Dewey

# working paper department of economics

Economic Equilibrium with Costly Marketing

by

c,3

Duncan K. Foley

Number 52 February 1970

massachusetts institute of technology
50 memorial drive
cambridge, mass. 02139

# Economic Equilibrium with Costly Marketing

by

c, 3

Duncan K. Foley

Number 52 February 1970

This paper was supported by the National Science Foundation (grant number GS 1585). The views expressed in this paper are the author's sole responsibility, and do not reflect those of the National Science Foundation, the Department of Economics, nor of the Massachusetts Institute of Technology.

Economic Equilibrium with Costly Marketing*

Duncan K. Foley
M.I.T.

## 1. Introduction

The traditional theory of general market equilibrium, the most famous rigorous presentation of which can be found in Debreu [1], remains the deepest scientific resource of economists, and is the basis of the most sophisticated attempts to study a wide range of economic problems. It is therefore disturbing that this theory, when applied to the complete problem of economic interaction over time, space and in the presence of chance, predicts the formation of numerous markets in time-dated, place-tagged, contingent commodities which do not actually exist. Corollary to this embarrassment is the prediction by the theory that economic agents will choose one plan of action good for all time and all contingencies, which they clearly do not.

The embarrassment does not come from the fact that the theory is too weak, but because it is too strong. Its assumptions are general and unexceptional but its conclusions awesomely specific. Confronted by this

---

* While writing this paper, I was a Ford Faculty Research Fellow. I would like to thank Ross Starr for numerous helpful conversations on this subject. Frank Hahn has independently developed a model very similar in structure to the one presented here.

problem, which has become acute only as an aftermath of the rigorous mathematical formulation of the theory, economists have tried to find weak links in the underlying assumptions. One such weakness is the absence from the theory of any real resource costs in information gathering and processing, or in the operation of "markets".

It is my purpose here to outline a very simple modification of the traditional model in which it is possible to analyze the consequences of costs in the operation of "markets". I believe this modification sacrifices none of the generality and rigor which make the theory of general equilibrium so splendid. The modification does, however, drastically alter the stylized picture the theory yields. In particular, markets will not generally exist in unlikely contingencies or for deliveries in the distant future, nor will economic agents find it useful or even possible to bind themselves to a single unchanging plan.

The key aspect of the modification I propose is an alteration in the notion of "price". In the present model there are two prices in each market: a buyer's price and a lower seller's price. The difference between these yields an income which compensates the real resources used up in the operation of the markets.

## 2. Consumers and Demands

Each of n consumers has a consumption set $X^i \subset E^m$ (since there are m commodities) which defines the consumer's biologically and technically feasible consumption plans. A point $x^i \in X^i$ includes provision of services and resources which the consumer owns as negative numbers. On the set $X^i$ there is a preference ordering $\succsim_i$.

The set $X = \sum_i X^i$ is the aggregate consumption set. $\hat{X}^i$ is the "attainable" consumption set for each consumer, the set of consumption plans for the consumer that the whole economy has resources and technical knowledge to provide.

I will make the following assumptions about $X^i$ and $\succsim_i$ :

a.1) The aggregate consumption set X has a lower bound. (This implies that each $X^i$ also has a lower bound).

a.2) For each i, $X^i$ is closed and convex.

b.1) For every consumption $\hat{x}^i \in \hat{X}^i$ there exists $x^i \in X^i$ with $x^i \succ \hat{x}^i$. (This assumption asserts that the full productive capacity of the economy is not sufficient to satiate any consumer completely).

b.2) For every $x^i \in X^i$ the sets

$$\{\bar{x}^i \in X^i \mid \bar{x}^i \succsim_i x^i\} \quad \text{and} \quad \{\bar{x}^i \in X^i \mid \bar{x}^i \precsim_i x^i\}$$

are closed in $X^i$.

b.3) For every $x^i \in X^i$ the set

$$\{\bar{x}^i \in X^i \mid \bar{x}^i \succsim_i x^i\} \quad \text{is convex.}$$

c.1) $0 \in X^i$ for all i.

The consumer faces two sets of prices, $p^S$ and $p^B$. (I will very often write $\pi = p^B - p^S$ to denote the difference between these). The cost of any consumption plan $x^i$ will depend on the two vectors $x^{iB}$ and $x^{iS}$ defined by:

$$x_j^{iB} = \max\,[x_j^i,\ 0]$$

$$x_j^{iS} = \min\,[x_j^i,\ 0]$$

$x^{iB}$ is the vector of purchases and $x^{iS}$ the vector of sales. The value of a plan $x^i$ is $p^B x^{iB} + p^S x^{iS}$.

