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ON TRIVIAL p-ADIC ZEROES FOR ELLIPTIC CURVES OVER KUMMER EXTENSIONS

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Abstract. *We prove the exceptional zero conjecture is true for semistable elliptic curves $E_{/\mathbb{Q}}$ over number fields of the form $F\big(e^{2\pi i/q^n}, \Delta_1^{1/q^n}, \ldots, \Delta_d^{1/q^n}\big)$ where F is a totally real field, and the split multiplicative prime $p \neq 2$ is inert in $F(e^{2\pi i/q^n}) \cap \mathbb{R}$.*

In 1986 Mazur, Tate and Teitelbaum [**9**] attached a p-adic L-function to an elliptic curve $E_{/\mathbb{Q}}$ with split multiplicative reduction at p. To their great surprise, the corresponding p-adic object $L_p(E,s)$ vanished at $s=1$ irrespective of how the complex L-function $L(E,s)$ behaves there. They conjectured a formula for the derivative

$$L_p'(E,1) \;=\; \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)} \times \frac{L(E,1)}{\text{period}} \qquad \text{where } \; E(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times / q_E^{\mathbb{Z}} \,,$$

and this was subsequently proven for $p \geq 5$ by Greenberg and Stevens [**6**] seven years later.

In recent times there has been considerable progress made on generalising this formula, both for elliptic curves over totally real fields [**10, 15**], and for their adjoint L-functions [**12**]. In this note, we outline how the techniques in [**3**] can be used to establish some new cases of the exceptional zero formula over solvable extensions $K/\mathbb{Q}$ that are not totally real.

1. Constructing the p-adic L-function

Let E be an elliptic curve defined over $\mathbb{Q}$, and $p \geq 3$ a prime of split multiplicative reduction. First we fix a finite normal extension $K/\mathbb{Q}$ whose Galois group is a semi-direct product

$$\mathrm{Gal}(K/\mathbb{Q}) \;=\; \Gamma \ltimes \mathcal{H}$$

where $\Gamma, \mathcal{H}$ are both abelian groups, with $\mathcal{H} = \mathrm{Gal}(K/K \cap \mathbb{Q}^{\mathrm{ab}})$ and likewise $\Gamma \cong \mathrm{Gal}(K \cap \mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Secondly we choose a totally real number field F disjoint from K, and in addition suppose:

(H1) the elliptic curve E is semistable over F;
(H2) the prime p is unramified in K;
(H3) the prime p is inert in the compositum $F \cdot k^+$ for all CM fields $k \subset K \cap \mathbb{Q}^{\mathrm{ab}}$.

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Now consider an irreducible representation of dimension > 1 of the form

$$\rho_{\chi,k}^{(\psi)} \ := \ \mathrm{Ind}_{F\cdot k}^{F}(\chi) \otimes \psi$$

where k is a CM field inside of $K \cap \mathbb{Q}^{\mathrm{ab}}$, the character $\chi : \mathrm{Gal}\big(F\cdot K/F\cdot k\big) \longrightarrow \mathbb{C}^{\times}$ induces a self-dual representation, and ψ is cyclotomic of conductor coprime to p. It is well known how to attach a bounded p-adic measure to the twisted motive $h^1(E_{/F}) \otimes \rho_{\chi,k}^{(\psi)}$, as we shall describe below.

By work of Shimura [**14**], there exists a parallel weight one Hilbert modular form $\mathbf{g}_{\chi}^{(\psi)}$ with the same complex L-series as the representation $\mathrm{Ind}_{F\cdot k}^{F\cdot k^+}(\chi) \otimes \mathrm{Res}_{F\cdot k^+}(\psi)$ over the field $F \cdot k^+$. The results of Hida and Panchiskin [**7, 11**] furnish us with measures interpolating

$$\int_{x\in\mathbb{Z}_p^{\times}} \varphi(x)\cdot \mathrm{d}\mu_{\mathbf{f}_E\otimes\mathbf{g}_{\chi}^{(\psi)}}(x) = \varepsilon_F\big(\rho_{\chi,k}^{(\psi)}\otimes\varphi\big)\times(\text{Euler factor at } p)\times\frac{L\big(\mathbf{f}_E\otimes\mathbf{g}_{\chi}^{(\psi)},\varphi^{-1},1\big)}{\langle \mathbf{f}_E,\mathbf{f}_E\rangle_{F\cdot k^+}}$$

where the character φ has finite order, $\mathbf{f}_E$ denotes the base-change to the totally real field $F\cdot k^+$ of the newform f_E associated to $E_{/\mathbb{Q}}$, and $\langle -,-\rangle_{F\cdot k^+}$ indicates the Petersson inner product.

