The Relative Contributions of Private Information Sharing and Public Information Releases to Information Aggregation*

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Abstract

We calculate learning rates when agents are informed through both public and private observation of other agents' actions. We provide an explicit solution for the evolution of the distribution of posterior beliefs. When the private learning channel is present, we show that convergence of the distribution of beliefs to the perfect-information limit is exponential at a rate equal to the sum of the mean arrival rate of public information and the mean rate at which individual agents are randomly matched with other agents. If, however, there is no private information sharing, then convergence is exponential at a rate strictly lower than the mean arrival rate of public information.

JEL Classification Numbers: D83, D82.

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1 Introduction

This paper calculates asymptotic learning rates when agents are informed through both public and private observation of other agents' actions.

We provide an explicit solution for the dynamics of the distribution of posterior beliefs for settings in which a large number of asymmetrically informed agents are randomly matched into groups over time, exchanging their information with each other when matched, and in which the information of randomly selected agents is also publicly revealed over time. We show that any agent's posterior beliefs converge in distribution to a common posterior at an exponential rate. With both public and private learning, the convergence rate is the sum of the mean arrival rate of public information and the mean rate at which an individual agent is matched with other agents. If, however, there is no private information sharing, then convergence is exponential at a rate strictly lower than the mean arrival rate of public information. We emphasize how the component of the asymptotic learning rate that is attributed to public announcements depends on the presence of private information sharing.

Our model works roughly as follows. A continuum of agents are initially endowed with signals that are informative about a random variable X. Given X, the signals endowed to one agent are independent of those endowed to another. Each agent enters private information sharing sessions at a mean rate of λ private meetings per year. At each such meeting, say an auction, other agents are randomly selected to attend. Each agent at the meeting reveals to the others a summary statistic of his or her posterior, such as a bid for an asset, reflecting the agent's originally endowed information and any information learned prior to the meeting. As an additional source of information, there are randomly timed public releases of the posterior beliefs of a randomly selected group of agents. Such public releases occur η times per year, in expectation.

Over time, as an agent gathers more and more information, the agent's posterior probability of the event that X has a particular outcome converges in distribution to one if the event is true, and to zero if the event is false. We calculate explicitly the probability distribution of an agent's posterior beliefs. With both private and public learning channels, we show that the convergence in distribution of the posterior is exponential at the rate $\lambda + \eta$, regardless of the sizes of the groups of agents that participate in meetings or have their information publicly revealed. If, however, there is no private information sharing, then the convergence rate is strictly lower than η, and depends non-trivially on the number of agents revealing information at each public release, the initial information

endowment, and the realization of X.

As argued by Hayek (1945) and Arrow (1974), an important role of markets and organizations is the aggregation of information that is dispersedly held by its participants. Information aggregation occurs through the public observation of variables that reflect other agents' actions (such as prices or public bids for an asset) or through the private observation of other agents' actions (such as bilateral bargaining in a decentralized market). Our results suggest that, in terms of rates of convergence, the private channel of learning is at least as effective as the public channel of learning. If private information sharing is active, any increases in the mean arrival rates η and λ of public and private information events are translated one for one into the belief convergence rate. Without the benefit of private information sharing, however, an increase in the mean rate η of public information releases is less than fully converted to an increase in the belief convergence rate.

Private information sharing is typical in functioning over-the-counter markets for many types of financial assets, including bonds and derivatives. In these markets, trades occur at private meetings in which counterparties offer prices that reveal information to each other, but not to other market participants. In addition to this form of private information sharing, many over-the-counter markets also have public releases of a selection of price quotations or executed trades. These releases can be found on dealer screens, or in financial news services such as Bloomberg, or in email broadcasts by dealers to market participants. In some markets, a sample of private trade executions are provided publicly through such post-trade price reporting systems as TRACE, although typically with a short time lag.[1] We do not consider the effects of time lags of public information releases.

Information aggregation has received significant attention in the economics literature. Several papers focus on public information. Grossman (1976), Townsend (1978), and Grossman and Stiglitz (1980) introduce the notion of rational-expectations equilibrium to capture the idea that prices aggregate information that is dispersed in the economy. Wilson (1977), Milgrom (1981), Vives (1993), Pesendorfer and Swinkels (1997), and Reny and Perry (2006) provide strategic foundations for the rational-expectations equilibrium concept. Another strand of literature investigates information aggregation when agents learn only through private interactions. For example, in over-the-counter

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[1]Some of the empirical implications for price behavior of TRACE-based public information sharing in over-the-counter bond markets has been considered by Edwards, Harris, and Piwowar (2007), Goldstein, Hotchkiss, and Sirri (2007), and Green, Burton and Schurhoff (2007).

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markets, agents learn from the bids of other agents in privately held auctions. Wolinsky (1990), Blouin and Serrano (2001), Duffie and Manso (2007), Duffie, Giroux, and Manso (2008), and Golosov, Lorenzoni, and Tsyvinski (2008) study information percolation in these markets. Word-of-mouth communication, studied for example by Banerjee and Fudenberg (2004), is another form of learning through private interactions. In contrast to the above papers, our paper studies information aggregation when learning occurs through both public and private interactions.

Other papers have studied rates of convergence to a common full-information posterior. Vives (1993) showed that when agents learn noisy public information from others, then they learn the truth at a slow speed of $t^{1/3}$ (where t is the number of periods of market interactions). The slow convergence result is due to an informational externality. The more informative the public signal is, as more periods accumulate, the less privately informed agents rely on their private signals, so that less information gets incorporated into the public signal, slowing down convergence. In their recent work, Amador and Weill (2008) show that when agents also learn noisy private information from others, then the rate of convergence is t. Our paper obtains a related discontinuity result without the informational externalities present in the above two papers.

Our search-and-matching technology is familiar from search-theoretic models that have provided foundations for models of competitive general equilibrium and for equilibrium in markets for labor, money, and financial assets.[2] Going beyond prior studies, we allow for information asymmetry about a common-value component, with learning from public and private interactions.

Section 2 provides the model setup. The dynamic equation for the distribution of posterior beliefs is derived in Section 3. Section 4 gives an explicit solution for the distribution of beliefs at each time. Section 5 obtains rates of convergence and discusses why the presence of private learning is crucial for the contribution of public announcements to the information convergence rate. Unless otherwise indicated, proofs are found in appendices.

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[2]Examples of theoretical work using random matching to provide foundations for competitive equilibrium include that of Rubinstein and Wolinsky (1985) and Gale (1987). Examples in labor economics include Pissarides (1985) and Mortensen (1986); examples in monetary theory include Kiyotaki and Wright (1993) and Trejos and Wright (1995); examples in finance include Duffie, Gârleanu, and Pedersen (2005), Lagos and Rocheteau (2008), and Weill (2008).

2 A Private-Public Model of Information Sharing

We study the evolution of the cross-sectional distribution of posterior beliefs in a large market with both private and public information releases. In prior work, Duffie and Manso (2007) and Duffie, Giroux, and Manso (2009) allowed only private information sharing. Further, rather than fixing the sizes of groups sharing information privately as in prior work, we allow randomly sized groups. This is natural; if the particular individuals that meet to share information are randomly selected from the population, one might suppose that the number of agents that meet is also uncertain.

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a "continuum" (a non-atomic finite measure space $(G, \mathcal{G}, \gamma)$) of agents are fixed. Without loss of generality, the total quantity $\gamma(G)$ of agents is 1. A random variable X of potential concern to all agents has two possible outcomes, H ("high") and L ("low"), with respective probabilities p_H and $p_L = 1 - p_H$.

Agents are informed by observing signals that may be correlated with X. Conditional on X, every pair of distinct signals is independent with outcomes 0 and 1. The signals need not have the same probability distributions. Each agent i is initially endowed with a finite sequence $\{s_1, \ldots, s_{N_i}\}$ of signals. We allow the number N_i of signals of agent i to be random, with N_i and N_j independent for $i \neq j$, and independent of signals. Without loss of generality, we suppose that

$$\mathbb{P}(s_i = 1 \mid H) \geq \mathbb{P}(s_i = 1 \mid L).$$

A signal s_i is *informative* if $\mathbb{P}(s_i = 1 \mid H) > \mathbb{P}(s_i = 1 \mid L)$. For any pair of agents, their sets of originally endowed signals are disjoint.

By Bayes' rule, the logarithm of the likelihood ratio between states H and L conditional on an arbitrary finite set $\{s_1, \ldots, s_n\}$ of distinct signals is

$$\log \frac{\mathbb{P}(X = H \mid s_1, \ldots, s_n)}{\mathbb{P}(X = L \mid s_1, \ldots, s_n)} = \log \frac{p_H}{p_L} + \theta, \tag{1}$$

where the "type" θ of this set of signals is

$$\theta = \sum_{i=1}^n \log \frac{\mathbb{P}(s_i \mid H)}{\mathbb{P}(s_i \mid L)}. \tag{2}$$

The higher the type θ of the set of signals, the higher the posterior probability that X is high.

