

Characterizing Vickrey allocation rule by anonymity

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Abstract We consider the problem of allocating finitely many units of an indivisible good among a group of agents when each agent receives at most one unit of the good and pays a non-negative price. For example, imagine that a government allocates a fixed number of licenses to private firms, or that it distributes equally divided lands to households. *Anonymity in welfare* is a condition of impartiality in the sense that it requires allocation rules to treat agents equally in welfare terms from the viewpoint of agents who are ignorant of their own valuations or identities. We show that the Vickrey allocation rule is the unique allocation rule satisfying *strategy-proofness*, *anonymity in welfare*, and *individual rationality*.

1 Introduction

We consider the problem of allocating finitely many units of an indivisible good among a group of agents when each agent receives at most one unit of the good and pays a non-negative price. For example, imagine that a government allocates a fixed number of licenses to private firms, or that it distributes equally divided lands to households.¹

¹ The vehicle ownership license system in Singapore is another example, where ownership licenses are distributed among residents through auction.

Two independent articles, [Ashlagi \(2008\)](#) and [Serizawa \(2006\)](#), were merged into this article.

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A number of allocation rules, including several forms of auction, have been proposed for various social purposes such as efficiency, revenue maximization, etc. For the purpose of efficiency, it is known that one rule has a remarkable feature. It is *the Vickrey (allocation) rule*.² First, the Vickrey rule allocates the good to those agents who value it most highly (*Efficiency*). Second, it extracts true information on agents' valuations from them (*Strategy-proofness*). Third, it induces agents' voluntary participation (*Individual rationality*). Most importantly, the Vickrey rule is the unique rule satisfying these three properties. It is also known that this rule satisfies a condition of impartiality, which we call *anonymity in welfare*. In this paper, we characterize the Vickrey rule by focusing on this impartiality condition instead of efficiency. Our characterization emphasizes that the Vickrey rule has a remarkable feature for the purpose of impartiality as well.

An (*allocation*) rule is generally formulated as a function from the set of agents' valuations to the feasible set. Given a rule, as the agents' private valuations are not known to the other agents, there may be incentives for agents to misrepresent their valuations in order to manipulate the final outcomes in their favor. As a result, the actual outcomes may not constitute a socially desirable allocation relative to the agents' true valuations. If a rule is immune to such behavior, that is, if each agent's dominant strategy is to announce his true valuation, then the rule is said to be *strategy-proof*. In addition, a condition of *individual rationality* is often imposed on rules to induce agents' participation; it says that a rule never selects an allocation that makes some agent worse off than he would be if he received no good and paid nothing. It is important to know what rules satisfy *efficiency*, *strategy-proofness*, and *individual rationality*. The *Vickrey rule* does; for this rule, the agents with the m highest valuations of the goods receive one unit each and pay the $(m + 1)$ -th valuation, while the other agents pay nothing, where m is the number of the units of the indivisible good to be allocated. It is proved in the literature mentioned later that *the Vickrey rule is the unique rule satisfying efficiency, strategy-proofness, and individual rationality*. This result emphasizes the distinguished importance of the Vickrey rule for the purpose of efficiency in the environment where *strategy-proofness* and *individual rationality* are indispensable.

However, society members are often more concerned with impartiality than with efficiency. *Anonymity in bundles* is a condition of impartiality in the sense that it requires rules to treat agents equally from the viewpoint of agents who are ignorant of their own valuations or identities; it states that when the valuations of two agents are swapped, their *bundles* under the rule are also swapped. *Anonymity in bundles* is employed as a impartiality condition in many literature such as [Sprumont \(1991\)](#), [Moulin \(1994\)](#), [Barberà and Jackson \(1995\)](#), [Serizawa \(1999\)](#), etc. However, no rule satisfies *anonymity in bundles* in our model. A condition embodying a similar idea of impartiality is *anonymity in welfare*; it states that when the valuations of two agents are swapped, their *welfares* under the rule are also swapped. This condition is weaker than *anonymity in bundle*, and is satisfied by many interesting rules including the Vickrey rule even in our model. Thus, *anonymity in welfare* is a suitable condition of impar-

² see [Vickrey \(1961\)](#)

tiality in this article. We establish that *the Vickrey rule is the unique rule satisfying strategy-proofness, anonymity in welfare, and individual rationality* (Theorem 2).