For any wealth $w^i$, the consumer is restricted to the set

$$B^i\,(p^S,\ p^B,\ w^i) = \{x^i \in X^i / x^i = x^{iB} + x^{iS} \text{ where } p^B X^{iB} + p^S X^{iS} \leq w^i\}$$

It is easy to see that if $p_j^B < p_j^S$ for some commodity j the set $B^i$ is unbounded because any consumer can buy and sell commodity j at a profit. In what follows I assume always that $p^B \geqq p^S$, that is, $\pi \geqq 0$, and that $p^S \geqq 0$.

In two dimensions, the consumer's budget set is the intersection of two price lines (see Fig. 1).

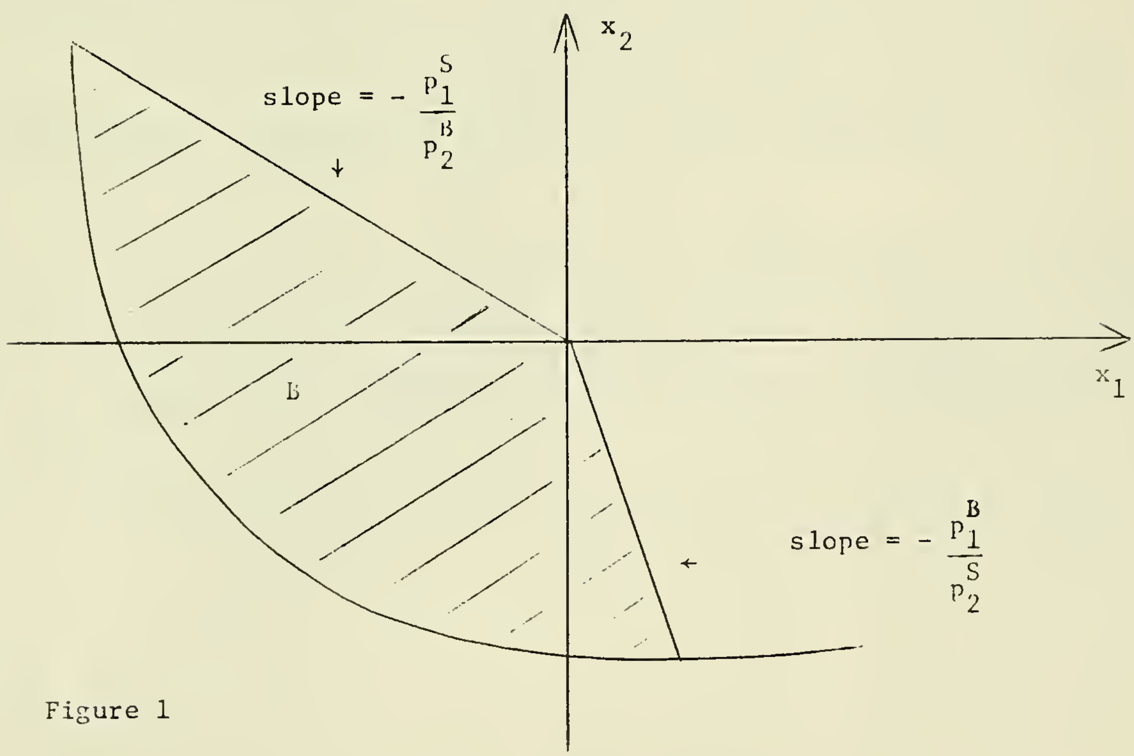


Figure 1

If every consumer has $w^i = 0$ and chooses a point $x^i$ with $p^B x^{iB} + p^S x^{iS} = 0$, then in the aggregate, $p^B x^B + p^S x^S = 0$ where $x^B = \sum_i x^{iB}$ and $x^S = \sum_i x^{iS}$, or writing $z = x^B + x^S$, $(p^B - p^S) x^B = -p^S z$.

This observation is analogous to Walras' Law, but has an interesting interpretation. The vector z is a vector of net consumer supplies and demands to producers. At any $p^B$, $p^S$ pair, consumers are willing to release resources just equal in value to the total premium they pay through higher buying prices. In a pure exchange economy without production, the vector z represents resources used up in the operation of the markets.

It is possible to show that the budget set $B^i$ is convex and then go on to prove that consumer demands in the two-price environment have the same continuity and convexity properties that hold in the one-price environment. I prefer to proceed by a shortcut, and to show that the consumer I have described is mathematically equivalent to another consumer in a one-price economy who satisfies the assumptions made above.

This useful way of describing consumer choice in the two-price environment is suggested by writing the budget constraint as

$$p^S \cdot x + (p^B - p^S) x^B = p^S \cdot x + \pi \cdot x^B \leq w.$$

The selling prices are applied to the entire bundle, with a premium for transactions at the buying prices. In fact, it is possible to define a new consumption set $\bar{X}^i \subset E^{2m}$ and new preferences on this set $\succsim_i'$ by the relations:

a) $\bar{X}^i = \{ (x, z) \mid x \in x^i, \ z_j \geq \max [x_j, 0] \ \text{ for } \ j = 1,\ldots,m \}$

b) if $(x, z)$ and $(\bar{x}, \bar{z}) \varepsilon \bar{X}^i$, then $(x, z) \succsim_i' (\bar{x}, \bar{z})$

if $x \succsim_i' \bar{x}$ in $X^i$.