Any particular agent is matched to other agents at each of a sequence of Poisson arrival times with a mean arrival rate (intensity) λ that is common across agents. At

each meeting time, $\ell - 1$ other agents are randomly selected. That is, each of the $\ell - 1$ matched agents is chosen at random from the population, without replacement, with the uniform distribution, which we can take to be the agent-space measure γ. Meeting group sizes are identically and pairwise independently distributed across meetings, and independent of all else. For each meeting size outcome l, we fix $q_l = \mathbb{P}(\ell = l)$. We assume that, for almost every pair of agents, the matching times and the counterparties of one agent are independent of those of the other. We do not show the existence of such a random matching process.[3] We assume throughout the joint measurability of agents' type processes $\{\theta_{it} : i \in G\}$ with respect to a σ-algebra on $\Omega \times G$ that allows us to apply the Fubini property that, for any measurable subset A of types,

$$\int_G \mathbb{P}(\theta_{it} \in A) \, d\gamma(i) = E\left(\int_G 1_{\theta_{it} \in A} \, d\gamma(i)\right).$$

This is consistent with the exact law of large numbers for a continuum of pairwise independent random variables under the technical assumptions of Sun (2006).

When agents meet they communicate to each other their posterior probabilities, given all of the information that they have collected up to the point of that encounter, of the event that X is high. Duffie and Manso (2007) provide an example of a market setting in which this revelation of beliefs occurs through the observation of bids submitted by risk-neutral investors in an auction for a forward contract on an asset whose payoff is X.

Proposition 3 of Duffie and Manso (2007) implies that whenever a collection of signals of type θ is combined with a disjoint collection of signals of type ϕ, the type associated with the combined set of signals is $\theta + \phi$. By induction, we have the following useful result.

Lemma 2.1 *Let $S_1, \ldots, S_n$ be disjoint sets of signals with respective types $\theta_1, \ldots, \theta_n$. Then the union $S_1 \cup \cdots \cup S_n$ of the signals has type $\theta_1 + \cdots + \theta_n$.*

In addition to private information sharing events, there are public information releases at random times $\{T_1, T_2, \ldots\}$ that are independent of all else. At the n-th public release, K_n randomly selected[4] agents reveal their posterior probabilities to all agents. The probability $p_k = \mathbb{P}(K_n = k)$ that k agents are selected is fixed.

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[3]For the case of groups of size $\ell = 2$, Duffie and Sun (2007) show existence for the discrete-time analogue of this random matching model.

[4]That is, the number and set of agents selected is independent of signals, of X, and of the outcomes of prior private and public information releases. The agents are selected by independent draws from the space G of all agents with the agent-distribution measure γ.

For simplicity, we assume symmetry in the initial distribution of information across agents. That is, given X, every agent's initial type has the same conditional probability distribution μ_0. We later comment on how to re-interpret our results without this symmetry assumption.

Under the technical assumptions of Sun (2006), the law of large numbers implies that, almost surely, for each outcome of X, the initial cross-sectional distribution of types is equal to each agent's conditional type distribution μ_0 given X. We assume that there is a positive probability that each agent has at least one informative signal. This implies that the first moment $\int x \, d\mu_0(x)$ of μ_0 is strictly positive on the event $\{X = H\}$, and strictly negative on the event $\{X = L\}$.

For any initial cross-sectional distribution m of types, we let $h(m, t)$ denote the new cross-sectional type measure that would apply in a model with no public releases after t units of time. We will later show how to compute $h(m, t)$ by extending the results of Duffie, Giroux, and Manso (2009). Almost surely, $h(\mu_0, t)$ has two outcomes, one on the event $\{X = H\}$, and the other on the event $\{X = L\}$.

For any measurable set $A \subset \mathbb{R}$ of types, we let $\mu_t(A)$ denote the fraction of agents whose posterior type at time t is in A. We can view μ as a stochastic process whose outcomes are probability measures on the space of types. In order to model the convergence of posterior beliefs, we will begin with an analysis of the evolution of μ_t.

At any time t before the first public information release, we know that $\mu_t = h(\mu_0, t)$. With the first public information release of K_1 agents' posterior beliefs at T_1, Lemma 2.1 implies that every agent's posterior type jumps by the sum Z_1 of the K_1 publicly revealed types. For a real number z and a type measure m, the translation $\mathcal{T}(m, z)$ of m by z is the measure defined, at any interval (a, b) of types, by

$$[\mathcal{T}(m, z)]((a, b)) = m((a - z, b - z)).$$

Thus,

$$\mu_{T_1} = \mathcal{T}(h(\mu_0, T_1), Z_1).$$

At any time $t \in [T_1, T_2)$, any agent's type θ may be viewed as the sum of Z_1 and the privately acquired type $\hat{\theta} = \theta - Z_1$. Thus, at such a time, when agents of respective types $\theta_1, \ldots, \theta_\ell$ meet and exchange their conditional probabilities of the event $\{X = H\}$, the i-th agent knows that the j-th agent's announced type θ_j can be viewed as the sum of the publicly revealed type Z_1 and the privately acquired type $\hat{\theta}_j = \theta_j - Z_1$. Thus, again by Lemma 2.1, all of the agents leave the meeting with a type equal to Z_1 plus the

sum of the privately acquired types $\hat{\theta}_1 + \hat{\theta}_2 + \cdots + \hat{\theta}_\ell$. Thus, for $t \in [T_1, T_2)$,

$$\mu_t = \mathcal{T}(h(\mathcal{T}(\mu_{T_1}, -Z_1), t - T_1), Z_1) = \mathcal{T}(h(\mu_0, t), Z_1).$$

More generally, the cross-sectional type measure evolves randomly according to the following rule.

Lemma 2.2 *At any time t between the times T_n and T_{n+1} of the n-th and $(n + 1)$-th public releases of information, almost surely, $\mu_t = \mathcal{T}(h(\mu_0, t), Z_n)$, where Z_n is the aggregate type revealed at T_n.*

This result follows from the fact that the aggregate type Z_n revealed publicly at time T_n is the sum $Z_1 + (Z_2 - Z_1) + \cdots + (Z_{n-1} - Z_{n-2}) = Z_{n-1}$ of the net aggregate type associated with previously revealed public information and the aggregate $Z_n - Z_{n-1}$ of the privately acquired types of the set of those agents who collectively reveal the new aggregate type Z_n at T_n. Thus, the incremental type associated with the information that is publicly revealed to all agents at time T_n is merely $Z_n - Z_{n-1}$. Thus, for any $t \in [T_n, T_{n+1})$,

$$\mu_t = \mathcal{T}(h(\mu_0, t), Z_{n-1} + (Z_n - Z_{n-1})) = \mathcal{T}(h(\mu_0, t), Z_n),$$

as claimed.

Lemma 2.2 gives a simple characterization: The belief types in a model with public releases of information are merely the translation of the belief types associated with a model with purely private information by the aggregate type Z_n revealed in very last public release of information. The distribution of Z_n is not obvious, because it incorporates the effects of information that was received before the latest public release through both private and public sources, which are recursively determined. Shortly, we will unravel the implications of this recursion.

We will eventually show that all agents' posterior beliefs converge in law to complete information, that is, to the posterior 1 on the event $\{X = H\}$, and to zero on the event $\{X = L\}$. Our particular concern is how the speed of convergence depends on the parameters $(\lambda, (q_k))$ of the private learning model and on the parameters $(\eta, (p_k))$ of the public learning model.

We pick an arbitrary agent, and let $p_H(t)$ denote that agent's posterior probability at time t of the event $\{X = H\}$. This posterior is a random variable that depends on the endowed signals of the agent as well as all signals publicly and privately observed by

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that agent until time t. We let F_t denote the cumulative distribution function (CDF) of $p_H(t)$ conditional on the event $\{X = H\}$. That is,

$$F_t(p) = \mathbb{P}(p_H(t) \leq p \mid X = H), \quad p \in [0, 1].\tag{3}$$

By our symmetry assumption on initial signal distributions, F_t does not depend on the identity of the agent. As time passes, the number of signals that are gathered by the agent is likely to get large, so we anticipate that F_t converges to the CDF F_∞ that places all mass on the posterior probability 1 that $X = H$. That is, $F_\infty(p) = 0$ for $p < 1$ and $F_\infty(1) = 1$. Our convergence analysis applies equally to the event $\{X = L\}$.

Because types and beliefs are one-to-one, using (2) we can calculate the belief distribution F_t from the conditional probability distribution ν_t of the type at time t of an arbitrary agent, given X. Specifically, on the event $\{X = H\}$,

$$F_t(p) = \nu_t \left(-\infty, \log \frac{p}{(1-p)} - \log \frac{p_H}{p_L} \right).\tag{4}$$

Lemma 2.3 *At any time t, $\nu_t = E(\mu_t \mid X)$.*

Proof. The claim is that, for each measurable subset A of types, $\nu_t(A) = E[\mu_t(A) \mid X]$. This follows from the fact that the probability $\nu_t(A)$ that the type θ_{it} of an arbitrary agent i is in A, given X, is

$$
\begin{aligned}
\mathbb{P}(\theta_{it} \in A \mid X) &= E\left(1_{\theta_{it} \in A} \mid X\right) \\
&= \int_G E\left(1_{\theta_{it} \in A} \mid X\right) d\gamma(i) \\
&= E\left(\int_G 1_{\theta_{it} \in A}\, d\gamma(i) \,\Big|\, X \right) \\
&= E\left(\mu_t(A) \mid X\right),
\end{aligned}
$$

using symmetry and the Fubini property, respectively. ∎

If we were to generalize by allowing that the agents do not get the same initial quality of information, then $E[\mu_t \mid X]$ is the probability distribution, given X, of the type of a randomly selected agent (that is, an agent randomly selected according to the probability measure γ on the agent space). Thus, even without our assumption of symmetry in the initial information across agents, one can view our convergence results as a characterization of the convergence of the beliefs of a "typical" agent.

