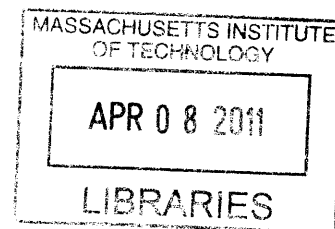


# Unitary Representations of Rational Cherednik Algebras and Hecke Algebras

by

Emanuel Stoica



Submitted to the Department of Mathematics  
in partial fulfillment of the requirements for the degree of

ARCHIVES

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2010

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## Abstract

We begin the study of unitary representations in the lowest weight category of rational Cherednik algebras of complex reflection groups. We provide the complete classification of unitary representations for the symmetric group, the dihedral group, as well as some additional partial results. We also study the unitary representations of Hecke algebras of complex reflection groups and provide a complete classification in the case of the symmetric group. We conclude that the KZ functor defined in [16] preserves unitarity in type A. Finally, we formulate a few conjectures concerning the classification of unitary representations for other types and the preservation of unitarity by the KZ functor and the restriction functors defined in [2].

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## Acknowledgments

I am deeply indebted to my advisor Pavel Etingof for his continuous guidance, support and patience. I would also like to thank Ivan Cherednik, David Vogan and Charles Dunkl for their helpful suggestions along the development of this project. I am also grateful to Iain Gordon and Roman Bezrukavnikov for their interest in the project. Even if it ever proves to have no deeper meaning, the project will remain that which catalyzed a central discovery about the author's own self.



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# Chapter 1

## Introduction

The classification of the unitary complex representations of groups and algebras is an important and often difficult problem in representation theory. This thesis will begin the study and classification of unitary highest weight representations of rational Cherednik algebras and will begin to connect them to the unitary representations of Hecke algebras.

The rational Cherednik algebra  $H_c(W, \mathfrak{h})$  is defined by a finite group  $W$ , a finite dimensional complex representation  $\mathfrak{h}$  of  $W$ , and a parameter function  $c$  on conjugacy classes of reflections in  $W$ . For any irreducible representation  $\lambda$  of  $W$ , one can define the irreducible lowest weight representation  $L_c(\lambda)$  of  $H_c(W, \mathfrak{h})$ . For a certain restriction of the parameter function, the representation  $L_c(\lambda)$  admits a unique non-degenerate contravariant Hermitian form (up to rescaling). We call  $L_c(\lambda)$  *unitary* if this form is can be normalized to be positive definite.

Given  $\lambda$  an irreducible representation of  $W$ , we describe the set  $U(\lambda)$  of parameters  $c$  such that the module  $L_c(\lambda)$  is unitary. We call it the unitarity locus of  $\lambda$ . The problem of describing the unitarity loci is motivated by harmonic analysis, and was formulated by I. Cherednik. In this thesis, we will solve the problem for certain families of reflections groups, such as the symmetric and dihedral groups, and also provide some partial results for other types of groups.

The main result of the first part of the thesis is Theorem 2.3.5 that describes the unitarity locus for the symmetric group (type A). We also describe completely

the unitarity locus for the dihedral group and the cyclic group. In type A, if  $\lambda$  is not 1-dimensional, the unitarity locus  $U(\lambda)$  consists of the interval  $[-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}]$  (“the continuous spectrum”) union with a certain finite set of points of the form  $\frac{1}{j}$  (“the discrete spectrum”) where  $j$  is an integer and  $L(\lambda)$  is the largest hook of the Young diagram corresponding to  $\lambda$ . In particular, this answers a question posed by I. Cherednik, proving that for  $c = \frac{1}{m}$  where  $2 \leq m \leq n$ , the irreducible subrepresentation of the polynomial representation  $M_c(\mathbb{C})$  is unitary. Furthermore, as it is shown in detail in [15], its unitary structure can be given by an integration pairing with the Macdonald-Mehta measure.

In the second part of the thesis, we begin the study of unitary representations of Hecke algebras of complex reflection groups. Given a complex reflection group  $W$  and the corresponding Hecke algebra  $\mathcal{H} = \mathcal{H}_q(W)$  over the complex numbers, we investigate the existence of a certain  $\mathcal{H}$ -invariant nondegenerate Hermitian form on a representation  $V$ . The defining property of the form is its invariance under the braid group  $B_W$  of  $W$ , namely  $(Tv, v') = (v, T^{-1}v')$  for all  $T \in B_W$  and  $v, v' \in V$ .

A representation of  $\mathcal{H}$  with such a Hermitian form will be called *unitary* if the form is positive definite. Since unitary representations are semisimple, we may restrict the study of unitarity to irreducible representations.

The main result of the second part of the thesis is Theorem 3.3.2 which provides a complete classification of the unitary irreducible representations of the Hecke algebra of the symmetric group. The proof uses the theory of tensor categories and properties of restriction functors for Hecke algebras.

The organization of this thesis is as follows. Chapter 2 focuses on the unitary representations of rational Cherednik algebras. Section 1 contains definitions and general results. Section 2 contains a few properties of unitarity loci. Section 3 describes the unitarity loci for the cyclic group, dihedral group and the symmetric group, along with some partial results.

Chapter 3 focuses on the unitary representations of Hecke algebras of complex reflection groups. Section 1 contains a few general definitions and results regarding the Hecke algebra of the complex reflection group  $G(r, 1, n)$ . Section 2 contains the

construction of a Hermitian form on Specht modules and their irreducible quotients which is invariant under the action of the corresponding braid group. Section 3 is focused on the classification of the unitary irreducible representations of the Hecke algebra of the symmetric group.

Finally, Chapter 4 contains a few closing remarks and conjectures.



# Chapter 2

## Unitary lowest weight representations of Cherednik algebras

### 2.1 Preliminaries

#### 2.1.1 Definition of rational Cherednik algebras

Let  $\mathfrak{h}$  be a finite dimensional complex vector space and a positive definite Hermitian inner product  $(,)$  on  $\mathfrak{h}$ . Let  $T : \mathfrak{h} \rightarrow \mathfrak{h}^*$  be the antilinear isomorphism given by  $(Ty)(y') = (y, y')$  for any  $y, y' \in \mathfrak{h}$ .

Let  $W$  be a finite group of linear transformations of  $\mathfrak{h}$  preserving the Hermitian inner product (unitary transformations). A reflection element in  $W$  is an element  $s \in W$  such that  $\text{rk}(s - 1)|_{\mathfrak{h}} = 1$ . Let  $S$  be the set of reflections in  $W$  and  $c : S \rightarrow \mathbb{C}$  be a  $W$ -equivariant function on  $S$  ( $W$  acts on  $S$  by conjugation). For every  $s \in S$ , let  $\zeta_s$  be its nontrivial eigenvalue in  $\mathfrak{h}^*$ ,  $\alpha_s \in \mathfrak{h}^*$  a generator of  $\text{Im}(s - 1)|_{\mathfrak{h}^*}$ , and  $\alpha_s^\vee \in \mathfrak{h}$  be the generator of  $\text{Im}(s - 1)|_{\mathfrak{h}}$ , normalized such that  $(\alpha_s, \alpha_s^\vee) = 2$ . If  $W$  is a group generated by its reflection elements, let  $d_1, \dots, d_{\dim \mathfrak{h}}$  be the degrees of the generators of the ring of  $W$ -invariant polynomials  $\mathbb{C}[\mathfrak{h}]^W$ .

**Definition 2.1.1.** (see e.g. [13, 14]) The rational Cherednik algebra  $H_c(W, \mathfrak{h})$  is

defined as the quotient of the algebra  $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$  by the ideal of commutation relations

$$[x, x'] = 0, [y, y'] = 0, [y, x] = (y, x) - \sum_{s \in S} c_s(y, \alpha_s)(x, \alpha_s^\vee)s,$$

for all  $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$ .

Let  $y_1, y_2, \dots$  be a basis of  $\mathfrak{h}$  and  $x_1, x_2, \dots$  the dual basis of  $\mathfrak{h}^*$ . The element

$$h = \sum_i x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s \in S} \frac{2c_s}{1 - \zeta_s} s$$

plays an important role in the representation theory of rational Cherednik algebras. It does not depend of the choice of the basis. Its most important properties are the identities

$$[h, x_i] = x_i, [h, y_i] = -y_i. \quad (2.1)$$

### 2.1.2 Verma modules, irreducible modules, Hermitian forms

Let  $\lambda$  be an irreducible representation of  $W$ . It can be extended to a representation of  $W \ltimes S\mathfrak{h}$  by letting  $\mathfrak{h}$  act by 0. Define the Verma module  $M_c(\lambda) := H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W \ltimes S\mathfrak{h}} \lambda$  as an induced module from  $W \ltimes S\mathfrak{h}$ . Every quotient of the Verma module  $M_c(\lambda)$  is called a lowest weight module with lowest weight  $\lambda$ . Let  $L_c(\lambda)$  be the unique irreducible quotient of the Verma module.

Denote by  $\mathcal{O}_c(W, \mathfrak{h})$  the category of  $H_c(W, \mathfrak{h})$ -modules which are finitely generated under the action of  $\mathbb{C}[\mathfrak{h}]$ , and locally nilpotent under the action of  $\mathfrak{h}$ . The Verma module  $M_c(\lambda)$  and all its quotients belong to this category. Note also that the element  $h$  acts locally finitely on any object of  $\mathcal{O}_c(W, \mathfrak{h})$  with finite dimensional generalized eigenspaces. In particular, it acts semisimply on any lowest weight module of lowest weight  $\lambda$ . The lowest eigenvalue is given by

$$h_c(\lambda) = \frac{\dim \mathfrak{h}}{2} - \sum_{s \in S} \frac{2c_s}{1 - \zeta_s} s|_\lambda.$$

The eigenvalues of  $h$  on  $M_c(\lambda)$  are of the form  $h_c(\lambda) + \mathbb{Z}_+$  which gives a  $\mathbb{Z}_+$ -grading on  $M_c(\lambda)$ .

Let  $M \in \mathcal{O}_c(W, \mathfrak{h})$ . A vector  $v \in M$  is called *singular* if  $yv = 0$  for any  $y \in \mathfrak{h}$ . It is clear that a lowest weight module  $M$  is irreducible if and only if it has no nonzero singular vectors of positive degree.

## 2.2 Unitarity loci and their properties

### 2.2.1 Definition of unitarity

Let  $C$  be the space of functions  $c$  on the set of reflections  $S$  such that  $c(s) = \bar{c}(s^{-1})$  for all  $s \in S$ .

Let  $\lambda$  be an irreducible representation of  $W$ , and  $(, )_\lambda$  a  $W$ -invariant Hermitian form normalized to be positive definite. If  $c \in C$ , we can extend it to modules of lowest weight  $\lambda$  in the category  $\mathcal{O}_c(W, \mathfrak{h})$  as the following result shows.

**Proposition 2.2.1.** (i) There exists a unique  $W$ -invariant Hermitian form  $\beta_{c,\lambda}$  on  $M_c(\lambda)$  which coincides with  $(, )_\lambda$  in degree zero, and satisfies the contravariance condition

$$(yv, v') = (v, Ty \cdot v')$$

for all  $v, v' \in M_c(\lambda), y \in \mathfrak{h}$ .

(ii) The kernel of  $\beta_{c,\lambda}$  is equal to the maximal proper submodule  $J_c(\lambda)$  of  $M_c(\lambda)$ , so  $\beta_{c,\lambda}$  descends to a nondegenerate form on the irreducible quotient  $L_c(\lambda) = M_c(\lambda)/J_c(\lambda)$ .

*Proof.* Straightforward. □

We will call  $\beta_{c,\lambda}$  the *contravariant* Hermitian form. It is unique up to rescaling. We can define  $*$  a semilinear anti-automorphism of  $H_c(W, \mathfrak{h})$  by

$$x^* = T^{-1}x, \quad y^* = Ty, \text{ and } w^* = w^{-1}$$

for all  $x \in \mathfrak{h}^*$ ,  $y \in \mathfrak{h}$ , and  $w \in W$ . Then the contravariant form  $\beta_{c,\lambda}$  satisfies

$$(fv, v') = (v, f^*v')$$

for all  $v, v' \in M_c(\lambda)$ ,  $f \in H_c(W, \mathfrak{h})$ .

**Definition 2.2.2.** (i) The representation  $L_c(\lambda)$  is said to be unitary if the form  $\beta_{c,\lambda}$  is positive definite on  $L_c(\lambda)$ .

(ii) Let  $U(\lambda)$  be the set of points  $c \in C$  such that  $L_c(\lambda)$  is unitary. We call  $U(\lambda)$  the *unitarity locus* of  $\lambda$ .

