Critical Sets of Full Latin squares

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Vaipuna Raass

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Abstract

This thesis explores the properties of critical sets of the full n-Latin square and related combinatorial structures including full designs, $(m, n, 2)$-balanced Latin rectangles and n-Latin cubes.

In Chapter 3 we study known results on designs and the analogies between critical sets of the full n-Latin square and minimal defining sets of the full designs.

Next in Chapter 4 we fully classify the critical sets of the full $(m, n, 2)$-balanced Latin square, by describing the precise structures of these critical sets from the smallest to the largest.

Properties of different types of critical sets of the full n-Latin square are investigated in Chapter 5. We fully classify the structure of any saturated critical set of the full n-Latin square. We show in Theorem 5.8 that such a critical set has size exactly equal to $n^3 - 2n^2 - n$. In Section 5.2 we give a construction which provides an upper bound for the size of the smallest critical set of the full n-Latin square. Similarly in Section 5.4, another construction gives a lower bound for the size of the largest non-saturated critical set. We conjecture that these bounds are best possible.

Using the results from Chapter 5, we obtain spectrum results on critical sets of the full n-Latin square in Chapter 6. In particular, we show that a critical set of each size between $(n-1)^3 + 1$ and $n(n-1)^2 + n - 2$ exists.

In Chapter 7, we turn our focus to the completability of partial k-Latin squares. The relationship between partial k-Latin squares and semi-k-Latin squares is used to show that any partial k-Latin square of order n with at most $(n-1)$ non-empty cells is completable.

As Latin cubes generalize Latin squares, we attempt to generalize some of the results we have established on k-Latin squares so that they apply to k-Latin cubes. These results are presented in Chapter 8.

To my beautiful wife ***Tili***

and our adorable kids ***Sina Jr*** *and* ***'Asa Jr,***

And in loving memory of my beloved parents

'Asa *and* ***Sina***

who both passed away while I was writing this thesis.

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I wish to thank my associate supervisors Tim Stokes and Kevin Broughan for their insightful advices and support, and for the occasional proofread of my thesis; Glenys, the rest of the staff, and my fellow postgraduate students for making the maths department the perfect environment for studying.

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‘Ofa lahi atu mo e hūfaki.

Statement of Originality

Unless otherwise acknowledged in the text, this thesis is the result of my own research.

Chapter 4 includes joint work with Nicholas Cavenagh. Section 4.2 to Section 4.5 of this chapter are published in [23] where I was chiefly responsible for the results in Section 4.4 and Section 4.5.

Chapter 5 includes joint work done with Nicholas Cavenagh published in [24].

Chapter 7 was submitted for publication [85].

To check the completability of partial k-Latin squares in Chapters 5, 6 and 7, we modified a computer program written by Professor Ian Wanless to generate completions of partial multi-Latin squares. The original program was written to complete partial Latin squares but since our focus in these chapters was determining which partial k-Latin squares were either critical sets or premature, the program was modified to rapidly complete a partial multi-Latin square (of order $n \leq 52$) and exit if it could not be completed, had exactly one completion, or after two completions were found.

Vaipuna Raass

Contents

Chapter 1

Introduction

The colourful history of Latin squares goes (at least) as far back as the year 1000, when they were being engraved on amulets by certain Arab and Indian communities and were deemed to have the power to cast out evil spirits [95]. It appears that the oldest book to contain Latin squares is "Shams al-Ma'arif al-Kubra" (or "The sun of great knowledge") by the Arab Sufi, Ahmad ibn Aliibn Yusuf al-Buni, published around the year 1200 [95]. Whether it was their supposedly super-natural powers or their entertaining value that was drawing attention to these combinatorial structures, Latin squares have, nevertheless, developed into a major subject in combinatorics with a number of useful applications.

The earliest referenced literature on Latin squares, however, was published in the eighteenth century. In 1723, the solution to the old card problem of arranging 16 cards of a deck of playing cards (consisting of the Aces, Kings, Queens and Jacks) in a 4×4 array so that each denomination and each suit appears only once in each row and column (equivalently, a pair of mutually orthogonal Latin squares of order 4), was published in a new edition of Ozanam's four-volume treatise [71]. In 1779, Euler posed The Problem of the 36 Officers in [49] which initiated a systematic development of the study of Latin squares. This was carried on by Cayley [25] in 1890, who showed that the multiplication table of a group is an appropriately bordered special type of Latin square.

The theory of Latin squares was also instrumental in the development of finite geometries which started early in the nineteenth century.

In the 1930s, a major application of Latin squares was opened by Fisher [51] who used them and other combinatorial structures in the design of statistical experiments. More recently, Latin squares have been used in processor scheduling for massively parallel computer systems [82], used as error-detecting and error-correcting codes in wireless message transmission [14], proposed for various cryptographic schemes [13, 37, 78, 67] and hash functions [92, 93], and put forward as a possible secret-sharing scheme [33, 87, 54].

Latin squares have also evolved into various natural generalizations; one of which is the main subject of this thesis.

1.1 Latin arrays

For convenience, we adopt the notation $N(a)$ for the set of positive integers $\{1, 2, ..., a\}$.

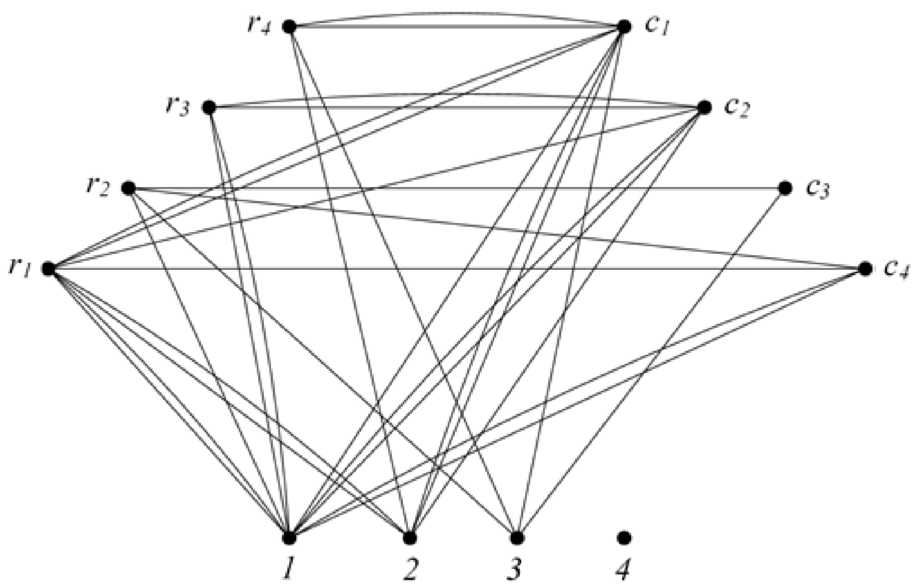

Figure 1.1: A tripartite multigraph

We assume the reader is familiar with standard definitions in graph theory (see Appendix 10.1). Let G be a tripartite multigraph with partite sets

$R = \{r_1, r_2, ..., r_m\}$, $C = \{c_1, c_2, ..., c_n\}$ and $S = N(t)$. We refer to any decomposition of G into triangles as a *Latin array.* Suppose that G is the tripartite graph in Figure 1.1 with partite sets $R = \{r_1, r_2, r_3, r_4\}$, $C = \{c_1, c_2, c_3, c_4\}$ and $S = \{1, 2, 3, 4\}$. Then the following triangle decomposition of G:

$$\begin{aligned} L(G) = \; & \{\{r_1, c_1, 2\}, \{r_1, c_1, 3\}, \{r_2, c_2, 1\}, \{r_2, r_2, 1\}, \{r_3, c_3, 3\}, \\ & \{r_3, c_4, 1\}, \{r_4, c_1, 1\}, \{r_4, c_1, 2\}, \{r_4, c_2, 2\}, \{r_4, c_4, 1\}\} \end{aligned}$$

corresponds to the array:

2,3			
	1,1		
		3	1
1,2	2		1

with rows and columns indexed by R and C. Here each triangle of the form $\{r_i, c_j, s\}$ corresponds to the occurrence of the symbol s in cell $L_{i,j}$, where $r_i \in R, c_j \in C$ and $s \in S$. We can thus think of $L(G)$ as either an array or as a set of ordered triples. We may also replace each triangle of the form $\{r_i, c_j, s\}$ with the ordered triple (i, j, s). For example, the Latin array above may also be represented as

$$\begin{aligned} L(G) = \; & \{(1, 1, 2), (1, 1, 3), (2, 2, 1), (2, 2, 1), (3, 3, 3), \\ & (3, 4, 1), (4, 1, 1), (4, 1, 2), (4, 2, 2), (4, 4, 1)\}. \end{aligned}$$

We switch freely between these equivalent representations.

A Latin array is such a very general structure that many of the main combinatorial structures in this thesis can be defined as a type of Latin array. If G is a simple complete tripartite graph with:

- $m \leq n$ and $n = t$, then $L(G)$ is a *Latin rectangle* of order n.

- $m = n = t$, then $L(G)$ is a *Latin square* of order n.

An example of a Latin square of order 5 is given below.

2	1	5	3	4
3	5	2	4	1
5	2	4	1	3
1	4	3	5	2
4	3	1	2	5

Note that each element of S occurs exactly once in each row and column of a Latin square.

Another generalization of Latin squares is formed if $m = n = t$ and there are k edges between each vertex from distinct partite sets of G. Here, $L(G)$ is a *multi-Latin square* of order n and index k (or a *k-Latin square* of order n). Equivalently, a k-Latin square of order n is an $n \times n$ array of multisets of cardinality k from $N(n)$ with each symbol occurring exactly k times in each row and exactly k times in each column (see [22]). The following is an example of a 3-Latin square of order 4.

1,2,4	1,2,3	2,3,4	1,3,4
1,1,3	2,2,4	1,3,4	2,3,4
2,4,3	1,3,4	1,2,2	1,3,4
4,2,3	1,3,4	1,3,4	1,2,2

Note that if G above is a complete tripartite simple graph (i.e. $k = 1$), $L(G)$ is a 1-Latin square or simply a Latin square. If $k = n$ and $L(G) = \{(r, c, s) \mid r \in R, c \in C, s \in S\} = R \times C \times S$, then we define the corresponding n-Latin square of order n containing $N(n)$ in each cell to be the *full n-Latin square*. The following is the full 4-Latin square.

1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4

The next structure corresponds to a complete tripartite multigraph, G, where each vertex of R is connected exactly t times to each vertex of C and exactly n times to each vertex of S; and each vertex of C is connected exactly m times to each vertex of S. Here, $L(G)$ is thus an $m \times n$ array of multisets of size t such that each element of $N(t)$ occurs n times in each row and m times in each column. We refer to $L(G)$ as a (m, n, t)-*balanced Latin rectangle* (or an $m \times n$ t-*balanced Latin rectangle*). The example below is a $(4, 5, 3)$-balanced Latin rectangle.

1,2,3	1,1,2	2,2,3	1,2,3	1,3,3
1,2,2	1,2,3	1,2,3	1,2,3	1,3,3
1,3,3	1,2,3	1,2,3	1,1,3	2,2,2
1,2,3	2,3,3	1,1,3	2,2,3	1,1,2

Thus, for an (m, n, t)-*balanced Latin rectangle*, $L(G)$ is a (possibly multi-)set of ordered triples $(r, c, s) \in N(m) \times N(n) \times N(t)$, such that:

- for each $r \in N(m)$ and $c \in N(n)$, there are t triples of the form (r, c, s);
- for each $r \in N(m)$ and $s \in N(t)$, there are n triples of the form (r, c, s);
- for each $c \in N(n)$ and $s \in N(t)$, there are m triples of the form (r, c, s).

In the array form, we may trivially construct an (m, n, t)-balanced Latin rectangle for any $m, n, t \geq 1$ by placing the set $N(t)$ in each cell of an $m \times n$ array, which gives the *full* (m, n, t)-balanced Latin rectangle. Clearly the full (n, n, n)-balanced Latin rectangle is also the full n-Latin square.

An (m,n,t)-balanced Latin rectangle is also known as an *exact* (n,m,t) *Latin rectangle* [3, 4].

The last Latin structure we define under the umbrella of Latin arrays is a semi-Latin square. A *semi-Latin square* of order n and index k (or a *semi-k-Latin square* of order n) is a triangle decomposition of a complete tripartite multigraph where $|R| = |C| = n$, $S = N(kn)$, k edges connect each vertex of R to each vertex of C, and each vertex of either R or C is connected exactly once to each vertex of S. In the array form, a *semi-Latin square* is an $n \times n$ array of sets of cardinality k (subsets of $N(kn)$) such that each element of $N(kn)$ occurs exactly once in each row and each column. The following example is a semi-3-Latin square of order 4.

1,4,5	2,6,10	3,7,8	9,11,12
2,3,11	1,4,9	5,6,12	7,8,10
6,7,9	3,8,12	1,10,11	2,4,5
8,10,12	5,7,11	2,4,9	1,3,6

Other combinatorial structures which are also Latin arrays include *school timetables* [63] and *match-tables* [65].

Since each of the above structures is a type of Latin array, we can give a general definition of a partially filled-in Latin array. A *partial Latin array* is any partial decomposition of G into triangles (that is, any set of edge-disjoint triangles in G). Equivalently, we may think of a partial Latin array as any triangle decomposition of a tripartite multigraph H where $H \subseteq G$. Thus:

- a *partial Latin square* of order n is an $n \times n$ array such that each element of $N(n)$ occurs at most once in each row and at most once in each column;

- a *partial k-Latin square* of order n is an $n \times n$ array of multisets of size at most k such that each element of $N(n)$ occurs at most k times in each row and at most k times in each column;

- a *partial (m,n,t)-balanced Latin rectangle* is an $m \times n$ array of multisets of size at most t such that each element of $N(t)$ occurs at most n times in each row and at most m times in each column; and

- a *partial semi-k-Latin square* is an $n \times n$ array of sets of size at most k such that each element of $N(kn)$ occurs at most once in each row and at most once in each column.

For any partial Latin array, we may ask whether or not it completes to a Latin array with the same parameters. That is: 'Does a partial decomposition of G into triangles complete to a decomposition of G into triangles?' If $L(G)$ is a partial Latin array and there is an unique Latin array $L'(G)$ such that $L(G) \subseteq L'(G)$, then we say that $L(G)$ is a *defining set* of $L'(G)$. If upon removing any triangle from $L(G)$, the partial Latin array formed is no longer a defining set, then we say that a defining set $L(G)$ is a *critical set* of $L'(G)$.

In particular, a defining set of the full n-Latin square has a unique completion to the n-Latin square L of order n that is the full n-Latin square.

We identify any partial Latin array as *saturated* if each cell is either empty or contains $N(t)$. Thus a critical set of the full n-Latin square is *saturated* if each cell is either empty or contains $N(n)$. Otherwise it is *non-saturated*. The following squares are examples of saturated and non-saturated critical sets respectively, each for the full 4-Latin square (see Section 5.1 and Section 5.2).

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	

1			
	2,3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4

We may similarly define critical sets for the full (m, n, t)-balanced Latin rectangle as being either saturated or non-saturated.

Because each cell of a multi-Latin square contains a multiset, we ask the reader to take note of the following multiset notations. If we denote the multiplicity of an element s in a multiset A by $\nu_A(s)$, then:

- $\nu_{A\cap B}(s) = \min\{\nu_A(s), \nu_B(s)\}$,

- $\nu_{A\cup B}(s) = \max\{\nu_A(s), \nu_B(s)\}$,

- $\nu_{A\setminus B}(s) = \max\{0, \nu_A(s) - \nu_B(s)\}$,

- $\nu_{A\uplus B}(s) = \nu_A(s) + \nu_B(s)$,

where $A \uplus B$ is the *multiset sum* of the multisets A and B. The *size* or the number of entries in a Latin array L, denoted by $|L|$, is the cardinality of the multiset sum of the multisets in each cell of L (i.e. the sum of multiplicities of each element over all the cells). The size of any partial Latin array $L(G)$ is also the number of edges between R and C used in triangles. Thus the above critical sets have sizes 36 and 28, respectively.

1.2 Trades in Latin arrays

We may think of a trade as one of a pair of disjoint (partial) triangle decompositions of a tripartite multigraph. Thus to construct a trade for a Latin array, we simply take two distinct triangle decompositions of the array and delete any triangles common to both sets. We formally define a trade in a Latin array as follows. For a tripartite multigraph G, if $L(G)$ is a triangle decomposition of G, then a *trade*, T, in $L(G)$ is defined as $T = L(G) \setminus L'(G)$ where $L'(G)$ is some triangle decomposition of G distinct to $L(G)$. The trade, $T' = L'(G) \setminus L(G)$, in $L'(G)$ is called the *disjoint mate* of T. We may refer to the pair (T, T') as a *bitrade*.

For example, let G be the tripartite graph in Figure 1.2 and let

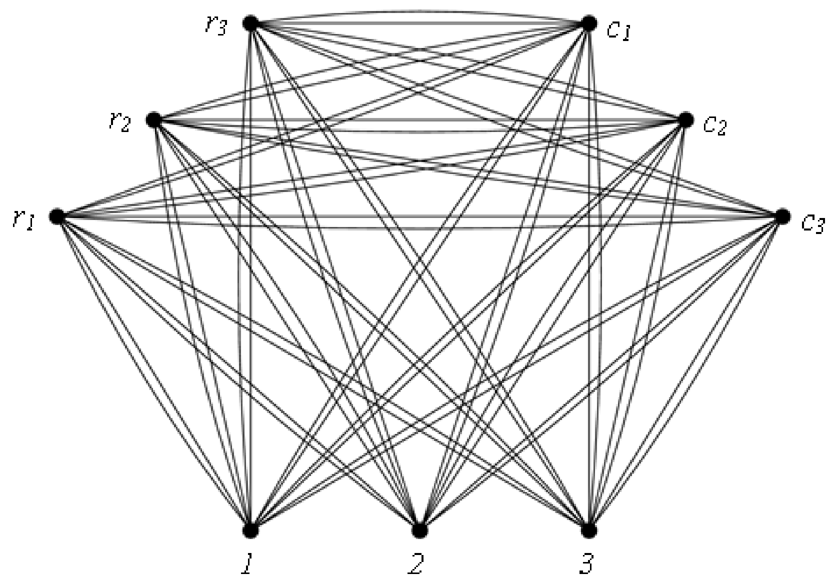

Figure 1.2: The graph G

$$\begin{aligned} L(G) = \ & \{(1,1,1),(1,1,2),(1,2,1),(1,2,3),(1,3,2),(1,3,3), \\ & (2,1,1),(2,1,2),(2,2,1),(2,2,3),(2,3,2),(2,3,3), \\ & (3,1,3),(3,1,3),(3,2,2),(3,2,2),(3,3,1),(3,3,1)\} \end{aligned}$$

=

1,2	1,3	2,3
1,2	1,3	2,3
3,3	2,2	1,1

and

$$\begin{aligned} L'(G) = \ & \{(1,1,3),(1,1,3),(1,2,1),(1,2,1),(1,3,2),(1,3,2), \\ & (2,1,1),(2,1,1),(2,2,2),(2,2,2),(2,3,3),(2,3,3), \\ & (3,1,2),(3,1,2),(3,2,3),(3,2,3),(3,3,1),(3,3,1)\} \end{aligned}$$

=

3,3	1,1	2,2
1,1	2,2	3,3
2,2	3,3	1,1

.

Then $T = L(G) \setminus L'(G)$ and $T' = L'(G) \setminus L(G)$ (in the array form below) is a bitrade.

1,2	3	3
2	1,3	2
3,3	2,2	

T

3,3	1	2
1	2,2	3
2,2	3,3	

T'

If $T = L(G) \setminus L'(G)$ contains only 4 elements of the form

$$\{(r, c, s), (r, c', s'), (r', c, s'), (r', c', s)\},$$

we refer to such a trade as an *intercalate*. Since by definition, there must be at least two symbols in each row and column of a trade, an intercalate is the smallest possible trade in any given Latin array.

Observe that in the array form above, we may define a bitrade as a pair of non-empty partial Latin arrays which are disjoint, have the same number of entries in corresponding cells, and whose corresponding rows and columns contain the same multisets of symbols. For this reason, replacing a trade by its disjoint mate in a Latin array yields another Latin array of the same graph. It also implies that if the complement of a partial Latin array contains a trade then it may be completed in more than one way. So naturally, there is a strong connection between trades and defining/critical sets of Latin arrays as shown by the following lemmas.

Lemma 1.1 *A partial Latin array D is a defining set of a Latin array L if and only if it intersects every trade in L.*

Proof. Suppose that D is a defining set of L and there exists a trade T in L such that $D \cap T = \emptyset$. Then D is also contained in the Latin array $(L \setminus T) \cup T'$ and thus is not a defining set. Conversely, suppose that D is a partial Latin array contained in both L and L' where $L \neq L'$. Then $T = L \setminus L'$ is a trade with disjoint mate $L' \setminus L$ and $D \cap T = \emptyset$. □

Lemma 1.2 *A partial Latin array C is a critical set of a Latin array L if C is a defining set of L and each element of C belongs to a trade in L.*

Proof. By definition, any critical set is a defining set. Suppose that $(r, c, s) \in C$ is not contained in any of the trades in L. Then (from the previous lemma) $C \setminus \{(r, c, s)\}$, which intersects every trade in L, is a smaller defining set and thus C is not a critical set. □

The above lemmas generalize an analogous result on Latin squares (see for example [19]).

1.3 Equivalence classes of Latin arrays

We now discuss equivalences of Latin arrays, that is, we look at ways in which two Latin arrays have identical structures (i.e. are combinatorially equivalent).

Let G and G' be isomorphic tripartite graphs (see Appendix 10.1). If G' can be obtained by permuting two or more elements in at least one of the partite sets (with the partite sets fixed) of G, then the Latin arrays $L(G)$ and $L(G')$ are *isotopic*. Simply put, we obtain an isotopic Latin array by permuting the rows, columns and/or symbols of the original array. For example, the two Latin arrays below are isotopic as we simply switch the first two rows of the first array to obtain the second.

2,3			
	1,1		
		3	1
1,2	2		1

$\equiv$

	1,1		
2,3			
		3	1
1,2	2		1

The set of all Latin arrays of the same order can be partitioned into equivalent classes called *isotopy classes*, such that two arrays in the same class are isotopic.

On the other hand, if we can obtain G' by permuting the partite sets of G then $L(G)$ and $L(G')$ are said to be *conjugates* or *parastrophes*. Equivalently, we can obtain the conjugates of a Latin array $L(G)$ by reordering the triples in $L(G)$. We formally define these six conjugates below.

- $L = L(G)$,
- $L* := \{(j, i, s)|(i, j, s) \in L\}$,
- ${}^{-1}L := \{(s, j, i)|(i, j, s) \in L\}$,
- $L^{-1} := \{(i, s, j)|(i, j, s) \in L\}$,
- ${}^{-1}(L^{-1}) := \{(j, s, i)|(i, j, s) \in L\}$,
- and $({}^{-1}L)^{-1} := \{(s, i, j)|(i, j, s) \in L\}$.

(The above notation is used in [38].)

Finally, if G' is obtained by a combination of the two types of permutations we defined above, then the corresponding arrays are said to be *main class equivalent*, *paratropic* or belong to the same *species* [94].

For example, the following Latin array $L(G)$ is obtained from the graph in Figure 1.1.

$$L(G) = \{(1,1,2),(1,1,3),(2,2,1),(2,2,1),(3,3,3), (3,4,1),(4,1,1),(4,1,2),(4,2,2),(4,4,1)\}$$

=

2,3			
	1,1		
		3	1
1,2	2		1

By swapping the sets R and C and swapping the symbols 1 and 2 in S we obtain:

$$L(G) = \{(1,1,1),(1,1,3),(2,2,2),(2,2,2),(3,3,3),$$
$$(4,3,2),(1,4,2),(1,4,1),(2,4,1),(4,4,2)\}$$

=

1,3			2
	2,2		1,2
		3	
		2	2

.

So $L(G')$ is isotopic to a conjugate of $L(G)$ and therefore $L(G')$ and $L(G)$ are main class equivalent.

Some Latin arrays are also equivalent to orthogonal arrays. An *orthogonal array* $OA(r,n,k)$ is a $r \times kn^2$ array such that in any two rows, each ordered pair from $N(n)$ occurs exactly k times.

Thus a Latin square L of order n is an orthogonal array $OA(3,n,1)$ which has a column (r,c,s) if and only if the cell $L_{r,c}$ contains s.

We can also use the idea of an orthogonal array to define multi-Latin squares. A k-Latin square L of order n is an orthogonal array $OA(3,n,k)$ which has a column (r,c,s) if and only if the cell $L_{r,c}$ contains s.

The orthogonal array $OA(3,3,1)$ below:

1	1	1	2	2	2	3	3	3
1	2	3	1	2	3	1	2	3
1	2	3	2	3	1	3	1	2

corresponds to the Latin square

1	2	3
2	3	1
3	1	2

while (the following orthogonal array) $OA(3,3,3)$

1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3

3	3	3	3	3	3	3	3	3
1	1	1	2	2	2	3	3	3
1	2	3	1	2	3	1	2	3

corresponds to the full 3-Latin square

1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3

.

Since the three rows of an orthogonal array of a Latin/multi-Latin square L corresponds to its rows, columns and symbols respectively, permuting the rows of the orthogonal array of L maps L to one of its conjugates.

For example, let L be the 3-Latin square of order 3 below:

1,2,2	1,2,3	1,3,3
1,1,3	1,2,3	2,2,3
2,3,3	1,2,3	1,1,2

.

Then the conjugates of L are:

$L* =^{-1} (L^{-1}) = (^{-1}L)^{-1} =$

1,2,2	1,1,3	2,3,3
1,2,3	1,2,3	1,2,3
1,3,3	2,2,3	1,1,2

,

$^{-1}L = L$, and

$L^{-1} =$

1,2,3	1,1,2	2,3,3
1,1,2	2,3,3	1,2,3
2,3,3	1,2,3	1,1,2

.

