq i < 2^k - 1$, (3.27) is

$$C^{2h}_{(k+1,i)} \equiv \sum_j C^{2h}_{(k,j)} (D^{2h}_{(k,i-j)} - D^{2h-1}_{(k,i-j)}) + 1_{\{i<2^{k-1}\}} C^{2h}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h}_{(k,i-2^{k-1})} (mod\ 2^{s(i)+1}) \tag{3.28}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ because $\sum_j C^{2h}_{(k,j)} \times 2D^{2h-1}_{(k,i-j)} \equiv \sum_{j_0} C^{2h}_{(k,j_0)} C^{2h}_{(k,i-j_0)} + 1_{\{i \leq 2^{k-1}-1\}} C^{2h}_{(k,i)} (2D^{2h-1}_{(k,0)} - C^{2h}_{(k,0)}) + 1_{\{i \geq 2^{k-1}\}} C^{2h}_{(k,i-2^{k-1})} \times 2D^{2h-1}_{(k,2^{k-1})} \equiv 1_{\{i<2^{k-1}\}} C^{2h}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h}_{(k,i-2^{k-1})} (mod\ 2^{s(i)+1})$.

From $D^{2h}_{(k,i-j)} - D^{2h-1}_{(k,i-j)} \equiv 0 (mod\ 2^{s(i-j)+1})$ by the inductive assumption,

$$\begin{aligned} C^{2h}_{(k+1,i)} &\equiv 1_{\{i<2^{k-1}\}} C^{2h}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h}_{(k,i-2^{k-1})} \\ &\equiv 2^{s(i)} (mod\ 2^{s(i)+1}) \end{aligned}$$

$C^{2h}_{(k+1,i)} \equiv C^{2h}_{(k,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ if $1 \leq i < 2^{k-1}$, and $C^{2h}_{(k+1,i)} \equiv 2C^{2h}_{(k,i-2^{k-1})} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ if $2^{k-1} \leq i < 2^k$.

Case 4: $C^{2h+1}_{(k+1,i)}$

Case 4 deals with $C^{2h+1}_{(k+1,i)}$ for $1 \leq i < 2^k$.

For $i = 1$, $C^{2h+1}_{(k+1,1)} = \sum_{0 \leq j \leq 2^{k+1}-3} b(2h+1+j) \equiv 0(mod\ 4)$. Therefore, $C^{2h+1}_{(k+1,1)} \equiv 0(mod\ 2^{s(1)+1})$.

For $i = 2^k - 1$, from Lemma 15, $C^{2h+1}_{(k+1,2^k-1)} \equiv C^{2h+1}_{(k,2^{k-1}-1)}(D^{2h}_{(k,2^{k-1})} + 2^k D^{2h}_{(k,2^{k-1})}) + (C^{2h+1}_{(k,2^{k-1}-1)} + 2^k C^{2h+1}_{(k,2^{k-1}-1)})D^{2h+1}_{(k,2^{k-1})} \equiv C^{2h+1}_{(k,2^{k-1}-1)}(D^{2h}_{(k,2^{k-1})} + D^{2h+1}_{(k,2^{k-1})}) \equiv 0(mod\ 2^{k+1})$ because $D^{2h}_{(k,2^{k-1})} + D^{2h+1}_{(k,2^{k-1})} \equiv \prod_{0 \leq j < 2^{k-1}} b(2h+2j) + \prod_{0 \leq j < 2^{k-1}} b(2h+1+2j) \equiv 2\prod_{0 \leq j < 2^{k-1}} b(2h+2j) \equiv 2(mod\ 4)$ from $b(2h+2j+1) \equiv b(2h+2j)(mod\ 4)$ for $0 \leq j < 2^{k-1}$. Therefore, $C^{2h+1}_{(k+1,2^k-1)} \equiv 0(mod\ 2^{s(2^k-1)+1})$.

Since $1 \leq s(i) < k$ for $2 \leq i < 2^k - 1$, the following recurrence relation modulo 2^k from Lemma 15 is enough:

$$C^{2h+1}_{(k+1,i)} \equiv \sum_j C^{2h+1}_{(k,j)}(D^{2h}_{(k,i-j)} + D^{2h+1}_{(k,i-j)}) - \sum_{j_0} C^{2h+1}_{(k,j_0)} C^{2h+1}_{(k,i-j_0)}(mod\ 2^k) \tag{3.29}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ and $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$.

For $1 \leq i < 2^k - 1$,

$$\sum_{j_0} C^{2h+1}_{(k,j_0)} C^{2h+1}_{(k,i-j_0)} \equiv 0(mod\ 2^{s(i)+1}) \tag{3.30}$$

can be shown. If $1 \leq i < 2^{k-1}$, the sum is over all $0 \leq j_0 \leq i$, and $\xi_2(C^{2h+1}_{(k,j_0)}) + \xi_2(C^{2h+1}_{(k,i-j_0)}) \geq s(j_0) + s(i-j_0) \geq s(i)$ with equality for $2^{s(i)}$ cases if i is even(in this case, j_0 and $i - j_0$ are even) and 0 cases if i is odd(in this case, one of j_0 and $i - j_0$ is odd and $\xi_2(C^{2h+1}_{(k,j_0)}) > s(j_0)$ or $\xi_2(C^{2h+1}_{(k,i-j_0)}) > s(i-j_0)$). If $2^{k-1} \leq i < 2^k - 1$, the sum is over all $i+1-2^{k-1} \leq j_0 \leq 2^{k-1}-1$, and $\xi(C^{2h+1}_{(k,j_0)}) + \xi(C^{2h+1}_{(k,i-j_0)}) \geq s(j_0) + s(i-j_0) > s(i)$ because $1 \leq j_0, i-j_0 < 2^{k-1}$(but $2^{k-1} \leq i < 2^k - 1$). For $1 \leq i < 2^k - 1$, (3.29) is

$$C^{2h+1}_{(k+1,i)} \equiv \sum_j C^{2h+1}_{(k,j)}(D^{2h}_{(k,i-j)} + D^{2h+1}_{(k,i-j)})(mod\ 2^{s(i)+1})$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$. For $i = 0$, it doesn't work because $C^{2h+1}_{(k+1,0)} = 1 \not\equiv 2 = C^{2h+1}_{(k,0)}(D^{2h}_{(k,0)} + D^{2h+1}_{(k,0)})(mod\ 2)$. For $i = 2^k - 1$, it works because $C^{2h+1}_{(k+1,2^k-1)} \equiv 0 \equiv C^{2h+1}_{(k,2^{k-1}-1)}(D^{2h}_{(k,2^{k-1})} + D^{2h+1}_{(k,2^{k-1})})(mod\ 2^{k+1})$. Therefore, for $0 \leq i \leq 2^k - 1$,

$$C^{2h+1}_{(k+1,i)} \equiv \sum_j C^{2h+1}_{(k,j)}(D^{2h}_{(k,i-j)} + D^{2h+1}_{(k,i-j)}) + 1_{\{i=0\}}(mod\ 2^{s(i)+1}) \tag{3.31}$$

where the sum is over all $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$.