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3 Dynamics of the Distribution of Beliefs

In order to calculate the type distribution ν_t, we first condition on the times $T_1, \ldots, T_{N(t)}$ at which public information has been revealed up until time t. Later, we will average over a particular joint distribution of the release times in order to calculate ν_t explicitly.

The aggregate type Z_1 of the initial public release has a probability distribution equal to that of the sum of K_1 independently drawn private types, which, given K_1, is $h(\mu_0, T_1)^{*K_1}$, using the superscript $*k$ to denote k-fold convolution. Thus,

$$E[\mu_{T_1} \,|\, T_1, X] = E[h(\mu_0, T_1) * h(\mu_0, T_1)^{*K_1} \,|\, T_1, X] = \sum_{k=0}^{\infty} p_k h(\mu_0, T_1)^{*k+1}.$$

Just before the second release at T_2, the expected cross-sectional distribution of types, given T_1 and T_2, is $h(h(\mu_0, T_1), T_2 - T_1) * \sum_{k=1}^{\infty} p_k h(\mu_0, T_1)^{*k}$. Thus,

$$E[\mu_{T_2} \,|\, T_1, T_2, X] = h(\mu_0, T_2) * \sum_{k=1}^{\infty} p_k h(\mu_0, T_1)^{*k} * \sum_{k=1}^{\infty} p_k h(\mu_0, T_2)^{*k}.$$

In general, letting $N(t)$ denote the number of public information releases that have occurred up to time t, induction implies the following characterization.

Lemma 3.1 *Almost surely,*

$$E[\mu_t \,|\, T_1, T_2, \ldots, T_{N(t)}, X] = h(\mu_0, t) * \Gamma_{n=1}^{N(t)} \sum_{k=1}^{\infty} p_k h(\mu_0, T_n)^{*k},$$

where, for any probability measures $\alpha_1, \ldots, \alpha_k$, we write $\Gamma_{i=1}^{k} \alpha_i = \alpha_1 * \alpha_2 * \cdots * \alpha_k$.

We now suppose that the counting process N for the number of public releases is a Poisson process with intensity $\eta > 0$. From Lemma 3.1 and the Poisson property of N, we have following result.

Theorem 3.2 *Given the variable X of common concern, the probability distribution of each agent's type at time t is $\nu_t = \alpha_t * \beta_t$, where $\alpha_t = h(\mu_0, t)$ is the type distribution in a model with no public releases of information, satisfying the differential equation*

$$\frac{d\alpha_t}{dt} = \lambda \left(\sum_{l=2}^{\infty} q_l \, \alpha_t^{*l} - \alpha_t \right), \qquad \alpha_0 = \mu_0, \tag{5}$$

and where β_t is the probability distribution over types that solves the differential equation

$$\frac{d\beta_t}{dt} = -\eta \beta_t + \eta \beta_t * \sum_{k=1}^{\infty} p_k \, \alpha_t^{*k}, \tag{6}$$

with initial condition given by the Dirac measure δ_0 at zero.

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We see that ν_t has two outcomes, one on the event $\{X = H\}$ and one on the event $\{X = L\}$, because it depends on μ_0, which likewise has two outcomes. The purely-private type distribution α_t is calculated explicitly by Duffie, Giroux, and Manso (2009) for cases in which the number ℓ of agents sharing information at each meeting is a fixed integer. The equation (5) for α_t is thus somewhat familiar from Duffie, Giroux, and Manso (2009). The equation (6) for β_t, folding in the effects of public information releases, reflects the characterization given by Lemma 3.1 as well as the mean rate η at which β_t gets replaced by a new public release. Corresponding to the public release at time t of the beliefs of k agents, β_t is replaced by the convolution of itself with α_t^{*k}.

4   Solving for Type Distributions as Wild Sums

In order to calcuate the probability distribution ν_t of an agent's type at time t, we first analyze the evolution of α_t and β_t.

For cases in which there is a fixed number n of agents at each private meeting, Duffie, Giroux and Manso (2009) prove that equation (5) has a unique solution, given explicitly by an expansion in convolution powers of α_0, in a form of summation originated by Wild (1951). We now provide a similar result for any distribution of meeting sizes.

Theorem 4.1 *The unique solution to the dynamic equation (5) for the distribution of types in a model with no public information is*

$$\alpha_t \;=\; e^{-\lambda t} \sum_{n=1}^{\infty} a_n(t)\, \mu_0^{*n}, \tag{7}$$

where the coefficients $a_n(t)$ are nonnegative, monotone increasing, and bounded, and can be defined recursively by $a_1(t) \;=\; 1$ and

$$a_j(t) \;=\; \lambda \sum_{k=2}^{j} \int_0^t e^{-\lambda(k-1)s}\, q_k \sum_{j_1+\cdots+j_k=j} \prod_{h=1}^{k} a_{j_h}(s)\, ds, \quad j \geq 2. \tag{8}$$

Moreover, $\lim_{t\to\infty} a_n(t) = \psi_n$ exists and the power series

$$f(z) \;=\; \sum_{n=1}^{\infty} \psi_n\, z^n$$

has a radius of convergence of 1.

We now turn to a characterization of β_t. Since (6) is linear, one can take Fourier transforms to show the following.

Proposition 4.2 *The unique solution to* (6) *is*

$$\beta_t = \exp\left(\eta\left(\int_0^t \sum_{k=1}^\infty p_k \, \alpha_s^{*k} \, ds - t\right)\right) \stackrel{\text{def}}{=} e^{-\eta t} \sum_{n=0}^\infty \frac{\eta^n}{n!} \left(\int_0^t \sum_{k=1}^\infty p_k \, \alpha_s^{*k} \, ds\right)^{*n}. \quad (9)$$

Thus,

$$\beta_t = e^{-\eta t} \sum_{n=0}^\infty b_n(t) \, \mu_0^{*n}, \quad (10)$$

where $b_0(t) = 1$ and

$$b_n(t) = \sum_{k=1}^n \frac{\eta^k}{k!} \sum_{i_1 + \cdots + i_k = n} d_{i_1}(t) \cdots d_{i_k}(t), \quad (11)$$

with

$$d_j(t) = \sum_{k=1}^j p_k \int_0^t \left(e^{-\lambda k s} \sum_{i_1 + \cdots + i_k = j} a_{i_1}(s) \cdots a_{i_k}(s)\right) ds. \quad (12)$$

Equation (9) has a simple interpretation. Public signals arrive at the rate η. For any time t and any number n of public releases, the public information arrival times are uniformly distributed on $[0, t]$. From (7) and (9), we obtain a representation of β_t as the Wild sum (10).

We now use the explicit solutions for α_t and β_t to characterize the probability distribution of an agent's type, for cases with both public and private signals. The main result of this section is the following.

Theorem 4.3 *The probability distribution of any agent's type at time t, given X, is*

$$\nu_t = e^{-(\lambda + \eta)t} \sum_{n=1}^\infty c_n(t) \, \mu_0^{*n}, \quad (13)$$

with coefficients $c_j(t)$ defined by $c_1 = 1$ and

$$c_n(t) = \sum_{k=1}^{n-1} a_k(t) \, b_{n-k}(t).$$

These coefficients are nonnegative and monotone increasing in t. The limit

$$\lim_{t \to +\infty} c_j(t) = \phi_j$$

exists for each j. Furthermore, the power series

$$g(z) = \sum_{j=1}^\infty \phi_j \, z^j$$

has a radius of convergence of 1.

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The Wild summation (13) implies that, at each point in time, the probability distribution of an arbitrary agent's type is a mixture of convolutions of the initial distribution μ_0. The coefficient $e^{-(\lambda+\eta)t} c_n(t)$ associated with the n-th convolution of μ_0 is the probability that the agent has observed the initially endowed information of $(n-1)$ other agents, whether through public or private interactions.

In Duffie, Giroux, and Manso (2009), the coefficients $\phi_1, \phi_2, \ldots$ are uniformly bounded. This is not generally true in our setting. We illustrate with the following result.

Proposition 4.4 *Suppose that the number of agents in any private information sharing meeting is 2, and that the number K_n of agents revealing their beliefs at any public information release is always 1. Then the probability distribution of an agent's type at time t, given X, has the Fourier transform*

$$\hat{\nu}_t = \frac{e^{-(\eta+\lambda)\,t}\,\hat{\mu}_0}{\left(1 - \hat{\mu}_0\,(1 - e^{-\lambda t})\right)^{\frac{\eta+\lambda}{\lambda}}},$$

where $\hat{\mu}_0$ is the Fourier transform of μ_0. Hence,

$$\nu_t = e^{-(\eta+\lambda)\,t} \sum_{n \geq 1} \frac{(\eta+\lambda)\,(\eta+2\lambda)\,\cdots\,(\eta+(n-1)\lambda)}{\lambda^{n-1}\,(n-1)!}\,(1 - e^{-\lambda t})^{n-1} \mu_0^{*n}.$$

In particular, if $\eta = \lambda$, then the probability distribution of any agent's type at time t, given X, is

$$\nu_t = e^{-2\lambda t} \sum_{n \geq 1} n(1 - e^{\lambda t})^{n-1} \mu_0^{*n}. \tag{14}$$

In the case treated by the proposition, we have a particularly simple explicit solution for the distribution of posterior beliefs, using (4). In this case, the limiting weight $\phi_n = n$ placed on acquisition of the information initially endowed to n agents grows linearly with n. It is possible to construct examples in which ϕ_n grows as any power of n.

5   Convergence Results

We now calculate the rate of convergence of an agent's posterior beliefs to the limit of perfect information. We divide our analysis into the cases with and without private information sharing.