It is easy to verify that if $X^i$ and $\succsim_i$ satisfy assumptions a.1), a.2), b.1), b.2), and b.3), then $\bar{X}^i$ and $\succsim_i$ will as well. The indifference curves in $\bar{X}^i$ will be thick, since the commodities z do not make the consumer any better off. Fortunately equilibrium analysis is sophisticated enough to handle this situation (cf. Debreu [2]).

The key to proving that $\bar{X}^i$ and $\succsim_i'$ satisfy the other assumptions made above is to show that $\bar{X}^i$ is convex and closed if $X^i$ is.

Theorem 2.1: If $X^i$ is convex and closed, then $\bar{X}^i$ is convex and closed.

Proof: First, take closure. Let $\{ (x^q, z^q) \} \to (\hat{x}, \hat{z})$ be a sequence of points such that $(x^q, z^q) \varepsilon \bar{X}^i$ for all q. Since $X^i$ is closed, $\hat{x} \varepsilon X^i$, being the limit of a sequence contained in $X^i$. The only other condition is that $\hat{z}_j \geq \max [\hat{x}_j, 0]$. Suppose for some j, $\max [\hat{x}_j, 0] - \hat{z}_j > \epsilon$. For large q, $| x_j^q - \hat{x}_j | < \frac{\epsilon}{4}$, $| z_j^q - \hat{z}_j | < \frac{\epsilon}{4}$, so that $\max [x_j^q, 0] - z_j^q > \frac{\epsilon}{2}$, which contradicts the assumption that $(x^q, z^q) \varepsilon \bar{X}^i$.

Next consider convexity. If $(\bar{x}, \bar{z})$ and $(\hat{x}, \hat{z}) \varepsilon \bar{X}^i$, and $(\tilde{x}, \tilde{z}) = \alpha (\bar{x}, \bar{z}) + (1 - \alpha) (\hat{x}, \hat{z})$ $(0 < \alpha < 1)$, convexity of $X^i$ implies that $\tilde{x} \varepsilon X^i$. The problem is to check that $\tilde{z}_j \geq \max [\tilde{x}_j, 0]$ for $j=1, \ldots\ldots, m$.

$$\begin{aligned} \tilde{z}_j = \alpha \bar{z}_j + (1 - \alpha) \hat{z}_j &\geq \alpha \max [\bar{x}_j, 0] + (1 - \alpha) \max [\hat{x}_j, 0] \\ &= \max [\alpha \bar{x}_j, 0] + \max [ (1 - \alpha) \hat{x}_j, 0] \\ &\geq \max [\alpha \bar{x}_j + (1 - \alpha) \hat{x}_j, 0] \\ &\geq \tilde{x}_j \qquad \text{Q.E.D.} \end{aligned}$$

The proofs that the $\bar{X}^i$ and $\succsim_i'$ also satisfy the other assumptions follow easily from this theorem.

It is also true that if the new consumer chooses $(x^{i*}, z^*)$ and prices $(p^{S*}, \pi^*)$ then the old consumer will choose $x^{i*}$ at $p^{S*}$ and $p^{B*} = p^{S*} + \pi^*$.

<u>Theorem 2.2:</u> Suppose $(x^{i*}, z^*) \varepsilon \bar{X}^i$ satisfies for $p^{S*}, p^{B*} \geq 0$, $\pi^* = p^{B*} - p^{S*} \geq 0$.

a) $p^{S*} x^{i*} + \pi^* z^* \leq w^i$

b) $(x^{i*}, z^*) \succsim_i' (x^i, z)$ for all $(x^i, z) \varepsilon \bar{X}^i$ with $p^{S*} x^i + \pi^* z \leq w^i$

c) $x^{i*} \varepsilon \hat{X}^i$

Then $x^{i*} \varepsilon X^i$ will satisfy

a$^1$) $p^{S*} x^{i*S} + p^{B*} x^{i*B} \leq w^i$

b$^1$) $x^{i*} \succsim_i x^i$ for all $x^i \varepsilon X^i$ with $p^{S*} x^{iS} + p^{B*} x^{iB} \leq w^i$.

<u>Proof</u>: Since $z_j^* \geq x_j^{i*B} = \max [x_j^{i*}, 0]$ and $p_j^B \geq 0$, part a$^1$) follows immediately from a).

Suppose there existed $\hat{x}^i \varepsilon X^i$ with $p^{S*} \hat{x}^{iS} + p^{B*} \hat{x}^{iB} \leq w^i$ and $\hat{x}^i \succ_i x^{i*}$. Then $(\hat{x}^i, \hat{x}^{iB}) \varepsilon \bar{X}^i$ by definition of $\bar{X}^i$, $p^{S*} \hat{x}^i + \pi^* \hat{x}^{iB} \leq w^i$, and $(\hat{x}^i, \hat{x}^{iB}) \succ_i' (x^{i*}, z^*)$, which is a contradiction. Q.E.D.