We now explain how to attach a p-adic L-function to E over the full compositum $F\cdot K$. Let us point out that by the representation theory of semi-direct products [**13**, Proposition 25], every irreducible $\mathrm{Gal}(F\cdot K/F)$-representation ρ must either be isomorphic to some $\rho_{\chi,k}^{(\psi)}$ above if $\dim(\rho) > 1$, otherwise $\rho = \psi$ for some finite order character ψ with prime-to-p conductor. For any normal extension $N/\mathbb{Q}$, at each character $\varphi : \mathrm{Gal}\big(N(\mu_{p^\infty})/N\big) \to \mathbb{C}^{\times}$ one defines

$$\mathfrak{M}_p(N,\varphi) \ := \ \prod_{\rho} \big(\varepsilon\text{-factor of } \rho\otimes\varphi\big)^{\mathfrak{m}(\rho)}$$

where the product ranges over all the irreducible representations ρ of the group $\mathrm{Gal}(N/\mathbb{Q})$, and $\mathfrak{m}(\rho)$ counts the total number of copies of ρ inside the regular representation.

Theorem 1. *There exists a bounded measure* $\mathrm{d}\mu_E^{(p)}$ *defined on the p-adic Lie group* $\mathrm{Gal}\big(F\cdot K(\mu_{p^\infty})/F\cdot K\big) \cong \mathbb{Z}_p^{\times}$*, interpolating the algebraic L-values*

$$\int_{x\in\mathbb{Z}_p^{\times}} \varphi(x)\cdot \mathrm{d}\mu_E^{(p)}(x) \ = \ \mathfrak{M}_p\big(F\cdot K,\varphi\big) \ \times \ \frac{L\big(E/F\cdot K,\ \varphi^{-1},\ 1\big)}{\big(\Omega_E^{+}\Omega_E^{-}\big)^{[F\cdot K:\mathbb{Q}]/2}}$$

at almost all finite order characters $\varphi \neq \mathbf{1}$*, while* $\int_{x\in\mathbb{Z}_p^{\times}} \mathrm{d}\mu_E^{(p)}(x) = 0$ *when* $\varphi = \mathbf{1}$ *is trivial (here the transcendental numbers* $\Omega_E^{\pm}$ *denote real and imaginary Néron periods for* $E_{/\mathbb{Z}}$*).*

To prove this result, we simply take a convolution of the measures $\mathrm{d}\mu_{\mathbf{f}_E\otimes\mathbf{g}_{\chi}^{(\psi)}}$ over the irreducible representations $\rho_{\chi,k}^{(\psi)}$ counted with multiplicity $[k:\mathbb{Q}]$, together with a convolution of ψ-twists of the p-adic Dabrowski [**2**] measure $\mathrm{d}\mu_{(\mathbf{f}_{E/F})\otimes\psi}$ for each (tame) character ψ of $\mathrm{Gal}(K\cap\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. After scaling by an appropriate ratio of automorphic periods $\prod\langle \mathbf{f}_E,\mathbf{f}_E\rangle$ to Néron periods $\Omega_E^{\pm}$, one duly obtains $\mathrm{d}\mu_E^{(p)}$ above.

At almost all finite twists by φ the Euler factor at p is trivial, so Theorem 1 now follows. For the full details we refer the reader to [**3**, Sections 5 and 6] where a proof is given for the number field $K = \mathbb{Q}(\mu_q, m^{1/q})$ with $q \neq p$; the argument is identical in the general case.