1.4 Aims and outcomes

A principal aim of this thesis is to examine the properties of critical sets of the full n-Latin square. This idea is motivated by the analogous concept of full designs (see [2, 47, 56, 77, 80]). For block size k, a *full design* simply consists of all the possible subsets of size k from the foundation set $N(v)$ (formal definitions of full designs are given in Chapter 3). In [47], it is shown that any minimal defining set for a design is the result of an intersection of the design with a minimal defining set of the full design of the same order (see Lemma 3.4 from Chapter 3). For Latin squares, in fact, the same result holds. We first show in the following theorem that the intersection of a Latin array L with a defining set of another Latin array L' of the same order where $L \subset L'$, is a defining set of L.

Theorem 1.3 *Let L and L' be two Latin arrays such that $L \subset L'$. If C is a defining set of L' Then $L \cap C$ is a defining set for L.*

Proof. Suppose there exist two distinct Latin arrays L and M based on the same tripartite graph G such that each contain $L \cap C$. Let $T = L \setminus M$ and $T' = M \setminus L$. Then $(L' \setminus T) \uplus T'$ is a Latin array based on the same tripartite graph as L' that contains C but is not equal to L'. Thus C is not a defining set, a contradiction. □

Since every Latin square of order n is a subset of the full n-Latin square, the following corollary is immediate.

Corollary 1.4 *Let C be a defining set of the full n-Latin square and let L be a Latin square of the same order. Then $L \cap C$ is a defining set for L.*

For example, let C and L respectively be the saturated critical set of the full 4-Latin square and 4×4 Latin square below:

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	

C

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

L

Then

$C \cap L =$

1	2	3	
4	1	2	
3	4	1	

is a defining set of L.

We can even go further and show that any critical set of a Latin square of order n is the result of the intersection of a defining set of the full n-Latin square and the Latin square.

Theorem 1.5 *Let L_n be the full n-Latin square and let L be a Latin square of the same order. If C is a critical set of L, then there exists a defining set, D_n of L_n such that $D_n \cap L = C$.*

Proof. Let $D_n = (L_n \setminus L) \cup C$. Since C is a critical set of L, D_n is a defining set of L_n and since $C \subset D_n$, $D_n \cap L = C$. □

The study of critical sets of the full n-Latin square thus has the potential to yield information on critical sets in Latin squares, which have been extensively studied (see Chapter 2 for a survey). Initially, one of the main objectives of this thesis was to study the defining sets (of Latin squares) that result from intersecting the critical sets of the full n-Latin square with Latin squares of the same order. In particular, we aimed to construct a defining set, D, of the full n-Latin square such that for some Latin square L of the same order, $D \cap L$ is a critical set of L with size $\lfloor n^2/4 \rfloor$; the conjectured optimal size of the smallest critical sets of Latin squares ([84, 8, 81]). It turned out that this is quite a simple task and the following corollary of Theorem 1.5 gives such a construction. In this thesis, a Latin square L of order n is said to be *back circulant* if $L = \{(r, c, r + c - 2 \pmod n) + 1) | r, c \in N(n)\}$.

Corollary 1.6 *Let L_n be the full n-Latin square and let B_n be the back circulant Latin square of the same order. If $C = \{(r, c, r + c - 1) | r + c \leq \lfloor n/2 \rfloor + 1\} \cup \{(r, c, r + c - n - 1) | r + c \geq n + \lfloor n/2 \rfloor + 2\}$, then $D_n := (L_n \setminus B_n) \cup C$ is a defining set of L_n, $D_n \cap B_n = C$ is a critical set of B_n and $|C| = \lfloor n^2/4 \rfloor$.*

We illustrate this result in the following example:

For $n = 4$,

$B_n =$

1	**2**	3	4
2	3	4	1
3	4	1	2
4	1	2	**3**

and $C =$

1	2		
2			
			3

$\Longrightarrow D_n = (L_n \setminus L_b) \cup C =$

1,2,3,4	1,**2**,3,4	1,2,4	1,2,3
1,**2**,3,4	1,2,4	1,2,3	2,3,4
1,2,4	1,2,3	2,3,4	1,3,4
1,2,3	2,3,4	1,3,4	1,2,**3**,4

thus $D_n \cap L_b = C$.

Since this was straightforward to show, we turned our focus to other properties of the critical sets of the full n-Latin square. We have at least provided partial answers to the following questions:

- What is the size of the smallest/largest critical set of the full n-Latin square?
- What are the combinatorial structures that generalize (or are generalized by) multi-Latin squares and which results have analogies between these structures?
- What are the properties of trades in these different combinatorial structures?
- For a given natural number n, what critical set sizes exist between the smallest and largest critical set of the full n-Latin square?

For the most part, the answers to the above questions became the various chapters of this thesis.

In Chapter 2, we summarize some of the results from the literature on (or related to) critical sets of Latin squares. As clearly implied by Lemma 1.1 and Lemma 1.2, any results including Latin trades were of particular interest in this chapter. We also give a brief account of the evolution of bounds on the sizes of the smallest/largest critical sets of this combinatorial structure.

Chapter 3 explores relevant results on designs. Here we explore analogies of critical sets of the full n-Latin square and minimal defining sets of the full designs. In the last section of this chapter, we provide a correction to the proof of one of the defining set constructions from [2].

In Chapter 4 we define sets of partial $m \times n$ arrays which completely describe the structure of all critical sets of the full $(m, n, 2)$-balanced Latin rectangle. Theorem 4.2 and Theorem 4.3 show that a partial $(m, n, 2)$-Latin rectangle is a critical set of the full $(m, n, 2)$-balanced Latin rectangle if and only if it is such an array. As (m, n, n)-balanced Latin rectangles are n-Latin rectangles, our aim here was to study the structure of the less complex critical sets of the full $(m, n, 2)$-balanced Latin rectangles then try to generalize these results to the full n-Latin rectangle. Although we did not fully succeed, we obtained significant partial results in Chapter 5.

In an analogous set-up to Chapter 4, we discuss the properties of the different critical sets of full n-Latin squares in Chapter 5. In Section 5.1, we fully classify the structure of any saturated critical set of the full n-Latin square. We show in Theorem 5.8 that such a critical set has size exactly equal to $n^3 - 2n^2 - n$. In Section 5.2 we give a construction which provides an upper bound of $(n-1)^3 + 1$ for the size of the smallest critical set of the full n-Latin square and we conjecture that this is best possible. Section 5.4 gives a lower bound of $n^3 - n^2 - 3n + 4$ for the size of the largest non-saturated critical set.

Chapter 6 is a survey of the spectrum of sizes of critical sets for the full n-Latin square. By studying and generalizing the structures of critical sets of small orders, we provide constructions that complete the lower half of the spectrum from $(n-1)^3 + 1$ to $n(n-1)^2 + 1$. Latter constructions in this chapter give sizes between $n(n-1)^2 + 2$ and $n^3 - n^2 - 3n + 4$ but there are still undetermined values in this part of the spectrum.

Motivated by the fact that a critical set of the full n-Latin square is a completable partial n-Latin square, we study completability of partial k-Latin squares in Chapter 7. In particular we give a generalization of Evans' conjecture for multi-Latin squares. We show that any partial multi-Latin square of order n and index k with at most $(n-1)$ entries or at most $(n-1)$ non-empty cells is also completable.

Chapter 8 discusses k-Latin cubes, which can be thought of as three dimensional generalizations of k-Latin squares. Our focus here is to attempt to generalize known results on Latin squares (and results from other chapters) so that they apply to Latin cubes. Furthermore we generalize these Latin cubes to multi-Latin cubes where each layer is a multi-Latin square.

Finally in Chapter 9 we summarize the main results of this thesis and suggest open problems for future research.

Chapter 2

Known results on Latin squares

In this chapter we review the literature on Latin squares with a focus on the critical sets of these combinatorial structures. The concept of a critical set in a Latin square was introduced by the statistician, J. A. Nelder [83], in 1977. In his note, he posed the question, "for a given Latin square of order n, what is the cardinality of a smallest critical set and, conversely, what is the cardinality of a largest critical set?" Over the years, Nelder's question became a much studied open problem and has been studied by various authors such as Curran and van Rees [35], Smetaniuk [88], C. Colbourn, M. Colbourn, and Stinson [31] between 1978 and 1983 and in recent years by Cooper, Donovan and Seberry [33], Keedwell [72], Mahmoodian [62] and Cavenagh [19].
Moreover, as alluded to by the authors of [9] and [43], critical sets may be a useful way of reducing the storage space required for the Latin squares.

2.1 Latin trades

The idea of a Latin trade was first used to study critical sets of Latin squares [72]. Just as Latin squares are Latin arrays, a Latin trade is a trade in a Latin array that is a Latin square. Informally, a trade in a Latin square is a subset T which can be removed and replaced with a (disjoint) subset T' to create a distinct Latin square. Thus we define a *Latin trade* (also known as a *Latin interchange*) as a non-empty partial Latin square T with a disjoint mate T'

such that they each occupy the same set of non-empty cells, and corresponding rows and columns contain the same sets of symbols. For example:

T

1		2	3
2	1	3	
3	2		1
	3	1	2

and

T'

3		1	2
1	3	2	
2	1		3
	2	3	1

Observe that if a Latin square L contains a trade T, then replacing T by T' gives another Latin square of the same order. A Latin trade on a 2×2 subsquare is called an *intercalate*. An intercalate is the smallest possible type of Latin trade and has only two symbols. That is,

$$\begin{aligned} T &= \{(r,c,s),(r,c',s'),(r',c,s'),(r',c',s)\} \\ \text{and} \quad T' &= \{(r,c,s'),(r,c',s),(r',c,s),(r',c',s')\} \end{aligned}$$

where $(r,c,s) \in L$. For example:

T

1	2
2	1

and

T'

2	1
1	2

2.2 $scs(n)$

For a Latin square of order n, the size of the smallest possible critical set is denoted by $scs(n)$ [83]. It is conjectured that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$ ([84, 8, 81]) and this conjecture has been verified computationally up to $n = 8$ in [84, 35, 8, 1] and [10]. The table below gives examples of critical sets which meet this bound.

<table>
<tr><th>n</th><th>conjectured $scs(n)$</th><th></th></tr>
<tr><td>2</td><td>1</td><td>
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<tr><td>⋮</td><td>⋮</td><td>⋮</td></tr>
<tr><td>n</td><td>$\left\lfloor \frac{n^2}{4} \right\rfloor$</td><td>
<table>
<tr><td>1</td><td>2</td><td>$\cdots$</td><td>k</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>$\cdots$</td><td>k</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
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</table>
where $k = \lfloor n/2 \rfloor$
</td></tr>
</table>

The general form of such a critical set is defined as follows:

$$L = \{(r, c, r+c-1) | r+c \leq \lfloor n/2 \rfloor +1\} \cup \{(r, c, r+c-n-1) | r+c \geq n+\lfloor n/2 \rfloor +2\}$$

In [42], Donovan and Cooper proved that such a partial Latin square is a critical set of the back circulant Latin square of the same order (see [27, 42]). One of the first lower bounds on $scs(n)$ was $scs(n) \geq n + 1$ for $n \geq 5$ [34]. Fu, Fu and Roger [53], improved this result by showing that $scs(n) \geq \frac{7n-3}{6}$ for $n > 20$.

As every critical set C of a Latin square L intersects every Latin trade in L (by Lemma 1.1), determining Latin trades within a Latin square is an important tool in the identification of its critical sets. In [19], Cavenagh uses this fact to show that any critical set with an empty row has at most one empty row, at most one empty column and at most one empty cell. Subsequently, he showed that $scs(n) \geq 2n-32$. In 2007, Cavenagh [20] established a superlinear lower bound for $scs(n)$ showing that

$$scs(n) \geq n \left\lfloor \frac{(\log n)^{\frac{1}{3}}}{2} \right\rfloor.$$

This result is the best known lower bound for $scs(n)$ for general n.

We consider the analogous function for the full $(m, n, 2)$-balanced Latin rectangle in Section 4.5 and for the full n-Latin square in Section 5.2 and Section 5.3. We denote the size of the smallest critical set of a k-Latin square of order n as $scs(n, k)$.

2.3 $lcs(n)$

The size of the largest possible critical sets of a Latin square of order n is denoted $lcs(n)$. The precise value of $lcs(n)$ is also undetermined for general n. Stinson and van Rees [91] showed that for all $n \geq 1$, $lcs(n) \geq 4^n - 3^n$. They established the result using a *doubling* construction method to yield the so-called *2-critical* sets. Each element of these 2-critical sets is contained in an intercalate within the Latin square that intersects the critical set precisely at that element. In [62] Hatami and Mahmoodian used a non-constructive method to show that:

$$\begin{aligned} lcs(n) &\geq n^2\left(1-\tfrac{2+\ln 2}{\ln n}\right)+n\left(1+\tfrac{2\ln 2+\ln(2\pi)}{\ln n}\right)-\tfrac{\ln 2}{\ln n} \\ &= n^2-\tfrac{(2+\ln 2)n^2}{\ln n}+o\left(\tfrac{n^2}{\ln n}\right) \end{aligned}$$

Ghandehari et al [55] improved this result to $lcs(n) \geq n^2 - (e + o(1))n^{\frac{5}{3}}$.

A trivial upper bound on $lcs(n)$ is $n^2 - n$. This result is a direct consequence of the fact that every row/column must have at least one empty cell. Bean and Mahmoodian, [12], improved this result to $lcs(n) \leq n^2 - 3n + 3$ and Horak and Dejter, [68], showed that $lcs(n) \leq n^2 - \frac{7n-\sqrt{n}-20}{2}$. The latter result is the best known upper bound on $lcs(n)$.

Other papers such as [11] and [44] introduce the idea of verifying the spectrum of possible sizes of critical sets. In [44], Donovan and Howse proved that for all n there exist critical sets of order n and size s, such that $\lfloor \frac{n^2}{4} \rfloor \leq s \leq \frac{n^2-n}{2}$, except if $s = \frac{n^2}{4} + 1$, n even [11]. A more recent result in [46] is that there exist critical sets of size s and order 2^n such that $4^{n-1} \leq s \leq 4^n - 3^n$.

In Section 4.5 we determine the exact size of the largest critical set of the full $(m, n, 2)$-balanced Latin rectangle, and a lower bound for that of the full n-Latin square is given in Section 5.4. We denote the size of the largest critical set of a k-Latin square of order n as $lcs(n, k)$.

2.4 Completable partial Latin squares

In general, determining if a partial Latin square is completable to a Latin square is NP-complete (see [30]). In 1945, M. Hall [60] used Hall's theorem [61] (also discussed in Appendix 10.2) to show that any Latin rectangle can be completed to a Latin square of the same order.

Theorem 2.1 [60] *Every $m \times n$ Latin rectangle of order n can be completed to a Latin square of order n*

In [64], Hilton and Johnson showed that if the non-empty cells of a partial Latin square form a rectangle with no empty cells, then Hall's Condition (see Appendix 10.2) is a necessary and sufficient condition for a completion. They achieved this result by showing that in such a case Hall's condition is equivalent to Ryser's condition [86], given by the following theorem.

Theorem 2.2 [86] *Let P be a partial Latin square of order n whose non-empty cells are those in the upper left $r \times s$ rectangle R, for some $r, s \in N(n)$. Then P is completable if and only if $\nu(\sigma) \geq r + s - n$ for each symbol $\sigma \in N(n)$, where $\nu(\sigma)$ is the number of times that σ appears in R.*

Hilton and Vaughan [66] extended Theorem 2.2 by showing that if in such an $r \times s$ rectangle, at most one cell in each column is empty, then Hall's Condition is once again a necessary and sufficient condition for a completion.

In 1960, Evans [50] conjectured that any partial Latin square of order n with at most $n-1$ entries can be completed. This well-known conjecture was proved by Smetaniuk [89] in 1981.

Theorem 2.3 [89] *If A is a partial Latin square of order n with at most $n-1$ entries, then A can be completed to a Latin square of order n.*

Alternate proofs for Theorem 2.3 are given by Häggkvist [59] for large n and Anderson and Hilton [5] for all n.

More results on completable partial Latin squares [7, 28, 58] were triggered by Daykin and Häggkvist when they conjectured in [36] that any $\frac{1}{4}$-dense partial Latin square can be completed (where an ϵ-dense partial Latin square is one in which each symbol, row, and column contains no more than ϵn-many non-empty cells). In [7], the author uses a novel technique derived from [70] to show that all 9.8×10^{-5}-dense partial Latin squares are completable. Very recently, Dukes [48] improved this result by showing that any 0.0288-dense partial Latin square can be completed to a Latin square.

The completability of partial k-Latin squares is studied in Chapter 7.

2.5 Premature partial Latin squares

A partial Latin square is *premature* if it cannot be completed to a Latin square of the same order, but is completable when a single entry of any one of its cells is removed. We thus may think of a premature partial Latin square as a minimally incompletable partial Latin square.

Since any partial Latin square of order n and size at most $n-1$ can be completed (by Theorem 2.3) and premature partial Latin squares of order n and size n exist [15], for example:

1		$\cdots$		
	1	$\cdots$		
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
		$\cdots$	1	
		$\cdots$		2

and

1	2	$\cdots$	n-1	
		$\cdots$		n
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
		$\cdots$		
		$\cdots$		

,

the following lemma is immediate.

Lemma 2.4 *The smallest premature partial Latin square of order n is of size n.*

Introduced in [15], several papers have been written on premature partial Latin squares (see [16, 26, 41]) mainly focusing on the size of a maximum premature partial Latin square. In [15] the authors show that a maximum premature partial Latin square of order n has size asymptotic to n^2 and always has at least $3n-4$ empty cells. In the same paper they conjectured that the number of empty cells in a premature partial Latin square is at least $n^{3/2}$. The authors of [16] showed that a premature partial Latin square containing a row with $n-1$ filled cells contains at least $4n-10$ empty cells. The lower bound on the number of empty cells is further improved in [26] to $7n/2-o(n)$.

In Chapter 7, we explore the more general premature partial k-Latin squares.

Chapter 3

Block Designs

A block design (or simply a design) is a combinatorial structure consisting of a set and a family of its subsets where the elements of these subsets meet certain regularity conditions modelling a certain application. Historically, this application is (related to) the design of statistical experiments [32, 90]. Other applications where block designs have been used include algebraic and finite geometry [69], software testing [29], and cryptography [18]. In this chapter we explore some of the known results on block designs which are precursors to results in this thesis. Section 3.1 discusses the different types of trade in block designs and how they may be constructed. In Section 3.2, we discuss the relationship between trades and defining sets of designs. Section 3.3 explores the properties of the defining sets of full designs. Here we highlight the analogous results to the defining sets of full designs since, in some sense, full designs are analogous to full n-Latin squares. Section 3.4 provides some constructions from the literature [2, 76, 47] that give minimal defining sets of the $(v, 3, v-2)$ full design. In the the last section we provide a correction to the proof of one of the defining set constructions from [2].

Formally, for $v \geq k \geq t > 0$, we define a t-(v, k, λ) design (or a t-design), as a pair (V, B) where $V = \{1, 2, ..., v\}$ (elements of V are called *points*) and B is a set of b k-subsets of V called *blocks*, where each point appears in exactly r blocks and each t-subset of V appears in exactly λ blocks.

A design is identified by the parameters t, v, k, λ, b and r. We refer to the parameters v, λ and b as the *order*, *index* and *size* of the design, respectively. The following lemma is well-known.

Lemma 3.1 *For a t-(v, k, λ) design, D:*

1. $rv = bk$,

2. $\lambda\binom{v}{t} = b\binom{k}{t}$.

Proof. We prove our first claim by counting in two ways the sum of the cardinalities of elements of B. The obvious way is to multiply the number of blocks b by the size k of each block. Alternatively, since each of the v-elements of the base set V appears r times in B, the sum of the cardinalities of elements of B is also rv.

For our second claim, we count the number of times each t-subset of V appears in B. Firstly, there are $\binom{v}{t}$ ways of choosing a t-subset from V and each of these t-subsets appears in exactly λ blocks. On the other hand, there are $\binom{k}{t}$ distinct t-subsets in each of the b blocks of B, hence $\lambda\binom{v}{t} = b\binom{k}{t}$. $\square$

We refer to a design as a *simple design* if it contains no repeated block. A simple design of index 1 is known as a *Steiner design*. The following 2-$(7, 3, 1)$ design is an example of a Steiner design.

$$V = \{1, 2, 3, 4, 5, 6, 7\} \text{ and}$$
$$B = \{\{1, 2, 4\}, \{1, 3, 7\}, \{1, 5, 6\}, \{2, 3, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{4, 5, 7\}\}.$$

In this chapter we focus only on simple designs with $t = 2$ in which case a design is simply referred to as a (v, k, λ) design.

3.1 Trades in block designs

We next discuss the idea of trades in a (v, k, λ) design. Let T_1 and T_2 be subcollections of m blocks of the set $V = N(v)$. We say that T_1 and T_2 are

2-*balanced* if each pair of V is contained in the same number of blocks of T_1 and of T_2. If T_1 and T_2 are disjoint and 2-balanced, then $T = \{T_1, T_2\}$ is a (v, k, λ) *trade* where T_1 and T_2 are said to be *trade mates.*

A simple way to construct a (v, k, λ) trade is by finding the set difference between two designs. Suppose that two (v, k, λ) designs are defined as $D_1 = (V, B_1)$ and $D_2 = (V, B_2)$ with $B_1 \neq B_2$. Similar to Latin arrays, the two collections defined by $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are disjoint and 2-balanced and are therefore trade mates (see Section 1.2). The following example demonstrates how such a trade can be constructed.

Let

$$\begin{aligned} V = \;& \{1,2,3,4,5,6\}, \\ B_1 = \;& \{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,6\},\{1,5,6\},\{2,3,6\}, \\ & \{2,4,5\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}, \quad \text{and} \\ B_2 = \;& \{\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{2,3,4\}, \\ & \{2,3,6\},\{2,4,5\},\{3,5,6\},\{4,5,6\}\}. \end{aligned}$$

Then clearly

$$\begin{aligned} & B_1 \cap B_2 = \{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}, \\ & T_1 = \{\{1,2,3\},\{1,2,4\},\{1,5,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}, \quad \text{and} \\ & T_2 = \{\{1,2,5\},\{1,2,6\},\{1,3,4\},\{2,3,4\},\{3,5,6\},\{4,5,6\}\}. \end{aligned}$$

Similar to a design, we refer to a trade as being a *simple trade* if none of its blocks is repeated and a *Steiner trade* if no pair of V occurs more than once in either trade mate. The trade constructed above is an example of one that is simple but not Steiner.

A Steiner trade of the form

$$T = \{\{u,w,x\},\{u,y,z\},\{v,w,y\},\{v,x,z\}\}$$

and

$$T' = \{\{v,w,x\},\{v,y,z\},\{u,w,y\},\{u,x,z\}\}$$

for distinct symbols u, v, w, x, y, z, is called a *Pasch trade*. This is the smallest possible trade in a block design.

3.2 Defining sets of designs

In this section we use the notation from [2].

The concept of a defining set of a (v, k, λ) design mirrors that of a Latin array (see Section 1.1). That is, a set of blocks that is a subset of an unique (v, k, λ) design D is a *defining set* of D. We denote this by dD. If no proper subset of dD is also a defining set of D, then dD is said to be a *minimal defining set* denoted by d_mD. A defining set of minimum size is called a *smallest defining set* denoted by d_sD.

Defining sets and trades of designs are closely linked. The following lemma (analogous to Lemma 1.1 and Lemma 1.2) introduced by Gray in [56] sums up this connection.

Lemma 3.2 *Let D_1 be a subset of the blocks of the design $D = (V, B)$. Then*

1. *D_1 is a defining set of $D = (V, B)$ if and only if D_1 intersects every trade contained in B.*

2. *D_1 is a minimal defining set of D if and only if D_1 is a defining set and for every block $b_1 \in D_1$, there is a trade T such that $T \cap D_1 = \{b_1\}$.*

Proof. 1. Suppose that D_1 is a defining set of D. If $T_1 \subseteq B$ and T_2 are trade mates and $D_1 \cap T_1 = \emptyset$, then $D_1 \subseteq B \setminus T_1$ and thus D_1 is also a subset of the design $T_2 \uplus (B \setminus T_1)$, a contradiction.

2. Suppose that D_1 is a minimal defining set of D and there exists a block $b^* \in D_1$ such that $T \cap D_1 \neq \{b^*\}$ for each trade T. Then $D_1 \setminus \{b^*\}$ also intersects every trade in D and thus a defining set with less blocks than D; a contradiction.

□

The following lemmas (analogous to Theorem 1.3 and Theorem 1.5 respectively), round up the properties of the defining sets of designs we need to discuss in this section.

Lemma 3.3 [57] *Let A and B be sets of blocks with $A \subseteq B$. If D is a defining set of B, then $D \cap A$ is a defining set of A.*

Proof. Any trade T contained in A is also contained in B. By Lemma 3.2, T intersects D and thus intersects $D \cap A$ also, so $D \cap A$ is a defining set of A. □

Lemma 3.4 [47] *Let A and B be sets of blocks with $A \subseteq B$. If D is a minimal defining set of A, then there exists a minimal defining set D^* of B such that $D^* \cap A = D$.*

Proof. Let $D^* = D \cup B \setminus A$. Clearly D^* is a defining set of B. For each $y \in D$ there is a trade, T, in A which intersects D only at y and thus $D^* \setminus \{y\}$ is not a defining set of B. Suppose that for each $x \in D^* \cap (B \setminus A)$, $D^* \setminus \{x\}$ is not a defining set of B. Then D^* is a critical set of B. Otherwise there exists $x \in D^* \cap (B \setminus A)$ such that $D^* \setminus \{x\}$ is a defining set of B. Remove x from D^*. Recursively apply the last step until D^* is a minimal defining set of B. □

As pointed out in [47], the two results above are of particular relevance when B is a full design and the subset A is a simple design of the same order.

3.3 Full Designs

In this section, we discuss and compare some of the known results and properties of full designs to those of the full n-Latin squares.

