Classification and enumeration of special classes of posets and polytopes

by

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Abstract

This thesis concerns combinatorial and enumerative aspects of different classes of posets and polytopes. The first part concerns the finite Eulerian posets which are binomial, Sheffer or triangular. These important classes of posets are related to the theory of generating functions and to geometry. Ehrenborg and Readdy [ER2] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets, where infinite posets are those posets which contain an infinite chain. We answer questions asked by R. Ehrenborg and M. Readdy [ER2]. We completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets; We give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases; In most cases above, we completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets. This work is also motivated by the work of R. Stanley about recognizing the *boolean lattice* by looking at smaller intervals.

In the second topic concerns lattice path matroid polytopes. The theory of matroid polytopes has gained prominence due to its applications in algebraic geometry, combinatorial optimization, Coxeter group theory, and, most recently, tropical geometry. In general matroid polytopes are not well understood. Lattice path matroid polytopes (LPMP) belong to two famous classes of polytopes, sorted closed matroid polytopes [LP] and polypositroids [Pos]. We study several properties of LPMPs and build a new connection between the theories of matroid polytopes and lattice paths. I investigate many properties of LPMPs, including their face structure, decomposition, and triangulations, as well as formulas for calculating their Ehrhart polynomial and volume.

Thesis Supervisor: Richard P Stanley
Title: Norman Levinson Professor of Applied Mathematics

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I am indebted to Professors Richard Ehrenborg and Margaret Readdy for various discussions which helped me to start the work on Chapter 3 of this thesis and continue their research. I am very grateful to Professor Matthias Beck for many discussions which helped me to get involved in polytopal combinatorics. I thank Dr. Henry Cohn and Professor Alex Postnikov for various interesting discussions, showing interest in my work and serving on my thesis committee.

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Chapter 1

Introduction

In this thesis, we study combinatorial and enumerative aspects of different classes of posets and polytopes. In particular, we study different aspects of Eulerian, binomial and Sheffer posets as well as lattice path matroid polytopes.

There are many theories which unify various aspects of enumerative combinatorics and generating functions. One such successful theory introduced by Doubilet, Rota and Stanley [DRS] is that of binomial posets. Classically binomial posets are infinite posets with the property that every two intervals of the same length have the same number of maximal chains. Doubilet, Rota and Stanley show this chain regularity condition gives rise to universal families of generating functions. Ehrenborg and Readdy [ER1], and independently Reiner [Rei], generalized the notion of a binomial poset to a larger class of posets, called Sheffer posets or upper binomial posets.

Ehrenborg and Readdy [ER2] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets, where infinite posets are those posets which contain an infinite chain. They posed the open questions of characterizing the finite case. Chapters 2 and 3 of this thesis based on my paper [Bid1] deal with these questions. We completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets; we give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases, In most cases above, we completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets.

In Chapter 4 of this thesis, based on my paper [Bid2], we study *lattice path matroid polytopes.* A *matroid* is a combinatorial construct which generalizes the notion of linear independence; matroids are very general objects with many applications throughout mathematics. They are found in matrices, graphs, transversals, point configurations, hyperplane arrangements, to name a few. For a thorough introduction to the subject we refer the reader to Oxley [Oxl]. For any matroid one can associate a matroid polytope by taking the convex hull of the incidence vectors of the bases of the matroid. The last few years have seen a flurry of research activities around matroid polytopes, in part because their combinatorial properties provide key insights into matroids and in part because they form an intriguing and seemingly fundamental class of polytopes which exhibit interesting geometric features. The theory of matroid polytopes has gained prominence due to its applications in algebraic geometry, combinatorial optimization, Coxeter group theory, and, most recently, tropical geometry. In general matroid polytopes are not well understood.

We study *lattice path matroid polytopes.* This special class of matroid polytopes belongs to two famous classes of polytopes, sorted closed matroid polytopes [LP], and polypositroids [Pos]. I study several properties of LPMPs and build a new connection between the theories of matroid polytopes and lattice paths. A good example of a lattice path matroid polytope is the Catalan matroid polytope [Ard], which has the Catalan number of vertices and many interesting properties. Matroid polytopes are generally hard to study and there are few general result about them, so we try to produce a clear picture of the combinatorial, geometric, and arithmetic properties of this special class of matroid polytopes.

Chapter 2

Definition of Several Classes of Posets

Standard terminology from the theory of partially ordered sets will be used throughout Chapters 2 and 3 of this thesis. We refer reader to [DRS, Sta4, Sta5] for notation and basic terminology concerning posets. In this chapter, we briefly reviews the definitions and essential features of binomial, Sheffer, triangular, simplicial, and Eulerian posets. We try to explain why these classes of posets play an important role in combinatorics.

2.1 Binomial posets

Binomial posets were introduced by Doubilet, Rota and Stanley [DRS] to explain why generating functions naturally occurring in combinatorics have certain forms. They are highly regular posets since the essential requirement is that every two intervals of the same length have the same number of maximal chains. As a result, many poset invariants are determined. For instance, the Möbius function is described by the generating function identity

$$\sum_{n\geq 0} \mu(n) \cdot \frac{t^n}{B(n)} = \left(\sum_{n\geq 0} \frac{t^n}{B(n)} \right)^{-1}, \tag{2.1}$$

where $\mu(n)$ is the Möbius function of an n-interval and $B(n)$ is the factorial function, that is, the number of maximal chains in an n-interval.

Definition 2.1.1. A locally finite poset P with $\hat{0}$ is called a *binomial poset* if it satisfies the following three conditions:

(i) P contains an infinite chain.

(ii) Every interval $[x, y]$ is graded; hence P has rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval.*

(iii) For all $n \in$ N, any two n-intervals contain the same number $B(n)$ of maximal chains. We call $B(n)$ the *factorial function* of P.

If P does not satisfy condition (i) and has a unique maximal element then we say P is a *finite binomial poset.*

$\diamondsuit$

Let P be a locally finite poset, and let $Int(P)$ denote the set of intervals of P. (Recall that the void set is not an interval.) Let K be a field, if $f : Int(P) \longrightarrow K$, then write $f(x, y) = f([x, y])$.

Definition 2.1.2. The *incidence algebra* $I(P, K)$ of P over K is the K-algebra of all functions $f : Int(P) \longrightarrow K$ (with the usual structure of a vector space over K), where multiplication (or *convolution*) is defined by $fg(x, y) = \Sigma_{x \leq z \leq y} f(x, z)g(z, y)$. $\diamondsuit$

The above sum is finite (and hence *fg* is defined) since P is locally finite. It is easy to see that $I(P, K)$ is an associative K-algebra with (two-sided) identity. Denoted by σ or 1, defined by $\sigma(x, y) = 1$ if $x = y$, and $\sigma(x, y) = 0$ if $x \neq y$.

The following is the main result concerning binomial posets.

Theorem 2.1.3. *Let P be a binomial poset with factorial function $B(n)$ and incidence algebra $I(P)$(over C). Define*

$R(P) = \{f \in I(P) : f(x, y) = f(\hat{x}, \hat{y}) \text{ if } l(x, y) = l(\hat{x}, \hat{y})\}$

If $f \in R(P)$ then write $f(n)$ for $f(x,y)$ when $l(x,y) = n$. Then $R(P)$ is a sub algebra of $I(P)$, and we have an algebra isomorphism $\phi : R(P) \longrightarrow \mathbb{C}[[t]]$ given by

$$\phi(f) = \sum_{n \geq 0} f(n) \frac{t^n}{n!} \tag{2.2}$$

Let P be a binomial poset and $f \in R(P)$. Suppose f^{-1} exists in $I(P)$, then $f^{-1} \in R(P)$.

Example 2.1.4. If $\mu(n)$ denotes the Möbius function $\mu(x,y)$ for n-interval of P, then from the above result we have

$$\sum_{n \geq 0} \mu(n) \cdot \frac{t^n}{B(n)} = \left(\sum_{n \geq 0} \frac{t^n}{B(n)} \right)^{-1} . \tag{2.3}$$

$\diamondsuit$

The number of elements of rank k in an n-interval is given by $B(n)/(B(k) \cdot B(n-k))$. In particular, an n-interval has $A(n) = B(n)/B(n-1)$ atoms (and coatoms). The function $A(n)$ is called the *atom function* and expresses the factorial function as $B(n) = A(n) \cdot A(n-1) \cdots A(1)$. Directly, we have $B(0) = B(1) = A(1) = 1$. Since the atoms of an $(n-1)$-interval are contained among the set of atoms of an n-interval, the inequality $A(n-1) \leq A(n)$ holds. Observe that if a finite binomial poset has rank j, the factorial and atom functions are only defined up to j.

Example 2.1.5. Let $\mathbb{B}$ be the collection of finite subsets of the positive integers ordered by inclusion. The poset $\mathbb{B}$ is a binomial poset with factorial function $B(n) = n!$ and atom function $A(n) = n$. An n-interval is isomorphic to the Boolean algebra B_n. This example is the *infinite Boolean algebra.* $\diamondsuit$

Example 2.1.6. Let $\mathcal{T}$ be the *infinite butterfly poset*, that is, $\mathcal{T}$ consists of the elements $\{\hat{0}\} \cup (\mathrm{P} \times \{1,2\})$ where P is an infinite chain. So, $(n,i) \prec (n+1,j)$ for all $i,j \in \{1,2\}$ and $\hat{0}$ is the unique minimal element. The poset $\mathcal{T}$ is a binomial poset. It has factorial function $B(n) = 2^{n-1}$ for $n \geq 1$ and atom function $A(n) = 2$ for $n \geq 2$. Let $\mathcal{T}_n$ denote an n-interval in $\mathcal{T}$. $\diamondsuit$

Example 2.1.7. Let $\mathbb{B}(q)$ be the lattice of all finite dimensional subspaces of a vector space of infinite dimension over $\mathbb{F}_q$, ordered by inclusion. Then $B(n) = (\mathbf{n})!$. $\diamondsuit$

2.2 Sheffer and Triangular Posets

Ehrenborg and Readdy [ER1], and independently Reiner [Rei], generalized the notion of a binomial poset to a larger class of posets, called Sheffer posets or upper binomial posets.

A Sheffer poset requires the number of maximal chains of an interval $[x, y]$ of length n to be given by $B(n)$ if $x > \hat{0}$ and $D(n)$ if $x = \hat{0}$. The upper intervals $[x, y]$ where $x > \hat{0}$ have the property of being binomial. Hence the interest is to understand the Sheffer intervals $[\hat{0}, y]$. Just like binomial posets, the Möbius function is completely determined.

$$\sum_{n\geq 1} \mu(n) \frac{t^n}{D(n)} = -\left(\sum_{n\geq 1} \frac{t^n}{D(n)}\right) \cdot \left(\sum_{n\geq 0} \frac{t^n}{B(n)}\right)^{-1}, \tag{2.4}$$

where μ is the Möbius function of a Sheffer interval of length n; See [ER2, Rei].

Definition 2.2.1. A locally finite poset P with $\hat{0}$ is called a *Sheffer poset* if it satisfies the following four conditions:

(i) P contains an infinite chain.

(ii) Every interval $[x, y]$ is graded; hence P has rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval.*

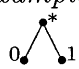
(iii) Two n-intervals $[\hat{0}, y]$ and $[\hat{0}, v]$, have the same number $D(n)$ of maximal chains.

(iv) Two n-intervals $[x, y]$ and $[u, v]$, such that $x \neq \hat{0}$, $u \neq \hat{0}$, have the same number $B(n)$ of maximal chains.

As in the finite Sheffer poset case, if P does not satisfy condition (i) and has a unique maximal element then we say P is a *finite Sheffer poset.* $\diamondsuit$

Example 2.2.2. For a poset P with a unique minimal element $\hat{0}$, let the *dual suspension* $\Sigma^*(P)$ be the poset P with two new elements a_1 and a_2. Let the order relations be as follows: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$ for all $y > \hat{0}$ in P and $i = 1, 2$. That is, the elements a_1 and a_2 are inserted between $\hat{0}$ and the atoms of P. Clearly if P is Eulerian then so is $\Sigma^*(P)$. Moreover, if P is a binomial poset then $\Sigma^*(P)$ is a Sheffer poset with the factorial function $D_{\Sigma^*(P)}(n) = 2 \cdot B(n-1)$ for $n \geq 2$. $\diamondsuit$

Example 2.2.3. Let P be the three element poset



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The poset $C_n = P^n \cup \{\hat{0}\}$ is the face lattice of the n-dimensional cube, also known as the *cubical lattice.* It is a finite Sheffer poset with factorial functions $B(k) = k!$ for $k \leq n$ and $D(k) = 2^{k-1} \cdot (k-1)!$ for $1 \leq k \leq n+1$. $\diamondsuit$

A larger class of posets to consider is the class of *triangular posets.* A triangular poset is a graded poset such that the number of maximal chains in each interval $[x, y]$ depends only on $\rho(x)$ and $\rho(y)$, where $\rho(x)$ and $\rho(y)$ are ranks of the elements x and y, respectively.

Definition 2.2.4. A finite poset P with $\hat{0}$ and $\hat{1}$ is called a *(finite) triangular poset* if it satisfies the following two conditions.

1. Every interval $[x, y]$ is graded; hence P has a rank function ρ.

2. Every two intervals $[x, y]$ and $[u, v]$ such that $\rho(x) = \rho(u) = m$ and $\rho(y) = \rho(v) = n$ have the same number $B(m, n)$ of maximal chains.

$\diamondsuit$

A non-trivial Eulerian example of a finite triangular poset is the face lattice of the 4-dimensional regular polytope known as the 24-cell.

2.3 Eulerian posets

The study of Eulerian partially ordered sets (posets) originated with Stanley [Sta1]. The foremost example of Eulerian posets are face lattices of convex polytopes and more generally, the face posets of regular CW-spheres. These include face lattices of convex polytopes, the Bruhat order on finite Coxeter groups, and the lattices of regions of oriented matroids. Hence there is much geometric and topological interest in understanding them. A partially ordered set (poset) is Eulerian if every interval contains the same number of elements of even rank as of odd rank. Another formulation of the Eulerian property is every interval satisfies the Euler-Poincaré formula.

Definition 2.3.1. A graded poset is *Eulerian* if every non-singleton interval satisfies the Euler-Poincaré relation. Equivalently, a poset P is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x)-\rho(y)}$ for all $x \leq y$ in P, where ρ denotes the rank function of P. $\diamondsuit$

There are certain numerical and polynomial invariants associated to Eulerian posets, we recommend reader to consult [Sta2] for a comprehensive survey about Eulerian posets.

2.4 Simplicial posets

A *simplicial poset* is a poset (partially ordered set) P that has a $\hat{0}$ element, (i.e., $\hat{0} \leq y$ for all $y \in P$), and equivalently:

1. For every $y \in P$, the interval $[\hat{0}, y]$ is a Boolean algebra.

2. Every interval $[x, y]$ is a Boolean algebra.

A simplicial poset is a generalization of a simplicial complex. $\Delta \subseteq 2^V$ is a simplicial complex on the set of vertices V if: $V \in \Delta$; and if $F \subset G \in \Delta$ then $F \in \Delta$. Since the face-poset, (poset of elements of Δ, called faces ordered by inclusion of a simplicial complex) is a simplicial poset. In fact, any simplicial poset that is also a meet-semilattice is the face-poset of a simplicial complex. Simplicial posets arise naturally as quotients of simplicial complexes under group actions [GS], but were first studied in their own right in [Sta3].

Chapter 3

Eulerian posets which are binomial, Sheffer and triangular

In this chapter, we study the following natural question that arises by looking at the classes of binomial, Eulerian and Sheffer posets. The question is which binomial and Sheffer posets are Eulerian? Ehrenborg and Readdy [ER2] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets, where infinite posets are those posets which contain an infinite chain. They posed the open questions of characterizing the finite case. This chapter is about my paper [Bid1], which answers these questions.

It is not hard to see that the Eulerian property can be determined by knowing the factorial function. In paper [ER2], Ehrenborg and Readdy classify the factorial functions of infinite Eulerian binomial posets. There are two possibilities, namely, for the factorial function to correspond to that of the infinite Boolean algebra or the infinite butterfly poset.

Notice that the classification of Ehrenborg and Readdy is on the level of the factorial function, not the poset itself.

They also completely classify the factorial functions of infinite Eulerian Sheffer posets. This classification is also up to factorial function. There are many Sheffer posets having the same factorial functions as the infinite cubical lattice.

In [Bid1], we completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets. I give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases; In the most of those cases, I completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets. We also briefly consider Eulerian triangular posets.

This chapter is organized as follows. In Section 3.1, we review Ehrenborg and Ready's Results on infinite Eulerian binomial and Sheffer posets. In Sections 3.2 and 3.3, we discuss our results in [Bid1] on the finite Eulerian binomial, Sheffer and triangular posets.

3.1 Previous Results on Infinite Eulerian binomial and Sheffer posets

In this section, we review Ehrenborg and Readdy results on the infinite Eulerian binomial and Sheffer posets.

Ehrenborg and Readdy in [ER2] provide the complete classification of the factorial functions of infinite Eulerian binomial posets as follows:

Theorem 3.1.1. *Let P be an Eulerian binomial poset with factorial function $B(n)$. Then either*

(i) the factorial function $B(n)$ is given by $B(n) = n!$ and every n-interval is isomorphic to the Boolean algebra B_n, or

(ii) the factorial function $B(n)$ is given by $B(0) = 1$ and $B(n) = 2^{n-1}$ and every n-interval is isomorphic to the butterfly poset T_n.

Example 3.1.3 demonstrates that their classification is unique up to factorial function. This example introduce different Eulerian binomial posets which have the same factorial function as the infinite Boolean algebra and the infinite butterfly poset $\mathcal{T}$.

Definition 3.1.2. Given two ranked posets P and Q, define the *rank product* $P * Q$ by

$$P * Q = \{(x, z) \in P \times Q \ : \ \rho_P(x) = \rho_Q(z)\}.$$

Define the order relation by $(x, y) \leq_{P*Q} (z, w)$ if $x \leq_P z$ and $y \leq_Q w$. If P and Q are binomial posets then so is the poset $P * Q$. It has the factorial function $B_{P*Q}(n) = B_P(n) \cdot B_Q(n)$. $\diamondsuit$

Example 3.1.3. Let $\mathcal{Q}$ be an infinite poset with a minimal element $\hat{0}$ contains an infinite chain such that every interval of the form $[\hat{0}, x]$ is a chain. Observe the poset $\mathcal{Q}$ is an infinite tree and, in fact, is a binomial poset with factorial function $B(n) = 1$. Thus we know that both $\mathcal{B} * \mathcal{Q}$ and $\mathcal{T} * \mathcal{Q}$ are Eulerian binomial posets. See Figure 3-1 for an example. When the poset $\mathcal{Q}$ is different from an infinite chain, we have that $\mathcal{B} * \mathcal{Q} \not\cong \mathcal{B}$ and $\mathcal{T} * Q \not\cong \mathcal{T}$. This follows since in the two posets $\mathcal{B}$ and $\mathcal{T}$ every pair of elements has an upper bound, that is, the two posets are confluent. This property does not hold in the tree $\mathcal{Q}$ and hence not in the rank products $\mathcal{B} * \mathcal{Q}$ and $\mathcal{T} * \mathcal{Q}$ either. $\diamondsuit$

Example 3.1.4. For each infinite cardinal κ there is a Boolean algebra consisting of all finite subsets of a set X of cardinality κ. We denote this poset by $\mathcal{B}_X$. Observe that different cardinals give rise to non-isomorphic Boolean algebras. $\diamondsuit$

Example 3.1.5. Let P be a binomial poset and I a nonempty lower order ideal of P. Construct a new poset by taking the Cartesian product of the poset P with the two element antichain $\{a, b\}$, and identify elements of the form (x, a) and (x, b) if x lies in the ideal I. The new poset is also binomial and has the same factorial function as P. $\diamondsuit$

The classic example of a Sheffer poset is the infinite cubical poset. In this case, every interval $[x, y]$ of length n, where x is not the minimal element $\hat{0}$, has $n!$ maximal chains. In fact, every such interval is isomorphic to a Boolean algebra. Intervals of the form $[\hat{0}, y]$ have $2^{n-1} \cdot (n-1)!$ maximal chains and are isomorphic to the face

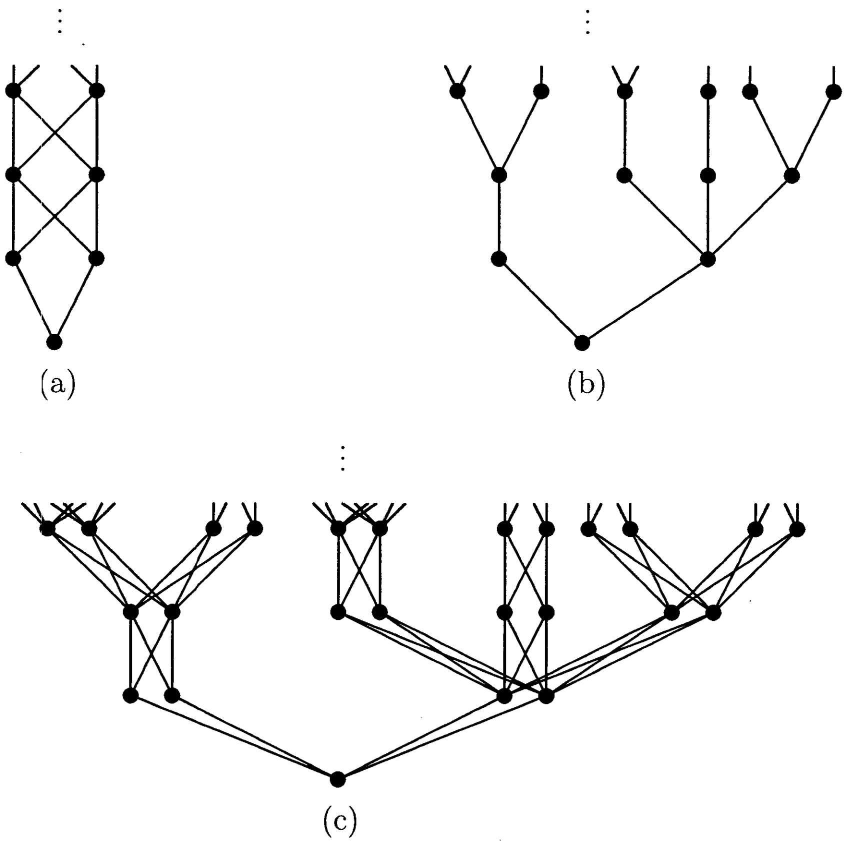


Figure 3-1: (a) the infinite butterfly poset $\mathcal{T}$, (b) an infinite tree $\mathcal{Q}$, (c) and the rank product $\mathcal{T} * \mathcal{Q}$, which has the same factorial function as the butterfly poset.

lattice of a finite dimensional cube.

Ehrenborg and Readdy [ER2] completely classify the factorial functions of Eulerian Sheffer posets, where the factorial function $B(n)$ follows from the classification of binomial posets.

Example 3.1.6. Let P be as in the previous example and let X be an infinite set. The poset $\mathcal{C}_X = P^X \cup \{\hat{0}\}$, that is, the poset P^X with a new minimal element adjoined, is a Sheffer poset. This example is precisely the *infinite cubical poset* with the factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. Similar to Example 3.1.4, for different infinite cardinalities of X we obtain non-isomorphic cubical posets. Note, however this poset is not a lattice since the two atoms $(0, 0, \ldots)$ and $(1, 1, \ldots)$ do not have a join. A Sheffer n-interval is isomorphic to the cubical lattice $\mathcal{C}_{n-1}$. Hence, every interval in the poset $\mathcal{C}_X$ is Eulerian. $\diamondsuit$

Example 3.1.7. Let $\mathcal{B} \cup \{\hat{0}\}$ be the infinite Boolean algebra with a new minimal element adjoined. This is a Sheffer poset with factorial functions $B(n) = n!$ and

$D(n) = (n-1)!$. Now consider the rank product $(\mathcal{B} \cup \{\hat{0}\}) * U_{2,2,\ldots}$. It has the factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. This poset has the same factorial functions as the infinite cubical poset and hence it is an Eulerian poset. $\diamondsuit$

Let P_{q_i}, $i = 1, \ldots, r$, be posets which contain a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. We define $\boxplus_{i=1,\ldots,r} P_{q_i}$ to be the poset which is obtained by identifying all of the minimal elements as well as identifying all of the maximal elements of the posets P_{q_i}. The following theorem characterizes Eulerian Sheffer posets with binomial factorial function $B(n) = 2^{n-1}$.

Theorem 3.1.8 (Ehrenborg, Readdy). *Let P be an Eulerian Sheffer poset with the binomial factorial function satisfying $B(0) = 1$ and $B(n) = 2^{n-1}$ for $n \geq 1$. Then the coatom function $C(n)$ and the poset P satisfy:*

(i) $C(3) \geq 2$, and a length 3 Sheffer interval is isomorphic to a poset of the form $P_{q_1,\ldots,q_r} = \boxplus_{i=1,\ldots,r} P_{q_i}$.