Definition 1. *For every $s \in \mathbb{Z}_p$, the p-adic L-function is given by the Mazur-Mellin transform*

$$\mathbf{L}_p\big(E/F \cdot K, s\big) \;:=\; \int_{x \in \mathbb{Z}_p^\times} \exp\big((s-1)\log_p x\big) \cdot \mathrm{d}\mu_E^{(p)} \;.$$

Since $\mathrm{d}\mu_E^{(p)}\big(\mathbb{Z}_p^\times\big) = 0$, it follows that $\mathbf{L}_p\big(E/F \cdot K, s\big)$ must vanish at the critical point $s = 1$. The p-adic Birch and Swinnerton-Dyer Conjecture then predicts

$$\mathrm{order}_{s=1}\Big(\mathbf{L}_p\big(E/F \cdot K, s\big)\Big) \;\stackrel{?}{=}\; \mathbf{e}_p(E/F \cdot K) + \dim_{\mathbb{R}}\Big(E(F \cdot K) \otimes \mathbb{R}\Big)$$

where $\mathbf{e}_p(E/F \cdot K)$ equals the number of places of $F \cdot K$ lying over p. Though its proof is beyond the range of current methods, one might hope instead to establish the weaker consequence

$$\mathrm{order}_{s=1}\Big(\mathbf{L}_p\big(E/F \cdot K, s\big)\Big) \;\geq\; \mathbf{e}_p(E/F \cdot K). \tag{1}$$

2. The Order of Vanishing at $s = 1$

Let $d \geq 1$ be an integer. We now restrict ourselves to studying the d-fold Kummer extension

$$K = \mathbb{Q}\Big(\mu_{q^n}, \Delta_1^{1/q^n}, \ldots, \Delta_d^{1/q^n}\Big) \quad \text{with } p \nmid \Delta_1 \times \cdots \times \Delta_d,$$

where $q \neq p$ is an odd prime, and the Δ_i's are pairwise coprime q-power free positive integers. Here $K_{\mathrm{ab}} := K \cap \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\mu_{q^n})$, and in our previous notation

$$\Gamma \cong (\mathbb{Z}/q^n\mathbb{Z})^\times \quad \text{and} \quad \mathcal{H} = \mathrm{Gal}\big(K/\mathbb{Q}(\mu_{q^n})\big) \cong (\mathbb{Z}/q^n\mathbb{Z})^{\oplus d}.$$

Note that the full Galois group is the semidirect product $\mathrm{Gal}(K/\mathbb{Q}) = \Gamma \ltimes \mathcal{H}$, where Γ acts on $\mathcal{H}$ through the cyclotomic character.

Recall that F was a totally real field disjoint from K over which the curve E is semistable. We now assume that p is inert in $F \cdot K_{\mathrm{ab}}^+$ and write $\mathfrak{p}^+$ to denote the prime ideal $p \cdot \mathcal{O}_{F \cdot K_{\mathrm{ab}}^+}$. In particular, conditions **(H1)-(H3)** hold. The strategy is to employ the factorisation

$$\mathbf{L}_p\big(E/F{\cdot}K, s\big) \;=\; \mathbf{L}_p\big(E/F{\cdot}K_{\mathrm{ab}}^+, s\big) \times \mathbf{L}_p\big(E{\otimes}\theta/F{\cdot}K_{\mathrm{ab}}^+, s\big) \times \prod_{\dim(\rho)>1} \mathbf{L}_p\big(E/F, \rho, s\big)^{\mathfrak{m}(\rho)} \tag{2}$$

where θ is the quadratic character of the field K_{ab} over its totally real subfield $K_{\mathrm{ab}}^+ = \mathbb{Q}(\mu_{q^n})^+$, and the product ranges over the irreducible $\mathrm{Gal}\big(F \cdot K/F\big)$-representations ρ of dimension ≥ 2 (we refer the reader to [**13**, Chapter 8] and [**5**, Section 2] for more details on the structure of these $(\mathbb{Z}/q^n\mathbb{Z})^\times \ltimes (\mathbb{Z}/q^n\mathbb{Z})^{\oplus d}$-representations).