For $2 \leq i < 2^k - 1$, $\sum_{j_0} C^{2h+1}_{(k,j_0)} C^{2h}_{(k,i-j_0)} \equiv 0(mod\ 2^{s(i)+1})$ can be shown. If $2 \leq i < 2^{k-1}$, the sum is over all $0 \leq j_0 \leq i$, and $\xi_2(C^{2h+1}_{(k,j_0)}) + \xi_2(C^{2h}_{(k,i-j_0)}) \geq s(j_0) + s(i - j_0) \geq s(i)$ with equality for $2^{s(i)}$ cases if i is even(in this case, j_0 and $i - j_0$ are even) and $2^{s(i)-1}$ cases if i is odd(in this case, j_0 is even and $s(i) - 1 > 0$ since i is odd and $i > 1$). If $2^{k-1} \leq i < 2^k - 1$, the sum is over all $i + 1 - 2^{k-1} \leq j_0 \leq 2^{k-1} - 1$, and $\xi_2(C^{2h+1}_{(k,j_0)}) + \xi_2(C^{2h}_{(k,i-j_0)}) \geq s(j_0) + s(i - j_0) > s(i)$ because $1 \leq j_0, i - j_0 < 2^{k-1}$ but $2^{k-1} \leq i < 2^k - 1$. For $2 \leq i < 2^k - 1$, (3.31) is

$$C^{2h+1}_{(k+1,i)} \equiv \sum_j C^{2h+1}_{(k,j)}(D^{2h+1}_{(k,i-j)} - D^{2h}_{(k,i-j)}) + 1_{\{i<2^{k-1}\}} C^{2h+1}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h+1}_{(k,i-2^{k-1})}(mod\ 2^{s(i)+1}) \tag{3.32}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ because $\sum_j C^{2h+1}_{(k,j)} \times 2D^{2h}_{(k,i-j)} \equiv \sum_{j_0} C^{2h+1}_{(k,j_0)} C^{2h}_{(k,i-j_0)} + 1_{\{i \leq 2^{k-1}-1\}} C^{2h+1}_{(k,i)}(2D^{2h}_{(k,0)} - C^{2h}_{(k,0)}) + 1_{\{i \geq 2^{k-1}\}} C^{2h+1}_{(k,i-2^{k-1})} \times 2D^{2h}_{(k,2^{k-1})} \equiv 1_{\{i<2^{k-1}\}} C^{2h+1}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h+1}_{(k,i-2^{k-1})}(mod\ 2^{s(i)+1})$.

From (3.32), if i is even, $C^{2h+1}_{(k+1,i)} \equiv 1_{\{i<2^{k-1}\}} C^{2h+1}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h+1}_{(k,i-2^{k-1})} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ because either j is odd or $i - j$ is even and $C^{2h+1}_{(k,j)}(D^{2h+1}_{(k,i-j)} - D^{2h}_{(k,i-j)}) \equiv 0(mod\ 2^{s(i)+1})$.

If i is odd,

$$\begin{aligned} C^{2h+1}_{(k+1,i)} &\equiv \sum_{\text{j is even}} C^{2h+1}_{(k,j)}(D^{2h+1}_{(k,i-j)} - D^{2h}_{(k,i-j)}) \\ &+ 1_{\{i \leq 2^{k-1}-1\}} C^{2h+1}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h+1}_{(k,i-2^{k-1})} \\ &\equiv 1_{\{i \leq 2^{k-1}-1\}} C^{2h+1}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{2h+1}_{(k,i-2^{k-1})} \\ &\equiv 0(mod\ 2^{s(i)+1}) \end{aligned}$$

because if j is even $\xi_2(C^{2h+1}_{(k,j)}(D^{2h+1}_{(k,i-j)} - D^{2h}_{(k,i-j)})) = s(j)+s(i-j) \geq s(i)$ with equality for $2^{s(i)-1}$ cases if $2 \leq i \leq 2^{k-1}$ and 0 cases if $2^{k-1} < i < 2^k$.

From *Case* 1, 2, 3, and 4, the proof is done. ■

Now, another proof of Theorem 7 is found.

Corollary 17 *Theorem 7 is true.*

Proof It is enough to show that $b(x+2^n) - b(x) \equiv 0(mod\ 2^{n+1})$ under the assumption $b(0) \equiv 1(mod\ 2)$ and

$$\Delta^r b(x) \equiv 0(mod\ 2^{r+1}) \tag{3.33}$$

for all $r \geq 1$ and $x \geq 0$.

It can be shown by induction on n. For $n = 0$, $b(x+1) - b(x) \equiv 0(mod\ 2^1)$ from $r = 1$ in (3.33). It is assumed that the statement is true for $n \leq k(k \geq 0)$, and it is proved for $n = k+1$.

$$\begin{aligned} b(x+2^{k+1}) - b(x) &\equiv (S^{2^{k+1}} - 1)(b(x)) \\ &\equiv (S-1)(S+1)(S^2+1)\cdots(S^{2^k}+1)(b(x))(mod\ 2^{k+2}) \end{aligned}$$

and $S^{2^t} + 1 = (S-1)g_t(S) + 2$ for $t \geq 0$, where $g_t(S) = S^{2^t-1} + \cdots + S + 1$. If we expand $(S^{2^{k+1}} - 1)$, each term is divisible by $2^t(S-1)^{k+1-t}$ and $2^t(S-1)^{k+1-t}(b(x)) \equiv 0(mod\ 2^{k+2})$ from $r = k+1-t$ in (3.33) for $0 \leq t \leq k$. The proof is done. ■

3.6 Remark

It is natural to extend Konvalinka conjecture for a general function $b(x)$ from $N \cup \{0\}$ to Z. The property of a polynomial $b(x)$ that we used is $b(x+2^k) - b(x) \equiv 0(mod\ 2^k)$ for $k \geq 2$ and $C^{x+2^k}_{(k,2^{k-1}-1)} \equiv C^x_{(k,2^{k-1}-1)}(mod\ 2^{k+1})$ for $x \geq 0$.

It is easy to show that $C^{x+2^k}_{(k,2^{k-1}-1)} \equiv C^{x}_{(k,2^{k-1}-1)}(mod\ 2^{k+1})$ is equivalent to

$$k_x + k_{x+4} + \cdots + k_{x+2^k-4} \equiv b(mod\ 2) \tag{3.34}$$

for some b and for all $x \geq 0$ if $b(x+2^k) - b(x) \equiv k_x 2^k (mod\ 2^{k+1})$.

Theorem 18 *For $b(x)$ from $N \cup \{0\}$ to Z, if $b(0) \equiv 1(mod\ 2)$, $b(0) \equiv b(1) \equiv (-1)^s b(2) \equiv (-1)^s b(3)(mod\ 4)$ for some $s \in N$, and $b(x+2^k)-b(x) \equiv k_x 2^k(mod\ 2^{k+1})$ for $x \geq 0$ and $k \geq 2$ with k_x satisfying (3.34),*

$$\xi(C_n^{b,1}) = \xi(C_n) = s(n+1) - 1$$

for all $n \geq 0$.

Proof The proof can be done by using the idea in Theorem 11 and Theorem 14. ■

But, it is uncertain whether the above condition is sufficient or not.

Conjecture 19 *$b(x)$ is a function from $N \cup \{0\}$ to Z. If $\xi(C_n^{b,1}) = \xi(C_n) = s(n+1)-1$ for all $n \geq 0$, there exists k_x satisfying (3.34) and $b(x+2^k) - b(x) \equiv k_x 2^k(mod\ 2^{k+1})$ for $x \geq 0$ and $k \geq 2$.*

It is interesting that the conjecture is true if $b(x+4) \equiv b(x)(mod\ 4)$.

