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WAITING TO PERSUADE

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Abstract. I analyze a sequential bargaining model in which players are optimistic about their bargaining power (measured as the probability of making offers), but learn as they play the game. I show that there exists a uniquely predetermined settlement date, such that in equilibrium the players always reach an agreement at that date, but never reach one before it. Given any discount rate, if the learning is sufficiently slow, the players agree immediately. I show that, for any speed of learning, the agreement is delayed arbitrarily long, provided that the players are sufficiently patient. Therefore, although excessive optimism alone cannot cause delay, it can cause long delays if the players are expected to learn.

KEYWORDS: Bargaining, Misperception, Optimism, Delay, Learning

1. Introduction

Bargaining delays are common, and frequently cause substantial losses to the bargaining parties. Often, agreements in labor negotiations are reached only after strikes or work slowdowns, and sometimes international conflicts last generations, costing lives and causing lifelong misery. (This might happen while the parties are officially negotiating a peace agreement, as in the case of Israeli-Palestinian conflict.) The usual game-theoretical explanation for these delays is based on asymmetric information: delay is a credible means for a player to communicate his private information that he has a strong position in bargaining, or a screening device to understand whether the other party is in a strong or weak position in bargaining (see Admati and Perry (1987) and Kennan and Wilson (1993)).

There is, however, a sense among researchers that agreement may be delayed even when the parties do not seem to have any asymmetric information about the payoffs. As an alternative cause of bargaining delays, many authors have proposed excessive

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optimism due to the lack of a common prior.[1] Based on surveys and experimental and field data, they have concluded that optimism is very common, and have attributed the bargaining delays to excessive optimism. Most of these authors do not have any formal model, but their arguments appear to be based on the two-period negotiation model by Landes (1971) and Posner (1972). Their reasoning seems to be the following. When each party is excessively optimistic about the share he would get tomorrow, there may not exist any settlement today that satisfies all parties' expectations. In another paper (Yildiz (2002)), I have shown that this argument relies critically on the artificial assumption that there are only two-periods; in a long horizon model there will be an immediate agreement whenever optimism is sufficiently persistent. The reason is that, if optimism is persistent, then the scope of trade is necessarily small, and thus the players cannot be very optimistic about their share in any agreement in the near future.

This paper provides a new rationale for delay when the parties are optimistic due to the lack of a common prior. Now there is no private information to convey; a player i simply believes that he has a strong position in bargaining, a belief the other player j does not share. Being a Bayesian, i must also believe that the events are likely to proceed in such a way that i will eventually be proven to be right. In that case, j will plausibly be convinced that i is right and thereby be persuaded to agree to i's terms. If j's initial beliefs are not too firm, this will happen so soon that i will find it worth waiting to persuade j. Of course, at the beginning, j does not believe that the events will proceed in that way; she probably thinks that i will be persuaded to agree to her terms in the near future. This leads to costly delays that are inefficient even under these optimistic beliefs.

As a formal model, I use the basic model of Yildiz (2002) but focus on the case that the players' initial beliefs are not too firm, allowing them to update their beliefs without restriction. (Yildiz (2002) focuses on the case that the players do not change their beliefs much as they play the game.) Using a canonical model of learning, I show that there exists a (unique) predetermined date t^* such that in equilibrium the players will never agree before t^* and reach an agreement at t^*. (Moreover, they would also have agreed at any date after t^*, had they not agreed before.) Notice that the settlement date t^* is common knowledge at the beginning and does not depend on what happens until then. This is surprising, because delay in usual bargaining models—whether caused by signaling, screening, or mixed strategies—is only a pos-

[1] See Hicks (1932), Landes (1971), Posner (1972), Gould (1973), Priest and Klein (1984), Neale and Bazerman (1985), Babcock et al (1995), Babcock and Loewenstein (1997). See also Farber and Bazerman (1989), who show that excessive optimism cannot explain the delay patterns in certain labor negotiations with conventional and final offer arbitration. Some other terms, such as overconfidence and self-serving biases, are also used for what can be called optimism in the present context.

sibility and there is immediate agreement with positive probability. Finding very tight bounds for the settlement time t^*, I further show that delay can be arbitrarily long as long as the players are sufficiently patient. This is true for any initial level of optimism and the firmness of beliefs. Therefore, although optimism alone cannot cause any delay, it can cause delays when it is combined with learning.

The intuition is as above. As is typical in Bayesian learning models, each player i updates his beliefs relatively quickly at the beginning of the process. When his bargaining partner j is patient enough, this entices j to wait so that i will observe the truth and hopefully agree to j's terms. After a while, having gained experience through observing some of the data, the players' learning will slow down, and it will no longer be worth waiting for them to change their minds. This is when they reach an agreement. Of course, in the mean time, as the players observe the same data, their beliefs become more similar and eventually optimism becomes negligible. Nevertheless, the upper bound for t^* implies that the players reach an agreement when the learning slows down—and much before optimism becomes negligible.

In the next section, I lay out the model and develop the main concepts. The main results are presented in Section 3. Section 4 extend these results to a continuous-time model, and Section 5 concludes. Most of the proofs are presented in a technical appendix, where I develop the notions and the tools that are necessary for these proofs.

2. Model

Take the set of all non-negative integers $T = \{0, 1, 2, \ldots\}$ as the time space. Take also $N = \{1, 2\}$ to be the set of players, and $U = \{u \in [0,1]^2 | u^1 + u^2 \leq 1\}$ to be the set of all feasible expected utility pairs. Designate dates $t, s \in T$ and players $i \neq j \in N$ as generic members.