All the usual results of demand theory based on the assumptions a.1), a.2), b.1), b.2), b.3), and c.1 can be applied to the consumer with consumption set $\bar{X}^i$ and preferences $\succsim_i'$ and through him to an ordinary consumer in a two-price environment.

If the consumer's preferences can be represented by a utility function $u(x)$, his demand problem is the constrained maximization problem:

$$\max \quad u(x)$$
$$\text{subj to} \quad p^B x^B + p^S x^S \leq w$$

The first order conditions are:

$$\partial u/\partial x_j - \lambda\, p_j^B = 0 \quad \text{if} \quad x_j > 0$$
$$\partial u/\partial x_j - \lambda\, p_j^S = 0 \quad \text{if} \quad x_j < 0$$

where $\lambda$ is the LaGrangean shadow price corresponding to the wealth constraint, which depends on the prices.

Since $p_j^B \geq p_j^S$, these conditions can be written

$p_j^S \leq (1/\lambda)\ (\partial u/\partial x_j) \leq p_j^B$, $j = 1,\ldots,m$ where the left-hand equality holds for $x_j < 0$ and the right-hand equality for $x_j > 0$.

## 3. Production and Marketing

While consumers have no choice but to buy at the buying price and sell at the selling price in a market, at least some producers can involve themselves in marketing activities, at a cost in real resources. The production plan of a producer can be written $(y^j, y^{Bj})$, where $y^j$ is the total net transaction the producer makes, and $y^{Bj}$ the vector of purchases and sales subject to the premium buying price. The profit arising from such a plan at prices $(p^S, p^B)$ will be:

$$\pi^j (p^S, \pi) = p^S y^j + \pi y^{Bj} \quad \text{where} \quad \pi = p^B - p^S.$$

Some producers will do no marketing of their own, hiring resources at the buying prices and selling their product at the selling price. Other producers may do all their marketing in all markets, hiring resources at the selling price and selling their output at the buying price. The second procedure, of course, will require more resources than the first, even if the total output is the same, if marketing is costly. Other producers may do some transactions at buying prices and others at selling prices.

What determines how much marketing a producer does for himself? The producer will market if the spread between buying and selling prices is large enough to make it profitable for him. If the spread is small a producer will prefer to leave the marketing to other producers.

If marketing activities were costless, there would be no spread between buying and selling prices and the economy would be the same as in the traditional models.

What are the costs involved in marketing activities? They include the effort required to inform buyers or sellers of the existence of a supply or demand for a commodity, and of the price. This may include advertising, costs of holding stocks for wide distribution, spoilage, breaking down commodities for retail sale, and product standardization and certification. The important feature of these costs is that they expend real resources without altering the characteristics of the delivered product.[1]

---

[1]This is a copout.

---

I assume that the set of feasible production plans for each producer, $Y^j$, has the following properties:

d.1) $0 \in Y^j$ for all $j$. (This assumption, together with (c.1) assures that there are feasible allocations for the economy.)

d.2) There is no $(y^j, y^{jB}) \in Y^j$ with $(y^j, y^{jB}) > 0$. (This rules out the possibility of free production or free marketing.)

d.3) $Y^j$ is a convex cone for all j.
(If $(\bar{y}^j, \bar{y}^B) \in Y^j$ and $(\hat{y}^j, \hat{Y}^{jB}) \in Y^j$ then
$(\alpha\, \bar{y}^j + \beta\, \hat{y}^j, \alpha\, \bar{y}^{jB} + \beta\, \hat{y}^{jB}) \in Y^j$ for $\alpha, \beta \geq 0$).

The last assumption rules out any set-up costs or indivisibilities in marketing activities. Many economists have argued that indivisibilities are characteristic of marketing activities.

## 4. Equilibrium

It is easy to generalize the notion of general competitive equilibrium to an economy with marketing costs.

Definition: A market equilibrium is a vector of prices $(p^{S*}, \pi^*)$, a vector $x^{i*} \in X^i$ for each consumer and a vector $(y^{j*}, y^{jB*}) \in Y^j$ for each producer such that:

a) $x^{i*}$ is maximal with respect to $\succsim_i$ in $B^i\,(p^{S*}, \pi^*, 0)$

b) $p^{S*}\, y^{j*} + \pi^*\, y^{jB*} \geq p^{S*}\, y^j + \pi^*\, y^{jB}$ for all $(y^j, y^{jB}) \in Y^j$, $j = 1,\ldots,k$.

c) $\sum_i (x^{i*}, x^{i*B}) - \sum_j (y^{j*}, y^{jB*}) = 0$

d) $p^{S*} \neq 0$, $\pi^* \geq 0$.