In our setting, it turns out that all agents' posterior beliefs converge in law to complete information. Without loss of generality, we characterize the speed of learning on the event $\{X = H\}$. An identical characterization applies on the event $\{X = L\}$.

We recall that F_t is the CDF of the posterior of an arbitrary agent, given $\{X = H\}$. By definition, F_t converges *in distribution* to the perfect-information CDF, F_∞, if, for all p, $F_t(p) \to F_\infty(p)$. (Because F_∞ is the CDF of a constant random variable, convergence in distribution is equivalent to convergence in probability.) We say that the convergence of beliefs to perfect information is exponential at the rate $r > 0$ if, for any p in $[0, 1]$, there are constants $\kappa_0 > 0$ and κ_1 such that,

$$ e^{-rt}\kappa_0 \leq |F_t(p) - F_\infty(p)| \leq e^{-rt}\kappa_1. $$

If there is a rate of convergence, it is unique.

Further, we say that the convergence of posterior beliefs to perfect information is exponential at "almost" the rate $r > 0$ if for any $\varepsilon > 0$ and p in $[0, 1]$, there are constants $\kappa_0 > 0$ and κ_1 such that

$$ e^{-(r+\varepsilon)t}\kappa_0 \leq |F_t(p) - F_\infty(p)| \leq e^{-rt}\kappa_1. $$

Thus, if there is an almost-rate of convergence, it is unique.

We will use the following technical assumption on the moment generating function $s \mapsto M(s) = \int e^{sx} d\mu_0(x)$ of the initial type distribution μ_0 on the event $\{X = H\}$.

Assumption 5.1 *There exists a constant $c > 0$ such that $M(s)$ is finite for $s \in [-c, 0]$.*

A focal point of the paper is the following result, which states that when both private and public learning channels are active, the rate of convergence of beliefs to perfect information is merely the sum $\lambda + \eta$ of the mean arrival rates of private and public learning events, and does not depend at all on the distribution of the number of agents releasing information at each of these types of events. This will be contrasted with the case of purely public learning.

Theorem 5.2 *Under Assumption 5.1, if the mean arrival rate λ of an agent's private information meetings is strictly positive, then the convergence of posterior beliefs to perfect information is exponential at the rate $\lambda + \eta$.*

We now study rates of convergence without private information sharing (that is, with $\lambda = 0$). In this case, $\alpha_t = \mu_0$ for all t and (9) implies that

$$\nu_t = \mu_0 * \left[\sum_{k=0}^{\infty} \frac{(\eta t)^k}{k!} e^{-\eta t} \left(\sum_{n=1}^{\infty} p_n \mu_0^{*n} \right)^{*k} \right]. \tag{15}$$

Under Assumption 5.1, the quantity

$$R = \sup_{y \in \mathbb{R}} \left(-\log M(y) \right). \tag{16}$$

is well-defined and finite.[5]

Because $M(0) = 1$, we see that $R > 0$. The importance of the quantity R is justified by a technical result based on Cramèr's Large Deviations Theorem.

Lemma 5.3 *Under Assumption 5.1, for any $a > 0$ and any $\varepsilon > 0$ there exist strictly positive constants κ_0 and κ_1 such that, for any $k \in \mathbb{N}$,*

$$\kappa_0 \, e^{-(R+\varepsilon)\,k} \;\leq\; \mu_0^{*k}((-\infty, a)) \;\leq\; \kappa_1 \, e^{-R\,k}.$$

We let

$$\Phi(z) = \sum_{n=1}^{\infty} p_n \, z^n,$$

and note that Φ maps $[0,1]$ onto $[0,1]$.

Theorem 5.4 *Under Assumption 5.1, if $\lambda = 0$ (that is, without private information sharing), the convergence in distribution of posterior beliefs to perfect information is exponential, at almost the rate*

$$\rho = \eta \left(1 - \Phi\big(e^{-R}\big) \right). \tag{17}$$

A consequence of Lemma 1 of Moscarini and Smith (2002) is that the exponential convergence characterized by this result is indeed only at rates that are arbitrarily close to the "almost rate" shown, but cannnot achieve exactly that rate.

In contrast to the case treated by Theorem 5.2, if there is no private information sharing, then the rate of convergence of beliefs to perfect information depends on the probability distribution of the number of agents' whose posteriors are revealed at each public information release. It also depends through R on the initial information endowment and on the realization of X. Moreover, as opposed to the case in which there is some private information sharing, the contribution of public information releases to the convergence rate is *less* than the mean rate η of arrivals of public information.

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[5]We set $M(y) = +\infty$ if it is not defined.

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n	ρ
1	0.025
2	0.049
3	0.073
4	0.097
5	0.120
6	0.142
7	0.164
8	0.185
9	0.205
10	0.226
100	0.923

Table 1: Almost-rates of convergence, ρ, for various cases of n, the number of agents whose posteriors are revealed at each arrival of public information. In this example, we assume no private information sharing and take the mean arrival rate η of public releases to be 1. Each agent i is initially endowed with one signal, say s_i, with $\mathbb{P}(s_i = 1 \mid H) = 2/3$ and $\mathbb{P}(s_i = 1 \mid L) = 1/3.5$

Example. We take the case $\eta = 1$ and suppose that any agent, say i, is initially endowed with one signal, say s_i, with $\mathbb{P}(s_i = 1 \mid H) = 2/3$ and $\mathbb{P}(s_i = 1 \mid L) = 1/3$. The initial distribution of types on the event $\{X = H\}$ is then $\mu_0 = 1/3\delta_{\{-\log 2\}} + 2/3\delta_{\{\log 2\}}$. It is straightforward to calculate that R, as defined by (16), is $\log\left(3/2\sqrt{2}\right)$. We suppose that a fixed number n of agents' posteriors are publicly revealed at each public information release. From Theorem 5.4, the probability distribution of any agent's posterior beliefs converges exponentially at almost the rate $\eta\left(1 - \left(2\sqrt{2}/3\right)^n\right)$. As opposed to the case in which there is private information sharing, Table 1 shows that the rate of convergence depends on the number n of agents whose posteriors are revealed at each public information release. Moreover, the rates of convergence shown are substantially lower than η for small n.

We now offer some intuition for the importance of non-zero private information sharing for the contribution of public information to belief convergence rates. From Theorem 3.2, information that is publicly released at time t has a type drawn from the distribution α_t. If the private matching intensity λ is strictly positive, then the privately-gathered type distribution α_t converge exponentially at the rate λ. Thus, regardless of the quality of the initially endowed information distribution μ_0, and regardless of the magnitude of λ so long as it is strictly positive, the distribution of publicly released

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posteriors are converging exponentially fast to perfect information. Thus, with $\lambda > 0$, the contribution to the overall information convergence rate of public information is the mean arrival rate η of public information releases.

In contrast, when the private matching intensity λ is zero, the privately acquired type measure α_t is merely μ_0 for all t. The informativeness of public information releases is then constant over time, and is merely a property of the quality of the initial distribution μ_0 of types, which is bounded away from perfect information. Thus, with $\lambda = 0$, it is not surprising that the contribution of public information releases to the rate of convergence depends on the initial distribution μ_0 of types and is strictly lower than η.

We can further analyze this discontinuity, at $\lambda = 0$, in the dependence of the information convergence rate on λ by examining the convergence of the moment generating function $M_t(\cdot)$ of the type distribution ν_t. For the case of no private information sharing, we have

$$M_t(y) = M_0(y) e^{-t\eta(1-\Phi(M_0(y)))}. \tag{18}$$

By definition, $M_0(y) \geq e^{-R}$. Thus

$$M_t(y) = M_0(y) e^{-t\eta(1-\Phi(M_0(y)))} \geq M_0(y) e^{-t\eta(1-\Phi(e^{-R}))}.$$

It follows that $M_t(y)$ cannot converge to zero any faster than the rate given in Theorem 5.4.

Now we compare to a setting with private information sharing. Because $\nu_t = \alpha_t * \beta_t$, we have

$$M_t(y) = M_t^\alpha(y) M_t^\beta(y), \tag{19}$$

where $M_t^\alpha(\cdot)$ and $M_t^\beta(\cdot)$ are the moment generating functions of α_t and β_t, respectively. We have

$$M_t^\beta(y) = e^{-\eta t + \eta \int_0^t \Phi(M_s^\alpha(y))\,ds}. \tag{20}$$

For imaginary $y \in i\mathbb{R}$ (that is, extending to the characteristic function), we have $|M_t^\alpha(y)| \leq K e^{-\lambda t}$ for some constant K, so $M_t^\alpha(y)$ converges to zero at the rate λ. The contribution of the term $\eta \int_0^t \Phi(M_s^\alpha(y))\,ds$ in the exponent of $M_t^\beta(y)$ is bounded, for $\lambda > 0$, by

$$\eta \left| \int_0^t \Phi(M_s^\alpha(y))\,ds \right| \leq \tilde{K} \int_0^\infty e^{-\lambda s}\,ds = \frac{\tilde{K}}{\lambda}, \tag{21}$$

for some constant $\tilde{K}$ that does not depend on t. Thus, for $\lambda > 0$, the term $\eta \int_0^t \Phi(M_s^\alpha(y))\,ds$ has no influence on the convergence rate η of $M_t^\beta(y)$. As we move from a non-zero rate

λ of private learning to the limit case of no private learning, however, this bound $\tilde{K}\,\lambda^{-1}$ explodes.