We discuss other literature related to our result. Holmström (1979) analyzes models with public alternatives and money, and establishes that a rule satisfies *efficiency* and *strategy-proofness* if and only if it is a rule of the class of rules, called “Groves rules.”³ Since allocating indivisibles can be interpreted as public alternatives, his result can be applied to our model⁴, and imply that the Vickrey rule is the unique rule satisfying *efficiency*, *strategy-proofness*, and *individual rationality*.⁵ Recently, Chew and Serizawa (2007) characterize the Vickrey rule similarly by induction logic.

There is also a body of literature analyzing the fairness of the Vickrey rule. Most of it focuses on *envy-freeness*. A rule is *envy-free* if no agent prefers another agent’s consumption bundle to his own. In a more general model with heterogeneous indivisible goods where nonquasi-linear preferences are accommodated, Svensson (1983) shows that *envy-freeness* of allocation implies *efficiency*.⁶ This result, together with Holmström (1979) and Chew and Serizawa (2007), implies that *the Vickrey rule is the unique rule satisfying strategy-proofness, envy-freeness, and individual rationality*. Many authors such as Ohseto (2006), Sakai (2008), etc., apply Svensson (1983) result to characterize the Vickrey rule. Besides, recently in problems of assigning objects, Atlamaz and Yengin (2008), and Moulin (2007) analyze variants of Vickrey rules by conditions of *k-fairness*. A rule is *k-fair* if all agents are always guaranteed $(1/n)$ -th of the *k*-th highest valuation, where *n* is the number of agents.

In this article, we do not impose *envy-freeness* on rules, and so we cannot apply Svensson (1983) result. Instead, we show that *strategy-proofness, anonymity in welfare, and individual rationality together imply efficiency* (Proposition). This result, together with Holmström (1979), and Chew and Serizawa (2007), implies our characterization of the Vickrey rule (Theorem 2). Focusing on *anonymity in welfare*, our characterization emphasizes that the Vickrey rule also has a remarkable impartiality property, and it complements Holmström (1979) result, Chew and Serizawa (2007), and those of Atlamaz and Yengin (2008), and Moulin (2010).

Some authors investigate the logical relationship of *anonymity* with other conditions in general models. For example, Pápai (2003) results entail that under *efficiency* and *strategy-proofness*, *envy-freeness* implies *anonymity in welfare*. Yengin (2010) shows that under *efficiency* and *strategy-proofness*, *equal welfare for equals* is equivalent to *anonymity in welfare*. Our article also complements those results.

Section 2 sets up the model, defines basic notions, and states the main result. Section 3 concludes by discussing open questions. Appendix provides the proof of the

³ see Groves (1973).

⁴ Holmström (1979) result can be applied to more general models where there are several indivisible goods and agents consume more than one indivisible good.

⁵ Similar characterizations of Groves’ rules in models with public alternatives have been previously established by Green and Laffont (1977) and Walker (1978). However, the characterizations of these two articles cannot be applied to allocation rules of indivisible goods because they assume that the class of admissible preferences includes preferences that are not admissible in the model of indivisible goods allocation.

⁶ See also Alkan et al. (1991), Maskin (1987), and Tadenuma and Thomson (1991).

Proposition that *strategy-proofness, anonymity in welfare, and individual rationality* together imply *efficiency*.

2 Model and main result

The set of agents is $N = \{1, \dots, n\}$. There are $m < n$ units of an indivisible good. An *item allocation* is a n -tuple $x = (x_1, \dots, x_n) \in \{0, 1\}^n$ such that $\sum x_i = m$, where for each $i \in N$, x_i is the number of units of the good agent i receives.⁷ Since $x_i = 0$ or $x_i = 1$ for each $i \in N$, agents can receive at most one unit of the good. We denote the set of item allocations by X , that is, $X = \{x = (x_1, \dots, x_n) \in \{0, 1\}^n : \sum x_i = m\}$. For each $i \in N$, we denote agent i 's payment by $p_i \in \mathbb{R}_+$. We assume that payments are nonnegative. The feasible set is $Z = X \times \mathbb{R}_+^n$. An allocation is a pair $z = (x, p) = (x_1, \dots, x_n; p_1, \dots, p_n) \in Z$, and agent i 's assignment of z is $z_i = (x_i, p_i)$.