In equilibrium, consumers are maximizing according to their preferences subject to the budget constraint, producers are maximizing profit both in ordinary production and in marketing activity, and all markets clear.

The easiest way to show the existence of such an equilibrium is to study an extended economy with 2m commodities, and apply to the extended economy known existence theorems. The best such theorem for my purpose is contained in Debreu [2]. I paraphrase that result here.

Theorem 4.1 [Existence of Quasi-Equilibrium]

An economy $\mathcal{E}$ defined by consumption sets $X^i$, preferences $\succsim_i$, production sets $Y^j$, and a vector $\{O_{ij}\}$ indicating the $i^{th}$ consumer's share of the profits of the $j^{th}$ producer (if any) which satisfies assumptions a.1), a.2), b.1), b.2), b.3), c.1), d.1), d.2), and d.3) has a quasi-equilibrium; that is, there exists $((x^{i*}), (y^{j*}), p^*) \subset ((X^i), (Y^j), L^m)$ such that:

a) $x^{i*}$ is maximal with respect to $\succsim_i$ in

$\{x^i \in X^i \mid p^* x^{i*} \leq \sum_j O_{ij}\, p^* y^{j*}\}$ or

$p^* x^{i*} = \text{Min}\; p^* X^i$, for every i.

b) $p^* y^{j*} = \text{Max}\; p^* Y^j$ for every j.

c) $\sum_i x^{i*} - \sum_j y^{j*} = 0$

d) $p^* \neq 0$.

I want to apply this theorem to the economy with consumption sets $\bar{X}^i$, preferences $\succsim_i'$, production sets $Y^j$ and an arbitrary vector $(\Theta_{ij})$ with $\Theta_{ij} \geq 0$, $\sum_i \sum_j \Theta_{ij} = 1$.

This yields a set of consumption and production plans $((x^{i*}, z^{ij}), (y^{j*}, y^{Bj*}), (p^{S*}, \pi^*))$ with the properties:

a) $(x^{i*}, z^{i*}) \varepsilon \bar{X}^i$ is maximal with respect to $\succsim_i'$ in

$\{ (x^i, z^i) \varepsilon X^i \mid p^S x^i + \pi z^i \leq \sum_j \Theta_{ij} (p^{S*} y^{j*} + \pi^* y^{Bj*}) \}$ or

$p^{S*} x^{i*} + \pi^* z^{i*} = \text{Min } (p^{S*}, \pi^*) \bar{X}^i$.

b) $p^{S*} y^{j*} + \pi^* y^{Bj*} = \text{Max } (p^{S*}, \pi^*) Y^j$

c) $\sum_i (x^{i*}, z^{i*}) - \sum_j (y^{j*}, y^{Bj*}) = 0$

d) $(p^{S*}, \pi^*) \neq 0$.

This is practically equivalent to the definition of market equilibrium given above. To establish complete equivalence I need to show that $\pi^* \geq 0$, and that the profits of all producers are zero. The latter proposition follows from profit maximization and the assumption that production sets are cones. $\pi^* \geq 0$ follows from the unboundedness of $\bar{X}^i$ in the $z^i$-components and assumption b.1), since if $\pi_j^* < 0$, a consumer could increase $z_j^i$ without limit and achieve a consumption outside $\hat{X}^i$ and preferred to $X^{i*}$.

This argument proves that a quasi-equilibrium exists with market costs under the assumptions of traditional equilibrium analysis. The most important restriction involving market costs is the assumption that marketing, like other productive activities, is not subject to increasing returns to scale or indivisibilities.

The difficulty that some consumers may in fact be at a minimum-wealth point in their consumption set and not at a preference-maximizing point remains. Debreu [3] notes that existence of a true equilibrium, in which this situation occurs for no consumer, can be assured by requiring that (a) the production set intersect the interior of the aggregate consumption set and that (b) if any consumer is at the minimum-wealth point in a quasi-equilibrium, all are. This theorem carries over to the economy with marketing costs, because this model is mathematically identical to the model studied by Debreu.

In some situations there may be difficulties in showing that the requirement (b) above is met, because of the existence of the new artificial retail commodities. For example, Debreu defines an "always desired" commodity as one such that every consumer can reach a preferred point in his consumption set by increasing his consumption of that commodity only. This will not be possible for bought commodities in the marketing costs model because increasing only that component of the consumption bundle takes the consumer out of his consumption set, unless the corresponding retail component is also increased.

With the assumptions that each consumer can dispose of a finite amount of all commodities, and that there are production activities which produce every commodity retail using only wholesale inputs it is possible to show that a quasi-equilibrium is a true equilibrium. If any consumer is at the minimum wealth point in quasi-equilibrium, all selling prices must be zero. But at least one buying premium is therefore positive. But this situation contradicts the property of profit-maximization in quasi-equilibrium, because there exists an activity with positive retail output of the commodity with a positive buying premium, and wholesale inputs, which would give positive profits. Because I assume production and marketing sets to be cones, a positive profit in any activity is not compatible with profit maximization.