As a further guide to understanding this discontinuity in information convergence, Appendix E provides a simplified variant of our model with information sharing, through pairwise random matching of the entire population, at each integer period. A single agent's information is revealed publicly at each period. In this setting, convergence to perfect information is exponential at *any* arbitrarily high rate. After removing the private information sharing, however, convergence is merely exponential at an almost-rate of R.

Appendices

A   Proof of Theorem 4.1.

We will start with

Lemma A.1 *Let $B(\mathbb{R})$ be the space of all signed measures γ on $\mathbb{R}$ of globally bounded variation*

$$\mathrm{Var}(\gamma) = \sup \sum_{i=1}^{N-1} |\gamma((x_i, x_{i+1}])|,$$

where the supremum is over all sequences $-\infty < x_1 < \cdots < x_N < \infty$ and all $N \in \mathbb{N}$. Then, the Fourier transform $\hat{\gamma}$, defined by

$$\hat{\gamma}(s) = \int_{\mathbb{R}} e^{ist} d\gamma(t),$$

is continuous as a map from $B(\mathbb{R})$ to $C(\mathbb{R})$, the set of continuous functions on $\mathbb{R}$ equipped with the supremum norm.

Proof. The proof follows from the standard inequality

$$|\hat{\gamma}_1 - \hat{\gamma}_2| \leq \mathrm{Var}(\gamma_1 - \gamma_2).$$

∎

Another important observation is

Lemma A.2 $\mathrm{Var}(\gamma_1 * \gamma_2) \leq \mathrm{Var}(\gamma_1)\,\mathrm{Var}(\gamma_2)$. *Further, for a positive measure γ,*

$$\mathrm{Var}(\gamma) = \gamma(\mathbb{R}).$$

Proposition A.3 *Suppose that there exists a unique solution $\hat{\alpha}_t$ to the equation*

$$\frac{d}{dt}\hat{\alpha}_t = -\lambda\,\hat{\alpha}_t + \lambda \sum_{k=2}^{\infty} q_k\,\hat{\alpha}_t^k, \tag{22}$$

for any initial condition $\hat{\alpha}_0$, $|\hat{\alpha}_0| \leq 1$, which is analytic in the disk

$$\mathbb{D} = \{\hat{\alpha}_0 \in \mathbb{C} : |\hat{\alpha}_0| < 1\}$$

and continuous on its closure. Let $\hat{\alpha}_0$ be the Fourier transform of μ_0 and

$$\hat{\alpha}_t = \sum_{j=0}^{\infty} B_j(t)\,\hat{\alpha}_0^j, \tag{23}$$

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where all coefficients $B_j(t)$ *are nonnegative. Then, the measure*

$$\alpha_t \;=\; \sum_{j=0}^{\infty} B_j(t)\, \alpha_0^{*j}$$

is the unique solution to (5).

Proof. Suppose that

$$\alpha_t \;=\; \int_0^t \left(-\lambda\, \alpha_s \;+\; \lambda \sum_{k=2}^{\infty} q_k\, \alpha_s^{*k} \right) ds.$$

Since $\sum_k q_k \;=\; 1$, the infinite series converges in the Var -norm, because

$$\mathrm{Var}\left(\sum_{k=2}^{\infty} q_k\, \alpha_t^{*k} \right) \;\leq\; \sum_{k=2}^{\infty} q_k\, \mathrm{Var}(\alpha_t^{*k}) \;=\; 1.$$

By the continuity of the Fourier transform,

$$\hat{\alpha}_t \;=\; \int_0^t \left(-\lambda\, \hat{\alpha}_s \;+\; \lambda \sum_{k=2}^{\infty} q_k\, \hat{\alpha}_s^{k} \right) ds.$$

Conversely, suppose that $\hat{\alpha}$ satisfies this equation and has the expansion (23). Then, define the measure

$$\alpha_t \;=\; \sum_{j=0}^{\infty} B_j(t)\, \alpha_0^{*j}.$$

Since, by assumption, $\sum_j B_j \;<\; \infty$, this indeed defines a measure. By continuity, the Fourier transform of this measure satisfies the above equation and, since the solution is unique, coincides with $\hat{\alpha}_t$. ∎

Thus, we first need to prove that the solution to (22) is analytic in the disc $\mathbb{D}$ and continuous in the closed disc as a function of the initial value $\hat{\alpha}_0$. We will start with the following

Lemma A.4 *Let* $f(z)$ *be analytic in the unit disc* $\mathbb{D}$. *Then the function* g *defined by*

$$g(z) \;=\; \int_0^z f(\xi)\, d\xi$$

is a well defined, analytic function in $\mathbb{D}$. *The power series*

$$g(z) \;=\; \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!}\, z^j$$

has the same radius of convergence as the power series for $f(z)$.

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Proof. The integral $\int_0^z f(\xi)\, d\xi$ does not depend on the path from 0 to z because, for an analytic function, $\int_\gamma f(\xi)\, d\xi = 0$ for any closed contour γ. Now, it is not difficult to check that

$$\frac{\partial g(z)}{dz} = f(z),$$

and hence g is analytic. ∎

Proposition A.5 *There exists an $\varepsilon > 0$ such that the solution to the equation (22) is an analytic function of $\hat\alpha_0$ for $\hat\alpha_0$ in $\mathbb{D}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$, for any $t \in \mathbb{R}_+$, and admits the expansion*

$$\hat\alpha_t = e^{-\lambda t} \sum_{j=0}^\infty a_j(t)\, \hat\alpha_0^j. \tag{24}$$

Proof. We have

$$\frac{d}{dt}\hat\alpha_t = (\hat\alpha_t\, Q(\hat\alpha_t) - 1)\, \lambda\, \hat\alpha_t,$$

where

$$Q(x) = \sum_{k=2}^\infty q_k\, x^{k-2}.$$

Integrating, we get

$$\int_{\hat\alpha_0}^{\hat\alpha} \frac{dx}{(x\, Q(x) - 1)\, x} = \lambda t.$$

Using the identity

$$\frac{1}{(x\, Q(x) - 1)\, x} = -\frac{1}{x} + \frac{Q(x)}{x\, Q(x) - 1},$$

and exponentiating, we get

$$e^{-\lambda t}\, \hat\alpha_0 \exp\left(-\int_0^{\hat\alpha_0} \frac{Q(x)\, dx}{x\, Q(x) - 1} \right) = \hat\alpha_t \exp\left(-\int_0^{\hat\alpha_t} \frac{Q(x)\, dx}{x\, Q(x) - 1} \right). \tag{25}$$

Now, the function

$$f(\hat\alpha) = \hat\alpha \exp\left(-\int_0^{\hat\alpha} \frac{Q(x)\, dx}{x\, Q(x) - 1} \right) \tag{26}$$

is analytic for $\hat\alpha \in \mathbb{D}$. The last claim follows because

$$|z\, Q(z)| < \sum_{k=2}^\infty q_k = 1$$

for all $z \in \mathbb{D}$, so

$$\frac{Q(z)}{z\, Q(z) - 1}$$

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is analytic in $\mathbb{D}$. Therefore, by Lemma A.4, $f(z)$ is also analytic. Now,

$$f'(0) = 1 \neq 0.$$

Therefore, by the implicit function theorem, there exists a unique function $\psi = \psi(z)$, analytic in a small disc $\mathbb{D}_\delta$ such that

$$f(\psi(z)) = \psi(f(z)) = z.$$

Since $f(0) = 0$, we can choose ε so small that $|f(z)| < \delta$ for $|z| < \varepsilon$. Then,

$$\hat{\alpha}_t(\hat{\alpha}_0) = \psi(e^{-\lambda t} f(\hat{\alpha}_0)) \tag{27}$$

is analytic for $|\hat{\mu}_0| < \varepsilon$, which is what had to be proved. ∎

To proceed further, we will get information about the Taylor-series coefficients of the analytic function $\hat{\alpha}_t(z)$. To this end, we will calculate higher derivatives of the right-hand side of (27). The combinatorial structure of these derivatives is quite complicated. We will make use of the Faa-di Bruno formula (see, Riordan (1958), pp. 35-37), providing an expression for the higher derivatives of a composition of two functions.

Lemma A.6 (Faa-di Bruno formula) *Let* $F \colon \mathbb{C} \to \mathbb{C}$ *and* $V \colon \mathbb{C} \to \mathbb{C}$ *be analytic. Then,*

$$(F(V(x)))^{(n)} = \sum_{k=1}^{n} F^{(k)}|_{V(x)} \sum_{Q(n,k)} n! \prod_{i=1}^{n} \frac{1}{(\lambda_i!)} \left(\frac{V^{(i)}}{i!} \right)^{\lambda_i},$$

$$\tag{28}$$

where

$$Q(n,k) = \left\{ (\lambda_1, \dots, \lambda_n) : \lambda_i \in \mathbb{N}_0, \sum_{i=1}^{n} \lambda_i = k, \sum_{i=1}^{n} i\lambda_i = n \right\}$$

and $\mathbb{N}_0$ *is the set of nonnegative integers.*

The following lemma is a direct consequence of the Faa-di Bruno formula.