Each agent $i \in N$ has a quasi-linear preference (relation) R_i on $\{0, 1\} \times \mathbb{R}_+$; that is, there is a valuation $v_i \in \mathbb{R}_+$ such that for all $(x_i, p_i) \in \{0, 1\} \times \mathbb{R}_+$, and all $(\hat{x}_i, \hat{p}_i) \in \{0, 1\} \times \mathbb{R}_+$, $(x_i, p_i) R_i (\hat{x}_i, \hat{p}_i)$ if and only if $v_i \cdot x_i - p_i \geq v_i \cdot \hat{x}_i - \hat{p}_i$. Given a preference R_i , we denote the associated strict and indifference relations by P_i and I_i respectively.⁸ We denote the set of agent i 's valuations by V_i . We assume $V_i = \mathbb{R}_+$.

Let $V = V_1 \times \dots \times V_n$. A valuation profile is an element $v = (v_1, \dots, v_n) \in V$. Given $v = (v_1, \dots, v_n) \in V$, $N' \subseteq N$, and $i \in N$, $v_{N'}$ denotes $(v_j)_{j \in N'}$ and v_{-i} denotes $(v_j)_{j \in N \setminus \{i\}}$. Given $v = (v_1, \dots, v_n) \in V$ and $\hat{v}_i \in V_i$, (\hat{v}_i, v_{-i}) denotes the valuation profile $(v_1, \dots, v_{i-1}, \hat{v}_i, v_{i+1}, \dots, v_n)$. Similarly, given $v \in V$, $N' \subseteq N$, and $\hat{v}_{N'} \in V_{N'} \equiv \prod_{i \in N'} V_i$, $(\hat{v}_{N'}, v_{-N'})$ denotes the valuation profile generated from v by replacing the valuations of the set N' of agents by $\hat{v}_{N'}$.

An (**allocation**) **rule** is a function f from V to Z . Given a rule f and a valuation profile $v \in V$, we write that $f(v) = (x(v), p(v))$, where $x(v) = (x_1(v), \dots, x_n(v))$ and $p(v) = (p_1(v), \dots, p_n(v))$ denote the outcome item allocation and payments of f for v , respectively. In addition, we write $f_i(v) = (x_i(v), p_i(v))$.

We introduce several conditions on rules. The first one is "efficiency." This condition states that a rule should maximize the total valuation.

Efficiency: for all $v \in V$, $x(v) \in \arg \max \{ \sum_{i \in N} v_i \cdot x_i : x \in X \}$.

Next, we introduce impartiality conditions. The first such condition is *equal bundles for equals*⁹; it states that a rule should give the same allocation to agents with the same valuation.

Equal bundles for equals: for all $v \in V$, $[v_i = v_j \implies f_i(v) = f_j(v)]$.

Equal bundles for equals is an impartiality condition that is often used in social choice theory. However, in the environment where there are fewer units to be allo-

⁷ We assume that all the units of the good are allocated to agents.

⁸ For all z_i and \hat{z}_i ,

$$\begin{aligned} z_i P_i \hat{z}_i &\Leftrightarrow [z_i R_i \hat{z}_i \ \&\not\&- (\hat{z}_i R_i z_i)], \\ z_i I_i \hat{z}_i &\Leftrightarrow [z_i R_i \hat{z}_i \ \&\hat{z}_i R_i z_i]. \end{aligned}$$

⁹ This condition is also called "strong symmetry."

cated than the number of agents, no rule satisfies *equal bundles for equals*. Thus, we introduce the following weaker condition of *equal welfares for equals*¹⁰; it states that a rule should give indifferent bundles to agents with the same valuation.

Equal welfares for equals: for all $v \in V$, $[v_i = v_j \implies f_i(v)I_i f_j(v)]$.

These two conditions say nothing about agents' bundles whose valuations are not equal. Thus, we introduce "anonymity conditions." *Anonymity in bundle* states that when the valuations of two agents are swapped, their bundles are also swapped. This condition requires a rule to treat agents equally from the viewpoints of agents who are ignorant of their own valuations or identities.