A stronger result than this is desirable and presumably discoverable.[1]

---

[1] Frank Hahn brought this difficulty to my attention.

---

## 5. Pareto Optimum and the Core

In a formal sense the traditional analysis of Pareto optima and the core[2] can be applied to the extended economy used above to study

---

[2] See Debreu [1], Debreu and Scarf [3].

---

the existence problem. But I think to do this would be to travel too fast.

The essential notion in studies of Pareto optimal and core allocations is the set of allocations achievable in some purely technological sense by a group of economic agents. In a pure exchange economy, for example, feasible allocations for any coalition (or for the economy) are those which sum to the total endowment of the coalition (zero in the present model since I measure all trades from the endowment point). The introduction of market cost is intended to reflect information costs involved in sustaining an allocation, but the assumptions made implicitly refer to the institutional environment, i.e. to markets. It is not obvious that radically different organizations of exchange would have the same type or magnitude of resource costs in the exchange process. A deeper and more satisfactory study of the core and Pareto optima would begin from a fundamental account of information costs of exchange without references to institutions and derive "markets" as one of a number of possible organizations of exchange. Only in such a theory could the "efficiency" of the equilibrium proposed here be studied other than trivially.

## 6. The Existence of Markets

Under what circumstances will there be trade in a given market at equilibrium with market costs? Put another way, when will the addition or elimination of a given market make no difference to equilibrium prices or to any consumer's demand in any other market?

I will treat this problem for a market in a commodity which is not an input or output of production. Producers are not either buyers or sellers of the commodity, but provide market services in the market if it pays them.

In describing consumer equilibrium in section 2, I showed that at the equilibrium trade to each consumer there corresponded a number $\lambda^i$ (in the case where consumer's preferences can be described by a differentiable utility function) such that:

$$p_j^S \leq (1/\lambda^i)\ (\partial u^i/\partial x_j) \leq p_j^B \qquad j = 1,\ldots,m.$$

If another market opens in a good not previously traded (but represented in utility functions) an individual will not trade if the prices $p_{m+1}^S$, $p_{m+1}^B$ in the new market satisfy

(1) $p_{m+1}^S \leq (1/\lambda^i)\ (\partial u^i/\partial x_{m+1}) \leq p_{m+1}^B$

at the original equilibrium demands. A large spread between buying and selling prices ensures that no consumer will trade.

Producers, on the other hand, will be induced by a large spread between prices to expend real resources in providing market services in the $(m + 1)$st market. Suppose that $\Delta$ is the marginal cost at equilibrium prices of expanding trading in the $(m + 1)$st market. Producers will be content with the previous equilibrium only if

(2) $p_{m+1}^B \leq p_{m+1}^S + \Delta$

If (1) holds simultaneously for all consumers, then

(3) $p_{m+1}^S \leq \min_i\ [\ (1/\lambda^i)\ (\partial u^i/\partial x_{m+1})\ ]$

(4) $p_{m+1}^B \geq \max_i\ [\ (1/\lambda^i)\ (\partial u^i/\partial x_{m+1})\ ]$

It will be possible to find $p_{m+1}^S$ and $p_{m+1}^B$ satisfying (2), (3) and (4) if and only if

(5) $\Delta \geq \max_i\ [\ (1/\lambda^i)\ (\partial u^i/\partial x_{m+1})\ ] - \min_i\ [\ (1/\lambda^i)\ (\partial u^i/\partial x_{m+1})\ ]$

This, put in commonsense language, means that the difference between the highest price at which any consumer would be willing to buy and the lowest price at which any consumer would be willing to sell is smaller than the cost of bringing about the transaction. If the (m + 1)st commodity were an input or output in production these conditions would be modified to exclude producers' trade as well. The highest price at which a producer is willing to buy the commodity is the maximum of the values of its marginal product to the producers; the lowest price at which a producer will sell is the minimum of the marginal costs of production.

In the case where the (m + 1)st commodity is traded only by consumers, a sufficient condition for no trade is

$$(6) \quad \Delta \geq \max_i \left[ (1/\lambda^i)(\partial u^i/\partial x_{m+1}) \right]$$

If the (m + 1)st market is in a commodity dated in the far distant future or a commodity in a contingency with low probability, its marginal utility will be small because of time preference in one case, and the small contribution it makes to expected utility in the other. The important idea here is that current actual resources are required to set up a market in futures or contingent commodities. If the consumers value these commodities little in relation to current actual resources, it will not pay anyone to set up a market to trade them.