## 2.2.2 Properties of the unitarity loci

Let us present some basic properties of the unitarity loci. Let  $\lambda$  be an irreducible representation of  $W$ .

**Proposition 2.2.3.** (i)  $U(\lambda)$  is a closed set in  $C$ .

(ii) The point 0 belongs to the interior of  $U(\lambda)$ .

(iii) The connected component of 0 in the set of all  $c$  for which  $M_c(\lambda)$  is irreducible is entirely contained in  $U(\lambda)$ .

*Proof.* (i)  $c \in U(\lambda)$  if and only if the contravariant form  $\beta_{c,\lambda}$  is positive semidefinite on  $M_c(\lambda)$ , which is a closed condition on  $c$ .

(ii) Because the Cherednik algebra satisfies the Poincare-Birkoff-Witt property, we have a natural identification of  $M_c(\lambda)$  with  $\lambda \otimes \mathbb{C}[\mathfrak{h}]$  as vector spaces. The form  $\beta_{0,\lambda}$  is simply the tensor product of the form  $(\cdot, \cdot)_\lambda$  on  $\lambda$  and the standard inner product on  $\mathbb{C}[\mathfrak{h}]$ , given by the formula  $(f, g) = (D_g f)(0)$  where  $D_g \in S\mathfrak{h}$  is the differential operator on  $\mathbb{C}[\mathfrak{h}]$  with constant coefficients corresponding to  $g \in S\mathfrak{h}^*$  via the operator  $T^{-1}$ . Thus  $\beta_{0,\lambda}$  is positive definite, as desired.

(iii) This property follows from the standard fact that a continuous family of nondegenerate Hermitian forms is positive definite if and only if one of them is positive definite. □



Below, we consider separately the case of constant functions  $c \in C$  (in this case,  $c$  is real). Let  $U^*(\lambda)$  be the subset of  $U(\lambda)$  containing constant functions. We can describe  $U^*(\lambda)$  as a subset of  $\mathbb{R}$ . In this case, Proposition 2.2.3 implies

**Corollary 2.2.4.** (i)  $U^*(\lambda)$  is a closed set in  $\mathbb{R}$ .

(ii) The point 0 belongs to the interior of  $U^*(\lambda)$  for any  $\lambda$ .

(iii) The connected component of 0 in the set of all  $c$  for which  $M_c(\lambda)$  is irreducible is entirely contained in  $U^*(\lambda)$ .

Let  $W_{\text{ab}}^\vee$  be the group of characters of  $W$ . It is easy to see that  $W_{\text{ab}}^\vee$  acts on the space  $C$  by multiplication. It also acts on representations of  $W$  by tensor multiplication.

**Proposition 2.2.5.** For any  $\chi \in W_{\text{ab}}^\vee$  one has  $U(\chi \otimes \lambda) = \chi U(\lambda)$ .

*Proof.* There exists a natural isomorphism  $i_\chi : H_c(W, \mathfrak{h}) \rightarrow H_{\chi^{-1}c}(W, \mathfrak{h})$  given by  $i_\chi(w) = \chi^{-1}(w)w$ ,  $i_\chi(x) = x$  and  $i_\chi(y) = y$  for all  $w \in W, x \in \mathfrak{h}^*$ , and  $y \in \mathfrak{h}$ . The pushforward by this isomorphism maps a representation  $\lambda$  to  $\chi \otimes \lambda$ . The desired conclusion follows.  $\square$

**Proposition 2.2.6.** Let  $c \in U(\lambda)$ . If  $\sigma$  is an irreducible subrepresentation of  $W$  inside  $\lambda \otimes \mathfrak{h}^*$ , then  $h_c(\sigma) \leq h_c(\lambda) + 1$ .

*Proof.* Let us regard  $\sigma$  as sitting in degree 1 part of  $M_c(\lambda)$ . The action of  $y \in \mathfrak{h}$  on the degree 1 part can be viewed as an operator  $\mathfrak{h} \otimes \mathfrak{h}^* \otimes \lambda \rightarrow \lambda$  which sends  $y \otimes x \otimes v$  to

$$(y, x)v - \sum_{s \in S} c_s(\alpha_s^\vee, x)(y, \alpha_s)s(v).$$

Equivalently, we obtain an endomorphism  $F_{c, \lambda, 1}$  of  $\mathfrak{h}^* \otimes \lambda$  given by

$$F_{c, \lambda, 1} = 1 - \sum_{s \in S} \frac{2c_s}{1 - \zeta_s} (1 - s) \otimes s, \quad (2.2)$$

because  $((1-s)x, y) = \frac{(1-\zeta_s)(\alpha_s^\vee, x)(y, \alpha_s)}{2}$ . Thus  $F_{c, \lambda, 1}$  acts on  $\sigma$  by the scalar

$$1 + h_c(\lambda) - h_c(\sigma).$$

Let us look at the restriction of the inner product  $\beta_{c, \lambda}$  to the irreducible  $W$ -subrepresentation  $\sigma$  sitting in the degree 1 part of  $M_c(\lambda)$ . This restriction must be of the form  $P(c)(\cdot, \cdot)_\sigma$ , where  $P(c)$  is a linear polynomial in  $c$ . Since  $P(c)$  is positive for  $c = 0$  (by Proposition 2.2.3(ii)), we conclude that  $P(c) = K(1 + h_c(\lambda) - h_c(\sigma))$  for some  $K > 0$ . This implies the statement.  $\square$

**Remark 2.2.7.** The representation  $\sigma$  sitting in degree 1 in  $M_c(\lambda)$  contains singular vectors if and only if  $h_c(\sigma) - h_c(\lambda) = 1$ .

Note that  $\sum_{s \in S}$  is a central element in  $\mathbb{C}[W]$ , and hence it acts by a scalar on any irreducible representation of  $W$ . Let  $T_\lambda$  be the scalar by which  $\sum_{s \in S} s$  acts on  $\lambda$ .

**Corollary 2.2.8.** Let  $c \in U^*(\lambda)$ . If  $\sigma$  is an irreducible subrepresentation of  $W$  inside  $\lambda \otimes \mathfrak{h}^*$ , then

$$c(T_\lambda - T_\sigma) \leq 1.$$

We will now generalize the operator  $F_{c, \lambda, 1}$  acting in degree 1 to higher degrees. For any  $c \in C$ , there is a unique self-adjoint operator  $F_{c, \lambda}$  on  $M_c(\lambda) = \lambda \otimes S\mathfrak{h}^*$ , given by the formula  $\beta_{c, \lambda}(v, v') = \beta_{0, \lambda}(F_{c, \lambda}v, v')$  for all  $v, v' \in M_c(\lambda)$ . We have  $F_{c, \lambda} = \bigoplus_{m \geq 0} F_{c, \lambda, m}$ , where  $F_{c, \lambda, m} : \lambda \otimes S^m \mathfrak{h}^* \rightarrow \lambda \otimes S^m \mathfrak{h}^*$  is an operator which is polynomial in  $c$  of degree at most  $m$ . Note that if  $F_{c, \lambda, m}$  is independent of  $c$ , then  $F_{c, \lambda, m} = 1$ , since  $F_{0, \lambda, m} = 1$ . The following proposition gives a recursive formula for  $F_{c, \lambda, m}$ .

**Proposition 2.2.9.** Let  $a_1, \dots, a_m \in \mathfrak{h}^*$ , and  $v \in \lambda$ . Then

$$\begin{aligned} F_{c, \lambda, m}(a_1 \dots a_m v) = & \\ & \frac{1}{m} \sum_{j=1}^m a_j F_{c, \lambda, m-1}(a_1 \dots a_{j-1} a_{j+1} \dots a_m v) - \\ & - \frac{1}{m} \sum_{j=1}^m \sum_{s \in S} \frac{2c_s}{1 - \zeta_s} (1 - s)(a_j) F_{c, \lambda, m-1}(a_1 \dots a_{j-1} s(a_{j+1} \dots a_m v)). \end{aligned}$$

*Proof.* For any  $y \in \mathfrak{h}$  we have

$$F_{c,\lambda,m-1}(ya_1\dots a_mv) = \partial_y F_{c,\lambda,m}(a_1\dots a_mv).$$

This follows from the defining property of  $F_{c,\lambda}$ . Let  $u = a_1\dots a_mv$ . Then, we obtain

$$F_{c,\lambda,m}(u) = \frac{1}{m} \sum_i x_i F_{c,\lambda,m-1}(y_i u).$$

We used the property that  $\sum_i x_i \partial_{y_i}$  acts by  $m$  on homogeneous polynomials of degree  $m$ . After computing  $y_i u$  using the commutation relations of the rational Cherednik algebra, we obtain the desired formula.  $\square$

**Remark 2.2.10.** Note that for  $m = 1$ , this formula reduces to formula (2.2).

**Corollary 2.2.11.** Assume that  $F_{c,\lambda,i}$  is constant (and hence equal to 1) for  $1 \leq i \leq m-1$ . Then on every irreducible  $W$ -subrepresentation  $\sigma$  of  $\lambda \otimes S^m \mathfrak{h}^*$ , the operator  $F_{c,\lambda,m}$  acts by the scalar  $1 + \frac{h_c(\lambda) - h_c(\sigma)}{m}$ .

Let us now assume that  $W$  is a (finite) group of real reflections. Its reflection representation  $\mathfrak{h}$  is the complexification of a real vector space  $\mathfrak{h}_{\mathbb{R}}$  with a positive definite symmetric inner product, which is extended to a Hermitian inner product on the complexification, and that  $W$  acts by orthogonal transformations on  $\mathfrak{h}_{\mathbb{R}}$ . In this case,  $s^2 = 1$  and  $\zeta_s = -1$  for any reflection  $s \in S$ . Also,  $c \in C$  iff  $c$  is real valued.

Let us choose  $y_1, y_2, \dots$  to be an orthonormal basis of  $\mathfrak{h}_{\mathbb{R}}$  and  $x_1, x_2, \dots$  its dual basis. Define the elements

$$e = -\frac{1}{2} \sum x_i^2, f = \frac{1}{2} \sum y_i^2.$$

It is easy to see that the element  $h$  introduced above satisfies in this case

$$h = \frac{1}{2} \sum (x_i y_i + y_i x_i).$$

The elements  $e, f, h$  form an  $\mathfrak{sl}_2$ -triple. Their construction does not depend on the

choice of the orthonormal basis. Note that  $e^* = -f$ ,  $f^* = -e$ , and  $h^* = h$ . The following theorem explains the usefulness of the  $\mathfrak{sl}_2$ -triple.

**Proposition 2.2.12.** (i) A unitary representation  $L_c(\lambda)$  of  $H_c(W, \mathfrak{h})$  restricts to a unitary representation of  $\mathfrak{sl}_2(\mathbb{R})$  from the lowest weight category  $\mathcal{O}$ . In particular,  $h_c(\lambda) = \frac{\dim \mathfrak{h}}{2} - \sum c_s s|_\lambda \geq 0$ .

(ii) A unitary representation  $L_c(\lambda)$  is finite dimensional iff  $L_c(\lambda) = \lambda$ .

(iii) A unitary representation  $L_c(\lambda)$  is finite dimensional iff  $h_c(\sigma) - h_c(\lambda) = 1$  for any irreducible  $W$ -subrepresentation  $\sigma$  contained in  $\lambda \otimes \mathfrak{h}^*$ . In this case,  $h_c(\lambda) = 0$ .

*Proof.* (i) Let  $v \in \lambda$  in the lowest weight  $h$ -eigenspace of  $L_c(\lambda)$ . Then  $fv = 0$ . Also

$$(ev, ev) = (v, -fev) = (v, (h - ef)v) = h_c(\lambda)(v, v).$$

If  $ev \neq 0$ , then  $h_c(\lambda) > 0$ . If  $ev = 0$  then  $hv = [e, f]v = 0$ , so  $h_c(\lambda) = 0$ . Hence  $h$  acts on the lowest weight vectors by a nonnegative scalar.

(ii) If  $L_c(\lambda)$  is finite dimensional, then by (i), it is a trivial representation of  $\mathfrak{sl}_2(\mathbb{R})$ . So  $h$  acts by 0, which combined with (2.1), implies any  $x \in \mathfrak{h}^*$  acts by 0.