Anonymity in bundle: for all $v \in V$, all $i \in N$, all $j \in N$, all $\widehat{v}_i \in V_i$, and all $\widehat{v}_j \in V_j$,

$$[\widehat{v}_i = v_j \text{ and } \widehat{v}_j = v_i] \implies [f_i(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = f_j(v) \text{ and } f_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = f_i(v)].$$

Note that *anonymity in bundle* implies *equal bundles for equals*, and that in our environment, no rule satisfies *anonymity in bundle*. Thus, we introduce a weaker condition of *anonymity in welfare*; it states that when the valuations of two agents are swapped, their welfare positions are also swapped.¹¹

Anonymity in welfare: for all $v \in V$, all $i \in N$, all $j \in N$, all $\widehat{v}_i \in V_i$, and all $\widehat{v}_j \in V_j$,

$$[\widehat{v}_i = v_j \text{ and } \widehat{v}_j = v_i] \implies [\widehat{v}_i \cdot \widehat{x}_i - \widehat{p}_i = v_j \cdot x_j - p_j \text{ and } \widehat{v}_j \cdot \widehat{x}_j - \widehat{p}_j = v_i \cdot x_i - p_i],$$

where $x_k = x_k(v)$, $p_k = p_k(v)$, $\widehat{x}_k = x_k(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}})$ and $\widehat{p}_k = p_k(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}})$ for $k = i, j$.

Note that *anonymity in welfare* implies *equal welfares for equals*. In Sect. 3, we employ this implication repeatedly to prove the Proposition below.

Envy-freeness requires that no agent should prefer another agent's bundle to his own. In contrast to the above anonymity conditions, this condition compares agents' welfare from the viewpoints of agents whose identities and valuations are specified.

Envy-freeness: for all $v \in V$, all $i \in N$, and all $j \in N$, $f_i(v)R_i f_j(v)$.

The next requirement, *strategy-proofness*, is that for each agent, revealing his true valuation should be a dominant strategy in the direct revelation game. Since the dominant strategy equilibrium is a strong equilibrium concept, *strategy-proofness* gives agents strong incentives to represent their true valuations.

Strategy-proofness: for all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, $f_i(v)R_i f_i(\widehat{v}_i, v_{-i})$.

Individual rationality induces agents to participate voluntarily by guaranteeing that a rule never assigns an outcome that makes some agent worse off than he would be at his status quo (0, 0).

Individual rationality: for all $v \in V$ and all $i \in N$, $f_i(v)R_i(0, 0)$.

¹⁰ This condition is also called "symmetry."

¹¹ An alternative definition of *anonymity in welfare* is that when the valuations of a group of agents are permuted, their welfare positions are permuted in the same way. However, since any permutation can be written as a product of simple transpositions, the two definitions of *anonymity in welfare* are equivalent.

Given a valuation profile $v \in V$, we rank agents' valuations, and denote agent i 's rank by $r(v, i)$. That is, if v_i is the highest in $\{v_1, \dots, v_n\}$, $r(v, i) = 1$; if it is the second highest, $r(v, i) = 2$; and so on. When several agents have the same valuation, ties are broken in a prespecified way.

One example of tie-breaker is to use their indices in increasing order. For instance, if a valuation profile is $v = (v_1, v_2, v_3, v_4) = (5, 4, 7, 5)$, then $r(v, 1) = 2$, $r(v, 2) = 4$, $i(v, 3) = 1$, and $r(v, 4) = 3$. Ranking agents with the same valuation according to their indices in decreasing order is another example. Thus, a ranking function $r(\cdot, \cdot)$ is associated with a certain tie-breaker. It is possible to construct ranking functions with more complex tie-breakers such as examples below.

Example 1 Let $n = 2$. The ranking function $r(\cdot, \cdot)$ is such that $r(v, 1) = 1$ if $v_1 = v_2 \geq 4$, but $r(v, 1) = 2$ if $v_1 = v_2 < 4$. In this ranking function, when the two agents have the same valuation, which agent is ranked first depends on the level of the common valuation.

Example 2 Let $n = 3$. The ranking function $r(\cdot, \cdot)$ is such that if $v_1 = v_2 \geq v_3$, then $r(v, 1) = 1$ and $r(v, 2) = 2$; if $v_1 = v_2 < v_3$, then $r(v, 1) = 3$ and $r(v, 2) = 2$. In this ranking function, when agents 1 and 2 have the same valuation, which of agent 1 or 2 is ranked higher depends on agent 3's valuation.

Given a valuation profile $v \in V$ and a ranking function $r(\cdot, \cdot)$, we denote the inverse of $r(v, \cdot)$ by $i(v, \cdot)$, that is, $i(v, k)$ is the agent whose valuation is the k -th highest. For instance, if a valuation profile is $v = (v_1, v_2, v_3, v_4) = (5, 4, 7, 5)$, and $r(v, 1) = 2$, $r(v, 2) = 4$, $r(v, 3) = 1$, and $r(v, 4) = 3$, then $i(v, 1) = 3$, $i(v, 2) = 1$, $i(v, 3) = 4$, and $i(v, 4) = 2$.