(ii) $C(2m) = 2$ for $m \geq 2$ and the two coatoms in a length $2m$ Sheffer interval cover exactly the same elements of rank $2m-2$.

(iii) $C(2m+1) = h$ is an even positive integer, for $m \geq 2$. Moreover, the set of h coatoms in a Sheffer interval of length $2m+1$ partitions into $h/2$ pairs, $\{c_1, d_1\}$, $\{c_2, d_2\}$, ..., $\{c_{h/2}, d_{h/2}\}$, such that c_i and d_i cover the same two elements of rank $2m-1$.

The following result in [ER2], classifies the factorial function of Eulerian Sheffer posets that have the binomial factorial function $B(n) = n!$, that is, every interval $[x, y]$, where $x > \hat{0}$, is a Boolean algebra.

Theorem 3.1.9. *Let P be an Eulerian Sheffer poset with binomial factorial function $B(n) = n!$. Then the Sheffer factorial function $D(n)$ satisfies one of the following three alternatives:*

(i) $D(n) = 2 \cdot (n-1)!$. In this case every Sheffer n-interval is of the form $\Sigma^*(B_{n-1})$.

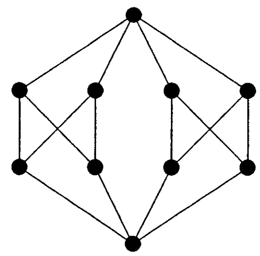


Figure 3-2: A finite Sheffer poset with the same factorial functions as the cubical lattice.

(ii) $D(n) = n!$. *In this case the poset is a binomial poset and hence every Sheffer n-interval is isomorphic to the Boolean algebra B_n.*

(iii) $D(n) = 2^{n-1} \cdot (n-1)!$. *If we furthermore assume that a Sheffer n-interval $[\hat{0}, y]$ is a lattice then the interval $[\hat{0}, y]$ is isomorphic to the cubical lattice C_n.*

The cubical posets of Example 3.1.6 and Example 3.1.7 demonstrate there is no straightforward classification of the non-lattice Sheffer intervals in case (iii) of Theorem 3.1.9. The following examples further illustrates Sheffer posets (both finite and infinite) having the same factorial functions as the cubical poset.

Example 3.1.10. Let C_n be the finite cubical lattice, that is, the face lattice of an $(n-1)$-dimensional cube. We are going to deform this lattice as follows. The 1-skeleton of the cube is a bipartite graph. Hence the set of atoms A has a natural decomposition as $A_1 \cup A_2$. Every rank 2 element (edge) covers exactly one atom in each A_i. Consider the poset

$$L_n = (C_n - A) \cup (A_1 \times \{1, 2\}) .$$

That is, we remove all the atoms and add in two copies of each atom from A_1. Define the cover relations for the new elements as follows. If a in A_1 is covered by b then let b cover both copies $(a, 1)$ and $(a, 2)$. The poset L_n is a Sheffer poset with the cubical factorial functions. $\diamondsuit$

The poset in Figure 3-2 is the atom deformed cubical lattice H_3. This poset is also obtained as length 3 Sheffer interval in Example 3.1.7.

Example 3.1.11. Let P and Q be two Sheffer posets (finite or infinite) having the cubical factorial functions $B(n) = n!$ and $D(n) = 2^{n-1} \cdot (n-1)!$. Their diamond product, namely $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$, also has the cubical factorial functions. $\diamondsuit$

Example 3.1.12. As an extension of the previous example, let P be a Sheffer poset (finite or infinite) having the cubical factorial functions. Then for a set X the poset $(P - \{\hat{0}\})^X \cup \{\hat{0}\}$ is a Sheffer poset with the cubical factorial functions. The cubical poset (Example 3.1.6) is an illustration of this. $\diamondsuit$

3.2 Finite Eulerian binomial posets

In this section, we discuss our result in [Bid1], which determines the structure and factorial function of finite Eulerian binomial posets. All the posets considered in this section are finite, so we simply remove the word finite.

Definition 3.2.1. Let Q_i, $i = 1, \ldots, k$, be posets which contain a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. We define $\boxplus_{i=1,\ldots,k} Q_i$ to be the poset which is obtained by identifying all of the minimal elements as well as identifying all of the maximal elements of the posets Q_i. We define the *k-summation* of P, denoted $\boxplus^k(P)$, to be $\boxplus_{i=1,\ldots,k} P$. This is also known as the banana product. $\diamondsuit$

Definition 3.2.2. Let P be a poset with $\hat{0}$. The *dual suspension* of P, denoted $\Sigma^*(P)$, is the poset P with two new elements a_1 and a_2 and with the following order relation: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$. $\diamondsuit$

In this section for an Eulerian binomial poset P of rank n we describe its structure as follows.

1. If $n = 3$, then $P \cong \boxplus_{i=1 \ldots r} P_{q_i}$ for some $q_1, \ldots, q_r$ such that $q_i \geq 2$, where P_{q_i} is the face lattice of q_i-gon.

2. If n is an even integer, then $P \cong B_n$ or T_n.

3. If n is an odd integer and $n \geq 5$, then there is an integer $k \geq 1$, such that $P \cong \boxplus^k(B_n)$ or $P \cong \boxplus^k(T_n)$ (see Definition 3.2.1).

First we provide some examples of finite binomial posets.

Example 3.2.3. The boolean lattice B_n of rank n is an Eulerian binomial poset with factorial function $B(k) = k!$ and atom function $A(k) = k$, $k \leq n$. Every interval of length k of this poset is isomorphic to B_k. $\diamondsuit$

Example 3.2.4. Let D_n be the chain containing $n+1$ elements. This poset has factorial function $B(k) = 1$ and atom function $A(k) = 1$, for each $k \leq n$. $\diamondsuit$

Example 3.2.5. The butterfly poset T_n of rank n is an Eulerian binomial poset with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$ and atom function $A(k) = 2$, for $2 \leq k \leq n$, and $A(1) = 1$. $\diamondsuit$

Example 3.2.6. Let $\mathbb{F}_q$ be the q-element field where q is a prime power, and let $V_n = V_n(q)$ be an n-dimensional vector space over $\mathbb{F}_q$. Let $L_n = L_n(q)$ denote the poset of all subspaces of V_n, ordered by inclusion. L_n is a graded lattice of rank n. It is easy to see that every interval of size $1 \leq k \leq n$ is isomorphic to L_k. Hence $L_n(q)$ is a binomial poset. This poset is not Eulerian for $q \geq 2$. $\diamondsuit$

The number of elements of rank k in a binomial interval of rank n is

$$\frac{B(n)}{B(k)B(n-k)}. \tag{3.1}$$

It is not hard to see that in any n-interval of an Eulerian binomial poset P with factorial function $B(k)$ for $1 \leq k \leq n$, the Euler-Poincaré relation is stated as follows:

$$\sum_{k=0}^{n} (-1)^k \cdot \frac{B(n)}{B(k)B(n-k)} = 0. \tag{3.2}$$

The following lemma can be found in [ER2].

Lemma 3.2.7. *Let P be a graded poset of odd rank such that every proper interval of P is Eulerian. Then P is an Eulerian poset.*

Lemma 3.2.8. *Let P be an Eulerian binomial poset of rank 3. Then the poset P and its factorial function $B(n)$ satisfy the following conditions:*

(i) *$B(2) = 2$ and $B(3) = 2q$, where q is a positive integer such that $q \geq 2$.*

(ii) *There is a list of integers $q_1, \ldots, q_r$, $q_i \geq 2$, such that $P = \boxplus_{i=1,\ldots,r} P_{q_i}$, where P_{q_i} is the face lattice of the q_i-gon.*

This result is [ER2, Example 2.5]. It is also a consequence of Lemma 3.3.3.

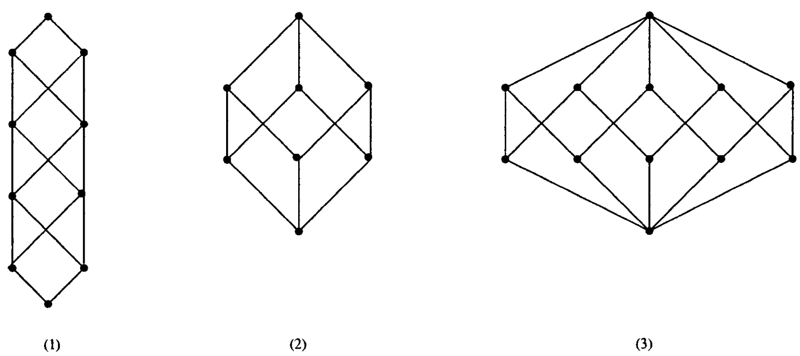


Figure 3-3: (1): T_5, (2): B_3 and (3): P_5, the face lattice of a 5-gon

R. Ehrenborg and M. Readdy proved the following two propositions. See [ER2, Lemma 2.17 and Prop. 2.15].

Let P be a binomial poset of rank n with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the butterfly poset T_n.

Let P be a binomial poset of rank n with factorial function $B(k) = k!$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the boolean lattice B_n of rank n.

The following is [ER2, Lemma 2.12].

Lemma 3.2.9. *Let P' and P be two Eulerian binomial posets of rank $2m+2$, $m \geq 2$, having atom functions $A'(n)$ and $A(n)$, respectively, which agree for $n \leq 2m$. Then the following equality holds:*

$$\frac{1}{A(2m+1)}\left(1 - \frac{1}{A(2m+2)}\right) = \frac{1}{A'(2m+1)}\left(1 - \frac{1}{A'(2m+2)}\right). \tag{3.3}$$

Lemma 3.2.10. *Every Eulerian binomial poset P of rank 4 is isomorphic to either T_4 or B_4.*

Proof. Applying Lemma 3.2.8 gives $B(3) = 2k$, where $k \geq 2$. Eq.(3.1) implies that the number of elements of rank one is the same as the number of elements of rank three in P. We denote this number by n. Hence

$$n = \frac{B(4)}{B(3)B(1)} = \frac{B(4)}{B(3)}. \tag{3.4}$$

We can also enumerate the number r of elements of rank 2 as follows:

$$r = \frac{B(4)}{B(2)B(2)}. \tag{3.5}$$

The Euler-Poincaré relation on intervals of length four is $2+r = 2n$. By enumerating the number of maximal chains, we conclude $B(4) = rB(2)B(2) = nB(3)$ and since always $B(2) = 2$, we have $2r = kn$. The Euler-Poincaré relation implies that $\frac{kn}{2} + 2 = 2n$, and so $k < 4$. We have the following cases.

(*i*) $k = 1$. Then $n = \frac{4}{3}$. This case is not possible.

(*ii*) $k = 2$. Then $n = 2$ and $r = 2$. We conclude that $B(k) = 2^{k-1}$, for $1 \leq k \leq 4$. By Proposition 3.2, $P \cong T_4$.

(*iii*) $k = 3$. Then $n = 4$ and $r = 6$. Thus $B(k) = k!$, for $1 \leq k \leq 4$. By Proposition 3.2, $P \cong B_4$.

□

In the following theorem we obtain the structure of Eulerian binomial posets of even rank.

Theorem 3.2.11. *Every Eulerian binomial poset of even rank $n = 2m \geq 4$ is isomorphic to either T_n or B_n (the butterfly poset of rank n or boolean lattice of rank n).*

Proof. We proceed by induction on m. The claim is true for $2m = 4$, by Lemma 3.2.10. Assume that the theorem holds for Eulerian binomial posets of rank $2m \geq 4$. We wish to show that it also holds for Eulerian binomial posets of rank $2m + 2$.

Let P be a Eulerian binomial poset of rank $2m + 2$. The factorial and atom functions of this poset are denoted by $B(n)$ and $A(n)$, respectively. By Lemma 3.2.10, every interval of size 4 is either isomorphic to B_4 or T_4. So the factorial function $B(3)$ of intervals of rank 3 can only take the values 4 or 6 and we have the following two cases:

- $B(3) = 6$. We wish to show that P is isomorphic to B_{2m+2} by induction on m. By Lemma 3.2.10, the claim is true for $2m = 4$. By the induction hypothesis, the claim holds for $n = 2m$, and we wish to prove it for $n = 2m + 2$. Let $P' = B_{2m+2}$, so P' has the atom function $A'(n) = n$ for $1 \leq n \leq 2m + 2$. By the induction hypothesis, $A(j) = A'(j) = j$ for $j \leq 2m$. Lemma 3.2.9 implies that Eq.(3.3) holds.

  Since $2m = A(2m) \leq A(2m + 2) < \infty$, we obtain the following inequalities:

  $$2m + 1 - \frac{2}{2m} < A(2m + 1) < 2m + 2. \tag{3.6}$$

  Thus $A(2m + 1) = 2m + 1$. Eq.(3.3) implies that $A(2m + 2) = 2m + 2$. By Proposition 3.2, the poset P is isomorphic to B_{2m+2}, as desired.

- $B(3) = 4$. We claim that the poset P of rank $n = 2m + 2$ is isomorphic to T_n. By the induction hypothesis, our claim holds for even $n \leq 2m$, and we would like to prove it for $n = 2m + 2$. Consider the poset T_{2m+2}. This poset has the atom function $A(n) = 2$ for $1 \leq n \leq 2m + 2$. By the induction hypothesis the intervals of length $2m$ in P are isomorphic to T_{2m}, so $A(j) = 2$ for $1 \leq j \leq 2m$.

Clearly $2 = A(2m) \leq A(2m+2) < \infty$. Eq.(3.3) implies that $2 \leq A(2m+1) < 4$. The case $A(2m+1) = 3$ is forbidden by a similar idea that appears in the proof of Theorem 2.16 in [ER2]: Assume that $A(2m+1) = 3$. Let $[x, y]$ be a $(2m+1)$-interval in P. For $1 \leq k \leq 2m$ there are $B(2m+1)/B(k) \cdot B(2m+1-k) = 3 \cdot 2^{2m-1}/(2^{k-1} \cdot 2^{2m-k}) = 3$ elements of rank k in this interval. Let c be a coatom. The interval $[x, c]$ has two atoms, say a_1 and a_2. Moreover, the interval $[x, c]$ has two elements of rank 2, say b_1 and b_2. Moreover we know that each b_j covers each a_i. Let a_3 and b_3 be the third atom, respectively the third rank 2 element, in the interval $[x, y]$. We know that b_3 covers two atoms in $[x, y]$. One of them must be a_1 or a_2, say a_1. However a_1 is covered by the three elements b_1, b_2 and b_3. This contradicts the fact that each atom is covered by exactly two elements. Hence this rules out the case $A(2m+1) = 3$.

Hence $A(2m+1) = A(2m+2) = 2$. Lemma 3.2 implies that P is isomorphic to T_{2m+2}.

$\square$

Theorem 3.2.12. *Let P be an Eulerian binomial poset of odd rank $n = 2m+1 \geq 5$. Then the poset P satisfies one of the following conditions:*

(i) There is a positive integer k such that P is the k-summation of the boolean lattice of rank n. In other words, $P \cong \boxplus^k(B_n)$.

(ii) There is a positive integer k such that P is the k-summation of the butterfly poset of rank n. In other words, $P \cong \boxplus^k(T_n)$.

Proof. Lemma 3.2.10 implies that every interval of length 4 is isomorphic either to B_4 or T_4. Thus the factorial function $B(3)$ can only take the values 4 or 6. Therefore we have the following two cases.

1. $B(3) = 6$. In this case we claim that there is a positive integer k such that $P \cong \boxplus^k(B_n)$. When we remove the $\hat{1}$ and $\hat{0}$ from P, the remaining poset is a

disjoint union of connected components. Consider one of them and add minimal element $\hat{0}$ and maximal element $\hat{1}$ to it. Denote the resulting poset by Q. It is not hard to see that Q is an Eulerian binomial poset, and also the posets P and Q have the same factorial functions and atom functions up to rank $2m$. Hence $B_Q(k) = B_P(k)$ and $A_Q(k) = A_P(k)$, for $1 \leq k \leq 2m$. Eq.(3.1) implies that in the poset Q the number of atoms and coatoms are the same. Denote this number by t. Let $x_1, \ldots, x_t$ and $a_1, \ldots, a_t$ be an ordering of the atoms and coatoms of Q, respectively. Also, let $c_1, \ldots, c_l$ be the set of elements of rank $2m-1$ in Q. For each element y of rank at least 2 in Q, let $S(y)$ be the set of atoms of Q that are below y. Set $A_i := S(a_i)$ for each element a_i of rank $2m$, $1 \leq i \leq t$, and also set $C_i := S(c_i)$ for each element c_i of rank $2m-1$, $1 \leq i \leq l$. By considering the factorial functions, Theorem 3.2.11 implies that the intervals $[\hat{0}, a_i]$ and $[x_j, \hat{1}]$ are isomorphic to B_{2m}, where $1 \leq i \leq t$ and $1 \leq j \leq t$. We conclude that any interval $[\hat{0}, c_k]$ of rank $2m-1$ is isomorphic to B_{2m-1}. As a consequence, $|A_i| = |S(a_i)| = 2m$, $1 \leq i \leq t$ and also $|C_k| = |S(c_k)| = 2m-1$, $1 \leq k \leq l$.

In the case that there are i and j such that $A_i \cap A_j \neq \phi$, where $1 \leq i, j \leq t$, we claim that $2m-1 \leq |A_i \cap A_j| \leq 2m$. Consider an atom $x_k \in A_i \cap A_j$, $1 \leq k \leq t$. Theorem 3.2.11 implies that $[x_k, \hat{1}] \cong B_{2m}$. Thus, there is an element c_h of rank $2m-2$ in this interval which is covered by a_i and a_j, $1 \leq h \leq l$. Notice that c_h is an element of rank $2m-1$ in Q. Therefore, $|C_h| = 2m-1 \leq |A_i \cap A_j| \leq |A_i| = |S(a_i)| = 2m$.

We claim that for all distinct pairs i and j, $1 \leq i, j \leq t$, we have $A_i \cap A_j \neq \emptyset$. Associate the graph G_Q to the poset Q as follows: $A_1, \ldots, A_t$ are vertices of this graph, and we connect vertices A_i and A_j if and only if $A_i \cap A_j \neq \phi$. Since $Q - \{\hat{0}, \hat{1}\}$ is connected, we conclude that G_Q is a connected graph. If $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are different edges of G_Q. We have that $\{A_i, A_k\}$ is also an edge of G_Q. $|A_i \cap A_j| \geq 2m-1$ as well as $|A_j \cap A_k| \geq 2m-1$. On other af, since $|A_i| = |A_j| = |A_k| = 2m$, we conclude that $A_i \cap A_k \neq \phi$. Therefore $\{A_i, A_k\}$ is also an edge of G_Q. As a consequence, the connected graph G_Q is a complete graph. Thus $A_i \cap A_j \neq \phi$ and also $2m-1 \leq |A_i \cap A_j| \leq 2m$, where $1 \leq i, j \leq t$ and $i \neq j$.

Now we show that $|A_i \cap A_j| = 2m-1$ for $i \neq j$. Suppose this claim does not hold. Then there are different i, j such that $|A_i \cap A_j| = 2m$. We claim that there are two elements of rank $2m-1$ in Q such that they both are covered by coatoms a_i and a_j. To prove this claim, consider an atom $x_f \in A_i \cap A_j$, so $[x_f, \hat{1}] \cong B_{2m}$. Hence, there is a unique element c_h of rank $2m-2$ in this interval which is covered by both a_i and a_j. By induction on m, Lemma 3.2.8, and the property that $|C_h| \leq |A_i \cap A_j| = 2m$, we conclude that $[\hat{0}, c_h]$ is isomorphic to B_{2m-1} and so $|C_h| = 2m-1$. Therefore there is an atom $x_d \in A_i \cap A_j \setminus C_h$. Since the interval $[x_d, \hat{1}]$ is isomorphic to B_{2m}, there is an element $c_k \neq c_h$ of rank $2m-1$ which is covered by coatoms a_i and a_j .

Since $|C_h| = |S(c_h)| = |C_k| = |S(c_k)| = 2m-1$ and C_k, C_h are both subsets of $A_i \cap A_j$, we conclude that there should be an atom $x_s \in C_k \cap C_h$. Therefore the interval $[x_s, \hat{1}]$ has two elements c_k and c_h of rank $2m-2$ such that they both are covered by two elements a_i and a_j of rank $2m-1$ in the interval $[x_s, \hat{1}]$. We know $[x_s, \hat{1}] \cong B_{2m}$ and there are no two elements of rank $2m-2$ covered by two elements of rank $2m-1$ in B_{2m}. This contradicts our assumption, and so $|A_i \cap A_j| = 2m-1$ for pairs i, j of distinct elements.

In summary:

(a) $|A_i| = 2m$ for $1 \leq i \leq t$,

(b) $|A_i \cap A_j| = 2m-1$ for all $1 \leq i < j \leq t$,

(c) $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$.

As a consequence, we have $t > 2m$.

Next, we are going to show that $t = 2m+1$. Without loss of generality, consider the three different sets $A_1 = S(a_1), A_2 = S(a_2)$ and $A_3 = S(a_3)$ associated with the three coatoms a_1, a_2 and a_3. We know that $|A_1| = |A_2| = |A_3| = 2m$ and

$|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 2m - 1$. Without loss of generality, let us that assume $A_1 = \{x_1, x_2, \ldots, x_{2m-1}, y_1\}$ and $A_2 = \{x_1, x_2, \ldots, x_{2m-1}, y_2\}$ where $y_i \neq x_1, \ldots, x_{2m-1}$ for $i = 1, 2$. We have two different cases. Either A_3 contains at least one of y_1 and y_2, or A_3 contains neither of them. First we study the second case, $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_3\}$ where $y_3 \notin \{y_1, y_2, x_1, \ldots, x_{2m-1}\}$. Considering the $t - 3$ other coatoms a_k, $4 \leq k \leq t$, there are different atoms y_k, $4 \leq k \leq t$, such that $y_k \notin \{y_1, y_2, y_3, x_1, \ldots, x_{2m-1}\}$ and $A_k = S(a_k) = \{x_1, x_2, \ldots, x_{2m-1}, y_k\}$. This implies that the number of atoms is $|\bigcup_{i=1}^{t} A_i| = t + 2m - 1$, which is a contradiction. Hence only the first case can happen and A_3 should contain one of y_1 or y_2. In this case $|A_2 \cap A_3| = |A_1 \cap A_3| = 2m - 1$ implies that $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_1, y_2\} \setminus \{x_j\} \subset A_1 \cup A_2$ for some x_j. Since A_3 was chosen arbitrarily, it follows that for each A_k we have $A_k \subset A_1 \cup A_2$. Hence

$$\bigcup_{i=1}^{t} A_k = \{x_1, \ldots, x_{2m-1}, y_1, y_2\}. \tag{3.7}$$

Thus the number of coatoms in the poset Q is $t = 2m + 1$. By Theorem 3.2.11, $B_Q(k) = k!$ for $1 \leq k \leq 2m$, therefore $B_Q(2m + 1) = (2m + 1)!$. By Proposition 3.2, Q is isomorphic to B_{2m+1} and so P is a union of copies of B_{2m+1} by identifying their minimal elements and their maximal elements. In other words, $P \cong \boxplus^k(B_{2m+1})$. It can be seen that P is binomial and Eulerian and the proof follows.

(*ii*) $B(3) = 4$. With the same argument as part (*i*), we construct the binomial poset Q by adding $\hat{1}$ and $\hat{0}$ to one of the connected components of $P - \{\hat{0}, \hat{1}\}$. We claim that Q is isomorphic to T_{2m+1}. Similar to part (*i*), let $a_1, \ldots, a_t$ and $x_1, \ldots, x_t$ denote coatoms and atoms of Q. Set $A_i = S(a_i)$. By Theorem 3.2.11, $|A_i| = 2$. It is easy to see that $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$. Define G_Q to be the graph with vertices $x_1, \ldots, x_t$ and edges $A_1, \ldots, A_t$. Since $Q \setminus \{\hat{0}, \hat{1}\}$ is a connected component, G_Q is a connected graph. Since $[x_i, \hat{1}] \cong T_{2m}$, the degree of each vertex of G_Q is 2 and G_Q is the cycle of length t. Therefore if $t > 2$,

$|A_i \cap A_j| = 1$ or 0, $1 \leq i < j \leq t$.

We claim $t = 2$. Suppose this claim does not hold, so $t > 2$. Consider an element c of rank 3 in Q. Lemma 3.2.8 and Theorem 3.2.11 imply that the both intervals $[\hat{0}, c]$ and $[c, \hat{1}]$ are isomorphic to butterfly posets. Hence there are two coatoms above c, say a_k and a_l, and similarly there are two atoms below c, say x_h and x_s, that is, $A_k = A_l = \{x_h, x_s\}$. This is not possible when $t > 2$. As a consequence, $t = 2$ and all the A_i's have 2 elements and $|\bigcup_1^t A_i| = |\{x_1, \ldots, x_t\}| = 2 = t$.