Lemma A.7 *The functions* a_j *are finite polynomials in* $e^{-\lambda t}$ *and satisfy*

$$\lim_{t \to \infty} a_j(t) = \frac{f^{(j)}(0)}{j!} \overset{def}{=} \psi_j,$$

where the function f *is given by (26).*

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Proof. Rewriting the identity (27) as

$$f(\hat{\alpha}_t(z)) = e^{-\lambda t} f(z)$$

and using the Fa-di Bruno formula at the point $\hat{\alpha}_0 = 0$, we get

$$\sum_{k=1}^{n} f^{(k)}(0) \sum_{Q(n,k)} n! \prod_{i=1}^{n} \frac{1}{(\lambda_i!)} \left(\frac{\hat{\alpha}_t^{(i)}(0)}{i!} \right)^{\lambda_i} = e^{-\lambda t} f^{(n)}(0).$$

Since $f'(0) = 1$,

$$\hat{\alpha}_t^{(n)}(0) = f'(0) \hat{\alpha}_t^{(n)}(0)$$

$$= e^{-\lambda t} f^{(n)}(0) - \sum_{k=2}^{n} f^{(k)}(0) \sum_{Q(n,k)} n! \prod_{i=1}^{n} \frac{1}{(\lambda_i!)} \left(\frac{\hat{\alpha}_t^{(i)}(0)}{i!} \right)^{\lambda_i}.$$

For $n = 1$,

$$\hat{\alpha}_t^{(1)}(0) = e^{-\lambda t}.$$

An induction argument then shows that for each $n \geq 2$, there exists a polynomial $P_n = P_n(z_1, \ldots, z_{n-1})$, not containing constant and linear terms, such that

$$\hat{\alpha}_t^{(n)}(0) = e^{-\lambda t} f^{(n)}(0) - P_n(\hat{\alpha}_t^{(1)}(0), \ldots, \hat{\alpha}_t^{(n-1)}(0)).$$

Consequently, as $t \to \infty$,

$$\hat{\alpha}_t^{(n)}(0) = O(e^{-\lambda t}).$$

Since P_n does not contain constant and linear terms, for any $n \geq 2$,

$$P_n(\hat{\alpha}_t^{(1)}(0), \ldots, \hat{\alpha}_t^{(n-1)}(0)) = O(e^{-2\lambda t})$$

as $t \to \infty$. Thus,

$$\lim_{t \to \infty} e^{\lambda t} \hat{\alpha}_t^{(n)}(0) = f^{(n)}(0),$$

as claimed. ∎

Lemma A.8 *The coefficients $a_n(t)$ in (7) are nonnegative, monotone increasing and bounded, and can be defined recursively as $a_1(t) = 1$ and*

$$a_j(t) = \lambda \sum_{k=2}^{j} \int_0^t e^{-\lambda(k-1)s} q_k \sum_{j_1 + \cdots + j_k = j} \prod_{h=1}^{k} a_{j_h}(s) \, ds, \qquad (29)$$

for all $j \geq 2$.

Proof. Let

$$\hat{\alpha}_t = \hat{A}_t \, e^{-\lambda t}.$$

Substituting $\hat{\alpha}_t$ into (22), we get that

$$\frac{d}{dt}\hat{A}_t = \sum_{k=2}^{\infty} e^{-\lambda(k-1)t} \, q_k \, \hat{A}_t^k.$$

Thus, $\hat{A}_t$ solves the equation

$$\hat{A}_t = F(\hat{A}_t),$$

where

$$F(\hat{A}_t) = \hat{A}_0 + \sum_{k=2}^{\infty} \int_0^t e^{-\lambda(k-1)s} \, q_k \, \hat{A}_s^k \, ds.$$

Substituting the power-series expansion

$$\hat{A}_s = \sum_{j=1}^{\infty} a_j(t) \, \hat{A}_0^j,$$

we get

$$
\begin{aligned}
F(\hat{A}_t) &= \hat{A}_0 + \sum_{k=2}^{\infty} \int_0^t e^{-\lambda(k-1)s} \, q_k \left(\sum_{j=1}^{\infty} a_j(s) \, \hat{A}_0^j \right)^k ds \\
&= \hat{A}_0 + \sum_{k=2}^{\infty} \int_0^t e^{-\lambda(k-1)s} \, q_k \sum_{j=1}^{\infty} \hat{A}_0^j \sum_{j_1+\cdots+j_k=j} \prod_{h=1}^{k} a_{j_h}(s) \, ds \\
&= \hat{A}_0 + \sum_{j=2}^{\infty} \hat{A}_0^j \sum_{k=2}^{j} \int_0^t e^{-\lambda(k-1)s} \, q_k \sum_{j_1+\cdots+j_k=j} \prod_{h=1}^{k} a_{j_h}(s) \, ds,
\end{aligned}
$$

where interchanging summation and integration is justified because of uniform convergence. Since $\hat{A}_t = F(\hat{A}_t)$, the coefficients in the power series expansions must coincide and the identity (29) follows. ■

Lemma A.9 *Let f be the function, defined in (26). The function $\hat{\alpha}_t(\hat{\alpha}_0)$ can be analytically continued to the whole disc $\mathbb{D}$ and*

$$|\hat{\alpha}_t(\hat{\alpha}_0)| \leq e^{-\lambda t} f(|\hat{\alpha}_0|),$$

for any $t \in \mathbb{R}_+$ and any $\hat{\alpha}_0 \in \mathbb{D}$.

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Proof. By Lemmas A.7 and A.8,

$$| \hat{\alpha}_t(\hat{\alpha}_0) | \;\leq\; e^{-\lambda t} \sum_{j=1}^{\infty} a_j(t) \, |\hat{\alpha}_0|^j \;\leq\; e^{-\lambda t} \sum_{j=1}^{\infty} \frac{f^{(j)}(0)}{j!} \, |\hat{\alpha}_0|^j \;=\; e^{-\lambda t} f(\, |\hat{\alpha}_0| \,).$$

The claim follows. ∎

We will also need the following auxiliary

Lemma A.10 *For any initial value* $r \in (0\,,\,1)\,,$ *the solution* k_t *to the equation*

$$\frac{d}{dt} k_t \;=\; -\lambda \, k_t \;+\; \lambda \sum_{k=2}^{\infty} q_k \, k_t^k \;,\;\; k_0 \;=\; r$$

exists on the whole half-line $\mathbb{R}_+$ *and satisfies*

$$r \;>\; k_t \;>\; \lim_{t \to +\infty} k_t \;=\; 0.$$

Proof. Note that the function

$$q(x) \;=\; -\lambda \, x \;+\; \lambda \sum_{k=2}^{\infty} q_k \, x^k \tag{30}$$

has no zeros in $(0\,,\,1)$ because, for $x \in (0,1)\,,$

$$\sum_{k=2}^{\infty} q_k \, x^{k-1} \;<\; 1.$$

That is, $q(x) \;<\; 0$ on $(0,1)\,.$ By the uniqueness theorem for ODE's, any solution k_t with initial data $r \in (0,1)$ must stay in the segment $(0,1)$ forever. Since k_t can not blow up to $\pm\infty\,,$ it exists on the whole $\mathbb{R}_+$ (Dieudonné (1960), Theorem 10.5.6). Since

$$\frac{d}{dt} k_t \;=\; q(k_t) \;<\; 0,$$

k_t is monotone decreasing in t and converges to zero. ∎

We are now ready to prove the final step of the proof of Theorem 4.1. By Proposition A.5, the power series $\hat{\alpha}_t(\hat{\alpha}_0)$ coincides with the unique solution to (22) when $|\hat{\alpha}_0|$ is sufficiently small. By Proposition A.3, to complete the proof of Theorem 4.1 it remains to show that $\hat{\alpha}_t$ solves (22) for all $\hat{\alpha}_0 \in \mathbb{D}$ and that $\hat{\alpha}_t(\hat{\alpha}_0)$ is continuous in the closure of $\mathbb{D}$. The next proposition shows that this is indeed true.

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Proposition A.11 *The unique solution to (22) coincides with $\hat{\alpha}_t(\hat{\alpha}_0)$ for any $\hat{\alpha}_0 \in \mathbb{D}$. The function $\hat{\alpha}_t(\hat{\alpha}_0)$ is analytic for $\hat{\alpha}_0 \in \mathbb{D}$ and is continuous on its closure, and maps $\mathbb{D}$ into itself. The radius of analyticity of $f(z)$ is exactly one.*

Proof. By Lemma A.9, $\hat{\alpha}_t(\hat{\alpha}_0)$ is analytic in $\mathbb{D}$. Since $q(x)$ defined in (30) is analytic in $\mathbb{D}$, it remains to show that $\hat{\alpha}_t$ maps $\mathbb{D}$ into itself. Indeed, if this is the case, the function $q(\hat{\alpha}_t(\hat{\alpha}_0))$ is analytic for $\hat{\alpha}_0 \in \mathbb{D}$ and, by Proposition A.5,

$$\hat{\alpha}_t(\hat{\alpha}_0) \;=\; \int_0^t g(\hat{\alpha}_s(\hat{\alpha}_0))\, ds \tag{31}$$

for all $\hat{\alpha}_0 \in \mathbb{D}_\varepsilon$. The left- and the right-hand sides are analytic functions in $\mathbb{D}$ and, by the uniqueness theorem for analytic functions, (31) holds for all $\hat{\alpha}_0 \in \mathbb{D}$. Furthermore, by Lemma A.8, $\hat{\alpha}_t(\hat{\alpha}_0)$ has nonnegative Taylor coefficients at zero and therefore, by Lemma A.10,

$$|\hat{\alpha}_t(\hat{\alpha}_0)| \;\leq\; \hat{\alpha}_t(|\hat{\alpha}_0|) \;\leq 1. \tag{32}$$

Inequality (32) and non-negativity of the coefficients imply that the power series is also well defined on the cirle $|\hat{\alpha}_0| = 1$ and therefore $\hat{\alpha}_t(\hat{\alpha}_0)$ is continuous on the closure of $\mathbb{D}$. Finally, $\hat{\alpha}_t(1) = 1$ implies

$$\sum_j a_j(t) \;=\; e^{\lambda t},$$

and therefore, by the monotone convergence theorem,

$$\sum_j \psi_j \;=\; \sum_j \lim_{t\to\infty} a_j(t) \;=\; \lim_{t\to\infty} \sum_j a_j(t) \;=\; +\infty.$$

Hence, the radius of analyticity of $f(z)$ is exactly one.