Under the Vickrey rule defined below, agents with the m highest valuations receive the goods and pay the $(m + 1)$ -th highest valuation, $v_{i(v, m+1)}$, and the other agents pay nothing. For instance, suppose that ties are broken according to agents' indices in increasing order. Consider the case in which $n = 3$, $m = 1$, and a valuation profile is $(v_1, v_2, v_3) = (1, 1, 0)$. Then, agents 1 receives the good, and since the 2-nd highest valuation $v_{i(v, 2)}$ is $v_2 = 1$, agents 1 pays 1. Or consider the case in which $n = 6$, $m = 3$, and a valuation profile is $(v_1, v_2, v_3, v_4, v_5, v_6) = (5, 3, 7, 5, 2, 3)$. Then, agents 1, 3, and 4 receive the good, and since the 4-th highest valuation $v_{i(v, 4)}$ is $v_2 = 3$, agents 1, 3 and 4 pay 3.

Definition A **Vickrey (allocation) rule** is a rule $f^* = (x^*(\cdot), p^*(\cdot))$ such that:

$$\text{for all } v \in V \text{ and all } i \in N, \quad (x_i^*(v), p_i^*(v)) = \begin{cases} (1, v_{i(v, m+1)}) & \text{if } r(v, i) \leq m, \text{ and} \\ (0, 0) & \text{otherwise.} \end{cases}$$

Since the Vickrey rules are accompanied by ranking functions, and since ranking functions are not unique, the Vickrey rules are not unique. In other words, there are as many Vickrey rules as ways of tie-breaking. However, unless several agents have the m -th highest valuation, regardless of ranking function, all the Vickrey rules give the same allocation. Furthermore, as Remark below states, all the Vickrey rules are welfare-equivalent. Therefore, we treat Vickrey rules as if they were unique.

Remark Let f^* and \widehat{f}^* be Vickrey rules. Then:

- (i) $\sum_{i \in N} x_i^*(v) = \sum_{i \in N} \widehat{x}_i^*(v)$ for any $v \in V$,
- (ii) $v_i \cdot x_i^*(v) - p_i^*(v) = v_i \cdot \widehat{x}_i^*(v) - \widehat{p}_i^*(v)$ for any $v \in V$ and any $i \in N$,

where $x_i^*(v)$ and $\widehat{x}_i^*(v)$ are, respectively, agent i 's item allocations under f^* and \widehat{f}^* for v , and $p_i^*(v)$ and $\widehat{p}_i^*(v)$ are, respectively, his payments under f^* and \widehat{f}^* for v .

By its definition, the Vickrey rule satisfies *efficiency*. It is straightforward that the Vickrey rule satisfies *strategy-proofness* and *individual rationality*. Theorem 1 below states that the Vickrey rule is the unique rule satisfying those three conditions.

Theorem 1 *A rule satisfies strategy-proofness, efficiency, and individual rationality if and only if it is the Vickrey rule.*

Theorem follows from Holmström (1979). The proof given below is a simplified version of the proof given by Chew and Serizawa (2007).

Proof We only show that if a rule satisfies *strategy-proofness*, *efficiency*, and *individual rationality*, then it is the Vickrey rule. Let a rule f satisfy *strategy-proofness*, *efficiency* and *individual rationality*. Suppose that f is not the Vickrey rule.

Since f satisfies *efficiency* and *individual rationality*, there are $v \in V$ and $i \in N$ such that $v_i \geq v_{i(v,m)}$, $x_i(v) = 1$ and $p_i(v) \neq v_{i(v,m+1)}$. Since f satisfies *individual rationality*, $p_i(v) \leq v_i$. Thus, $v_{i(v,m+1)} < p_i(v) \leq v_i$, or $p_i(v) < v_{i(v,m+1)}$.