Similar to part (i), $B_Q(k) = 2^{k-1}$ for $1 \leq k \leq 2m+1$. By Proposition 3.2, we conclude that Q is isomorphic to T_{2m+1}. Therefore, there is an integer $k > 0$ such that $P \cong \boxplus^k(T_n)$.

3.3 Finite Eulerian Sheffer posets

In this section, we give an almost complete classification of the factorial functions and the structure of finite Eulerian Sheffer posets.

First, we provide some examples of Eulerian Sheffer posets. We study Eulerian Sheffer posets of ranks $n = 3$ and 4 in Lemmas 3.3.3 and 3.3.4. By these two lemmas, we reduce the values of $B(3)$ to 4 or 6. In Subsection 3.3.1, Lemma 3.3.5 and Theorems 3.3.6, 3.3.10, 3.3.11, 3.3.12 deal with Eulerian Sheffer posets with $B(3) = 6$. Finally in Subsection 3.3.2, Theorems 3.3.13, 3.3.14, 3.3.15 deal with Eulerian Sheffer posets with $B(3) = 4$.

The results of this section are summarized below.

Let P be a Eulerian Sheffer poset of rank n. Then P satisfies one of following conditions.

(a) $n = 3$. $P \cong \boxplus_{i=1\ldots k} P_{q_i}$ for some $q_1, \ldots, q_r$ such that $q_i \geq 2$.

(b) $n = 4$. The complete classification of factorial functions of the poset P follows from Lemma 3.3.4.

(c) n is odd and $n \geq 4$. Then one of the following is true:

i. $B(3) = D(3) = 6$. Then $P \cong \boxplus^\alpha(B_n)$ for some α.

ii. $B(3) = 6, D(3) = 8$. This case is open.

iii. $n = 5$, $B(3) = 6, D(3) = 10$. This case remains open.

iv. $B(3) = 6, D(3) = 4$. Then $P \cong \boxplus^\alpha(\Sigma^*(B_{n-1}))$ for some α.

v. $B(3) = 4$. The classification follows from Theorems 3.11 and 3.13 in [ER2].

(d) n is even and $n \geq 6$. Then one of the following is true:

i. $B(3) = D(3) = 6$. Then $P \cong B_n$.

ii. $B(3) = 6, D(3) = 8$. The poset P has the same factorial function as the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$ and $B(k) = k!$.

iii. $B(3) = 6, D(3) = 4$. Then $P \cong \Sigma^*(\boxplus^\alpha(B_{n-1}))$ for some α.

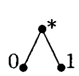
iv. $B(k) = 2^{k-1}$, for $1 \leq k \leq 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$ for some $\alpha > 1$. In this case P is isomorphic to $\Sigma^* \boxplus^\alpha(T_{2m+1})$.

v. $B(k) = 2^{k-1}$, $1 \leq k \leq 2m+1$. The classification follows from Theorems 3.11 and 3.13 in [ER2]

It is clear that every binomial poset is also a Sheffer poset. Here are some other examples of Sheffer posets, some of which appear in [ER2, Rei].

Example 3.3.1. Let P be a binomial poset of rank n with the factorial functions $B(k)$. By adjoining a new minimal element $\widehat{-1}$ to P, we obtain a Sheffer poset of rank $n+1$ with binomial factorial functions $B(k)$ for $1 \leq k \leq n$ and Sheffer factorial functions, $D(k) = B(k-1)$ for $1 \leq k \leq n+1$. $\diamondsuit$

Example 3.3.2. Let T be the following three element poset:



.

Let T^n be the Cartesian product of n copies of the poset T. The poset $C_n = T^n \cup \{\hat{0}\}$ is the face lattice of an n-dimensional cube, also known as the *cubical lattice*. The cubical lattice is a Sheffer poset with $B(k) = k!$ for $1 \leq k \leq n$ and $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq n+1$. $\diamondsuit$

The number of elements of rank k in a Sheffer interval of rank n is

$$\frac{D(n)}{D(k)B(n-k)}. \tag{3.8}$$

Let P be an Eulerian Sheffer poset of rank n. The Euler-Poincaré relation for every m-Sheffer interval, $2 \leq m \leq n$, becomes

$$1 + \sum_{k=1}^{m} (-1)^k \cdot \frac{D(m)}{D(k)B(m-k)} = 0. \tag{3.9}$$

It is clear that B_2 is the only Eulerian Sheffer poset of rank 2.

In the next lemma, we characterize the structure of Eulerian Sheffer posets of rank 3. The characterization of the factorial function is an immediate consequence.

Lemma 3.3.3. *Let P be a Eulerian Sheffer poset of rank* 3.

- *(i) The poset P has the factorial function $D(2) = 2$ and $D(3) = 2q$, where q is a positive integer such that $q \geq 2$.*
- *(ii) There is a list of integers $q_1, \ldots, q_r$, $q_i \geq 2$ such that $P \cong \boxplus_{i=1,\ldots,r} P_{q_i}$, where P_{q_i} denotes the face lattice of a q_i-gon.*

Proof. Consider an Eulerian Sheffer poset P of rank 3. Now $P - \{\hat{0}, \hat{1}\}$ consists of elements of rank 1 and rank 2 of P. By the Euler-Poincaré relation, it is easy to see that $B(2) = 2$ and every interval of length 2 in P is isomorphic to B_2. So in $P - \{\hat{0}, \hat{1}\}$, every element of rank 2 is connected to two elements of rank 1 and vice-versa. Therefore, the Hasse diagram of $P - \{\hat{0}, \hat{1}\}$ decomposes as the disjoint union of cycles of even lengths $2q_1, \ldots, 2q_r$ where $q_i \geq 2$. We conclude that the poset P is obtained by identifying all minimal elements of the posets $P_{q_1}, \ldots, P_{q_r}$ and identifying all of their maximal elements. Hence $P \cong \boxplus_{i=1,\ldots,r} P_{q_i}$ and $D(3) = 2(q_1 + \cdots + q_r)$. Thus every Eulerian Sheffer poset of rank 3 has the factorial functions $D(3) = 2q$ where $q \geq 2$ and $B(2) = D(2) = 2$. $\square$

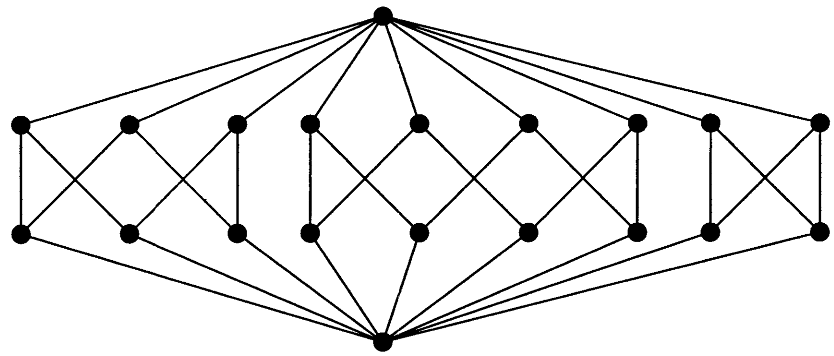


Figure 3-4: $P \cong \boxplus_{k=2,\ldots,4} P_k$.

Lemma 3.3.4 deals with Eulerian Sheffer posets of rank 4.

Lemma 3.3.4. *Let poset P be an Eulerian Sheffer poset of rank 4. Then one of the following conditions hold.*

(a) $B(3) = 2b$, $D(3) = 4$, $D(4) = 4b$, *where* $b \geq 2$.

(b) $B(3) = 8$, $D(3) = 3!$, $D(4) = 2^3 \cdot 3!$.

(c) $B(3) = 10$, $D(3) = 3!$, $D(4) = 5!$.

(d) $B(3) = 4$, $D(3) = 3!$, $D(4) = 2 \cdot 3!$.

(e) $B(3) = 3!$, $D(3) = 3!$, $D(4) = 4!$.

(f) $B(3) = 3!$, $D(3) = 4$, $D(4) = 2 \cdot 3!$.

(g) $B(3) = 3!$, $D(3) = 10$, $D(4) = 5!$.

(h) $B(3) = 3!$, $D(3) = 8$, $D(4) = 2^3 \cdot 3!$.

(i) $B(3) = 4$, $D(3) = 2b$, $D(4) = 4b$ *where* $b \geq 2$.

Proof. Let P be an Eulerian Sheffer poset of rank 4. Note that for every Eulerian Sheffer poset $B(1) = D(1) = 1$ as well as $B(2) = D(2) = 2$. The variables a, b and c denote the number of elements of rank 1, 2 and 3 of P, respectively. By the Euler-Poincaré relation $2 + b = a + c$. The number of maximal chains in P is given by $4b = B(3)a = D(3)c$. Lemma 3.3.3 implies that there are positive integers k_1, k_2 such that $D(3) = 2k_2$ and $B(3) = 2k_1$. Thus $b + 2 = (\frac{2}{k_1} + \frac{2}{k_2})b$. We conclude that $\frac{2}{k_1} + \frac{2}{k_2} > 1$; therefore the case k_1 and k_2 cannot both be greater than 3. Next we study the remaining cases as follows.

(1) $k_2 = 1$ or $k_1 = 1$. If $k_2 = 1$, then $c = 2b$ and $2b \leq b+2$. Therefore $b = 1, 2$, and we have the following cases:

   i. $b = 1$. Then $a = 1$ and $c = 2$, so the Sheffer interval of length 2 in P does not satisfy the Euler-Poincaré relation. This case is not possible.

   ii. $b = 2$. Then $c = 4$ and $a = 0$, which is not possible.

   Similarly, the case $k_1 = 1$ is not possible.

(2) $k_2 = 2$ or $k_1 = 2$. If $k_2 = 2$, then $2b = 2c$, so $c = b$, $a = 2$ and $k_1 = b$. The fact that every interval of rank 2 is isomorphic to B_2 implies that $b \geq 2$. Thus $B(1) = 1$, $B(2) = 2$ and $B(3) = 2b$, as well as $D(1) = 1$, $D(2) = 2$, $D(3) = 4$, and $D(4) = 4b$. The poset $T \cong \Sigma^*(P_b)$, where P_b is the lattice of b-polygon, is an Eulerian Sheffer poset with the described factorial functions.

   Similarly, in case $k_1 = 2$, the poset P has the same factorial functions as $\Sigma(P_b)$. That is $B(3) = 4$, $D(3) = 2b$ and $D(4) = 4b$.

(3) $k_2 = 3$. The equation $b + 2 = a + c = (\frac{2}{k_1} + \frac{2}{3})b$ implies that $k_1 < 6$, so we need to consider the following five cases.

   i. $k_1 = 5$. Then $b + 2 = \frac{2}{5}b + \frac{2}{3}b$, so $\frac{1}{15}b = 2$, $b = 30$, $c = 20$ and $a = 12$. Thus P has the factorial functions $B(3) = 10$, $D(3) = 3!$ and $D(4) = 5!$. The face lattice of icosahedron is an Eulerian Sheffer poset with the same factorial functions.

   ii. $k_1 = 4$. The poset P has the same factorial functions as the dual of the cubical lattice of rank 4, that is, $B(3) = 8$, $D(3) = 3!$ and $D(4) = 2^3 \cdot 3!$.

   iii. $k_1 = 3$. The poset P has the factorial functions $B(3) = 3!$, $D(3) = 3!$ and $D(4) = 4!$. Thus P is isomorphic to B_4.

iv. $k_1 = 2$. The poset P has the same factorial functions as $\Sigma(B_3)$, $B(3) = 4$, $D(3) = 3!$ and $D(4) = 2 \cdot 3!$.

(4) $k_1 = 3$. Then $b + 2 = (\frac{2}{k_1} + \frac{2}{k_2})b$ implies that $k_2 < 6$, so we have the following five cases.

i. $k_2 = 5$. Similar to the case $k_1 = 5$ and $k_2 = 3$, the poset P has the same factorial functions as the face lattice of a dodecahedron, $B(3) = 3!$, $D(3) = 10$ and $D(4) = 5!$.

ii. $k_2 = 4$. Similar to the case $k_1 = 4$ and $k_2 = 3$, the poset P has the same factorial functions as the cubical lattice of rank 4. That is, $B(3) = 3!$, $D(3) = 8$ and $D(4) = 2^3 \cdot 3!$.

iii. $k_2 = 3$. The poset P has the factorial functions $B(3) = 3!$, $D(3) = 3!$ and $D(4) = 4!$. So $P \cong B_4$.

iv. $k_2 = 2$. Similar to the case $k_1 = 2$ and $k_2 = 3$, the poset P has the same factorial functions as $\Sigma^*(B_3)$, $B(3) = 3!$, $D(3) = 4$ and $D(4) = 2 \cdot 3!$.

□

3.3.1 Characterization of the factorial functions and structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$.

In this section we consider Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 3!$. Lemma 3.3.5 shows that for any such poset of rank $n \geq 5$, the Sheffer factorial function $D(3)$ can only take the values $4, 6$ or 8. In Subsections 3.3.1, 3.3.1 and 3.3.1, we consider the three different cases $D(3) = 4, 6, 8$, respectively.

Lemma 3.3.5. *Let P be a Eulerian Sheffer poset of rank $n \geq 6$ with $B(3) = 3!$. Then $D(3)$ can take only the values $4, 6, 8$.*

Proof. By Lemma 3.3.4, the Sheffer factorial function of poset P for Sheffer 3-intervals can take the following values $D(3) = 4, 6, 8, 10$. We claim that the case $D(3) = 10$ is not possible. Suppose there is an Eulerian Sheffer poset P of rank of at least 6 with the factorial functions $D(3) = 10$ and $B(3) = 3!$. By Lemma 3.3.4, the poset P has the following factorial functions: $D(1) = 1$, $D(2) = 2$, $D(3) = 10$, $D(4) = 5!$, $B(1) = 1$, $B(2) = 2!$ and $B(3) = 3!$. Set $C(6) = E$, $C(5) = F$, where $C(5)$ and $C(6)$ are coatom functions of P. By Theorems 3.2.11 and 3.2.12, we conclude that there is an integer $\alpha > 0$ such that $B(4) = 4!$ and $B(5) = \alpha \cdot 5!$. The Euler-Poincaré relation implies that

$$1 + \sum_{k=1}^{6} (-1)^k \cdot \frac{D(6)}{D(k)B(6-k)} = 0,$$

therefore, by substituting the values in above equation, we have

$$2 = \frac{EF}{\alpha} - EF + E, \alpha(E-2) = (\alpha - 1)EF. \tag{3.10}$$

There are two cases to consider:

(a) $\alpha = 1$. Eq.(3.10) implies that $E = 2$. However, $E \geq A(5) = 5$ where $A(5)$ is an atom function of B_5. This case is not possible.

(b) $\alpha > 1$. By Eq.(3.10),

$$\left(\frac{\alpha}{\alpha - 1}\right)\left(\frac{E-2}{E}\right) = F.$$

$E \geq A(5) = 5$ implies that $F < 4$. On the other hand, since $F \geq A(4) \geq 4$. This case is also not possible.

We conclude that there is no Eulerian Sheffer poset of rank at least 6 with $D(3) = 10$ and $B(3) = 3!$, as desired. $\square$

Characterization of the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$.

In this subsection, we study the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$. Theorem 3.3.6 characterizes the factorial functions of such posets of even rank. However, the question of characterizing the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geq 5$ with $B(3) = 3!$ and $D(3) = 8$ remains open.

Theorem 3.3.6. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 \geq 4$ with $B(3) = 3!$ and $D(3) = 8$. Then P has the same factorial functions as C_n, the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$, $1 \leq k \leq n$ and $B(k) = k!$, $1 \leq k \leq n-1$.*

In order to prove Theorem 3.3.6, we establish the following two lemmas.

Lemma 3.3.7. *Let Q be an Eulerian Sheffer poset of odd rank $2m+1$, $m \geq 2$, with $B(3) = 3!$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$ for $2 \leq n \leq 2m$ and $C(2m+1) \neq 4m+1$.*

Proof. We proceed by contradiction. Assume Q is such a poset. Theorem 3.2.11 implies that Q has the binomial factorial function $B(k) = k!$ for $1 \leq k \leq 2m$. By Eq.(3.8) we enumerate the number elements of ranks 1, $2m-1$ and $2m$ in this Sheffer poset. Let $\{a_1, \ldots, a_{4m+1}\}$, $\{e_1, \ldots, e_{(4m+1)(2m-1)}\}$ and $\{x_1, \ldots, x_t\}$ denote the sets of elements of rank $2m$, $2m-1$ and 1 in Q, respectively, where $t = \frac{4m+1}{2m} \cdot 2^{2m-1}$. For each element y of rank at least 2, let $S(y)$ be the set of atoms in $[\hat{0}, y]$. Set $A_j = S(a_j)$ for each element a_j of rank $2m$ and also $E_j = S(e_j)$ for each element e_j of rank $2m-1$. Eq.(3.8) implies that $|S(y)| = 2^{r-1}$ for any element y of rank $2 \leq r \leq 2m$.

Assume that $A_i \cap A_j \neq \phi$ for some $1 \leq i, j \leq 4m+1$. Consider $x_a \in A_i \cap A_j$, then the interval $[x_a, \hat{1}]$ is isomorphic to B_{2m}. Therefore, there is an element $\alpha_{i,j}$ which is covered by both a_i and a_j. Assume that there is $x_b \in A_i \cap A_j$ so that $x_b \neq x_a$. Thus, there is an element $\beta_{i,j} \in [x_b, \hat{1}]$ which is covered by both a_i and a_j. In case $\alpha_{i,j} \neq \beta_{i,j}$, we can see $S(\alpha_{i,j})$ and $S(\beta_{i,j})$ are disjoint sets, therefore $A_i = A_j = S(\alpha_{i,j}) \cup S(\beta_{i,j})$. Informally speaking, when $A_i \cap A_j \neq \phi$ and $A_i \neq A_j$, we denote the unique $\alpha_{i,j}$ by $a_i \wedge a_j$ which is covered by both a_i and a_j. In this case $A_i \cap A_j$ is the set of atoms which are below $a_i \wedge a_j$. Thus,

$$|A_i \cap A_j| = |S(a_i \wedge a_j)| = 2^{rank(a_i \wedge a_j)-1} = 2^{2m-2}. \tag{3.11}$$

We claim that for all pairs i, j satisfying $i \neq j$ and $1 \leq i, j \leq 4m+1$, the intersection of the atom sets satisfies $A_i \cap A_j \neq \phi$. Suppose this claim does not hold. Then there exist two different s, l such that $|A_s \cap A_l| = 0$ where $1 \leq s, l \leq 4m+1$. Since $|A_s| + |A_l| < t$, there is a set $A_k = S(a_k)$ such that $A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l) \neq \phi$, $1 \leq k \leq 4m+1$.

Therefore we have $|A_l \cap A_k|, |A_s \cap A_k| \leq 2^{2m-2}$. Furthermore, since $|\{x_1, \ldots, x_t\}| = t = \frac{4m+1}{2m} \cdot 2^{2m-1}$, $|A_l| = |A_s| = |A_k| = 2^{2m-1}$ and $|A_l \cap A_s| = 0$, we conclude that

$$|A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l)| \leq |\{x_1, \ldots, x_t\} - A_s \cup A_l| = \frac{2^{2m-1}}{2m}. \tag{3.12}$$

By Eq.(3.12), $|A_l \cap A_k|, |A_s \cap A_k| \neq 0$ and $A_k \neq A_l, A_s$. Therefore, Eq.(4.5.3) implies that $|A_l \cap A_k| = 2^{2m-2}$ and $|A_s \cap A_k| = 2^{2m-2}$.

We have seen $|A_s \cap A_k| = 2^{2m-2}$, $|A_l \cap A_k| = 2^{2m-2}$ and $|A_k| = 2^{2m-1}$. Moreover, since we assumed $A_s \cap A_l = \phi$, we conclude that $A_k = A_s \cup A_l$. On the other hand $A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l) \neq \phi$, which is not possible when $A_k = A_s \cup A_l$. This contradicts our assumption. Therefore $|A_i \cap A_j| \neq 0$ for $1 \leq i, j \leq 4m+1$. So for every distinct pair a_i and a_j, there is an atom $x_h \in A_i \cap A_j$. As above

$[x_h, \hat{1}] \cong B_{2m}$, so there is at least one element of rank $2m - 2$ in this interval, denoted by e_k, $1 \leq k \leq (4m+1)(2m-1)$, and it is covered by both a_i and a_j. In addition, for every element e_l of rank $2m-1$ in the poset Q the interval $[e_l, \hat{1}]$ is isomorphic to B_2. As a consequence, for every e_l there is exactly one pair a_i, a_j such that e_l is covered by them. Hence, the number of the disjoint pairs of elements of rank $2m$ in the poset Q is at most the number of elements of rank $2m-1$. That is, $(4m+1)(2m-1) \geq (4m+1)(2m)$ which is not possible. This contradicts the assumption. So there is no poset Q with the described factorial and coatom functions, as desired. $\square$

Lemma 3.3.7 implies the following. Let P be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$, with $B(k) = k!$, for $1 \leq k \leq 2m$. Then the coatom function of P must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$, $2 \leq n \leq 2m$, $C(2m+1) \neq 4m+1$ and $C(2m+2) \neq 4(2m+1)$.

Lemma 3.3.8. *Let Q be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$, with $B(k) = k!$ for $1 \leq k \leq 2m$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$, $2 \leq n \leq 2m$, $C(2m+1) \neq 4m-1$ and $C(2m+2) \neq \frac{4}{3}(2m+1)$.*

Proof. We proceed by contradiction. Suppose Q is such a poset of rank $2m+2$ with the described factorial functions. We enumerate the number of elements of rank $2m+2-k$ in Q as follows:

$$\frac{D(2m+2)}{B(k)D(2m+2-k)} = \frac{C(2m+2)\cdots C(2m+2-k+1)}{k!}. \tag{3.13}$$

Thus, $\{a_1, \ldots, a_{\frac{4}{3}(2m+1)}\}$ and $\{e_1, \ldots, e_{\frac{4}{6}(2m+1)(4m-1)}\}$ are the sets of elements of rank $2m+1$ and $2m$ in Q, respectively. For every element e_i of rank $2m$, the interval $[e_i, \hat{1}]$ is isomorphic to B_2. So each element of rank $2m$ covered by exactly two different elements of rank $2m+1$.

There are exactly $\frac{4}{6}(2m+1)(4m-1)$ elements of rank $2m$ in Q, and we also know that there are $(\frac{4}{6}(2m+1))(\frac{4}{3}(2m+1)-1)$ different pairs of coatoms $\{a_i, a_j\}$ in Q, $1 \leq i < j \leq \frac{4}{3}(2m+1)$. We conclude there are at least two different coatoms a_k, a_l such that they both cover two different elements e_i, e_j of rank $2m$. The interval $T = [\hat{0}, a_k]$ has binomial factorial function $B_T(k) = k!$ for $1 \leq k \leq 2m$ and coatom function $C_T(n) = 2(n-1)$ for $2 \leq n \leq 2m$ and $C_T(2m+1) = 4m-1$. Let $\{y_1, \ldots, y_t\}$ be the set of atoms in the poset T where $t = \frac{(4m-1)}{2m} \cdot 2^{2m-1}$. Thus $A_k = \{y_1, \ldots, y_t\}$. Set $E_j = S(e_j), E_i = S(e_i)$, so $E_j, E_i \subset A_k$. By Eq.(3.8) $|E_i| = |E_j| = 2^{2m-1}$, therefore $|E_i| + |E_j| > |A_k|$. We conclude that there is at least one atom $y_1 \in T$ which is below the elements e_i, e_j and a_k.

Proposition 3.2 implies that $[y_1, a_k] \cong B_{2m}$. By the boolean lattice properties, there is an element c of rank $2m-2$ in the interval $[y_1, a_k]$ such that c is covered by the elements e_i and e_j. By Proposition 3.2, $[c, \hat{1}] \cong B_3$. Consider the interval $[c, \hat{1}]$. Let a_k and a_l be two elements of rank 2 in this interval which both cover two elements e_i and e_j of rank 1. This contradicts the fact that $[c, \hat{1}] \cong B_3$. We conclude that $[c, \hat{1}] \ncong B_3$, giving the desired contradiction. Hence there is no poset Q with describe conditions.

□

The following lemma can be obtained by applying the proof of Lemma 4.8 in [ER2].