Theorem 20 *If $b(x+4) \equiv b(x)(mod\ 4)$ for $x \geq 0$, Conjecture 19 is true.*

Proof First of all, it is shown that $b(x)$ is odd for $x \in N \cup \{0\}$. If there exists some x that $b(x)$ is even, $W_{n,k} \equiv 0(mod\ 2)$ for $k \geq x+1$ and $A_x(W_{n,0}, W_{n,1}, \cdots, W_{n,x}) \equiv$

$(W_{n+1,0}, W_{n+1,1}, \cdots, W_{n+1,x})(mod\ 2)$, where

$$\mathbf{A_x} := \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ b(0) & 0 & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & b(x-2) & 0 & 1 \\ \vdots & \vdots & 0 & b(x-1) & 0 \end{pmatrix}$$

There exist $m \geq 0$ and $t > 0$ that $(W_{m,0}, \cdots, W_{m,x}) \equiv (W_{m+t,0}, \cdots, W_{m+t,x})(mod\ 2)$ because there are at most 2^{x+1} possible combinations of $(W_{n,0}, W_{n,1}, \cdots, W_{n,x})(mod\ 2)$. Then, $W_{n+t,0} \equiv W_{n,0}(mod\ 2)$ for $n \geq m$ and $C_n^{b,1}(mod\ 2)$ is periodic for $n \geq m$. But, $C_n^{b,1} \equiv C_n \equiv 1(mod\ 2)$ if and only if $n = 2^k - 1$ for $k \geq 0$, and $C_n^{b,1}(mod\ 2)$ is not periodic for $n \geq m$. Therefore, $b(x)$ is odd for $x \in N \cup \{0\}$.

Similar to the proof of the backward direction of Konvalinka conjecture, $b(1) \equiv b(0)(mod\ 4)$ and $b(3) \equiv b(2)(mod\ 4)$. Therefore, we can divide the proof into two cases.

Case 1 : $b(4x) \equiv b(4x+1) \equiv b(4x+2) \equiv b(4x+3) \equiv a(mod\ 4)$ for $x \geq 0$, where $a \equiv 1\ or\ 3(mod\ 4)$

It is shown that $C_{(n,i)}^h \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \leq i \leq 2^{n-1} - 1$ and $D_{(n,i)}^h \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^{n-1}$ by induction on n.

For $n = 1$, $D_{(1,1)}^h = b(h) \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$.

For $n = 2$, $C_{(2,1)}^h = b(h) + b(h+1) \equiv 2b(h) \equiv 2^{s(1)}(mod\ 2^{s(1)+1})$, $D_{(2,1)}^h = b(h) + b(h+1) + b(h+2) \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$, and $D_{(2,2)}^h = b(h)b(h+2) \equiv 2^{s(2)-1}(mod\ 2^{s(2)})$.

It is assumed that the statement is true for $n \leq k (k \geq 2)$, and it is proved for $n = k+1$.

(1): $D_{(k+1,i)}^h \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^k$

From (3.5), $d_{(2^k,i)}^h = D_{(k,i)}^{h+1} + b(h)C_{(k,i-1)}^{h+2}$ and $d_{(2^k,i)}^h = D_{(k,i)}^h + b(h+2^k-1)C_{(k,i-1)}^h$. Therefore, $D_{(k,i)}^{h+1} - D_{(k,i)}^h = b(h+2^k-1)C_{(k,i-1)}^h - b(h)C_{(k,i-1)}^{h+2}$. Since $C_{(k,i-1)}^{h+2} \equiv C_{(k,i-1)}^h \equiv$

$2^{s(i-1)}(mod\ 2^{s(i-1)+1})$,

$$D^{h+1}_{(k,i)} - D^{h}_{(k,i)} = b(h+2^k-1)C^{h}_{(k,i-1)} - b(h)C^{h+2}_{(k,i-1)} \equiv 0(mod\ 2^{s(i-1)+1}) \qquad (3.35)$$

Similarly, from (3.5), $d^{h}_{(2^k+1,i)} = D^{h+2}_{(k,i)} + b(h)D^{h+2}_{(k,i-1)} + b(h+1)C^{h+3}_{(k,i-1)}$ and $d^{h}_{(2^k+1,i)} = D^{h}_{(k,i)} + b(h+2^k)D^{h}_{(k,i-1)} + b(h+2^k-1)C^{h}_{(k,i-1)}$. Since $D^{h+2}_{(k,i)} \equiv D^{h+1}_{(k,i)} \equiv D^{h}_{(k,i)}(mod\ 2^{s(i-1)+1})$ from (3.35) and $C^{h+3}_{(k,i-1)} \equiv C^{h}_{(k,i-1)} \equiv 2^{s(i-1)}(mod\ 2^{s(i-1)+1})$, $D^{h+2}_{(k,i-1)} - D^{h}_{(k,i-1)} \equiv 0(mod\ 2^{s(i-1)+1})$ and

$$D^{h+2}_{(k,i)} - D^{h}_{(k,i)} \equiv 0(mod\ 2^{s(i)+1}) \qquad (3.36)$$

Similar to Lemma 13,

$$D^{h}_{(k+1,i)} = \sum_{j'} D^{h}_{(k,j')}D^{h+2^k}_{(k,i-j')} + b(h+2^k-1)\sum_{j''} C^{h}_{(k,j'')}C^{h+2^k+1}_{(k,i-1-j'')} \qquad (3.37)$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

For $i = 1$, $D^{h}_{(k+1,1)} = D^{h}_{(k,0)}D^{h+2^k}_{(k,1)} + D^{h}_{(k,1)}D^{h+2^k}_{(k,0)} + b(h+2^k-1)C^{h}_{(k,0)}C^{h+2^k+1}_{(k,0)} \equiv 1(mod\ 2)$. Therefore, $D^{h}_{(k+1,1)} \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$.

For $2 \leq i \leq 2^k$, $\sum_{j''} C^{h}_{(k,j'')}C^{h+2^k+1}_{(k,i-1-j'')} \equiv 0(mod\ 2^{s(i)})$ can be shown by *Case* 1 in Lemma 12. From (3.36), $D^{h+2^k}_{(k,i)} - D^{h}_{(k,i)} \equiv 0(mod\ 2^{s(i)+1})$ and $s(j)-1+s(i-j)+1 \geq s(i)$, (3.37) is

$$D^{h}_{(k+1,i)} \equiv \sum_{j'} D^{h}_{(k,j')}D^{h}_{(k,i-j')}(mod\ 2^{s(i)}) \qquad (3.38)$$

Since (3.38) is same as (3.12), $D^{h}_{(k+1,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^k$ by *Case* 1 in Lemma 12.

(2): $C^{h}_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \leq i < 2^k$(i is even)

Similar to Lemma 13,

$$C^{h}_{(k+1,i)} = \sum_{j}(C^{h}_{(k,j)}D^{h-1+2^k}_{(k,i-j)} + D^{h}_{(k,i-j)}C^{h+2^k}_{(k,j)}) - \sum_{j_0} C^{h}_{(k,j_0)}C^{h+2^k}_{(k,i-j_0)} \qquad (3.39)$$

where the sum is over all $max\{0, i-2^{k-1}\} \le j \le min\{i, 2^{k-1}-1\}$ and $max\{0, i+1-2^{k-1}\} \le j_0 \le min\{i, 2^{k-1}-1\}$.