7. An Example

A simple numerical example may help to clarify the notion of equilibrium proposed above. Suppose an economy exists in which the only goods are consumption dated at successive future dates, and that there are two types of consumers with similar utility functions who discount utility 100% per period:

$$u^i(c_0, c_1, \ldots, c_m) = \sum_{j=0}^{m} [\ln(c_j + w_j^i)]/2^j \qquad \text{where}$$

$w_j^i$ is the endowment of the $i^{th}$ consumer in the period j, and $c_j$ is his net purchase or sale of consumption in period j.

The demand functions for this utility function are well known. The problem is to

$$\max \sum_{j}^{m} [\ln(c_j + w_j)]/2^j$$

$$\text{subj to } \sum_{j}^{m} p_j\, c_j = 0$$

The first order conditions are:

$$1/[2^j(c_j + w_j)] - \lambda\, p_j = 0 \qquad j = \phi, \ldots, m.$$

or

$$1/[\lambda(2^j)] = (p_j\, c_j + p_j\, w_j) \qquad j = 0, \ldots, m.$$

Adding these for all j:

$$[\sum_{j=0}^{m} (1/2^j)]\,[1/\lambda] = \sum_{j=0}^{m} p_j\, w_j, \quad \text{or} \quad 1/\lambda = [\sum_{j=0}^{m} p_j\, w_j]/[\sum_{j=0}^{m} (1/2^j)]$$

This gives the demand functions:

$$c_{j'} = [(1/2^{j'})/\sum_{j}(1/2^j)]\,[\sum_{j=0}^{m} p_j\, w_j]/p_{j'} - w_{j'} \qquad j' = 0, \ldots, m$$

For the case of two markets (m = 1) the equilibrium price vector

$p = (1, 1/2)$, and the equilibrium demands are:

$$c_0^1 = 1/3 \qquad c_1^1 = -2/3 \qquad\qquad w^1 = (1, 2)$$

when

$$c_0^2 = -1/3 \qquad c_1^2 = 2/3 \qquad\qquad w^2 = (2, 1)$$

If a market in period 2 is added the equilibrium price vector is $(1, 1/2, 1/4)$ and if $w^1 = (1, 2, 1.4)$ and $w^2 = (1, 2, 1.6)$ then equilibrium demands are

$$c_0^1 = .34 \qquad c_1^1 = -.66 \qquad c_2^1 = -.06$$

$$c_0^2 = -.34 \qquad c_1^2 = .66 \qquad c_2^2 = .06$$

The existence of the extra market changes each type of consumer's demand for goods in other periods.

Suppose now that trading a unit of any good costs .1 units of consumption in period 0. The production set is a cone containing the vectors $(-.1, 0; 1, 0)$ and $(-.1, 0; 0, 1)$. In the case $m=2$, the cone contains the three vectors: $(-.1, 0, 0; 1, 0, 0)$, $(-.1, 0, 0; 0, 1, 0)$ and $(-.1, 0, 0; 0, 0, 1)$.

The consumer problem becomes

$$\max \sum_{j=0}^{m} [\ln(c_j + w_j)]/2^j$$

$$\text{subj to} \quad \sum_{j=0}^{m} p_j^S c_j^S + \sum_{j=0}^{m} p_j^B c_j^B = 0 \qquad c_j^S = \min [c_j, 0]$$

$$c_j^B = \max [c_j, 0]$$

The first order conditions become

$$1/[2^j(c_j + w_j)] - \lambda\, p_j^S = 0 \qquad \text{if} \qquad c_j < 0$$

$$1/[2^j(c_j + w_j)] - \lambda\, p_j^B = 0 \qquad \text{if} \qquad c_j > 0$$

These can be written

$$(1/2^j)\ (1/\lambda) = p_j^S\, c_j + p_j^S\, w_j \qquad \text{if} \qquad c_j < 0$$

$$(1/2^j)\ (1/\lambda) = p_j^B\, c_j + p_j^B\, w_j \qquad \text{if} \qquad c_j > 0$$

Add up the terms which apply:

$$(1/\lambda) \sum_j (1/2^j) = \sum_{j=0}^{m} p_j^S\, c_j^S + \sum_{j=0}^{m} p_j^B\, c_j^B + \sum_{j=0}^{m} \left( \begin{matrix} p_j^B \\ p_j^S \end{matrix} \right) w_j$$

where $\left( \begin{matrix} p_j^B \\ p_j^S \end{matrix} \right)$ is $p_j^B$ or $p_j^S$ depending on whether $c_j \gtrless 0$.

The demand functions can be written:

$$c_{j'} = [(1/2^{j'})/\sum_j (1/2^j)] \quad [\sum_j \left( \begin{matrix} p_j^B \\ p_j^S \end{matrix} \right) w_j]/p_{j'}^B \ - w_{j'}, \quad c_{j'} > 0$$

$$= [(1/2^{j'})/\sum_j (1/2^j)] \quad [\sum_j \left( \begin{matrix} p_j^B \\ p_j^S \end{matrix} \right) w_j]/p_{j'}^S \ - w_{j'}, \quad c_{j'} < 0.$$

In searching for an equilibrium with marketing costs I assume that it will be sufficiently near the regular equilibrium that type 1 will be a buyer in period 0 and a seller in period 1 and vice versa for type 2.