We start by defining the more general *quasi-full design* which is any $(v, k, \binom{v-2}{k-2})$ design. A *full design* is the unique simple quasi-full design where B is made up of all possible k-tuples of the elements of V. Below is an example of the full design of order 5.

Example 3.5 Let $v = 5$ and $k = 3$. Thus $\lambda = \binom{3}{1} = 3$ and

$$B = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\},\{2,3,4\}\}.$$

Since all possible k-tuples are contained in the blocks of a full design, it follows that every simple (v, k, λ) design is contained in a full design of the same order, and by Lemma 3.3, any defining set of the full design will contain a defining set of any design of the same order. Furthermore, by Lemma 3.4, if all defining sets of a full design can be classified then intersected with any design of the same order, we get a list of all the defining sets of the design. Of course, if we can determine the minimal defining sets of the full design directly, then fewer steps would be needed in this process. With that said, it is clear that studying the minimal defining sets of full designs (as in [2, 47, 80]) gives valuable information about the minimal defining sets of all (v, k, λ) designs.

3.4 Constructions of defining sets of full designs

For this section we give a survey of constructions for the minimal defining sets of the full $(v, 3, v-2)$ design.

Construction 3.6 [2]

Let $V = \{0, 1, ..., v-1\}$ and

$$F(V) = \{\{x, y, z\} \mid x, y, z \in V, \quad x, y, z \text{ distinct}\}$$

where $v \geq 6$. Define

$$D_1 = F(V \setminus \{0\}) \setminus \{\{1, 2, x\} \mid 3 \leq x \leq v-1\}.$$

The set D_1 contains $\binom{v}{3} - (v^2 - v - 4)/2 = (v^3 - 6v^2 + 5v + 12)/6$ blocks.

Example 3.7 For $v = 6$, we exclude the ten blocks containing 0 and the blocks $\{1,2,3\}$, $\{1,2,4\}$, and $\{1,2,5\}$. Thus

$$D_1 = \{\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}.$$

Construction 3.8 [76]

Let $V = \{0, 1, ..., v-1\}$ and

$$F(V) = \{\{x,y,z\} \mid x,y,z \in V, \quad x,y,z \text{ distinct}\}$$

where $v \geq 5$. Define

$$D_2 = F(V \setminus \{0\}) \setminus F(\{1,2,3,4\}).$$

The set D_2 contains $\binom{v}{3} - (v^2 - 3v + 10)/2 = (v^3 - 6v^2 + 11v - 30)/6$ blocks.

Example 3.9 For $v = 6$, we exclude the ten blocks containing 0 and the blocks

$$F(\{1,2,3,4\}) = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\},$$

thus

$$D_2 = \{\{1,2,5\},\{1,3,5\},\{1,4,5\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}.$$

In a way, the exclusion of every block containing 0 in the above constructions is similar to removing the entries of a row or column of a full n-Latin square. In Section 5.1 of Chapter 5, we will show that a saturated partial n-Latin square is a defining set of the full n-Latin square if and only if it contains no cycle of empty cells. Thus, although not a critical set, such a partial n-Latin square is a defining set of the full n-Latin square.

Three other constructions of minimal defining sets of the $(v, 3, v-2)$ full design are given in [47] where Construction 3 gives the same minimal defining set as D_1 and Constructions 4 and 5 are for $v > 6$. In [2], the authors give minimal defining sets for $3 \leq v \leq 5$ and smallest defining sets for $v = 6, 7$. These results are summarized below.

Let $d_3(v)$ be the size of the full $(v, 3, v-2)$ design. Then

- For $3 \leq v \leq 5$, $d_3(v) = 0$;
- $d_3(6) = 6$; and
- $d_3(7) \leq 15$.

The last result is correct as stated in Theorem 6 of [2], however the proof has an error which we correct in the next section.

3.5 A correction to a defining set construction

Firstly, the defining set of the full $(7, 3, 5)$ design given in the proof as

$$\begin{aligned} S &= \{\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\}, \\ &\quad \{2,3,5\},\{2,3,6\},\{2,3,7\},\{2,4,5\},\{2,4,6\}, \\ &\quad \{2,4,7\},\{2,5,7\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\} \end{aligned}$$

does not intersect the Pasch trade

$$T = \{\{1,2,3\},\{2,5,6\},\{1,6,7\},\{3,5,7\}\}$$

and

$$T' = \{\{1,7,3\},\{7,5,6\},\{1,6,2\},\{3,5,2\}\}.$$

So by Lemma 3.2, S cannot be a defining set. An alternative proof to Theorem 6 of [2] is given below.

Theorem 3.10 $d_3(7) \leq 15$.

Proof. Changing the block {1,4,5} of S to {2,5,6} yields a correct defining set as we now show.

Let

$$V = \{1, 2, ..., 7\}$$

and

$$\begin{aligned} S &= \{\{1,3,6\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\}, \\ &\quad \{2,3,6\},\{2,3,7\},\{2,4,5\},\{2,4,6\},\{2,4,7\}, \\ &\quad \{2,5,6\},\{2,5,7\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}. \end{aligned}$$

To complete the proof we simply need to show that S is a defining set of the full $(7,3,5)$ design, $D=(V,B)$. We start by assuming that S can be extended to a quasi-full design $D'=(V,B')$ and then we show that $D'=D$. Let $S'=B'-S$, s'_i be the number of blocks of S' containing the element i, and ${s'}_{i,j}$ be the number of blocks of S' containing the pair i,j. We observe that:

- ${s'}_6=6, {s'}_{6,7}=5, {s'}_{1,6}={s'}_{2,6}=2$ so $\{1,6,7\},\{2,6,7\}\in S'$.
- ${s'}_2=6, {s'}_{1,2}=5, {s'}_{2,3}={s'}_{2,4}={s'}_{2,5}=1, {s'}_{2,6}={s'}_{2,7}=2$ so the five remaining blocks containing 2 also contain 1 and thus $\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,2,7\}\in S'$.
- ${s'}_{3,6}={s'}_{4,6}={s'}_{5,6}=1$ so the three remaining blocks of S' containing 6 are $\{3,6,7\},\{4,6,7\},\{5,6,7\}$.

Consider

$$\begin{aligned} S'' &= S'\setminus\{\{1,6,7\},\{2,6,7\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\\ &\qquad \{1,2,7\},\{3,6,7\},\{4,6,7\},\{5,6,7\}\}. \end{aligned}$$

For $v=\{2,6\}$, $s''_v=0$ so only five elements appear in the blocks of S''. Also $B\setminus P_3(V)\subseteq S''$ where $P_3(V)$ is the set of all the 3-subsets of set V. Hence the blocks of $T=(B\setminus P_3(V), P_3(V)\setminus B)$ are based on at most five elements. But since every trade must be based on at least six elements, $B\setminus P_3(V)=\emptyset$ and thus $D=D'$. $\square$

Chapter 4

Critical sets of $(m, n, 2)$-balanced Latin rectangles

In this chapter we turn our focus to critical sets of (m, n, t)-balanced Latin rectangles. Any (m, n, t)-balanced Latin rectangle R used in this chapter may be represented in two ways. Firstly, as an $m \times n$ array of multisets, with the set in cell $R_{r,c}$ containing λ occurrences of element s if and only if the triple (i, j, s) has multiplicity λ in R (see the Introduction). We may also represent an (m, n, t)-balanced Latin rectangle as an edge-coloured bipartite graph. Given an (m, n, t)-balanced Latin rectangle R, such a graph B_R has partite sets $N(m)$ and $N(n)$, with λ edges of colour s between vertices r and c whenever the triple (r, c, s) has multiplicity λ in R. Figure 4.1 gives the two representations of the following $(2, 3, 3)$-balanced Latin rectangle.

$$\begin{aligned} R = \ & \{(1,1,1),(1,1,1),(1,1,2),(1,2,1),(1,2,2),(1,2,3),\\ & (1,3,2),(1,3,3),(1,3,3),(2,1,2),(2,1,3),(2,1,3),\\ & (2,2,1),(2,2,2),(2,2,3)(2,3,1),(2,3,1),(2,3,2)\}. \end{aligned}$$

We will switch freely between these equivalent representations in this chapter, using whichever form makes proofs easier to follow.

As (n, n, n)-balanced Latin rectangles are n-Latin squares of order n, our aim is to study the structure of the critical sets of the full $(m, n, 2)$-balanced

1,1,2	1,2,3	2,3,3
2,3,3	1,2,3	1,1,2

$\equiv$

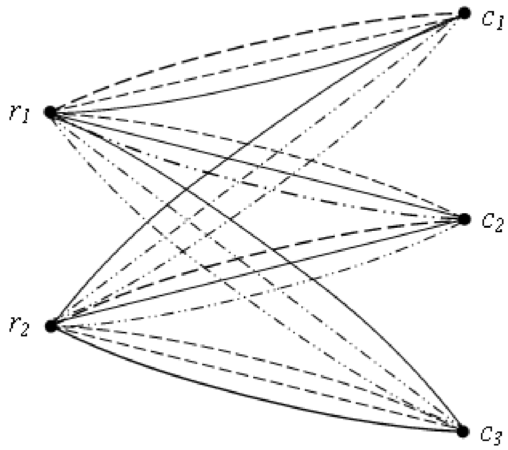

Figure 4.1: A $(2,3,3)$-balanced Latin rectangle

Latin rectangle and apply these results to the full n-Latin square in Chapter 5.

In this chapter, we focus on the case $t = 2$; F is always the full $(m,n,2)$-balanced Latin rectangle and a *saturated* partial $(m,n,2)$-balanced Latin rectangle is one where each non-empty cell contains the entries $\{1,2\}$. The next result, shown in [23], is analogous to Theorem 1.3.

Here we define $\mathcal{A}(R,S)$ (where R and S are integral vectors of orders m and n respectively) as a set of all $m \times n$ 2-Latin rectangles with entries either 1 or 2, and with row and column sums prescribed by R and S.

Theorem 4.1 *Let C be a defining set of F and let $A \in \mathcal{A}(R,S)$ be an $(m,n,2)$-balanced Latin rectangle with $|R| = m$ and $|S| = n$. Then $A \cap C$ is a defining set for A.*

Proof. Suppose that $A \cap C$ is not a defining set for A. Then there exists an $(m,n,2)$-balanced Latin rectangle $A' \in \mathcal{A}(R,S)$ such that $A' \neq A$ and $A \cap C \subset A'$. Let $T = A \setminus A'$ and $T' = A' \setminus A$. Since $A \cap C \subset A \cap A'$, $T \cap C = \emptyset$.

Now T and T' are partial $(m,n,2)$-balanced Latin rectangles with the same set of occupied cells, with 1 and 2 occurring the same number of times in each row and column. Thus $G := (F \setminus T) \uplus T'$ is an $(m,n,2)$-balanced Latin rectangle. Furthermore since $T \cap C = \emptyset$, $C \subset G$. Since $G \neq F$, C is not a defining set for F, a contradiction. $\square$

In [23], the symbol 2 of an element of $\mathcal{A}(R, S)$ is replaced by 0 to give a $(0,1)$-matrix. Thus the results in this chapter also give us useful information in identifying defining sets in $(0,1)$-matrices.

In Section 4.1, we give examples of critical sets of full $(n, n, 2)$-balanced Latin rectangles of small orders highlighting the similarities in their structures. We then generalize these examples to give the precise structure of a critical set of the full $(m, n, 2)$-balanced Latin rectangle in Section 4.2. We do this by describing sets of partial $m \times n$ arrays $A[\boldsymbol{a}, \boldsymbol{b}]$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of non-negative integers satisfying certain properties. Then in Theorem 4.2 and Theorem 4.7, we show that if $(\boldsymbol{a}, \boldsymbol{b})$ is a pair of "good" vectors (see Definition 4.2), then a partial $(m, n, 2)$-balanced Latin rectangle is a critical set of the full $(m, n, 2)$-balanced Latin rectangle if and only if it is an element of $A[\boldsymbol{a}, \boldsymbol{b}]$. This gives a classification of such critical sets. Section 4.3 gives the analogous result (on $(m, n, 2)$-balanced Latin rectangles) to Lemma 1.2 and we use this result in Section 4.4 to prove Theorem 4.2. In Section 4.5 we show that when $m, n \geq 2$, the size of the smallest critical set of the full $(m, n, 2)$-balanced rectangle is $(m-1)(n-1)+1$ and the size of the largest critical set of the full $(m, n, 2)$-balanced Latin rectangle is $2(m-1)(n-1)$. To conclude this chapter we generalize the definition of $A[\boldsymbol{a}, \boldsymbol{b}]$ for the full $(m, n, 2)$-balanced Latin rectangle (in Section 4.2) to construct critical sets of the (m, n, t)-balanced Latin rectangle.

Section 4.2 to Section 4.5 is joint work with Nicholas Cavenagh and is published in [23].

4.1 Examples of critical sets of the full $(n, n, 2)$-balanced Latin rectangle

In this section we give examples of critical sets of the full $(n, n, 2)$-balanced rectangle for $2 \leq n \leq 4$. We then describe some patterns which we generalize formally in the next section.

1,2	

	2
1	

1,2	1,2	
1,2	1,2	

1,2	1,2	
1,2		
	1,2	

1,2	1,2	
1,2		
		1,2

1,2	1,2	
1	1	1

1,2		
	1,2	
1	1	1

1	1,2	
1		
1	1	1

1	1	1
1	1	1

	2	2
1	1,2	
1		

	2	2
1		2
1	1	

		2
1	1	
1	1	

For $n = 4$, eight of the critical sets are saturated and some of the non-saturated critical sets are given below.

1,2	1,2	1,2	
1,2	1,2	1,2	
1	1	1	1

1,2	1,2	1,2	
1,2	1,2		
		1,2	
1	1	1	1

1,2		1,2	
1,2	1,2		
	1,2	1,2	
1	1	1	1

1,2	1,2	1,2	
1,2	1,2		
			1,2
1	1	1	1

1,2	1,2	1,2	
1,2			
	1,2		1,2
1	1	1	1

1,2	1,2		
1,2		1,2	1,2
		1,2	
1	1	1	1

1,2		1,2	
	1,2		1,2
1,2	1,2		
1	1	1	1

1,2	1,2		
1,2			1,2
		1,2	1,2
1	1	1	1

1,2	1,2	1,2	
1	1	1	1
1	1	1	1

1,2	1,2		
		1,2	
1	1	1	1
1	1	1	1

1	1,2	1,2	
1	1,2		
1		1,2	
1	1	1	1

	2	2	2
1	1,2	1,2	
1	1,2		
1		1,2	

1	1,2	1,2	
1	1,2		
1			1,2
1	1	1	1

	2	2	2
1	1,2	1,2	
1	1,2		
1			1,2

1	1,2	1,2	
1			
1	1	1	1
1	1	1	1

1,2	1,2		2
			2
1	1	1	
1	1	1	

1	1,2		
1		1,2	
1	1	1	1
1	1	1	1

1,2			2
	1,2		2
1	1	1	
1	1	1	

1	1	1,2	
1	1		
1	1	1	1
1	1	1	1

1	1,2		2
1			2
1	1	1	
1	1	1	

1,2		2	2
		2	2
1	1	2	2
1	1		

	2	2	2
1		2	2
1	1	1,2	
1	1		

1	1	1	1
1	1	1	1
1	1	1	1

	2	2	2
1		2	2
1	1		2
1	1	1	

		2	2
1	1		2
1	1	1	
1	1	1	

			2
1	1	1	
1	1	1	
1	1	1	

We make the following observations. For $n = 3, 4$, a saturated critical set contains the most entries in each case and the last critical sets for both values of n contain the smallest number of entries with sizes $2(n-1)^2$ and $(n-1)^2+1$

respectively. For $n = 5$,

1,2	1,2	1,2	1,2	
1,2	1,2	1,2	1,2	
1,2	1,2	1,2	1,2	
1,2	1,2	1,2	1,2	

and

				2
1	1	1	1	
1	1	1	1	
1	1	1	1	
1	1	1	1	

are also the largest and smallest critical sets of the full $(5, 5, 2)$-balanced Latin square. We can generalize this observation as follows:

1. The partial $(n, n, 2)$-Latin square

$1,2$	$1,2$	$\cdots$	$1,2$	
$1,2$	$1,2$	$\cdots$	$1,2$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$1,2$	$1,2$	$\cdots$	$1,2$	
		$\cdots$		

   of order n and size $2(n-1)^2$ is a critical set of the full $(n, n, 2)$-balanced Latin square.

2. The partial $(n, n, 2)$-Latin square

		$\cdots$		2
1	1	$\cdots$	1	
1	1	$\cdots$	1	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1	1	$\cdots$	1	

   of order n and size $(n-1)^2+1$ is a critical set of the full $(n, n, 2)$-balanced Latin square.

We will later prove that these are the largest and smallest critical sets, respectively, in Section 4.5.

Observe that each critical set example presented so far in this chapter may be described as in Figure 4.2 where each grey block is a saturated critical set of the corresponding full 2-balanced Latin rectangle of the same size. We make use of this observation in the next two sections to fully describe the critical sets of the full $(m, n, 2)$-balanced rectangle (see [23]).

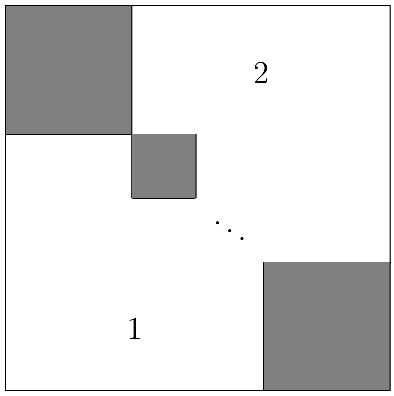

Figure 4.2: Critical set of the full $(m, n, 2)$-balanced Latin rectangle

4.2 The general structure of a critical set

In this section, we describe arrays which will ultimately classify all critical sets of the full $(m, n, 2)$-balanced rectangle (up to a reordering of the rows and columns).

Henceforth, $(\boldsymbol{a}, \boldsymbol{b}) = ((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k))$ is always a pair of integral non-negative vectors such that $\sum_{i=1}^{k} a_i = m$ and $\sum_{i=1}^{k} b_i = n$. We use the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ to define partitions of the sets $N(m)$ and $N(n)$. For each $I \in N(k)$, let $R_I = N(\sum_{i=1}^{I} a_i) \setminus N(\sum_{i=1}^{I-1} a_i)$ and $C_I = N(\sum_{i=1}^{I} b_i) \setminus N(\sum_{j=1}^{I-1} b_i)$. Note that if $a_I = 0$ ($b_I = 0$) then R_I (respectively, C_I) is empty.

Definition 4.1 *We define $A[\boldsymbol{a}, \boldsymbol{b}]$ to be the set of all $m \times n$ arrays A with the following structure. Let $r \in R_I$ and $c \in C_J$ where $I, J \in N(k)$.*

- *If $I > J$, cell $A_{r,c} = 1$.*

- *If $I < J$, cell $A_{r,c} = 2$.*
- *If $I = J$, cell $A_{r,c}$ is either empty or $A_{r,c} = \{1, 2\}$, subject to the following. Let B_I be a bipartite graph with partite sets given by R_I and C_I, with edge $\{r, c\}$, $r \in R_I$, $c \in C_I$ existing if and only if cell $A_{r,c}$ is empty. Then B_I is a tree.*

The following is an example of two elements of $A[(3, 2, 1), (3, 1, 3)]$; both critical sets of the full $(6, 7, 2)$-balanced Latin rectangle.

1,2	1,2		2	2	2	2
1,2	1,2		2	2	2	2
			2	2	2	2
1	1	1		2	2	2
1	1	1		2	2	2
1	1	1	1			

1,2		1,2	2	2	2	2
1,2			2	2	2	2
		1,2	2	2	2	2
1	1	1		2	2	2
1	1	1		2	2	2
1	1	1	1			

We sometimes describe an array A in $A[\boldsymbol{a}, \boldsymbol{b}]$ in terms of *blocks* so that each cell of a block is either empty or contains the same entries. For each $I, J \in N(k)$, the block $A_{I,J}$ is the subarray of A induced by the rows R_I and columns C_J. That is, $A_{I,J} = \{A_{r,c} \mid r \in R_I, c \in C_J\}$. The blocks of the form $A_{I,I}$ are said to form the *main diagonal blocks*. (Observe from Theorem 5.6, each block $A_{I,I}$ is a critical set of the full 2-balanced Latin square of the same size). Thus all cells below the main diagonal blocks contain 1 and all those above contain 2.

Definition 4.2 *We say that a pair of vectors $(\boldsymbol{a}, \boldsymbol{b})$ is* good *if:*

(C1) *for each $i \in N(k)$, $a_i > 0$ or $b_i > 0$;*

(C2) *if $a_i = 0$ then $a_{i-1} \geq 2$, $b_{i-1} \geq 1$ (if $i > 1$) and $a_{i+1} \geq 2$, $b_{i+1} \geq 1$ (if $i < k$); and*

(C3) *if $b_i = 0$ then $b_{i-1} \geq 2$, $a_{i-1} \geq 1$ (if $i > 1$) and $b_{i+1} \geq 2$, $a_{i+1} \geq 1$ (if $i < k$).*

For a good pair of vectors, observe that for each I at least one of R_I or C_I is non-empty.

Next we show that given a pair of good vectors $(\boldsymbol{a}, \boldsymbol{b})$, a partial $(m, n, 2)$-Latin rectangle is a critical set of the full $(m, n, 2)$-balanced Latin rectangle if and only if it is an element of $A[\boldsymbol{a}, \boldsymbol{b}]$.

Theorem 4.2 *Let $(\boldsymbol{a},\boldsymbol{b})$ be a pair of good vectors. Then any element of $A[\boldsymbol{a}, \boldsymbol{b}]$ is a critical set of the full $(m, n, 2)$-balanced Latin rectangle.*

Theorem 4.3 *Up to a reordering of the rows and columns, any critical set of the full $(m, n, 2)$-balanced Latin rectangle is an element of $A[\boldsymbol{a}, \boldsymbol{b}]$ for some pair $(\boldsymbol{a}, \boldsymbol{b})$ of good vectors.*

Theorem 4.8 of Section 4.4 proves Theorem 4.2. A proof of Theorem 4.3 is given in [23]. Together these theorems give a classification of the critical sets of the full $(m, n, 2)$-balanced Latin rectangle.

4.3 Trades in $(m, n, 2)$-balanced Latin rectangles

As with other combinatorial designs, trades play an important role in identifying defining sets and critical sets of $(m, n, 2)$-balanced Latin rectangles. (see, for example, [21]).

Recall from Section 1.2 that a *trade* in the full $(m, n, 2)$-balanced rectangle F is some non-empty $T \subset F$ such that there exists a *disjoint mate* T' where $T' \cap T = \emptyset$ and $(F \setminus T) \uplus T'$ is an $m \times n$ balanced 2-rectangle (which is clearly not full). An *intercalate* in the full $(m, n, 2)$-balanced rectangle is the set $T = \{(r, c, s), (r, c', s'), (r', c, s'), (r', c', s)\}$ where $s \neq s'$ and $s, s' \in \{1, 2\}$.

The following lemma is simply a special case of Lemma 1.1 and Lemma 1.2.

Lemma 4.4 *Let F be the full $(m, n, 2)$-balanced rectangle. Then the set D is a defining set of F if and only if $D \subseteq F$ and D intersects every trade* *within F. A defining set D of the full $(m, n, 2)$-balanced rectangle is in turn a critical set if and only if, for each $(r, c, e) \in D$, there is a trade T in the full $(m, n, 2)$-balanced rectangle such that $T \cap D = \{(r, c, e)\}$.*

Let B_F denote the bipartite edge-coloured graph corresponding to F. For a full $(m, n, 2)$-balanced rectangle, B_F can be thought of as a bipartite graph with partite sets $N(m)$ and $N(n)$ with one red edge and one blue edge (corresponding to entries 1 and 2, respectively) between each pair of vertices from different partite sets.

Consider the subgraphs B_T and $B_{T'}$ of B_F equivalent to a trade T with disjoint mate T' (respectively) in F. Since $B_{T'}$ is obtained from B_T by switching the colour on each edge of B_T, it follows that each vertex in T is incident with the same number of blue and red edges. Suppose that there exist vertices v and w such that there exist both a red edge and a blue edge in T of the form $\{v, w\}$. Then there also exists an edge of each colour on $\{v, w\}$ in T', contradicting the fact that $T \cap T' = \emptyset$. It follows that B_T is the union of properly edge-coloured cycles. A *properly edge-coloured* cycle is an even cycle with edges coloured alternately red and blue (i.e. each vertex in the cycle is adjacent to one red edge and one blue edge). We call a properly edge-coloured even cycle (and the corresponding array) a *trade cycle* and we have proven the following.

Lemma 4.5 *Any trade within F is the union of cell-disjoint trade cycles.*

Corollary 4.6 *Any minimal trade is a trade cycle.*

4.4 Existence of critical sets

In this section we give a proof of Theorem 4.2. That is, we show that any element of $A[\boldsymbol{a}, \boldsymbol{b}]$ is indeed a critical set of the full $(m, n, 2)$-balanced Latin rectangle, where $(\boldsymbol{a}, \boldsymbol{b})$ is a good pair of vectors. We make use of the theory of trades developed in the previous section.

Lemma 4.7 *Let $(\boldsymbol{a}, \boldsymbol{b})$ be a good pair of vectors. Let A be an array in $A[\boldsymbol{a}, \boldsymbol{b}]$ and suppose that cell $A_{r,c} = 1$ only. Then there exists a row $r' < r$ and a column $c' > c$ satisfying one of the following cases:*

1. *$A_{r',c} = 1$ and $A_{r,c'}, A_{r',c'}$ are each empty;*

2. *$A_{r,c'} = 1$ and $A_{r',c}, A_{r',c'}$ are each empty;*

3. *$A_{r',c'} = 2$ and $A_{r,c'}, A_{r',c}$ are each empty; or*

4. *$A_{r',c'}$ is empty, $A_{r',c} = 1$ and $A_{r,c'} = 1$.*

Proof. We illustrate the above cases below.

	c	c'
r'	1	
r	1	

	c	c'
r'		
r	1	1

	c	c'
r'		2
r	1	

	c	c'
r'	1	
r	1	1

Let $r \in R_I$ and $c \in C_J$; by definition $I > J$ (see Section 4.2). First, suppose that both $|R_J| \geq 1$ and $|C_I| \geq 1$. Then selecting $r' \in R_J$ and $c' \in C_I$ such that $A_{r',c}$ and $A_{r,c'}$ are empty, cell $A_{r',c'}$ must contain 2. This results in Case 3 above.