From (3.14), $\sum_{j_0} C^h_{(k,j_0)} C^{h+2^k}_{(k,i-j_0)} \equiv 0(mod\ 2^{s(i)+1})$. If j is odd($i-j$ is odd), $s(j)+s(i-j)-1 > s(i)$ and $C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} \equiv 0(mod\ 2^{s(i)+1})$. Therefore, (3.39) is

$$C^h_{(k+1,i)} \equiv \sum_{j:\ even} (C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^h_{(k,i-j)} C^{h+2^k}_{(k,j)})(mod\ 2^{s(i)+1}) \tag{3.40}$$

From (3.35) and (3.36),

$$\begin{aligned} & D^{h+1}_{(k,i+1)} - D^h_{(k,i+1)} = b(h+2^k-1)C^h_{(k,i)} - b(h)C^{h+2}_{(k,i)} \\ \equiv\ & D^{h+3}_{(k,i+1)} - D^{h+2}_{(k,i+1)} = b(h+2^k+1)C^{h+2}_{(k,i)} - b(h+2)C^{h+4}_{(k,i)}(mod\ 2^{s(i+1)+1}) \end{aligned}$$

If i is even, $C^h_{(k,i)} - C^{h+2}_{(k,i)} \equiv C^{h+2}_{(k,i)} - C^{h+4}_{(k,i)}(mod\ 2^{s(i)+2})$ and

$$C^h_{(k,i)} \equiv C^{h+4}_{(k,i)}(mod\ 2^{s(i)+2}) \tag{3.41}$$

If i is even, from (3.35),

$$D^{h+1}_{(k,i)} \equiv D^h_{(k,i)}(mod\ 2^{s(i)+1}) \tag{3.42}$$

since $s(i-1) \ge s(i)$. Therefore, (3.40) is

$$\begin{aligned} C^h_{(k+1,i)} &\equiv \sum_{j:\ even} 2C^h_{(k,j)} D^h_{(k,i-j)} \\ &\equiv \sum_j 2C^h_{(k,j)} D^h_{(k,i-j)} \\ &\equiv \sum_{j_0} C^h_{(k,j_0)} C^h_{(k,i-j_0)} + 1_{\{i \le 2^{k-1}-1\}} C^h_{(k,i)}(2D^h_{(k,0)} - C^h_{(k,0)}) + 1_{\{i \ge 2^{k-1}\}} C^h_{(k,i-2^{k-1})} 2D^h_{(k,2^{k-1})} \\ &\equiv 1_{\{i<2^{k-1}\}} C^h_{(k,i)} + 1_{\{i \ge 2^{k-1}\}} 2C^h_{(k,i-2^{k-1})} \\ &\equiv 2^{s(i)}(mod\ 2^{s(i)+1})(mod\ 2^{s(i)+1}) \end{aligned}$$

(3): $C^h_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \le i < 2^k$(i is odd)

From (3.40),

$$
\begin{aligned}
C^h_{(k+1,i)} &\equiv \sum_j (C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^h_{(k,i-j)} C^{h+2^k}_{(k,j)}) \\
&\equiv \sum_j C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} - D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} + 2D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} (mod\ 2^{s(i)+1})
\end{aligned}
$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ because $2D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} \equiv \sum_{j_0} C^h_{(k,j_0)} C^h_{(k,i-j_0)} + 1_{\{i \leq 2^{k-1}-1\}} C^h_{(k,i)} (2D^h_{(k,0)} - C^h_{(k,0)}) + 1_{\{i \geq 2^{k-1}\}} C^h_{(k,i-2^{k-1})} 2D^h_{(k,2^{k-1})} \equiv 1_{\{i<2^{k-1}\}} C^h_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^h_{(k,i-2^{k-1})} \equiv 2^{s(i)} (mod\ 2^{s(i)+1})$ where the sum is over all $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$. Therefore, all we need to show is

$$
\sum_j (C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} - D^h_{(k,i-j)} C^{h+2^k}_{(k,j)}) \equiv 0 (mod\ 2^{s(i)+1}) \tag{3.43}
$$

From (3.36), (3.41), and (3.42), (3.43) is same as

$$
\begin{aligned}
&\sum_j C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} - D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} \\
\equiv &\sum_{j:even} C^h_{(k,j)} (D^{h+1}_{(k,i-j)} - D^h_{(k,i-j)}) + \sum_{j:odd} D^h_{(k,i-j)} (C^h_{(k,j)} - C^{h+2^k}_{(k,j)}) \\
\equiv &\sum_j C^h_{(k,j)} (D^{h+1}_{(k,i-j)} - D^h_{(k,i-j)}) + \sum_j D^h_{(k,i-j)} (C^h_{(k,j)} - C^{h+2^k}_{(k,j)}) (mod\ 2^{s(i)+1})
\end{aligned}
$$

because i is odd.

It is shown that $C^{h+2^{k-1}}_{(k-1,i)} - C^h_{(k-1,i)} \equiv 0 (mod\ 2^{s(i)+2})$ and $D^{h+1}_{(k-1,i)} - D^h_{(k-1,i)} \equiv 0 (mod\ 2^{s(i)+1})$ if i is odd. If $k = 2$, $D^{h+1}_{(1,1)} - D^h_{(1,1)} \equiv b(h+1) - b(h) \equiv 0 (mod\ 2^{s(1)+1})$. If $k > 2$, by the inductive assumption with (3.41) and (3.42), for $1 \leq i_0 \leq 2^{k-1} - 1$,

$$
\sum_{j:even} C^h_{(k-1,j)} (D^{h+1}_{(k-1,i_0-j)} - D^h_{(k-1,i_0-j)}) + \sum_{j:odd} D^h_{(k-1,i_0-j)} (C^h_{(k-1,j)} - C^{h+2^{k-1}}_{(k-1,j)}) \equiv 0 (mod\ 2^{s(i_0)+1}) \tag{3.44}
$$

For $1 \leq i \leq 2^{k-2} - 1$, take $i_0 = i + 2^{k-2}$ in (3.44). Then, if $s(j) + s(i_0 - j) > s(i_0)$, $C^h_{(k-1,j)} (D^{h+1}_{(k-1,i_0-j)} - D^h_{(k-1,i_0-j)}) \equiv 0 (mod\ 2^{s(i_0)+1})$ and $D^h_{(k-1,i_0-j)} (C^h_{(k-1,j)} - C^{h+2^{k-1}}_{(k-1,j)}) \equiv 0 (mod\ 2^{s(i_0)+1})$. Therefore, $D^h_{(k-1,2^{k-2})} (C^h_{(k-1,i)} - C^{h+2^{k-1}}_{(k-1,i)}) \equiv 0 (mod\ 2^{s(i_0)+1})$

and $C^{h+2^{k-1}}_{(k-1,i)} - C^h_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+2})$. (3.44) is

$$\sum_{j:even} C^h_{(k-1,j)}(D^{h+1}_{(k-1,i_0-j)} - D^h_{(k-1,i_0-j)}) \equiv 0(mod\ 2^{s(i_0)+1}) \tag{3.45}$$

If i_1 is the minimum that satisfies $D^{h+1}_{(k-1,i_1)} - D^h_{(k-1,i_1)} \not\equiv 0(mod\ 2^{s(i_1)+1})$, $\sum_{j:even} C^h_{(k-1,j)}(D^{h+1}_{(k-1,i_1-j)} - D^h_{(k-1,i_1-j)}) \equiv D^{h+1}_{(k-1,i_1)} - D^h_{(k-1,i_1)} \not\equiv 0(mod\ 2^{s(i_1)+1})$ and it contradicts (3.45). Therefore, $D^{h+1}_{(k-1,i)} - D^h_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+1})$ for $1 \le i \le 2^{k-2}$.