(iii) Using (ii), this is equivalent to  $L_c(\lambda) = \lambda$ , which is equivalent to  $y \in \mathfrak{h}$  acts by 0 on any subrepresentation  $\sigma$  in  $\lambda \otimes \mathfrak{h}^*$ . By 2.2.7, this is, in turn, equivalent to  $h_c(\sigma) - h_c(\lambda) = 1$ .  $\square$

Let  $W$  be an irreducible Coxeter group, and  $\mathfrak{h}$  be its reflection representation. Recall that the representations  $\wedge^i \mathfrak{h}$  are irreducible. The representation  $\wedge^{\dim \mathfrak{h}} \mathfrak{h}$  is isomorphic to the sign representation  $\mathbb{C}_-$  of  $W$ . The following theorem describes the unitarity locus of the exterior powers  $\wedge^i \mathfrak{h}$ .

**Proposition 2.2.13.** (i) Let  $\tau$  be an irreducible representation of the Coxeter group  $W$  and  $H$  be its Coxeter number. Then  $[-\frac{1}{H}, \frac{1}{H}] \subset U^*(\tau)$ .

(ii)  $U^*(\mathbb{C}) = (-\infty, \frac{1}{H}]$ , and  $U^*(\mathbb{C}_-) = [-\frac{1}{H}, +\infty)$ .

(iii) For  $0 < i < \dim \mathfrak{h}$ ,  $U^*(\wedge^i \mathfrak{h}) = [-\frac{1}{H}, \frac{1}{H}]$ .

*Proof.* (i) If  $c \in (-\frac{1}{H}, \frac{1}{H})$ , then  $c$  is a regular value ([10, 16]), namely the category  $\mathcal{O}_c(W, \mathfrak{h})$  is semisimple. Therefore, the Verma module  $M_c(\tau)$  is irreducible, and the desired conclusion follows from Proposition 2.2.3(iii).

(ii) Let  $c \in U^*(\mathbb{C})$ . Then  $h_c(\mathbb{C}) = \frac{\dim \mathfrak{h}}{2} - c|S| = |S|(\frac{1}{H} - c)$ . By Proposition 2.2.12,  $h_c(\mathbb{C}) \geq 0$ , hence  $c \leq \frac{1}{H}$ . On the other hand, for any negative  $c$ , the module  $M_c(\mathbb{C})$  is irreducible, and hence unitary. The first statement of (ii) now follows from (i). To obtain the second statement, we use Proposition 2.2.5.

(iii) The “ $\supset$ ” inclusion follows from part (i). To prove the “ $\subset$ ” inclusion, we note that the irreducible representation  $\wedge^{i+1} \mathfrak{h}$  sits naturally in the degree 1 part of  $M_c(\wedge^i \mathfrak{h})$ . Let us compute  $T_{\wedge^i \mathfrak{h}}$ . The trace of a reflection in  $\wedge^i \mathfrak{h}$  is

$$\binom{\dim \mathfrak{h} - 1}{i} - \binom{\dim \mathfrak{h} - 1}{i - 1}.$$

Thus, we have

$$T_{\wedge^i \mathfrak{h}} = |S| \frac{\binom{\dim \mathfrak{h} - 1}{i} - \binom{\dim \mathfrak{h} - 1}{i - 1}}{\binom{\dim \mathfrak{h}}{i}} = |S| \left(1 - \frac{2i}{\dim \mathfrak{h}}\right).$$

Hence,

$$h_c(\wedge^{i+1} \mathfrak{h}) - h_c(\wedge^i \mathfrak{h}) = 2c|S|/\dim \mathfrak{h} = cH.$$

Thus from proposition 2.2.6, we conclude that if  $c \in U^*(\wedge^i \mathfrak{h})$  then  $cH \leq 1$ , namely  $c \leq \frac{1}{H}$ . The inequality  $c \geq -\frac{1}{H}$  follows from Proposition 2.2.5.  $\square$

## 2.3 Description of unitarity loci

### 2.3.1 The cyclic group

Let  $W = \mathbb{Z}/n\mathbb{Z}$ . All its irreducible representations are 1-dimensional. By Proposition 2.2.5 it is sufficient to determine the unitarity locus for one such representation. For instance, let  $\mathfrak{h}$  be given by  $j \rightarrow \zeta^{-j}$  where  $\zeta = e^{2\pi i/n}$ .

The module  $M_c(\mathbb{C})$  has basis  $x^k$ ,  $k \geq 0$ . Let  $a_k := \beta_{c, \mathbb{C}}(x^k, x^k)$  (we can normalize

the form such that  $a_0 = 1$ ). We obtain by a straightforward computation

$$a_k = a_{k-1} \left( k - 2 \sum_{j=1}^{n-1} \frac{1 - \zeta^{jk}}{1 - \zeta^j} c_j \right)$$

where  $c_j = c(j)$ ,  $j = 1, \dots, n-1$ .

Let

$$b_k := 2 \sum_{j=1}^{n-1} \frac{1 - \zeta^{jk}}{1 - \zeta^j} c_j,$$

for any  $k \geq 0$ . Note that  $b_0 = 0$ , and  $b_{n+m} = b_n$ . If  $c \in C$  then  $b_j$  are real, and it is easy to see that  $b_1, \dots, b_{n-1}$  form a real linear system of coordinates for the parameter space  $C$ . This follows from the basic fact that the matrix with entries  $\frac{1 - \zeta^{jk}}{1 - \zeta^j}$ ,  $1 \leq j, k \leq n-1$ , is nonsingular.

The following proposition and corollary describe  $L_c(\mathbb{C})$  and its unitarity locus.

**Proposition 2.3.1.** (i)  $M_c(\mathbb{C})$  is irreducible iff  $k - b_k \neq 0$  for any  $k \geq 1$ . It is irreducible and unitary iff  $k - b_k > 0$  for all  $k = 1, \dots, n-1$ .

(ii) If  $M_c(\mathbb{C})$  is reducible, then let  $r$  is the smallest positive integer such that  $r = b_r$ . Then  $L_c(\mathbb{C})$  has dimension  $r$  (which can be any number not divisible by  $n$ ), and basis  $1, x, \dots, x^{r-1}$ . The representation  $L_c(\mathbb{C})$  is unitary iff  $r < n$  and  $b_k < k$  for all  $1 \leq k < r$ .

**Corollary 2.3.2.**  $U$  is the set of vectors  $(b_1, \dots, b_{n-1})$  such that in the vector  $(1 - b_1, 2 - b_2, \dots, n - 1 - b_{n-1})$  all the entries preceding the first zero entry are positive (if there is no zero, all entries must be positive).

If  $n = 2$ , then  $c_1 = c$  and  $b_1 = 2c$ , and we obtain that  $U = (-\infty, \frac{1}{2}]$ . At  $c = \frac{1}{2}$  the unitary representation is 1-dimensional.

## 2.3.2 The dihedral group

In this subsection we will describe the unitarity loci for the irreducible representations of the dihedral group.

First, let  $W = D_n$  be an odd dihedral where  $n = 2d + 1$  (the group of symmetries of the regular  $(2d + 1)$ -gon). The group  $W$  has only one conjugacy class of reflections (so  $C = \mathbb{R}$ ). It has two 1-dimensional representations,  $\mathbb{C}$  and  $\mathbb{C}_-$ , and  $d$  irreducible representations of dimension 2,  $\lambda_l$ ,  $1 \leq l \leq d$ . The representation  $\lambda_l$  can be described in matrix form by

$$s_1 = \begin{pmatrix} 0 & \zeta^l \\ \bar{\zeta}^l & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then  $\mathfrak{h} = \lambda_1$  is the reflection representation.

**Proposition 2.3.3.** (i)  $U(\mathbb{C}) = (-\infty, \frac{1}{2d+1}]$ .

(ii)  $U(\lambda_l) = [-\frac{l}{2d+1}, \frac{l}{2d+1}]$  for all  $1 \leq l \leq d$ .

*Proof.* (i) Already proved in 2.2.13.

(ii) Let us look at the decomposition  $S^k \lambda_1 = \lambda_k \oplus \lambda_{k-2} \oplus \dots$  (the last summand is  $\mathbb{C}$  if  $k$  is even). By tensoring this decomposition with  $\lambda_l$ , we obtain only 2-dimensional summands if  $k < l$ , while 1-dimensional summands appear first when  $k = l$ . Since  $h_c(\lambda) = 1$  for any 2-dimensional  $\lambda$ , using Corollary 2.2.11, it follows by induction on  $k$  that the operator  $F_{c, \lambda_l, k}$  is constant in  $c$  for  $k < l$  (and hence equal to 1). Thus, again by Corollary 2.2.11,  $F_{c, \lambda_l, l}(X) = (1 \pm \frac{2d+1}{l}c)X$  if  $X$  belongs to the sign, respectively trivial subrepresentation of  $\lambda_l \otimes S^l \lambda_1$ . If  $c \in U(\lambda_l)$ , this implies  $c \in [-\frac{l}{2d+1}, \frac{l}{2d+1}]$ .

It remains to show that  $M_c(\lambda_l)$  is irreducible if  $(2d + 1)|c| < l$ . This is proved in [6]. It can also be proved directly. It follows from the above that  $M_c(\lambda_l)$  contains no singular vectors of degree at most  $l$ . Assume now  $c > 0$ . Then any singular vector would be in the sign representation. Let  $k \geq l$  be the degree of this vector. Then we get  $h_c(\mathbb{C}_-) - h_c(\lambda_l) = k$ , which implies that  $(2d + 1)c = k \geq l$ , as desired. The case of negative  $c$  is similar.  $\square$

Let now  $W = D_n$  be an even dihedral group where  $n = 2d$ ,  $d \geq 2$  (the group of symmetries of a regular  $2d$ -polygon). This group has two conjugacy classes of reflections, represented by the Coxeter generators  $s_1, s_2$  that satisfy  $(s_1 s_2)^{2d} = 1$ . The 1-dimensional representations are  $\mathbb{C}$ ,  $\mathbb{C}_-$ , and also the representations  $\varepsilon_1$  and  $\varepsilon_2$ , given by the formulas

$$\varepsilon_1: \begin{cases} s_1 \longrightarrow -1 \\ s_2 \longrightarrow 1 \end{cases} \quad \varepsilon_2: \begin{cases} s_1 \longrightarrow 1 \\ s_2 \longrightarrow -1 \end{cases}$$

There are also  $d - 1$  irreducible representations  $\lambda_l$  of dimension 2,  $1 \leq l \leq d - 1$ , defined by the same formulas as in the odd case. As before,  $\mathfrak{h} = \lambda_1$  is the reflection representation. We will extend the notation  $\lambda_l$  to all integer values of  $l$ , so that we have  $\lambda_l = \lambda_{-l}$  and  $\lambda_{d-l} = \lambda_{d+l}$ ,  $\lambda_0 = \mathbb{C} \oplus \mathbb{C}_-$ , and  $\lambda_d = \varepsilon_1 \oplus \varepsilon_2$ . Note that  $\lambda_l \otimes \varepsilon_i = \lambda_{d-l}$  and  $\lambda_l \otimes \mathbb{C}_- = \lambda_l$ .

Let  $c_1$  and  $c_2$  be the values of the two parameters corresponding to the two conjugacy classes. The unitarity locus  $U(\lambda)$  will be a subspace of the real plane  $\mathbb{R}^2$ . By Proposition 2.2.5, it suffices to find  $U(\lambda)$  for  $\lambda = \mathbb{C}$  and  $\lambda = \lambda_l$  for some  $1 \leq l < \frac{d}{2}$ .

**Proposition 2.3.4.** (i)  $U(\mathbb{C})$  is the union of the region defined by the inequalities  $c_1 + c_2 < \frac{1}{d}$ ,  $c_1 \leq \frac{1}{2}$  and  $c_2 \leq \frac{1}{2}$  with the line  $c_1 + c_2 = \frac{1}{d}$ .

(ii) If  $1 \leq l \leq \frac{d}{2}$  then  $U(\lambda_l)$  is the rectangle defined by the inequalities  $|c_1 + c_2| \leq \frac{l}{d}$  and  $|c_1 - c_2| \leq \frac{d-l}{d}$ .

*Proof.* (i) The operator  $F_{c,\mathbb{C},1}$  acts by the scalar  $1 - (c_1 + c_2)d$ . This implies the condition  $c_1 + c_2 \leq \frac{1}{d}$  for  $c \in U(\mathbb{C})$ .