The problem is to clear the two markets. Clearly the relation between buying and selling price in each market must be $p_j^B = p_j^S + .1p_0^S$. Taking $p_0^S = 1$, $p_0^B = 1.1$, $p_1^B = p_0^S + .1$, and the problem reduces to finding $p_0^S$ that will clear the period 1 market. Adding the demand functions where $w^1 = (1, 2)$ and $w^2 = (2, 1)$:

$$c_1^1 + c_1^2 = \ [1/3(1.1 + 2p_1^S/p_1^S] - 2 \ + \ [1/3[2 + (p_1^S + .1)]/p_1^S + .1] - 1 = 0$$

This gives a quadratic equation in $p_1^S$:

$6p_1^{S^2} - 2.5p_1^S - .11 = 0$ with a positive root $p_1^S = .457$, implying $p_1^B = .557$.

The demands are

$c_0^1 = .22$ $\qquad\qquad$ $c_1^1 = -.53$

$c_0^2 = -.30$ $\qquad\qquad$ $c_0^2 = .53$

It is instructive to compare this equilibrium with the one where there was no trading cost. In the usual case the interest rate for both borrowers and lenders implied by the prices is 100%. With transaction costs type 1 people, who are borrowers, face an interest rate equal to

$$(p_0^B/p_1^S) - 1 = (1.1/.457) - 1 = 1.41,$$

i.e. 141%. The type 2 lenders, on the other hand, are receiving an interest rate of only $(p_0^S/p_1^B) - 1 = 1/.557 - 1 = .79$, i.e. 79%.

To illustrate the point made in section 6, consider adding another market in period 2 consumption, with $w^1 = (1, 2, 1.4)$ and $w^2 = (2, 1, 1.6)$.

The $\lambda$ multipliers implied by the previous equilibrium are $\lambda' = .745$, $\lambda^2 = .588$, and $(1/\lambda')\ (\partial u'/\partial c_2) = .242$, $(1/\lambda^2)\ (\partial u^2/\partial c_2) = .251$. In this case $\Delta = .1$, obviously, and the condition

$$\Delta \geq \max_i [(1/\lambda^i)(\partial u^i/\partial x_{m+1})] - \min_i [(1/\lambda^i)(\partial u^i/\partial x_{m+1})]$$

is met since $\Delta = .1 \geq .251 - .242 = .009$.

If in the new market the selling price is established at .2 and the buying price at .3 neither consumer will want to trade and the price differential will not be sufficient to draw resources into setting up a market.

The implied two-period borrowing and lending rates can be found by solving $(1 + r_B)^2 = (p_0^S/p_2^B)$ and $(1 + r_2)^2 = (p_0^B/p_2^S)$.

For the prices suggested above, the borrowing rate is 144% and the lending rate 83%, in contrast to the equilibrium without market costs where both rates are equal to 100%.

8. Conclusions

Although the model described above is logically consistent, it is not as satisfactory in one important respect as the traditional model of equilibrium with futures and contingency markets. Consumers and producers in the traditional model have all the information they need to make once and for all a complete consumption plan. There is no reason why they should not simply exploit to the limit their trading opportunities at equilibrium prices, then rest content and simply carry out their predetermined plan. If markets were reopened no trade would take place; the equilibrium price system would remain the same. Given the environment of the traditional model with full contingency and futures markets, there is no reason why a sensible consumer should not behave as the model predicts.

In the present model, however, prices will change when markets reopen and consumers know it. The reason is that the production sets change with time in a particular way: it is not possible to use period m resources in setting up markets until period m actually arrives. In the next period, new markets will open and price spreads in other markets will change. The consumer, then, has reason not to behave in the way I have postulated. He may choose not to exploit fully his trading opportunities at this moment, but to defer some trades to the future when other markets will exist and price spreads may be more favorable to him.[1]

---

[1] Another reason for changing prices is that some contingency which everyone judged very unlikely may come to pass. Since no trade was done in that contingency or any subset of it, the price spreads will be large, initially. A somewhat similar problem of reopened markets arises in Radner [4] where the problem is that new information changes some consumption sets.

---

Thus consideration of marketing costs in this simple model leads directly to the study of sequential trading. In these models there will be both futures and spot markets, and the interaction of prices in these various markets becomes the focus of interest.

References

[1] Debreu, Gerard, *Theory of Value,* New York: John Wiley & Sons, 1959.

[2] ______________, "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*, 3 (September, 1962), pp. 257-273.

[3] Debreu, Gerard and Herbert Scarf, "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4 (September, 1963), pp. 235-246.

[4] Radner, Roy, "Competitive Equilibrium under Uncertainty," *Econometrica*, 36 (January, 1968), pp. 31-58.