Similar to Lemma 13,

$$\begin{aligned} d^h_{(2^k+2^{k-1}-2,i)} &= \sum_{j_1} C^h_{(k-1,i-j_1)} D^{h-1+2^{k-1}}_{(k,j_1)} + \sum_{j_2} D^h_{(k-1,i-j_2)} C^{h+2^{k-1}}_{(k,j_2)} - \sum_{j_3} C^h_{(k-1,i-j_3)} C^{h+2^{k-1}}_{(k,j_3)} \\ &= \sum_{j_2} C^h_{(k,j_2)} D^{h-1+2^k}_{(k-1,i-j_2)} + \sum_{j_1} D^h_{(k,j_1)} C^{h+2^k}_{(k-1,i-j_1)} - \sum_{j_3} C^h_{(k,j_3)} C^{h+2^k}_{(k-1,i-j_3)} \end{aligned}$$

where $max\{0, i+1-2^{k-2}\} \le j_1 \le min\{i, 2^{k-1}\}$, $max\{0, i-2^{k-2}\} \le j_2 \le min\{i, 2^{k-1}-1\}$, and $max\{0, i+1-2^{k-2}\} \le j_3 \le min\{i, 2^{k-1}-1\}$. Since $C^{h+2^{k-1}}_{(k-1,i)} - C^h_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+2})$ and $D^{h+1}_{(k-1,i)} - D^h_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+1})$,

$$\sum_{j_1} C^h_{(k-1,i-j_1)}(D^{h+1}_{(k,j_1)} - D^h_{(k,j_1)}) \equiv \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^{k-1}}_{(k,j_2)} - C^h_{(k,j_2)})(mod\ 2^{s(i)+1}) \tag{3.46}$$

From (3.46),

$$\begin{aligned} & \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) \\ \equiv\ & \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^{h+2^{k-1}}_{(k,j_2)}) + \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^{k-1}}_{(k,j_2)} - C^h_{(k,j_2)}) \\ \equiv\ & \sum_{j_1} 2^{s(i-j_1)}(D^{h+1}_{(k,j_1)} - D^h_{(k,j_1)} + D^{h+1}_{(k,j_1)} - D^h_{(k,j_1)}) \\ \equiv\ & 0(mod\ 2^{s(i)+1}) \end{aligned}$$

and

$$\sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) \equiv 0(mod\ 2^{s(i)+1}) \tag{3.47}$$

It is shown that

$$\sum_j D^h_{(k,i-j)}(C^{h+2^k}_{(k,j)} - C^h_{(k,j)}) \equiv 0(mod\ 2^{s(i)+1}) \tag{3.48}$$

where $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ with (3.47).

For $0 \leq i \leq 2^{k-2}-1$, $0 \leq j_2 \leq i$, and $0 \leq j \leq i$. From (3.47),

$$\begin{aligned}\sum_j D^h_{(k,i-j)}(C^{h+2^k}_{(k,j)} - C^h_{(k,j)}) &\equiv \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) \\ &\equiv 0(mod\ 2^{s(i)+1})\end{aligned}$$

For $2^{k-2} \leq i \leq 2^{k-1}-1$, $i-2^{k-2} \leq j_2 \leq i$ and $0 \leq j \leq i$. From (3.47) with $0 \leq i-2^{k-2} \leq 2^{k-2}-1 (s(i) = s(i-2^{k-2})+1)$,

$$\begin{aligned}&\sum_j D^h_{(k,i-j)}(C^{h+2^k}_{(k,j)} - C^h_{(k,j)}) \\ \equiv& \sum_{j_2} D^h_{(k,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) + \sum_{0\leq j_2'\leq i-2^{k-2}-1} D^h_{(k,i-j_2')}(C^{h+2^k}_{(k,j_2')} - C^h_{(k,j_2')}) \\ \equiv& \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) + \sum_{0\leq j_2'\leq i-2^{k-2}} 2D^h_{(k-1,i-2^{k-2}-j_2')}(C^{h+2^k}_{(k,j_2')} - C^h_{(k,j_2')}) \\ \equiv& 0(mod\ 2^{s(i)+1})\end{aligned}$$

For $2^{k-1} \leq i \leq 2^{k-1}+2^{k-2}-1$, $i-2^{k-2} \leq j_2 \leq 2^{k-1}-1$ and $i-2^{k-1} \leq j \leq 2^{k-1}-1$. From (3.47) and $1+s(i-2^{k-2}-j_2')-1+s(j_2')+1 \geq s(i-2^{k-2})+1 = s(i)+1 (2^{k-2} \leq i-2^{k-2} \leq 2^{k-1}-1)$,

$$\begin{aligned}&\sum_j D^h_{(k,i-j)}(C^{h+2^k}_{(k,j)} - C^h_{(k,j)}) \\ \equiv& \sum_{j_2} D^h_{(k,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) + \sum_{i-2^{k-1}\leq j_2'\leq i-2^{k-2}-1} D^h_{(k,i-j_2')}(C^{h+2^k}_{(k,j_2')} - C^h_{(k,j_2')}) \\ \equiv& \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) + \sum_{i-2^{k-1}\leq j_2'\leq i-2^{k-2}-1} 2D^h_{(k-1,i-2^{k-2}-j_2')}(C^{h+2^k}_{(k,j_2')} - C^h_{(k,j_2')}) \\ \equiv& 0(mod\ 2^{s(i)+1})\end{aligned}$$

For $2^{k-1}+2^{k-2} \le i \le 2^k-1$, $i-2^{k-1} \le j \le 2^{k-1}-1$. From (3.47) with $2^{k-1} \le i-2^{k-2} \le 2^{k-1}+2^{k-2}-1 (s(i) = s(i-2^{k-2})+1)$,

$$\begin{aligned}
&\sum_j D^h_{(k,i-j)}(C^{h+2^k}_{(k,j)} - C^h_{(k,j)}) \\
\equiv& \sum_{(i-2^{k-2})-2^{k-2} \le j_2' \le 2^{k-1}-1} 2D^h_{(k-1,i-2^{k-2}-j_2')}(C^{h+2^k}_{(k,j_2')} - C^h_{(k,j_2')}) \\
\equiv& \ 0 (mod\ 2^{s(i)+1})
\end{aligned}$$

Finally, (3.43) is $\sum_{j:even} C^h_{(k,j)}(D^{h+1}_{(k,i-j)} - D^h_{(k,i-j)})(mod\ 2^{s(i)+1})$. It is shown that $C^0_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \le i \le 2^k-1$. From (3.39), it is obvious that $C^0_{(k+1,i)} \equiv 0(mod\ 2^{s(i)})$. If i_1 is the minimum that $C^0_{(k+1,i_1)} \equiv 0(mod\ 2^{s(i_1)+1})$, then $\xi_2(C_n) \ge s(n+1)$ for $n = 2^k-1+i_1$ from *Case* 2 in Theorem 10 because $\xi_2(C^0_{(k+1,j)}C_{n-j}) \ge s(n+1)-1$ with equality for $2^{s(n+1)-1}-2$ cases and $s(n+1) = s(2^k+i_1) \ge 2$. This is because $\xi_2(C^0_{(k+1,j)}C_{n-j}) \ge s(j)+s(n+i_1-j)-1 > s(n+1)-1$ if $j > i_1 (n+1-j < 2^k)$, and $\xi_2(C^0_{(k+1,i_1)}C_{n-i_1}) \ge s(i_1)+1+s(2^k)-1 = s(n+1)$. Since $C^0_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \le i \le 2^k-1$, $\sum_{j:even} C^0_{(k,j)}(D^1_{(k,i-j)} - D^0_{(k,i-j)})(mod\ 2^{s(i)+1})$. Since $C^h_{(k,j)} \equiv C^0_{(k,j)}(mod\ 2^{s(i)+1})$ and $D^{h+2}_{(k,i-j)} \equiv D^h_{(k,i-j)}(mod\ 2^{s(i)+1})$ in (3.37),

$$\sum_{j:even} C^h_{(k,j)}(D^{h+1}_{(k,i-j)} - D^h_{(k,i-j)}) \equiv C^0_{(k,j)}(D^1_{(k,i-j)} - D^0_{(k,i-j)}) \equiv 0(mod\ 2^{s(i)+1})$$

for $1 \le i \le 2^k-1$, and the proof is done.