I will analyze the following perfect-information game. At each $t \in T$, Nature recognizes a player $i \in N$; i offers a utility pair $u = (u^1, u^2) \in U$; if the other player accepts the offer, then the game ends, yielding a payoff vector $\delta^t u = (\delta^t u^1, \delta^t u^2)$ for some $\delta \in (0, 1)$; otherwise, the game proceeds to date $t+1$. If the players never agree, each gets 0. I assume that the players' beliefs have beta distributions, a tractable distribution that is widely used in statistical learning models. Fixing any positive integers $\bar{m}_1$, $\bar{m}_2$, and n with $1 \leq \bar{m}_2 < \bar{m}_1 \leq n-2$, I assume that, for any given dates t and s with $s \geq t$, at the beginning of date t, if a player i observes that player 1 has made m offers (and player 2 has made $t - m$ offers), then he assigns probability

$$\frac{\bar{m}_i + m}{t + n} \tag{1}$$

to the event that player 1 will make an offer at date s. This belief structure arises when each player believes that recognition at different dates are identically and independently distributed with some unknown parameter μ measuring the probability of player 1 making an offer at any date t, and μ is distributed with a beta distribution with parameters $\bar{m}_i$ and n. I assume that everything described in this paragraph is common knowledge.

As in Yildiz (2002), this model differs from the Rubinstein-Stahl framework by allowing different probability distributions for different players. This difference in beliefs about the recognition process can be taken as the difference in beliefs about each player's bargaining power. This is because in sequential bargaining models, including the present one, a player's bargaining power is ultimately determined by the recognition process, as the following two results suggest. First, Lemma 5 below establishes that a player's equilibrium payoff is the present value of all rents he expects to extract when he makes offers in the future. Second, under the assumptions of this paper, a player i becomes better off in equilibrium whenever each player comes to believe that i has a higher probability of recognition in the future.

Measuring optimism. Towards measuring optimism, write $\Delta = \bar{m}_1 - \bar{m}_2$. While n measures the firmness of the players' prior beliefs, Δ/n will be shown to measure the initial level of optimism. Notice that the beliefs about s at t depend only on t —not s. Hence, optimism will be measured at the time the beliefs are held without distinguishing which future recognition these beliefs are about. Write (m, t) for the history (at the beginning of date t) in which player 1 has made m offers and player 2 has made $t - m$ offers. Write $p_t^i(m)$ for the probability player i assigns at (m, t) to the event that he will be recognized at any fixed date $s \geq t$. By (2) below, $p_t^1(m) + p_t^2(m) > 1$, and hence each player thinks at (m, t) that the probability that he will be recognized at date s is higher than what the other player assesses. As explained above, this means that they are *optimistic at* (m, t). Write

$$y_t(m) = p_t^1(m) + p_t^2(m) - 1$$

for *the level of optimism at* (m, t). By (1),

$$y_t(m) = \frac{\bar{m}_1 - \bar{m}_2}{t + n} \equiv \frac{\Delta}{t + n} > 0. \tag{2}$$

Note that y_t is *deterministic*, i.e., y_t does not depend on m; so m will be suppressed. This determinism is due to the assumption that the players' beliefs are equally firm, i.e., n is same for both players. This will simplify the analysis dramatically.

Negligible levels of optimism. Let us say that a level y_t of optimism is negligible if and only if y_t can never cause a disagreement at $t - 1$. Now, the best a player can expect at $t - 1$ from the future is to take the whole dollar at t if he is recognized at t. Hence, he can expects at most p_t^i from the future. Thus, the players must agree at $t - 1$ whenever $\delta(p_t^1 + p_t^2) = \delta(1 + y_t) \leq 1$, i.e., whenever $y_t \leq (1 - \delta)/\delta$.

Moreover, if the game were expected to end at t, the recognized player would take the whole dollar at t. In that case, there would be disagreement at $t-1$ whenever $y_t > (1-\delta)/\delta$. Therefore, I will say that *optimism is negligible* at t if and only if $y_t \leq (1-\delta)/\delta$. Optimism becomes negligible in this sense at

$$t_0 \equiv \frac{\delta}{1-\delta}\Delta - n,$$

and remains negligible thereafter.

3. Agreement and Delay

In this section, I show that there exists a predetermined t^* such that the players will never agree before t^*, and agree at t^* (and thereafter if they had not yet agreed). Finding very tight bounds for the settlement date t^*, I show that (i) the agreement can be delayed arbitrarily long, provided that the players are sufficiently patient, and (ii) the agreement is reached when learning slows down, and much before optimism becomes negligible (i.e., $t^* < \sqrt{t_0}$).

Towards this end, I first present two agreement results in the spirit of Yildiz (2002), who proves similar results under the restrictive assumption that the players do not learn. The first result states that there will be immediate agreement if the optimism is expected to drop slowly.

Theorem 1. *For any t with $y_t - y_{t+1} \leq (1-\delta)/\delta$, there is an agreement regime at $t-1$.*

Proof. Most proofs are in the Appendix. ■

That is, in equilibrium there is an immediate agreement as long as it is known that the level of optimism will not drop dramatically in the near future, i.e., the learning will be slow. Theorem 1 implies that the players will agree immediately whenever $y_1 - y_2 \leq (1-\delta)/\delta$. As an immediate corollary, this further implies that if the players' beliefs are sufficiently firm, they will reach an agreement immediately — independent of the initial level of optimism, extending another agreement result in Yildiz (2002):

Corollary 1. *Let $\Delta = ny_o$ so that the initial level of optimism remains constant at y_o. Then, there exists some integer $\bar{n}$ such that the players reach an agreement immediately whenever $n \geq \bar{n}$.*

Proof. By (2), $y_1 - y_2 = \frac{y_o n}{(n+1)(n+2)}$, which converges to zeros a $n \to \infty$. Therefore, there exists some integer $\bar{n}$ such that, whenever $n \geq \bar{n}$, $y_1 - y_2 \leq (1-\delta)/\delta$, yielding an immediate agreement by Theorem 1. ■

The main focus of the present paper is on the case that the players' initial beliefs are not firm, and hence they update their beliefs substantially as they observe how players are recognized. In that case, the players may delay the agreement for a while, as the next theorem will imply.