Now recall that  $S^k \lambda_1 = \lambda_k \oplus \lambda_{k-2} \oplus \dots$  (the last summand is  $\mathbb{C}$  for even  $k$ ). In particular,  $S^d \lambda_1$  contains one copy of  $\varepsilon_1$  and  $\varepsilon_2$ . The operator  $F_{c,\mathbb{C}}$  restricted to the 1-dimensional subrepresentation  $\varepsilon_i$  acts by a scalar. This scalar is given by

$$Q(c) = (1 - 2c_i) \prod_{j=1}^{d-1} \left(1 - \frac{d}{j}(c_1 + c_2)\right). \quad (2.3)$$

It follows from [6] that for  $c_i = \frac{1}{2}$ , the representation  $\varepsilon_i$  contain singular vectors. Also that if  $c_1 + c_2 = \frac{j}{d}$  for some  $1 \leq j \leq d - 1$ , there is a singular vector in degree  $j$  in the representation  $\lambda_j$ , and that the subrepresentation generated by this vector contains  $\varepsilon_i$  in degree  $d$  of  $M_c(\lambda_l)$ . This implies that  $Q(c)$  is divisible by right hand product (2.3). In the meantime,  $Q(c)$  is a polynomial of degree at most  $d$  and  $Q(0) = 1$ . This proves the desired equality for  $Q(c)$ .



Formula (2.3) and the inequality  $c_1 + c_2 \leq \frac{1}{d}$  imply that if a unitary representation  $L_c(\mathbb{C})$  contains  $\varepsilon_i$  in degree  $d$ , then  $c_i < \frac{1}{2}$ . It remains to consider unitary representations  $L_c(\mathbb{C})$  that do not contain  $\varepsilon_i, i = 1, 2$ . This means that either this  $\varepsilon_i$  is singular in  $M_c(\mathbb{C})$ , which implies  $c_i = \frac{1}{2}$ , or,  $\lambda_1$  is singular in degree 1, which implies  $c_1 + c_2 = \frac{1}{d}$ .

(ii) Let  $l < \frac{d}{2}$ . Using a similar argument as in the odd case, there is no 1-dimensional subrepresentation of  $W$  in  $M_c(\lambda_l)$  in degrees  $k < l$ , while the trivial and sign representations sit in degree  $l$ . This implies by Corollary 2.2.11 and induction on  $k$  that  $F_{c, \lambda_l, i} = 1$  for  $i < l$ , and  $F_{c, \lambda_l, l}(X) = (1 \pm \frac{d}{l}(c_1 + c_2))X$ , if  $X$  belongs to the sign, respectively trivial subrepresentation of  $\lambda_l \otimes S^l \lambda_1$ . If  $c \in U(\lambda)$ , we obtain  $|c_1 + c_2| \leq \frac{l}{d}$ .

Let us now prove that  $|c_1 - c_2| \leq \frac{d-l}{d}$ . By [6], at  $c_1 - c_2 = \pm \frac{d-l}{d}$  there are singular vectors in  $\varepsilon_1$  or  $\varepsilon_2$  in degree  $d - l > l$ . We can show inductively that  $F_{c, \lambda}$  restricted to  $\lambda_{l+j}$  sitting in degree  $j$  for  $0 \leq j \leq d - l - 1$  acts by 1. Hence, we conclude that  $F_{c, \lambda_l, d-l}$  acts on  $\varepsilon_i$  by the scalars  $1 \pm \frac{d}{d-l}(c_1 - c_2)$ , which proves the desired inequality for unitary representations.

Finally, if both inequalities are satisfied strictly, then  $M_c(\lambda_l)$  is irreducible [6], and thus the rectangle defined by these inequalities is contained in  $U(\lambda_l)$ , as desired. This finishes the proof.  $\square$

### 2.3.3 The symmetric group

We will now study the case  $W = \mathfrak{S}_n$ ,  $n \geq 2$ , and  $\mathfrak{h} = \mathbb{C}^n$ . In this case we have only one conjugacy class of reflections, so  $C = \mathbb{R}$ .

Irreducible representations of  $\mathfrak{S}_n$  are labeled by Young diagrams  $\lambda$  (=partitions). We will denote by  $\lambda^\vee$  the conjugate partition of  $\lambda$ . Note that if we identify representations with Young diagrams, then  $\lambda \otimes \text{sgn} = \lambda^\vee$ .

We let  $L(\lambda)$  be the length of the largest hook of the Young diagram  $\lambda$ ,  $b(\lambda)$  denote the multiplicity of the largest part of  $\lambda$ , and set  $\ell(\lambda) = L(\lambda) - b(\lambda) + 1$ .

The main result of the first part of the thesis is the following

**Theorem 2.3.5.** Let  $\lambda \neq (n), (1^n)$  be a partition. The unitarity locus is given by  $U(\lambda) = (-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}) \cup \{ \frac{1}{k} \mid \ell(\lambda) \leq k \leq L(\lambda) \text{ or } -L(\lambda) \leq k \leq -\ell(\lambda^\vee) \}$ .

The proof of Theorem 2.3.5 will be contained in the propositions 2.3.6, 2.3.7, 2.3.9.

The eigenvalue  $T_\lambda$  of  $\sum_{s \in S} s$  on  $\lambda$  equals the content of the Young diagram  $\lambda$ ,

$$\text{ct}(\lambda) = \sum_{\text{cell } (i,j) \in \lambda} i - j.$$

**Proposition 2.3.6.** Let  $\lambda \neq (n), (1^n)$ . Then  $(-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}) \subset U(\lambda)$ .

*Proof.* Let  $q = e^{2\pi ic}$  and  $\mathcal{H}_n(q)$  be the Hecke algebra of  $\mathfrak{S}_n$  with parameter  $q$ , and  $S^\lambda$  be the Specht module corresponding to  $\lambda$ , as defined in [7]. It follows from Theorem 4.11 in [8] that  $S^\lambda$  is irreducible if  $c \in (-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)})$ . By the properties of the KZ functor introduced in [16] this implies that  $M_c(\lambda)$  is irreducible in this range.  $\square$

**Proposition 2.3.7.** Let  $\lambda \neq (n), (1^n)$ . The unitarity locus  $U(\lambda) \subset (-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}) \cup \{ \frac{1}{k} \mid \ell(\lambda) \leq k \leq L(\lambda) \text{ or } -L(\lambda) \leq k \leq -\ell(\lambda^\vee) \}$ .

*Proof.* Let  $\ell = \ell(\lambda)$ ,  $L = L(\lambda)$ ,  $b = b(\lambda)$  (so  $L = \ell + b - 1$ ).

Let us first show that  $U(\lambda) \subset [-\frac{1}{\ell(\lambda^\vee)}, \frac{1}{\ell(\lambda)}]$ . Recall that  $\lambda \otimes \mathfrak{h}^*$  is the sum of representations whose Young diagrams  $\mu$  is obtained from  $\lambda$  by removing and then adding a corner cell. Hence  $h_c(\mu) - h_c(\lambda) = c(ct(\lambda) - ct(\mu))$ . We note that  $\ell(\lambda)$  and  $-\ell(\lambda^\vee)$  are the maximum and minimum values of  $ct(\lambda) - ct(\mu)$ . From Corollary 2.2.8 we obtain  $U(\lambda) \subset [-\frac{1}{\ell(\lambda^\vee)}, \frac{1}{\ell(\lambda)}]$ .

Let us now prove that the intervals  $I_k = (\frac{1}{\ell+k}, \frac{1}{\ell+k-1})$ ,  $k = 1, \dots, b-1$  do not intersect  $U(\lambda)$ . Denote by  $\lambda_i$ ,  $i = 1, \dots, b$  the partition of  $n$  obtained by reducing  $i$  copies of the largest part of  $\lambda$  by 1, and then adding  $i$  copies of part 1. It follows from the rule of tensoring by  $\mathfrak{h}^*$  that  $\lambda \otimes S^i \mathfrak{h}^*$  contains a unique copy of  $\lambda_i$ . Consequently, for any  $c$ ,  $M_c(\lambda)$  contains a unique copy of  $\lambda_i$  in degree  $i$ . Let us show that  $\beta_{c,\lambda}|_{\lambda_i} = f_{i,\lambda}(c)(\cdot)_{\lambda_i}$ , where  $(\cdot)_{\lambda_i}$  does not depend on  $c$ .

**Lemma 2.3.8.** Up to rescaling,  $f_{\lambda,i}(c) = (1 - (\ell + i - 1)c) \dots (1 - \ell c)$ .

*Proof.* We will proceed by induction on  $i$ . The case  $i = 0$  is automatic. Let us assume that the statement is proved for  $i \leq m - 1$  and prove it for  $i = m$ . By the induction assumption, at  $c = \frac{1}{\ell+j-1}$  for some  $j = 1, \dots, m - 1$ , the module  $M_c(\lambda)$  has a singular vector  $u$  sitting in  $\lambda_j$  in degree  $j$ . Indeed, the contravariant form on  $\lambda_j$  is zero at such  $c$ , and there can be no singular vectors of lower degree, because if one moves at most  $i < j$  corner cells of  $\lambda$  to get a partition  $\sigma$ , then  $T_\lambda - T_\sigma \leq i(\ell + i - 1) < i(\ell + j - 1)$ , so  $c(T_\lambda - T_\sigma) < i$ .

Note that  $\lambda \otimes S^m \mathfrak{h}^*$  contains a unique copy of  $\lambda_{m-1}$  in degree  $m - 1$  and a unique copy of  $\lambda_m$  in degree  $m$  and that  $\lambda_{m-1} \otimes \mathfrak{h}^*$  contains a copy of  $\lambda_m$ . It can be proved that the vectors of  $\lambda_{m-1}$  generate a copy of  $\lambda_m$  in degree  $m$ , which implies that  $f_{\lambda,m}$  is divisible by  $f_{\lambda,m-1}$ . Also,  $\deg f_{\lambda,m} \leq \deg f_{\lambda,m-1} + 1$ .

Thus, to complete the induction step, it suffices to show that

$$f'_{\lambda,m}(0) = f'_{\lambda,m-1}(0) - \ell - m + 1.$$

To prove this formula, let us differentiate the equation of Proposition 2.2.9 with respect to  $c$  at  $c = 0$ . We get

$$F'_{0,\lambda,m}(a_1 \dots a_m v) = \frac{1}{m} \sum_{j=1}^m \left( a_j F'_{0,\lambda,m-1}(a_1 \dots a_{j-1} a_{j+1} \dots a_m v) - \sum_{s \in S} [a_1 \dots a_m, s] v \right).$$

This can be rewritten, using tensor notation, as

$$F'_{0,\lambda,m} = \frac{1}{m} \sum_{j=1}^m (F'_{0,\lambda,m-1})_{\hat{j}} - \frac{1}{m} (T_\lambda - T_{\lambda_m}),$$

where the subscript  $\hat{j}$  means that the operator acts in all components of the tensor product but the  $j$ -th. We obtain

$$f'_{\lambda,m}(0) = f'_{\lambda,m-1}(0) - \frac{1}{m} (T_\lambda - T_{\lambda_m}) = f'_{\lambda,m-1}(0) - \ell - m + 1,$$

as desired. □

Now the theorem follows easily from Lemma 2.3.8. Namely, we see that  $L_c(\lambda)$  is not unitary on the interval  $I_k$  because the polynomial  $f_{\lambda,k+1}(c)$  is negative on this interval, and hence the form  $\beta_{c,\lambda}$  is negative definite on  $\lambda_{k+1}$ .  $\square$

The following proposition contains the rest of proof of the Theorem 2.3.5.

**Proposition 2.3.9.**  $L_c(\lambda)$  is unitary at  $c = \frac{1}{k}$  for any integer  $k \leq -\ell(\lambda^\vee)$  or  $k \geq \ell(\lambda)$ .

*Proof.* We will follow S. Griffeth's argument in [15] based on [22]. Let us first introduce the Cherednik-Dunkl subalgebra  $E \subset H_c(\mathfrak{S}_n, \mathbb{C}^n)$  generated by

$$\epsilon_i = x_i y_i - c \sum_{1 \leq j < i} s_{ij}$$

for all  $1 \leq i \leq n$ . This algebra is well-known to be commutative (for instance [11], [18]). Let us also define the intertwiners

$$\begin{aligned} \sigma_i &= s_i - \frac{1}{\epsilon_i - \epsilon_{i+1}}, \\ \phi &= x_1 s_1 \cdots s_{n-1}, \text{ and} \\ \psi &= y_n s_{n-1} \cdots s_1, \end{aligned}$$

for  $1 \leq i \leq n-1$ . They satisfy the following intertwining relations

$$\begin{aligned} \epsilon_i \sigma_j &= \sigma_j \epsilon_{s_j(i)}, \text{ for all } 1 \leq i \leq n, 1 \leq j \leq n-1, \\ \epsilon_i \phi &= \phi \epsilon_{i-1}, \text{ for all } 2 \leq i \leq n, \\ \epsilon_1 \phi &= \phi(\epsilon_n + 1). \end{aligned}$$

In addition,

$$\begin{aligned} \sigma_i^2 &= \frac{(\epsilon_i - \epsilon_{i+1})^2 - c^2}{(\epsilon_i - \epsilon_{i+1})^2}, \\ \psi \phi &= \epsilon_n. \end{aligned}$$

For an  $n$ -tuple of complex numbers  $\alpha = (\alpha_1, \dots, \alpha_n)$  we define the eigenspace

$$L_c(\lambda)_{(\alpha_1, \dots, \alpha_n)} = \{m \in L_c(\lambda) \mid \epsilon_i^\vee m = \alpha_i m \text{ for } 1 \leq i \leq n\}.$$

If  $v \in L_c(\lambda)_\alpha$  where  $\alpha_i \neq \alpha_{i+1}$  for  $1 \leq i \leq n-1$ , then  $\sigma_i(v)$ ,  $\phi(v)$ , and  $\psi(v)$  are also eigenvectors of  $E$  (non necessarily nonzero).