Next, suppose that $|R_J| = 0$. Since $I > J$, $J < k$, so since $(\boldsymbol{a}, \boldsymbol{b})$ is a good pair of vectors, $|R_{J+1}| \geq 2$ and $|C_{J+1}| \geq 1$. If $I = J + 1$, since B_{J+1} is a tree and is thus connected, there exists $r' \in R_{J+1}$ and $c' \in C_{J+1}$ such that $A_{r',c'}$ and $A_{r,c'}$ are each empty. Then cell $A_{r',c}$ lies in block $A_{I,J}$ and thus contains entry 0 only. This results in Case 1. If $|R_J| = 0$ and $I > J + 1$, then let $A_{r',c'}$ be an empty cell in block $A_{J+1,J+1}$. Then $A_{r,c'}$ and $A_{r',c}$ are in blocks $A_{I,J+1}$ and $A_{J+1,J}$, each of which contains only 1's. Case 4 follows.

Otherwise $|C_I| = 0$. Then $|C_{I-1}| \geq 2$ and $|R_{I-1}| \geq 1$. Block $A_{I,I}$ is empty and each other block of the form $A_{I,J'}$ with $J' < I$ contains only 1's. Suppose that $J = I - 1$. Thus, since B_{I-1} is a tree, there is a row $r' \in R_{I-1}$ and a column $c' \neq c$ in C_{I-1} such that $A_{r',c'}$ and $A_{r',c}$ are empty. Since $A_{r,c'}$ contains only entry 1 we have Case 2. If $J < I - 1$, let $c' \in C_{I-1}$ and let $r' \in R_{I-1}$

where $A_{r',c'}$ is empty. Then $A_{r,c'}$ and $A_{r',c}$ each only contain entry 1, so we have Case 4. □

Theorem 4.8 *Let A be an element of $A[\boldsymbol{a}, \boldsymbol{b}]$ where $(\boldsymbol{a}, \boldsymbol{b})$ is a good pair of vectors. Then A is a critical set of the full $(m, n, 2)$-balanced Latin rectangle.*

Proof. Suppose first that A is not a defining set of the full $(m, n, 2)$-balanced Latin rectangle. Then there exists an $(m, n, 2)$-balanced Latin rectangle $F' \neq F$ such that $A \subset F'$. Let $T = F \setminus F'$ and $T' = F' \setminus F$. Observe that the non-empty cells of T (and T') are precisely the cells of F' which contain either $\{1, 1\}$ or $\{2, 2\}$.

Without loss of generality, let the cell $F'_{i,j}$ contain $\{1, 1\}$. Then, since the total number of 1's and 2's in each row and column is fixed, there exists a column $j' \neq j$ such that $F'_{i,j'} = \{2, 2\}$. Similarly, there exists a row $i' \neq i$ such that $F'_{i',j'} = \{1, 1\}$.

By finiteness there exists a list of distinct cells

$$F'_{i(1),j(1)}, F'_{i(1),j(2)}, F'_{i(2),j(2)}, \ldots, F'_{i(\mu),j(\mu)}, F'_{i(\mu),j(\mu+1)=j(1)}$$

where $F'_{i(a),j(a)} = \{1, 1\}$ and $F'_{i(a),j(a+1)} = \{2, 2\}$ for each $a \in N(\mu)$.

Because of the structure of A, for each $a \geq 1$ either $i(a) \in R_I$ and $i(a+1) \in R_J$ for some I and J with $J > I$ *or* cells $A_{i(a),j(a+1)}$ and $A_{i(a+1),j(a+1)}$ belong to the same main diagonal block. Moreover, either $j(a) \in C_I$ and $j(a+1) \in C_J$ for some $J > I$ *or* cells $A_{i(a),j(a)}$ and $A_{i(a),j(a+1)}$ belong to the same main diagonal block. Since $\mu > 1$, either $i(\mu) \in R_J$ and $i(1) \in R_I$ with $I < J$ or the entire trade is contained within a main diagonal block. If the former holds, cell $A_{i(\mu),j(1)}$ contains 1 only, a contradiction, and if the latter holds there is a cycle in the graph B_I (for some I), also a contradiction.

We next show that A is a minimal defining set. That is, we remove each entry from A and show that it is no longer a defining set.

Case 1: The cell $A_{r,c}$ contains one element. By symmetry we may assume without loss of generality, that $A_{r,c} = 1$ and let $A' = A \setminus \{A_{r,c}\}$. In each of the

Cases 1 to 4 given in Lemma 4.7, $F \setminus A'$ contains a trade cycle (on the four cells given by that lemma). So by Lemma 4.4, A' is not a critical set of F.

Case 2: The cell $A_{r,c}$ contains two elements. Thus $A_{r,c}$ belongs to a block of the form $A_{I,I}$. Since B_I is a tree, there is a trade cycle using either $A_{r,c} = 1$ or $A_{r,c} = 2$ and entries in cells which are empty in A.

By Lemma 4.4, A is a critical set of F. □

4.5 The smallest and largest critical set

Having now classified the structure of any critical set in the full $(m, n, 2)$-balanced Latin rectangle F, we now determine the smallest and largest possible size of such a structure.

To this end, for $m, n > 1$ we define R^1_{mn} and R^2_{mn} to be the unique elements of $A[m-1, 1; 1, n-1]$ and $A[1, m-1; n-1, 1]$, respectively. Below is the partial array R^1_{34}.

	2	2	2
	2	2	2
1			

From Theorem 4.8, both R^1_{mn} and R^2_{mn} are critical sets of the full $(m, n, 2)$-balanced rectangle. Observe that they each have size $(m-1)(n-1)+1$. We next show that R^1_{mn} and R^2_{mn} are critical sets of the full $(m, n, 2)$-balanced rectangle of minimum size and are unique in this property.

Lemma 4.9 *If $m, n > 1$ then the size of the smallest critical set of the full $(m, n, 2)$-balanced Latin rectangle is $(m-1)(n-1)+1$. Up to a reordering of rows and columns, R^1_{mn} and R^2_{mn} are the unique critical sets with this property.*

Proof. Let C be a critical set of the full $(m, n, 2)$-balanced Latin rectangle and let e be the number of empty cells in C. From Theorem 4.2, the graph

B_0 corresponding to the empty cells forms a forest, so $e \leq m+n-1$. If $e = m+n-1$, then B_0 is a tree on $m+n$ vertices. Thus each non-empty cell contains 2 elements and $|C| \geq 2(mn-m-n+1) > (m-1)(n-1)+1$, with equality only possible in the case $m = n = 2$. Otherwise, $e \leq m+n-2$ and $|C| \geq mn-m-n-2 = (m-1)(n-1)+1$, with equality only possible if B_0 has two components and no cells containing $\{1,2\}$.

Next, suppose that C is a critical set of the full $(m,n,2)$-balanced Latin rectangle of size $(m-1)(n-1)+1$ where $(m,n) \neq (2,2)$. From above, B_0 has precisely two components, each of which is a complete bipartite graph and thus a star (see Appendix 10.1). So $C \in A[(a_1,a_2),(b_1,b_2)]$ and either $a_1 = b_2 = 1$ or $a_2 = b_1 = 1$. □

Observe that in the case $m = n = 2$, any element of $A[(2),(2)]$ also gives a critical set of minimum possible size.

We next show that a saturated critical containing exactly $2(m-1)(n-1)$ entries is a largest critical set of the full $(m,n,2)$-balanced Latin rectangle.

Lemma 4.10 *If $m,n > 1$ then the size of the largest critical set of the full $(m,n,2)$-balanced Latin rectangle is $2(m-1)(n-1)$.*

Proof. Let C be a critical set of the full $(m,n,2)$-balanced Latin rectangle and without loss of generality, let $m \leq n$. By inspection, in the case $m = 2$, $|C| = n$ or $2(n-1)$ thus $|C| \leq 2(m-1)(n-1)$. Observe also that for $m,n \geq 3$, if each non-empty cell of C contains only one entry then from Section 4.2, $|C| \leq mn-3) < 2(m-1)(n-1)$. We therefore only need to prove the lemma in the case where at least one of the non-empty cells of C contains $\{1,2\}$ and $m,n \geq 3$.

Suppose that C contains only one $x \times y$ main diagonal block with $x,y \geq 2$. Then by Section 4.2, every cell outside the main diagonal block of C is either empty or contain only one entry, thus:

$$\begin{aligned}
|C| &\leq 2(x-1)(y-1) + mn - xy \\
&= mn + xy - 2x - 2y + 2 \\
&= mn + (m-k)(n-l) - 2(m-k) - 2(n-l) + 2 \\
&\quad (\text{where} \;\; k = m - x \leq m - 2 \;\; \text{and} \;\; l = n - y \leq n - 2) \\
&= 2mn - 2m - 2n + 2 + kl - kn - lm + 2k + 2l \\
&\leq 2mn - 2m - 2n + 2 + l(m-2) - kn - lm + 2k + 2l \\
&= 2mn - 2m - 2n + 2 - kn + 2k \\
&\leq mn + mn - 2m - 2n + 2 \\
&= 2(m-1)(n-1),
\end{aligned}$$

with equality possible when $x = m$ and $y = n$.

Otherwise, there exist two blocks in C, one containing an $x_1 \times y_1$ main diagonal block and the other an $x_2 \times y_2$ main diagonal with $x_1, x_2, y_1, y_2 \geq 2$. In which case

$$\begin{aligned}
|C| &\leq 2mn - 2(x_1 + y_1 - 1) - 2(x_2 + y_2 - 1) - x_1(n - y_1) - x_2(n - y_2) \\
&\leq 2mn - 2(2 + y_1 - 1) - 2(2 + y_2 - 1) - 2(n - y_1) - 2(n - y_2) \\
&= 2mn - 4n - 4 \\
&< 2mn - 2m - 2n + 2 \\
&= 2(m-1)(n-1).
\end{aligned}$$

□

We illustrate the last part of the above proof in the following example.

Let the partial $(x_1 + x_2, n, 2)$-Latin rectangle in Figure 4.3 be $x_1 + x_2$ rows of a critical set C of the full $(m, n, 2)$-balanced Latin rectangle, containing an $x_1 \times y_1$ and an $x_2 \times y_2$ main diagonal blocks.

				y_1 *columns*				y_2 *columns*							
x_1 *rows*	1	⋯	1	1,2	⋯	1,2		2	⋯	2	2	2	⋯	2
	⋮	⋱	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮
	1	⋯	1	1,2	⋯	1,2		2	⋯	2	2	2	⋯	2
	1	⋯	1		⋯			2	⋯	2	2	2	⋯	2
x_2 *rows*	1	⋯	1	1	⋯	1	1	1,2	⋯	1,2		2	⋯	2
	⋮	⋱	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮
	1	⋯	1	1	⋯	1	1	1,2	⋯	1,2		2	⋯	2
	1	⋯	1	1	⋯	1	1		⋯			2	⋯	2

Figure 4.3: $x_1 + x_2$ rows of C

Then observe that in these rows, $x_1 + x_2 + y_1 + y_2 - 2$ cells are empty and $x_1(n - y_1) + x_2(n - y_2)$ cells contain only one entry. Furthermore, if $m > x_1 + x_2$ then every other cell in the $y_1 + y_2$ columns containing these two main diagonal blocks contain only one entry as well (see Section 4.2), thus $|C| \leq 2mn - 2(x_1 + y_1 - 1) - 2(x_2 + y_2 - 1) - x_1(n - y_1) - x_2(n - y_2)$.

4.6 Critical sets of full (m, n, t)-balanced Latin rectangles

We conclude this chapter with a generalization of Theorem 4.8 to the full (m, n, t)-balanced Latin rectangle. For this purpose we redefine $A[\boldsymbol{a}, \boldsymbol{b}]$ as follows:

Definition 4.3 *Let $r \in R_I$ and $c \in C_J$ where $I, J \in N(k)$.*

- *If $I > J$, cell $A_{r,c} = N(k-1)$.*
- *If $I < J$, cell $A_{r,c} = N(k) \setminus \{1\}$.*
- *If $I = J$, cell $A_{r,c}$ is either empty or contains $N(k)$, subject to the following. Let B_I be a bipartite graph with partite sets given by R_I and C_I,*

with edge $\{r, c\}$, $r \in R_I$, $c \in C_I$ *existing if and only if cell* $A_{r,c}$ *is empty. Then* B_I *is a tree.*

The following theorem follows from Theorem 4.8.

Theorem 4.11 *Let* A *be an element of* $A[\boldsymbol{a}, \boldsymbol{b}]$*. Then* A *is a critical set of the full* (m, n, t)*-balanced Latin rectangle.*

For example, the element of $A[(2,1,2),(4,1,2)]$ below is a critical set of the full $(5,7,4)$-balanced Latin rectangle.

1,2,3,4	1,2,3,4	1,2,3,4		2,3,4	2,3,4	2,3,4
				2,3,4	2,3,4	2,3,4
1,2,3	1,2,3	1,2,3	1,2,3		2,3,4	2,3,4
1,2,3	1,2,3	1,2,3	1,2,3	1,2,3	1,2,3,4	
1,2,3	1,2,3	1,2,3	1,2,3	1,2,3		

Note that not every critical set of the full n-Latin squares is an element of $A[\boldsymbol{a}, \boldsymbol{b}]$ (see Chapter 6 for other examples).

Chapter 5

Critical sets of the full n-Latin square

This chapter explores properties of the different critical sets of the full n-Latin square. In Section 5.1, we investigate first the more structured critical sets where each non-empty cell contains $N(n)$. Recall from the Introduction that any partial n-Latin rectangle with this property is referred to as being *saturated*. By studying the saturated critical sets of the more general full balanced rectangles, we determine the exact size of these structures. In Theorem 5.8, we show that a saturated critical set for the full n-Latin square has size exactly equal to $n^3 - 2n^2 - n$.

In Section 5.2, we turn our focus on the non-saturated case. We calculate a lower bound for the smallest size of any $2 \times n$ sub-rectangle of a critical set of the full n-Latin square. Using this result, we show that a lower bound for the smallest size of a critical set of the full n-Latin square is $(n^3 - 2n^2 + 2n)/2$. In Section 5.3 we generalize the smallest critical set of the full 2-balanced Latin square from Section 4.5 to give a construction which provides an upper bound of $(n-1)^3 + 1$ for the size of the smallest critical set of the full n-Latin square; we conjecture this is best possible. Finally in Section 5.4, we give a lower bound for the size of the largest critical set of the full n-Latin square. This lower bound of $n^3 - n^2 - 3n + 4$ is the size of largest critical set of the full-n-Latin square we were able to construct (see Section 4.6).

Parts of Section 5.1, Section 5.2 and Section 5.3 are published in [24].

5.1 Saturated critical sets

In this section, we present the exact size of a saturated critical set of the full n-Latin square by showing that these critical sets always contain exactly $2n-1$ empty cells.

The following squares are examples of saturated critical sets of the full 4-Latin square.

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	

		1,2,3,4	1,2,3,4
1,2,3,4			1,2,3,4
1,2,3,4	1,2,3,4		
1,2,3,4	1,2,3,4	1,2,3,4	

Listed below are the saturated critical sets of the full n-Latin squares ($n = 2, 3, 4$) up to main class equivalence.

$n = 2$

1,2	

$n = 3$

1,2,3	1,2,3	
1,2,3	1,2,3	

1,2,3	1,2,3	
1,2,3		1,2,3

1,2,3	1,2,3	
1,2,3		
		1,2,3

$n = 4$

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4		
		1,2,3,4	

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4		1,2,3,4	
1,2,3,4	1,2,3,4		
	1,2,3,4	1,2,3,4	

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4		
			1,2,3,4

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4			
	1,2,3,4		1,2,3,4

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4		
1,2,3,4		1,2,3,4	1,2,3,4
		1,2,3,4	

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4		1,2,3,4	
	1,2,3,4		1,2,3,4
1,2,3,4	1,2,3,4		

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4		
1,2,3,4			1,2,3,4
		1,2,3,4	1,2,3,4

The number of saturated critical sets for full n-Latin squares appears to grow rapidly as n increases. Nevertheless some important properties of these saturated critical sets are outlined in the following lemmas.

Let C be a saturated critical set of the full n-Latin square, L. Then the following results hold:

Lemma 5.1 *Every row/column of C must have at least one empty cell.*

Proof. Consider a row, with $n-1$ full cells, in a saturated defining set. This means that each symbol of the set $N(n)$ has occurred $n-1$ times in this row so, by the definition of a multi-Latin square, the empty cell is forced to contain $N(n)$ also. It follows then that any defining set with a row containing no empty cell cannot be a critical set as removing the entries of a cell from this row results in another defining set. □

Corollary 5.2 *Any row and any column of C contain, in total, at least two empty cells.*

Proof. Without loss of generality, let row r_1 and column c_1 of a defining set of the full n-Latin square contain, in total, two empty cells with r_1 containing only one empty cell. By methods used in the proof of Lemma 5.1, the empty cell of r_1 is forced to contain $N(n)$ and the other empty cell, being the only empty cell of c_1, is forced to follow suit. □

Motivated by the properties highlighted by the results above, the following construction will always provide a saturated partial n-Latin square, P, with each row/column containing at least one empty cell with the number of empty cells in any row and column combined at least 2.

Construction 5.3

For an $n \times n$ array P

- Leave cell $P_{1,1}$ empty.
- For $2 \leq i \leq n$, leave cells $P_{i,i-1}$ and $P_{i,i}$ empty.
- Fill all other cells with $N(n)$.

For $n = 4$, we leave $P_{1,1}$, $P_{2,1}$, $P_{2,2}$, $P_{3,2}$, $P_{3,3}$, $P_{4,3}$ and $P_{4,4}$ empty as shown below.

	1,2,3,4	1,2,3,4	1,2,3,4
		1,2,3,4	1,2,3,4
1,2,3,4			1,2,3,4
1,2,3,4	1,2,3,4		

Saturated partial n-Latin squares constructed this way are always of size $n^3 - 2n^2 - n$. In Corollary 5.7, we will show that not only is P a saturated critical set of the full n-Latin square; the size of any saturated critical set of a full n-Latin square is $n^3 - 2n^2 - n$.

Next we remind the reader of an array of multisets that generalizes the full n-Latin square (see the Introduction and Chapter 4). A *partial* (m, n, t)-*balanced Latin rectangle* is an $m \times n$ array of multisets of size t such that each symbol of $N(t)$ occurs at most n times in each row and at most n times in each column. If each symbol of $N(t)$ occurs exactly n times in each row and exactly n times in each column, then such an array is a (m, n, t)-*balanced Latin rectangle*. An example of a $(4, 5, 3)$-balanced Latin rectangle is given below.

1,1,2	1,1,3	2,3,3	1,2,3	2,2,3
1,2,3	2,2,3	1,3,3	2,2,3	1,1,1
1,2,2	1,2,3	1,1,2	1,2,3	3,3,3
3,3,3	1,2,3	1,2,2	1,1,3	1,2,2

If each cell contains $N(k)$, we refer to such a rectangle as a *full* (m, n, t)-*balanced Latin rectangle*. Thus the full n-Latin square is the full (n, n, n)-balanced Latin rectangle.

Given a saturated partial (m, n, t)-balanced Latin rectangle, L, with rows $r_1, r_2, ..., r_l$ and columns $c_1, c_2, ..., c_m$, we define $G_e(L) = (V_1 \cup V_2, E)$ (where $V_1 = \{r_1, r_2, ..., r_l\}$ and $V_2 = \{c_1, c_2, ..., c_m\}$ to be the bipartite graph that corresponds to the empty cells of L. That is, edge $\{r_i, c_j\}$ is in $G_e(L)$ if and only if the cell $L_{i,j}$ of L is empty. We say that there is a cycle in L if and only if there is a cycle in $G_e(L)$.

For example, let L be the partial 3×3 3-balanced Latin rectangle below.

	1,2,3	
		1,2,3
1,2,3		

The empty cells $L_{1,1}, L_{2,1}, L_{2,2}, L_{3,2}, L_{3,3}, L_{1,3}, L_{1,1}$ form a cycle in L represented by the bipartite graph:

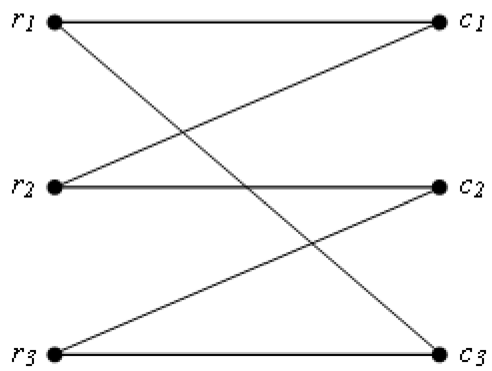

Figure 5.1: The graph $G_e(L)$

We first show that any saturated partial (m, n, t)-balanced Latin rectangle with each non-empty cell containing $N(t)$ (as in the above example) is a defining set if and only if it contains no cycle of empty cells.

Theorem 5.4 *Let $k \geq 2$. A saturated partial (m, n, t)-balanced Latin rectangle is a defining set if and only if it contains no cycle of empty cells.*

Proof. Suppose that a saturated defining set of a (m, n, t)-balanced Latin rectangle, D, contains a cycle of empty cells and let these empty cells be

$$D_{i(1),j(1)}, D_{i(1),j(2)}, D_{i(2),j(2)}, ..., D_{i(\mu),j(\mu)}, D_{i(\mu),j(1)}$$

such that

$$\{r_{i(1)}, c_{j(1)}\}, \{r_{i(1)}, c_{j(2)}\}, \{r_{i(2)}, c_{j(2)}\}, ..., \{r_{i(\mu)}, c_{j(\mu)}\}, \{r_{i(\mu)}, c_{j(1)}\}$$

is the edge sequence of the corresponding cycle in $G_e(D)$. We need to show that D now also completes to a non-full (m, n, t)-balanced Latin rectangle. Such a completion can be achieved by filling $D_{i(1),j(1)}, D_{i(2),j(2)}, ..., D_{i(\mu),j(\mu)}$ with the multiset $(N(t) \setminus \{x\}) \uplus \{y\}$ and the remaining empty cells of the cycle with $(N(t) \setminus \{y\}) \uplus \{x\}$, where $x \neq y : x, y \in N(t)$. Thus D can not be a defining set.

Conversely, let D be a saturated partial (m, n, t)-balanced Latin rectangle with no cycle of empty cells. Then $G_e(D)$ also contains no cycles and has at least one vertex of degree 1 by Theorem 10.1. So D has a row or column with only one empty cell, and this empty cell is forced to contain $N(t)$, obtaining another saturated partial (m, n, t)-balanced Latin rectangle with no cycle of empty cells. Recursively, since D is finite, D has an unique completion. □

Observe that the saturated partial Latin square,

	$1, 2, 3$	
		$1, 2, 3$
$1, 2, 3$		

not only completes to the full 3-Latin square; labeling the empty cells $E_1, E_2, ..., E_6$,

E_1	$1, 2, 3$	E_6
E_2	E_3	$1, 2, 3$
$1, 2, 3$	E_4	E_5

and choosing $x = 1$ and $y = 2$, it also completes to

$2, 2, 3$	$1, 2, 3$	$1, 1, 3$
$1, 1, 3$	$2, 2, 3$	$1, 2, 3$
$1, 2, 3$	$1, 1, 3$	$2, 2, 3$

.

Consequently, a saturated critical set of the (m, n, t)-balanced Latin rectangle, being a defining set, must contain no cycle. We next show that adding an empty cell to such a critical set creates a cycle, thus increasing the number of possible completions it has.

Lemma 5.5 *Let C be a saturated critical set of the full (m, n, t)-balanced Latin rectangle. Then deleting the entries of any cell which contains $N(t)$ creates a cycle in C.*

Proof. Suppose that a saturated partial (m, n, t)-balanced Latin rectangle, D, uniquely completes to the full (m, n, t)-balanced Latin rectangle and we can add another empty cell to D (i.e. delete the entries of a cell which contains $N(t)$) without forming a cycle. Then by Theorem 5.4, the new saturated partial rectangle is also a defining set with a smaller size, so D is not a critical set by definition. □

If a simple connected bipartite graph, G, with no cycle contains the maximum number of edges for this property, then G is by definition a tree. Since $G_e(C)$ for a saturated critical set, C, fits into this category, we use this property in the next theorem to determine the number of empty cells in a saturated critical set.

Theorem 5.6 *Let C be a saturated critical set of the (m, n, t)-balanced Latin rectangle. If $|E|$ is the number of empty cells in C, then*

$$|E| = m + n - 1.$$

Proof. By Theorem 5.4 and Lemma 5.5, $G_e(C)$ is connected but has no cycle i.e. $G_e(C)$ is a tree with $m + n$ vertices. Thus, by Theorem 10.2, $G_e(C)$ has $m + n - 1$ edges. □

Since the full n-Latin square is also an (n, n, n)-balanced Latin rectangle, the following theorem is immediate.

Theorem 5.7 *Let C be a saturated critical set of the full n-Latin square. If $|E|$ is the number of empty cells in C, then $|E| = 2n - 1$.*

And since an n-Latin square of order n contains n^3 entries, we can easily work out the size of any saturated critical set of order n from the above theorem.

Theorem 5.8 *A saturated critical set of of the full n-Latin square has size $s = n^3 - 2n^2 - n$.*

Note that in this section we have also classified the structure of any saturated critical set of the full n-Latin square.