From (1), (2), and (3), $C^h_{(n,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \le i \le 2^{n-1}-1$ and $D^h_{(n,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \le i \le 2^{n-1}$.

For $k \ge 2$, from (3.35), $D^{h+2}_{(k,2^{k-1})} \equiv D^h_{(k,2^{k-1})}(mod\ 2^k)$. Therefore,

$$b(h+2^k) - b(h) \equiv 0(mod\ 2^k) \qquad (3.49)$$

because $D^{h+2}_{(k,2^{k-1})} - D^h_{(k,2^{k-1})} = (b(h+2^k) - b(h))b(h+2)b(h+4)\cdots b(h+2^k-2)$.
Since $C^{h+2^k}_{(k,i)} - C^h_{(k,i)} \equiv 0(mod\ 2^{s(i)+2})$, $C^{h+2^k}_{(k,2^{k-1}-1)} \equiv C^h_{(k,2^{k-1}-1)}(mod\ 2^{k+1})$. From (3.49), $b(h+2^k)-b(h) \equiv k_h 2^k(mod\ 2^{k+1})$ for some k_h. From (3.49), $b(h+2^{k+1})-b(h) \equiv 0(mod\ 2^{k+1})$ and $k_{h+2^k} \equiv k_h(mod\ 2)$.
Since $C^{h+2^k}_{(k,2^{k-1}-1)} - C^h_{(k,2^{k-1}-1)} \equiv 2^k(k_h+k_{h+1}+k_{h+4}+k_{h+5}+\cdots+k_{h+2^k-4}+k_{h+2^k-3})(mod\ 2^{k+1})$,

$$k_h + k_{h+1} + k_{h+4} + k_{h+5} + \cdots + k_{h+2^k-4} + k_{h+2^k-3} \equiv 0(mod\ 2) \tag{3.50}$$

and it is equivalent to (3.34).

Case 2 : $b(4x) \equiv b(4x+1) \equiv (-1)b(4x+2) \equiv (-1)b(4x+3) \equiv a(mod\ 4)$ for $x \geq 0$, where $a \equiv 1$ *or* $3(mod\ 4)$

It is shown that $C^{2h}_{(n,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \leq i \leq 2^{n-1}-1$, $C^{2h+1}_{(n,2i)} \equiv 2^{s(2i)}(mod\ 2^{s(2i)+1})$ for $1 \leq 2i \leq 2^{n-1}-1$, $C^{2h+1}_{(n,2i+1)} \equiv 0(mod\ 2^{s(2i+1)+1})$ for $1 \leq 2i+1 \leq 2^{n-1}-1$, and $D^h_{(n,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^{n-1}$ by induction on n.
For $n=1$, $D^h_{(1,1)} = b(h) \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$.
For $n=2$, $C^{2h}_{(2,1)} = b(2h) + b(2h+1) \equiv 2b(2h) \equiv 2^{s(1)}(mod\ 2^{s(1)+1})$, $C^{2h}_{(2,1)} = b(2h) + b(2h+1) \equiv 0(mod\ 2^{s(1)+1})$, $D^h_{(2,1)} = b(h) + b(h+1) + b(h+2) \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$, and $D^h_{(2,2)} = b(h)b(h+2) \equiv 2^{s(2)-1}(mod\ 2^{s(2)})$.
It is assumed that the statement is true for $n \leq k (k \geq 2)$, and it is proved for $n = k+1$.

(1): $D^h_{(k+1,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^k$
From (3.35),

$$D^{h+1}_{(k,i)} - D^h_{(k,i)} = b(h+2^k-1)C^h_{(k,i-1)} - b(h)C^{h+2}_{(k,i-1)} \equiv 0(mod\ 2^{s(i-1)+1}) \tag{3.51}$$

Similar to (3.36), if i is even,

$$D^{h+2}_{(k,i)} - D^h_{(k,i)} \equiv 0(mod\ 2^{s(i)+1}) \tag{3.52}$$

and if i is odd,

$$D^{h+2}_{(k,i)} - D^{h}_{(k,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1}) \tag{3.53}$$

Similar to Lemma 13,

$$D^{h}_{(k+1,i)} = \sum_{j'} D^{h}_{(k,j')} D^{h+2^k}_{(k,i-j')} + b(h+2^k-1) \sum_{j''} C^{h}_{(k,j'')} C^{h+2^k+1}_{(k,i-1-j'')} \tag{3.54}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j' \leq min\{i, 2^{k-1}\}$ and $max\{0, i-2^{k-1}\} \leq j'' \leq min\{i-1, 2^{k-1}-1\}$.

For $i = 1$, $D^{h}_{(k+1,1)} = D^{h}_{(k,0)} D^{h+2^k}_{(k,1)} + D^{h}_{(k,1)} D^{h+2^k}_{(k,0)} + b(h+2^k-1) C^{h}_{(k,0)} C^{h+2^k+1}_{(k,0)} \equiv 1(mod\ 2)$. Therefore, $D^{h}_{(k+1,1)} \equiv 2^{s(1)-1}(mod\ 2^{s(1)})$.

For $2 \leq i \leq 2^k$, $\sum_{j''} C^{h}_{(k,j'')} C^{h+2^k+1}_{(k,i-1-j'')} \equiv 0(mod\ 2^{s(i)})$ can be shown by *Case* 1 in Lemma 16. From (3.52) and (3.53), $D^{h+4}_{(k,i)} - D^{h}_{(k,i)} \equiv 0(mod\ 2^{s(i)+1})$ and $s(j) - 1 + s(i-j) + 1 \geq s(i)$, (3.54) is

$$D^{h}_{(k+1,i)} \equiv \sum_{j'} D^{h}_{(k,j')} D^{h}_{(k,i-j')} (mod\ 2^{s(i)}) \tag{3.55}$$

Since (3.55) is same as (3.12), $D^{h}_{(k+1,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^k$ by *Case* 1 in Lemma 12.

(2): $C^{h}_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \leq i < 2^k$(i is even)

Similar to Lemma 13,

$$C^{h}_{(k+1,i)} = \sum_{j} (C^{h}_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^{h}_{(k,i-j)} C^{h+2^k}_{(k,j)}) - \sum_{j_0} C^{h}_{(k,j_0)} C^{h+2^k}_{(k,i-j_0)} \tag{3.56}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$ and $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$.

From (3.14) and (3.30), $\sum_{j_0} C^{h}_{(k,j_0)} C^{h+2^k}_{(k,i-j_0)} \equiv 0(mod\ 2^{s(i)+1})$. If j is odd($i-j$ is odd), $s(j) + s(i-j) - 1 > s(i)$ and $C^{h}_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^{h}_{(k,i-j)} C^{h+2^k}_{(k,j)} \equiv 0(mod\ 2^{s(i)+1})$.