Let us now assume  $c = \frac{1}{k}$  where  $k \leq -\ell(\lambda^\vee)$  is an integer (the case when  $k \geq \ell(\lambda)$  is similar). The crucial property we will use (see Theorem 4.12 in [22]) is that  $L_c(\lambda)$  has a basis of eigenvectors of  $E$ , and any eigenvalue  $\alpha$  has properties  $\alpha_i \in c\mathbb{Z}_{<0}$  and  $\alpha_i \neq \alpha_{i+1}$ .

We can now show that  $L_c$  is unitary. Any  $x_i$  can be expressed as a polynomial in  $\sigma_j, \phi$  and  $\epsilon_j$ . Since  $L_c(\lambda)$  is irreducible, any vector in  $L_c(\lambda)$  can, therefore, be obtained by applying a polynomial operator in  $\sigma_j, \phi$  and  $\epsilon_j$  to degree 0 vectors in  $\lambda$  (remember that  $L_c(\lambda) = \mathbb{C}[\mathfrak{h}] \otimes \lambda$  as a vector space). Furthermore, since  $\sigma_j$  and  $\phi$  map eigenvectors to eigenvectors and eigenspaces are 1-dimensional, any eigenvector can be obtained by applying a monomial in these variables to a degree 0 eigenvector in  $\lambda$ . Eigenvectors of different eigenvalues are orthogonal, since  $\sigma_i$  are self-adjoint. Hence, it is sufficient to show that, if  $f$  is an eigenvector of  $E$  with  $(f, f)$  positive, then  $(\sigma_i f, \sigma_i f)$  and  $(\phi f, \phi f)$  are positive. We use the basic properties  $\epsilon_i^* = \epsilon_i$ ,  $\sigma_i^* = \sigma_i$ , and  $\phi^* = \psi$ . From the contravariance of the Hermitian form, we obtain

$$(\phi f, \phi f) = (f, \psi \phi f) = (f, \epsilon_n f) = \alpha_n (f, f),$$

$$(\sigma_i f, \sigma_i f) = (f, \sigma_i^2 f) = \frac{(\alpha_i - \alpha_{i+1})^2 - c^2}{(\alpha_i - \alpha_{i+1})^2} (f, f).$$

for  $1 \leq i \leq n-1$ . The coefficients  $\alpha_n$  and  $\frac{(\alpha_i - \alpha_{i+1})^2 - c^2}{(\alpha_i - \alpha_{i+1})^2}$  are nonnegative, since the  $\alpha_i \in c\mathbb{Z}_{<0}$ ,  $c < 0$ , and  $\alpha_i \neq \alpha_{i+1}$ . On the other hand, the inner products on 1-dimensional eigenspaces cannot be zero, because the form is nondegenerate. Hence the form is positive definite on each eigenspace, and, therefore, on  $L_c(\lambda)$ .  $\square$

Theorem 2.3.5 now follows from the propositions 2.3.6, 2.3.7, and 2.3.9.

**Remark 2.3.10.** In [22], the Cherednik algebra of the symmetric group  $\mathfrak{S}_n$  is defined by the commutation relations

$$y_i x_j = \begin{cases} x_j y_i - s_{ij} & \text{if } i \neq j \\ x_i y_i + \kappa + \sum_{k \neq i} s_{ik} & \text{if } i = j \end{cases}$$

where  $\kappa = -\frac{1}{c}$ . To connect our definition and operators to those in [22], we send  $y_i \rightarrow y_i, x_i \rightarrow -x_i/c$  and  $\epsilon_i \rightarrow (-c)\epsilon_i^\vee$ .

### 2.3.4 Unitarity and the integral representation of the Gaussian inner product on $M_c(\mathbb{C})$ .

In this subsection, we attempt to motivate the study of unitary lowest weight representations of rational Cherednik algebras. For this purpose, we define the Gaussian inner product on a Verma module  $M_c(\lambda)$ , introduced by Cherednik in [4].

**Definition 2.3.11.** The Gaussian inner product  $\gamma_{c,\lambda}$  on  $M_c(\lambda)$  is given by the formula

$$\gamma_{c,\lambda}(v, v') = \beta_{c,\lambda}(\exp(f)v, \exp(f)v').$$

This inner product is well defined, since the operator  $f$  acts locally nilpotently on  $M_c(\lambda)$ . We note that the kernel of  $\gamma_{c,\lambda}$  is  $J_c(\lambda)$ , the same as the kernel of  $\beta_{c,\lambda}$ . Therefore, the inner product descends to any lowest weight module with lowest weight  $\lambda$ , in particular to the irreducible module  $L_c(\lambda)$ , on which it is nondegenerate. Furthermore, it is positive definite on  $L_c(\lambda)$  if and only if  $\beta_{c,\lambda}$  is positive definite.

**Proposition 2.3.12.** (i) The form  $\gamma_{c,\lambda}$  satisfies the condition

$$\gamma_{c,\lambda}(xv, v') = \gamma_{c,\lambda}(v, xv'), \quad x \in \mathfrak{h}_{\mathbb{R}}^*.$$

(ii) The form  $\gamma_{c,\lambda}$  is the unique Hermitian form, up to rescaling, that is  $W$ -invariant

and satisfies the condition

$$\gamma_{c,\lambda}((-y + Ty)v, v') = \gamma_{c,\lambda}(v, yv'), \quad y \in \mathfrak{h}_{\mathbb{R}}.$$

*Proof.* The most important ingredient in the proof is the identity

$$fx = xf + T^{-1}x.$$

This implies  $\exp(f)x = (x + T^{-1}x)\exp(f)$  and also  $\exp(f)(y - Ty) = Ty\exp(f)$ .

(i) We have

$$\begin{aligned} \gamma_{c,\lambda}(xv, v') &= \beta_{c,\lambda}(\exp(f)xv, \exp(f)v') = \\ \beta_{c,\lambda}((x + T^{-1}x)\exp(f)v, \exp(f)v') &= \beta_{c,\lambda}(\exp(f)v, (T^{-1}x + x)\exp(f)v') = \\ \beta_{c,\lambda}(\exp(f)v, \exp(f)xv') &= \gamma_{c,\lambda}(v, xv'). \end{aligned}$$

(ii) The proof that  $\gamma_{c,\lambda}$  satisfies the condition (ii) is entirely similar. Let us show uniqueness. If  $\gamma$  is any  $W$ -invariant Hermitian form satisfying the condition in (ii), then let  $\beta(v, v') = \gamma(\exp(-f)v, \exp(-f)v')$ . It is easy to show now that  $\beta(yv, v') = \beta(v, Tyv')$  for all  $v, v'$ . By Proposition 2.2.1,  $\beta$  is a multiple of  $\beta_{c,\lambda}$ , hence  $\gamma$  is a multiple of  $\gamma_{c,\lambda}$ .  $\square$

We will need the following known result (see [12], Theorem 3.10).

**Proposition 2.3.13.** We have

$$\gamma_{c,\mathbb{C}}(f, g) = K(c)^{-1} \int_{\mathfrak{h}_{\mathbb{R}}} f(z) \overline{g(z)} d\mu_c(z) \quad (2.4)$$

where

$$d\mu_c(z) := e^{-|z|^2/2} \prod_{s \in S} |\alpha_s(z)|^{-2c_s} dz,$$

and

$$K(c) = \int_{\mathfrak{h}_{\mathbb{R}}} d\mu_c(z), \quad (2.5)$$

provided that the integral (2.5) is absolutely convergent.

*Proof.* From Proposition 2.3.12, the form  $\gamma_{c,\mathbb{C}}$  is uniquely determined, up to rescaling, by the condition that it is invariant under the anti-involution  $g \rightarrow g^{-1}$ ,  $x \rightarrow x$ , and  $y \rightarrow Ty - y$  for all  $x \in \mathfrak{h}^*$ ,  $y \in \mathfrak{h}$  and  $g \in W$ . These properties are easy to check for the right hand side of (2.4), using the fact that the action of  $y$  is given by Dunkl operators.  $\square$

The integral formula extends analytically to arbitrary complex  $c$ . The constant  $K(c)$  is given by the following Macdonald-Mehta product formula, as proved by E. Opdam [21] for Weyl groups and by F. Garvan for  $H_3$  and  $H_4$ . Given an irreducible reflection group  $W$  and a constant parameter  $c$ ,

$$K(c) = K_0 \prod_{j=1}^{\dim \mathfrak{h}} \frac{\Gamma(1 - d_j c)}{\Gamma(1 - c)}$$

where  $d_j$  are the degrees of generators of  $\mathbb{C}[\mathfrak{h}]^W$ . It follows that for constant  $c$  the first pole of  $K(c)$  occurs at  $c = \frac{1}{H}$ , which gives another proof of Corollary 2.2.13(ii).

Let  $c$  be a constant positive function, which is a singular value for  $W$  (i.e., it is rational and has as denominator a divisor of some degree  $d_i$  of  $W$ ). Let  $N_c$  be the minimal nonzero submodule of the polynomial Verma module  $M_c(\mathbb{C})$ . I. Cherednik observed that

**Proposition 2.3.14.** If  $N_c \in L^2(\mathfrak{h}_{\mathbb{R}}, d\mu_c)$ , then  $N_c$  is a unitary representation.

*Proof.* As in the proof of Proposition 2.3.13, the integral gives a  $W$ -invariant form  $\gamma$  on  $N_c$  that satisfies the condition in Proposition 2.3.12 (ii). This implies that  $\gamma$  is a multiple of  $\gamma_{c,\lambda_c}$ . The positive definiteness is straightforward.  $\square$

Let  $W$  be an irreducible Coxeter group,  $\mathfrak{h}$  its reflection representation, and  $c = \frac{1}{d_i}$ . Motivated by the previous observation and a number of examples, I. Cherednik asked whether  $N_c$  is contained in  $L^2(\mathfrak{h}_{\mathbb{R}}, d\mu_c)$  and, in particular, if it is unitary.

Let  $W = \mathfrak{S}_n$  and  $c = \frac{1}{m}$  where  $1 \leq m \leq n$ . Using Enomoto's theorem, reproved differently and in slightly more generality in our paper [15],  $N_{\frac{1}{m}} = L_{\frac{1}{m}}(\lambda)$  where  $\lambda = (m-1, \dots, m-1, s)$  and  $n = a(m-1) + s$  and  $0 \leq s \leq m-1$ . Theorem 2.3.5



shows that  $L_{\frac{1}{m}}(\lambda)$  is unitary. Moreover, it can be shown that  $N_{\frac{1}{m}}$  is contained in  $L^2(\mathfrak{h}_{\mathbb{R}}, d\mu_c)$ , which gives another proof of unitarity (see [15] for details).



# Chapter 3

## Unitary representations of Hecke algebras

### 3.1 Preliminaries

Let  $r$  and  $n$  be positive integers. We will first review the definition of the Hecke algebra of the complex reflection group  $G(r, 1, n)$ , namely the wreath product  $(\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$ . This algebra is also known as the *Ariki-Koike algebra*.

**Definition 3.1.1.** Given a commutative domain  $R$  and a family of parameters  $\underline{q} = (q, q_1, \dots, q_r) \in R^{r+1}$ , the Hecke algebra  $\mathcal{H}_{\underline{q}}(W)$  of the complex reflection group  $W = (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$  is the algebra generated by  $T_0, T_1, \dots, T_{n-1}$  and relations

$$\begin{aligned}(T_0 - q_1) \dots (T_0 - q_r) &= 0, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ (T_i + 1)(T_i - q) &= 0 \quad \text{for all } i = 1, \dots, n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad \text{for all } 1 \leq i \leq n-2, \\ T_i T_j &= T_j T_i \quad \text{for all } 0 \leq i \leq j-2 \leq n-3.\end{aligned}$$

Note that  $T_1, \dots, T_{n-1}$  generate a subalgebra of  $\mathcal{H}_{\underline{q}}(W)$  isomorphic to the Hecke algebra  $\mathcal{H}_q(\mathfrak{S}_n)$  of the symmetric group. It has a natural basis  $\{T_w \mid w \in \mathfrak{S}_n\}$  over  $R$

indexed by permutations, such that every  $T_w$  is given by certain a product of elements  $T_i$  for  $1 \leq i \leq n-1$ .