Theorem 2. *There exists a $t^* \in T$ such that, at each $t \geq t^*$, the players reach an agreement immediately if they have not reached an agreement yet, and they do not reach an agreement before t^*.*

Theorem 2 establishes that there exists a uniquely predetermined settlement date t^*. In a moment I will also provide bounds for t^* and show that t^* can be arbitrarily large when players are sufficiently patient. Hence, Theorem 2 implies that, unless the players' initial beliefs are very firm, agreement will be delayed for a while. This is because, typically, at the beginning of a learning process players are more open to new information, in the sense that they update their beliefs substantially as they observe which player gets a chance to make an offer. Knowing this, each player waits, believing that the events are very likely to proceed in such a way that his opponent will change his mind. As time passes, they become experienced. In this way, two things occur simultaneously, both facilitating agreement. Firstly, having similar experiences, the discrepancy in their beliefs diminishes. More importantly, each player i becomes so closed minded that his opponent j loses her hope to convince i and thus becomes more willing to agree to i's terms. Therefore, after a while, they reach an agreement. It will be clear that in this process, the latter effect leads the players to an agreement much before the former effect starts playing a role, i.e., $y_t - y_{t+1}$ becomes smaller than $(1-\delta)/\delta$ much before y_t does (see the discussion after Lemma 1).

At the beginning of the game it is common knowledge that they will not reach an agreement until t^*, when they will reach an agreement no matter what happens up to that point.[2] How they will share the pie at t^* will depend on how many times each player will have been recognized. Since they disagree about how many times each player is likely to be recognized by t^*, there is no consensus among the players on *how* they can better each of them by agreeing on a decision at the beginning. Therefore, they wait until t^* even though there *is* a consensus among them that there is an agreement at the beginning that would leave each player better off.

How long will they delay the agreement? To answer this question, the next result provides tight bounds for t^*.

Lemma 1. *The settlement time t^* satisfies*

$$\max\{0, t_l\} \leq t^* \leq \max\{0, t_u\} \tag{3}$$

[2] In contrast, typically, delay is only a possibility in models with asymmetric information, such as Admati and Perry (1987).

where

$$t_u = \frac{\sqrt{1 + \frac{4\delta\Delta}{1-\delta}} - 1}{2} - n \tag{4}$$

and t_l is the highest integer t such that $s = t + n$ satisfies the cubic inequality

$$f \equiv (1-\delta)\, s\, (s+1)\, (s+2) - 2\, (s+1)\, \delta\Delta + \delta s \delta\Delta + (\delta\Delta)^2 \leq 0. \tag{5}$$

Lemma 1 provides tight bounds for delay. These bounds have two important implications for this paper. Firstly, the upper bound t_u implies that the players settle much before $t_0 = \frac{\delta\Delta}{1-\delta} - n$, when optimism becomes negligible. This is stated in the next result.

Theorem 3. *The settlement time t^* satisfies*

$$t^* \leq \max\{0, t_u\} < \sqrt{t_0}$$

whenever $t_0 > 0$.

Proof. If $t_u < 0$, then the inequality is trivially true. Assume that $t_u \geq 0$. By definition, $t_u + n = \sqrt{t_0 + n + 1/4} - 1/2 < \sqrt{t_0 + n}$. Hence, $(t_u + n)^2 < t_0 + n$, yielding $t_u^2 < t_0 - (2t_u + n - 1)\, n < t_0$. ■

That is, the agreement is reached when the learning slows down, not when optimism becomes negligible. This observation is also supported by the fact that there is immediate agreement when optimism remains always high (cf. Section 4). Therefore, in reaching an agreement, considerations about learning seem to be more important than optimism itself.

Second, as the players become very patient, the lower bound goes to ∞, yielding arbitrarily long delays:

Theorem 4. *For all (t, n, Δ), there exists $\bar{\delta} \in (0,1)$ such that $t^* \geq t$ whenever $\delta > \bar{\delta}$.*

Proof. Given any (t, n, Δ), since $n \geq \Delta$, we have

$$\lim_{\delta \to 1} f = -\Delta\, (t + n + 2 - \Delta) < 0,$$

where f is as defined in (5). Then, there exists $\bar{\delta} \in (0,1)$ such that, whenever $\delta > \bar{\delta}$, we have $f < 0$, and thus, by Lemma 1, $t \leq t_l \leq t^*$. ■

Intuitively, as the players become patient, the efficiency loss due to delay becomes negligible, while each player's individual gain from proving his bargaining power remains substantial, enticing the players to wait arbitrarily long. To see this consider the limiting case that $y \equiv 0$. In that case, the per-period efficiency loss due to delay is $1-\delta$, approaching 0 as $\delta \to 1$. On the other hand, by (9) in the Appendix, the continuation value of a player i at any (m,t) is $p_t^i(m)$, and hence any increase in $p_t^i(m)$ is translated to the equilibrium payoff of i, without vanishing as $\delta \to 1$.