Lemma 3.3.9. *Let P and P' be two Eulerian Sheffer posets of rank $2m+2$, $m \geq 2$, such that their binomial factorial functions and coatom functions agree up to rank $n \leq 2m$. That is, $B(n) = B'(n)$ and $C(n) = C'(n)$, where $m \geq 2$. Then the following equation holds:*

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{C'(2m+1)}\left(1 - \frac{1}{C'(2m+2)}\right). \quad (3.14)$$

Proof of Theorem 3.3.6. Let $C(k)$ and $C^{'}(k) = 2(k-1)$ respectively be the coatom functions of the Eulerian Sheffer poset P and C_{2m+2}, the cubical lattice of rank $2m+2$, for $2 \leq k \leq 2m+2$. We only need to show that $C(k) = C^{'}(k) = 2(k-1)$ for $2 \leq k \leq 2m+2$. We prove this claim by induction on m. By Lemma 3.3.4, $C(4) = C^{'}(4) = 6$ and the claim holds for $m = 1$. By the induction hypothesis $C(k) = C^{'}(k) = 2(k-1)$ for $2 \leq k \leq 2m$. Set $F = C(2m+1)$ and $E = C(2m+2)$. Theorem 3.2.12 implies that $B(k) = k!$ for $1 \leq k \leq 2m$ and there is a positive integer α such that $B(2m+1) = \alpha(2m+1)!$. We know that $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq 2m$, so $D(2m+1) = F2^{2m-1}(2m-1)!$ and $D(2m+2) = EF2^{2m-1}(2m-1)!$. Since P is an Eulerian Sheffer poset, the Euler-Poincaré relation implies that,

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{3.15}$$

By substituting the values of the factorial functions, we have

$$2 - E + \frac{EF}{2}\left[\frac{1}{2m} - \frac{1}{2m(2m+1)} + \frac{2^{2m}}{2m(2m+1)} - \frac{2^{2m}}{2\alpha m(2m+1)}\right] = 0. \tag{3.16}$$

Thus,

$$E\left(1 - F\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right)\right) = 2. \tag{3.17}$$

In case $\alpha \geq 2$, it is easy to verify that $\left(\frac{2\alpha m+(\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right) > \frac{1}{2m}$. Since $F \geq A(2m) \geq 2m$, the left-hand side of Eq.(3.17) becomes negative in this case. Therefore, $\alpha = 1$ and the posets P and C_{2m+2} have the same binomial factorial functions. Since $2m+1 = A(2m+1) \leq C(2m+2) < \infty$, Lemma 3.3.9 implies that $4m-1 \leq C(2m+1) = F \leq 4m+1$. Since $\alpha = 1$, Eq.(3.17) implies that $2 - E + \frac{EF}{4m+2} = 0$. Thus E and F must satisfy one of the following cases:

(1) $F = 4m-1$ and $E = \frac{4}{3}(2m+1)$.

(2) $F = 4m$ and $E = 4m+2$.

(3) $F = 4m + 1$ and $E = 4(2m + 1)$.

As we have discussed in Corollary 3.3.1 and Lemma 3.3.8, the cases (1) and (3) are not possible. Case (2) occurs in the cubical lattice of rank $2m + 2$, C_{2m+2}. Thus, the poset P has the same factorial functions as C_{2m+2}, as desired. $\square$

Classification of the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geq 5$ with $B(3) = 6$ and $D(3) = 8$ remains open. Let α be a positive integer and set $Q_\alpha = \boxplus^\alpha(C_{2m+1})$. It can be seen that Q_α is an Eulerian Sheffer poset and it has the following factorial functions: $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq n-1$, $D(n) = \alpha \cdot 2^{n-1}(n-1)!$ and $B(k) = k!$ for $1 \leq k \leq n-1$. We ask the following question:

Question: Let P be an Eulerian Sheffer poset of odd rank $n = 2m + 1 \geq 5$ with $B(3) = 6, D(3) = 8$. Is there a positive integer α such that the poset P has the same factorial functions as poset $Q_\alpha = \boxplus^\alpha(C_{2m+1})$, where C_{2m+1} is a cubical lattice of rank $2m + 1$?

Characterization of the structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$, and $D(3) = 3! = 6$.

In this section, we prove the following:

Theorem 3.3.10. *Let P be an Eulerian Sheffer poset of rank $n \geq 3$ with $B(3) = D(3) = 3! = 6$ for 3-intervals. P satisfies one of the following cases:*

(i) There is an integer $k \geq 1$ such that $P \cong \boxplus^k(B_n)$, where n is odd.

(ii) $P \cong B_n$, where n is even.

Proof. We proceed by induction on n. Theorem 3.3.3 and Lemma 3.3.4 imply that this theorem holds for $n = 3$ and 4. Assume that Theorem 3.3.10 holds for $n \leq m$, we wish to show that it also holds for $n = m + 1 \geq 5$. This problem divides into the following cases:

(i) $n = m + 1$ is odd. Consider the poset Q obtained by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0},\ \hat{1}\}$. So Q is an Eulerian Sheffer poset with $B(3) = D(3) = 3! = 6$. By induction hypothesis, every interval of rank $k \leq m$ is isomorphic to B_k. So the Sheffer and binomial factorial functions of Q and the boolean lattice of rank $m + 1$ agree up to rank $m = n - 1$. Therefore Q and also P are binomial posets. Theorem 3.2.12 implies there is a positive integer k such that $P \cong \boxplus^k(B_n)$, as desired.

(ii) $n = m + 1$ is even. We proceed by induction on n, the rank of P. Let $C(k)$ and $C'(k) = k$ be the coatom functions of the posets P and B_n, respectively, where $k \leq n$. By induction hypothesis $C(k) = C'(k)$ for $k \leq n - 2$. So, Lemma 3.3.9 implies that

$$\frac{1}{C(n-1)}\left(1 - \frac{1}{C(n)}\right) = \frac{1}{C'(n-1)}\left(1 - \frac{1}{C'(n)}\right). \qquad (3.18)$$

By the induction hypothesis, there is a positive integer α such that $C(n-1) = \alpha(n-1)$. Moreover, we know that $C'(n-1) = n - 1$ and $C'(n) = n$. Eq.(3.18) implies that $\alpha = 1$ and $C(n) = n$, so the poset P has the same factorial functions as B_n and $P \cong B_n$, as desired.

□

Characterization of the structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 4$.

Let P be an Eulerian Sheffer poset of rank $n \geq 5$, with $B(3) = 3!$ and $D(3) = 4$. In this section we show that in the case $n = 2m + 2$ the poset P satisfies $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$ for some integer $\alpha \geq 1$ and in the case $n = 2m + 1$, $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$, for some integer $\alpha \geq 1$.

Theorem 3.3.11. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 \geq 4$ with $B(3) = 3!$ and $D(3) = 4$. Then $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, where $\alpha = \frac{B(2m+1)}{(2m+1)!}$*

is a positive integer. Consequently the poset P has the following binomial and Sheffer factorial functions as (i) and (ii).

(i) $B(k) = k!$ *for* $1 \leq k \leq 2m$, *and* $B(2m+1) = \alpha(2m+1)!$,

(ii) $D(1) = 1$, $D(k) = 2(k-1)!$ *for* $2 \leq k \leq 2m+1$, *and* $D(2m+2) = 2\alpha(2m+1)!$.

Proof. By Theorem 3.2.12 we know that there is a positive integer α such that P has the binomial factorial function $B(2m+1) = \alpha(2m+1)!$ and $B(k) = k!$, for $1 \leq k < n = 2m+1$. We proceed by induction on m. The case $m = 1$ implies that $\alpha = 1$, $B(3) = 3!$ and $D(3) = 4$. By applying Lemma 3.3.4, it can be seen that the poset P has the same factorial functions as $\Sigma^*(B_3)$. Therefore, the poset P has two atoms and its binomial 3-intervals are isomorphic to B_3. We conclude that $P \cong \Sigma^*(B_3)$ and so the theorem holds for $m = 1$. In the case $m > 1$, by Theorem 3.2.12, the poset $Q \cong \boxplus^{\alpha}(B_{2m+1})$ is the only Eulerian binomial poset of rank $2m+1$ with the binomial factorial function $B(k) = k!$ for $1 \leq k \leq 2m$ and $B(2m+1) = \alpha(2m+1)!$, where α is a positive integer. Set $P' = \Sigma^*(Q) = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$. It can be seen that P' is an Eulerian Sheffer poset of rank $2m+2$ with coatom function $C'(2m+2) = \alpha(2m+1)$ and $C'(k) = (k-1)$ for $3 \leq k \leq 2m+1$, as well as $C'(2) = 2$.

By the induction hypothesis, the theorem holds. We wish to show it also holds for the coatom function of a poset P of rank $2m+2$ which satisfies the conditions of the theorem. By the induction hypothesis the coatom function is $C(k) = k-1$ for $3 \leq k \leq 2m$ and $C(2) = 2$. By substituting the values of $C'(2m+2)$ and $C'(2m+1)$ in Eq.(3.14) of Lemma 3.3.9, we have

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{2m}\left(1 - \frac{1}{\alpha(2m+1)}\right). \tag{3.19}$$

The poset P has the binomial factorial function $B(2m+1) = \alpha(2m+1)!$, where α is a positive integer, and $B(k) = k!$ for $1 \leq k < 2m+1$. We conclude that $A(2m+1) = \alpha(2m+1)$ and $A(2m) = 2m$. So $C(2m+2) \geq A(2m+1) =$

$\alpha(2m+1)$, as well as $C(2m+1) \geq A(2m) = 2m$. We claim that Eq.(3.19) implies that $C(2m+1) = 2m$ and $C(2m+2) = \alpha(2m+1)$. Assume that there are $k, s \geq 1$ so that $C(2m+1) = 2m+k$ and $C(2m+2) = \alpha(2m+1)+s$. (Note that $k = 0$ implies $s = 0$ and vice-versa.) Eq.(3.19) implies that

$$\frac{2m}{2m+k} = \left(\frac{\alpha(2m+1)-1}{\alpha(2m+1)}\right)\left(\frac{\alpha(2m+1)+s}{\alpha(2m+1)+s-1}\right). \tag{3.20}$$

It is easy to verify that in case $k, s \geq 1$, the right-hand side of Eq.(3.20) is always greater than the left-hand side. Thus, $k, s = 0$ and $C(2m+1) = 2m$ as well as $C(2m+2) = \alpha(2m+1)$. By the induction hypothesis, $D(k) = 2(k-1)!$ for $2 \leq k \leq 2m$. Since $C(2m+1) = 2m$ as well as $C(2m+2) = \alpha(2m+1)$, we conclude that P has the same factorial functions as the poset $P' = \Sigma^*(\boxplus^\alpha(B_{2m+1}))$.

Applying Eq.(3.8), the poset P has $\frac{D(2m+2)}{B(2m+1)} = 2$ elements of rank 1. Call them $\hat{0}_1$ and $\hat{0}_2$. Using Eq.(3.8), the number of elements of rank $1 \leq k \leq 2m+1$ in the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ is

$$\frac{\alpha(2m+1)!}{k!(2m+1-k)!}. \tag{3.21}$$

The intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ both have the factorial functions, $B(k) = k!$ for $1 \leq k \leq 2m$ and $B(2m+1) = \alpha(2m+1)!$. It can be seen that the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ satisfy the Euler-Poincaré relation and these intervals are Eulerian and binomial. Applying Theorem 3.2.12, one sees that both intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ are isomorphic to the poset $\boxplus^\alpha(B_{2m+1})$. Since the poset P has the same factorial functions as poset $P' = \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, Eq.(3.8) yields that the number of elements of rank $k+1$ in P is the same as the number of elements of rank k in the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ for $1 \leq k \leq 2m+1$, that is

$$\frac{\alpha(2m+1)!}{k!(2m+1-k)!}. \tag{3.22}$$

In summary, we have

(1) $[\hat{0}_1, \hat{1}] \cong [\hat{0}_2, \hat{1}] \cong Q \cong \boxplus^{\alpha}(B_{2m+1})$.

(2) The number of elements of rank $k+1$ in the poset P is the same as the number of elements of rank k in the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$, $1 \leq k \leq 2m+1$.

(3) The poset P has only two atoms $\hat{0}_1, \hat{0}_2$.

Statements $(1), (2), (3)$ imply that $P \cong P' = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$, as desired.

□

Theorem 3.3.12. *Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6$ and $D(3) = 4$. Then $P \cong \boxplus^{\alpha}(\Sigma^*(B_{2m}))$ for some positive integer α.*

Proof. We obtain the poset Q by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0},\ \hat{1}\}$. It is easy to see that Q is an Eulerian Sheffer poset and also P and Q have the same factorial functions and coatom function up to rank $2m$. That is, $B_Q(k) = B_P(k)$ and $D_Q(k) = D_P(k)$ for $1 \leq k \leq 2m$. We use $B(k)$, $D(k)$, $C(k)$, $A(k)$ to denote the factorial functions and the coatom function and atom function of Q.

By Theorem 3.2.11, the poset Q has the binomial factorial function $B(k) = k!$ for $1 \leq k \leq 2m$. We have $C(2m+1) \geq A(2m) = 2m$. Since every interval of rank 2 in the Q is isomorphic to B_2, it has at least two coatoms. For every coatom a_i in Q, Theorem 3.3.11 implies that the interval $[\hat{0}, a_i] \cong \Sigma^*(\boxplus^{\alpha}(B_{2m-1}))$. By considering the factorial functions we conclude that $\alpha = 1$ as well as $[\hat{0}, a_i] \cong \Sigma^*(B_{2m-1})$. Since Q is obtained by adding $\hat{0}, \hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$, we conclude that there are at least two particular coatoms a_1, a_2 such that there is an element $c \in [\hat{0}, a_1]$, $[\hat{0}, a_2]$ where $c \neq \hat{0}$. By considering the factorial functions of the interval $[c, \hat{1}]$, Theorems 3.2.11 and 3.2.12 imply that there is a positive integer k such that $[c, \hat{1}] \cong B_k$. Therefore, there is an element b of rank $k-2$ in $[c, \hat{1}]$ such that $b = a_1 \wedge a_2$. The element b is also an element of rank $2m-2$ in Q. The interval $[\hat{0}, b]$ is a subinterval of $[\hat{0}, a_1]$, so $[\hat{0}, b] \cong \Sigma^*(B_{2m-2})$. We conclude that the interval $[\hat{0}, b]$ only has two atoms, say x_1, x_2. Since $[\hat{0}, a_1] \cong [\hat{0}, a_2] \cong \Sigma^*(B_{2m-1})$, the intervals $[\hat{0}, a_1]$ and $[\hat{0}, a_2]$ only have two atoms x_1 and x_2.

Define a graph G_Q as follows; vertices of G_Q are coatoms of poset Q and two vertices (coatoms) a_i and a_j are adjacent in G_Q if and only if there is an element $d \neq \hat{0}$ such that $d \in [\hat{0}, a_i], [\hat{0}, a_j]$. Since Q is obtained by adding $\hat{0}, \hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$, G_Q is a connected graph. Thus, every coatom of rank $2m$ in Q is just above two atoms x_1, x_2 in Q. Hence the number of elements of rank 1 in poset Q is 2, and by Eq.(3.8),

$$\frac{C(2m+1)D(2m)}{B(2m)} = 2. \tag{3.23}$$

Thus, $C(2m+1) = 2m$ and also Q has the same factorial functions as $\Sigma^*(B_{2m})$. By the same argument as Theorem 3.3.11, we conclude that $Q \cong \Sigma^*(B_{2m})$. So $P \cong \boxplus^{\alpha}(\Sigma^*(B_{2m}))$ for some positive integer α, as desired. □

3.3.2 Characterization of the structure and factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 4$.

In this section, we characterize Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 4$. Let P be an Eulerian Sheffer poset of rank $n \geq 5$ with $B(3) = 4$. It can be seen that the poset P satisfies one of the cases:

(a) P has the following binomial factorial function $B(k) = 2^{k-1}$, where $1 \leq k \leq n-1$;

(b) n is even and there is a positive integer $\alpha > 1$ such that poset P has the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n-2$ and $B(n-1) = \alpha \cdot 2^{n-2}$ for some positive integer α.

As a consequence of Theorems 3.11 and 3.12 in [ER2], we can characterize posets in the case (i).

Theorem 3.3.15 deals with the case (ii). It shows that if the Eulerian Sheffer posets P of rank $n = 2m + 2$ has the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq 2m$ and $B(2m + 1) = \alpha \cdot 2^{2m}$ for some positive integer α, then $P \cong \Sigma^*(\boxplus^\alpha(T_{2m+1}))$. See Figure 3-5.

Given two ranked posets P and Q, define the rank product $P * Q$ by

$$P * Q = \{(x, z) \in P \times Q \ : \ \rho_P(x) = \rho_Q(z)\}.$$

Define the order relation by $(x, y) \leq_{P*Q} (z, w)$ if $x \leq_P z$ and $y \leq_Q w$. The rank product is also known as the Segre product; see [?].

Theorem 3.3.13. *[Consequence of Theorem 3.11 [ER2]] Let P be an Eulerian Sheffer poset of rank $n \geq 4$ with the binomial factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n - 1$. Then its coatom function $C(k)$ and P satisfy the following conditions:*

(i) $C(3) \geq 2$, *and a length 3 Sheffer interval is isomorphic to a poset of the form $P_{q_1,\ldots,q_r}$, as described before.*

(ii) $C(2k) = 2$, *for $\lfloor \frac{n}{2} \rfloor \geq k \geq 2$ and the two coatoms in a length $2k$ Sheffer interval cover exactly the same element of rank $2k - 2$.*

(iii) $C(2k + 1) = h$ *is an even positive integer for $\lfloor \frac{n-1}{2} \rfloor \geq k \geq 2$. Moreover, the set of h coatoms in a Sheffer interval of length $2k + 1$ partitions into $\frac{h}{2}$ pairs, $\{c_1, d_1\}, \{c_2, d_2\}, \ldots, \{c_{\frac{h}{2}}, d_{\frac{h}{2}}\}$, such that c_i and d_i cover the same two elements of rank $2k - 1$*

Theorem 3.3.14. *[Consequence of Theorem 3.12 [ER2]] Let P be an Eulerian Sheffer poset of rank $n > 4$ with the binomial factorial function $B(k) = 2^{k-1}$,*

$1 \leq k \leq n-1$ *and the coatom function* $C(k)$, $1 \leq k \leq n$. *Then a Sheffer* k*-interval* $[\hat{0}, y]$ *of* P *factors in the rank product as* $[\hat{0}, y] \cong (T_{k-2} \cup \{\hat{0}, \hat{-1}\}) * Q$, *where* $T_{k-2} \cup \{\hat{0}, \hat{-1}\}$ *denotes the butterfly interval of rank* $k-2$ *with two new minimal elements attached in order, and* Q *denotes a poset of rank* k *such that*

(*i*) *each element of rank* 2 *through* $k-1$ *in* Q *is covered by exactly one element,*

(*ii*) *each element of rank* 1 *in* Q *is covered by exactly two elements,*

(*iii*) *each element of even rank* 4 *through* $2\lfloor \frac{k}{2} \rfloor$ *in* Q *covers exactly one element,*

(*iv*) *each element of odd rank* r *from* 5 *through* $2\lfloor \frac{k}{2} \rfloor + 1$ *in* Q *covers exactly* $\frac{C(r)}{2}$ *elements, and*

(*v*) *each* 3*-interval* $[\hat{0}, x]$ *in* Q *is isomorphic to a poset of the form* $P_{q_1,\ldots,q_r}$, *where* $q_1 + \cdots + q_r = C(3)$.

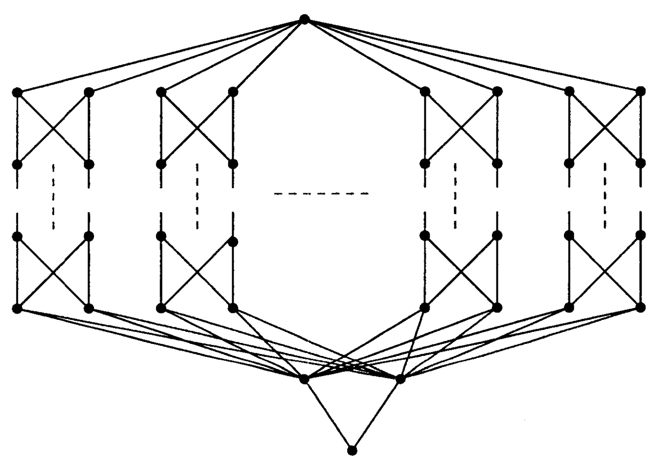


Figure 3-5: $P = \Sigma^*(\boxplus^\alpha(T_{2m+1}))$

In the following theorem we study the only remaining case (*ii*).

Theorem 3.3.15. *Let* P *be an Eulerian Sheffer poset of even rank* $n = 2m+2 > 4$ *with the binomial factorial function* $B(k) = 2^{k-1}$ *for* $1 \leq k \leq 2m$, *and* $B(2m+1) = \alpha \cdot 2^{2m}$, *where* $\alpha > 1$ *is a positive integer. Then* $P \cong \Sigma^*(\boxplus^\alpha(T_{2m+1}))$.

Proof. Let $D(k)$ where $1 \leq k \leq 2m+2$, and also $B(k)$ where $1 \leq k \leq 2m+1$, be the Sheffer and binomial factorial functions of the poset P, respectively. The Euler-Poincaré relation for an interval of length $2m+2$ is:

$$1 + \sum_{k=1}^{2m+2} (-1)^k \cdot \frac{D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{3.24}$$

The above Euler-Poincaré relation for the interval of even rank $2m+2$ can also be stated as follows:

$$\frac{2}{D(2m+2)} + \sum_{k=1}^{2m+1} \frac{(-1)^k}{D(k)B(2m+2-k)} = 0. \tag{3.25}$$

This is [ER2, Eq.(3.2)]. By expanding the left side of Eq.(3.25), we have:

$$\frac{(-1)}{\alpha \cdot 2^{2m}} + \sum_{k=2}^{2m+2} \frac{(-1)^k}{D(k) \cdot 2^{2m+2-k-1}} = 0. \tag{3.26}$$

Here, Eq.(3.25) for Sheffer $2m$-intervals can be stated as follows,

$$\sum_{k=1}^{2m} \frac{(-1)^k}{D(k) \cdot 2^{2m-1-k}} = 0. \tag{3.27}$$

Thus,

$$\frac{1}{2^{2m}} = \sum_{k=2}^{2m} \frac{(-1)^k}{D(k) \cdot 2^{2m+1-k}}. \tag{3.28}$$

It follows from Eq.(3.26) and Eq.(3.28) that

$$\frac{-1}{\alpha \cdot 2^{2m}} + \frac{1}{2^{2m}} + \frac{-1}{D(2m+1)} + \frac{2}{D(2m+2)} = 0. \tag{3.29}$$

Let k be the number of atoms in a Sheffer interval of size $2m+1$ and $c = C(2m+2)$, we thus have $D(2m+1) = k \cdot 2^{2m-1}$ and $D(2m+2) = c \cdot k \cdot 2^{2m-1}$. Therefore

$$\frac{1}{2^{2m}} - \frac{1}{\alpha \cdot 2^{2m}} = \frac{1}{k \cdot 2^{2m-1}} - \frac{1}{\frac{c \cdot k}{2} \cdot 2^{2m-1}}. \tag{3.30}$$

Thus,

$$\frac{1}{2} - \frac{1}{2\alpha} = \frac{1}{k} - \frac{1}{\frac{c}{2} \cdot k}. \tag{3.31}$$

Comparing coatom and atom functions of the Sheffer and binomial intervals, we have $k \geq 2$ as well as $c \geq 2\alpha$.

In case $k \geq 4$, we have

$$\frac{1}{2} - \frac{1}{k} > \frac{2}{c \cdot k} - \frac{1}{2\alpha} \tag{3.32}$$

which is not possible. In case $k = 3$, Eq.(3.31) only holds when $c = 2$ and $\alpha = 1$. Since $\alpha > 1$, this case is also not possible. Therefore, we conclude that $k = 2$ and so $c = 2\alpha$. Thus, every Sheffer j-interval has two atoms for $1 \leq j \leq 2m+1$.

We thus have $D(k) = 2B(k-1) = 2^{k-1}$ for $2 \leq k \leq 2m+1$ as well as $D(2m+2) = \alpha \cdot 2^{2m+1}$. Let $\hat{0}_1$, $\hat{0}_2$ be atoms of P. By Theorem 3.2.12, both of the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ are isomorphic to the poset $Q = \boxplus^{\alpha}(T_{2m+1})$. It follows from Eq.(3.8) that the number of elements of rank $k-1$ in the intervals $[\hat{0}_1, \hat{1}] \cong [\hat{0}_2, \hat{1}] \cong Q$ is the same as the number of elements of rank k in poset P and it can be computed as follows,

$$\frac{D(2m+2)}{D(k)B(2m+2-k)} = \frac{B(2m+1)}{B(k)B(2m+1-k)}. \tag{3.33}$$

We know that $\hat{0}_1$,$\hat{0}_2$ are the only atoms in P, so by the above fact we conclude that $P \cong \Sigma^* Q = \Sigma^*(\boxplus^{\alpha}(T_{2m+1}))$, as desired.

□

3.4 Eulerian Triangular posets

As we discussed before, the larger class of posets to consider are triangular posets. For definitions regarding triangular posets, see Chapter 2. A non-Eulerian example of triangular poset is the the face lattice of the 4-dimensional regular polytope known as the 24-cell. In the following theorem, we characterize the Eulerian triangular posets of rank $n \geq 4$ such that $B(k, k+3) = 6$ for $1 \leq k \leq n-3$.

Theorem 3.4.1. *Let P be an Eulerian triangular poset of rank $n \geq 4$ such that for every $0 \leq k \leq n-3$, $B(k, k+3) = 6$. Then P can be characterized as follows:*

(a) n is odd, there is an integer $\alpha \geq 1$ such that $P \cong \boxplus^{\alpha}(B_n)$.