In our thesis, we will assume  $R = \mathbb{C}$ . For 'most' complex values of the parameters  $\underline{q} = (q, q_1, \dots, q_r)$ , the Hecke algebra  $\mathcal{H}_{\underline{q}}(W)$  is semisimple. We will call these values *generic*.

We assume all parameters are invertible. We will use extensively the *involution*  $\sigma$ , a semilinear involution of  $\mathcal{H} = \mathcal{H}_{\underline{q}}$  that sends  $a \mapsto \bar{a}$ ,  $q \mapsto q^{-1}$ ,  $q_i \mapsto q_i^{-1}$ ,  $T_0 \mapsto T_0^{-1}$ , and  $T_i \mapsto T_i^{-1}$ , for all  $a \in \mathbb{C}$  and  $1 \leq i \leq n-1$ . This defines an involution for the universal Hecke algebra, where  $\underline{q}$  are taken to be (invertible) formal parameters. When we specialize the parameters to complex numbers on the unit circle, this map can be extended to a  $\mathbb{C}$ -semilinear involution of the special Hecke algebra. Note that we must assume the parameters are on the unit circle to be able to define this involution for specializations to complex numbers.

We also define the  $\mathbb{C}$ -linear *anti-involution*  $*$  on  $\mathcal{H}$  that sends  $q \mapsto q$ ,  $q_i \mapsto q_i$ ,  $T_0 \mapsto T_0$ , and  $T_i \mapsto T_i$ , for  $1 \leq i \leq n-1$ . This determines an anti-involution for any specialization of variables to complex numbers.

We will follow [9] in defining for any multipartition  $\lambda = (\lambda^1, \dots, \lambda^r)$  of  $n$  a Specht module  $S^\lambda$ . Given two multipartitions  $\lambda$  and  $\mu$ , we say that  $\lambda$  *dominates*  $\mu$  if  $\sum_{i=1}^{k-1} |\lambda^i| + \sum_{i=1}^j \lambda_i^k \geq \sum_{i=1}^{k-1} |\mu^i| + \sum_{i=1}^j \mu_i^k$  for all  $1 \leq k \leq r$  and  $j \geq 1$ . If  $\lambda$  dominates  $\mu$  and  $\lambda \neq \mu$  we write  $\lambda \triangleright \mu$ .

Let  $\mathbf{t}^\lambda$  be the standard  $\lambda$ -tableau filled with  $1, 2, \dots, n$  in order in the first row, second, and so on, in  $\lambda^1, \dots, \lambda^r$ . For any standard  $\lambda$ -tableau  $\mathfrak{s}$ , define  $d(\mathfrak{s}) \in \mathfrak{S}_n$  to be the permutation such that  $\mathbf{t}^\lambda d(\mathfrak{s}) = \mathfrak{s}$ . It is straightforward that the set  $\{d(\mathfrak{s}) | \mathfrak{s} \text{ standard } \lambda\text{-tableau}\}$  is in one-to-one correspondence with (right) coset representatives of  $\mathfrak{S}_\lambda$  in  $\mathfrak{S}_n$  where  $\mathfrak{S}_\lambda = \mathfrak{S}_{|\lambda^1|} \times \mathfrak{S}_{|\lambda^2|} \times \dots \times \mathfrak{S}_{|\lambda^r|}$  is a Young subgroup of  $\mathfrak{S}_n$ .

Given the multipartition  $\lambda$  of  $n$ , let  $\underline{a} = (a_1, \dots, a_r)$  where  $a_i = \sum_{j=1}^{i-1} |\lambda^j|$  for any  $1 \leq i \leq r$ . Let also  $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$  and

$$L_m = q^{1-m} T_{m-1} \dots T_1 T_0 T_1 \dots T_{m-1}$$

for  $m = 1, \dots, n$ . We define  $m_\lambda = (\prod_{k=1}^r \prod_{m=1}^{a_k} (L_m - q_k))x_\lambda \in \mathcal{H}$ . For any pair of standard  $\lambda$ -tableaux  $(\mathfrak{s}, \mathfrak{t})$ , set  $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}^* m_\lambda T_{d(\mathfrak{t})}$ .

Now, let us define the vector space

$$\bar{N}^\lambda = \mathbb{C}\{m_{\mathfrak{s}\mathfrak{t}} | \mathfrak{s}, \mathfrak{t} \text{ standard } \mu\text{-tableaux for a multipartition } \mu \text{ of } n, \mu \triangleright \lambda\}$$

which is a two-sided ideal in  $\mathcal{H}$  (see [9] Proposition 3.22).

**Definition 3.1.2.** Define  $z_\lambda = (\bar{N}^\lambda + m_\lambda)/\bar{N}^\lambda$  to be an element in  $\mathcal{H}/\bar{N}^\lambda$  and the Specht module  $S^\lambda = \mathcal{H}z_\lambda$ .

**Remark 3.1.3.** In [9], the Specht module  $S^\lambda$  is defined as a right module  $z_\lambda \mathcal{H}$ , but we prefer the opposite definition. The anti-involution  $*$  maps one to the other, since  $m_\lambda^* = m_\lambda$  and  $(\bar{N}^\lambda)^* = \bar{N}^\lambda$ .

It is well known that for generic values of the parameters, the Specht modules corresponding to multipartitions of  $n$  are irreducible and exhaust all irreducible representations of  $\mathcal{H}$ .

## 3.2 Hermitian representations of the cyclotomic Hecke algebra of type $G(r, 1, n)$

In this section, we assume  $\mathcal{H} = \mathcal{H}_q(W)$  is the Hecke algebra of the complex reflection group  $G(r, 1, n)$  defined above. We prove that if parameters are complex numbers on the unit circle, any irreducible representation  $V$  of  $\mathcal{H}$  is Hermitian, namely, it admits a nondegenerate Hermitian form such that  $\langle Tv, v' \rangle = \langle v, \sigma(T^*)v' \rangle$  for all  $v, v' \in V$  and  $T \in \mathcal{H}$ . Equivalently, the form is invariant under the corresponding braid group.

In order to construct the Hermitian form, we will use the symmetric bilinear form  $\langle, \rangle$  on the Specht module  $S^\lambda$  defined in [9], which satisfies  $\langle Tv, v' \rangle = \langle v, T^*v' \rangle$  for every  $v, v' \in S^\lambda$ , and  $T \in \mathcal{H}$ . If the Hecke algebra is semisimple, the Specht module is irreducible and the form is nondegenerate. If the Hecke algebra is not semisimple, the form may degenerate, and the quotient  $D^\lambda = S^\lambda / \text{rad}\langle, \rangle$  will be either irreducible

or zero. The family  $\{D^\lambda \mid \lambda \text{ multipartition of } n, D^\lambda \neq 0\}$  forms a complete set of irreducible representations of  $\mathcal{H}$  (for more details, see [9]).

For any multipartition  $\lambda$  of  $n$ , the following holds

**Lemma 3.2.1.** (i) The involution  $\sigma$  preserves  $\bar{N}^\lambda$ , and  $\sigma(m_\lambda) = m_\lambda u$  for some invertible element  $u \in \mathcal{H}$ ;

(ii) The involution  $\sigma$  preserves the Specht modules, namely  $\sigma(S^\lambda) = S^\lambda$  for any  $\lambda$ .

*Proof.* Let us first prove that  $\sigma(x_\lambda) = q^s x_\lambda$  for some integer  $s$ . It is easy to observe from the definition of  $x_\lambda$  that we can reduce the problem to the case  $x = \sum_{w \in \mathfrak{S}_n} T_w$ . Note that  $T_w x = q^{l(w)} x$  for all  $w \in \mathfrak{S}_n$  where  $l(w)$  is the length of  $w$ . It is well known that  $x$  is unique with such property, up to scaling (see, for example, [7], section 3). Applying  $\sigma$ , we obtain  $T_w \sigma(x) = q^{l(w)} \sigma(x)$  which, combined with the uniqueness of  $x$ , gives  $\sigma(x) = P x$  where  $P \in \mathbb{C}[q, q^{-1}]$ . Since  $\sigma$  is an involution,  $P(q)P(q^{-1}) = 1$  which implies  $P = q^s$  for some integer  $s$ .

Since  $m_\lambda = \prod_{k=1}^r (\prod_{m=1}^{a_k} (L_m - q_k)) x_\lambda \in \mathcal{H}$ , a straightforward computation taking into account the commutativity of  $L_m$  for  $1 \leq m \leq n$  gives  $\sigma(m_\lambda) = m_\lambda u$  where  $u = q^t \prod_{k=1}^r L_k^{b_k} q_k^{-a_k}$  for some integers  $t, a_k, b_k$ , and  $1 \leq k \leq r$ . Note that  $u \in \mathcal{H}$  is invertible. Finally, we note that  $\bar{N}^\lambda = \sum_{\nu \triangleright \lambda} \mathcal{H} m_\nu \mathcal{H}$ , since it is a two-sided ideal in  $\mathcal{H}$ . Hence,  $\sigma$  preserves  $\bar{N}^\lambda$ , and also  $S^\lambda$ .  $\square$

**Proposition 3.2.2.** If the complex parameters  $q$  are on the unit circle, then any irreducible representation  $D^\lambda \neq 0$  has a nondegenerate Hermitian form such that

$$(Tv, v') = (v, \sigma(T^*)v') \quad (3.1)$$

for all  $v, v' \in D^\lambda$  and  $T \in \mathcal{H}$ .

**Remark 3.2.3.** For any  $0 \leq i \leq n-1$ ,  $\sigma(T_i^*) = T_i^{-1}$ . Moreover,  $\sigma(T^*) = T^{-1}$  for any  $T$  from the group generated by elements  $T_i$ .

*Proof.* First, let us recall the existence of a symmetric bilinear form  $\langle, \rangle$  on  $S^\lambda$  such

that  $\langle Tv, v' \rangle = \langle v, T^*v' \rangle$  for any  $v, v' \in D^\lambda$  and  $T \in \mathcal{H}$ . This form descends to a nondegenerate form on  $D^\lambda$ .

Let us note that there is a semilinear involution on the Specht module  $S^\lambda$ , which by abuse of notation we will also denote by  $\sigma$ , such that  $\sigma(T)\sigma(v) = \sigma(Tv)$  for any  $T \in \mathcal{H}$  and  $v \in S^\lambda$ . On  $S^\lambda$ , we construct a new form  $\langle v, w \rangle_1 = \langle v, \sigma(w) \rangle$ . Clearly, the form  $\langle, \rangle_1$  is sesquilinear and descends to a nondegenerate form on  $D^\lambda$ . Since the form  $\langle, \rangle_1$  is unique up to rescaling, it follows that  $\langle v, v' \rangle_1 = c \overline{\langle v', v \rangle_1}$  for all  $v, v' \in D^\lambda$  and a complex constant  $c$ . Note that  $c$  must be on the unit circle, hence  $c = \exp(i\theta)$  for some  $\theta \in \mathbb{R}$ . Define now  $(v, v') = \alpha \langle v, v' \rangle_1$  where  $\alpha = \exp(i\frac{\theta}{2})$ . We conclude immediately that  $(,)$  is Hermitian.  $\square$

We end this section with the following

**Definition 3.2.4.** Given a multipartition  $\lambda$  such that  $D^\lambda \neq 0$ , we say the  $D^\lambda$  is *unitary* if the nondegenerate Hermitian form defined above can be normalized to be positive definite.

### 3.3 Unitary representations of the Hecke algebra of type A

The rest of the thesis will focus on the Hecke algebra of the symmetric group  $\mathfrak{S}_n$ . In this case, the Hecke algebra depends only on one parameter  $q$ . For a complex  $q$ , let  $e$  be the smallest positive integer such that  $1 + q + \dots + q^{e-1} = 0$ . If such an integer does not exist, we set  $e = \infty$ .

The irreducible representations  $D^\lambda$  of  $\mathcal{H}_q$  are indexed by partitions of  $n$ . As it is well known,  $D^\lambda \neq 0$  if and only if  $\lambda^\vee$  is  $e$ -restricted where  $\lambda^\vee$  is the conjugate partition. In other words,  $\lambda_i^\vee - \lambda_{i+1}^\vee < e$  for all  $i$ . In this section, we will denote  $D_\lambda := D^{\lambda^\vee}$ . Hence,  $D_\lambda \neq 0$  if and only if  $\lambda$  is  $e$ -restricted. Similarly, we will denote  $S_\lambda = S^{\lambda^\vee}$ . As mentioned in the previous section,  $D_\lambda = S_\lambda / \text{rad}(,)$ .