5.2 Sub-rectangles of critical sets of the full n-Latin square

We next analyze critical sets for the full n-Latin square which are not necessarily saturated. Here, we determine the upper bound for the smallest size of any $2 \times n$ sub-rectangle of a critical set of the full n-Latin square. Using this result, we show in Theorem 5.21 that a lower bound for the smallest size of the the critical sets of the full n-Latin square is

$$|C| \geq \frac{n^3 - 2n^2 + 2n}{2}.$$

To work towards determining this upper bound, we explore the properties of the sub-rectangles within these critical sets. A necessary condition for a sub-rectangle of any critical set to guarantee an unique completion is that its complement contains no trade (see Section 1.2). It is therefore a sensible first step to determine the properties a 2×2 sub-square of a critical set of the full n-Latin square must possess to satisfy this condition. We remind the reader that a trade in a 2×2 sub-array is of the form:

$$\{(r, c, s), (r, c', s'), (r', c, s'), (r', c', s)\},$$

($r \neq r'$, $c \neq c'$ and $s \neq s'$) and is known as an *intercalate.*

Lemma 5.9 *Any 2×2 sub-array within a defining set of the full n-Latin square must contain no intercalates in its complement.*

Proof. If not, the intercalate may be replaced with the set of triangles

$$\{(r, c, s'), (r, c', s), (r', c, s), (r', c', s')\}$$

and more than one completion is possible. □

This idea is illustrated in the following example.

Let A be the the partial full-Latin square below.

1	
	1

Then its complement,

2	**1**,2
1,2	**2**

contains an intercalate and thus A has two completions:

1,1	2,2
2,2	1,1

and

1,2	1,2
1,2	1,2

.

It is therefore necessary to avoid the occurrence of an intercalate in the complement of any 2×2 sub-array of any defining set of the full n-Latin square. We can achieve this in two ways; either the union of the elements of at least one of the diagonally opposite pairs of cells is the set $N(n)$ or the union of the elements of the two diagonally opposite pairs of cells are equal with size $n-1$. The 2×2 sub-arrays below are examples of each case;

1,2,...,n	

,

1,2,...,x	1,2,...,y
$y+1$,$y+2$,...,$n-1$	$x+1$,$x+2$,...,$n-1$

for some $x, y \leq n-1$.

Thus the minimum size for one of these 2×2 sub-arrays is n. We formally present this idea in the following lemmas.

Lemma 5.10 *Let S be a 2×2 sub-array of a defining set of the full n-Latin square with size at most $n-1$. Then the complement, $\bar{S}$, of S contains an intercalate.*

Proof. At least one symbol s appears in each cell of $\bar{S}$ and at least one symbol $s' \neq s$ appears in at least three cells of $\bar{S}$. Thus, $\bar{S}$ contains an intercalate. □

The following is an example of when $|S| = n - 1$. Let $n = 5$ and let

S =

1,2,3	
	4

.

Then

$\bar{S}$ =

4,5	1,2,3,4,5
1,2,3,4,5	1,2,3,5

contains the intercalate:

5	4
4	5

and S completes to

1,2,3,4,4	1,2,3,5,5
1,2,3,5,5	1,2,3,4,4

as well.

Based solely on the above lemma, a weak lower bound for the size of a partial n-Latin square, P, with no intercalates in its complement (with $\binom{n}{2}^2$ distinct 2×2 sub-arrays), is $\frac{n^3}{4}$ (since each cell occurs a total of $(n-1)^2$ times in these sub-arrays).

We next work on improving this bound. More useful properties are given in the following lemma.

Lemma 5.11 *Let S be the 2×2 sub-array of a defining set of the full n-Latin square, with cells $S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}$ as shown below.*

$S_{1,1}$	$S_{1,2}$
$S_{2,1}$	$S_{2,2}$

Then $\bar{S}$ *contains no intercalate if and only if at least one of the following holds:*

(1) $|S_{1,1} \cup S_{2,2}| = n$,

(2) $|S_{2,1} \cup S_{1,2}| = n$,

(3) $S_{1,1} \cup S_{2,2} = S_{2,1} \cup S_{1,2}$ *and* $|S_{1,1} \cup S_{2,2}| = |S_{2,1} \cup S_{1,2}| = n - 1$.

Proof. Suppose $\bar{S}$ contains an intercalate. Then $|\bar{S}_{1,1} \cap \bar{S}_{2,2}| \geq 1$, $|\bar{S}_{1,2} \cap \bar{S}_{2,1}| \geq 1$, and either $\bar{S}_{1,1} \cap \bar{S}_{2,2} \neq \bar{S}_{1,2} \cap \bar{S}_{2,1}$ or $|\bar{S}_{1,1} \cap \bar{S}_{2,2}|, |\bar{S}_{1,2} \cap \bar{S}_{2,1}| \geq 2$. Thus $|S_{1,1} \cup S_{2,2}| \leq n - 1$, $|S_{1,2} \cup S_{2,1}| \leq n - 1$ and either $S_{1,1} \cup S_{2,2} \neq S_{1,2} \cup S_{2,1}$ or at least one of these sets is of size at most $n - 2$.

Conversely, suppose that conditions (1), (2) and (3) of the lemma are false. If $|S_{1,1} \cup S_{2,2}| = |S_{1,2} \cup S_{2,1}| = n - 1$ then $S_{1,1} \cup S_{2,2} \neq S_{1,2} \cup S_{2,1}$. Otherwise, either $|S_{1,1} \cup S_{2,2}|$ or $|S_{1,2} \cup S_{2,1}|$ is of size of at most $n - 2$. In both cases, there exists two distinct symbols $a, b \in N(n)$ such that $a \notin S_{1,1} \cup S_{2,2}$ and $b \notin S_{1,2} \cup S_{2,1}$; thus $\bar{S}$ contains the intercalate

a	b
b	a

.

□

Corollary 5.12 *Let* C *be a defining set of the full* n*-Latin square. If each sub-array of* C *is required to satisfy one of the conditions of Lemma* 5.11*, then* C *is a critical set.*

We now take the union of the two cells in each column of the 2×2 subsquare described in the above lemma to form two sets, and we deduce the relationship between these two sets in the following corollary. This result will be useful in exploring $2 \times m$ sub-rectangles.

Corollary 5.13 *Let S be a 2×2 sub-array of a critical set of the full-Latin square with cells $S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}$ (as in the statement of Lemma 5.11) and let $S_1 = S_{1,1} \cup S_{2,1}$ and $S_2 = S_{1,2} \cup S_{2,2}$. Then either $|S_1 \cup S_2| = n$ or $|S_1 \cup S_2| = n - 1$ and $|S| \geq 2(n-1)$.*

We now apply this result to more general $2 \times m$ sub-rectangles of defining sets of full n-Latin squares. First we describe the structure of these sub-rectangles in the cases when one of the columns has at most one entry. In the results below, without loss of generality, we assume the rows and columns of the sub-rectangle are indexed by $N(2)$ and $N(m)$. We also define $S_j = S_{1,j} \cup S_{2,j}$ $(j \in N(m))$.

Corollary 5.14 *Let S be two rows of a critical set of the full n-Latin square, $n \geq 2$. For any $j \in N(n)$, if:*

(1) $|S_j| = 0$ *, then either $|S_k| = n$ or $|S_{1,k}| = |S_{2,k}| = n - 1$ for all $k \neq j$.*
(2) $|S_j| = 1$ *, then $|S_k| \geq n - 1$ for all $k \neq j$.*

Lemma 5.15 *Let S be two rows of a critical set of the full-Latin square of order n, $n \geq 2$, such that for all $j, k \in N(n)$, $|S_j \cup S_k| = n - 1$. Then $|S| \geq n(n-1)$.*

Proof. By Corollary 5.13, each 2×2 sub-array of S is of size of at least $2(n-1)$. Now, there are $\binom{n}{2}$ sub-arrays in S with each column occurring $n-1$ times in these sub-arrays. So

$$|S| \geq \frac{\frac{n!}{(n-2)!2!} \times 2(n-1)}{n-1} = n(n-1)$$

□

For the more general cases, we will use the following results which apply to sets in general.

Lemma 5.16 *Let S_1 and S_2 be subsets of $N(n)$. If $|S_1|, |S_2| \geq n - 1$ then either $|S_1 \cap S_2| \geq n - 1$ or $|S_1 \cup S_2| = n$.*

Proof. The lemma is clearly true if at least one set is of size n. If both sets are of size $n-1$ then either $S_1 = S_2$ or $N(n) \setminus S_1 \subset S_2$ and thus $|S_1 \cup S_2| = n$.
□

The following technical lemma will be useful later in determining the size of the $2 \times n$ sub-rectangles.

Lemma 5.17 *Let $S = \{S_1, S_2, ..., S_m\}$ be a set of subsets of $N(n)$ such that:*

$$|\bigcap_{i=1}^{m} S_i| = n - x;$$

for some $x \in N(n)$, and

$$S_i \cup S_j = N(n)$$

for all $i, j \in N(m)$ such that $i \neq j$. Then

$$\sum_{i=1}^{m} |S_i| = mn - x.$$

Proof. Let $T = \bigcap_{i=\mu}^{m} S_i$. Then every element of $N(n)$ not in T is an element of all but one element of S. Thus

$$\sum_{i=1}^{m} |S_i| = m(n-x) + (m-1)x = mn - x.$$

□

Corollary 5.18 *Let S be a $2 \times m$ sub-rectangle of a critical set of the full-Latin square of order $n \geq 2$, such that for all $j, k \in N(n)$, $|S_j \cup S_k| = n$. Then $|S| \geq (m-1)n$.*

Proof. Without loss of generality, let $|S_1| = x$. Then $|\bigcap_{i=2}^{m} S_i| \geq n - x$ so by Lemma 5.17, $\sum_{i=2}^{m} |S_i| \geq (m-1)n - x$ and thus $|S| \geq (m-1)n$. □

We now show that the size $|S|$ of any two rows of a critical set of the full n-Latin square is at least $(n-1)^2 + 1$.

Lemma 5.19 *Let S be two rows of a critical set of the full-Latin square of order $n \geq 2$. Then $|S| \geq (n-1)^2 + 1$.*

Proof. Let the columns of S be indexed with c_j, $j \in N(n)$, and let m be the largest integer such that there exists m columns, each with at most $n-2$ entries in S. We split our proof according to the cases $m = 0$, $m = 1$ and $2 \leq m \leq n$.

Case 1: $m = 0$.

Then clearly $|S| \geq n(n-1) \geq (n-1)^2 + 1$.

Case 2: $m = 1$.

Without loss of generality, let $|c_1| = x$, $0 \leq x \leq n-2$. If $x = 0$, then by Corollary 5.14, $|S| \geq n(n-1)$. Else $|S| \geq x + (n-1)^2 \geq (n-1)^2 + 1$.

Case 3: $2 \leq m \leq n$

Without loss of generality, let $|c_j| \leq n-2$ for all $j \in N(m)$. By Corollary 5.13, $|S_x \cup S_y| = n$ for all $x, y \in N(m)$, $x \neq y$. So by Corollary 5.18, $\sum_{i=1}^{m} |S_i| \geq n(m-1)$, and thus

$$\begin{aligned} |S| &= \sum_{i=1}^{m} |c_i| + \sum_{i=m+1}^{n} |c_i| \\ &\geq \sum_{i=1}^{m} |S_i| + (n-m)(n-1) \\ &\geq n(m-1) + (n-m)(n-1) \\ &= (n-1)^2 + m - 1 \\ &\geq (n-1)^2 + 1. \end{aligned}$$

□

The following is an example of two rows of the full 5-Latin square containing exactly 17 entries:

1				
	2,3,4,5	2,3,4,5	2,3,4,5	2,3,4,5

.

In fact, for $n \geq 3$, any two rows of the full n-Latin square with $(n-1)^2 + 1$ entries has this structure. We prove this claim in the next lemma.

Lemma 5.20 *Let S be two rows of a critical set of the full n-Latin square, where $n \geq 3$. If $|S| = (n-1)^2 + 1$ then S is conjugate to:*

1		$\cdots$		
	$2,3,\ldots,n$	$\cdots$	$2,3,\ldots,n$	$2,3,\ldots,n$

.

Proof. Without loss of generality, let m be the number of columns with size at most $n-2$ and let $|c_j| \leq n-2$ for all $j \in N(m)$. By Corollary 5.13, $|S_x \cup S_y| = n$ for all $x, y \in N(m)$, $x \neq y$. So by Corollary 5.18, $\sum_{i=1}^{m} |S_i| \geq n(m-1)$ and thus

$$n(m-1) + (n-m)(n-1) \leq |S| = (n-1)^2 + 1 \implies m \leq 2.$$

We next show that $m = 1$. Consider the first three columns (c_1, c_2 and c_3 respectively) of S. Since $|c_k| \geq n-1$ for all k such that $4 \leq k \leq n$, $|c_1| + |c_2| + |c_3| \leq 2n - 1$. Suppose that $m = 2$. Then $|c_1| + |c_2| \leq n$ and by Corollary 5.13, $|c_1| + |c_2| = n \implies |c_3| = n - 1 \implies |S_x \cup S_y| = n$ for all $x, y \in N(3)$, $x \neq y$. Thus by Corollary 5.18, $|c_1| + |c_2| + |c_3| \geq 2n$, a contradiction. Now suppose that $m = 0$. Then clearly $|c_1|+|c_2|+|c_3| \geq 3(n-1)$. Thus $m = 1$, $|S_1| = 1$ and by the first two conditions of Lemma 5.11, S assumes the prescribed structure. □

To conclude this section, we use the above results to formulate a lower bound for the size of the smallest critical set of the full n-Latin square.

Theorem 5.21 *Let C be a critical set of the full n-Latin square. Then*

$$|C| \geq \frac{n^3 - 2n^2 + 2n}{2}.$$

Proof. Being an $n \times n$ array, C has $\binom{n}{2}$ distinct pairs of rows with each row occurring in $n-1$ pairs. Thus by Lemma 5.19,

$$|C| \geq \frac{\binom{n}{2}((n-1)^2+1)}{n-1} = \frac{n}{2}((n-1)^2+1) = \frac{n^3 - 2n^2 + 2n}{2}$$

□

5.3 An upper bound for the size of the smallest non-saturated critical set

We begin this section by constructing a non-saturated critical set of the full n-Latin square; a generalization of the smallest critical set of the full 2-balanced Latin square (see Section 4.5). This critical set has size $(n-1)^3+1$ with the final row and column combined, containing only the symbol n occurring in the cell where they intersect and any cell neither in the final row nor the final column containing $N(n-1)$. Let C be such a critical set.

1,2,...,$n-1$	1,2,...,$n-1$	$\cdots$	1,2,...,$n-1$	
1,2,...,$n-1$	1,2,...,$n-1$	$\cdots$	1,2,...,$n-1$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1,2,...,$n-1$	1,2,...,$n-1$	$\cdots$	1,2,...,$n-1$	
		$\cdots$		n

Figure 5.2: The critical set C.

Clearly C has size $(n-1)^3+1$. We only need to show that it is a critical set of the full n-Latin square.

Theorem 5.22 *C is a critical set of the full n-Latin square.*

Proof. We first show that C is a defining set of the full n-Latin square. Suppose we try to complete one of the first $n-1$ cells of column m, $m \in N(n-1)$, with the symbol s, $s \in N(n-1)$. Then cell $C_{n,m}$ is now forced to contain at least two copies of the symbol n and for some $k \in N(n-1)$, $k \neq m$, cell $C_{n,k}$, must not contain the symbol n. This implies that column k can only have at most $n-1$ n's so C cannot complete to a multi-Latin square. Thus for all $i, j \in N(n-1)$, $C_{i,j}$ is forced to contain $N(n)$. Cells in the final row and column follow suit.

We next show that C is a minimal defining set. Suppose we remove the symbol x, $x \in N(n-1)$, from cell $C_{p,q}$, $p, q \in N(n-1)$. Then the 2×2 sub-array formed by the cells $C_{p,q}$, $C_{p,n}$, $C_{n,q}$, and $C_{n,n}$ does not meet the conditions stated by Corollary 5.13 and thus C cannot be a critical set of the full n-Latin square. On the other hand, if the symbol n is removed from $C_{n,n}$, then again, by Corollary 5.13, any 2×2 sub-array containing $C_{n,n}$ cannot be contained in a critical set of the full n-Latin square. □

We next show that the size of a smallest critical set of the full n-Latin square is exactly $(n-1)^3 + 1$ for $n = 2, 3$.

Lemma 5.23 *The exact size of a smallest critical set of the full n-Latin square for $n \in \{2, 3\}$ is $(n-1)^3 + 1$.*

Proof. For $n = 2$, by observation the only critical sets of the full 2-Latin squares are:

1,2	

,

1	1

and

1	
	2

up to main class equivalence.

For $n = 3$, we have seen earlier in the chapter that a critical set of size 9 exists. Now, suppose that C is a critical set of the full 3-Latin square with only 8 entries. Then two of the rows of C contain at most 5 of these entries and by Lemma 5.20, one of these two rows contains only one entry. This implies that the third row must contain 4 entries, a contradiction. □

In the special case when a critical set, C, of the full n-Latin square contains only $n-1$ elements of $N(n)$, C contains at least $n(n-1)^2$ entries. An example of such a critical set of the full 5-Latin square is given below.

1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4

Lemma 5.24 *Let C be a critical set of the full n-Latin square containing only $n-1$ elements of $N(n)$. Then $|C| \geq n(n-1)^2$.*

Proof. Without loss of generality, suppose that the symbol 1 does not occur in C and the symbol 2 occurs at most $n(n-1)-1$ times in C. Then there exists two cells, $C_{i,j}$ and $C_{i',j'}$ of C, such that $i \neq i'$, $j \neq j'$ and neither cell contains the symbol 2. Thus the cells $C_{i,j}$, $C_{i,j'}$, $C_{i',j'}$ and $C_{i',j}$ form an intercalate in the complement of C. So each symbol occurring in C occurs at least $n(n-1)$ times and thus $|C| \geq n(n-1)^2$. □

Motivated by the above results, we conjecture that the size of a smallest critical set of the full n-Latin square is indeed $(n-1)^3+1$.

Conjecture 5.25 *The size of a smallest critical set of the full n-Latin square is $(n-1)^3+1$.*

5.4 A lower bound for the size of the largest non-saturated critical set

For the last section of this chapter we remind the reader of the critical sets of the full n-Latin square from Theorem 4.11 which generalizes Theorem 4.8 to the full (m,n,t)-balanced Latin rectangle. By defining $A[\boldsymbol{a},\boldsymbol{b}]$ as we did in this section, any element of this array where $m=n=t$ is a critical set of the full n-Latin square. In particular, the following partial n-Latin square P, an element of $A[(2,n-2),(2,n-2)]$, is a critical set and moreover, P is the largest critical set we have constructed for the full n-Latin square. While the following lemma is in fact a corollary of Theorem 4.11, we also exhibit a proof for completeness.

Lemma 5.26 *Let P be the partial n-Latin square of order n below:*

1,2,...,n		2,3,...,n	$\cdots$	2,3,...,n
		2,3,...,n	$\cdots$	2,3,...,n
2,3,...,n	2,3,...,n	2,3,...,n	$\cdots$	2,3,...,n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
2,3,...,n	2,3,...,n	2,3,...,n	$\cdots$	2,3,...,n

Then P is a critical set of the full n-Latin square with size $n^3 - n^2 - 3n + 4$.

Proof. Observe that all but 4 cells of P contain $n - 1$ entries and one is saturated. Thus $|P| = (n^2 - 4)(n - 1) + n = n^3 - n^2 - 3n + 4$ as claimed. To complete the proof, we need to show that P is a minimal defining set. We first confirm that P completes uniquely to the full n-Latin square.

Since each cell of row 3 to row n contains $\{2, 3, ..., n\}$, each is forced to contain $N(n)$. Similarly, so do the remaining cells of columns 3 to n. Thus each empty cell is forced to follow suit.

Finally, we show that each entry of P is necessary for its unique completion. Observe that each cell containing $\{1, 2, ..., n\}$ is an element of the subsquare

1,2,...,n	

and each cell containing $\{2, 3, ..., n\}$ is an element of

2,...,n	2,...,n

,

	2,...,n
	2,...,n

or

	2,...,n
2,...,n	2,...,n

Thus by Corollary 5.12, P is a critical set of the full n Latin square. □

Corollary 5.27 *Let $lcs(n,n)$ be the largest critical set of the full n-Latin square. Then $lcs(n,n) \geq n^3 - n^2 - 3n + 4$.*

After exploring many different examples of critical sets of the full n-Latin square, we conjecture that:

Conjecture 5.28 *The size of the largest critical set of the full n-Latin square is $n^3 - n^2 - 3n + 4$.*

Chapter 6

Spectrum of critical sets of the full n-Latin square

In this chapter we study a spectrum of possible sizes of critical sets for the full n-Latin square. We focus on determining the possible sizes between the conjectured smallest and largest critical set of the full n-Latin square of sizes $n^3 - 3n^2 + 3n$ and $n^3 - n^2 - 3n + 4$ respectively (refer to Conjecture 5.25 and Conjecture 5.28). This gives a potential spectrum of size at least $(n-1)^2 + (n-2)^2$. We will show that from size $n^3 - 3n^2 + 3n$ to $n(n-1)^2 + n - 2$ the spectrum is complete. However, for sizes at least $n(n-1)^2 + n - 1$, it remains an open problem whether each size exists. Lemma 6.5 only gives some of the sizes in this part of the spectrum.

We start by reminding the reader of the general structure of the smallest critical set (see Section 5.3) and the largest critical set (see Section 4.6) of the full n-Latin square that we have constructed in this thesis.

1,...,$n-1$	1,...,$n-1$	$\cdots$	1,...,$n-1$	
1,...,$n-1$	1,...,$n-1$	$\cdots$	1,...,$n-1$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1,...,$n-1$	1,...,$n-1$	$\cdots$	1,...,$n-1$	
		$\cdots$		n

1,...,n		2,...,n	$\cdots$	2,...,n
		2,...,n	$\cdots$	2,...,n
2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n

The examples below are critical sets of the full Latin square of orders 3 and 4 respectively. In both cases, we present a critical set of each size in the range claimed by the above conjecture except for a critical set of size 39 for the full 4-Latin square.

$n = 3$:

1		
	2,3	2,3
	2,3	2,3

1		1,2
	2,3	1,2
3	2,3	

1,2	1	
1	3	2,3
	2,3	2,3

	1,2,3	1,2,3
	1,2,3	1,2,3

1,2,3		2,3
		2,3
2,3	2,3	2,3

$n = 4$:

1			
	2,3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4

1			1,2,3
	2,3,4	2,3,4	1,2,3
	2,3,4	2,3,4	1,2,3
4	2,3,4	2,3,4	

1,2	1		
1	3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4

1,2	1,2	1	
1	3,4	3,4	2,3,4
	2,3,4	2,3,4	2,3,4
	2,3,4	2,3,4	2,3,4

1,2	1,2	1	
	3,4	3,4	2,3,4
2	2,3,4	2,3,4	2,3,4
2	2,3,4	2,3,4	2,3,4

1,2,3	1		
	4	2,3,4	2,3,4
2,3	2,3,4	2,3,4	2,3,4
2,3	2,3,4	2,3,4	2,3,4

1,2,3	1,2,3	1	
	4	4	2,3,4
2,3	2,3,4	2,3,4	2,3,4
2,3	2,3,4	2,3,4	2,3,4

1,2,3,4	1		
		2,3,4	2,3,4
2,3,4	2,3,4	2,3,4	2,3,4
2,3,4	2,3,4	2,3,4	2,3,4

	1,2,3,4	1,2,3,4	1,2,3,4
	1,2,3,4	1,2,3,4	1,2,3,4
	1,2,3,4	1,2,3,4	1,2,3,4

1,2,3,4	1,2,3,4		2,3,4
1,2,3,4	1,2,3,4		2,3,4
			2,3,4
2,3,4	2,3,4	2,3,4	2,3,4

1,2,3,4		2,3,4	2,3,4
1,2,3,4		2,3,4	2,3,4
		2,3,4	2,3,4
2,3,4	2,3,4	2,3,4	2,3,4

1,2,3,4		2,3,4	2,3,4
		2,3,4	2,3,4
2,3,4	2,3,4	2,3,4	2,3,4
2,3,4	2,3,4	2,3,4	2,3,4

The above examples (particularly for $n = 4$) not only gave us most of the sizes we were after, but they all have generalizable structures which form the basis for the main results of this chapter.

First we prove the following lemma which is also useful for later results in this chapter.

Lemma 6.1 *Let P be a partial n-Latin square of order n such that for some m, $1 \le m \le n-1$:*

$$P_{i,1} = \{1\} \quad (1 \le i \le m);$$

$$\textit{and} \quad P_{i,j} = \{2,3,...,n\} \quad (m \le i \le n, \quad 2 \le j \le n);$$

1		$\cdots$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$
1		$\cdots$	
	2,3,...,n	$\cdots$	2,3,...,n
$\vdots$	$\vdots$	$\ddots$	$\vdots$
	2,3,...,n	$\cdots$	2,3,...,n

Then in any completion of P to the n-Latin square of order n, each cell of the last $n-m$ rows of P contains $N(n)$.

Proof. Since the last $n-1$ cells of each of the last m rows of P can only contain at most $n-1$ occurrences of 1, each empty cell of the first column contains at least one 1. However, since there are already m occurrences of 1 in the first column, each of these empty cells contain exactly one 1. Subsequently, each cell of the last m rows is forced to contain $N(n)$. □

Henceforth in this chapter, we will show that the partial n-Latin square, P, we use in each result (Lemma 6.2 to Lemma 6.5) is not only a critical set of the full n-Latin square but these chosen critical sets span a consecutive set of integers in our spectrum. For convenience we denote the contents of a non-empty cell of P by P_{NE}, and we will show in each lemma that there exists a sub-square, S, of P such that $P_{NE} \in S$ and each entry of P_{NE} satisfies Corollary 5.12.