First, consider the case where $v_{i(v,m+1)} < p_i(v) \leq v_i$. Let $\widehat{v}_i \in (v_{i(v,m+1)}, p_i(v))$. Then, since f satisfies *efficiency* and *individual rationality*, $x_i(\widehat{v}_i, v_{-i}) = 1$ and $p_i(\widehat{v}_i, v_{-i}) \leq \widehat{v}_i < p_i(v)$. Thus, $v_i \cdot x_i(\widehat{v}_i, v_{-i}) - p_i(\widehat{v}_i, v_{-i}) > v_i \cdot x_i(v) - p_i(v)$, contradicting *strategy-proofness*.

Next, consider the case where $p_i(v) < v_{i(v,m+1)}$. Let $\widehat{v}_i \in (p_i(v), v_{i(v,m+1)})$. Then, since f satisfies *efficiency* and *individual rationality*, $x_i(\widehat{v}_i, v_{-i}) = 0$ and $p_i(\widehat{v}_i, v_{-i}) = 0$. Thus, $\widehat{v}_i \cdot x_i(v) - p_i(v) > \widehat{v}_i \cdot x_i(\widehat{v}_i, v_{-i}) - p_i(\widehat{v}_i, v_{-i})$, contradicting *strategy-proofness*.

Hence, f is the Vickrey rule. □

No rule satisfies *anonymity in bundle* in our model, but all the Vickrey rules satisfy *anonymity in welfare*. The Proposition below says that the condition of *anonymity in welfare*, together with *strategy-proofness* and *individual rationality*, implies *efficiency*. We prove this Proposition in Appendix.

Proposition *If a rule satisfies strategy-proofness, anonymity in welfare, and individual rationality, it also satisfies efficiency.*

As Svensson (1983) shows, *envy-freeness* alone implies *efficiency*. As Example 3 illustrates, *anonymity in welfare* alone does not imply *efficiency*.

Example 3 Let $m = 1$ and $n = 2$. Let f be the rule such that if $v_1 \leq v_2$, $f(v) = ((1, 0); (v_2, 0))$, and if $v_1 > v_2$, $f(v) = ((0, 1); (0, v_1))$. Then, f satisfies *anonymity in welfare* but not *efficiency*. The rule f also violates *strategy-proofness* and *individual rationality*.

As we mentioned above, the Vickrey rule satisfies *strategy-proofness*, *individual rationality* and *anonymity in welfare*. Thus, Theorem 1 and the Proposition together imply Theorem 2 below. Theorem 2 states that the Vickrey rule is the unique rule satisfying those three conditions.

Theorem 2 *A rule satisfies strategy-proofness, anonymity in welfare, and individual rationality if and only if it is the Vickrey rule.*

The three examples below illustrate that the three axioms in Theorem 2 are independent.

Example 4 Let $m = 1$ and $n = 2$. Let f be the rule such that if $v_1 \geq 1$, $f(v) = ((1, 0); (1, 0))$, and if $v_1 < 1$, $f(v) = ((0, 1); (0, 0))$. Then, f satisfies *strategy-proofness* and *individual rationality*, but not *anonymity in welfare*.

Example 5 Let $m = 1$ and $n = 2$. Let f be the rule such that if $v_1 \geq v_2$, $f(v) = ((1, 0); (v_1, 0))$, and if $v_1 < v_2$, $f(v) = ((0, 1); (0, v_2))$. Then, f satisfies *anonymity in welfare*, and *individual rationality*, but not *strategy-proofness*.

Example 6 Let $m = 1$ and $n = 2$. Let f be the rule such that if $v_1 \geq v_2$, $f(v) = ((1, 0); (2v_2, v_1))$, and if $v_1 < v_2$, $f(v) = ((0, 1); (v_2, 2v_1))$. Then, f satisfies *strategy-proofness*, and *anonymity in welfare*, but not *individual rationality*.

3 Concluding remarks

In this article, we considered the problem of allocating finitely many units of an indivisible good when each agent receives at most one unit of the good and pays a non-negative price. We established that *strategy-proofness*, *anonymity in welfare*, and *individual rationality* together imply *efficiency*, and that *the Vickrey rule is the unique rule satisfying strategy-proofness, anonymity in welfare, and individual rationality*. We close this article by mentioning open questions we left.

As we discussed in Introduction, Svensson (1983) showed that *envy-freeness* of allocation implies *efficiency* in a more general model with heterogeneous indivisible goods where even nonquasi-linear preferences are accommodated. The results of Holmström (1979) and Chew and Serizawa (2007) can be applied to general models with heterogeneous indivisible goods where agents may consume more than one indivisible good as long as preferences are quasi-linear.¹² That is, the Vickrey rule is the unique rule satisfying *strategy-proofness*, *efficiency*, and *individual rationality* in such general models. We conjecture that (i) *strategy-proofness*, *anonymity in welfare*, and *individual rationality* together also imply *efficiency* in general models, and that (ii) *the Vickrey rule is also the unique rule satisfying strategy-proofness, anonymity in welfare, and individual rationality* in general models.