We define the unitarity locus of  $\lambda$  to be the set

$$U(\lambda) = \{c \in (-\frac{1}{2}, \frac{1}{2}] \mid D_\lambda \text{ is nonzero and unitary at } q = \exp(2\pi ic)\}.$$

**Proposition 3.3.1.** (i) For any  $n \geq 2$ ,  $U(1^n) = (-\frac{1}{2}, \frac{1}{2}]$  and

$$U(n) = (-\frac{1}{2}, \frac{1}{2}] \setminus \{\pm \frac{r}{m} \mid 1 \leq r, m \leq n, \gcd(r, m) = 1\}.$$

(ii) If  $n \geq 3$  and  $\lambda = (n - k, 1^k)$  for  $1 \leq k \leq n - 2$ , then  $U(\lambda) = [-\frac{1}{n}, \frac{1}{n}]$ .

*Proof.* Given  $c \in (-\frac{1}{2}, \frac{1}{2}]$ , we set  $q = \exp(2\pi ic)$  and  $e$  the smallest positive integer such that  $1 + q + \dots + q^{e-1} = 0$ .

If  $\lambda = (n)$  is  $e$ -restricted, then  $D_{(n)} \neq 0$  is one-dimensional and, therefore, unitary. Similarly,  $D_{(1^n)}$  is one-dimensional and unitary.

Let us, now, assume that  $n \geq 3$  and  $\lambda = (n - k, 1^k)$  for some  $1 \leq k \leq n - 2$ . The form on  $S_\lambda$  is nondegenerate, if  $c \in (-\frac{1}{n}, \frac{1}{n})$ , and positive definite at  $c = 0$ , hence  $(-\frac{1}{n}, \frac{1}{n}) \subset U(\lambda)$ . At  $c = \pm \frac{1}{n}$ , the form is positive definite on the quotient  $D_\lambda = S_\lambda / \text{rad}(\cdot, \cdot)$ , therefore  $[-\frac{1}{n}, \frac{1}{n}] \subset U(\lambda)$ .

We will use induction on  $n$  and the restriction functor  $\text{Res}$  given by the natural inclusion  $\mathcal{H}(\mathfrak{S}_{n-1}) \subset \mathcal{H}(\mathfrak{S}_n)$  to prove that  $U(\lambda) \subset [-\frac{1}{n}, \frac{1}{n}]$ .

First, we will use the induction hypothesis to show that  $U(\lambda) \subset [-\frac{1}{n-1}, \frac{1}{n-1}]$ . For  $n = 3$ , this is automatic, so we may assume  $n > 3$ . Let  $c \in U(\lambda)$  and assume  $\lambda$  is  $e$ -restricted. At least one of the partitions  $(n - k - 1, 1^k)$  and  $(n - k, 1^{k-1})$  is different from  $(n - 1)$  and  $(1^{n-1})$ . Let us denote one such partition by  $\nu$ . We observe that  $D_\nu \neq 0$ , since  $\nu$  is  $e$ -restricted. Using Theorem 2.5 in [3], the module  $D_\nu$  is contained in the composition series of  $\text{Res} D_\lambda$ . Since  $D_\lambda$  is unitary, so is  $D_\nu$ . By the induction hypothesis,  $c \in [-\frac{1}{n-1}, \frac{1}{n-1}]$ .

We will now show that  $D_\lambda$  is not unitary for  $c \in [-\frac{1}{n-1}, -\frac{1}{n}) \cup (\frac{1}{n}, \frac{1}{n-1}]$ . Using an argument similar to that in 3.2.2, we can define a Hermitian form  $(\cdot, \cdot)$  on the universal Specht module  $\tilde{S}_\lambda$  for the universal Hecke algebra. It will be related to the symmetric form  $\langle \cdot, \cdot \rangle$  defined in [7, 8] by  $(v, v') = q^{-k/2} \langle v, \sigma(v') \rangle$  for any  $v, v' \in \tilde{S}_\lambda$  and a fixed



integer  $k$  depending on  $\lambda$ . Specializing at generic complex values of  $q$ , we obtain the Hermitian form on  $S_\lambda$ , up to rescaling. Since for  $c \in (-\frac{1}{n}, \frac{1}{n})$  the module  $S_\lambda$  is irreducible and its form is nondegenerate, we can normalize the form on  $\tilde{S}_\lambda$ , such that it is positive definite upon specialization in this interval.

Note that  $S_\lambda$  is irreducible for any  $c \in \pm(\frac{1}{n}, \frac{1}{n-1}]$  because the  $e$ -core of  $\lambda$  is itself (see [8], Theorem 4.13 for details). Hence, the signature of the form on  $S_\lambda$  does not change when  $c \in \pm(\frac{1}{n}, \frac{1}{n-1}]$ , because the form on  $\tilde{S}_\lambda$  does not degenerate when specialized in this interval. It suffices to prove that this form is not positive definite upon specialization in a small neighborhood  $\pm(\frac{1}{n}, \frac{1}{n} + \epsilon)$ . Let us look at the signature of the form in this interval. Let  $\zeta = e^{2\pi i/n}$  and  $M_i = \{v \in \tilde{S}_\lambda | (q - \zeta)^i \text{ divides } (v, -)\}$ . The Jantzen filtration of the Hermitian form given by  $\tilde{S}_\lambda = M_0 \supset M_1 \supset \dots$  is the same as for the symmetric form  $\langle, \rangle$ . Looking at the determinant of the latter computed in [8] Theorem 4.11, we observe that it has a root  $\zeta$  of multiplicity  $\binom{n-2}{k}$ . This equals the dimension of  $D_\mu$  at  $c = \pm\frac{1}{n}$  where  $\mu = (n - k - 1, 1^{k+1})$ . At  $c = \pm\frac{1}{n}$ , it is known that  $S_\lambda$  has length 2, its socle is  $D_\mu$ , and its head  $D_\lambda$ . Hence, specializing at  $c = \pm\frac{1}{n}$ , the Jantzen filtration becomes  $0 = M_2 \subsetneq M_1 = D_\mu \subsetneq S_\lambda$ . Therefore, the signature of the form changes when crossing  $c = \pm\frac{1}{n}$ , without becoming negative definite. We conclude that the form ceases to be positive definite in the interval  $\pm(\frac{1}{n}, \frac{1}{n} + \epsilon)$ , which finishes the proof.  $\square$

We will now state and prove the main theorem. Let  $\lambda$  be a partition,  $L(\lambda)$  its largest hook, and  $b(\lambda)$  the multiplicity of the largest part. If  $\lambda$  is rectangular with  $b > 1$ , let  $\ell^*(\lambda) = L(\lambda) - b(\lambda) + 2$ , otherwise, let  $\ell^*(\lambda) = L(\lambda) - b(\lambda) + 1$ . For any  $\lambda$ , we will call the integers  $\ell^*(\lambda) \leq k \leq L(\lambda)$  the *main hooks* of  $\lambda$ . Note that if  $k$  is a main hook of  $\lambda \neq (n)$ , then  $\lambda$  is  $k$ -restricted.

**Theorem 3.3.2.** If  $\lambda \neq (n), (1^n)$ , the unitarity locus is given by  $U(\lambda) = (-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}) \cup \{\pm\frac{1}{k} | \ell^*(\lambda) \leq k \leq L(\lambda)\}$ .

*Proof.* The proof will be contained in the following two propositions.  $\square$

**Proposition 3.3.3.** If  $\lambda \neq (n), (1^n)$ , then  $U(\lambda)$  is contained in the set  $(-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}) \cup \{\pm\frac{1}{k} | \ell^*(\lambda) \leq k \leq L(\lambda)\}$ .

*Proof.* Let us assume  $n \geq 4$  and  $\lambda \neq (n - k, 1^k)$  for all  $0 \leq k \leq n - 1$ , otherwise the result follows from 3.3.1. We will use the restriction functor coming from the natural inclusion  $\mathcal{H}(\mathfrak{S}_{n-1}) \subset \mathcal{H}(\mathfrak{S}_n)$ . Let  $c \in U(\lambda)$ ,  $q = \exp(2\pi ic)$  and  $e$  the smallest positive integer such that  $1 + q + \dots + q^{e-1} = 0$ .

Let  $L = L(\lambda)$ . First, assume that  $e > L$ . We will prove that  $c \in (-\frac{1}{L}, \frac{1}{L})$ . Using the restriction functor, we obtain  $\text{Res} S_\lambda = \oplus_{\nu \rightarrow \lambda} S_\nu$  where the arrow indicates the addition of a box. Let  $\nu$  be a partition in this sum obtained by removing a box from  $\lambda$  situated away from the largest hook. Since  $S_\lambda$  is unitary, so is  $S_\nu$ . The largest hook of  $\nu$  is  $L$ , hence  $c \in (-\frac{1}{L}, \frac{1}{L})$  by the induction step.

Let us now assume that  $2 \leq e \leq L$ ,  $c = \frac{r}{e}$  for some integer  $r$  relatively prime to  $e$ . Note that  $\lambda$  is  $e$ -restricted. From Theorem 2.6 in [3], the socle of the restriction of  $D_\lambda$  to  $\mathcal{H}(\mathfrak{S}_{n-1})$  is

$$\text{Soc}(\text{Res} D_\lambda) = \oplus_{\mu \xrightarrow{\text{good}} \lambda} \mu$$

where the arrow indicates the addition of a *good* box (see [3] for details). For all partitions  $\mu$  in the sum,  $D_\mu \neq 0$  and unitary.

We distinguish the following cases. First, we assume there exists a good corner not situated on the largest hook of  $\lambda$ . Let  $\nu$  be the partition obtained by removing this corner. Applying the induction hypothesis to  $D_\nu$ , we conclude that  $l \leq e \leq L$  and  $r = \pm 1$ .

Assume, now, that all good corners are on the largest hook. If there is a good corner on the first column, we remove it and obtain a partition  $\nu$ . Applying the induction hypothesis, we obtain  $r = \pm 1$  and  $l - 1 \leq e \leq L$ . Assuming  $e = l - 1$ , we note that the highest corner of  $\lambda$  is good, hence it must be on the largest hook. In addition,  $\lambda$  has no corners outside the largest hook, because they would be good. Hence,  $\lambda$  is of the form  $(n - k, 1^k)$ , a contradiction. Therefore,  $l \leq e \leq L$ .

Assume, now, there is a good corner only on the first column. By removing this corner, we obtain a partition  $\nu$  such that  $D_\nu$  is unitary. Hence  $r = \pm 1$  and  $a \leq e \leq L$  where  $a = \ell(\nu)$  is the smallest main hook of  $\nu$ . Assume, first, that the highest corner of  $\nu$  is also a corner for  $\lambda$ . This must be a *normal* corner of  $\lambda$  (see [3] for the definition).

It is not situated on the largest hook of  $\lambda$ , otherwise,  $\lambda = (2, 1^{n-1})$ . Let us remove this corner and obtain a partition  $\nu'$ . Note that  $D_{\nu'} \neq 0$ . Using Theorem 2.5 in [3],  $D_{\nu'}$  belongs to the composition series of  $\text{Res} D_\lambda$ , hence it is unitary. Applying the induction hypothesis, we obtain the desired result.

Finally, if the highest corner of  $\nu$  is not a corner of  $\lambda$ , then  $\lambda_1 - \lambda_2 \geq 2$  and  $L - 1 \leq e \leq L$ . Then there is no corner outside of the largest hook, because it would be good. Hence,  $\lambda$  is of the form  $(n - k, 1^k)$ , a contradiction. This finishes the proof.  $\square$

**Proposition 3.3.4.** If  $\lambda \neq (n), (1^n)$ , then  $(-\frac{1}{L(\lambda)}, \frac{1}{L(\lambda)}) \cup \{\pm \frac{1}{k} \mid \ell^*(\lambda) \leq k \leq L(\lambda)\}$  is contained in the unitarity locus  $U(\lambda)$ .

*Proof.* Let  $L = L(\lambda)$ . First note that  $(-\frac{1}{L}, \frac{1}{L}) \in U(\lambda)$ , since the form on  $S_\lambda$  is nondegenerate on this interval and it is positive definite at  $c = 0$ . We will now prove that  $D_\lambda$  is unitary at  $\pm \frac{1}{l+k}$  for all nonnegative integers  $k$  where  $l = \ell(\lambda)$  is the smallest main hook.