Lemma 6.2 *For $n \geq 3$, let P be a partial n-Latin square of order n such that:*

$$P_{1,1} = \{1\}; \quad P_{n,1} = \{n\};$$
$$P_{i,j} = \{2, 3, ..., n\} \quad (2 \leq i \leq n, \quad 2 \leq j \leq n-1);$$
$$\text{and} \quad P_{i,n} = \{1, 2, ..., n-1\} \quad (2 \leq i \leq n-1).$$

1		$\cdots$		1,2,...,$n-1$
	2,3,...,n	$\cdots$	2,3,...,n	1,2,...,$n-1$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	2,3,...,n	$\cdots$	2,3,...,n	1,2,...,$n-1$
n	2,3,...,n	$\cdots$	2,3,...,n	

Then P is a critical set of the full n-Latin square with size $(n-1)^3+2$.

Proof. We first show that P is a defining set of the full n-Latin square. By Lemma 6.1, the last $n-1$ columns forms a saturated sub-rectangle of P and thus P completes uniquely to the full n-Latin square.

Next we show that each entry in P is necessary for this unique completion. If:

$P_{NE} = \{1\}$, $S =$

1	
	2,3,...,n

;

$P_{NE} = \{n\}$, $S =$

	1,2,...,$n-1$
n	

;

$P_{NE} = \{2,3,...,n\}$, $S =$

2,3,...,n	2,3,...,n

;

$P_{NE} = \{1,2,...,n-1\}$, $S =$

1	1,2,...,$n-1$
n	

;

or

$S =$

	1,2,...,$n-1$
n	

;

thus by Corollary 5.12, P is a critical set.

The size of P is clearly as stated so the proof is complete. □

Lemma 6.3 *For $n \geq 3$, let P be a partial n-Latin square of order n such that for some m, $1 \leq m \leq n-2$:*

$$
\begin{aligned}
& P_{1,m+1} = P_{2,1} = \{1\}; \\
& P_{1,j} = \{1,2\} \quad (1 \leq j \leq m); \\
& P_{2,j} = \{3,4,...,n\} \quad (2 \leq j \leq m+1); \\
& P_{2,j} = \{2,3,...,n\} \quad (m+2 \leq j \leq n); \\
\textit{and} \quad & P_{i,j} = \{2,3,...,n\} \quad (3 \leq i \leq n, \quad 2 \leq j \leq n).
\end{aligned}
$$

1,2	1,2	1,2	$\cdots$	1,2	1		$\cdots$		
1	3,...,n	3,...,n	$\cdots$	3,...,n	3,...,n	2,...,n	$\cdots$	2,...,n	2,...,n
	2,...,n	2,...,n	$\cdots$	2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n	2,...,n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	2,...,n	2,...,n	$\cdots$	2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n	2,...,n

Then P is a critical set of the full n-Latin square and $|P| = (n-1)^3 + m + 2$.

Proof. We first show that P is a defining set of the full n-Latin square.

By Lemma 6.1, each cell of the last $n-2$ rows and that of the last $n-m-2$ columns is forced to contain $N(n)$. This forces each of the cells containing $\{3,4,...,n\}$ to contain exactly one 1 (and the square to be a superset of the partial n-Latin square in Lemma 6.1) thus by Lemma 6.1, P completes uniquely to the full n-Latin square.

Next we show that each entry in P is necessary for this unique completion. If:

$P_{NE} = \{1\}$, $S =$

1	
3,...,n	2,...,n

or

1	3,...,n
	2,...,n

;

$P_{NE} = \{1, 2\}$ *or* $\{3, ..., n\}$, $S =$

1,2	1
1	3,...,n

or

1,2	1
3,...,n	3,...,n

;

$P_{NE} = \{2, ..., n\}$, $S =$

2,...,n	2,...,n

or

	2,...,n
	2,...,n

;

thus by Corollary 5.12, P is a critical set.

Observe that since m cells contain $\{1, 2\}$, 2 cells contain $\{1\}$, m cells contain $\{3, ..., n\}$, and $(n - 2)(n - 1) + n - m - 1$ cells contain $\{2, ..., n\}$,

$$\begin{aligned} |P| &= 2m + 2 + m(n - 2) + [(n - 2)(n - 1) + n - m - 1](n - 1) \\ &= (n - 1)^3 + m + 2. \end{aligned}$$

□

Another critical set is formed if cell $P_{1,2}$ of the above partial n-Latin square is empty and $P_{1,3} = P_{1,4} = ... = P_{1,n} = \{2\}$ as below:

1,2	1,2	1,2	$\cdots$	1,2	1		$\cdots$	
	3,...,n	3,...,n	$\cdots$	3,...,n	3,...,n	2,...,n	$\cdots$	2,...,n
2	2,...,n	2,...,n	$\cdots$	2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
2	2,...,n	2,...,n	$\cdots$	2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n

Observe that for such a partial n-Latin square,

$$\begin{aligned} |P| &= 2m + n - 1 + m(n - 2) + [(n - 2)(n - 1) + n - m - 1](n - 1) \\ &= (n - 1)^3 + m + n - 1, \end{aligned}$$

thus $(n-1)^3 + n \leq |P| \leq (n-1)^3 + 2n - 3$.

We next generalize the above observation in the following lemma by replacing the entries $\{2\}$, $\{1,2\}$ and $\{3,...,n\}$ by $\{2,3,...,n\}$, $\{1,2,...,k\}$ and $\{k+1, k+2, ..., n\}$ respectively. In doing so we add another $n^2 - 3n + 2$ sizes (in succession) to our spectrum.

Lemma 6.4 *For $n \geq 3$, let P be a partial n-Latin square of order n such that for some m $(1 \leq m \leq n-2)$ and k $(2 \leq k \leq n)$:*

$$P_{1,m+1} = \{1\};$$
$$P_{1,j} = \{1,2,...,k\} \quad (1 \leq j \leq m);$$
$$P_{2,j} = \{k+1, k+2, ..., n\} \quad (2 \leq j \leq m+1);$$
$$P_{2,j} = \{2,3,...,n\} \quad (m+2 \leq j \leq n);$$
$$P_{i,1} = \{2,3,...,k\} \quad (3 \leq i \leq n);$$
$$\text{and} \quad P_{i,j} = \{2,3,...,n\} \quad (3 \leq i \leq n, \quad 2 \leq j \leq n).$$

1,...,k	1,...,k	$\cdots$	1,...,k	1		$\cdots$	
	k+1,...,n	$\cdots$	k+1,...,n	k+1,...,n	2,...,n	$\cdots$	2,...,n
2,...,k	2,...,n	$\cdots$	2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
2,...,k	2,...,n	$\cdots$	2,...,n	2,...,n	2,...,n	$\cdots$	2,...,n

Then P is a critical set of the full n-Latin square and $|P| = (n-1)^3 + kn - n - 2k + m + 3$.

Proof. We first show that P completes uniquely to the full n-Latin square. By Lemma 6.1, the last $n-m-1$ columns complete to form a saturated subrectangle of P. This in turn forces each of the other non-empty cells of the second row to contain at most one copy of each element of $\{2,3,...,k\}$ which gives a superset of the critical set in Theorem 5.22.

Next, we show that each entry of P is necessary for this unique completion.

If:

$P_{NE} = \{1\}$, $S =$

1	
k+1,...,n	2,...,n

;

$P_{NE} = \{2,...,k\}$, $S =$

	k+1,...,n
2,...,k	2,...,n

;

$P_{NE} =$ $\{1,...,k\}$ *or* $\{k+1,...,n\}$ $S =$

1,...,k	1
	k+1,...,n

;

or

$S =$

1,...,k	1
k+1,...,n	k+1,...,n

;

$P_{NE} = \{2,...,n\}$, $S =$

2,...,n	2,...,n

;

or

$S =$

	k+1,...,n
2,...,k	2,...,n

;

thus by Corollary 5.12, P is a critical set.

To determine the size of P, observe that m cells contain $\{1,...,k\}$, 1 cell contains $\{1\}$, m cells contain $\{k+1,...,n\}$, $(n-2)$ cells contain $\{2,...,k\}$ and $(n-2)(n-1)+n-m-1$ cells contain $\{2,...,n\}$, thus

$$\begin{aligned}|P| &= km+1+m(n-k)+(n-2)(k-1)\\ &\quad +[(n-2)(n-1)+n-m-1](n-1)\\ &= n^3-3n^2+kn+2n-2k+m+2\\ &= (n-1)^3+kn-n-2k+m+3.\end{aligned}$$

□

We next explore a generalized structure of the critical set we discussed in Section 5.4. This structure gives many of the sizes in the last $(n-2)^2$ sizes of the spectrum (i.e. sizes between $n(n-1)^2+1$ and n^3-n^2-3n+4).

Lemma 6.5 *For $n \geq 3$, let P be a partial n-Latin square of order n such that for $2 \leq l, m \leq n-1$:*

$$P_{i,j} = \{1,2,...,n\} \quad (1 \leq i \leq l-1, \quad 1 \leq j \leq m-1);$$
$$\textit{and} \quad P_{i,j} = \{2,3,...,n\} \quad (l+1 \leq i \leq n, \quad m+1 \leq j \leq n).$$

1,2,...,n	$\cdots$	1,2,...,n		2,3,...,n	$\cdots$	2,3,...,n
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
1,2,...,n	$\cdots$	1,2,...,n		2,3,...,n	$\cdots$	2,3,...,n
	$\cdots$			2,3,...,n	$\cdots$	2,3,...,n
2,3,...,n	$\cdots$	2,3,...,n	2,3,...,n	2,3,...,n	$\cdots$	2,3,...,n
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
2,3,...,n	$\cdots$	2,3,...,n	2,3,...,n	2,3,...,n	$\cdots$	2,3,...,n

Then P is a critical set of the full n-Latin square with size $n(n-1)^2+(n-l)(n-m)$.

Proof. P is clearly a defining set of the full n-Latin square thus we first need to prove it is also a critical set. Since each cell containing $\{1,2,...,n\}$ is contained in a subsquare

1,2,...,n	

and each cell containing $\{2,3,...,\text{n}\}$ is contained in a subsquare conjugate to

2,3,...,n	2,3,...,n

or the bottom right cell of

	2,3,...,n
2,3,...,n	2,3,...,n

then by Corollary 5.12, P is a critical set of the full n-Latin square.

Next observe that $(l-1)(m-1)$ cells contain $\{1, 2, ..., n\}$ and $n^2 - lm$ cells contain $\{2, 3, ..., n\}$, thus:

$$\begin{aligned} |P| &= (l-1)(m-1)n + (n^2 - lm)(n-1) \\ &= n^3 - n^2 + n - ln - mn + lm \\ &= n(n^2 - 2n + 1) + n^2 - ln - mn + lm \\ &= n(n-1)^2 + (n-l)(n-m). \end{aligned}$$

□

By combining Lemma 6.2, Lemma 6.3, Lemma 6.4 and Lemma 6.5, we have the following spectrum result.

Theorem 6.6 *For all n there exists a critical set C of the full n-Latin square with size $|C|$ where $(n-1)^3 + 1 \leq |C| \leq n(n-1)^2 + n - 2$.*

Proof. By Theorem 5.22 and Lemma 6.2, critical sets of sizes $(n-1)^3+1$ and $(n-1)^3+2$ exist. By Lemma 6.3, $|C| = (n-1)^3 + m + 2$ $(1 \leq m \leq n-2)$, so a critical set of each size from $(n-1)^3 + 3$ to $(n-1)^3 + n$ also exists. By Lemma 6.4, $|C| = (n-1)^3 + kn - n - 2k + m + 3$ $(1 \leq m \leq n-2$ and $2 \leq k \leq n)$, thus for:

- $m = 1$, $k = 2$, $|C| = (n-1)^3 + n$;
- $m = n-2$, $k = n$, $|C| = n(n-1)^2$ $(= (n-1)^3 + (n-1)^2)$.

With each size between $(n-1)^3 + n$ and $n(n-1)^2$ given by different values m and k, we have now established the existence of each the first $(n-1)^2$ sizes in the spectrum. The final part is given by Lemma 6.5 where $|C| =$

$n(n-1)^2 + (n-l)(n-m)$. Since $n-l, n-m \in N(n-2)$, $(n-l)(n-m) = x$ for all $x \in N(n-2)$. □

Furthermore, the dependence of $|P|$ on the product $(n-l)(n-m)$ $(2 \leq l, m \leq n-1)$, implies that only $(n-2)(n-1)/2$ out of the $(n-2)^2$ sizes in the top part of the potential spectrum are given by the above structure; leaving $(n^2 - 5n + 6)/2$ sizes undetermined. For example, when $n = 4$, $|P| \neq 39$ and when $n = 5$, three of the nine values of $|P|$ (i.e. $85, 87$ and 88) are undetermined.

We summarize this chapter with the following table:

Order (n)	Size of potential spectrum	Number of undetermined holes
3	5	0
4	13	1
5	25	3
6	41	6
7	61	10
8	85	15
9	113	21
10	145	28
⋮	⋮	⋮
n	$(n-1)^2 + (n-2)^2$	$(n^2 - 5n + 6)/2$

Chapter 7

Completability of partial multi-Latin squares

As we discussed in Section 2.4, any Latin square of order n with at most $n-1$ entries can be completed. This result has since been generalized to semi-Latin squares in [39] and a partial generalization to Latin cubes is given in [40]. On the other hand, a partial Latin square of order n filled with n or more symbols may be incompletable. For example:

1					
	1				
		1			
			1		
				1	
					2

As a direct consequence of the optimal size of the completable Latin square, incompletable Latin squares contain at least n entries.

This chapter explores the completability of partial k-Latin squares (i.e. whether a given partial multi-Latin square is completable or incompletable) which, even for $k = 1$, is an *NP-complete* problem [30]. In Section 7.1, we generalize Evans' conjecture to include multi-Latin squares. We show that any partial multi-Latin square of order n and index k with at most $(n-1)$ entries is also completable. In Section 7.2, we investigate the incompletable partial Latin squares with minimal sizes. We present constructions of incompletable partial k-Latin squares of order n and size $k(n-1)+1$ in Section 7.2 and we conjecture this to be the optimal size.

Subsection 7.1.1 and Section 7.2 have been submitted for publication ([85]).

An observation while exploring partial Latin squares with no completions is that they all contain more than one symbol. In fact, it is easy to show that a partial Latin square L filled in with only one symbol s is completable. In the next section we will show that a partial k-Latin square containing only one symbol is also completable.

7.1 Completable partial multi-Latin squares

In this section we first show that a partial multi-Latin square containing only one symbol also completes to a multi-Latin square of the same order; then we generalize Theorem 2.3 to a partial k-Latin square of order n with at most $n-1$ entries.

Lemma 7.1 *Let P be a partial k-Latin square of order n containing only one symbol s which occurrs at most $kn-1$ times. Then there exists a partial k-Latin square P' of the same order and size kn, also containing only the symbol s, such that $P \subseteq P'$.*

Proof. We first observe that, by the definition of a k-Latin square, each symbol s occurs at most k times in each cell, at most k times in each row, and at most k times in each column of P. Moreover, s occurs kn times in a partial k-Latin square if and only if it occurs k times in each row and k times in each column.

Without loss of generality, let $s=1$ and let r_i be a row of P that contains at most $k-1$ occurrences of the symbol 1. This implies that we have at most $kn-1$ occurrences of the symbol 1 in P and that at least one column of P

contains at most $k-1$ occurrences of the symbol 1. Let c_j be one such column. Then we can add a 1 to the intersection of r_i and c_j i.e. cell $L_{i,j}$.

The process can be repeated until each row/column of P contains k occurrences of 1. □

The next construction completes any partial k-Latin square P' from the above construction to a k-Latin square L of the same order, by recursively permuting the columns cyclically, adding one modulo n to each symbol.

Construction 7.2 *Let P be a partial k-Latin square of order n and size kn containing only one symbol s. We complete P to a k-Latin square L as follows.*

For $1 \leq i, j \leq n$ and $1 \leq h \leq n-1$, if s occurs λ times in cell $P_{i,j}$ then $s+h \pmod{n}$ occurs λ times in cell $L_{i,j+h \pmod{n}}$.

We use the above construction to complete the following partial 3-Latin square containing only the symbol 1. For $k = n = 3$ and $s = 1$, if

$P' =$

1	1,1	
		1,1,1
1,1	1	

then

$L =$

1,3,3	1,1,2	2,2,3
2,2,2	3,3,3	1,1,1
1,1,3	1,2,2	2,3,3

.

Combining Lemma 7.1 and Construction 7.2, we can now show that any partial k-Latin square containing only one symbol completes to a k-Latin square of the same order.

Theorem 7.3 *A partial multi-Latin square P filled in with only one symbol s is completable.*

Proof. By Lemma 7.1, we only need to show that our claim is true when P has size kn i.e. it suffices to show that the multi-Latin square L from Construction 7.2 is indeed valid.

We shall divide our proof to two parts; showing that:

1. each symbol occurs k times in each row and k times in each column of L.
2. each cell of L contains k symbols.

Let $|s_{i,j}|$ be the number of times the symbol s occurs in cell $L_{i,j}$ and without loss of generality, let $s = 1$. We observe that, in each row,

$$\sum_{j=1}^{n} |1_{i,j}| = k, \quad for \;\; 1 \leq i \leq n,$$

and in each column,

$$\sum_{i=1}^{n} |1_{i,j}| = k, \quad for \;\; 1 \leq j \leq n.$$

Next,

$$\begin{aligned} & |1_{i,j}| = |2_{i,j+1 \pmod n}| = |3_{i,j+2 \pmod n}| = ... = |n_{i,j+n-1 \pmod n}| \\ \implies & |1_{i,j-1 \pmod n}| = |2_{i,j}|, |1_{i,j-2 \pmod n}| = |3_{i,j}|, ..., |1_{i,j-n+1 \pmod n}| = |n_{i,j}| \end{aligned}$$

for $1 \leq i,j \leq n$. Thus the number of occurrences of each symbol in any row of L is given by

$$\sum_{j=1}^{n} |s_{i,j}| = \sum_{j=1}^{n} |1_{i,j-s+1 \pmod n}| = k, \quad for \;\; 1 \leq i \leq n;$$

and in each column

$$\sum_{i=1}^{n} |s_{i,j}| = \sum_{i=1}^{n} |1_{i,j-s+1 \pmod n}| = k, \quad for \;\; 1 \leq j \leq n.$$

Finally, Construction 7.2 implies that each cell $L_{i,j}$ contains $|1_{i,j}|$ occurrences of symbol 1, $|2_{i,j}| = |1_{i,(j+1) \pmod n}|$ occurrences of symbol 2, ..., and $|n_{i,j}| = |1_{i,(j+n-1) \pmod n}|$ occurrences of symbol n. That is, each cell contains

$$\sum_{j=1}^{n} |1_{i,j}| = k$$

symbols. □

7.1.1 A generalization of Evans' conjecture

We finish this section with a generalization of Evans' conjecture for multi-Latin squares. Here, we are particularly interested in [39], where Kuhl and Denley generalized Evans' conjecture to include semi-Latin squares.

Recall that a *partial semi-Latin square* of order n and index k is an $n \times n$ array of sets of size at most k, such that each symbol of a set of size nk occurs at most once in each row and at most once in each column. Such an array is a *semi-Latin square* if each set is of size k, and thus each symbol occurs exactly once in each row and in each column (see Section 1.1). The following is an example of a semi-Latin square of order 4 and index 3.

1,4,5	2,6,10	3,7,8	9,11,12
2,3,11	1,4,9	5,6,12	7,8,10
6,7,9	3,8,12	1,10,11	2,4,5
8,10,12	5,7,11	2,4,9	1,3,6

Henceforth, we will refer to a semi-Latin square with index k as a *semi-k-Latin square*. (This clashes with the definition of multi-Latin squares given in [39] - see also below. Since semi-Latin square is by far the most common term in the literature, we adhere to it.) Semi-Latin squares are a generalization of Latin squares, since a semi-1-Latin square is simply a Latin square.

Kuhl and Denley's generalization of Evans' conjecture is given below:

Theorem 7.4 *Let S be a partial semi-k-Latin square of order n with at most $n-1$ non-empty cells each containing k entries. Then S can be completed.*

We approach this generalization by transforming a k-Latin square of order n with at most $n-1$ entries into a semi-Latin square of order n and index k, and show that under certain conditions it completes to a semi-Latin square, using Kuhl and Denley's result. We first illustrate these transformations with the following example.

Let M be the partial 3-Latin square of order 4 given below:

1	3		
1			

.

1. Transform M to a partial semi-3-Latin square $S = M^*$ of the same order by replacing each occurrence of a symbol s with a symbol from $\{s^1, s^2, ..., s^k\}$ so that each symbol of this set occurs at most once in each row and in each column of M^*.

1^1	3^1		
1^2			

2. Transform S to a partial semi-3-Latin square S' by filling in each non-empty cell of S.

$1^1, 1^3, 2^1$	$3^1, 1^2, 2^2$		
$1^2, 2^2, 2^3$			

3. Complete S' to a semi-k-Latin square using Theorem 7.4.

$1^1, 1^3, 2^1$	$3^1, 1^2, 2^2$	$2^3, 3^2, 3^3$	$4^1, 4^2, 4^3$
$1^2, 2^2, 2^3$	$2^1, 3^2, 3^3$	$4^1, 4^2, 4^3$	$1^1, 1^3, 3^1$
$3^1, 3^2, 3^3$	$4^1, 4^2, 4^3$	$1^1, 1^2, 1^3$	$2^1, 2^2, 2^3$
$4^1, 4^2, 4^3$	$1^1, 1^3, 2^3$	$2^1, 2^2, 3^1$	$1^2, 3^2, 3^3$

4. Replace each occurrence of the symbols from $\{s^1, s^2, ..., s^k\}$ with the symbol s to form a completion of M to a 3-Latin square.

1,1,2	3,1,2	2,3,3	4,4,4
1,2,2	2,3,3	4,4,4	1,1,3
3,3,3	4,4,4	1,1,1	2,2,2
4,4,4	1,1,2	2,2,3	1,3,3

The next step is to show that we can always perform the above transformations to any partial k-Latin square M of order n and size at most $n-1$. Thus we first show that M can be transformed into a partial semi-k-Latin square of the same order. As in the example, we aim to do this by replacing each occurrence of a symbol s with a symbol from $\{s^1, s^2, ..., s^k\}$ so that each symbol of this set occurs at most once in each row and in each column. We denote such a transformed partial semi-k-Latin square by M^*.

Lemma 7.5 *Let M be a partial k-Latin square of order n. Then M can be transformed into a partial semi-k-Latin square of the same order by replacing each occurrence of a symbol s with a symbol from $\{s^1, s^2, ..., s^k\}$ so that each symbol of this set occurs at most once in each row and in each column of M^*.*

Proof. Let the rows and columns of M be indexed with $\{r_1, r_2, ..., r_n\}$ and $\{c_1, c_2, ..., c_n\}$ respectively and let $G_s(M) = (V, E)$ be a bipartite multigraph with $V_1 = \{r_1, r_2, ..., r_n\}$ and $V_2 = \{c_1, c_2, ..., c_n\}$ such that the multiplicity of edge $\{r_i, c_j\}$ of $G_s(M)$ is equal to the number of times the symbol s occurs in cell $M_{i,j}$. Then $G_s(M)$ is of degree at most k and therefore is k-edge-colourable by Theorem 10.5. Thus the lemma holds as each occurrence of the symbol s in any row or column is represented by an unique colour in the proper colouring of $G_s(M)$. □

The proof of our main result (Theorem 7.7) will require the use of Theorem 7.4, which states that a semi-k-Latin square of order n with at most $(n-1)$ non-empty cells each containing k entries can be completed. With Lemma 7.5, a partial k-Latin square of order n with at most $n-1$ entries transforms into a semi-k-Latin square of order n with at most $n-1$ non-empty cells. We now show that we can always fill in these non-empty cells so that each contains k entries.

Lemma 7.6 *Let S be a partial semi-k-Latin square of order n with at most $n-1$ non-empty cells. Then S is a subset of a partial semi-k-Latin square S' of the same order such that if cell $S_{i,j}$ is non empty then cell $S'_{i,j}$ is of cardinality k.*

Proof. Suppose that cell $S_{i,j}$ is non-empty with cardinality of at most $k-1$. Then in S', there are at most $k(n-1)-1$ entries in the same row or column as cell $S'_{i,j}$. So at least $k+1$ symbols do not occur in either row i or column j hence cell $S'_{i,j}$ can be filled in. □

With the above lemma completing the preliminary results needed, we now prove our main result in this chapter.

Theorem 7.7 *Let M be a partial k-Latin square of order n with at most $n-1$ entries from $N(n)$. Then M can be completed to a k-Latin square of order n.*

Proof. By Lemma 7.5 we transform M to a partial semi-k-Latin square M^* of order n with at most $k(n-1)$ from the set $\bigcup_{s=1}^{n}\{s^1, s^2, ..., s^k\}$. By Lemma 7.6, we fill in the non-empty cells of M^* so that each contains k symbols from the set $\bigcup_{s=1}^{n}\{s^1, s^2, ..., s^k\}$. We complete this partial semi-k-Latin square to a semi-Latin square by Theorem 7.4. Then replacing each occurrence of the symbols from $\{s^1, s^2, ..., s^k\}$ with the symbol s, for $1 \leq s \leq n$, gives a completion of M. □

Based on Construction 7.12 and Construction 7.13, which will show the existence of a non-completable partial k-Latin square of order n and size $k(n-1)+1$, and motivated by the fact that a k-Latin square of order n contains k times more entries than a Latin square of the same order, we make the following conjecture:

Conjecture 7.8 *A partial k-Latin square of order n with at most $k(n-1)$ entries can be completed.*

In the special case where all of the $k(n-1)$ entries are contained in $n-1$ (full) cells, the partial k-Latin square is certainly completable. The following result which formalizes this particular case is analogous to Theorem 7.4 and generalizes Theorem 7.7. The proof is very similar to that of Theorem 7.7 and is therefore omitted.