(b) n is even, then $P \cong B_n$.

Proof. We proceed by induction on the rank of poset n.

- $n = 4$. A triangular poset of rank 4 is also a Sheffer poset. Since $B(1,4) = 6$, by Lemma 3.3.4 we conclude that $P \cong B_4$.
- $n = 2m+1$. By induction hypothesis, every interval of rank $k \leq 2m$ in P is isomorphic to B_k. Hence P is a Sheffer poset and Theorem 3.3.10 implies that $P \cong \boxplus^{\alpha}(B_n)$, where $\alpha \geq 1$ is a positive integer.
- $n = 2m+2$. Let r and t be the number of elements of rank 1 and $2m+1$ in P. By induction hypothesis, there are positive integers k_t and k_r such that $B(1, 2m+2) = k_t(2m+1)!$ and $B(0, 2m+1) = k_r(2m+1)!$. Therefore, $B(0, 2m+2) = tk_r(2m+1)! = rk_t(2m+1)!$ and also $B(n, n+k) = k!$, where $1 \leq k \leq 2m+1-n$ and $n \geq 1$. The Euler-Poincaré relation for interval of size $2m+2$ state as follows,

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k B(0, 2m+2)}{B(0,k)B(k, 2m+2)} = 0. \tag{3.34}$$

By substituting the values in Eq.(3.34), we have

$$1 + tk_r \left(\sum_{k=2}^{2m} \frac{(-1)^k (2m+1)!}{k!(2m+2-k)!} \right) + \frac{-tk_r(2m+1)!}{k_t(2m+1)!} + \frac{-tk_r(2m+1)!}{k_r(2m+1)!} + 1 = 0. \tag{3.35}$$

Eq.(3.35) lead us to

$$2 - t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r}\right) + tk_r\left(\sum_{k=2}^{2m}\left(\frac{(-1)^k(2m+1)!}{k!(2m+2-k)!}\right)\right) = 0, \tag{3.36}$$

so,

$$2 = t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r} + \frac{-(k_r)(4m+2)}{2m+2}\right). \tag{3.37}$$

Without loss of generality, let us assume that $k_r \geq k_t \geq 1$. Therefore,

$$2 = t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r} + \frac{-(k_r)(4m+2)}{2m+2}\right) \leq t\left(k_r + 1 - \left(\frac{4m+2}{2m+2}\right)k_r\right) \leq t\left(1 - \frac{2m}{2m+2}k_r\right). \tag{3.38}$$

The right-hand side of the above equation is positive only if $k_r = 1$. So $k_r = 1$ and since $k_r \geq k_t \geq 1$, we conclude that $k_t = 1$. Therefore, $2 = t\frac{2}{2m+2}$ and so $t = 2m+2$. Similarly, we conclude that $r = 2m+2$. Thus, P has the same factorial function as B_{2m+2} and by Proposition 3.2, this poset is isomorphic to B_{2m+2}, as desired.

□

3.5 Eulerian Simplicial posets

Stanley suggested the following problem.

Problem 3.5.1. Suppose L is an Eulerian poset of rank $d+1$ that is a boolean algebra up to rank $1 + [d/2]$. Is L a boolean algebra? If the answer is negative, what happen in case L is a lattice?

The motivation is the fact that if the face lattice of a d-polytope is a boolean algebra up to rank $[d/2] + 1$, then it is a boolean algebra.

Consider the inverse of the poset $\Sigma^* B_n$, this poset shows that the above question is negative for general posets. I also prove the following result for lattices.

Theorem 3.5.2. *Suppose L is an Eulerian lattice of rank d that is a boolean algebra up to rank $1 + [d/2]$. Then $L \cong B_d$.*

The key tool is to study this problem is to consider the face lattice of Cyclic polytope together with the Dehn-Sommerville relation. I will pursue some natural continuations of this work, to find some sufficient conditions on Eulerian posets so that the above result holds.

The following result known as the *Dehn-Sommervile Equation* for Eulerian simplicial posets,

Theorem 3.5.3. *Let P be an Eulerian simplicial poset of rank $n + 1$ with h-vector $(h_0, h_1, \cdots, h_n)$. Then $h_i = h_{n-i}$ for all i.*

Lemma 3.5.4. *Let P be an Eulerian simplicial poset of rank d, so that $f_i = \binom{n}{i+1}$ for $0 \leq i < [d/2]$, then $f_{[d/2]}, \cdots, f_{d-1}$ can be determined uniquely as follows:*

- *$n = d-1$, P has the same f-vector as $\Sigma^* B_{d-1}$ (P is isomorphic to $\Sigma^* B_{d-1}$).*
- *$n = d$, P has the same f-vector as B_d (P is isomorphic to B_d).*
- *$n > d$, P has the same f-vector as cyclic polytope $C(n, d)$, as a conclusion $f_{[d/2]} \neq \binom{n}{[d/2]+1}$*

Proof. Since the poset P is simplicial, so $n \geq d - 1$. By theorem 3.5.3, f-vector of the poset P (as well as the h-vector) satisfy $[d/2]$ Dehn-Sommervile equations. We also know that $f_i = \binom{n}{i+1}$ for $0 \leq i < [d/2]$. Therefore, the f-vector of the poset P can be uniquely determined

On the other hand, $\Sigma^* B_{d-1}$, B_d and $C(n, d)$ are examples of Eulerian simplicial posets of rank d with $f_i = \binom{n}{i+1}$ for $0 \leq i < [d/2]$ for the cases $n = d - 1$, $n = d$ and $n > d$. □

Lemma 3.5.5. *Suppose L is an Eulerian lattice of rank d that is a boolean algebra up to rank $1 + [d/2]$. Then $L = B_d$.*

Proof. We proceed by induction on d. In case $d = 2$, B_2 is the only Eulerian lattice (poset) of rank 2.

Suppose that the lemma holds for $d \leq k-1$. Assume that lattice L has n atoms $x_1, \cdots, x_n$ and m coatoms $a_1, \cdots, a_m$. L is a lattice, so $[\hat{0}, a_i]$ is.

Consider the interval $[\hat{0}, a_i]$ where for $1 \leq i \leq m$. Without losing generality, assume that a_i cover the atoms $x_1, \cdots, x_t, t \leq n$. Each $\lfloor \frac{d}{2} \rfloor + 1$ subset of $x_1, \cdots, x_t$, such as $x_{i_1}, ..., x_{i_{\lfloor \frac{d}{2} \rfloor + 1}}$, covered by unique element of rank $\lfloor \frac{d}{2} \rfloor + 1$ in L and so they covered by unique element of rank $\lfloor \frac{d}{2} \rfloor + 1$ in $[\hat{0}, a_i]$. We conclude that $[\hat{0}, a_i]$ is isomorphic boolean algebra generated by $x_1, \cdots, x_t$ up to rank $1 + [k - 1/2]$. Therefore, by induction hypothesis $[\hat{0}, a_i] = B_{k-1}$. So, $[\hat{0}, a_i] = B_{k-1}$ for all $1 \leq i \leq m$.

We conclude that L is simplicial Eulerian poset of rank d. By Lemma 3.5.4, we conclude that P has the same f-vector as B_d, and P is isomorphic to B_d.

$\square$

Chapter 4

Matroid Polytopes Associated to Lattice Paths

In this chapter, we describe the results in [Bid2] regarding the lattice path matroid polytopes. We investigate properties of these polytopes, including their face structure, their decomposition and triangulation as well as a formula for calculating their Ehrhart polynomial and volume.

4.1 Matroids and Matroid Polytopes

A *matroid* is a combinatorial construct which generalizes the notion of linear independence; matroids are very general objects with many applications throughout mathematics. They are found in matrices, graphs, transversals, point configurations, hyperplane arrangements, greedy optimization, and pseudosphere arrangements to name a few. For a thorough introduction to the subject we refer the reader to Oxley [Oxl].

Definition 4.1.1. A pair $(\mathcal{S}, \mathcal{I})$ is called a *matroid* if $\mathcal{S}$ is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of $\mathcal{S}$ satisfying:

(a) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,

(b) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I \cup z \in \mathcal{I}$ for some $z \in J \setminus I$.

$\diamondsuit$

For a matroid $\mathcal{M} = (\mathcal{S}, \mathcal{I})$, $I \subseteq \mathcal{S}$ is *independent* if $I \in \mathcal{I}$ and dependent otherwise. The set $B \subseteq \mathcal{S}$ is called a *base* or *basis* if it is an inclusion maximal independent set. That is, $B \in \mathcal{I}$ and there does not exist $A \in \mathcal{I}$ such that $B \subset A \subseteq S$.

Example 4.1.2. A matroid of rank r is a *uniform matroid* if all r-element subsets of the ground set are bases. There is, up to isomorphism, exactly one uniform matroid of rank r on an m-element set; this matroid is denoted by $U_{r,m}$. $\diamondsuit$

Example 4.1.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be graph with vertices $\mathcal{V}$ and edges $\mathcal{E}$. The collection of spanning forests of $\mathcal{G}$ are the bases of the graphical matroid on ground set $\mathcal{V}$, denoted $\mathcal{M}_{\mathcal{G}}$. We can verify the basis exchange axiom by considering two spanning forests $\mathcal{F}_1, \mathcal{F}_2$. If $e \in \mathcal{F}_2 \setminus \mathcal{F}_1$, then $\mathcal{F}_1 + e$ contains a cycle $\mathcal{C}$ in $\mathcal{G}$. Also there exists $f \in \mathcal{C} \setminus \mathcal{F}_2$ and thus $F_1 \setminus f + e$ is a spanning forest. $\diamondsuit$

The particular matroids of interest in this chapter arise from lattice paths, to which we now turn. The lattice paths we consider use steps $E = (1, 0)$ and $N = (0, 1)$. We will often treat lattice paths as words in the alphabets $\{E, N\}$ and we will use the notation α^n to denote the concatenation of n letters, or strings of letters, α. If $P = s_1 s_2 \cdots s_n$ is a lattice path, then its *reversal* is defined as $P^\rho = s_n s_{n-1} \cdots s_1$. The *length* of a lattice path $P = s_1 s_2 \cdots s_n$ is n, the number of steps in P.

Lattice path matroids provide a bridge between matroid theory and the theory of lattice paths, as it demonstrated in [BdMN, Bid2]. These matroids can lead to a mutually enriching relationship between the two subjects. In [BdMN], Bonin, Mier and Noy give a combinatorial interpretation for each coefficient of the Tutte polynomial of a lattice path matroid. Computing the Tutte polynomial of an arbitrary matroid is known to be #P-complete; the same is true even within special classes such as graphic or transversal matroids. However, by using the path interpretation of the coefficients in the case of lattice path matroids, they show that the Tutte polynomial of such a matroid can be computed in polynomial time. On the lattice path side, their interpretation of the coefficients along with easily computed examples of the Tutte polynomial can suggest new theorems about lattice paths.

The lattice path matroid is a special class of transversal matroids. Let $\mathcal{A} = (A_j : j \in J)$ be a set system. That is, a multiset of subsets of a finite set S. A *transversal* (or system of distinct representatives) of $\mathcal{A}$ is a set $\{x_j : j \in J\}$ of $|J|$ distinct elements such that $x_j \in A_j$ for all j in J. A *partial transversal* of $\mathcal{A}$ is a transversal of a set system of the form $(A_k : k \in K)$ with K a subset of J. The following theorem is a fundamental result due to Edmonds and Fulkerson.

Theorem 4.1.4. *The partial transversals of a set system $\mathcal{A} = (A_j : j \in J)$ are the independent sets of a matroid on S.*

We say that $\mathcal{A}$ is a *presentation* of the transversal matroid. The bases of a transversal matroids are the maximal partial transversals of $\mathcal{A}$. For more on transversal matroids see [Oxl, Section 1.6].

Definition 4.1.5. Let $P = p_1p_2 \dots p_{m+r}$ and $Q = q_1q_2 \dots q_{m+r}$ be two lattice paths from $(0,0)$ to (m,r) with P never going above Q. Let $\{p_{u_1}, \dots, p_{u_r}\}$ be the set of North steps of P with $u_1 < u_2 < \cdots < u_r$; similarly, let $\{q_{l_1}, \dots, q_{l_r}\}$ be the set of North steps of Q with $l_1 < l_2 < \cdots < l_r$. Let N_i be the interval $[l_i, u_i]$ of integers. Let $\mathcal{M}[P,Q]$ be the transversal matroid that has ground set $[m+r]$ and presentation $(N_i : i \in [r])$; the pair (P,Q) is a *presentation of* $\mathcal{M}[P,Q]$. A *lattice path matroid* is a matroid $\mathcal{M}$ that is isomorphic to $\mathcal{M}[P,Q]$ for some such pair of lattice paths P and Q. $\diamondsuit$

We think of $1, 2, \dots, m+r$ as the first step, the second step, etc. Observe that the set N_i contains the steps that can be the i-th North step in a lattice path from $(0,0)$ to (m,r) that remains in the region bounded by P and Q. When thought of as arising from the particular presentation using bounding paths P and Q, the elements of the matroid are ordered in their natural order, i.e., $1 < 2 < \cdots < m+r$. However, this order is not inherent in the matroid structure; the elements of a lattice path matroid typically can be linearly ordered in many ways so as to correspond to steps in lattice paths. We associate a lattice path $P(X)$ with each subset X of the ground set of a lattice path matroid as specified in the next definition.

Definition 4.1.6. Let X be a subset of the ground set $[m+r]$ of the lattice path matroid $\mathcal{M}[P,Q]$. The lattice path $P(X)$ is the word

$$s_1 s_2 \ldots s_{m+r}$$

in the alphabet $\{E, N\}$ where

$$s_i = \begin{cases} N, & \text{if } i \in X; \\ E, & \text{otherwise.} \end{cases}$$

$\diamondsuit$

Thus, the path $P(X)$ is formed by taking the elements of $\mathcal{M}[P,Q]$ in the natural linear order and replacing each by a North step if the element is in X and by an East step if the element is not in X. The fundamental connection between the transversal matroid $\mathcal{M}[P,Q]$ and the lattice paths that stay in the region bounded by P and Q is the following theorem which says that the bases of $\mathcal{M}[P,Q]$ can be identified with such lattice paths.

Theorem 4.1.7 (Bonin et al.). *A subset B of $[m+r]$ with $|B| = r$ is a basis of $\mathcal{M}[P,Q]$ if and only if the associated lattice path $P(B)$ stays in the region bounded by P and Q.*

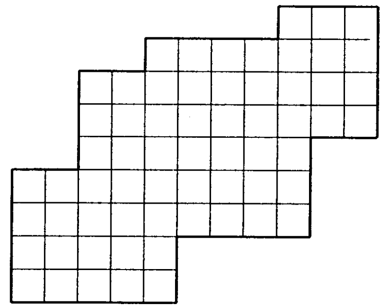


Figure 4-1: The bases of $\mathcal{M}[P,Q]$, associated to the lattice paths lie in the region between paths $P = E^5N^2E^4N^3E^2N^4$ and $Q = N^4E^2N^3E^2N^1E^4N^1E^3$ with the same endpoints.

4.1.1 Matroid Polytopes

For any matroid one can associate a matroid polytope by taking the convex hull of the incidence vectors of the bases of the matroid. The last few years has seen a flurry of research activities around matroid polytopes, in part because their combinatorial properties provide key insights into matroids and in part because they form an intriguing and seemingly fundamental class of polytopes which exhibit interesting geometric features. The theory of matroid polytopes has gained prominence due to its applications in algebraic geometry, combinatorial optimization, Coxeter group theory, and most recently, tropical geometry. In general matroid polytopes are not well understood.

We define *bases* to be maximal independent sets of a matroid. Let $\mathcal{B}$ be the set of bases of a matroid $\mathcal{M}$. If $B = \{\sigma_1, \ldots, \sigma_r\} \in \mathcal{B}$, we define the *incidence vector of* B as $e_B := \sum_{i=1}^{r} e_{\sigma_i}$, where e_j is the standard elementary jth vector in $\mathbb{R}^n$. The *matroid polytope* of $\mathcal{M}$ is defined as $\mathcal{P}(\mathcal{M}) := \mathrm{conv}\{\, e_B \mid B \in \mathcal{B}\,\}$, where $\mathrm{conv}(\cdot)$ denotes the convex hull.

In [Bid2], we study *lattice path matroid polytopes* (LPMPs). This special class of matroid polytopes belongs to two famous classes of polytopes, sorted closed matroid polytopes [LP] and polypositroids [Pos]. In this chapter, we study several properties of LPMPs and build a new connection between the theories of matroid polytopes and lattice paths. A good example of a lattice path matroid polytope is the Catalan matroid polytope, which has the Catalan number of vertices and many interesting properties. The rest of this chapter, discuss the results in [Bid2] concerning lattice path matroid polytopes. Matroid polytopes are generally hard to study and there are few general result about them. So, we try to produce a clear picture of the combinatorial, geometric, and arithmetic properties of a special class of matroid polytopes.

Here, we recall the facts about the enumeration of lattice paths.

Lemma 4.1.8. *For a fixed positive integer k, the number of lattice paths from $(0,0)$ to (kn,n) that use steps E and N and that never pass above the line $y = x/k$ is the n-th k-Catalan number*

$$C_n^k = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

In particular, the number of paths from $(0,0)$ to (n,n) that never pass above the line $y = x$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

We now turn to a special class of lattice path matroids, the generalized Catalan matroids, as well as to various subclasses that exhibit a structure that is simpler than that of typical lattice path matroids.

Definition 4.1.9. A lattice path matroid $\mathcal{M}$ is a *generalized Catalan matroid* if there is a presentation (P,Q) of $\mathcal{M}$ with $P = E^m N^r$. In this case we simplify the notation $\mathcal{M}[P,Q]$ to $\mathcal{M}[Q]$. In case, in addition the upper path Q is $(E^k N^l)^n$ for some positive integers k, l, and n, we say that $\mathcal{M}$ is the (k,l)-*Catalan matroid* $\mathcal{M}_n^{k,l}$. In place of $\mathcal{M}_n^{k,1}$ we write $\mathcal{M}_n^k$; such matroids are called *k-Catalan matroids.* In turn, we simplify the notation $\mathcal{M}_n^1$ to $\mathcal{M}_n$; such matroids are called *Catalan matroids.* $\diamondsuit$

4.2 Faces and Dimension of Lattice Path Matroid Polytopes.

In this section, we study the faces and dimension of lattice path matroid polytopes. In general, faces of matroid polytopes are not well understood. The following is the main result on faces of matroid polytopes. Gelf́and, Goresky, MacPherson and Serganova [GGMS, Thm 4.1] show the following characterization of matroid polytopes. Let $\mathcal{M}$ be a matroid, then

Lemma 4.2.1. *[Gel'fand et al.] Two vertices e_{B_1} and e_{B_2} are adjacent in $\mathcal{P}(\mathcal{M})$ if and only if $e_{B_1} - e_{B_2} = e_i - e_j$ for some i, j.*

The circuit exchange axiom gives rise to the following equivalence relation on the ground set $[n]$ of the matroid $\mathcal{M}$, i and j are *equivalent* if there exists a circuit C with $\{i, j\} \subseteq C$. The equivalence classes are the *connected components* of $\mathcal{M}$. Let $c(\mathcal{M})$ denote the number of connected components of $\mathcal{M}$. We say that $\mathcal{M}$ is connected if $c(\mathcal{M}) = 1$. The following proposition has been shown in [FS] by Feichtner and Sturmfels.

Proposition 4.2.2. *[Feichtner, Sturmfels] The dimension of the matroid polytope $\mathcal{P}(\mathcal{M})$ equals to $n - c(\mathcal{M})$.*

Let $P = p_1p_2 \dots p_{m+r}$ and $Q = q_1q_2 \dots q_{m+r}$ be two lattice paths from $(0, 0)$ to (m, r) with P never going above Q.

Proposition 4.2.3 (Bonin *et al.*). *The class of lattice path matroids is closed under direct sums. Furthermore, the lattice path matroid $\mathcal{M}[P, Q]$ is connected if and only if the bounding lattice paths P and Q meet only at $(0, 0)$ and (m, r).*

Applying Propositions 4.2.2 and 4.2.3, we have the following lemma:

Lemma 4.2.4. *The dimension of the lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P, Q])$ is $m + r - k + 2$, where k is the number of intersection vertices of the paths P and Q.*

The Catalan matroid polytope $\mathcal{P}(\mathcal{M}_n)$, for any $n \geq 2$, has $c(\mathcal{M}_n) = 3$ connected components and its dimension is $2n-3$.

Applying Lemma 4.2.1, we give combinatorial interpretation of the number of edges of generalized Catalan matroid polytope as follows:

Lemma 4.2.5. *Consider the lattice path matroid polytope $\mathcal{P}(\mathcal{M}[E^mN^r, Q]) = \mathcal{P}(\mathcal{M}[Q])$. The number of edges of this polytope is the total area between the paths from $(0,0)$ to (m,r) which do not exceed P and the path E^mN^r.*

Proof. We know that the vertices of the generalized Catalan matroid polytope $\mathcal{P}(\mathcal{M}[E^mN^r, Q])$ are the paths with m E steps and r N steps which does not exceed Q. By Lemma 4.2.1, the number of edges of this polytope is equal to the number of paths P and P' in this region which differ in one N step and one E step consecutively. Without loss of generality, we can assume that $P = P_1NP_2EP_3$ and $P' = P_1EP_2NP_3$.

For each path P in $[E^mN^r, Q]$, we can always switch ordered pairs of N step and E step to one other pair of E step and N step and obtain the other path P' in $[E^mN^r, Q]$. Clearly, the vertices associated P and P' in $\mathcal{M}[E^mN^r, Q]$ are adjacent to each other. We only need to count the number of all pairs N and E steps which comes after each other in all paths in this region. It is not hard to see that this number is equal to the total area between all the paths in $[E^mN^r, Q]$ and the path E^mN^r consisting of N and E steps. $\square$

Lemma 4.2.6. *The number of edges of Catalan matroid polytope $\mathcal{P}(\mathcal{M}_n) = \mathcal{P}(\mathcal{M}[E^nN^n, (EN)^n]) = \mathcal{P}(\mathcal{M}[E^{n-1}N^{n-1}, (NE)^{n-1}])$, $a(n)$ is the the total area below paths consisting of steps east $(1,0)$ and north $(0,1)$ from $(0,0)$ to (n,n), that stay weakly below $y = x$. So we can calculate $a(n)$ as follows*

$$a(n) = \frac{n^2}{2}\frac{1}{n+1}\binom{2n}{n} - \frac{4^n}{2} - \frac{1}{4}\binom{2n+2}{n+1}. \tag{4.1}$$

Proof. Let S_n denote the total area below paths consisting of steps E and N from $(0,0)$ to (n,n) that stay weakly below $y = x$. Furthermore, let A_n be the total area between the paths consisting of steps E and N from $(0,0)$ to (n,n) and the line $x = y$. The nth Catalan number, C_n, is the number of paths from $(0,0)$ to (n,n) that stay weakly below $y = x$.

We proceed by induction, it is not hard to verify that

$$A_{n+1} = 2\sum_{k=0}^{n}(k+\frac{1}{2})C_kC_{n-k} + \sum_{k=0}^{n} A_kC_{n-k} + \sum_{k=0}^{n} A_{n-k}C_k. \tag{4.2}$$

The last two sums of Eq. 4.2 are equal, and so we have

$$A_{n+1} = 2\sum_{k=0}^{n} A_kC_{n-k} + \frac{1}{2}\sum_{k=0}^{n} C_kC_{n-k} + \sum_{k=0}^{n} kC_kC_{n-k}. \tag{4.3}$$

Let $C(t)$ and $A(t)$ be the generating functions for C_n and A_n, Therefore, we have

$$\frac{A(t)}{t} = 2A(t)C(t) + \frac{1}{2}C(t)^2 + tC'(t)C(t). \tag{4.4}$$

By differentiating, we obtain the following generating function for $A(t)$

$$A(t) = \frac{1-2t-\sqrt{1-4t}}{4t(1-4t)}. \tag{4.5}$$

Therefore, we obtain the value for A_n as follows:

$$A_n = \frac{4^n}{2} - \frac{1}{4}\binom{2n+2}{n+1}. \tag{4.6}$$

From the definition of A_n and S_n, we have,

$$S_n = \frac{n^2}{2}\frac{1}{n+1}\binom{2n}{n} - \frac{4^n}{2} - \frac{1}{4}\binom{2n+2}{n+1}. \tag{4.7}$$

The number of edges of Catalan matroid polytope $\mathcal{P}(\mathcal{M}_n)$, $a(n) = S_n$. We are done. $\square$

Consider the connected lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$, where P

and Q are paths from $(0,0)$ to (m,r). Therefore, $P = E^{\alpha_1}N^{\alpha_2}\cdots N^{\alpha_{2k}}$ and also $Q = N^{\beta_1}E^{\beta_2}\cdots E^{\beta_{2r}}$. As we discuss before, any bases of matroid $\mathcal{M}[P,Q]$ associated to the vector $X = x_1\cdots x_{m+r}$. Where vector X is base for $\mathcal{M}[P,Q]$ if and only if $P(X)$ lies in the region $[P,Q]$. Let p_i and q_i be the number of N steps occur in the first i steps of paths P and Q, where $1 \leq i \leq m+r$. Clearly, $p_{m+r} = q_{m+r} = m$. Therefore, $P(X)$ lies in the region $[P,Q]$ if and only if $p_i \leq x_1 + \cdots + x_i \leq q_i$ for all $1 \leq i \leq m+r$.