As the players become very patient, although delay becomes arbitrarily long, the efficiency loss due to delay becomes negligible—as Theorem 3 implies:

Corollary 2. *For all n and Δ, $\lim_{\delta \to 1} \delta^{t^*} = 1$, where t^* is as defined in Theorem 2.*

Proof. By (4), as $\delta \to 1$, $\log \delta^{\sqrt{t_0}} \cong \sqrt{\Delta/(1-\delta)} \log \delta \to 0$. Hence, by Theorem 3, $1 \geq \lim_{\delta \to 1} \delta^{t^*} \geq \lim_{\delta \to 1} \delta^{\sqrt{t_0}} = 1$. ■

This corollary is due to the fact that when the players are patient, it costs arbitrarily little to wait until $t_u \cong \sqrt{t_0}$, when learning slows down. In contrast, for patient players, the cost of waiting until t_0 (when optimism becomes negligible) is bounded away from zero.

4. Delay in a continuous-time limit

In this section, I take a continuum of real times τ as the primitive, and approximate it with a grid of index-times t. The players' time preferences and the level of optimism are given by the real time, and do not depend on the grid. Using Lemma 1, I find bounds for the real-time limit τ^* of the settlement date t^*as the grid approaches continuum. I show that the results in the previous section extend to this model: although there is immediate agreement when the optimism is very persistent or instantaneously vanishing, there is delay in between. As the players become sufficiently patient, the real-time delay becomes arbitrarily long.

Taking a continuum of real times τ, let the level of optimism at τ be

$$y(\tau) = \frac{y_o}{1+\tau/\pi} \tag{6}$$

where y_o is the initial level of optimism and $\pi > 0$ is a parameter measuring the persistence of optimism. Given any $\tau > 0$, $y(\tau)$ decreases to zero as π approaches 0, and increases to y_o as $\pi \to \infty$. Each player's utility from getting x at τ is $e^{-r\tau}x$ where $r > 0$ is the real-time impatience. Now consider a grid of index times t where each index t corresponds to a real time $\tau(t,k) = t/k$, and $k > 0$ measures the fineness of the grid. The discount rate is $\delta(k) = e^{-r/k}$. Take also $n = \pi k$ and $\Delta = y_o n$ so that the level of optimism at a given real time $\tau(t,k)$ is $\Delta/(n+t) = y_o/(1+\tau(t,k)/\pi) = y(\tau(t,k))$ as in (6). Given any k, let $t^*(k)$ be the settlement time, defined in Theorem 2 for the parameters $\delta = e^{-r/k}$, $n = \pi k$ and $\Delta = y_o n$. Write also τ^* for the limit of $\tau(t^*(k), k)$ as $k \to \infty$. Building on Lemma 1, the next theorem provides bounds for the real-time delay as the discrete-time grid approaches the continuum.

Theorem 5. *In the model of this section, the settlement time* $\tau^* \equiv \lim_{k\to\infty} \tau(t^*(k), k)$ *in the continuous-time limit satisfies*

$$\tau^* \leq \tau_u \equiv \max\left\{\sqrt{\frac{\pi y_o}{r}} - \pi, 0\right\}.$$

Moreover, if $y_o \pi r < 4/27$, *then*

$$\tau^* \geq \sqrt{\frac{\pi y_o}{3r}} - \pi.$$

Finally, given any y_o *and any* π, $\tau^* \to \infty$ *as* $r \to 0$.

Firstly, consistent with the agreement results, the upper bound τ_u implies that there is immediate agreement (i.e., $\tau^* = 0$) whenever $\pi \geq y_o/r$, i.e., when optimism is very persistent. When $0 < \pi < y_o/r$, there may be delay, although delay must become arbitrarily short as π approaches 0. The lower bound implies that there will be delay whenever $0 < \pi < y_o/(3r)$.

Second, both of the upper and the lower bounds for delay are weakly increasing in y_o/r. This is intuitive because y_o scales the speed of learning as well as the level of optimism, while r measures the players' impatience. The settlement time is determined by when the learning slows down in terms of the players' patience. As r decreases, the players become patient, increasing the length of delay. As r approaches 0, the discount rate approaches 1, and τ^* goes to infinity for any $\pi y_o > 0$, extending Theorem 4 to the present setup. Therefore, in the continuous-time limit, there will be very long real-time delays if the players are patient, optimistic, *and* can learn about their bargaining power in the process of bargaining.

5. Conclusion

This paper presents a new rationale for bargaining delays based on optimism and learning. It observes that when two optimistic, Bayesian players negotiate, each player i believes that the events are likely to proceed in such a way that i will eventually be proven right. If the other player j's initial beliefs are not too firm, this will entice i to wait in the hopes that j will quicky learn that i is right and thereby be persuaded to agree to i's terms. This yields costly delays that may be arbitrarily long and are inefficient even under these optimistic beliefs. In this reasoning the considerations about learning seem to be more salient than optimism itself. In fact, the players settle when the players' learning slows down, and much before optimism becomes negligible. Moreover, they will settle immediately whenever the level of optimism is expected to remain high for a long while. In conclusion, although excessive optimism alone cannot cause delays, it can cause long delays when the players are expected to learn in the future.

A. Appendix: A more technical exposition

Notation. Designate a utility pair $u = (u^1, u^2) \in U$ as a generic member, and write $\rho = (\rho_t)_{t \in T}$ for the recognition process. Write $P^i(\cdot|m,t)$ for the probability assessment of a player i at any history (m,t) and $E^i(\cdot|m,t)$ for the corresponding expectation operator.

A.1. Preliminary Results.

This section contains certain results that are necessary for the main results in the text. Here, I describe the subgame-perfect equilibria (henceforth, simply equilibria), and I find simple expressions for the equilibrium payoffs. The first result is taken from Yildiz (2002):

Lemma 2. *Given any (m,t,i), there exists a unique $V_t^i(m) \in [0,1]$ such that, at any subgame-perfect equilibrium, the continuation value of i at (m,t) is $V_t^i(m)$.*

That is, there is a unique equilibrium payoff-vector. All equilibria yield the same outcome, so the trivial multiplicity of equilibria will be ignored. Towards describing the equilibrium behavior, write $S_t = V_t^1 + V_t^2$ for the perceived size of the pie at the beginning of date t. The next lemma simplifies the analysis substantially.