Suppose now that, for some $T > 0$, $\hat{\alpha}_T$ does not map $\mathbb{D}$ into itself, that is, $|\hat{\alpha}_T(z_0)| > 1$ for some $z_0 \in \mathbb{D}$. Let $Z(r) = \max_{t\in[0,T]} \hat{\alpha}_t(r)$. Standard compactness arguments imply that $Z(r)$ is well-defined, increasing in r and continuous. By (32), $Z(|z_0|) > 1$ and $r_0 = \inf\{r \in (0,1): Z(r) \geq 1\}$ is well-defined. By Proposition A.5 and Lemma A.10, $r_0 > 0$. Furthermore, by definition, $Z(r) < 1$ for all $r \in [0, r_0)$ and, by compactness of $[0,T]$ and continuity of $\hat{\alpha}_t$, there exists a $T_0 \in [0,T]$ such that $\hat{\alpha}_{T_0}(r_0) = 1$. By (32) and the argument in the previous paragraph, $\hat{\alpha}_t(\hat{\alpha}_0)$ solves (22) for all $|\hat{\alpha}_0| \leq r_0$ and $t \in [0, T_0)$. But, by Lemma A.10, $\hat{\alpha}_t(r_0)$ is monotone decreasing in t and therefore $\hat{\alpha}_{T_0}(r_0) < \hat{\alpha}_0(r_0) = r_0 < 1$, which is a contradiction. ∎

B   Proof of Proposition 4.2

Proof. An argument used in Proposition 3 of Duffie, Giroux and Manso (2009) implies that β_t solves (6) if and only if its Fourier transform $\hat{\beta}_t$ solves

$$\frac{d\hat{\beta}_t}{dt} \;=\; -\eta\hat{\beta}_t \;+\; \eta\,\hat{\beta}_t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds. \tag{33}$$

This is a linear ordinary differential equation whose unique solution, with $\hat{\beta}_0 \;=\; 1$, is

$$\hat{\beta}_t \;=\; \exp\left(\eta\left(\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds \;-\; t \right) \right).$$

Using the Taylor series for e^x,

$$\hat{\beta}_t \;=\; e^{-\eta t} \sum_{n=0}^{\infty} \frac{\eta^n}{n!} \left(\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds \right)^n. \tag{34}$$

Now, by (7),

$$(\hat{\alpha}_t)^k \;=\; e^{-\lambda k t} \left(\sum_{l=1}^{\infty} a_l(t)\,\hat{\mu}_0^l \right)^k \;=\; e^{-\lambda k t} \sum_{l=k}^{\infty} \sum_{i_1+\cdots+i_k=l} a_{i_1}(t) \cdots a_{i_k}(t)\,\hat{\mu}_0^l. \tag{35}$$

Therefore,

$$\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds \;=\; \sum_{j=1}^{\infty} d_j(t)\,\hat{\mu}_0^j,$$

with d_j defined by (12). Substituting (35) into (34), we obtain

$$\hat{\beta}_t \;=\; e^{-\eta t} \sum_{n=0}^{\infty} b_n(t)\,\hat{\mu}_0^n,$$

with b_n defined by (11). Taking the inverse Fourier transform of this identity, we arrive at (10). ∎

C   Proofs of Theorem 4.3 and Proposition 4.4

We will use the well known Montel Theorem (Titchmarsh (1960), p. 170).

Theorem C.1 (Montel Theorem) *Let $D \subset \mathbb{C}$ be an open set. A uniformly bounded set A of analytic functions on D is compact. That is, if there exists a constant C such that $|f(z)| \leq K$ for all z in D and all $f \in A$, then for any infinite sequence $\{f_k(z)\} \subset A$ there exists subsequence $\{f_{n_k}\}$ and a function $f(z)$, analytic in D, such that $f_{n_k}(z) \to f(z)$ and $f_{n_k}^{(m)}(z) \to f^{(m)}(z)$ uniformly on compact subsets of D for any $m \geq 0$.*

Proposition C.2 *The function*

$$e^{(\lambda+\eta)t}\hat{\nu}_t(z) \;=\; e^{\lambda t}\,\hat{\alpha}_t(z)\,\exp\left(\eta\left(\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k(z)\,ds\right)\right)$$

is analytic in the disc $\mathbb{D}$ for any $t > 0$, and the family $\{e^{(\lambda+\eta)t}\hat{\nu}_t(z)\,,\ t > 0\}$ is uniformly bounded on compact subsets of $\mathbb{D}$.

Proof. By Proposition A.11, the function $\alpha_s(z)$ maps $\mathbb{D}$ into itself, the infinite series

$$l_s(z) \;=\; \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k(z)$$

converges absolutely and satisfies

$$\left|\sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k(z)\,ds\right| \;\leq\; \sum_{k=1}^{\infty} p_k\,|\hat{\alpha}_s^k(z)| \;\leq\; 1.$$

By the Montel Theorem, $l_s(z)$ is analytic in $\mathbb{D}$. Again, since $l_s(z)$ is uniformly bounded by one, $\int_0^t l_s(z)\,ds$ is analytic in $\mathbb{D}$ by the Montel Theorem. The analyticity of $\hat{\nu}_t(z)$ follows. Now, by Lemma A.9,

$$|\hat{\alpha}_s^k(z)| \;\leq\; e^{-\lambda k t}\,f(|z|)^k.$$

Pick a $T > 0$ so large that $e^{-\lambda T}f(|z|) \;<\; 1$. Then, for all $t > T$,

$$\left|\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k(z)\,ds\right| \;\leq\; T + \int_T^t \sum_{k=1}^{\infty} p_k\,e^{-\lambda k(s-T)}(e^{-\lambda T}f(|z|))^k\,ds$$

$$\leq\; T + \int_T^{\infty} \sum_{k=1}^{\infty} p_k\,e^{-\lambda k(s-T)}\,ds$$

$$=\; T + \sum_{k=1}^{\infty} \frac{p_k}{\lambda k} \;<\; \infty.$$

Thus, we have the uniform boundedness on compact subsets of $\mathbb{D}$ and Montel's theorem implies the required analyticity. ∎

Lemma C.3 *Let*

$$\hat{\nu}_t(z) \;=\; e^{-(\lambda+\eta)t}\sum_{j=1}^{\infty} c_j(t)\,z^j.$$

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Then, the coefficients $c_j(t)$ are nonnegative, monotone increasing, bounded from above and satisfy

$$\lim_{t \to \infty} c_j(t) \;=\; \phi_j.$$

The function

$$\phi(z) \;=\; \sum_{k=1}^{\infty} \phi_j \, z^j \tag{36}$$

is analytic in $\mathbb{D}$.

Proof. By Lemma A.8, the coefficients $a_j(t)$ in the expansion

$$e^{\lambda t} \, \hat{\alpha}_t(z) \;=\; \sum_{j=1}^{\infty} a_j(t) \, z^j$$

are nonnegative and monotone increasing in t. Therefore, the coefficients of the Taylor expansion of $(\hat{\alpha}_t(z))^k$ are nonnegative for any k and hence

$$\sum_{k=1}^{\infty} p_k \, \hat{\alpha}_s^k(z) \;=\; \sum_{j=1}^{\infty} D_j(t) \, z^j,$$

for some nonnegative $D_j(t)$. Therefore, the Taylor coefficients of

$$\int_0^t \sum_{k=1}^{\infty} p_k \, \hat{\alpha}_s^k(z) ds \;=\; \sum_{j=1}^{\infty} \left(\int_0^t D_j(s) ds \right) z^j$$

are nonnegative and monotone increasing in t. Since multiplying and adding nonnegative, increasing functions generates nonnegative, increasing functions, we immediately get that $c_j(t)$ are monotone increasing and nonnegative. By Proposition C.2,

$$c_j(t)(0.5)^j \;\leq\; \sum_{k=1}^{\infty} c_k(t)(0.5)^k \;<\; K,$$

for some constant K, independent of time. Hence, $c_j(t)$ is increasing and bounded from above for each j, so the limit $\lim_{t \to \infty} c_j(t) \;=\; \phi_j$ exists for each j.

Since the functions $e^{(\lambda+\eta)t} \, \hat{\nu}_t(z)$ are uniformly bounded on compact subsets of $\mathbb{D}$, this family of functions is compact by the Montel Theorem and there exists a subsequence $e^{(\lambda+\eta)t_j} \, \hat{\nu}_{t_j}(z)$ converging uniformly on compact subsets of $\mathbb{D}$ to a function $g(z)$, analytic in $\mathbb{D}$. By the Montel Theorem, the Taylor coefficients also converge, so the Taylor coefficients of $g(z)$ are given by $\lim_{j \to \infty} c(t_j) \;=\; \phi_j$. That is, $g(z)$ is given by (36), as stipulated. ∎

Proof of Proposition 4.4. Using the solution for α_t from equation (7) of Duffie and Manso (2007), we note that

$$\int_0^t \hat{\alpha}_s \, ds \;=\; -\frac{1}{\lambda} \log \left(1 - \hat{\mu}_0 \left(1 - e^{-\lambda t} \right) \right). \tag{37}$$

Therefore,

$$\hat{\beta}_t \;=\; \frac{e^{-\eta t}}{\left(1 - \hat{\mu}_0 \left(1 - e^{-\lambda t} \right) \right)^{\frac{\eta}{\lambda}}}$$

and

$$\hat{\nu}_t = \hat{\alpha}_t \, \hat{\beta}_t \;=\; \frac{e^{-(\eta+\lambda)t} \, \hat{\mu}_0}{\left(1 - \hat{\mu}_0 \left(1 - e^{-\lambda t} \right) \right)^{\frac{\eta+\lambda}{\lambda}}}. \tag{38}$$

∎

D   Proofs of Convergence Rates

Proof of Theorem 5.2.   The argument is analogous to that of Propositions 2 and 4 in Duffie, Giroux, and Manso (2008). We will provide the rate of convergence to zero of $\nu_t((-\infty, a))$ on the event $\{X = H\}$. A like argument gives the same rate of convergence to 1 on the event $\{X = L\}$.