Theorem 7.9 *A partial k-Latin square of order n with at most $n-1$ non-empty cells can be completed.*

A strategy that we explored towards possibly proving the above conjecture for the general case was to try to partition a partial k-Latin square into k partial Latin squares. Observe that if a partial k-Latin square P of order n with at most $k(n-1)$ entries is fully separable into k partial Latin squares each of order n and size $n-1$ then each one of these partial Latin squares complete to a Latin square by Theorem 2.3. Moreover, the join of the completions of these partial Latin squares give a completion of P. The *join* of two k-Latin squares L' and L'', denoted by $L' \oplus L''$, refers to the $2k$-Latin square L of the same order, such that $L_{i,j} = L'_{i,j} \uplus L''_{i,j}$. We say that a k-Latin square L is *fully separable* if there exist k Latin squares $L_1, L_2, ..., L_k$, all of the same order, such that $L = L_1 \oplus L_2 \oplus ... \oplus L_k$ (see [22]).

Lemma 7.10 *Let P be a fully separable partial k-Latin square of order n and size $k(n-1)$ that separates to k partial Latin squares each of size $n-1$. Then P completes to a k-Latin square of order n.*

Proof. By Theorem 2.3, each partial Latin square completes to the Latin squares $\{L_1, L_2, ..., L_k\}$ respectively and $L = L_1 \uplus L_2 \uplus ... \uplus L_n$ is a completion of P to a k-Latin square of the same order. □

We illustrate the proof of the above lemma with the following example. Let P be the fully separable partial 3-Latin square below.

$P =$

1,1		
	2	1,1,2
	3	2,2

Then

$P = P^1 \oplus P^2 \oplus P^3 =$

1		
	2	

$\oplus$

1		
		2

$\oplus$

	3	2

.

Now P^1, P^2, and P^3 complete to the Latin squares

1	3	2
3	2	1
2	1	3

$\oplus$

1	2	3
2	3	1
3	1	2

$\oplus$

2	1	3
3	2	1
1	3	2

respectively. And

$P^1 \oplus P^2 \oplus P^3 =$

1,1,2	1,2,3	2,3,3
2,3,3	2,2,3	1,1,2
1,2,3	1,1,3	2,2,3

is a completion of P.

Unfortunately the above approach cannot always succeed directly. Consider the example below of a partial 3-Latin square, L of order 6 and size 13.

1,1,2					
1	1,1,3				
2,2	3,3	2,3			

Observe that in order to partition L into 3 partial Latin squares L_1, L_2 and L_3 we should be able to replace each occurrence of a symbol s in L with a symbol s^k to form a semi-Latin square L^*, where $k \in N(3)$ and s^k is put in L_k in the partition of L. Without loss of generality, let $L^*_{1,1} = \{1^1, 1^2, 2^3\}$, then L^* is forced to contain:

$1^1, 1^2, 2^3$		
1^3	$1^1, 1^2, 3^3$	
$2^1, 2^2$	$3^1, 3^2$	$2^3, 3^3$

.

Then L cannot be fully separated since the entries of $L_{3,3}$ are forced by the structure of L to be put in the same cell of L_3. For similar reasons, the following partial k-Latin (sub-)square, P of order n and size $4k+1$ (which generalizes the one above) also cannot be separated into k partial Latin squares.

$\underbrace{s_1, \ldots, s_1}_{(k-1)\text{-times}}, s_2$		
s_1	$\underbrace{s_1, \ldots, s_1}_{(k-1)\text{-times}}, s_3$	
$\underbrace{s_2, \ldots, s_2}_{(k-1)\text{-times}}$	$\underbrace{s_3, \ldots, s_3}_{(k-1)\text{-times}}$	s_2, s_3

This example motivates the following conjecture:

Conjecture 7.11 *A partial k-Latin square of order n and size at most $4k$ is fully separable and thus completable.*

7.2 Premature partial k-Latin squares

Recall from Section 2.5 that a *premature* partial Latin square is a partial Latin square that cannot be completed to a Latin square of the same order, but is completable upon the removal of any one of its entries. In this section we explore the more general premature partial multi-Latin squares.

Following a similar strategy to finding critical sets of the full n-Latin square, we first studied premature partial $(n, n, 2)$-balanced Latin squares of small orders. Listed below are some such structures for $n = 2, 3, 4, 5$.

$n = 2$

1	
	2,2

$n = 3$

1,1		
	2,2	2

1,1		
	1,1	
		2,2

$n = 4$

1,1			
1			
	2,2	2,2	

$n = 5$

1,1				
1,1				
	2,2	2,2	2	

The first example for $n = 3$ and the ones for $n = 2, 4,$ and 5 are incompletable since the intersection of the first column and the non-empty row is forced by the row to be filled in with the entry $\{1, 1\}$ forcing the first column to contain $n + 1$ occurrences of the symbol 1. As for the other example for order 3, it is easy to see that both the empty cells of row/column 3 are forced to have entries $\{1, 2\}$ making it impossible to complete. It can also be shown that deleting any symbol from any cell results in a completable partial k-Latin square.

While exploring premature partial k-Latin squares for $k \geq 2$, we discovered other interesting patterns. Among the examples found, two patterns stood out and are given below for $n = 2, 3, 4$.

$n = k = 2$

1,1	
	2

$n = k = 3$

1,1,1		
	2,2,2	3

,

1,1,1		
	1,1,1	
		2

$n = k = 4$

1,1,1,1			
	2,2,2,2	3,3,3,3	4

,

2,2,2,2			
	2,2,2,2		
		2,2,2,2	
			4

The following constructions generalize the examples above.

Construction 7.12

For an $n \times n$ array, P:

- Fill in cell $P_{1,1}$ with the entry $\underbrace{1, 1, ..., 1}_{k-times}$.
- For $2 \leq i \leq n-1$, fill in cell $P_{2,i}$ with the entry $\underbrace{i, i, ..., i}_{k-times}$.
- Fill in cell $P_{2,n}$ with the entry n.

Construction 7.13

For an $n \times n$ array, Q:

- For $1 \leq i \leq n-1$, fill in cell $Q_{i,i}$ with the entry $\underbrace{x, x, ..., x}_{k-times}$, $x \in N(k)$.
- Fill in cell $Q_{n,n}$ with the entry y, $y \neq x; y \in N(k)$.

We next show that any partial Latin square constructed by the above constructions is indeed a premature partial Latin square. Observe that any partial k-Latin square from either construction has size $k(n-1)+1$.

Lemma 7.14 *Let P be a partial k-Latin square of order n generated by Construction* 7.12*. Then P is a premature partial k-Latin square of order n.*

Proof. Observe that P is of the form:

$\underbrace{1, ..., 1}_{k \text{ times}}$			$\cdots$		
	$\underbrace{2, ..., 2}_{k \text{ times}}$	$\underbrace{3, ..., 3}_{k \text{ times}}$	$\cdots$	$\underbrace{n-1, ..., n-1}_{k \text{ times}}$	n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$

We first show that P is incompletable. By the definition of a row of a partial k-Latin square, cell $P_{2,1}$ is forced to contain at least one occurrence of the symbol 1 which already occurs k times in column 1, so P is incompletable. We next show that removing any one of the entries of P guarantees completability. Let s be the the entry we remove from P. We split this part of the proof into the cases $s = 1$, $2 \leq s \leq n-1$ and $s = n$.

Case 1: $s = 1$.

P completes to the k-Latin square below:

1,...,1,n	1,n,...,n	$\cdots$	n-2,...,n-2	n-1,...,n-1
1,n,...,n	2,...,2	$\cdots$	n-1,...,n-1	1,...,1,n
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n-2,...,n-2	n-1,...,n-1	$\cdots$	n-4,...,n-4	n-3,...,n-3
n-1,...,n-1	1,...,1,n	$\cdots$	n-3,...,n-3	n-2,...,n-2

Case 2: $2 \leq s \leq n-1$.

Without loss of generality, let $s = 2$. Then P completes to the k-Latin square below:

1,...,1	2,n,...,n	2,...,2,n	3,...,3	$\cdots$	n-1,...,n-1
2,n,...,n	1,2,...,2	3,...,3	4,...,4	$\cdots$	1,...,1,n
2,...,2,n	3,...,3	4,...,4	5,...,5	$\cdots$	2,n,...,n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n-2,...,n-2	n-1,...,n-1	1,...,1	2,n,...,n	$\cdots$	n-3,...,n-3
n-1,...,n-1	1,...,1,n	2,n,...,n	1,2,...,2	$\cdots$	n-2,...,n-2

Case 3: $s = n$.

For this case, P completes by Theorem 7.9. □

Lemma 7.15 *Let Q be a partial k-Latin square of order n generated by Construction* 7.13*. Then Q is a premature partial k-Latin square of order n.*

Proof. Any partial k-Latin square constructed by Construction 7.13 has the following general structure:

$\underbrace{1, ..., 1}_{k \text{ times}}$		$\cdots$		
	$\underbrace{1, ..., 1}_{k \text{ times}}$	$\cdots$		
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
		$\cdots$	$\underbrace{1, ..., 1}_{k \text{ times}}$	
		$\cdots$		2

.

Since each of the first $n-1$ rows and columns of Q contain k occurrences of the symbol 1, cell $Q_{n,n}$ which already contains the symbol 2, is forced to contain all the k occurrences of the symbol 1 for the n^{th} row (column), thus Q is incompletable. We next show that Q is a subset of at least one k-Latin square of the same order if we remove any one of its entries. By the symmetry of Q we only have two cases to prove.

Case 1: $s = 1$.

Without loss of generality, we remove an entry from cell $Q_{1,1}$. Then Q completes to the following k-Latin square:

1,...,1,2	2,...,2,n	3,...,3	$\cdots$	$n-1$,...,$n-1$	1,n,...,n
n,...,n	1,...,1	2,...,2	$\cdots$	$n-2$,...,$n-2$	$n-1$,...,$n-1$
$n-1$,...,$n-1$	2,n,...,n	1,...,1	$\cdots$	$n-3$,...,$n-3$	$n-2$,...,$n-2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
3,...,3	4,...,4	5,...,5	$\cdots$	1,...,1	2,...,2,n
1,2...,2	3,...,3	4,...,4	$\cdots$	n,...,n	1,...,1,2

.

Case 2: $s = 2$

Q completes to a k-Latin square by Theorem 7.9. □

In Section 7.1, we showed in Theorem 7.7 that like Latin squares, any k-Latin square of order n and size at most $n-1$ is completable. We thus have the following result:

Corollary 7.16 *The size of the smallest premature k-Latin square of order n, $spm(k,n)$ satisfies the inequality*

$$spm(k,n) \geq n.$$

Since premature partial k-Latin squares of order n and size $k(n-1)+1$ exist for all $n \geq 2$, we get an upper bound for the size of the smallest premature partial k-Latin square.

Theorem 7.17 *The size of the smallest premature k-Latin square of order n, $spm(k,n)$ satisfies the inequality*

$$spm(k,n) \leq k(n-1)+1.$$

It may just be pure coincidence that:

- a partial Latin square of order n and size at most $n-1$ is completable [89];

- our conjectured sufficient condition for a completable partial k-Latin square (with k times more entries than a Latin square of the same order) is to contain at most $k(n-1)$ entries(see Section 7.1); and

- the smallest premature partial k-Latin square of order n found has size $k(n-1)+1$;

but it is enough motivation for the next conjecture.

Conjecture 7.18 *The size of the smallest premature k-Latin square of order n is $k(n-1)+1$.*

Chapter 8

Latin cubes and multi-Latin cubes

In this chapter we discuss a three dimensional generalization of a Latin square referred to in the literature as a *Latin cube*. As Latin cubes generalize Latin squares, our focus here is to attempt to generalize known results on Latin squares (and results from other chapters) so that they apply to Latin cubes. Furthermore we generalize these Latin cubes to multi-Latin cubes where each layer is a multi-Latin square.

In Section 8.1, we briefly summarize some of the relevant known results on Latin cubes from the literature. Trades in Latin cubes are explored, with analogous results to Lemma 1.1 and Lemma 1.2 given in Section 8.2. We discuss, in Section 8.3, equivalences in Latin cubes by generalizing Section 1.3. In Section 8.4 we investigate partial Latin cubes with at least one completion to a Latin cube of the same order. In particular, we show that the size of any critical set of a Latin cube of order at least 3 is bounded below by its order. We then conclude the chapter with some results on partial multi-Latin cubes (where the layers are partial multi-Latin squares) with a focus on the critical sets of the full n-Latin cube.

8.1 Known results on Latin cubes

Informally, we may think of a Latin cube of order n as n layers of Latin squares of order n stacked on top of each other so that none of the symbols is directly above or below itself. That is, in a Latin cube L of order n with layers L^1, L^2,...,L^n; the set $\{L^1_{i,j}, L^2_{i,j}, ..., L^n_{i,j}\}$ has size n; such a set is referred to as a *file*.

We first introduce a more general stack of Latin squares of orders n. For $1 \leq k \leq n$, a $n \times n \times k$ *Latin cuboid* (or *Latin parallelepiped*) of order n is an $n \times n \times k$ (i.e. n rows, n columns and k layers) array such that each symbol from a set of size n occurs exactly once in each row, exactly once in each column and at most once in each file. An $n \times n \times n$ Latin cuboid, where each symbol occurs exactly once in each file, is a *Latin cube* of order n. Observe that an $n \times n \times 1$ Latin cuboid is equivalent to a Latin square of order n.

The examples below are Latin cubes of orders 2 and 3 respectively.

1	2
2	1

2	1
1	2

1	2	3
3	1	2
2	3	1

2	3	1
1	2	3
3	1	2

3	1	2
2	3	1
1	2	3

For the Latin cube of order 3 above, removing at least one layer results in a Latin cuboid.

Displaying each layer of a Latin cube side by side may be effective for small orders, but this becomes unwieldy for large orders. We therefore sometimes record Latin cubes by indices (as used in [6]). This way, we can write the Latin cube of order 2 in the above example as $L = (L_{i,j,k})$ with

$$L_{1,1,1} = L_{1,2,2} = L_{2,2,1} = L_{2,1,2} = 1 \text{ and } L_{1,2,1} = L_{1,1,2} = L_{2,1,1} = L_{2,2,2} = 2.$$

We interpret $L_{i,j,k}$ as the symbol occurring in the i^{th} row, j^{th} column and k^{th} layer. Alternatively, we may also represent a Latin cube as a set of ordered quadruples, (i, j, k, s), where $L_{i,j,k} = s$.

While every Latin rectangle completes to a Latin square (see [60]), the analogous result does not apply to Latin cuboids. Kochol in [73, 74, 75] constructs Latin cuboids that cannot complete to Latin Cubes. In [75], he proved the following result:

Theorem 8.1 [75] *For any k and n satisfying $\frac{n}{2} < k \leq n-2$ there is a non-completable $n \times n \times k$ Latin cuboid.*

Kochol also conjectured that all non-completable $n \times n \times k$ Latin cuboids are made up of at least $\frac{n}{2}+1$ layers. This was disproved by the authors of [17] with the following results:

Theorem 8.2 [17] *For all $m \geq 4$, there exists a non-completable $2m \times 2m \times m$ Latin cuboid.*

Theorem 8.3 [17] *For all even $m \notin \{2, 6\}$, there exists a non-completable $(2m-1) \times (2m-1) \times (m-1)$ Latin cuboid.*

8.2 Trades in Latin cubes

Formally, a *trade* in a Latin cube L of order n is some non-empty partial Latin cube $T \subset L$ such that there exists a *disjoint mate* T' where $T' \cap T = \emptyset$ and $(L \setminus T) \uplus T'$ is a Latin cube, L' of the same order. A trade is therefore the set difference between two Latin cubes of the same order i.e. $T = L \setminus L'$ and $T' = L' \setminus L$. For example, let L be the Latin cube of order 3 in Section 8.1 and let L' be the Latin cube of order 3 below:

1	2	3		3	1	2		2	3	1
2	3	1		1	2	3		3	1	2
3	1	2		2	3	1		1	2	3

.

Then:

$T =$

				2	3	1		3	1	2
3	1	2						2	3	1
2	3	1		3	1	2				

and

$$T' = \begin{array}{|c|c|c|}\hline & & \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline & & \\ \hline 2 & 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline & & \\ \hline \end{array} \; .$$

Note that if T is a trade in a Latin cube, each non-empty layer of T is a Latin trade in the corresponding layers of L.

An *intercalate* in a Latin cube is the set

$$\begin{aligned} T \;=\; & \{(i,j,k,s),(i,j',k,s'),(i',j,k,s'),(i',j',k,s), \\ & (i,j,k',s'),(i,j',k',s),(i',j,k',s),(i',j',k',s')\} \end{aligned}$$

where $s \neq s'$ and $s, s' \in N(n)$. Structurally, an intercalate is a two-layered Latin cube where each layer is an intercalate of a Latin square. An Example of this structure is the Latin cube of order 2.

Regardless of the difference in structure, trades are equally essential in identifying defining sets and critical sets of a Latin cube. The following results are analogous to Lemma 1.1 and Lemma 1.2 with analogous proofs (which we omit).

Lemma 8.4 *A partial Latin cube D is a defining set of a Latin cube L if and only if it intersects every trade in L.*

Lemma 8.5 *A partial Latin cube C is a critical set of a Latin cube L, if C is a defining set of L and each element of C belongs to a trade in L.*

8.3 Equivalences in Latin cubes

In this section we describe equivalences in Latin cubes. These are similar to the two types of equivalence classes in Latin squares (discussed in Section 1.3) but since a Latin cube is, by definition, an ordered quadruple, the equivalence classes here contain more elements.

We say that two Latin cubes are in the same *isotopy class* if one can be obtained by permuting the rows/columns/layers/symbols of the other.

Furthermore, the *conjugates* of a Latin cube are described by the different permutations of its ordered quadruple. That is, for a Latin cube L consisting of quadruples of the form (i, j, k, s), re-ordering each quadruple, gives all the 23 other Latin cubes in the *conjugacy class* of L. For example, if L is the 3×3 Latin cube below:

1	2	3		2	3	1		3	1	2
3	1	2		1	2	3		2	3	1
2	3	1		3	1	2		1	2	3

,

then replacing each quadruple (i, j, k, s) with (j, k, s, i) gives:

1	2	3		3	1	2		2	3	1
2	3	1		1	2	3		3	1	2
3	1	2		2	3	1		1	2	3

which is conjugate to L.

Thus in a Latin cube, any statement that is true for the rows also applies to the columns, layers and symbols of the cube. We exploit this property throughout this chapter.

8.4 Completable partial Latin cubes

As with Latin squares, we are interested in partially filled in Latin cubes. A *partial Latin cube* of order n is an $n \times n \times n$ cube such that each symbol from a set of size n occurs at most once in each row, at most once in each column and at most once in each file. For a partial Latin cube to be completable, at the very least, the partial Latin squares forming its layers must each be completable.

Thus, an interesting conjecture on completing partial Latin cubes of order n is obtained by generalizing Evans' conjecture to Latin cubes.

Conjecture 8.6 [79] *If P is a partial Latin cube of order n with at most $n-1$ entries, then P can be completed.*

The above conjecture has not been proved for all values of n but it is certainly true for partial Latin cubes of orders 3 and 4 as we show below.

Lemma 8.7 *A partial Latin cube P of order 3 with size at most 2 can be completed to a Latin cube of the same order.*

Proof. For the proof of this lemma, it suffices to look only at the case when $|P| = 2$. Thus we only need to break this proof into two cases. In Case 1, both entries are contained in the same layer/row/column index of P. In Case 2, the two entries are in different layers, different rows and different columns.

Case 1.

Without loss of generality, let both entries be contained in the same layer. Then, by Theorem 2.3, this layer completes to a Latin square of order 3, and the other two complete to isotopies of this Latin square obtained from permuting either its rows or columns.

Case 2. The two entries are in different layers, different rows and different columns.

Let the non-empty layers be P_1 and P_2 and without loss of generality, let $P_{1,1,1}$ and $P_{2,2,2}$ be the non-empty cells of P. Then by Theorem 2.3, there exists a Latin square, L_1 of order 3, such that $P_1 \subseteq L_1$ and $L_{2,2,1} \neq P_{2,2,2}$. Since P_2 only contains one entry, permuting the appropriate rows of L_1 gives a completion of P_2. By default, the cells of P_3 are filled with the remaining symbols from each file. □

Lemma 8.8 *A partial Latin cube P of order 4 with size at most 3 can be completed to a Latin cube of the same order.*

Proof. We may assume that $|P| = 3$; (otherwise add extra entries to P). We break this proof into three cases. In Case 1, all the entries are contained in the same layer/row/column index of P. In Case 2, exactly two layers (equivalently, rows or columns) of P are non-empty, and in Case 3, each entry is unique and contained in a different layer, different row and different column. From Section 8.3, these cases are exhaustive.

Case 1.

In a similar way to Case 1 of the previous proof, one way of completing P is by completing the non-empty layer first then permuting the rows/columns of this Latin square to get a completion for each of the other layers.

Case 2.

Let the non-empty layers of P be P_1 and P_2 with P_1 containing two of its entries. If the non-empty cell of P_2 is $P_{i,j,2}$ for some $i, j \in N(3)$, then by Theorem 2.3, there exists a completion L_1 of P_1 where $L_{i,j,1} \neq P_{i,j,2}$. By permuting the appropriate rows of L_1 we can obtain a completion for each of the other layers.

Case 3.

Without loss of generality, let 1,2 and 3 be in cells $P_{1,1,1}$, $P_{2,2,2}$ and $P_{3,3,3}$ respectively. Then the Latin cube:

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

2	3	4	1
1	2	3	4
4	1	2	3
3	4	1	2

3	4	1	2
2	3	4	1
1	2	3	4
4	1	2	3

4	1	2	3
3	4	1	2
2	3	4	1
1	2	3	4

is a completion of P. □

For orders at least 5, the approach we used in the previous two lemmas becomes too complicated. The closest results in attempts to prove Corollary 8.6 for the general case are by Denley and Kuhl [79] and by Denley and Öhman [40]. These results are given below.

Theorem 8.9 [79] *Let P be a partial Latin cube of order n with at most $n-1$ entries such that all filled cells appear in distinct files. Then P can be completed to a Latin cube of order n.*

Theorem 8.10 [40] *Let P be a partial Latin cube all of whose at most $n-1$ entries are contained in either a single layer or the same column of each non-empty layer. Then P is completable.*

Our next step was to try and construct critical sets of Latin cubes of small orders. Below is an example of a critical set of order 3.

Example 8.11

1		
	2	

2		

From initial observations, a partial Latin cube of order 3 with at most 2 entries seems likely to have more than one completion. This is certainly true if both entries are in a single layer as permuting the two Latin squares formed when completing the two empty layers, forms a second completion. The following lemma is immediate.

Lemma 8.12 *Any completable partial Latin cube with at least two empty layers has more than one completion.*

The above result also applies to completable partial Latin cubes with at least two rows/columns of the same indices, empty in each layer. Of course, such a partial Latin cube may be obtained by simply permuting the rows/columns and the layers of the first Latin cube. The following lemma confirms that partial Latin cubes of order 3 and size at most 2 do indeed have more than one completion. More importantly it tells us that the size of a smallest critical set of a Latin cube of order n ($n \geq 3$) is at least n. Observe that in the case of a Latin cube of order 2, the size of the smallest critical set is 1.

Lemma 8.13 *A critical set P of a Latin cube of order $n \geq 3$ contains at least n entries.*

Proof. If $|P| \leq n-2$ then by Lemma 8.12, if P is completable then it has at least two completions. Thus we only need to show that there are also at least two completions when $|P| \leq n-1$. Again by Lemma 8.12, if $|P| \leq n-1$, each entry is unique and no pair of entries are contained in the same layer or file. So without loss of generality, we may assume that symbol i is contained in cell $P_{i,i,i}$ for all $i \in N(n-1)$. After filling in the cells of the first column of the first layer P_1 so that $P_{1,j,1} = j$ for all $j \in N(n)$, then there are at least two completions of P_1. Observe that these completions do not contain symbol i in cell $P_{i,i,j}$ $(i \neq j)$. Let L_1 and L_2 be two distinct such completions of P_1. Then, cyclically permuting the columns of L_1 or L_2 gives two distinct completions of P to a Latin cube. □

In the following example, we illustrate the proof of the above lemma for a partial Latin cube of order 3 containing 2 entries.

Let

$P =$

1		

P_1

	2	

P_2

P_3

.

Then

$L_1 =$

1	2	3
2	3	1
3	1	2

or

1	3	2
2	1	3
3	2	1

;

$L_2 =$

3	1	2
1	2	3
2	3	1

or

2	1	3
3	2	1
1	3	2

;

$L_3 =$

2	3	1
3	1	2
1	2	3

or

3	2	1
1	3	2
2	1	3

.

Thus the critical set of order 3 given in Example 8.11 is a smallest critical set for Latin cubes of this order.

Corollary 8.14 *The size of the smallest critical set of a Latin cube of order* 3 *is* 3.

We next show an example of a partial Latin cube of order 4 and size 6 that is also a critical set, and may even be a critical set of minimum size for order 4. The following is an example of one such partial Latin cube.

1	3		
2			

			2
4			

			1

Lemma 8.15 *Let P be the partial Latin cube of order* 4 *and size* 6 *above. Then P is a critical set of a Latin cube of the same order.*

Proof. By inspection, P has a unique completion to the Latin cube below:

1	3	2	4	3	1	4	2	4	2	1	3	2	4	3	1
2	4	3	1	1	3	2	4	3	1	4	2	4	2	1	3
4	2	1	3	2	4	3	1	1	3	2	4	3	1	4	2
3	1	4	2	4	2	1	3	2	4	3	1	1	3	2	4

so P is a defining set. To complete the proof; we need to show the existence of at least one more completion upon the removal of any single entry. The following table gives a subsequent (second) completion for the removal of the entries in cells $P_{1,1,1}$, $P_{1,4,2}$, and $P_{4,1,2}$ respectively.

removed entry	second completion of P															
$P_{1,1,1}$	4	3	2	1	1	4	3	2	3	2	1	4	2	1	4	3
	2	1	4	3	3	2	1	4	1	4	3	2	4	3	2	1
	3	4	1	2	2	3	4	1	4	1	2	3	1	2	3	4
	1	2	3	4	4	1	2	3	2	3	4	1	3	4	1	2
$P_{1,4,2}$	1	3	4	2	2	1	3	4	4	2	1	3	3	4	2	1
	2	4	1	3	3	2	4	1	1	3	2	4	4	1	3	2
	4	2	3	1	1	4	2	3	3	1	4	2	2	3	1	4
	3	1	2	4	4	3	1	2	2	4	3	1	1	2	4	3
$P_{4,1,2}$	1	3	2	4	4	1	3	2	2	4	1	3	3	2	4	1
	2	4	3	1	1	2	4	3	3	1	2	4	4	3	1	2
	4	2	1	3	3	4	2	1	1	3	4	2	2	1	3	4
	3	1	4	2	2	3	1	4	4	2	3	1	1	4	2	3

If either $P_{1,2,1}$, $P_{2,1,1}$ or $P_{4,4,3}$ is removed, then a second completion of P is given by Lemma 8.12. □

A critical set of this size is relatively small considering the size of the smallest critical set of a Latin square of the same order is 4. Combined with the fact that at least 3 of the layers must be non-empty, it seems reasonable to conjecture that any critical set of a Latin cube of order 4 has size at least 5.