Case I - The prime $\mathfrak{p}^+$ is inert in $F \cdot K_{\mathrm{ab}}\big/F \cdot K_{\mathrm{ab}}^+$:

Let $n_t := \mathrm{ord}_q \frac{\big[\mathbb{Q}\big(\mu_{q^n},\, \Delta_t^{1/q^n}\big) : \mathbb{Q}(\mu_{q^n})\big]}{\big[\mathbb{Q}_p\big(\mu_{q^n},\, \Delta_t^{1/q^n}\big) : \mathbb{Q}_p(\mu_{q^n})\big]}$, so that $\prod_{t=1}^d q^{n_t}$ is the number of places of K above p. The Artin representations $\rho_{\chi,k}^{(\psi)}$ that produce an exceptional zero in

$h^1(E_{/F}) \otimes \rho_{\chi,k}^{(\psi)}$ at p are precisely those where $\psi = \mathbf{1}$ and the character χ factors through the quotient group

$$\mathcal{H}^\dagger \;=\; \frac{\mathcal{H}}{\bigoplus_{t=1}^d \frac{q^{n_t}\mathbb{Z}}{q^n\mathbb{Z}}}\,.$$

Moreover $\mathfrak{m}\big(\rho_{\chi,k}^{(\mathbf{1})}\big) = \dim\big(\rho_{\chi,k}^{(\mathbf{1})}\big) = \phi\big([\mathcal{H}^\dagger : \mathrm{Ker}(\chi)]\big)$, which equals the number of generators for the image of χ; therefore

$$\sum_{\substack{\dim(\rho)>1,\\ h^1(E)\otimes\rho \text{ exc'l}}} \mathfrak{m}(\rho)\cdot\mathrm{order}_{s=1}\Big(\mathbf{L}_p\big(E,\rho,s\big)\Big) \ge \sum_{\substack{\dim(\rho_{\chi,k}^{(\mathbf{1})})>1,\\ \chi:\mathcal{H}^\dagger\to\mathbb{C}^\times}} \mathfrak{m}\big(\rho_{\chi,k}^{(\mathbf{1})}\big) = \sum_{r=1}^n \#\big\{\chi : \mathcal{H}^\dagger \twoheadrightarrow \mu_{q^r}\big\} = \#\mathcal{H}^\dagger - 1.$$

We must also include the order of $\mathbf{L}_p\big(E/F\cdot K_{\mathrm{ab}}^+, s\big)$ at $s=1$ which is at least one, hence

$$\mathrm{order}_{s=1}\Big(\mathbf{L}_p\big(E/F\cdot K, s\big)\Big) \;\ge\; 1 + \big(\#\mathcal{H}^\dagger - 1\big) \;=\; \prod_{t=1}^d q^{n_t}.$$

Case II - The prime $\mathfrak{p}^+$ splits in $F\cdot K_{\mathrm{ab}}\big/F\cdot K_{\mathrm{ab}}^+$:
There are $2\times\prod_{t=1}^d q^{n_t}$ places of K above p. The rest of the calculation is the same as Case I except that both of $\mathbf{L}_p\big(E/F\cdot K_{\mathrm{ab}}^+, s\big)$ and $\mathbf{L}_p\big(E\otimes\theta/F\cdot K_{\mathrm{ab}}^+, s\big)$ have trivial zeroes at $s=1$, whilst $\mathrm{order}_{s=1}\big(\mathbf{L}_p(E/F,\rho,s)\big) \ge 2$ by [**3**, Thm 6.3]. Consequently we obtain the lower bound

$$\mathrm{order}_{s=1}\Big(\mathbf{L}_p\big(E/F\cdot K, s\big)\Big) \;\ge\; 1+1+2\times\big(\#\mathcal{H}^\dagger - 1\big) \;=\; 2\times\prod_{t=1}^d q^{n_t}.$$

Combining both cases together, we have shown

Theorem 2. *If p is inert in $F(\mu_{q^n})^+$, then*

$$\mathrm{order}_{s=1}\big(\mathbf{L}_p(E/F\cdot K, s)\big) \ge\; \mathbf{e}_p(E/F\cdot K).$$

In other words, the inequality in Equation (1) holds true for these number fields.