Let $Y_1, Y_2, \ldots$ be random variables that, given X, are independent with distribution $\mu_0 = \alpha_0$. By Theorem 4.3,

$$
\begin{aligned}
\nu_t((-\infty \,, a)) \;&=\; e^{-(\lambda+\eta)t} \sum_{n=1}^{\infty} c_n(t) \, \mu_0^{*n}((-\infty, a)) \\
&=\; e^{-(\lambda+\eta)t} \sum_{n=1}^{N} c_n(t) \, \mathbb{P}\left[\sum_{i=1}^{n} \left(Y_i - \frac{a}{n} \right) \le 0 \right] \\
&\quad + e^{-(\lambda+\eta)t} \sum_{n=N+1}^{\infty} c_n(t) \, \mathbb{P}\left[\sum_{i=1}^{n} \left(Y_i - \frac{a}{n} \right) \le 0 \right] \\
&\le\; \beta \, e^{-(\lambda+\eta)\,t} \;+\; e^{-(\lambda+\eta)\,t} \sum_{n=N+1}^{\infty} \phi_n \, e^{ac} \, \gamma^n \\
&\le\; e^{-(\lambda+\eta)\,t} \left(\beta + e^{ac} \, g(\gamma) \right),
\end{aligned}
$$

and the proof is complete. ∎

Proof of Lemma 5.3.   Let $\{Y_i\}$ be an *iid* sequence of random variables with the distribution of μ_0 on the event $\{X = H\}$. By Cramèr's Large Deviations theorem

(Deuschel and Stroock (1989), p. 6),

$$\mu_0^{*k}((-\infty, a)) \;=\; \mathbb{P}(Y_1 + \cdots + Y_k < a) \;=\; e^{-k\,(R+o(1))}$$

as $k \to \infty$. The lower bound immediately follows. For the upper bound, we will use the Chernoff (1953) bound, stating that

$$\mathbb{P}[Y_1 + \cdots + Y_k < a] \;\leq\; e^{-k\,S(a/k)},$$

where

$$S(x) \;=\; \sup_{y \in \mathbb{R}} \left(y\,x \;-\; \log E[e^{yY}] \right).$$

It is known (for example, Deuschel and Stroock (1989), p. 6) that $S(\cdot)$ is a strictly convex function, attaining its minimal value 0 at $x = E[Y]$. Therefore, $S(x)$ is monotone decreasing on $[0, E[Y]]$ and $R = S(0) > S(a/k)$. But, since convex functions are locally Lipschitz, $S(0) - S(a/k) < C\,a/k$ for some constant C. Therefore,

$$e^{-k\,S(a/k)} \;\leq\; e^{-k\,(R-C\,a/k)} \;=\; e^{Ca}\,e^{-kR},$$

and the proof is complete. ∎

Proof of Theorem 5.4. By Theorem 4.3,

$$\nu_t \;=\; \sum_{n=1}^{\infty} e^{-\eta t}\, c_n(t)\, \mu_0^{*n}$$

with nonnegative $c_n(t)$. By Lemma 5.3,

$$\nu_t((-\infty, a)) \;\leq\; \kappa_1 \sum_{n=1}^{\infty} e^{-\eta t}\, c_n(t)\, e^{-Rn}.$$

Now, gathering the terms and using representation (15),

$$
\begin{aligned}
\sum_{n=1}^{\infty} e^{-\eta t}\, c_n(t)\, e^{-Rn} \;&=\; e^{-R}\left[\sum_{k=0}^{\infty} \frac{(\eta t)^k}{k!} e^{-\eta t}\left(\sum_{n=1}^{\infty} p_n\, e^{-Rn}\right)^k\right] \\
&=\; e^{-R}\left[\sum_{k=0}^{\infty} \frac{(\eta t)^k}{k!} e^{-\eta t}\left(\Phi(e^{-R})\right)^k\right] \\
&=\; e^{-R} \exp\left(-\,\eta\left(1 - \Phi(e^{-R})\right)t\right).
\end{aligned}
$$

The lower bound is proved similarly. ∎

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E   Discrete Time Example

This appendix presents a simplified model whose purpose is to illustrate the discontinuity in the rates of convergence with respect to the private meeting intensity, at the point at which that intensity reaches zero.

Each agent from a continuum of agents of mass 1 starts with one signal. In this setting, however, time is discrete. At each integer period $t \geq 1$, the posterior of a randomly selected agent in the population is revealed publicly. At every k-th period, beginning at period k, there is a random full matching of the population into pairs. That is, each agent is matched with another agent that is randomly selected from the population. (A rigorous construction of such a random matching, obeying the law of large numbers used below, is provided by Duffie and Sun (2007).) Agents reveal their posteriors to each other in these two-agent meetings. Across periods, the random matchings are independent.

At time t, almost surely, all agents in the population will have collected the same number of signals. Some signals will have been collected from public releases of information while other signals will have been collected from private meetings with other agents. In particular, at time t, the number of signals that each agent will have collected is

$$(2^{\text{Int}(t,k)} - 1) + \sum_{s=1}^{\text{Int}(t,k)} k * 2^{s-1} + \text{Mod}(t,k)2^{\text{Int}(t,k)}, \tag{39}$$

where $\text{Int}(t,k)$ is the integer part of the division of t by k, and where $\text{Mod}(t,k)$ is the modulus after division of t by k.

The first term in (39) represents the number of signals that each agent has collected through private meetings up to time t. At time t, each agent will have participated in $\text{Int}(t,k)$ private meetings, and therefore, each agent will have collected $(2^{\text{Int}(t,k)} - 1)$ signals through private meetings up to time t.

The last two terms in (39) represent the number of signals that each agent will have collected through public information releases by time t. A public information release at time t provides agents with $2^{\text{Int}(t,k)}$ new signals. Therefore, by time t, each agent will have collected $\sum_{s=1}^{\text{Int}(t,k)} k*2^{s-1}+\text{Mod}(t,k)2^{\text{Int}(t,k)}$ signals from public information releases. The number of signals collected during the first $\text{Int}(t,k)k$ periods is $\sum_{s=1}^{\text{Int}(t,k)} k*2^{s-1}$. The number collected during the last $\text{Mod}(t,k)$ periods $\text{Mod}(t,k)2^{\text{Int}(t,k)}$.

From (39), it follows that the probability distribution of an agent's type at time t is

$$\nu_t = \mu_0^{*(2^{\text{Int}(t,k)}+\sum_{s=1}^{\text{Int}(t,k)} k*2^{s-1}+\text{Mod}(t,k)2^{\text{Int}(t,k)})}. \tag{40}$$

We can use Lemma 5.3 to prove the following proposition.

Proposition E.1 *Under Assumption 5.1, on the event $\{X = H\}$, for any $\varphi > 0$ there exists some $\kappa_0 > 0$ such that, for any positive integer time t,*

$$\nu_t((-\infty, a)) \leq \kappa_0 e^{-\varphi t}. \tag{41}$$

Proof of Proposition E.1. Take t high enough that $2^{\mathrm{Int}(t,k)} > 2\varphi/R$. Then,

$$2^{\mathrm{Int}(t,k)} + \sum_{s=1}^{\mathrm{Int}(t,k)} k * 2^{s-1} + \mathrm{Mod}(t,k)2^{\mathrm{Int}(t,k)} > \mathrm{Int}(t,k)k\frac{\varphi}{R} + \mathrm{Mod}(t,k)\frac{\varphi}{R} = t\frac{\varphi}{R}.$$

Using Lemma 5.3, we obtain that for t such that $2^{\mathrm{Int}(t,k)} > 2\varphi/R$,

$$\nu_t((-\infty, a)) \leq \kappa e^{-\varphi t}. \tag{42}$$

On the other hand, for t such that $2^{\mathrm{Int}(t,k)} \leq 2\varphi/R$, we can always select a high enough constant κ' such that

$$\nu_t((-\infty, a)) \leq \kappa' e^{-\varphi t}. \tag{43}$$

Making $\kappa_0 = \max\{\kappa, \kappa'\}$, we obtain the result. ■

This proposition implies that with public and private sources of information, the exponential convergence rate is higher than φ for any $\varphi > 0$.

On the other hand, if we assume that agents learn only through public information releases (that is, $k = \infty$ in the model discussed in this section), then by time t, each agent will have collected exactly t signals. Therefore the probability distribution of an agent's type at time t is

$$\nu_t = \mu_0^{t+1}. \tag{44}$$

Using Lemma 5.3 it is easy to see that, under Assumption 5.1, convergence is exponential at an "almost rate" of R.

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