Lemma 3. *For each t, S_t is deterministic, i.e., $S_t(m) = S_t(m')$ for all m and m'.*

Proof. (*Sketch—see Yildiz (2001) for a complete but tedious proof.*) The infinite-horizon game here can be truncated at some $\bar{t}$, by assigning $(0,0)$ as the payoff vector at $\bar{t}$. Moreover, it can be seen from Lemma 5 below that, if S_s is deterministic for each $s > t$, so is S_t. By induction, S_t must be deterministic in the truncated game. Letting $\bar{t} \to \infty$, one obtains the lemma. ■

Equilibrium behavior and bargaining power. Assume that $\delta S_{t+1} \leq 1$. Now, if the players agree at t, then the total gain from trade is 1, while it is only δS_{t+1} if they delay the agreement to the next date. Hence, the recognized player i has all the bargaining power on realizing the gain of size $1-\delta S_{t+1}$ that they can get by not delaying the agreement. He uses this temporal monopoly power to extract $1 - \delta S_{t+1}$ as a rent. He gives the other player j her continuation value δV_{t+1}^j, and keeps the rent $1 - \delta S_{t+1}$ plus his continuation value δV_{t+1}^i for himself. His share sums up to $1 - \delta V_{t+1}^j$. When $\delta S_{t+1} > 1$, there cannot be any agreement at t that satisfies both players' expectations, hence they disagree at t. There is no rent in that case. I will say that *there is an agreement* (resp., *disagreement*) *regime at t* iff $\delta S_{t+1} \leq 1$ (resp., $\delta S_{t+1} > 1$).

Write

$$R_t = \max\{1 - \delta S_{t+1}, 0\}$$

for the rent at t and

$$\Lambda_t = \sum_{s=t}^{\infty} \delta^{s-t} R_s \tag{7}$$

for the present value of all future rents. Under this notation V satisfies the simple difference equation in the next lemma. This lemma immediately follows from a result in Yildiz (2002); a complete proof is provided, because the proof explains the equilibrium behavior in detail.

Lemma 4. *Given any* (m,t) *and* i,

$$V_t^i(m) = P^i(\rho_t = i|m,t)\, R_t + \delta E^i\left(V_{t+1}^i|m,t\right). \tag{8}$$

Proof. There are two cases. First consider the case that $\delta S_{t+1} > 1$, when $R_t = 0$. Assume that Player 1 is recognized. Now, both players are willing to agree on a division $u = (u^1, u^2)$ only if $u^1 \geq \delta V_{t+1}^1(m+1)$ and $u^2 \geq \delta V_{t+1}^2(m+1)$, requiring that $u^1 + u^2 \geq \delta V_{t+1}^1(m+1) + \delta V_{t+1}^2(m+1) = \delta S_{t+1} > 1$, an impossibility. Therefore the players cannot agree. Similarly, they cannot agree when Player 2 is recognized either. Hence, $V_t^i(m) = P^i(\rho_t = 1|m,t)\,\delta V_{t+1}^i(m+1) + P^i(\rho_t = 2|m,t)\,\delta V_{t+1}^i(m) = \delta E^i\left(V_{t+1}^i|m,t\right)$.

Now consider the case $\delta S_{t+1} \leq 1$. Assume that Player 1 is recognized. Player 2 accepts an offer u iff $u^2 \geq \delta V_{t+1}^2(m+1)$. Since $1 - \delta V_{t+1}^2(m+1) \geq \delta V_{t+1}^1(m+1)$, Player 1 now offers $\left(1 - \delta V_{t+1}^2(m+1), \delta V_{t+1}^2(m+1)\right)$, and the offer is accepted. Likewise, if Player 2 is recognized, he offers $\left(\delta V_{t+1}^1(m), 1 - \delta V_{t+1}^2(m)\right)$, which is accepted. The continuation value of Player 1 at (m,t) is

$$\begin{aligned} V_t^1(m) &= P^1(\rho_t = 1|m,t)\left(1 - \delta V_{t+1}^2(m+1)\right) + P^1(\rho_t = 2|m,t)\,\delta V_{t+1}^1(m) \\ &= P^1(\rho_t = 1|m,t)\,(1 - \delta S_{t+1}) + \delta E^1\left(V_{t+1}^1|m,t\right) \\ &= P^1(\rho_t = 1|m,t)\, R_t + \delta E^1\left(V_{t+1}^1|m,t\right). \end{aligned}$$

Similarly, $V_t^2(m) = P^2(\rho_t = 2|m,t)\, R_t + \delta E^2\left(V_{t+1}^2|m,t\right)$. ■

That is, the continuation value of a player i at the beginning of t is the rent he expects to extract at t, plus the present value of his continuation value at the beginning of the next date. The expectations are taken using his own beliefs. This leads to simple expressions for V and S in terms of Λ:

Lemma 5. *Given any* (m,t) *and* $i \in N$,

$$V_t^i(m) = p_t^i(m)\Lambda_t \tag{9}$$
$$S_t = (1 + y_t)\,\Lambda_t. \tag{10}$$

Proof. By the law of iterated expectations, the solution to the difference equation (8) is

$$V_t^i(m) = \sum_{s=t}^{\infty} \delta^{s-t} P^i(\rho_s = i|m,t)\, R_s = \sum_{s=t}^{\infty} \delta^{s-t} p_t^i(m)\, R_s = p_t^i(m)\Lambda_t,$$

where the second and the third equalities are due to the definitions of $p_t^i(m)$ and Λ_t, respectively. Summing up (9) over the players, one can obtain (10). (See Yildiz (2000) for details.) ■