Lemma 4.2.7. *The polytope $\mathcal{P}(\mathcal{M}[P,Q])$ can be determined by the following inequalities,*

(a) $p_i \leq x_1 + \cdots + x_i \leq q_i$ for all $1 \leq i \leq m+r$, where $x_1 + \cdots + x_{m+r} = m$,

(b) $0 \leq x_i \leq 1$.

where p_i and q_i be the number of N steps occur in the first i steps of paths P and Q.

Proof. All the vertices of $\mathcal{P}(\mathcal{M}[P,Q])$ satisfy the conditions (a) and (b). Therefore, every point in $\mathcal{P}(\mathcal{M}[P,Q])$ satisfy the both of these conditions. Now consider the point $A = a_1\cdots a_{m+r}$ satisfying inequalities (a) and (b). In case there exists $1 \leq i \leq m+r$ so that $a_i = 0$, we can proceed by induction on $m+r$. The point A is in convex hull of the vertices in $\mathcal{P}(\mathcal{M}[P,Q])$ whose ith vertices are 0, so it lies in $\mathcal{P}(\mathcal{M}[P,Q])$. Otherwise, let a_i be the minimum value of vector in a and let X_i be a vertex whose ith coordinate is 1. We define the vector $B = \dfrac{A - a_iX_i}{1-a_i}$. It satisfies the inequalities conditions and it has zero entries. Therefore, we can proceed with induction.

$\square$

Lemma 4.2.8. *Consider the connected lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$, where P and Q are paths from $(0,0)$ to (m,r), so that $P = E^{\alpha_1}N^{\alpha_2}\cdots N^{\alpha_{2l}}$ and also $Q = N^{\beta_1}E^{\beta_2}\cdots E^{\beta_{2s}}$. The affine hull of this polytope is $x_1+\cdots+x_{m+r} = r$. In case $\beta_1 > 1$ and $\alpha_{2l} > 1$, the facets in the affine hull $x_1 + \cdots + x_{m+r} = r$ can be described as follows,* *(a)* $x_1 + \cdots + x_{\beta_1+\cdots+\beta_{2k}} \leq \beta_1 + \beta_3 + \cdots + \beta_{2k-1}$ *for* $1 \leq k < s$.

(b) $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k}} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k}$ *for* $1 \leq k \leq l-1$.

(c) $x_i \geq 0$ *for* $i = 1, \ldots, m+r$, *except for* i*'s so that we have the facets* $x_1 + \cdots + x_i \geq j$ *and* $x_1 + \cdots + x_{i-1} \leq j$ *in* (*a*) *and* (*b*) *descriptions.*

(d) $x_i \leq 1$ *for* $i = 1, \ldots, m+r$, *except for* i*'s so that we have the facets* $x_1 + \cdots + x_i \geq j$ *and* $x_1 + \cdots + x_{i+1} \leq j+1$ *in* (*a*) *and* (*b*) *descriptions.*

In case $\beta_1 = 1$ *and* $\alpha_{2l} > 1$, *the facets in the affine hull* $x_1 + \cdots + x_{m+r} = r$ *can be described as follows,*

(a) $x_1 + \cdots + x_{\beta_1+\cdots+\beta_{2k}} \leq \beta_1 + \beta_3 + \cdots + \beta_{2k-1}$ *for* $1 \leq k < s$,

(b) $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k}} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k}$ *for* $1 \leq k \leq l-1$,

(c) $x_i \geq 0$ *for* $i = 1, \ldots, m+r$, *except for* i*'s so that we have the facets* $x_1 + \cdots + x_i \geq j$ *and* $x_1 + \cdots + x_{i-1} \leq j$ *in* (*a*) *and* (*b*) *descriptions.*

(d) $x_i \leq 1$ *for* $i = 1, \ldots, m+r$, *except for* $i \leq 1 + \beta_2$ *and also for* i*'s so that we have the facets* $x_1 + \cdots + x_i \geq j$ *and* $x_1 + \cdots + x_{i+1} \leq j+1$ *in* (*a*) *and* (*b*) *descriptions.*

In case $\beta_1 = 1$ *and* $\alpha_{2l} = 1$ *the facets in the affine hull can be described as follows:*

(a) $x_1 + \cdots + x_{\beta_1+\cdots+\beta_{2k}} \leq \beta_1 + \beta_3 + \cdots + \beta_{2k-1}$ *for* $1 \leq k < s$,

(b) $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k}} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k}$ *for* $1 \leq k \leq l-1$,

(c) $x_i \geq 0$ *for* $i = 1, \ldots, m+r$, *except for* i*'s so that both facets* $x_1 + \cdots + x_i \geq j$ *and* $x_1 + \cdots + x_{i-1} \leq j$ *in the above descriptions* (*a*) *and* (*b*).

(d) $x_i \leq 1$ *for* $i = 1, \ldots, m+r$, *except* $i \leq 1 + \beta_2$ *and for* $i \geq \alpha_1 + \cdots +_{2l-2}$. *and also for* i*'s so that we have the facets* $x_1 + \cdots + x_i \geq j$ *and* $x_1 + \cdots + x_{i+1} \leq j+1$, *in the above descriptions* (*a*) *and* (*b*).

In case $\beta_1 > 1$ *and* $\alpha_{2l} = 1$ *the facets in the affine hull* $x_1 + \cdots + x_{m+r} = r$ *can be described as follows,* *(a)* $x_1 + \cdots + x_{\beta_1+\cdots+\beta_{2k}} \leq \beta_1 + \beta_3 + \cdots + \beta_{2k-1}$ *for* $1 \leq k < s$.

(b) $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k}} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k}$ *for* $1 \leq k \leq l-1$.

(c) $x_i \geq 0$ *for* $i = 1, \ldots, m+r$, *except for* i*'s so that both facets* $x_1+\cdots+x_i \geq j$ *and* $x_1 + \cdots + x_{i-1} \leq j$ *in the above descriptions* (a) *and* (b).

(d) $x_i \leq 1$ *for* $i = 1, \ldots, m+r$, *except for* i*'s so that both facets* $x_1+\cdots+x_i \geq j$ *and* $x_1 + \cdots + x_{i+1} \leq j+1$ *in the above descriptions* (a) *and* (b), *and also for* $i \geq \alpha_1 + \cdots + \alpha_{2l-2}$.

Proof. Let us recall that each polytope is intersection of a finite family of half spaces in its affine hull. The minimal such family determines the facets of polytope. The polytope $\mathcal{P}(\mathcal{M}[P,Q])$ lies on the affine hull $x_1 + \cdots + x_{m+r} = r$. So we only need to verify that the polytope $\mathcal{P}(\mathcal{M}[P,Q])$ obtain by the described half spaces in affine hull $x_1 + \cdots + x_{m+r} = r$ and this set is minimal. As we describe in Lemma 4.2.7, the polytope $\mathcal{P}(\mathcal{M}[P,Q])$ can describe as intersection of the following hyperplanes,

(a) $0 \leq x_i$, $x_i \leq 1$ for $1 \leq i \leq m+r$,

(b) $x_1+\cdots+x_{\beta_1+\cdots+\beta_{2k}+t} \leq \beta_1+\beta_3+\cdots+\beta_{2k-1}+t$ for $1 \leq k < s$ and $t \leq \beta_{2k+1}$,

(c) $x_1+\cdots+x_{\beta_1+\cdots+\beta_{2k-1}+t} \leq \beta_1+\beta_3+\cdots+\beta_{2k-1}$ for $1 \leq k \leq s$, where $t \leq \beta_{2k}$

(d) $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k-1}+t} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k-2} + t$, where $0 \leq t \leq \alpha_{2k}$ and $1 \leq k \leq l$.

(e) $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k-2}+t} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k-2}$, where $0 \leq t \leq \alpha_{2k-1}$ and $1 \leq k \leq l$.

It is not hard to verify that, the hyperplanes $x_i \leq 1$ and $x_i \geq 0$ for $i = 1, \ldots, m+r$, $x_1 + \cdots + x_{\beta_1+\cdots+\beta_{2k}} \leq \beta_1 + \beta_3 + \cdots + \beta_{2k-1}$ for $1 \leq k < s$ and $x_1 + \cdots + x_{\alpha_1+\cdots+\alpha_{2k}} \geq \alpha_2 + \alpha_4 + \cdots + \alpha_{2k}$ for $1 \leq k \leq l-1$, generate all the hyperplanes stated above.

(a) The hyperplanes $x_1 + \cdots + x_i \geq j$ and $x_1 + \cdots + x_{i-1} \leq j$, implies that $x_i \geq 0$, so we can omit the hyperplane $x_i \geq 0$ for such i's.

(b) The hyperplanes $x_1+\cdots+x_i \geq j$ and $x_1+\cdots+x_{i+1} \leq j+1$, implies that $x_i \leq 1$, so we can omit the hyperplane $x_i \leq 1$ for such i's.

(c) In case $\beta_1 = 1$, the hyperplame $x_1 + \cdots + x_{1+\beta_2} \leq 1$ implies that $x_i \leq 1$ for $i \leq \beta_2 + 1$.

(d) In case $\alpha_{2l} = 1$, the equality $x_1 + \cdots + x_{m+r} = r$ and implies that $x_i \leq 1$ for $i \geq \alpha_1 + \cdots + \alpha_{2l-2}$.

Therefore, the hyperplanes stated on the theorem generate $\mathcal{P}(\mathcal{M}[P,Q])$ in the affine hull $x_1 + \cdots + x_{m+r} = r$. It is not hard to verify that the following hyperplanes, will generate the same polytope and this set is minimal. □

Lemma 4.2.9. *The Catalan matroid polytope* $\mathcal{P}(\mathcal{M}_{n+1}) = \mathcal{P}(\mathcal{M}[E^{n+1}N^{n+1},(EN)^{n+1}]) = \mathcal{P}(\mathcal{M}[E^nN^n,(NE)^n])$, *for any* $n \geq 2$, *has* $5n-5$ *facets which lies in the following hyperplanes:*

- $x_3, \ldots, x_{2n-1} \leq 1$,
- $x_1, x_2, \ldots, x_{2n-2} \geq 0$,
- $\sum_{i=1}^{2k-2} x_i \leq k-1$ *for* $2 \leq k \leq n$,
- $\sum_{i=1}^{2n-1} x_i \geq n-1$ *or equivalently* $x_{2n} \leq 1$.

Lemma 4.2.10. *The Catalan matroid polytope* $\mathcal{P}(\mathcal{M}_n^r) = \mathcal{P}(E^{rn}N^n,(E^rN)^n) = \mathcal{P}(E^{r(n-1)}N^{n-1},(NE^r)^{n-1})$, *for any* $n \geq 2$.

- $x_{r+2}, \ldots, x_{r(n-1)} \leq 1$,
- $x_1, x_2, \ldots, x_{r(n-1)} \geq 0$,
- $\sum_{i=1}^{k(r+1)} x_i \leq k$ *for* $1 \leq k \leq n-1$,

Theorem 4.2.11. *All the faces of lattice path matroid polytope are lattice path matroid polytopes*

Proof. Without lost of generality, we may assume that the lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$ is connected, where P and Q are paths from $(0,0)$ to (m,r), so that $P = E^{\alpha_1}N^{\alpha_2}\cdots N^{\alpha_{2k}}$ and also $Q = N^{\beta_1}E^{\beta_2}\cdots E^{\beta_{2s}}$.

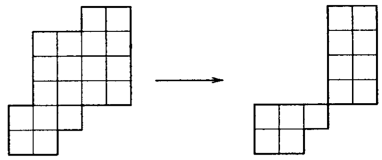


Figure 4-2: Faces of lattice path matroid polytope

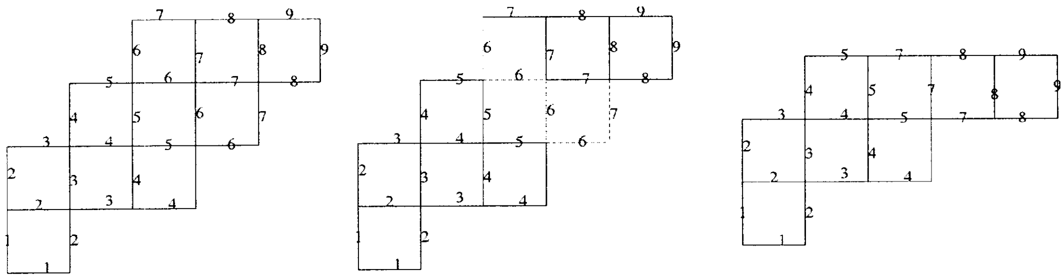


Figure 4-3: Faces of lattice path matroid polytope

We are going to show that all the facets of this polytope are lattice path matroid polytope. Clearly, the vertices in facet of the form $x_1 + \cdots + x_i \leq q_i$, $x_1 + \cdots + x_i \geq p_i$ are the path which go through the ith vertex of the paths Q and P, respectively. Thus, these facets are lattice path matroid polytope which a direct sum of two other lattice path matroid polytope. We wish to show that facets which obtain by equalities $x_i = 0, 1$ are also lattice path matroid polytope.

Consider the facet of polytope $x_i = 1$. Vertices of this facet are paths with N step on their ith step. We just delete all ith steps of $\mathcal{M}[P, Q]$ and the $(i+1)$th steps which are only connected to E ith steps in $\mathcal{M}[P, Q]$. As we see in Figure 3.2, we connect two obtained regions by moving right hand side region one step down. The result is lattice path matroid associated to this facet. Similarly, the vertices with $x_i = 0$, form a lattice path matroid polytope. $\square$

$\square$

4.3 Decomposition of Lattice Path Matroid Polytope

In this section, we study the decomposition of lattice path matroid polytopes. Billera, Jia and Reiner in [BJR] defined a matroid polytope decomposition of $\mathcal{P}(\mathcal{M})$ to be a decomposition $\mathcal{P}(\mathcal{M}) = \bigcup_{i=1}^{t} \mathcal{P}(\mathcal{M}_i)$ where each $\mathcal{P}(\mathcal{M}_i)$ is also a matroid polytope for some matroid $\mathcal{M}_i$, and for each $1 \leq i \neq j \leq t$, the intersection $\mathcal{P}(\mathcal{M}_i) \cap \mathcal{P}(\mathcal{M}_j)$ is a face of both $\mathcal{P}(\mathcal{M}_i)$ and $\mathcal{P}(\mathcal{M}_j)$. The polytope $\mathcal{P}(\mathcal{M})$ is said to be decomposable if it has a matroid polytope decomposition for $t \geq 2$, and indecomposable otherwise. A decomposition is called hyperplane split if $t = 2$. A hyperplane split $\mathcal{P}(\mathcal{M}) = \mathcal{P}(\mathcal{M}_1) \cup \mathcal{P}(\mathcal{M}_2)$ is said to be trivial if one of the two polytopes $\mathcal{P}(\mathcal{M}_i)$ for $i = 1, 2$ is a face of the other (that is, if one of the $\mathcal{P}(\mathcal{M}_i)$ is a face of $\mathcal{P}(\mathcal{M})$) and nontrivial otherwise. We notice that if $\mathcal{P}(\mathcal{M}) = \mathcal{P}(\mathcal{M}_1) \cup \mathcal{P}(\mathcal{M}_2)$ is a nontrivial hyperplane split then $\mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2)$ must be a facet of both $\mathcal{P}(\mathcal{M}_1)$ and $\mathcal{P}(\mathcal{M}_2)$, and the dimension of $\mathcal{P}(\mathcal{M}_i)$ for $i = 1, 2$ is the same as that of $\mathcal{P}(\mathcal{M})$.

Let $\mathcal{M} = (E, B)$ be a matroid of rank r and let $A \subseteq E$. We recall that the independent set of the restriction matroid of $\mathcal{M}$ to A, denoted by $\mathcal{M}|_A$, is given by $\mathcal{I}(\mathcal{M}|_A) = \{I \subseteq A : I \in \mathcal{I}(\mathcal{M})\}$. Let (E_1, E_2) be a partition of E, that is, $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$. Let $r_i > 1$, $i = 1, 2$ be the rank of $\mathcal{M}|_{E_i}$. We say that (E_1, E_2) is a *good partition* if there exist integers $0 < a_1 < r_1$ and $0 < a_2 < r_2$ with the following properties

(P1) $r_1 + r_2 = r + a_1 + a_2$,

(P2) For all $X \in \mathcal{I}(\mathcal{M}|_{E_1})$ with $|X| \leq r_1 - a_1$ and all $Y \in \mathcal{I}(\mathcal{M}|_{E_2})$ with $|Y| \leq r_2 - a_2$, we have $X \cup Y \in \mathcal{I}(\mathcal{M})$.

Lemma 4.3.1. *[Alfonsin, Chatelain] Let $\mathcal{M} = (E, B)$ be a matroid of rank r and let (E_1, E_2) be a good partition of E. Let $\mathcal{B}(\mathcal{M}_1) = \{B \in \mathcal{B}(\mathcal{M}) : |B \cap E_1| \leq r_1 - a_1\}$ and $\mathcal{B}(\mathcal{M}_2) = \{B \in \mathcal{B}(\mathcal{M}) : |B \cap E_2| \leq r_2 - a_2\}$. Where r_i is the rank of matroid $\mathcal{M}|_{E_i}$ for $i = 1, 2$ and a_1 and a_2 are integers verifying properties $(P1)$ and $(P2)$. Then $\mathcal{B}(\mathcal{M}_1)$ and $\mathcal{B}(\mathcal{M}_2)$ are the collections of bases of matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.*

Theorem 4.3.2. *[Alfonsin, Chatelain] Let $\mathcal{M} = (E, B)$ be a matroid of rank r and let (E_1, E_2) be a good partition of E. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids given in Lemma 4.3.1. Then, $\mathcal{P}(\mathcal{M}) = \mathcal{P}(\mathcal{M}_1) \cup \mathcal{P}(\mathcal{M}_2)$ is a nontrivial hyperplane split.*

Corollary 4.3.3. *[Alfonsin, Chatelain] Let $\mathcal{M}[P, Q]$ be the transversal matroid on $\{1, \ldots, m+r\}$ and presentation $(N_i : i \in \{1, \ldots, r\})$, where N_i denotes the interval $[s_i, t_i]$ of integers. Suppose that there exists integer x such that $s_j < x < t_j$ and $s_{j+1} < x+1 < t_{j+1}$ for some $1 \leq j \leq r-1$. Then, $\mathcal{P}(\mathcal{M}[P, Q])$ has a nontrivial hyperplane split.*

Proof. Consider the integer x such that $s_j < x < t_j$ and $s_{j+1} < x+1 < t_{j+1}$ for some $1 \leq j \leq r-1$. Let $E_1 = \{1, \ldots, x\}$ and $E_2 = \{x+1, \ldots, m+r\}$. Then, $\mathcal{M}|_{E_1}$ (resp. $\mathcal{M}|_{E_2}$) is the transversal matroid with representation ${N_i}^1 : i \in \{1, \ldots, r\}$ where ${N_i}^1 = N_i \cap E_1$ (resp. with representation ${N_i}^2 : i \in \{1, \ldots, r\}$ where ${N_i}^2 = N_i \cap E_2$). Let r_1 and r_2 be the ranks of $\mathcal{M}|_{E_1}$ and $\mathcal{M}|_{E_2}$, respectively. We have that ${N_i}^1 \neq \phi$, for all $i \leq j+1$ (since the smallest element in N_i is strictly smaller than $x+1$). Therefore, $r_1 \geq j+1$. Similarly, we have ${N_i}^2 \neq \phi$, for all $i \geq r-j+1$. Since the largest element in N_i is larger than $x+1$, Therefore, $r_2 \geq r-j+1$ and the partition (E_1, E_2) verifies property $(P1)$ by taking integers a_1 and a_2 such that $r_1 - a_1 = j$ and $r_2 - a_2 = r - j$. Moreover, the sets $\mathcal{B}(\mathcal{M}_1) = \{B \in \mathcal{B}(\mathcal{M}) : |B \cap E_1| \leq r_1 - a_1\}$ and $\mathcal{B}(\mathcal{M}_2) = \{B \in \mathcal{B}(\mathcal{M}) : |B \cap E_2| \leq r_2 - a_2\}$ are the collections of bases of matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. □

Definition 4.3.4. Let $\mathcal{M}[P, Q]$ be a connected lattice path matroid so that the region between P and Q is connected border strip and let p be a path whose vertices are boxes of border strip and its edges are connected boxes, we call $\mathcal{M}[P, Q]$ to be a border strip matroid and we denote it by $S(p)$. ◇

Lemma 4.3.5. *Let $\mathcal{P}(\mathcal{M}[P, Q])$ be a connected lattice path matroid polytope of rank r on $\{1, \ldots, m+r\}$ which is not a border strip path matroid polytope. Then the polytope $\mathcal{P}(\mathcal{M}[P, Q])$ can be decomposed into border strip matroid polytope $\mathcal{P}(S(p))$ where p is range over all paths contained in $\mathcal{M}[P, Q]$.*

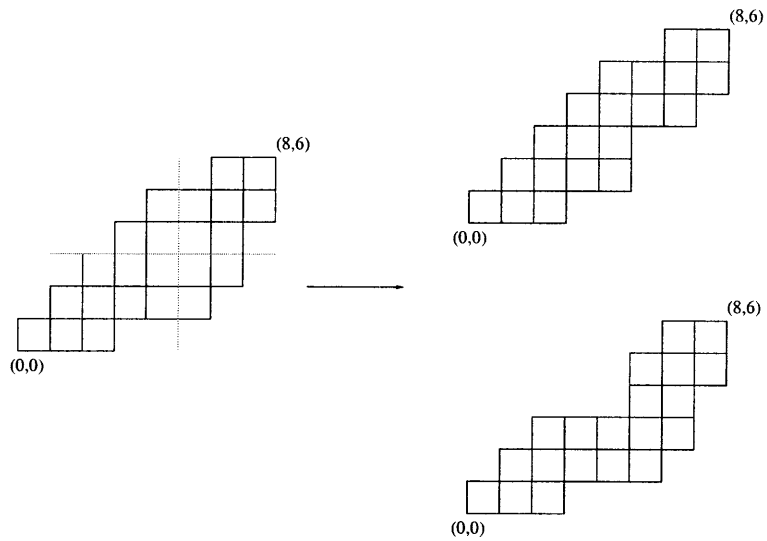


Figure 4-4: Decompositions of matroid polytopes

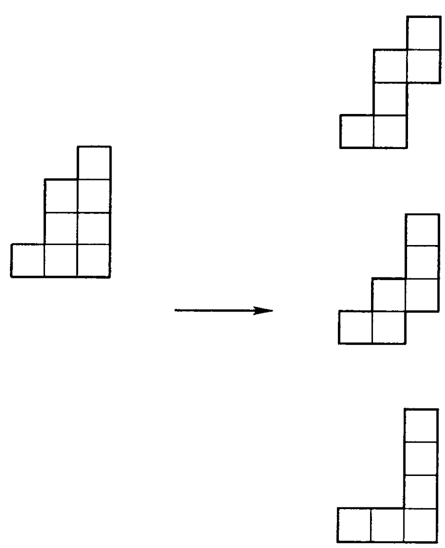


Figure 4-5: Decompositions of matroid polytope to border strip matroid polytopes

Proof. We consider $\mathcal{M}_1$ and $\mathcal{M}_2$ which are lattice path matroids, as described in Corollary 4.3.3. Consider the point $(\alpha+1-j, j)$ on the region of lattice path matroid $\mathcal{M}[P,Q]$. The region of $\mathcal{M}_1$ is obtained by removing boxes of which are above the lines $y=j$ and on the left hand side of the line $x=\alpha-j$. The region of $\mathcal{M}_2$ is also obtained by removing the boxes which are on the right hand side of the vertical line $x=\alpha-j$ and below $y=j$.

Let k be the least positive integer so that $t_k-k\geq\alpha-j$. Let P_1 be the path of length $m+r$ with the following set of North steps $\{t_1,\ldots,t_j,\alpha,\alpha+1,\ldots,\alpha+k-1,t_k,\ldots,t_r\}$. Let be a path Q_1 of length $m+r$ with r North steps $\{s_1,\ldots,s_l,\alpha-l,\ldots,\alpha,s_{k+1},\ldots,s_r\}$, where l is the greatest element so that $s_l-l\leq\alpha-j$. It is easy to verify to see that $\mathcal{M}_1=\mathcal{M}[P_1,Q]$ and $\mathcal{M}_2=\mathcal{M}[P,Q_1]$. See Figure 3.2.