3. A Higher Derivative Formula

Henceforth we shall assume that $p\ge 5$ is inert in K_{ab}, corresponding to Case I mentioned on the previous page; this condition is equivalent to ensuring that p is a primitive root modulo q^2. Let us write $\mathcal{E}_p(X)\in\mathbb{Z}[X]$ for the characteristic polynomial of a geometric Frobenius element at p, acting on the regular representation of $\mathrm{Gal}(F\cdot K/\mathbb{Q})$, such that the highest power of $X-1$ has already been divided out of the polynomial (it is tautologically non-zero at $X=1$).

Theorem 3. *If $p\ge 5$ is inert in $F(\mu_{q^n})$, then*

$$\frac{1}{\mathbf{e}_p!}\cdot\left.\frac{\mathrm{d}^{\mathbf{e}_p}\mathbf{L}_p\big(E/F\cdot K, s\big)}{\mathrm{d}s^{\mathbf{e}_p}}\right|_{s=1} \;=\; \mathcal{L}_p(E)\times\,\mathcal{E}_p(1)\times\frac{\sqrt{\mathrm{disc}(F\cdot K)}\cdot L\big(E/F\cdot K,1\big)}{\big(\Omega_E^+\Omega_E^-\big)^{[F\cdot K:\mathbb{Q}]/2}}. \tag{3}$$

*where $\mathcal{L}_p(E) := \prod_{\wp|p}\frac{\log_p(q_{E,\wp})}{\mathrm{ord}_\wp(q_{E,\wp})}$ denotes Jones' $\mathcal{L}$-invariant [**8**], with the product taken over the primes of $F\cdot K$ lying above p.*

The proof follows identical lines to the $d = 1$ situation in [**3**, Section 6] – more precisely:

- the special values $\mathbf{L}_p\big(E\otimes\theta/F\cdot K_{\rm ab}^{+}, 1\big)$ and $\mathbf{L}_p\big(E/F, \rho, 1\big)$ at the non-exceptional ρ's can be computed directly from their interpolation properties;
- the derivative $\mathbf{L}'_p\big(E/F\cdot K_{\rm ab}^{+}, 1\big)$ is given by Mok's formula [**10**, Thm 1.1] since $p \geq 5$;
- the derivatives $\mathbf{L}'_p\big(E/F, \rho, 1\big)$ at those exceptional ρ's are calculated using [**3**, Thm 6.2].

Lastly the terms can then be multiplied together as in Equation (2), and the result follows. Needless to say, the hard work is contained in [**3**, Thm 6.2] and requires us to extend the deformation theory approach of Greenberg and Stevens to ρ-twisted Hasse-Weil L-functions. The main ingredient is the construction of an "improved" p-adic L-function à la [**6**, Prop 5.8] (a conjectural p-adic interpolation rule for such an object can be found in [**4**, §§4.4]).

In fact Jones' $\mathcal{L}$-invariant is non-vanishing by [**1**] as the elliptic curve E is defined over $\mathbb{Q}$. Therefore if one considers Theorems 2 and 3 in tandem, one immediately obtains the

Corollary 1. *If the prime $p \geq 5$ is inert in $F(\mu_{q^n})$, then*

$$L(E/F\cdot K, 1) \neq 0 \quad \textit{if and only if} \quad \mathrm{order}_{s=1}\big(\mathbf{L}_p(E/F\cdot K, s)\big) = \mathbf{e}_p(E/F\cdot K).$$

More generally, one can replace the requirement that "E be an elliptic curve defined over $\mathbb{Q}$" with the statement that

> "$\mathbf{f}$ is a primitive HMF over F of parallel weight 2, that is Steinberg at the primes $\mathfrak{p}\big|p$"

and everything works fine, except that there is no longer a nice description for the $\mathcal{L}$-invariant. Likewise one can accommodate weight two Hilbert modular forms with non-trivial nebentypus, providing the primes above p do not divide its conductor.

Of particular interest in non-commutative Iwasawa theory is to extend Theorems 2 and 3 to the situation where $q = p$, i.e. for the p-ramified extensions $F\big(\mu_{p^n}, \Delta_1^{1/p^n}, \ldots, \Delta_d^{1/p^n}\big)/\mathbb{Q}$. The obstacles appear to be technical rather than conceptual, and a higher derivative formula should certainly be possible in this context (work in progress of Antonio Lei and the author).

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