Let  $N \geq 2$  and  $a$  be positive integers and  $Q$  a primitive root of unity of order  $2(a + N)$ . Then one can define the fusion category  $\mathcal{C}_Q$  of representations of the quantum group  $U_Q(\mathfrak{sl}_N)$ . It is a modular tensor category whose simple objects are highest weight representations with dominant integral highest weight  $\mu$  such that  $(\mu, \theta) \leq a$  where  $\theta = (1, 0, \dots, 0, -1)$  is the highest root. For details, we refer to [1] and the references therein.

The category  $\mathcal{C}_Q$  is unitary for  $Q = \exp(\pm \pi i / (a + N))$  and Hermitian for any  $Q$ . For this fact, and the general notions of Hermitian and unitary categories, see [19, 20] and the references therein. Since  $\mathcal{C}_Q$  is unitary for such  $Q$ , the braid group representations on multiplicity spaces of tensor powers of an object are also unitary. In particular, if  $V$  is the vector representation, this is true for the power  $V^{\otimes n}$ .

Now, the braid group representations on multiplicity spaces of this tensor power factor through the Hecke algebra  $\mathcal{H}_{Q^2}$ . From the quantum Schur-Weyl duality we obtain

$$V^{\otimes n} = \bigoplus_{\mu: (\mu, \theta) \leq a} \pi_\mu \otimes D_{\mu^\vee}$$

where  $\pi_\mu$  are simple objects of  $\mathcal{C}_Q$  and  $D_{\mu^\vee}$  are irreducible representations of the Hecke algebra  $\mathcal{H}_{Q^2}$ .

If we think of  $\mu$  as a partition, then it is a partition of  $n$  with at most  $N$  non-zero parts such that  $\mu_1 - \mu_N \leq a$ . Equivalently,  $\lambda = \mu^\vee$  is a partition of  $n$  with largest part at most  $N$ , smallest main hook  $l$ , and  $l - N \leq a$ .

Returning to our problem, given a partition  $\lambda \neq (n), (1^n)$  with smallest main hook  $l$ , let  $N \geq 2$  be its largest part and  $a = l - N + k$  for a nonnegative integer  $k$ . Note that  $a > 0$ . It follows from above that  $D_\lambda$  is unitary at  $q = \exp(\pm 2\pi i/(l + k))$ . This finishes the proof.  $\square$

# Chapter 4

## Preservation of unitarity and several conjectures

Let  $W$  be a reflection group and  $\mathfrak{h}$  its reflection representation. The KZ functor defined in [16] maps representations from the category  $\mathcal{O}_c(W, \mathfrak{h})$  of the rational Cherednik algebra  $H_c(W)$  to representations of the Hecke algebra  $\mathcal{H}_q(W)$ .

Let  $W = \mathfrak{S}_n$  and  $q = \exp(2\pi ic)$ . The KZ functor maps the irreducible lowest weight module  $L_c(\lambda)$  to the irreducible  $\mathcal{H}_q$ -representation  $D_\lambda$ , if  $c > 0$ , and to  $D_{\lambda^\vee}$ , if  $c < 0$ . Note that  $L_c(\lambda)$  is mapped to 0 if  $\lambda$  is not  $e$ -restricted.

Using theorems 2.3.5 and 3.3.2 we obtain the following

**Corollary 4.0.5.** The KZ functor maps unitary representations from the category  $\mathcal{O}_c(\mathfrak{S}_n, \mathfrak{h})$  to unitary representations of the Hecke algebra  $\mathcal{H}_q(\mathfrak{S}_n)$  or to zero. Moreover, all unitary representations of the Hecke algebra are obtained in this way.

**Remark 4.0.6.** The only irreducible unitary representations mapped to zero by the KZ functor are those corresponding to rectangular partitions  $(l, l, \dots, l)$  at  $c = \frac{1}{l}$ .

As in type A, for  $W = G(r, p, n)$  and  $\mathfrak{h}$  its reflection representation, the question of unitarity is expected to be closely related to Cherednik-Dunkl semisimplicity, namely the existence for every lowest weight irreducible representation of a basis given by eigenvectors of the Cherednik-Dunkl subalgebra of  $H_c(W, \mathfrak{h})$ .

Let  $W = \mathbb{Z}/2\mathbb{Z} \ltimes \mathfrak{S}_n$  (type  $B$ ) and  $\mathfrak{h}$  its reflection representation. The irreducible representations in category  $\mathcal{O}_c(W, \mathfrak{h})$  are parametrized by 2-partitions. Let us describe below a conjecture we have formulated for the unitarity loci in type  $B$ . It is a subspace of the real plane.

Let  $\Lambda = (\lambda, \mu)$  be a 2-partition of  $n$ . Let also  $A(\lambda)$  and  $a(\lambda)$  the largest and smallest parts of  $\lambda$ , respectively. Let  $L = A(\lambda) + A(\mu^\vee) - 1$  and  $\ell = A(\lambda) + A(\mu^\vee) - a(\lambda^\vee)$ . Set  $\Lambda^\vee = (\lambda^\vee, \mu^\vee)$  and  $\Lambda^{op} = (\mu, \lambda)$ . Define, now,  $\ell^\vee = \ell(\Lambda^\vee)$ ,  $\ell^{op} = \ell(\Lambda^{op})$ ,  $\ell^{\vee op} = \ell(\Lambda^{\vee op})$ , and, similarly,  $L^{op}$ .

Let us introduce the following regions of  $\mathbb{R}^2$ . Let us define

$$\begin{aligned} r(\Lambda) &= \{(c_1, c_2) \mid \ell c_1 + c_2 \leq \frac{1}{2}, \ell^{op} c_1 - c_2 \leq \frac{1}{2}, -\ell^\vee c_1 + c_2 \leq \frac{1}{2}, -\ell^{\vee op} c_1 - c_2 \leq \frac{1}{2}\}, \\ R(\Lambda) &= \{(c_1, c_2) \mid |L c_1 + c_2| < \frac{1}{2} \text{ and } |L^{op} c_1 - c_2| < \frac{1}{2}\}, \\ d(\Lambda) &= \{(c_1, c_2) \mid s c_1 + c_2 = \frac{1}{2}, \text{ for some } \ell \leq s \leq L\} \cap r(\Lambda), \end{aligned}$$

and, similarly,  $d^{op}(\Lambda)$ ,  $d^\vee(\Lambda)$  and  $d^{\vee op}(\Lambda)$ .

**Conjecture 4.0.7.** Let  $\Lambda = (\lambda, \mu)$  a 2-partition of  $n$  such that  $\lambda, \mu \neq \emptyset$ . The unitarity locus is given by  $U(\Lambda) = R(\Lambda) \cup d(\Lambda) \cup d^\vee(\Lambda) \cup d^{op}(\Lambda) \cup d^{\vee op}(\Lambda)$ .

A similar conjecture to 4.0.7 can be formulated when  $\lambda$  or  $\mu$  are empty.

We also conjecture that Corollary 4.0.5 can be generalized to the restriction functor defined in [2] for all complex reflection groups. Let  $W$  be a complex reflection group,  $\mathfrak{h}$  its reflection representation, and  $W'$  a parabolic subgroup of  $W$ , namely the stabilizer of a point in  $\mathfrak{h}$ . Let also  $H_c(W, \mathfrak{h})$  be the corresponding rational Cherednik algebra,  $c'$  the restriction of the parameter  $c$  to  $W'$ , and  $Loc(\mathfrak{h}_{reg}^{W'})$  the category of local systems on  $\mathfrak{h}_{reg}^{W'}$ . There is a restriction functor

$$Res: \mathcal{O}_c(W, \mathfrak{h}) \rightarrow \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}) \boxtimes Loc(\mathfrak{h}_{reg}^{W'}).$$

Composing it with the functor  $Mon: Loc(\mathfrak{h}_{reg}^{W'}) \rightarrow Rep \pi_1(\mathfrak{h}_{reg}^{W'})$  which sends local systems on  $\mathfrak{h}_{reg}^{W'}$  to representations of the fundamental group  $\pi_1(\mathfrak{h}_{reg}^{W'})$ , we obtain the

functor

$$Res': \mathcal{O}_c(W, \mathfrak{h}) \rightarrow \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}) \boxtimes Rep \pi_1(\mathfrak{h}_{reg}^{W'}).$$

Let  $I$  be a finite set. If  $M_i \in \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})$  and  $V_i \in Rep \pi_1(\mathfrak{h}_{reg}^{W'})$  for all  $i \in I$ , we say that the object  $\oplus_{i \in I} M_i \boxtimes V_i \in \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}) \boxtimes Rep \pi_1(\mathfrak{h}_{reg}^{W'})$  is *unitary* if and only if  $M_i$  and  $V_i$  are unitary in their categories.

**Conjecture 4.0.8.** If  $M \in \mathcal{O}_c(W, \mathfrak{h})$  is unitary then so is  $Res'(M)$ .

In this thesis, we have proved the conjecture for  $W = S_n$ ,  $\mathfrak{h}$  its reflection representation and  $W' = 1$ .





# Bibliography

- [1] Bojko Bakalov and Jr. Kirillov, Alexander. Lectures on tensor categories and modular functors. University Lecture Series. *American Mathematical Society, Providence, RI*, 21, 2001.
- [2] Roman Bezrukavnikov and Pavel Etingof. Parabolic induction and restriction functors for rational Cherednik algebras. *Adv. Math.*, 14(3-4):397–425, 2009.
- [3] J. Brundan. Modular branching rules and the Mullineux map for Hecke algebras of type  $A$ . *Proc. London Math. Soc. (3)*, 77(3):551–581, 2009.
- [4] Ivan Cherednik. Towards harmonic analysis for DAHA: integral formulas for canonical traces. *www-math.mit.edu/~etingof/hadaha.pdf*.
- [5] Ivan Cherednik. *Double affine Hecke algebras*. London Mathematical Society Lecture Note Series, volume 319. Cambridge University Press, 2005.
- [6] Tatyana Chmutova. Representations of the rational Cherednik algebras of dihedral type. *J. Algebra*, 297(2):542–565, 2006.
- [7] Richard Dipper and Gordon James. Representations of Hecke algebras of general linear groups. *Proc. London Math. Soc. (3)*, 52(1):20–52, 1986.
- [8] Richard Dipper and Gordon James. Blocks and idempotents of Hecke algebras of general linear groups. *Proc. London Math. Soc. (3)*, 54(1):57–82, 1987.
- [9] Richard Dipper, Gordon James, and Andrew Mathas. Cyclotomic  $q$ -Schur algebras. *Math. Z.*, 229(3):385–416, 1998.
- [10] C. F. Dunkl, M. F. E. de Jeu, and E. M. Opdam. Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.*, 346:237–256, 1994.
- [11] C. F. Dunkl and E. M. Opdam. Dunkl operators for complex reflection groups, *Proc. London Math. Soc.* 86 (3): 70-108, 2003.
- [12] Charles Dunkl. Integral kernels with reflection group invariance. *Can.J.Math.*, 43:1213–1227, 1991.
- [13] Pavel Etingof. *Calogero-Moser Systems and Representation Theory*. European Mathematical Society, 2007.

- [14] Pavel Etingof and Victor Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.*, 147(2):243–348, 1991.
- [15] Pavel Etingof and Emanuel Stoica. Unitary representations of rational Cherednik algebras. With an appendix by Stephen Griffeth. *Represent. Theory*, 13:349–370, 2009.
- [16] Victor Ginzburg, Nicolas Guay, Eric Opdam, and Raphaël Rouquier. On the category  $\mathcal{O}$  for rational Cherednik algebras. *Invent. Math.*, 154(3):617–651, 2003.
- [17] Stephen Griffeth. Orthogonal functions generalizing Jack polynomials. *arXiv:0707.0251*.
- [18] Stephen Griffeth. Towards a combinatorial representation theory for the rational Cherednik algebra of type  $G(r, p, n)$ . *arXiv:math/0612733*.
- [19] Alexander A. Kirillov. On an inner product in modular tensor categories. *J. Amer. Math. Soc.*, 9(4):1135–1169, 1996.
- [20] Alexander A. Kirillov. On an inner product in modular tensor categories II. Inner product on conformal blocks and affine inner product identities. *Adv. Theor. Math. Phys.*, 2(1):155–180, 1998.
- [21] Eric Opdam. Some applications of hypergeometric shift operators. *Invent. Math.*, 98(1):1–18, 1998.
- [22] Takeshi Suzuki. Cylindrical combinatorics and representations of Cherednik algebras of type  $A$ . *arXiv:math/0610029*.