(b) By induction we know that $\mathcal{P}(\mathcal{M}[P_1,Q])$, $\mathcal{P}(\mathcal{M}[P,Q_1])$ can be decompose to border strip path matroid polytopes $\mathcal{P}(S(p))$ for all $p\in[P_1,Q]$ and for all $p\in[P_1,Q]$, respectively.

Since all the paths contain in the region $[P,Q]$ is either in the region $[P,Q_1]$ or in $[P_1,Q]$. Therefore, $\mathcal{P}(\mathcal{M}[P,Q])$ can decompose into to border strip lattice path matroid polytopes $\mathcal{P}(S(p))$ for all $p\in[P,Q]$.

□

Lemma 4.3.6. *Each border strip path matroid polytope can be decomposed into product of simplices via hyperplane splits.*

Proof. Considering the path p and border strip matroid polytope, $\mathcal{M}(S(p))$. We proceed by induction on the length of the path p, we have the following cases:

1. $p=N^{a_1}E^{a_2}\cdots N^{a_k}$ and $p=N^{a_1}E^{a_2}\cdots E^{a_k}$ where $a_1\geq 1$ and $a_2\geq 2$. As we see in Part (1) of Figure 3.2, the hyperplane $x_1+\cdots+x_{a_1+1}=a_1$, split $\mathcal{M}(S(p))$ into two other lattice path matroid polytopes, where each of them is product of a simplex and one other border strip matroid polytope with shorter length, therefore we can proceed by induction.

2. $p=E^{a_1}N^{a_2}\cdots E^{a_k}$ and $p=E^{a_1}N^{a_2}\cdots N^{a_k}$, where $a_1\geq 1$ and $a_2\geq 2$.

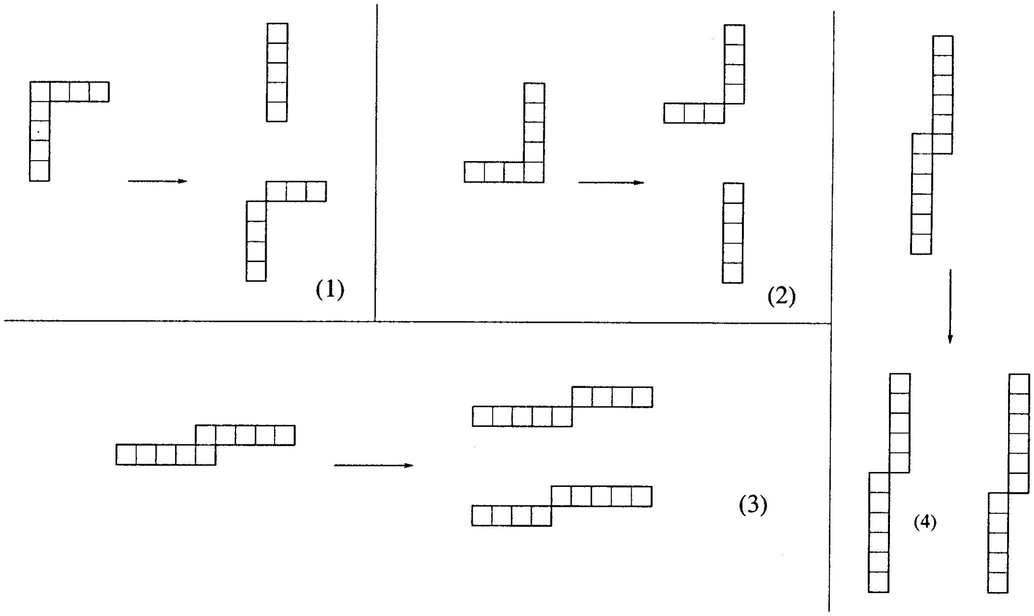


Figure 4-6: Decomposition of border strip matroid polytopes with hyperplane split

As we see in Part (2) of Figure 3.2, the hyperplane $x_{a_1} + \cdots + x_{a_1+a_2} = a_2$, split $\mathcal{M}(S(p))$ into two other lattice path matroid polytopes, where each of them is product of a simplex and one border strip matroid polytope with shorter length, so we can proceed by induction.

3. $p = E^{a_1}N^{a_2}\cdots E^{a_k}$ and $p = E^{a_1}N^{a_2}\cdots N^{a_k}$, where $a_1 \geq 1$ and $a_2 = 1$

4. $p = N^{a_1}E^{a_2}\cdots N^{a_k}$ and $p = N^{a_1}E^{a_2}\cdots E^{a_k}$, where $a_1 \geq 1$ and $a_2 = 1$

   In the above cases, the hyperplane $x_1 + \cdots + x_{a_1-1} + x_{a_1+1} + \cdots + x_{a_k} = r$, where r is the height of p, the rank of matroid $S(p)$, split $\mathcal{M}(S(p))$ into two other lattice path matroid polytopes. Each of these lattice path matroid polytopes is a product of a simplex and one other single lattice path matroid polytope with shorter length. Therefore, we can proceed by induction. See parts (3) and (4) of Figure 3.2.

□

Theorem 4.3.7. *Each lattice path matroid polytope can be decompose into product of simplices using hyperplane splits.*

4.4 Triangulation of Lattice Path Matroid Polytope

A k-simplex is $conv(S) := conv(s_1, \cdots, s_{k+1})$, where $S = \{s_1, \cdots, s_{k+1}\}$ is the collection of $k+1$ affinely independent points in $\mathbb{R}^n$. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope with vertices $v_1, \cdots, v_p$. A triangulation $\mathcal{T}$ of $\mathcal{P}$ is a collection of simplices on the vertices $v_1, \cdots, v_p$ of $\mathcal{P}$ such that

1. If $A \in \mathcal{T}$, then all faces of A are in $\mathcal{T}$,

2. If $A, B \in \mathcal{T}$, then $A \cap B$ is a face of both A and B,

3. $\bigcup_{A \in \mathcal{T}} \text{conv}(A) = \mathcal{P}$.

It is sufficient to list only the highest dimensional simplices of a triangulation. A triangulation $\mathcal{T}$ is unimodular if the volume of every highest dimensional simplex of $\mathcal{T}$ are the same. We also say a simplex is unimodular if its normalized volume is one [DLJF]. Recall that two elements $s, t \in \mathcal{S}$ of a matroid $\mathcal{M} = (\mathcal{S}, \mathcal{I})$ are connected if there exists a circuit that contains s and t. This defines an equivalence relation on $\mathcal{S}$ and moreover $\mathcal{M}$ can be written as a direct sum of restrictions to the connected components of $\mathcal{M}$ [Oxl]. This implies a matroid polytope $\mathcal{P}(\mathcal{M})$ can be written as a direct product of connected matroid polytopes. We note that if a polytope $\mathcal{P}$ is a direct product of polytopes $Q_1, \cdots, Q_k$ which each have a unimodular triangulation, then P has a unimodular triangulation [DLJF]. Hence, we need only consider connected matroids when investigating unimodular triangulations.

The simplex of $\mathcal{P}(\mathcal{M})$ is unimodular if and only if the determinant of the vertices is $rank(\mathcal{M})$. Neil White proposed an algebraic conjecture on the toric ideal determined by a matroid [Whi]. Let $\mathcal{F}$ be a field and define the polynomial ring $S_M := \mathcal{F}[y_B | B \in \mathcal{B}_{\mathcal{M}}]$. Let the toric ideal $I_{\mathcal{M}}$ be defined as the kernel of the homomorphism $S_{\mathcal{M}} \to \mathcal{F}[x_1, \cdots, x_n]$ given by $y_B \to \prod_{i \in B} x_i$. Given any pair of bases $B_1, B_2 \in \mathcal{B}$ the symmetric exchange property states that for every $x \in B_1$ there exists $y \in B_2$ such that $B_1 - x + y$ and $B_2 - y + x$ are bases and it is said that $x \in B_1$ double swaps into B_2. If $B_1 - x + y$ and $B_2 - y + x$ are bases they are called a *double swap*. White [Whi], proposed the following conjecture:

Conjecture 4.4.1. For any matroid $\mathcal{M}$, the toric ideal $I_{\mathcal{M}}$ is generated by the quadratic binomials $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ such that the pair of bases D_1, D_2 can be obtained from the pair B_1, B_2 by a double swap.

A stronger conjecture is that the previous quadratic binomials are in fact a Gröbner bases for some term ordering. This implies that there exists a regular unimodular triangulation of any matroid polytope. The following is a geometric variation of Whites famous conjecture:

Conjecture 4.4.2. Let $\mathcal{M}$ be a matroid of rank r on n elements. There exists a unimodular triangulation of $\mathcal{P}(\mathcal{M})$.

The *hypersimplex* $\Delta_{k,n} \subset \mathbb{R}^n$ is the convex polytope defined as the convex hull of the points ϵ_I, for $I \in \binom{[n]}{k}$. All these $\binom{n}{k}$ points are actually vertices of the hypersimplex because they are obtained from each other by permutations of the coordinates. This $(n-1)$-dimensional polytope can also be defined as

$$\Delta_{k,n} = \{(x_1, \ldots, x_n) \mid 0 \leq x_1, \ldots, x_n \leq 1;\ x_1 + \cdots + x_n = k\}.$$

The following unimodular triangulation of hypersimplex introduced by Sturmfels [Stu].

Let S be a multiset of elements from $[n]$. We define *sort*(S) to be the unique nondecreasing sequence obtained by ordering the elements of S. Let I and J be two k-element subsets of $[n]$, and let *sort*$(I \cup J) = (a_1, a_2, \ldots, a_{2k})$. Then we set $U(I,J) = \{a_1, a_3, \ldots, a_{2k-1}\}$ and $V(I,J) = \{a_2, a_4, \ldots, a_{2k}\}$. For example, for $I = \{1,2,3,5\}$, $J = \{2,4,5,6\}$, we have *sort*$(I \cup J) = (1,2,2,3,4,5,5,6)$, $U(I,J) = \{1,2,4,5\}$ and $V(I,J) = \{2,3,5,6\}$.

We say that an ordered pair (I,J) is *sorted* if $I = U(I,J)$ and $J = V(I,J)$. We call an ordered collection $\mathcal{I} = (I_1, \ldots, I_r)$ of k-subsets of $[n]$ *sorted* if (I_i, I_j) is sorted for every $1 \leq i < j \leq r$. Equivalently, if $I_l = \{I_{l1} < \cdots < I_{lk}\}$ for $l = 1, \ldots, r$, then $\mathcal{I}$ is sorted if and only if $I_{11} \leq I_{21} \leq \cdots \leq I_{r1} \leq I_{12} \leq I_{22} \leq \cdots \leq I_{rk}$. For such a collection $\mathcal{I}$, let $\nabla_{\mathcal{I}}$ denote the $(r-1)$-dimensional simplex with the vertices $\epsilon_{I_1}, \ldots, \epsilon_{I_r}$.

Theorem 4.4.3 (Sturmfels). *The collection of simplices $\nabla_{\mathcal{I}}$, where $\mathcal{I}$ varies over all sorted collections of k-element subsets in $[n]$, is a simplicial complex that forms a triangulation of the hypersimplex $\Delta_{k,n}$.*

Corollary 4.4.4. *The normalized volume of the hypersimplex $\Delta_{k,n}$ is equal to the number of maximal sorted collections of k-subsets in $[n]$.*

Definition 4.4.5. A collection $\mathcal{M}$ of k-subsets of $[n]$ is *sort-closed* if for every two elements I and J in $\mathcal{M}$, the subsets $U(I,J)$ and $V(I,J)$ are both in $\mathcal{M}$. $\diamondsuit$

A sorted subset of $\mathcal{M}$ is a subset of the form $\{I_1, \ldots, I_r\} \subset \mathcal{M}$ such that $(I_1, \ldots, I_r)$ is a sorted collection of k-subsets of $[n]$.

Theorem 4.4.6 (Lam, Postnikov). *The triangulation $\Gamma_{k,n}$ of the hypersimplex induces a triangulation of the polytope $\mathcal{P}_{\mathcal{M}}$ if and only if $\mathcal{M}$ is sort-closed. The normalized volume of $\mathcal{P}_{\mathcal{M}}$ is equal to the number of sorted subsets of $\mathcal{M}$ of size* $\dim(\mathcal{P}_{\mathcal{M}})+1$.

Theorem 4.4.7. *The lattice path matroid is sorted closed matroid, so the unimodular triangulation of hypersimplex is induces triangulation of lattice path matroid polytope.*

Proof. Let $\mathcal{M}[P,Q]$ be a lattice path matroid so it can described as transversal matroid on $\{1, \ldots, m+r\}$ with presentation $(N_i : i \in \{1, \ldots, r\})$, where N_i denotes the interval $[s_i, t_i]$ of integers. Let $C = c_1 < \cdots < c_r$ and $B = b_1 < \cdots < b_r$ be two bases of this matroid. Therefore, $s_1 \leq c_1, b_1 \leq t_1$, $\ldots$, $s_r \leq c_k, b_k \leq t_r$. Let $A = sort(B \cup C) = (a_1, \ldots, a_{2k})$. It is not hard to see that $s_i \leq a_{2i-1}, a_{2i} \leq t_i$, therefore, $U(B,C), V(B,C) \in \mathcal{M}[P,Q]$. $\square$

Theorem 4.4.8. *Let $\mathcal{M}[P,Q]$ be a connected lattice path matroid polytope so that P and Q are paths from $(0,0)$ to (m,r). Let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ be positions of*

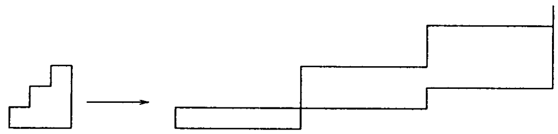


Figure 4-7: The volume of the polytope $\mathcal{P}(\mathcal{C}_3) = \mathcal{P}(\mathcal{M}[E^3N^3, (NE)^3])$, the number of lattice paths between $P' = E^6N^1E^6N^1E^6N^4$ and $Q' = N^1E^6N^2E^6N^2E^6N$ with no consecutive 6, E steps.

north steps in paths P and Q, we define the paths P' and Q' from $(0,0)$ to $(\alpha_r, r(m+r))$ as follows:

1. $P' = E^{\alpha_1}N^{m+r}E^{\alpha_2-\alpha_1}N^{m+r}\cdots E^{\alpha_r-\alpha_{r-1}}N^{m+r}$.

2. $Q' = E^{\beta_1}N^{m+r}E^{\beta_2-\beta_1}N^{m+r}\cdots E^{\beta_r-\beta_{r-1}}N^{m+r}E^{\alpha_r-\beta_r}$.

The normalized volume of the matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$ is the number of lattice paths between P' and Q', so that they don't have $r+m$ consecutive N steps.

Proof. The connected lattice path matroid $\mathcal{M}[P,Q]$ has rank r and the matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$ has a dimension $m+r-1$. Therefore, the normalized volume of $\mathcal{M}[P,Q]$ is the number of sequences $I_{11} \leq I_{21} \leq \cdots \leq I_{m+r1} \leq I_{12} \leq I_{22} \leq \cdots \leq I_{m+r,r}$, so that $\beta_k \leq I_{jk} \leq \alpha_k$ and $I_l = \{I_{l1} < \cdots < I_{lr}\}$ for $1 \leq l \leq m+r$.

It is easy to verify that simplices described above are in bijection with the number of lattice paths between P' and Q' from $(0,0)$ to $(\alpha_r, r(m+r))$ with no consecutive $m+r$, N steps. So, the normalized volume of the matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$ is the number of lattice paths between P' and Q' so that it does not have $r+m$ consecutive N steps. □

Corollary 4.4.9. *The normalized volume of the Catalan matroid polytope $\mathcal{P}(\mathcal{C}_n) = \mathcal{P}(\mathcal{M}[E^nN^n, (NE)^n])$ is the number of paths from $(0,0)$ to $(2n, 2n^2)$ with no consecutive $2n$ North steps which lies between path P and Q. Where $Q = E^1N^{2n}E^2N^{2n}\cdots E^2N^{2n}E$ and $P = E^{n+1}N^{2n}E^1N^{2n}\cdots E^1N^{2n}$ which are the paths paths from $(0,0)$ to $(2n, 2n^2)$.*

Consider a lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$. As we discuss in Lemma 4.2.7, this polytope can be defined as follows,

1. $p_i \leq x_1 + \cdots + x_i \leq q_i$ for all $1 \leq i \leq m+r$, where $x_1 + \cdots + x_{m+r} = m$,

2. $0 \leq x_i \leq 1$.

By applying the result of Lam and Postnikov regarding the volume of alcoved polytope, we can obtain another combinatorial interspersion for the volume of lattice path matroid polytopes:

Lemma 4.4.10 (Lam, Postnikov). *Consider the alcoved polytope $\mathcal{P} \subset \mathbb{R}^n$ is defined by the hyperplane $x_1 + x_2 + \cdots + x_n = k$, the inequalities $0 \leq x_i \leq 1$ together with inequalities of the form*

$$b_{ij} \leq x_{i+1} + \cdots + x_j \leq c_{ij}$$

for integer parameters b_{ij} and c_{ij} for each pair (i, j) satisfying $0 \leq i < j \leq n-1$.

Let $W_{\mathcal{P}} \subset S_{n-1}$ be the set of permutations $w = w_1 w_2 \cdots w_{n-1} \in S_{n-1}$ satisfying the following conditions:

1. *w has $k-1$ descents.*
2. *The sequence $w_i \cdots w_j$ has at least b_{ij} descents. Furthermore, if $w_i \cdots w_j$ has exactly b_{ij} descents, then $w_i < w_j$.*
3. *The sequence $w_i \cdots w_j$ has at most c_{ij} descents. Furthermore, if $w_i \cdots w_j$ has exactly c_{ij} descents, then we must have that $w_i > w_j$.*

Consider the lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$, where P and Q are paths from $(0,0)$ to (m,r). Therefore, $P = E^{\alpha_1} N^{\alpha_2} \cdots N^{\alpha_{2k}}$ and also $Q = N^{\beta_1} E^{\beta_2} \cdots E^{\beta_{2s}}$. As we discuss before, the polytope $\mathcal{P}(\mathcal{M}[P,Q])$ can be determined by the following inequalities,

1. $p_i \leq x_1 + \cdots + x_i \leq q_i$ for all $1 \leq i \leq m+r$, where $x_1 + \cdots + x_{m+r} = r$,
2. $0 \leq x_i \leq 1$,

where p_i and q_i be the number of N steps occur in the first i steps of paths P and Q.

Corollary 4.4.11. *Consider the alcoved polytope $\mathcal{P}(\mathcal{M}[P,Q])$ is defined by the hyperplane $x_1 + x_2 + \cdots + x_{m+r} = r$, the inequalities $0 \leq x_i \leq 1$ together with inequalities of the form*

$$p_i \leq x_1 + \cdots + x_i \leq q_i$$

for integer parameters p_i and q_i for each i satisfying $1 \leq i \leq m+r-1$.

Let $W_{\mathcal{P}} \subset S_{m+r-1}$ be the set of permutations $w = w_1 w_2 \cdots w_{m+r-1} \in S_{m+r-1}$ satisfying the following conditions:

1. *w has $r-1$ descents.*

2. *The sequence $w_1 \cdots w_j$ has at least p_j descents. Furthermore, if $w_1 \cdots w_j$ has exactly p_j descents, then $w_1 < w_j$.*

3. *The sequence $w_1 \cdots w_j$ has at most q_j descents. Furthermore, if $w_1 \cdots w_j$ has exactly q_j descents, then we must have that $w_1 > w_j$.*

4.5 Ehrhart Polynomial and Volume of Lattice Path Matroid Polytope

4.5.1 Ehrhart Polynomial and Volume of Matroid Polytopes

To state the results of this section; let us recall that given an integer $k > 0$ and a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, we define $k\mathcal{P} := \{k\alpha | \alpha \in \mathcal{P}\}$ and the function $i(\mathcal{P}, k) := \mathcal{N}(k\mathcal{P} \cap \mathbb{Z}^n)$, where we define $i(\mathcal{P}, 0) := 1$. It is well known that for integral polytopes, as in the case of matroid polytopes, $i(\mathcal{P}, k)$ is a polynomial, called the Ehrhart polynomial of $\mathcal{P}$. Moreover the leading coefficient of the Ehrhart polynomial is the normalized volume of $\mathcal{P}$, where a unit is the volume of the fundamental domain of the affine lattice spanned by $\mathcal{P}$ [Sta3].

The *Ehrhart series* of a polytope $\mathcal{P}$ is the infinite series $\sum_{k=0}^{\infty} i(\mathcal{P}, k) t^k$. We recall the following classic result about Ehrhart series (see e.g., [Hib, Sta3]). Let $\mathcal{P} \subseteq \mathbb{R}^n$ be an integral convex polytope of dimension d. Then it is known that its Ehrhart series is a rational function of the form

$$\sum_{k=0}^{\infty} i(\mathcal{P}, k) t^k = \frac{h_0^* + h_1^* t + \cdots + h_{d-1}^* t^{d-1} + h_d^* t^d}{(1-t)^{d+1}}. \tag{4.8}$$

The numerator is often called the *h^*-polynomial* of $\mathcal{P}$, and we define the coefficients of the polynomial in the numerator of Eq. 4.8, $h_0^* + h_1^* t + \cdots + h_{d-1}^* t^{d-1} + h_d^* t^d$, as the h^*-*vector* of $\mathcal{P}$, which we write as $h^*(\mathcal{P}) := (h_0^*, h_1^*, \ldots, h_{d-1}^*, h_d^*)$.

A vector $(c_0, \ldots, c_d)$ is *unimodal* if there exists an index p, $0 \leq p \leq d$, such that $c_{i-1} \leq c_i$ for $i \leq p$ and $c_j \geq c_{j+1}$ for $j \geq p$. Due to its algebraic implications, several authors have studied the unimodality of h^*-vectors (see [Hib] and [Sta3]).

It is well-known that if the Ehrhart ring of an integral polytope $\mathcal{P}$, $A(\mathcal{P})$, is Gorenstein, then $h^*(\mathcal{P})$ is unimodal, and symmetric [Hib, Sta3]. Nevertheless, the vector $h^*(\mathcal{P})$ can be unimodal even when the Ehrhart ring $A(\mathcal{P})$ is not Gorenstein. For matroid polytopes, their Ehrhart ring is indeed often not Gorenstein. For instance, De Negri and Hibi [De] prove explicitly when the Ehrhart ring of a uniform matroid polytope is Gorenstein or not. De Loera et al computed the explicit Ehrhart polynomials of matroid polytopes, they observe that their coefficients are always positive. De Loera et al conjecture that:

Conjecture 4.5.1. Let $\mathcal{P}(M)$ be the matroid polytope of a matroid M.

(A) The h^*-vector of $\mathcal{P}(\mathcal{M})$ is unimodal.

(B) The coefficients of the Ehrhart polynomial of $\mathcal{P}(\mathcal{M})$ are positive.

They verify the Conjecture 4.5.1 for all uniform matroids up to 75 elements as well as for a finite variety of non-uniform matroids which are collected at [Haw].

They show that

Theorem 4.5.2.

(1) Conjecture 4.5.1 is true for all uniform matroids up to 75 *elements and all uniform matroids of rank 2. It is also true for all matroids listed in [Haw].*

(2) Let $\mathcal{P}(U^{3,n})$ be the matroid polytope of a uniform matroid of rank 3 *on n elements, and let I be a non-negative integer. Then there exists $n(I) \in \mathbb{N}$ such that for all $n \geq n(I)$ the h^*-vector of $\mathcal{P}(U^{3,n})$, $(h_0^*, \ldots, h_n^*)$, is non-decreasing from index 0 to I. That is, $h_0^* \leq h_1^* \leq \cdots \leq h_I^*$.*

4.5.2 Sections of order Cone and Stanley-Pitman Polytope

Let P be a partial ordering of the set $\{\alpha_1, \dots, \alpha_p\}$, such that if $\alpha_i < \alpha_j$ then $i < j$. A *linear extension* of P is an order-preserving bijection $\pi : P \longrightarrow [p]$, so that if $z < z'$ in P, then $\pi(z) < \pi(z')$. We will identify π with the permutation $a_1 \cdots a_p$ of $[p]$ defined by $\pi(\alpha_{a_i}) = i$. In particular, the identity permutation $12 \cdots p$ is a linear extension of P. Given $\pi = a_1 \cdots a_p \in \mathcal{L}(P)$ define A_π to be the set of all order preserving maps $f : P \longrightarrow \mathbb{R}$ such that

1. $f(\alpha_{a_1}) \leq f(\alpha_{a_2}) \leq \cdots \leq f(\alpha_{a_p})$, and
2. $f(\alpha_{a_j)} < f(\alpha_{a_{j+1}})$ if $a_j > a_{j+1}$.

Theorem 4.5.3 (Stanley-Pitman). *The set of all order-preserving maps $f : P \longrightarrow \mathbb{R}$ is the disjoint union of sets A_π as π ranges over $\mathcal{L}(P)$.*

Define the *order cone* $\mathcal{L}(P)$ of the poset P to be the set of all order-preserving maps $f : P \longrightarrow \mathbb{R}_{\geq 0}$. Thus $\mathcal{L}(P)$ is a pointed polyhedral cone in the space $\mathbb{R}^P$. Assume that P has unique maximal element $\hat{1}$, and let $t_1 < \cdots < t_n = \hat{1}$ be a chain C in P.