8.5 Multi-Latin cubes

In this section we introduce the idea of a multi-Latin cube; a natural generalization of Latin cubes where we allow the cells of the cube to contain multiple entries. A *partial multi-Latin cube* of order n and index k is an $n \times n \times n$ array of multisets of size at most k, such that each symbol from a set of size n occurs at most k times in each cell, at most k times in each row, at most k times in each column and at most k times in each file. A *multi-Latin cube* of order n and index k (or a k-Latin cube of order n) is a partial k-Latin cube such that each symbol from a set of size n occurs k times in each row, k times in each column and k times in each file. In a k-Latin cube of order n, each layer is a k-Latin square of the same order. The following is an example of a 2-Latin cube of order 3.

1,1	2,3	2,3
2,2	1,3	1,3
3,3	1,2	1,2

2,2	1,3	1,3
3,3	1,2	1,2
1,1	2,3	2,3

3,3	1,2	1,2
1,1	2,3	2,3
2,2	1,3	1,3

An n-Latin cube of order n where each cell contains $N(n)$, is referred to as the *full n-Latin cube*. Similarly, each layer of the full n-Latin cube is the full n-Latin square. For example, the full 3-Latin cube is given below.

1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3

1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3

1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3
1,2,3	1,2,3	1,2,3

The following result generalizes Theorem 1.3. We also omit this proof as it is analogous to the proof of that theorem.

Theorem 8.16 *Let C be a defining set of the full n-Latin cube and let L be any Latin cube of order n. Then $L \cap C$ is a defining set for L.*

8.5.1 Critical sets of the full n-Latin cube

Compared to critical sets of Latin cubes, critical sets of the full n-Latin cube were much easier to construct. The following lemmas give constructions for critical sets for any full Latin cube of order n. We start by letting one layer of the critical set empty.

Lemma 8.17 *Let P be a partial n-Latin cube of order n with one empty layer and let each of the $n-1$ non-empty layers of P be a critical set of the full n-Latin square. Then P is a critical set of the full n-Latin cube.*

Proof. Let L_n be the full n-Latin square. Then each of the non-empty layers of P completes to L_n, forcing the empty layer to complete to L_n as well, so P is a defining set of the full n-Latin cube. So now we need to show that P is a minimal defining set. Suppose that P_i is one of the non-empty layers of P and remove an entry from one of the non empty cells of P_i such that it now also completes to a non-full n-Latin square, L. If $T = L_n \setminus L$ and $T' = L \setminus L_n$, then by completing P_i to L, the empty layer to $(L_n \setminus T') \uplus T$, and the remaining layers to L_n gives a non-full n-Latin cube completion of P. Thus P is a critical set of the full n-Latin cube. □

An important part of the second half of the above proof is showing that a Latin trade T and its disjoint mate T' are contained in two of the layers;

guaranteeing a second completion. Since the reason for the existence of this second completion is clearly stated in this proof, we simply present this result as the following corollary.

Corollary 8.18 *Let P be a partial n-Latin cube with layers $P_1, P_2, ..., P_n$ such that each cell of P is a subset of $N(n)$. Let $\bar{P}_i = L_n \setminus P_i$ where L_n is the full n-Latin square. If there exists two partial n-Latin squares T and T' such that:*

1. *the pair (T, T') is a bitrade,*

2. *$T \subseteq \bar{P}_i$ and $T' \subseteq \bar{P}_{i'}$,*

then P has at least two completions.

Equivalences in multi-Latin cubes are similar to those in Latin cubes discussed in Section 8.3. Thus, for a partial n-Latin cube P, if we define $E(\bar{P})$ to be the equivalence class of $\bar{P}$ then any element of $E(\bar{P})$ can replace $\bar{P}$ in Corollary 8.18. We illustrate this idea in the following example:

Let $\bar{P}^r = \bar{P}_{k,j,i}$ be the conjugate of $\bar{P}$ formed by permuting its rows and layers. For the partial 3-Latin cube of order 3 below:

2,3	1,2,3	1,3	1,2,3	1,3	2,3	1,3	2,3	1,2,3
1,3	1,2,3	2,3	1,2,3	2,3	1,3	2,3	1,3	1,2,3
1,2,3	1,2,3		1,2,3	1,2,3				

,

$\bar{P}$ =

1		2		2	1	2	1	
2		1		1	2	1	2	
		1,2,3			1,2,3	1,2,3	1,2,3	1,2,3

contains no bitrade while in

$\bar{P}^r =$

1		2
	2	1
2	1	

2		1
	1	2
1	2	

		1,2,3
		1,2,3
1,2,3	1,2,3	1,2,3

,

$T = \bar{P}^r_1$ and $T' = \bar{P}^r_2$. As none of the entries of the cells of P is a multiset the obvious completion of P is to the full 3-Latin cube. However, replacing T with T' and vice versa in the full 3-Latin cube gives another completion which we display below.

2,2,3	1,2,3	1,1,3
1,1,3	1,2,3	2,2,3
1,2,3	1,2,3	1,2,3

1,2,3	1,1,3	2,2,3
1,2,3	2,2,3	1,1,3
1,2,3	1,2,3	1,2,3

1,1,3	2,2,3	1,2,3
2,2,3	1,1,3	1,2,3
1,2,3	1,2,3	1,2,3

Another result on full n-Latin square that we can use here is Theorem 5.22. Again, if we let $n-1$ layers of a partial Latin cube of order n be the critical sets described in Theorem 5.22 with the remaining layer empty, then by Lemma 8.17, the partial Latin cube is a critical set of the full n-Latin cube.

Corollary 8.19 *Let P be a partial n-Latin cube of order n with one empty layer and let each of the $n-1$ non-empty layers of P be of the form:*

$1, 2, ..., n-1$	$\cdots$	$1, 2, ..., n-1$	
$\vdots$	$\ddots$	$\vdots$	$\vdots$
$1, 2, ..., n-1$	$\cdots$	$1, 2, ..., n-1$	
	$\cdots$		n

.

Then P is a critical set of the full Latin cube of order n.

Each non-empty layer in the above critical set is the critical set of the full n-Latin square with the smallest size we have found. So this analogous result for the full n-Latin cube follows.

Corollary 8.20 *The size of the smallest critical set of a full Latin cube of order n satisfies the inequality $scs \leq (n-1)[(n-1)^3+1]$.*

8.5.2 Saturated critical sets

If each non-empty cell of a critical set of the full n-Latin cube contains $N(n)$, then by Theorem 5.7 and Lemma 8.17, the following construction gives one type of these critical sets.

Construction 8.21

	$\cdots$		
$\vdots$	$\ddots$	$\vdots$	$\vdots$
	$\cdots$		
	$\cdots$		

1,2,...,n	$\cdots$	1,2,...,n	
$\vdots$	$\ddots$	$\vdots$	$\vdots$
1,2,...,n	$\cdots$	1,2,...,n	
	$\cdots$		

...

1,2,...,n	$\cdots$	1,2,...,n	
$\vdots$	$\ddots$	$\vdots$	$\vdots$
1,2,...,n	$\cdots$	1,2,...,n	
	$\cdots$		

Corollary 8.22 *The partial n-Latin cube of order n given by Construction* 8.21 *is a saturated critical set of the full n-Latin cube.*

We have also found other forms of these types of saturated critical sets. In particular, the following is a saturated critical set of the full 3-Latin cube.

1,2,3		
	1,2,3	

	1,2,3	
		1,2,3

		1,2,3
	1,2,3	1,2,3

Lemma 8.23 *Let P be the saturated partial* 3*-Latin cube above. Then P is a critical set of the full* 3*-Latin cube.*

Proof. We first show that P is a defining set of the full n-Latin cube. Without loss of generality, suppose that $P_{1,1,3} = \{1,1,2\}$. Then the third layer must complete to:

1,1,2	2,3,3	1,2,3
2,3,3	1,1,2	1,2,3
1,2,3	1,2,3	1,2,3

.

Thus $P_{1,2,1} = \{1,1,2\}$. This in turn forces the first layer to be:

1,2,3	1,1,2	2,3,3
2,3,3	1,2,3	1,1,2
1,1,2	2,3,3	1,2,3

,

which violates the definition of a 3-Latin cube as the entry 3 occurs more than three times in the file $\{P_{2,1,1}, P_{2,1,2}, P_{2,1,3}\}$. Thus $P_{1,1,3} = \{1,2,3\}$. Hence, the third layer completes to the full 3-Latin square forcing the other two layers to follow suit. We now show that P is a minimal defining set. Observe that other than $P_{3,3,3}$, every other non-empty cell is contained in a partial sub-cube of P conjugate to:

1,2,3				

.

Thus if we remove any of the entries of any one of these cells, then by Corollary 8.18, P has at least two completions. On the other hand, if we remove an entry of $P_{3,3,3}$, then without loss of generality, another completion of P is given below.

1,2,3	1,2,3	1,2,3		2,2,3	1,2,3	1,1,3		1,1,3	1,2,3	2,2,3
2,2,3	1,2,3	1,1,3		1,1,3	1,2,3	2,2,3		1,2,3	1,2,3	1,2,3
1,1,3	1,2,3	2,2,3		1,2,3	1,2,3	1,2,3		2,2,3	1,2,3	1,1,3

□

Corollary 8.24 *The size of the smallest saturated critical set of the full* 3*-Latin cube is at most* 21*.*

Since the size of the critical set of the full 3-Latin cube given by Construction 8.21 is 24, Lemma 8.23 raises the interesting open question below:

Open question: *What is the size of smallest saturated critical set of the full* n*-Latin cube?*

In the following results, P is a saturated partial n-Latin cube. We construct a partial n-Latin square $f(P) = P'$ where symbol $k \in P'_{i,j}$ if and only if $P_{i,j,k} = N(n)$. Note that P' is a subset of the full n-Latin square and thus the process is reversible, i.e. f^{-1} is well-defined.

Theorem 8.25 *Let* P' *be a partial* n*-Latin square. Suppose that there is a Latin trade* T *and disjoint mate* T' *such that* T *and* T' *contain two distinct entries and* $T \cup T' \subseteq \bar{P}'$*. Then* $f^{-1}(P')$ *is not a defining set for the full* n*-Latin cube.*

Proof. This theorem follows from Corollary 8.18. □

Corollary 8.26 *If* P' *is a defining set for the full* n*-Latin square then* $f^{-1}(P')$ *is a defining set for the full* n*-Latin cube.*

We illustrate this corollary with the following example:

Let

$P' =$

1		
	2,3	2,3
	2,3	2,3

.

Then

$P =$

1,2,3										
					1,2,3	1,2,3			1,2,3	1,2,3
					1,2,3	1,2,3			1,2,3	1,2,3

which is clearly a defining set of the full n-Latin cube.

The converse of Corollary 8.26 however, is not true. The following partial 3-Latin square:

1	2	
	1	3
	3	2,3

which corresponds to the saturated partial 3-Latin cube from Lemma 8.23 also completes to:

1,2,3	1,2,3	1,2,3
2,2,3	1,2,3	1,1,3
1,1,3	1,2,3	2,2,3

and thus not a defining set of the full 3-Latin square.

Chapter 9

Conclusion

In this chapter I summarize the main results in this thesis. I also include here the main conjectures and open questions that have been raised during my research, with the hope that they will stir enough curiosity and hence research interests in this subject.

In a nutshell, my thesis explored the properties of the critical sets of the full n-Latin square and other combinatorial structures that either generalize or are generalized by multi-Latin squares. The original inspiration was the following well-known result on designs (a special case of Lemma 3.4) which largely motivated my interest in the critical sets of the analogous full n-Latin squares.

Lemma 9.1 *Let A be a design and let B be a full design of the same order. If D is a minimal defining set of A, then there exists a minimal defining set D^* of B such that $D^* \cap A = D$.*

It was already known then that the analogous result for Latin squares gives a defining set of the Latin square (see Theorem 1.3). The question was whether the intersection can be a critical set of the Latin square. This question did not turn out to be difficult to answer. In Chapter 1, we showed that we can easily construct a defining set, D, of the full n-Latin square such that for some Latin square, L of the same order, $D \cap L$ is a critical set of L with minimum size.

In fact we showed that any critical set of a Latin square of order n is the result of the intersection of a defining set of the full n-Latin square and the Latin square.

Theorem 9.2 *Let L_n be the full n-Latin square and let L be a Latin square of the same order. If C is a critical set of L, then there exists a defining set, D_n of L_n such that $D_n \cap L_b = C$.*

Having solved the initial research question, we turned our attention to studying the size and structure of critical sets of the full n-Latin square. Strategically we approached this task by exploring the structure of critical sets of not only full n-Latin squares with small orders, but also those of full Latin rectangles with cells containing less than n entries.

In Chapter 4 we were able to fully classify the critical sets of the full $(m, n, 2)$-balanced Latin rectangle by Theorem 4.2 and Theorem 4.3. Here we used the notion of a pair of good vectors $(\boldsymbol{a}, \boldsymbol{b})$ (see Definition 4.2) to give the necessary and sufficient condition for a critical set of the full $(m, n, 2)$-balanced Latin rectangle. That is, a partial $(m, n, 2)$-balanced Latin rectangle is a critical set of the full $(m, n, 2)$-balanced Latin rectangle if and only if it is an element of the set of arrays $A[\boldsymbol{a}, \boldsymbol{b}]$ described in Definition 4.1.

Building on the results of Chapter 4 we showed in Chapter 5 that the size of a saturated critical set is always equal to $n^3 - 2n^2 - n$. In addition to that, we showed that for the critical sets of the full n-Latin square, $(n^3 - 2n^2 + 2n)/2 \leq scs(n, n) \leq (n-1)^3 + 1$ and $n^3 - n^2 - 3n + 4 \leq lcs(n, n) \leq n^3 - 3$.

We thus made the following conjectures:

Conjecture 5.25 *The size of the smallest critical set of the full n-Latin square is $(n-1)^3 + 1$.*

Conjecture 5.28 *The size of the largest critical set of the full n-Latin square is $n^3 - n^2 - 3n + 4$.*

Next in Chapter 6 we attempted to establish a spectrum of critical sets of all sizes between the conjectured bounds above. We presented constructions of critical sets of the full n-Latin square and showed that from size $n^3 - 3n^2 + 3n$ to $n(n-1)^2+n-2$ the spectrum is complete. It remains an intriguing problem though to determine whether each size exists for sizes between $n(n-1)^2+n-1$ and $n^3 - n^2 - 3n + 4$. Many of the sizes in this part of the spectrum are given by the construction in Lemma 6.5.

A critical set of the full n-Latin square is of course a completable partial n-Latin square; Chapter 7 explored the completability of partial k-Latin squares. After showing in Lemma 7.5 that any partial k-Latin square can be transformed into a semi-k-Latin square of the same order, we were able to show in Theorem 7.7 that any partial k-Latin square of order n with at most $n - 1$ entries from $N(n)$ can be completed to a k-Latin square of order n.

Furthermore, we showed in Theorem 7.9 that any partial k-Latin square of order n with at most $n - 1$ non-empty cells can be completed.

We also looked into premature partial k-Latin squares in this chapter and our results here led to the following conjecture:

Conjecture 7.18 *The size of the smallest premature k-Latin square of order n is $k(n-1)+1$.*

Of course if this conjecture is true then the conjecture below which we made earlier in Chapter 7 follows.

Conjecture 7.8 *A partial k-Latin square of order n with at most $k(n-1)$ entries can be completed.*

In Chapter 8, we focused on generalizing known results on Latin squares (and results from other chapters) to Latin cubes. For instance, 'Can we solve (the equivalent of) Evans' conjecture for Latin cubes?'. We considered the following conjecture:

Conjecture 8.6 [79] *If P is a partial Latin cube of order n with at most $n-1$ entries, then P can be completed.*

In Section 8.4 we showed that a critical set of a Latin cube of order $n \geq 3$ must contain at least n entries.

By generalizing the results on the critical sets of the full n-Latin square from Chapter 5, we also showed that the size of the smallest critical set of a full Latin cube of order n is at most $(n-1)[(n-1)^3+1]$.

However if the critical set is saturated, then Lemma 8.23 raises the interesting open question below:

Open question: *What is the size of smallest saturated critical set of the full n-Latin cube?*

Unlike n-Latin squares, this appears to be a more difficult question.

Chapter 10

Appendices

10.1 Appendix A: Some graph theory definitions and results

A *graph* is an ordered pair, $G = (V, E)$, where V is a finite set of the vertices or nodes of the graph and E is the set of unordered pairs of elements of V, i.e. $E \subseteq \{\{u, v\} : u, v \in V\}$, called the edges of the graph. Visually, we represent each vertex by a point and each edge by a line connecting two points. Two vertices v_1 and v_2 of a garph are *neighbours* or *adjacent* if connected by an edge and the number of edges incident on a vertex v is called the *degree* of v denoted by $\delta(v)$.

We say that a graph $G = (V, E)$:

- is *k-regular* if $\delta(v) = k$ for all $v \in V$.
- is *simple* if and only if $\{\forall v \in V : \{v, v\} \notin E\}$ (i.e. no loop edges) and each element of E is distinct (i.e. no multiple edges).
- contains a *perfect matching* if there exists a set of distinct edges of G in which every vertex is incident to exactly one edge.
- has a *proper edge colouring* if there is a mapping f from E into some finite set C such that two edges of E which are incident to the same vertex have different colours.

- is *k-edge-colourable* if there exists a finite set C such that $|C| \leq k$.

- *isomorphic* to a graph H if there exists an edge-preserving bijection, $f : V(G) \to V(H)$ (between their respective vertex sets) such that any two vertices u and v of G are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in H.

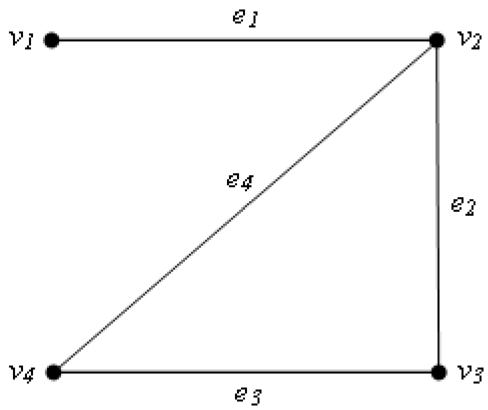

Figure 10.1: The graph $G = (V, E)$

Let $e_1, e_2, ..., e_{n-1}$ be edges of a graph $G = (V, E)$ for which there is a sequence $v_1, v_2, ..., v_n \in V$ such that $e_i = \{v_i, v_{i+1}\}$ for $i = 1, 2, ..., n-1$. The sequence of edges $e_1, e_2, ..., e_{n-1}$ is called a *path* in G and the sequence of vertices $v_1, v_2, ..., v_n$ is called the *vertex sequence* of the path. A path with vertex sequence $v_1, v_2, ..., v_n, v_1$ (where the only repeated vertex in the vertex sequence is v_1) is called a *cycle*.

A graph $G = (V, E)$ is:

- connected if every pair of distinct vertices is joined by a path.

- a *tree* if it is simple, connected and contains no cycle.

We next discuss the idea of a subgraph. A graph, $G' = (V', E')$, is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. For example:

Let G be the graph in Figure 10.1, and $G' = (V', E')$ be a simple graph with $V' = \{v_1, v_2, v_3\}$ and $E' = \{\{v_1, v_2\}, \{v_2, v_3\}\}$. Then G' is a subgraph of

G.

A set of triangle subgraphs of G such that each edge of G belongs to exactly one triangle is called a *triangle decomposition* of G (see Section 1.1 for an example).

In this thesis, a Latin structure may also be represented as a bipartite graph or a tripartite graph. We define a *k-partite graph* as a graph whose vertices can be divided into k disjoint sets such that no two vertices from the same set is joined by an edge. 2-partite and 3-partite graphs are called *bipartite* and *tripartite* graphs respectively. A complete k-partite graph is one in which there is an edge between every pair of vertices from different disjoint sets. These graphs are denoted by $K_{s_1,s_2,\ldots,s_k}$ where s_i is the size of each disjoint set. The graph $K_{1,k}$ is called a *star*.

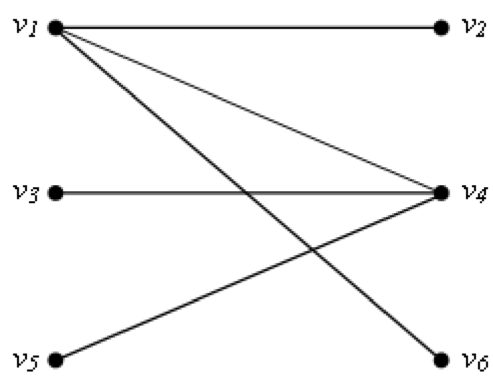

Figure 10.2: A bipartite graph

with $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$.

10.1.1 Some known results on trees

The following are known results on trees that we exploited in this thesis.

Theorem 10.1 *Every tree has at least one node of degree* 1.

Proof. Let T_n be a tree with n vertices, v_i, where $n \geq 1$ and $1 \leq i \leq n$.

If $n = 1$, then $\delta(v_1) = 1$ since by definition a tree is simple and cannot have loop edges.

Otherwise if $n > 1$ then we show that the longest path, P, in T_n has a node of degree 1.

Assume that P is the longest path in T with vertex sequence

$$\upsilon_1, \upsilon_2, ..., \upsilon_{k-1}, \upsilon_k.$$

Since P is a subgraph of T_n, P is simple. Suppose that $\delta(\upsilon_k) > 1$. Then, υ_k has another neighbor $\omega \neq \upsilon_{k-1}$. If ω is a vertex in P then we have a cycle in T_n, but this is impossible since T_n is a tree. On the other hand, if ω is not in P, then there exists a path Q, with vertex sequence $\upsilon_1, \upsilon_2, ..., \upsilon_k, \omega$, longer then P. In both cases, we have a contradiction, so $\delta(\upsilon_k) = 1$. □

Theorem 10.2 *If T is a tree with n vertices then T has $n-1$ edges.*

Proof. We prove this theorem by induction.

Let T_n be a tree with n vertices and $E(n) = n - 1$, be the number of edges of T. Then for $n = 1$, $E(1) = 0$ is obviously true.

Now we need to show that if $E(k)$, $k \geq 1$, is true then $E(k+1)$ is also true. So our induction hypothesis is that $E(k) = k - 1$.

Let T_{k+1} be any tree with $k+1$ vertices. Since T_{k+1} is a tree and trees have no cycles, then by Theorem 10.1, at least one vertex of T_{k+1} must be of degree 1. Suppose that v is one such vertex. Then removing v from T_{k+1} leaves us with T_k which, by the induction hypothesis, is also a tree with $k-1$ edges. So $E(k+1) = E(k) + 1 = k - 1 + 1 = k$. □

10.2 Appendix B: Hall's theorem

Let $R_1, R_2, ..., R_n$ be subsets of $N(n)$. We define a *system of distinct representatives* (denoted by *SDR*) for the family $(R_1, R_2, ..., R_n)$ to be an ordered set $(r_1, r_2, ..., r_n)$ of elements of the set $N(n)$ such that:

- $r_i \in R_i$ for all $i \in N(n)$;
- $r_i \neq r_j$ for $i \neq j$.

For example, if $R_1 = \{1, 2\}$, $R_2 = \{1, 2, 3\}$ and $R_3 = \{2, 3\}$. Then the possible *SDR*s of (R_1, R_2, R_3) are $(1, 2, 3)$, $(1, 3, 2)$ and $(2, 1, 3)$.

Let $R_1, R_2, ..., R_n$ be sets with symbols from the set $N(n)$ and let

$$R(I) = \bigcup_{i \in I} R_i$$

for $I \subseteq N(n)$. Then the family $(R_1, R_2, ..., R_n)$ satisfies *Hall's condition* if for all $I \subseteq N(n)$;

$$|R(I)| \geq |I|.$$

The following theorem was proved by P. Hall [61] in 1935.

Theorem 10.3 [61] *Let $R_1, R_2, ..., R_n$ be sets with symbols from the set $N(n)$. Then there exists an SDR for $(R_1, R_2, ..., R_n)$ if and only if Hall's condition holds.*

For a bipartite graph $G = (V, E)$ with partite sets V_1 and V_2 where $N(A)$ (for some set $A \subseteq V_1, V_2$) is the set of neighbours of A, Hall's theorem may be phrased as follows:

Theorem 10.4 *Let $G = (V, E)$ be a bipartite graph with partite sets V_1 and V_2. Then G has a perfect matching if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.*

The theorem below is a well-known result on bipartite multigraphs that we use in the proof of Lemma 7.5.

Theorem 10.5 *A bipartite multigraph G of degree at most k is k-edge-colourable.*

Proof. Observe that if G is not k-regular then we can simply add dummy edges to form a bipartite multigraph G' that is k-regular. It suffices therefore, to show that the theorem holds if G is k-regular. Clearly, Hall's condition is satisfied thus G contains a perfect matching by Theorem 10.4. We colour all edges of this matching with one colour then remove them from G to obtain a $(k-1)$-regular bipartite graph. We repeat this process k times and we are done. □

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