Equation (9) establishes that a player's bargaining power is determined by the recognition process, justifying the modeling of optimism in this paper. It states that the continuation value of a player at any (m,t) is the present value of the rents he expects to extract when he is recognized in the future. Due to the specific belief structure assumed in (1), player i assigns the same probability $p_t^i(m)$ for his recognition at each date in the future. Hence this present value is simply the probability $p_t^i(m)$ times the present value Λ_t of all the future rents. This further implies (10), which states that the perceived size S_t of the pie at (m,t) is the present value Λ_t of the rents in the future, inflated by the optimism parameter $(1+y_t)$. Note that $V_t^i(m)$ depends on m, while S_t is deterministic.

Characterizing agreement. By (10), there is an agreement regime at any $t-1 \in T$ if and only if

$$\Lambda_t \le \frac{1}{\delta(1+y_t)} \equiv D_t. \tag{11}$$

Recall that there is an agreement regime at each $t \ge t_0-1$. Write $PA = \{t \in T | \Lambda_s \le D_s \forall s > t\}$ for the interval of the dates t such that there is an agreement regime at each $s \ge t$. As $t_0 - 1 \in PA$, PA is non-empty.

Bounds for equilibrium payoffs. Define

$$\bar{B}_t \equiv \frac{1}{1+\delta y_{t+1}} \quad \text{and} \quad \underline{B}_t \equiv 1 - \delta y_{t+1}\bar{B}_{t+1} = \frac{1-\delta(y_{t+1}-y_{t+2})}{1+\delta y_{t+2}}.$$

In the rest of this subsection I will show that $\underline{B}_t < \Lambda_t < \bar{B}_t$ at each $t \in PA$. This is the main technical step in this paper. The bounds are plotted in Figure 1. Notice that the bounds are very tight. This is generally true, as $\bar{B}_{t-1} < \underline{B}_t < \bar{B}_t$. Moreover, the bounds are valid only when $t \in PA$, because the proof is based on the following recursive equation, which holds only at agreement regimes.

Lemma 6. *For any t, if $\delta S_{t+1} \le 1$, then $\Lambda_t = 1 - \delta y_{t+1}\Lambda_{t+1}$.*

Proof. Assume $\delta S_{t+1} \le 1$. Then, $R_t = 1 - \delta S_{t+1}$. Hence (10) yields $R_t = 1 - \delta(1+y_{t+1})\Lambda_{t+1}$. Hence, by (7), $\Lambda_t = R_t + \delta\Lambda_{t+1} = 1 - \delta y_{t+1}\Lambda_{t+1}$. ■

Define the sequences B and C by $B_t = \bar{B}_{t-1}$ and $C_t = \frac{y_{t-1}}{y_t}\frac{1}{1+\delta y_{t-1}}$, respectively. Note that $B_t < \underline{B}_t < \bar{B}_t < C_t$ at each t. Using Lemma 6, one can easily prove the following lemma.

Lemma 7. *Given any $t \in PA$, and $b, c \in \mathbb{R}$, we have*

$$\Lambda_{t+1} = B_{t+1} - b \iff \Lambda_t = \bar{B}_t + \delta y_{t+1} b, \tag{12}$$
$$\Lambda_{t+1} = C_{t+1} + c \iff \Lambda_t = B_t - \delta y_{t+1} c. \tag{13}$$

By Lemma 7, $\Lambda_t < \bar{B}_t$ whenever $\Lambda_{t+1} > B_{t+1}$, and $\Lambda_t > B_t$ whenever $\Lambda_{t+1} < C_{t+1}$. Hence,

Lemma 8. *Given any $t \in PA$, if $B_{t+1} < \Lambda_{t+1} < C_{t+1}$, then $B_t < \Lambda_t < \bar{B}_t$.*

Lemma 9. *For any $t \in PA$, $B_t < \Lambda_t < C_t$.*

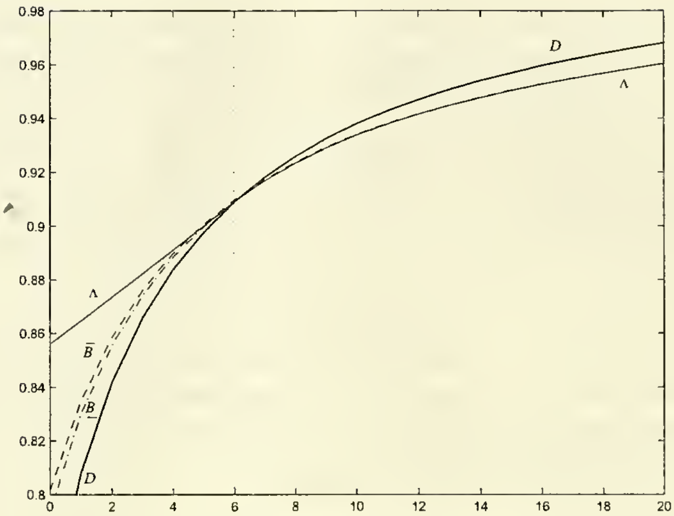

Figure 1: Functions D, $\underline{B}$, $\bar{B}$, and Λ ($\delta = 0.99$, $n = 3$, $\Delta = 1$; $t_l = t^* = 6$, and $t_u \cong 6.46$.)