Therefore, (3.56) is

$$C^{h}_{(k+1,i)} \equiv \sum_{j:\,even} (C^{h}_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^{h}_{(k,i-j)} C^{h+2^k}_{(k,j)}) (mod\ 2^{s(i)+1}) \tag{3.57}$$

From (3.51) and (3.53),

$$\begin{aligned} & D^{h+1}_{(k,i+1)} - D^{h}_{(k,i+1)} = b(h+2^k-1) C^{h}_{(k,i)} - b(h) C^{h+2}_{(k,i)} \\ \equiv\ & D^{h+3}_{(k,i+1)} - D^{h+2}_{(k,i+1)} = b(h+2^k+1) C^{h+2}_{(k,i)} - b(h+2) C^{h+4}_{(k,i)} (mod\ 2^{s(i+1)+1}) \end{aligned}$$

If i is even, $C^{h}_{(k,i)} + C^{h+2}_{(k,i)} \equiv -C^{h+2}_{(k,i)} - C^{h+4}_{(k,i)} (mod\ 2^{s(i)+2})$ and

$$C^{h}_{(k,i)} \equiv C^{h+4}_{(k,i)} (mod\ 2^{s(i)+2}) \tag{3.58}$$

If i is even, from (3.51),

$$D^{h+1}_{(k,i)} \equiv D^{h}_{(k,i)} (mod\ 2^{s(i)+1}) \tag{3.59}$$

since $s(i-1) \geq s(i)$. Therefore, (3.57) is

$$\begin{aligned} C^{h}_{(k+1,i)} &\equiv \sum_{j:\,even} 2C^{h}_{(k,j)} D^{h}_{(k,i-j)} \\ &\equiv \sum_{j} 2C^{h}_{(k,j)} D^{h}_{(k,i-j)} \\ &\equiv \sum_{j_0} C^{h}_{(k,j_0)} C^{h}_{(k,i-j_0)} + 1_{\{i \leq 2^{k-1}-1\}} C^{h}_{(k,i)} (2D^{h}_{(k,0)} - C^{h}_{(k,0)}) + 1_{\{i \geq 2^{k-1}\}} C^{h}_{(k,i-2^{k-1})} 2D^{h}_{(k,2^{k-1})} \\ &\equiv 1_{\{i<2^{k-1}\}} C^{h}_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^{h}_{(k,i-2^{k-1})} \\ &\equiv 2^{s(i)} (mod\ 2^{s(i)+1}) (mod\ 2^{s(i)+1}) \end{aligned}$$

(3): $C^{2h}_{(k+1,i)} \equiv 2^{s(i)} (mod\ 2^{s(i)+1})$ and $C^{2h+1}_{(k+1,i)} \equiv 0 (mod\ 2^{s(i)+1})$ for $1 \leq i < 2^k$(i is odd)

For $i = 1$, $C^{2h}_{(k+1,1)} = \sum_{0 \leq j \leq 2^{k+1}-3} b(2h+j) \equiv 2(mod\ 4)$ and $C^{2h+1}_{(k+1,1)} = \sum_{0 \leq j \leq 2^{k+1}-3} b(2h+1+j) \equiv 0(mod\ 4)$. Therefore, $C^{2h}_{(k+1,1)} \equiv 2^{s(1)} (mod\ 2^{s(1)+1})$ and $C^{2h}_{(k+1,1)} \equiv 0 (mod\ 2^{s(1)+1})$.

For $3 \leq i < 2^k (i \text{ is odd})$, from (3.57),

$$\begin{aligned} C^h_{(k+1,i)} &\equiv \sum_j (C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} + D^h_{(k,i-j)} C^{h+2^k}_{(k,j)}) \\ &\equiv \sum_j (C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} - D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} + 2D^h_{(k,i-j)} C^{h+2^k}_{(k,j)})(mod\ 2^{s(i)+1}) \end{aligned}$$

where the sum is over all $max\{0, i-2^{k-1}\} \leq j \leq min\{i, 2^{k-1}-1\}$. Since $2D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} \equiv \sum_{j_0} C^{2h}_{(k,j_0)} C^h_{(k,i-j_0)} + 1_{\{i \leq 2^{k-1}-1\}} C^h_{(k,i)} (2D^h_{(k,0)} - C^{2h}_{(k,0)}) + 1_{\{i \geq 2^{k-1}\}} C^h_{(k,i-2^{k-1})} 2D^h_{(k,2^{k-1})} \equiv 1_{\{i<2^{k-1}\}} C^h_{(k,i)} + 1_{\{i \geq 2^{k-1}\}} 2C^h_{(k,i-2^{k-1})} \equiv C^h_{(k+1,i)} (mod\ 2^{s(i)+1})$, where the sum is over all $max\{0, i+1-2^{k-1}\} \leq j_0 \leq min\{i, 2^{k-1}-1\}$. Therefore, all we need to show is

$$\sum_j C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} - D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} \equiv 0(mod\ 2^{s(i)+1}) \tag{3.60}$$

For $i = 1$, if $h = 2h'$, it also works.

From (3.52), (3.53), (3.58), and (3.59),

$$\begin{aligned} &\sum_j C^h_{(k,j)} D^{h-1+2^k}_{(k,i-j)} - D^h_{(k,i-j)} C^{h+2^k}_{(k,j)} \\ \equiv &\sum_{j:even} C^h_{(k,j)} (D^{h-1}_{(k,i-j)} - D^h_{(k,i-j)}) + \sum_{j:odd} D^h_{(k,i-j)} (C^h_{(k,j)} - C^{h+2^k}_{(k,j)}) \\ \equiv &\sum_j C^h_{(k,j)} (D^{h-1}_{(k,i-j)} - D^h_{(k,i-j)}) + \sum_j D^h_{(k,i-j)} (C^h_{(k,j)} - C^{h+2^k}_{(k,j)})(mod\ 2^{s(i)+1}) \end{aligned}$$

because i is odd.

It is shown that $C^{h+2^{k-1}}_{(k-1,i)} - C^h_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+2})$, $D^{2h}_{(k-1,i)} - D^{2h-1}_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+1})$, and $D^{2h+1}_{(k-1,i)} - D^{2h}_{(k-1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ if i is odd and $k > 2$. By the inductive assumption with (3.58) and (3.59), for $3 \leq i_0 \leq 2^{k-1} - 1$,

$$\sum_{j:even} C^h_{(k-1,j)} (D^{h-1}_{(k-1,i_0-j)} - D^h_{(k-1,i_0-j)}) + \sum_{j:odd} D^h_{(k-1,i_0-j)} (C^h_{(k-1,j)} - C^{h+2^{k-1}}_{(k-1,j)}) \equiv 0(mod\ 2^{s(i_0)+1}) \tag{3.61}$$

For $1 \leq i \leq 2^{k-2} - 1$, take $i_0 = i + 2^{k-2}$ in (3.61). Then, if $s(j) + s(i_0 - j) > s(i_0)$, $C^h_{(k-1,j)} (D^{h-1}_{(k-1,i_0-j)} - D^h_{(k-1,i_0-j)}) \equiv 0(mod\ 2^{s(i_0)+1})$ and $D^h_{(k-1,i_0-j)} (C^h_{(k-1,j)} -$

$C^{h+2^{k-1}}_{(k-1,j)}) \equiv 0(mod\ 2^{s(i_0)+1})$. Therefore, $D^h_{(k-1,2^{k-2})}(C^h_{(k-1,i)} - C^{h+2^{k-1}}_{(k-1,i)}) \equiv 0(mod\ 2^{s(i_0)+1})$ and $C^{h+2^{k-1}}_{(k-1,i)} - C^h_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+2})$. For $2h$, (3.61) is

$$\sum_{j:even} C^{2h}_{(k-1,j)}(D^{2h-1}_{(k-1,i_0-j)} - D^{2h}_{(k-1,i_0-j)}) \equiv 0(mod\ 2^{s(i_0)+1}) \tag{3.62}$$

If i_1 is the minimum that satisfies $D^{2h-1}_{(k-1,i_1)} - D^{2h}_{(k-1,i_1)} \not\equiv 0(mod\ 2^{s(i_1)+1})$, $\sum_{j:even} C^{2h}_{(k-1,j)}(D^{2h-1}_{(k-1,i_0-j)} - D^{2h}_{(k-1,i_0-j)}) \equiv D^{2h-1}_{(k-1,i)} - D^{2h}_{(k-1,i)} \not\equiv 0(mod\ 2^{s(i)+1})$ and it contradicts (3.62). Therefore, $D^{2h}_{(k-1,i)} - D^{2h-1}_{(k-1,i)} \equiv 0(mod\ 2^{s(i)+1})$ for $1 \leq i \leq 2^{k-2}$. From (3.53), $D^{2h+1}_{(k-1,i)} - D^{2h}_{(k-1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ for $1 \leq i \leq 2^{k-2}$.