Let $x_1, \dots, x_n$ be nonnegative real numbers, set $u_i = x_1 + \cdots + x_i$ and $u = (u_1, \dots, u_n)$. Let W_u denote the subspace of $\mathbb{R}^P$ defined by $f(t_i) = u_i$ for $1 \leq i \leq n$. Define the *order cone section* $\mathcal{L}_C(P, u)$ to be the intersection $\mathcal{L}(P) \cap W_u$ restricted to the coordinates $P - C$.

Equivalently, $\mathcal{L}_C(P, u)$ is the set of all order-preserving maps $f : P - C \longrightarrow \mathbb{R}_{\geq 0}$ such that the extension of f to P defined by $f(t_i) = u_i$ remains order-preserving. Note that $\mathcal{L}_C(P, u)$ is bounded since for all $s \in P - C$ and all $f \in \mathcal{L}_C(P, u)$. We have $0 \leq f(s) \leq u_n$, therefore, $\mathcal{L}_C(P, u)$ is a convex polytope in $\mathbb{R}^{P-C}$. Moreover, dim $\mathcal{L}_C(P, u) = |P - C|$ provided each $x_i > 0$. (Or in certain other situation, such as when no element of $P - C$ is greater than t_1)

Given any integer polytope $P \subset \mathbb{R}^m$, write $\mathcal{N}(P) = \#(P \cap \mathbb{Z}^m)$. The lattice point enumerator $\mathcal{N}(xP)$ is called the *Ehrhart Polynomial* of integer polytope P and is denoted by $i(P, x)$.

We are going to explain Stanley formula for the number of integer points in $\mathcal{L}_C(P, u)$.

Given $\pi = a_1 \cdots a_p \in \mathcal{L}(P)$, where $h_i(\pi)$ for the *height* t_i in π, i.e., $t_i = \pi^{-1}(a_{h_i(\pi)})$. Thus $1 \leq h_1(\pi) < \cdots < h_n(\pi) = p$. Also write

$$d_i(\pi) = \{j : h_{i-1}(\pi) \leq j < h_i(\pi), a_j > a_{j+1}\}, \tag{4.9}$$

where we set $h_0(\pi) = 0$ and $a_0 = 0$. Thus $d_i(\pi)$ is the number of *descents* of π appearing between $h_{i-1}(\pi)$ and $h_i(\pi)$. The number of ways to choose k objects with repetition from a set of n objects is given by $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$. Pitman and Stanley in [SP] have the following results,

Theorem 4.5.4 (Pitman, Stanley).

$$\mathcal{N}(\mathcal{L}_C(P,u)) = \sum_{\pi \in \mathcal{L}(P)} \prod_{i=1}^{n-1} \left(\binom{x_i - d_i(\pi) + 1}{h_i(\pi) - h_{i-1}(\pi) - 1}\right).$$

Proof. Fix $\pi = a_1 \cdots a_p \in \mathcal{L}(P)$. Write $h_i = h_i(\pi)$ and $d_i = d_i(\pi)$. Let $f : P \to \mathbb{R}$ be an order-preserving map such that (a) $f \in A_\pi$, $(b) f(t_i) = u_i = x_1 + \cdots + x_i$ (c) The restriction $f|_{P-C}$ of f to $P - C$ satisfies $f|_{P-C} \in \mathcal{L}_C(P,u)$, where f satisfies (a) and (b), are given by

$$\begin{aligned} 0 \leq c_1 \leq c_2 \leq \cdots \leq c_{h_1} = x_1 \leq c_{h_1+1} \leq \cdots \leq c_{h_2} = x_1 + x_2 \\ \leq \cdots \leq c_p = x_1 + \cdots + x_n. \end{aligned} \tag{4.10}$$

$$c_j < c_{j+1} \text{ if } a_j > a_{j+1}. \tag{4.11}$$

Let $\alpha, \beta, m \in \mathbb{N}$ and $0 \leq j_1 < j_2 < \cdots < j_q \leq m$. Elementary combinatorial reasoning shows that the number of integer vectors $(r_1, \ldots, r_m)$ satisfying $\alpha = r_0 \leq r_1 \leq \cdots \leq r_m \leq r_{m+1} = \alpha + \beta$, $r_{j_i} < r_{j_i+1}$ for $1 \leq i \leq q$ is equal to $\left(\binom{\beta-q+1}{m}\right)$. Hence the number of integer sequences satisfying is given by

$$\left(\binom{x_1 - d_1 + 1}{h_1 - 1}\right)\left(\binom{x_2 - d_2 + 1}{h_2 - h_1 - 1}\right) \cdots \left(\binom{x_n - d_n + 1}{h_n - h_{n-1} - 1}\right)$$

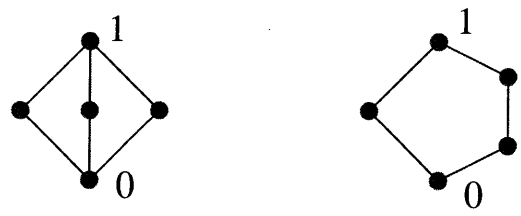


Figure 4-8: Poset $Q_3 = \mathbf{2} \times \mathbf{3}$

summing over all $\pi \in \mathcal{L}(P)$. $\square$

The polytope $\Pi_n(\mathbf{x})$ defined as follows is known as Stanley-Pitman polytope,

$$\Pi_n(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : y_i \geq 0 \text{ and } y_1 + \cdots + y_i \leq x_1 + \cdots + x_i, \text{ for all } 1 \leq i \leq n\}. \tag{4.12}$$

Let us say that two integer polytopes $P \subset \mathbb{R}^k$ and $Q \subset \mathbb{R}^m$ are *integrally equivalent* if there is an affine transformation $\phi : \mathbb{R}^k \to \mathbb{R}^m$ whose restriction to P is a bijection $\phi : P \to Q$, and such that if $\mathcal{A}$ denotes affine span, then ϕ restricted to $\mathbb{Z}^k \cap \mathcal{A}(P)$ is a bijection, $\phi : \mathbb{R}^k \cap \mathcal{A}(P) \to \mathbb{R}^m \cap \mathcal{A}(Q)$. It follows that P and Q have the same combinatorial type and the same integral structure and hence the same volume, Ehrhart polynomial, etc.

For arbitrarily $\mathbf{x} = (x_1, \ldots, x_n)$, we are going to apply the above theory to $\Pi_n(\mathbf{x})$, and explain the results in [SP] on $\Pi_n(\mathbf{x})$. Now let $\mathbf{i}$ denote an i-element chain, and let $Q_n = \mathbf{2} \times \mathbf{n}$, the product of a two element chain with an n element chain. We regard the elements of Q_n as $\alpha_1, \ldots, \alpha_{2n}$ with $\alpha_1 < \cdots < \alpha_n$, $\alpha_{n+1} < \cdots < \alpha_{2n}$, and $\alpha_i < \alpha_{n+i}$ for $1 \leq i \leq n$. Let $t_i = \alpha_{n+i}$ and C be the chain $t_1 < \cdots < t_n$. Let $x_1, \ldots, x_n \geq 0$ and set $u_i = x_1 + \cdots + x_i$. The polytope $\mathcal{L}_C(Q_n, u) \subset \mathbb{R}^{Q_n - C} \cong \mathbb{R}^n$ thus by definition we have the following inequalities,

$$0 \leq f_1 \leq \cdots \leq f_n$$
$$f_i \leq u_i,\ 1 \leq i \leq n.$$

Let $y_i = f_i - f_{i-1}$ (with $f_0 = 0$). Then the above inequalities for $\Pi_n(\mathbf{x})$ become,

- $y_i \geq 0$ for $1 \leq i \leq n$,
- $y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$.

The transformation $y_i = f_i - f_{i-1}$ induces an integral equivalence between $\mathcal{L}_C(Q_n, u)$ and $\Pi_n(\mathbf{x})$. Hence, the result of the above section, when specialized to $P = Q_n$ are directly applicable to $\Pi_n(\mathbf{x})$.

Consider $\pi = a_1 \cdots a_p \in \mathcal{L}(Q_n)$. Write $j_i = h_i(\pi)$ and $d_i = d_i(\pi)$, let $f : Q_n \to \mathbb{R}$ be an order-preserving map such that (a) $f \in A_\pi$, $(b) f(t_i) = u_i = x_1 + \dots + x_i$ (c) The restriction $f|_{Q_n - C}$ to $Q_n - C$ satisfies $f|_{Q_n-C} \in \mathcal{L}_C(Q_n, u)$, where f satisfies (a) and (b), are given by

$$0 \le c_1 \le c_2 \le \dots \le c_{j_1} = x_1 \le c_{j_1+1} \le \dots \le c_{j_2} = x_1+x_2 \le \dots \le c_{j_n} = x_1+\dots+x_n. \tag{4.13}$$

where $1 \le j_1 < \dots < j_n = 2n$ and $j_i \le 2i$. The number of such sequences is just the Catalan number $C_n = \dfrac{1}{n+1}\binom{2n}{n}$. If we set $k_i = j_i - j_{i-1} - 1$(with $j_0 = 0$) then the sequence $\mathbf{k} = (k_1, \dots, k_n)$ are satisfy the following equation:

$$K_n = \left\{ \mathbf{k} \in \mathbb{N}^n : \sum_{i=1}^{j} k_i \ge j \text{ for all, } 1 \le j \le n-1 \text{ and} \sum_{i=1}^{n} k_i = n \right\}. \tag{4.14}$$

Moreover, in the linear extension $a_1 \cdots a_{2n}$ there are no descents to the left of $n+1$, there is exactly one descent between $n+i$ and $n+i+1$ provided that $k_i \ge 1$. If $k_i = 0$ then there re no descent between $n+i$ and $n+i+1$. So $d_i \ge 1$ if and only if $k_i \ge 1$ and $i > 1$. Therefore we conclude that

$$\mathcal{N}(\Pi_n(\mathbf{x})) = \sum_{k \in K_n} \left(\!\!\binom{x_1+1}{k_1}\!\!\right) \prod \left(\!\!\binom{x_i}{k_i}\!\!\right). \tag{4.15}$$

4.5.3 The Ehrhart Polynomial and the Volume of Generalized Catalan Matroid Polytope

The generalized Catalan matroid polytope $\mathcal{P}(\mathcal{C}_a) = \mathcal{P}(\mathcal{M}[(E^m N^r, N^{a_1} E^{a_2} N^{a_3} \cdots E^{a_{2k}}])$ can be described as follows, where $x_i = 1$ for $a_{2j} + 1 \le i \le a_{2j+1}$ and $x_i = 0$ for $a_{2j-1} + 1 \le i \le a_{2j}$.

1. $0 \leq y_i \leq 1$ for $1 \leq i \leq m+r$,

2. $y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$,

3. $y_1 + \cdots + y_{m+r} = x_1 + \cdots + x_{m+r} = r$.

Let $\mathbf{x} = (x_1, \ldots, x_{m+r})$ and $u_i = x_1 + \cdots + x_i$. It is not hard to see that $\mathcal{P}(\mathcal{C}_a) = \Pi_{m+r}(\mathbf{x}) \cap \Delta_{m+r,r}$. Let $\mathcal{M}_C(Q_{m+r}, u)$ be a subpolytope of $\mathcal{L}_C(Q_{m+r}, u)$ so that $c_{m+r} = r$ and $0 \leq c_{i+1} - c_i \leq 1$, which is the intersection of $\mathcal{L}_C(Q_{m+r}, u)$ with alcoved polytope $c_{m+r} = r$ and $0 \leq c_{i+1} - c_i \leq 1$, where $c_i = y_1 + \cdots + y_i$. It is easy to verify that the polytopes $\mathcal{M}_C(Q_{m+r}, u)$ and $\mathcal{P}(\mathcal{C}_a)$ are integrally equivalent.

Let us consider $c_i = y_1 + \cdots + y_i$ for $1 \leq i \leq m+r$. We can order c_i's for $i = 1, \ldots, m+r$ as follows:

$0 \leq c_1 \leq \cdots \leq c_{k_1} \leq 1 < c_{k_1+1} \leq \cdots \leq c_{k_1+k_2} \leq 2 < \cdots \leq r-1 < c_{k_1+\cdots+k_{r-1}+1} \leq \cdots \leq c_{k_1+\cdots+k_r} = r$.

Let us p_i to be the number of steps on the path $N^{a_1}E^{a_2}N^{a_3}\cdots E^{a_{2k}}$ which are not exceed row i. It easy to verify that $k_1 + \cdots + k_r = m+r$ and $k_1 + \cdots + k_i \geq p_i$ for $i = 1, \cdots, r$. Therefore, the number of lattice points in $\mathcal{M}_C(Q_{m+r}, u)$ and so $\mathcal{P}(C_a)$ can be counted as follows,

Lemma 4.5.5. *The number of lattice points in generalized Catalan matroid polytope can be determined as:*

$$\sum_{k \in K^a_{m+r}} (k_1 - 1), \tag{4.16}$$

where

$$K^a_{m+r} = \left\{ \boldsymbol{k} \in \mathbb{N}^{m+r} : \sum_{i=1}^{j} k_i \geq p_i \text{ for all } 1 \leq j \leq m+r-1, k_i \geq 1 \text{ and} \sum_{i=1}^{r} k_i = m+r \right\}.$$

Proof. For each sequence of $\mathbf{k} \in K^a_{m+r}$. The number of subsequences of

$0 \leq c_1 \leq c_2 \leq \cdots \leq c_{k_1-1} = 1$ is $k_1 - 1$ and the number subsequences of $i-1 < c_{k_{i-1}+1} \leq \cdots \leq c_{k_i} = i$ is one. $\square$

For Catalan matrioid polytope we have $p_i = 2i$. Therefore,

$$K_{2n} = \left\{ \mathbf{k} \in \mathbb{N}^{2n} : \sum_{i=1}^{j} k_i \geq 2j \text{ for all } 1 \leq j \leq 2n-1, k_i \geq 1 \text{ and } \sum_{i=1}^{n} k_i = 2n \right\}.$$

So, we have

Lemma 4.5.6. *The number of lattice points in Catalan matroid polytope can be determined as:*

$$\sum_{k \in K_{2n}} (k_1 - 1), \tag{4.17}$$

We can describe integer points in $\mathcal{N}(tC_a)$ to be the sequence $c_1 \leq \cdots \leq c_{m+r}$ satisfy the Eq. 4.18, together with the additional condition $0 \leq c_{i+1} - c_i \leq t$ for all $1 \leq i \leq m+r-1$.

$$\begin{aligned} 0 \leq c_1 \leq c_2 \leq \cdots \leq c_{k_1} \leq t < c_{k_1+1} \leq \cdots \leq c_{k_1+k_2} \leq 2t \\ \leq \cdots \leq c_{k_1+\cdots+k_r} = rt. \end{aligned} \tag{4.18}$$

The following conditions partially include above and they help us to obtain the upper bound for Ehrhart polynomial of generalized Catalan matroid polytopes.

1. $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{k_1} = t$, where $c_i - c_{i-1} \leq t$.
2. $(i-1)t < c_{k_1+\cdots+k_{i-1}+1} \leq \cdots \leq c_{k_1+\cdots+k_i} = (i)t$.

The number of sequences satisfy Part 1 of the above equation is $\left(\binom{t+1}{k_1}\right)$, and the number sequences that satisfy part 2 is $\left(\binom{t}{k_i}\right)$.

Theorem 4.5.7. *The upper bound for Ehrhart polynomial of generalized Catalan matroid polytope can be described as follows:*

$$\sum_{k \in K^a_{m+r}} \left(\binom{t+1}{k_1}\right)\left(\binom{t}{k_2}\right)\left(\binom{t}{k_3}\right)\cdots\left(\binom{t}{k_r}\right).$$

As a conclusion we have,

Theorem 4.5.8. *The upper bound for the volume of generalized Catalan polytope can be described as follows:*
$\sum_{k \in K^a_{m+r}} \frac{1}{(k_1)!(k_2)!(k_3)! \cdots (k_r)!}$.

Corollary 4.5.9. *The upper bound for Ehrhart polynomial of Catalan matroid polytope can be described as follows:*

$$\sum_{k \in K_{2n}} \left(\binom{t+1}{k_1}\right)\left(\binom{t}{k_2}\right)\left(\binom{t}{k_3}\right) \cdots \left(\binom{t}{k_n}\right).$$

As a conclusion we have the upper bound for the volume of Catalan polytope can be described as follows:
$\sum_{k \in K_{2n}} \frac{1}{(k_1)!(k_2)!(k_3)! \cdots (k_n)!}$.

Now, we are going to calculate the Ehrhart polynomial of generalized Catalan matroid polytope. As we discuss before, for any integer t we can calculate $\mathcal{N}(tC_a)$ by enumerating the number of following sequences:

1. $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{k_1} = t - s_1$, where $c_i - c_{i-1} \leq t$, the number of such sequences is $\left(\binom{t+1-s_1}{k_1}\right)$.

2. For $1 < i < r$, $(i-1)t < (i-1)t + s_{2i-2} = c_{k_1+\cdots+k_{i-1}+1} \leq \cdots \leq c_{k_1+\cdots+k_i} = (it) - s_{2i-1} \leq (i)t$, the number of such sequences are $\left(\binom{t-s_{2i-1}-s_{2i-2}}{k_i}\right)$.

3. For $i = r$, $(r-1)t < (r-1)t + s_{2r-2} = c_{k_1+\cdots+k_{r-1}+1} \leq \cdots \leq c_{k_1+\cdots+k_r} = (r)t$, the number of such sequences are $\left(\binom{t-s_{2r-2}}{k_r}\right)$.

Since the distance of each two consecutive elements are at most t, therefore we have $s_1 \leq t, s_1 + s_2 \leq t$, $s_2 + s_3 \leq t, \cdots, s_{2(r-2)} + s_{2(r-1)} \leq t, s_{2(r-1)} \leq t$.

Let us define the set

$$S_{m+r}(t) = \{s = (s_1, \dots, s_{2(r-1)}) \text{ so that } s_1 \le t, s_1 + s_2 \le t, \\ \dots, s_{2(r-1)-1} + s_{2(r-1)} \le t, s_{2(r-1)} \le t\}. \tag{4.19}$$

By above discussion, we have,

Theorem 4.5.10. *Ehrhart polynomial of generalized Catalan matroid polytope can be computed as follows:*

$$\sum_{s \in S_{m+r}(t)} \sum_{k \in K^a_{m+r}} \left(\!\binom{t+1-s_1}{k_1}\!\right)\left(\!\binom{t-s_2-s_3}{k_2}\!\right) \cdots \left(\!\binom{t-s_{2r-1}}{k_r}\!\right). \tag{4.20}$$

$$K^a_{m+r} = \left\{ k \in \mathbb{N}^{m+r} : \sum_{i=1}^{j} k_i \ge p_i \text{ for all } 1 \le j \le m+r-1, k_i \ge 1 \text{ and,} \sum_{i=1}^{r} k_i = m+r \right\}.$$

Corollary 4.5.11. *Ehrhart polynomial of generalized Catalan matroid polytope can be computed as follows:*

$$\sum_{s \in S_{2n}(t)} \sum_{k \in K_{2n}} \left(\!\binom{t+1-s_1}{k_1}\!\right)\left(\!\binom{t-s_2-s_3}{k_2}\!\right) \cdots \left(\!\binom{t-s_{2n-1}}{k_n}\!\right). \tag{4.21}$$

where,

$$K_{2n} = \left\{ k \in \mathbb{N}^{2n} : \sum_{i=1}^{j} k_i \ge 2j \text{ for all } 1 \le j \le 2n-1, k_i \ge 1 \text{ and} \sum_{i=1}^{n} k_i = 2n \right\}.$$

4.5.4 Ehrhart Polynomial and Volume of Lattice Path Matroid polytope

Considering the lattice path matroid polytope $\mathcal{P}(\mathcal{M}[P,Q])$, where P and Q, are paths from $(0,0)$ to (m,r). Let p_i and q_i be the number of N steps occur in the first i steps of paths P and Q, respectively, where $1 \leq i \leq m+r$. Clearly, $p_{m+r} = q_{m+r} = r$. $\mathcal{P}(X)$ lies in the region $[P,Q]$ if and only if $p_i \leq x_1+\cdots+x_i \leq q_i$ for all $1 \leq i \leq m+r$. Therefore, the polytope $\mathcal{P}(\mathcal{M}[P,Q])$ can be determined by the following inequalities:

1. $p_i \leq x_1 + \cdots + x_i \leq q_i$ for all $1 \leq i \leq m+r$, where $x_1 + \cdots + x_{m+r} = r$,

2. $0 \leq x_i \leq 1$.

Let us denote $x_1, x_1+x_2, \ldots, x_1+\cdots+x_{m+r}$ by $c_1, \ldots, c_{m+r}$ so $p_i \leq c_i \leq q_i$ and $c_{m+r} = r$.

Consider $a_1, \ldots, a_{r-1}$ and $b_1, \ldots, b_{r-1}$, so that $a_k + 1 = \min\{i, p_i \geq k+1\}$ and $b_k + 1 = \min\{i, q_i \geq k+1\}$. We define the set of arrays of positive integers $\Gamma(P,Q)$ as follows:

1. $\alpha_1 + \cdots + \alpha_r = m + r$,

2. $a_i \leq \alpha_1 + \cdots + \alpha_i \leq b_i$ for $i \leq r-1$,

3. $\alpha_i \geq 1$.

Consider the point $\mathbf{x} = (x_1, \ldots, x_{m+r})$ and $c_i = x_1 + \cdots + x_i$. It is easy to verify that the integer point $\mathbf{x} = (x_1, \ldots, x_{m+r})$ is in $\mathcal{P}(\mathcal{M}[P,Q])$ if and only if for some $\alpha = (\alpha_1, \ldots, \alpha_r) \in \Gamma(P,Q)$ we have that

$$0 \leq c_1 \leq \cdots \leq c_{\alpha_1} \leq 1 < c_{\alpha_1+1} \leq \cdots \leq c_{\alpha_1+\alpha_2} \leq 2$$
$$< \cdots \leq c_{\alpha_1+\cdots+\alpha_{r-1}} \leq r-1 < \cdots \leq c_{\alpha_1+\cdots+\alpha_r} = r. \tag{4.22}$$

and $c_{\alpha_1} \geq 1$. Therefore, we have

Theorem 4.5.12. *The number of lattice points in $\mathcal{P}(\mathcal{M}[P,Q])$ is $|\Gamma(P,Q)|$.*

We define the set $S(t)$ be the set of $(s_1, s_2, \ldots, s_{2r-2})$ so that $s_i + s_{i+1} \leq t$. The integer points in $t\mathcal{P}(\mathcal{M}[P,Q])$ are in bijection with the following set of sequences. For any $\alpha \in \Gamma(P,Q)$, the sequences

1. $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{\alpha_1} = t - s_1$, where $c_i - c_{i-1} \leq t$, the number of such sequences is $\left(\binom{t+1-s_1}{\alpha_1}\right)$.
2. For $1 < i < r$, $(i-1)t \leq (i-1)t + s_{2i-2} = c_{\alpha_1+\cdots+\alpha_{i-1}+1} \leq \cdots \leq c_{\alpha_1+\cdots+\alpha_i} = (i)t - s_{2i-1} \leq (i)t$, the number of such sequences are $\left(\binom{t-s_{2i-1}-s_{2i-2}}{\alpha_i}\right)$.
3. For $i = r$, $(r-1)t \leq (r-1)t + s_{2r-2} = c_{\alpha_1+\cdots+\alpha_{r-1}+1} \leq \cdots \leq c_{\alpha_1+\cdots+\alpha_r} = (r)t$, the number of such sequences are $\left(\binom{t-s_{2r-2}}{\alpha_r}\right)$.

Since the distance of each two consecutive elements are at most t, therefore we have $s_1 \leq t, s_1 + s_2 \leq t$, $s_2 + s_3 \leq t, \cdots, s_{2(r-2)} + s_{2(r-1)} \leq t, s_{2(r-1)} \leq t$.

Let us consider the following set

$$S_{m+r}(t) = \{s = (s_1, \ldots, s_{2(r-1)}) \text{ so that } s_1 \leq t, s_1 + s_2 \leq t,$$
$$\ldots, s_{2(r-1)-1} + s_{2(r-1)} \leq t, s_{2(r-1)} \leq t\}. \tag{4.23}$$

Observing the above facts, we can compute the Ehrhart polynomial of lattice path matroid polytope as follows:

Theorem 4.5.13.

$$\sum_{\alpha \in \Gamma(P,Q)} \sum_{s \in S(t)} \left(\binom{t+1-s_1}{\alpha_1}\right)\left(\binom{t-s_2-s_3}{\alpha_2}\right)\cdots\left(\binom{t-s_{2r-2}}{\alpha_r}\right). \tag{4.24}$$

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