Proof. Take any $t \in PA$, and define θ_t^s by setting $\theta_t^t = 1$ and $\theta_t^s = \prod_{k=t+1}^{s}(-\delta y_k)$ at each $s > t$. Using Lemma 7 and mathematical induction on l, one can easily check that

$$\Lambda_t = C_t + \theta_t^{t+2l}\left[\Lambda_{t+2l} - C_{t+2l}\right] - \sum_{0 \le k \le l-1} \theta_t^{t+2k}\left[C_{t+2k} - \bar{B}_{t+2k}\right]. \tag{14}$$

for each $t \in PA$, and $l \geq 0$, where I use the convention that summation over the empty set is zero.

Equation (14) implies that $\Lambda_t < C_t$ when l is sufficiently large. To see this, note first that, since $|-\delta y_t| < 1$, as $l \to \infty$, $\theta_t^{t+2l} \to 0$. Since $|\Lambda_{t+2l} - C_{t+2l}| < 1$ at each t, l, it follows that, as $l \to \infty$, $\theta_t^{t+2l}\left[\Lambda_{t+2l} - C_{t+2l}\right] \to 0$. Second, $\theta_t^{t+2k} > 0$ for each k, as it consists of multiplication of evenly many negative numbers. Since $C_{t+2k} - \bar{B}_{t+2k}$ is always positive, it follows that $\sum_{0 \le k \le l-1} \theta_t^{t+2k}\left[C_{t+2k} - \bar{B}_{t+2k}\right]$ is positive, increasing in l, and hence bounded away from zero. Therefore, there exists a non-negative integer l' such that

$$\theta_t^{t+2l}\left[\Lambda_{t+2l} - C_{t+2l}\right] - \sum_{0 \le k \le l-1} \theta_t^{t+2k}\left[C_{t+2k} - \bar{B}_{t+2k}\right] < 0 \tag{15}$$

whenever $l \geq l'$, whence $\Lambda_t < C_t$ by (14).

On the other hand, using (14) at $t+1$ and (13), one can also obtain

$$\Lambda_t = B_t - \delta y_{t+1}\theta_{t+1}^{t+2l+1}\left[\Lambda_{t+2l+1} - C_{t+2l+1}\right] + \delta y_{t+1} \sum_{0 \le k \le l-1} \theta_{t+1}^{t+1+2k}\left[C_{t+1+2k} - \bar{B}_{t+1+2k}\right]. \tag{16}$$

Of course, by (15), there exists some non-negative integer l'' such that

$$-\delta y_{t+1}\theta_{t+1}^{t+2l+1}\left[\Lambda_{t+2l+1}-C_{t+2l+1}\right]+\delta y_{t+1}\sum_{0\leq k\leq l-1}\theta_{t+1}^{t+2k+1}\left[C_{t+2k+1}-\bar{B}_{t+2k+1}\right]>0 \quad (17)$$

whenever $l\geq l''$, whence $\Lambda_t > B_t$ by (16). Therefore, for any $l\geq \max\{l',l''\}$, inequalities (15) and (17) simultaneously hold. Hence, by (14) and (16), $B_t<\Lambda_t<C_t$. ■

Lemma 10. *For all $t\in PA$, $B_t<\Lambda_t<\bar{B}_t$.*

Proof. Take any $t\in PA$. By Lemma 9, $B_{t+1}<\Lambda_{t+1}<C_{t+1}$, hence by Lemma 8, $B_t<\Lambda_t<\bar{B}_t$. ■

Lemma 11. *For each $t\in PA$, $\underline{B}_t<\Lambda_t<\bar{B}_t$.*

Proof. For any $t\in PA$, observe that $t+1\in PA$, and hence, by the last lemma, $\Lambda_{t+1}<\bar{B}_{t+1}$, and therefore by Lemma 6, $\Lambda_t=1-\delta y_{t+1}\Lambda_{t+1}>\underline{B}_t$. ■

A.2. Proof of Theorem 1. First observe that, by definition,

$$\bar{B}_t\leq D_t \iff y_t-y_{t+1}\leq\frac{1-\delta}{\delta}. \quad (18)$$

Since y_t-y_{t+1} is decreasing in t and approaches 0 as $t\to\infty$, there exists some real number t_u such that $\bar{B}_t\leq D_t$ if and only if $t\geq t_u$. Now, assume $y_t-y_{t+1}\leq(1-\delta)/\delta$. Then, $t\geq t_u$, and hence $\Lambda_s<\bar{B}_s\leq D_s$ for each $s\geq t$, showing that $t-1\in PA$.

A.3. Proof of Theorem 2. Take $t^*\equiv\min PA$. By definition, there is an agreement regime at each $t\geq t^*$, and hence it suffices to show that there is a disagreement regime at each $t<t^*$. If $t^*=0$, this is vacuously true, so assume that $t^*>0$. In that case, $t^*-1<t_u$, and there is a disagreement regime at t^*-1. Now I will show that, whenever there is a disagreement regime at any $t<t_u$, there will also be a disagreement regime at $t-1$, showing by mathematical induction that there is a disagreement regime at each $s\leq t^*-1$. To this end, take some $t<t_u$ with a disagreement regime so that $S_{t+1}>1/\delta$. Since $R_t=0$, $\Lambda_t=\delta\Lambda_{t+1}$. By (10), this yields

$$S_t=(1+y_t)\,\delta\Lambda_{t+1}=\frac{\delta(1+y_t)}{1+y_{t+1}}S_{t+1}.$$

Hence, $S_t\geq S_{t+1}$ whenever $\delta(1+y_t)\geq 1+y_{t+1}$, i.e., whenever $y_t-\frac{1}{\delta}y_{t+1}\geq\frac{1-\delta}{\delta}$. But this is true: $y_{t+1}\geq 0$ and $t<t_u$, hence $y_t-\frac{1}{\delta}y_{t+1}\geq y_t-y_{t+1}\geq\frac{1-\delta}{\delta}$. Therefore, $S_t\geq S_{t+1}>1/\delta$, and hence there is a disagreement regime at $t-1$.