Similar to (3.46), for $3 \leq i \leq 2^{k-2}$,

$$\sum_{j_1} C^h_{(k-1,i-j_1)}(D^{h-1}_{(k,j_1)} - D^h_{(k,j_1)}) \equiv \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^{k-1}}_{(k,j_2)} - C^h_{(k,j_2)})(mod\ 2^{s(i)+1}) \tag{3.63}$$

From (3.63),

$$\begin{aligned}
&\sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) \\
\equiv\ & \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^{h+2^{k-1}}_{(k,j_2)}) + \sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^{k-1}}_{(k,j_2)} - C^h_{(k,j_2)}) \\
\equiv\ & \sum_{j_1} 2^{s(i-j_1)}(D^{h+1}_{(k,j_1)} - D^h_{(k,j_1)} + D^{h+1}_{(k,j_1)} - D^h_{(k,j_1)}) \\
\equiv\ & 0(mod\ 2^{s(i)+1})
\end{aligned}$$

and

$$\sum_{j_2} D^h_{(k-1,i-j_2)}(C^{h+2^k}_{(k,j_2)} - C^h_{(k,j_2)}) \equiv 0(mod\ 2^{s(i)+1}) \tag{3.64}$$

It is shown that

$$\sum_{j} D^h_{(k,i-j)}(C^{h+2^k}_{(k,j)} - C^h_{(k,j)}) \equiv 0(mod\ 2^{s(i)+1}) \tag{3.65}$$

where $max\{0, i - 2^{k-1}\} \leq j \leq min\{i, 2^{k-1} - 1\}$ by (3.48) because (3.64) is same as (3.47). It is easy to check that it also works for $k = 2$.

Finally, (3.60) is $\sum_{j:even} C^h_{(k,j)}(D^{h-1}_{(k,i-j)} - D^h_{(k,i-j)})(mod\ 2^{s(i)+1})$. $\sum_{j:even} C^0_{(k,j)}(D^3_{(k,i-j)} - D^4_{(k,i-j)})(mod\ 2^{s(i)+1})$ because $C^0_{(k+1,i)} \equiv 2^{s(i)}(mod\ 2^{s(i)+1})$ by using the idea in *Case* 1. Since $C^{2h}_{(k,j)} \equiv C^0_{(k,j)}(mod\ 2^{s(i)+1})$ and $D^{h+4}_{(k,i-j)} \equiv D^h_{(k,i-j)}(mod\ 2^{s(i)+1})$ in (3.52) and (3.53),

$$\sum_{j:even} C^{2h}_{(k,j)}(D^{2h-1}_{(k,i-j)} - D^{2h}_{(k,i-j)}) \equiv C^0_{(k,j)}(D^3_{(k,i-j)} - D^4_{(k,i-j)}) \equiv 0(mod\ 2^{s(i)+1})$$

for $3 \leq i \leq 2^k - 1$ and

$$\sum_{j:even} C^{2h+1}_{(k,j)}(D^{2h}_{(k,i-j)} - D^{2h+1}_{(k,i-j)}) \equiv C^0_{(k,j)}(D^3_{(k,i-j)} - D^4_{(k,i-j)} + 2^{s(i-j)}) \equiv 0(mod\ 2^{s(i)+1})$$

for $3 \leq i \leq 2^k - 1$. The proof is done.

From (1), (2), and (3), $C^h_{(n,2i)} \equiv 2^{s(2i)}(mod\ 2^{s(2i)+1})$ for $1 \leq 2i \leq 2^{n-1} - 1$, $C^{2h}_{(n,2i+1)} \equiv 2^{s(2i+1)}(mod\ 2^{s(2i+1)+1})$ and $C^{2h+1}_{(n,2i+1)} \equiv 0(mod\ 2^{s(2i+1)+1})$ for $1 \leq 2i+1 \leq 2^{n-1} - 1$, and $D^h_{(n,i)} \equiv 2^{s(i)-1}(mod\ 2^{s(i)})$ for $1 \leq i \leq 2^{n-1}$.

For $k \geq 2$, from (3.51), $D^{h+2}_{(k,2^k-1)} \equiv D^h_{(k,2^k-1)}(mod\ 2^k)$. Same as *Case* 1 with (3.49) and (3.50), k_h satisfies (3.34). ■

For $n+1 = \sum_{i=j}^k n_i 3^i$ where $0 \leq n_i < 3$ for $j \leq i \leq k$ and $n_j, n_k > 0$, $\xi_3(C_n) = \#\{j < i \leq k | n_i = 2\}$ is proved in the next theorem.

Theorem 21 *If $n+1 = \sum_{i=j}^k n_i 3^i$ where $0 \leq n_i < 3$ for $j \leq i \leq k$ and $n_j, n_k > 0$,*

$$\xi_3(C_n) = \#\{j < i \leq k | n_i = 2\}$$

Proof By definition, $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$. Since $\xi_p(n!) = \lfloor\frac{n}{p}\rfloor + \lfloor\frac{n}{p^2}\rfloor + \lfloor\frac{n}{p^3}\rfloor + \cdots$,

$$\xi_3((n+1)!) = \sum_{i=j}^{k} n_i(1 + 3 + \cdots + 3^{i-1})$$

Since $n = \sum_{i=j}^{k} n_i 3^i - 1 = \sum_{i=0}^{j-1} 2 \cdot 3^i + (n_j - 1)3^j + \sum_{i=j+1}^{k} n_i 3^i$,

$$\begin{aligned}\xi_3(n!) &= \sum_{i=0}^{j-1} 2(1+3+\cdots+3^{i-1}) + (n_j-1)(1+3+\cdots+3^{j-1}) \\ &+ \sum_{i=j+1}^{k} n_i(1+3+\cdots+3^{i-1})\end{aligned}$$

and

$$\begin{aligned}\xi_3((2n)!) &= \sum_{i=0}^{j-1}(1+2\cdot 2(1+3+\cdots+3^{i-1})) + (2(n_j-1)(1+3+\cdots+3^{j-1})) \\ &+ \sum_{i=j+1}^{k}(\lfloor\frac{2n_i}{3}\rfloor + 2n_i(1+3+\cdots+3^{i-1}))\end{aligned}$$

Therefore,

$$\begin{aligned}\xi_3(C_n) &= \xi_3((2n)!) - \xi_3((n+1)!) - \xi_3(n!) \\ &= \sum_{i=0}^{j-1}(1+2(1+3+\cdots+3^{i-1})) - (1+3+\cdots+3^{j-1}) + \sum_{i=j+1}^{k}\lfloor\frac{2n_i}{3}\rfloor \\ &= \sum_{i=0}^{j-1} 3^i - (1+3+\cdots+3^{j-1}) + \sum_{i=j+1}^{k}\lfloor\frac{2n_i}{3}\rfloor \\ &= \sum_{i=j+1}^{k}\lfloor\frac{2n_i}{3}\rfloor\end{aligned}$$

and $\xi_3(C_n) = \#\{j < i \leq k | n_i = 2\}$. ■

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