A.4. Proof of Lemma 1. The upper bound t_u is computed by setting $\frac{\Delta}{(t_u+n)(t_u+n+1)} = (1-\delta)/\delta$. Since $y_t - y_{t+1} = \frac{\Delta}{(t+n)(t+n+1)}$, (18) yields $t^* \leq \max\{0, t_u\}$. To compute t_l, use the lower bound $\underline{B}_t$ for Λ_t. Check that $\underline{B}_t \geq D_t$ if and only if t satisfies (5). By Lemma 11, $s = t + n$ satisfies (5) only if $t \leq t_u$, hence there exists the largest integer t_l that satisfies (5), and $t_l \leq t_u$. Clearly, $\Lambda_{t_l} > \underline{B}_{t_l} > D_{t_l}$, yielding disagreement at $t_l - 1$. Hence, $t_l - 1 < t^*$ and therefore $t_l \leq t^*$.

A.5. Proof of Theorem 5. Given any k, let $t^*(k)$, $t_l(k)$, and $t_u(k)$ be the settlement time and its lower and upper bounds, respectively, defined in Lemma 1 for the parameters $\delta = e^{-r/k}$, $n = \pi k$ and $\Delta = y_o n$. Write τ^*, τ_l, and τ_u for the limits of $\tau(t^*(k), k)$, $\tau(t_l(k), k)$, and $\tau(t_u(k), k)$, respectively, as $k \to \infty$. Define $\omega = \pi y_o / r$.

Towards proving the first statement, note that $\delta = e^{-r/k} \cong 1 - r/k$ for large values of k. Hence, by (4), $t_u(k) \cong (\sqrt{1 + 4\omega k^2} - 1)/2 - \pi k$ so that $\tau_u = \lim_{k\to\infty} t_u(k)/k = \sqrt{\omega} - \pi$ as claimed. To prove the second statement, for any given t, write $s \equiv t + n$ and $\sigma \equiv s/k = \tau(t,k) + \pi$. When k is large, we also have $\delta = e^{-r/k} \cong 1 - r/k$ and $(k\sigma + 1)/k \cong \sigma \cong (k\sigma + 2)/k$. Substituting these in (5), one can check that $f \cong rk^2\left[\sigma^3 - \omega\sigma + \omega^2 r\right]$. Then, by Theorem 1, $\tau^* \geq \sigma - \pi$ whenever

$$\phi \equiv \sigma^3 - \omega\sigma + \omega^2 r \leq 0.$$

Note that ϕ has a local minimum at $\underline{\sigma} = \sqrt{\omega/3}$. If $y_o \pi r < 4/27$, then ϕ is negative at $\underline{\sigma}$, showing that $\tau^* > \underline{\sigma} - \pi$ as desired.[3] The last statement in the theorem follows from the fact that as $\pi \to 0$, $y_o \pi r \to 0$ and thus $\tau^* > \sqrt{\omega/3} - \pi \to \infty$.

References

[1] Admati, A. and M. Perry (1987): "Strategic delay in bargaining," *Review of Economic Studies* 54: 345-364.

[2] Babcock, L. and G. Loewenstein (1997): "Explaining Bargaining Impasse: The Role of Self-Serving Biases," *The Journal of Economic Perspectives*, 11-1, 109-126.

[3] Babcock, L., G. Loewenstein, S. Issacharoff, and C. Camerer (1995): "Biased Judgements of Fairness in Bargaining," *The American Economic Review*, 85-5, 1337-1343.

[4] Farber, H. and M. Bazerman (1989): "Divergent expectations as a cause of disagreement in bargaining: Evidence from a comparison of arbitration schemes," *Quarterly Journal of Economics*, 104, 99-120.

[5] Gould, J.P. (1973): "The Economics of Legal Conflict," *Journal of Legal Studies*, 2, 235.

[6] Hicks, J. (1932): *The Theory of Wages*, New York: Macmillan Publishing Co.

[3]Note that $\tau_l + \pi$ is the largest solution to the cubic equation $\phi = 0$, which is greater than $\underline{\sigma}$. When $y_o \pi r > 4/27$, ϕ has a unique root, which is negative.

[7] Kennan, J. and R. Wilson (1993): "Bargaining with private information," *Journal of Economic Literature*, 31-1, 45-104.

[8] Landes, W. M. (1971): "An economic analysis of the courts," *Journal of Law and Economics*, 14, 61-107.

[9] Neale, M. and M. Bazerman (1985): "The Effects of Framing and Negotiator Overconfidence on Bargaining Behaviors and Outcomes," *The Academy of Management Journal*, 28-1, 34-49.

[10] Priest, G. and B. Klein (1984): "The Selection of Disputes for Litigation," *Journal of Legal Studies*, 13, 1-55.

[11] Posner, R. (1972): "The Behavior of Administrative Agencies," *Journal of Legal Studies*, 1, 305.

[12] Rubinstein, A. (1982): "Perfect equilibrium in a bargaining model," *Econometrica*, 50-1, 97-110.

[13] Stahl, I. (1972): *Bargaining Theory*, Stockholm School of Economics.

[14] Yildiz, M. (2000): *Essays on Sequential Bargaining*, Ph.D. Thesis, Stanford University.

[15] Yildiz, M. (2001): "Sequential Bargaining without a Common Prior on the Recognition Process," *MIT Department of Economics Working Paper* No. 01-43.

[16] Yildiz, M. (2002): "Bargaining without a Common Prior — An Agreement Theorem," *Econometrica*, forthcoming.