Hashimoto and Saitoh (2009) recently studied a queuing model in which preferences are quasi-linear, and are represented by constant unit waiting cost. They applied

¹² Saitoh and Serizawa (2008) showed a parallel result in the model with homogenous indivisible goods and nonquasi-linear preferences. See also Roth and Sotomayor (1990)

the proof techniques of our article to show that *strategy-proofness* and *anonymity in welfare* imply *efficiency*. Since positions in queue can be interpreted as heterogeneous indivisibles, their model is a model of heterogeneous indivisibles. Although their generalization is limited in the sense that preferences are represented by single parameters, Hashimoto and Saitoh (2009) result suggests that the techniques we developed in our article will be useful in generalized models if the techniques are modified suitably. However, such modifications need considerable amount of effort, and so we leave them in this article although we believe that they are interesting questions in future research.

As Pápai (2003) showed, when agents may consume more than one indivisible good and valuations are not restricted, no rule in the class of Groves (including the Vickrey rule) satisfies *envy-freeness*.¹³ On the other hand, the Vickrey rule satisfies *anonymity in welfare* even when agents may consume more than one indivisible good. Thus, the conjecture (ii) above will contrast *anonymity in welfare* with *envy-freeness* if it is established.

Appendix: proof of proposition

In this appendix, we prove the Proposition stated in Sect. 2. First, we state and prove several lemmas as preliminary results.

Lemma 1 *Let f be an individually rational rule. For all $v \in V$ and all $i \in N$, if $x_i(v) = 0$, then $p_i(v) = 0$.*

Lemma 1 directly follows from *individual rationality*. Thus, we omit its proof.

Lemma 2 *Let f be a strategy-proof rule. For all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, if $x_i(\widehat{v}_i, v_{-i}) = x_i(v)$, then $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$.*

Proof Let $v \in V$, $i \in N$, and $\widehat{v}_i \in V_i$ be such that $x_i(\widehat{v}_i, v_{-i}) = x_i(v)$. If $p_i(\widehat{v}_i, v_{-i}) > p_i(v)$, $f_i(v) \widehat{P}_i f_i(\widehat{v}_i, v_{-i})$, and if $p_i(\widehat{v}_i, v_{-i}) < p_i(v)$, $f_i(\widehat{v}_i, v_{-i}) \widehat{P}_i f_i(v)$. Both cases contradict *strategy-proofness*. Thus, $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$. □

Lemma 3 *Let f be a strategy-proof and individually rational rule.*

- (a) *For all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, if $x_i(v) = 1$ and $\widehat{v}_i > v_i$, then $x_i(\widehat{v}_i, v_{-i}) = 1$ and $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$.*
- (b) *For all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, if $x_i(v) = 0$ and $\widehat{v}_i < v_i$, then $x_i(\widehat{v}_i, v_{-i}) = 0$ and $p_i(\widehat{v}_i, v_{-i}) = 0$.*

Proof of a: Let $v \in V$, $i \in N$, and $\widehat{v}_i \in V_i$ be such that $x_i(v) = 1$ and $\widehat{v}_i > v_i$. Suppose by contradiction that $x_i(\widehat{v}_i, v_{-i}) = 0$. Then, by Lemma 1, $p_i(\widehat{v}_i, v_{-i}) = 0$. By *individual rationality*, $p_i(v) \leq v_i < \widehat{v}_i$. Thus, $\widehat{v}_i \cdot x_i(v) - p_i(v) > 0 = \widehat{v}_i \cdot x_i(\widehat{v}_i, v_{-i}) - p_i(\widehat{v}_i, v_{-i})$. This contradicts *strategy-proofness*. Therefore, $x_i(\widehat{v}_i, v_{-i}) = 1 = x_i(v)$, and by Lemma 2, $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$. □

¹³ This negative result holds when preferences are not restricted unless they are quasi-linear.

Proof of b: Let $v \in V, i \in N$, and $\widehat{v}_i \in V_i$ be such that $x_i(v) = 0$ and $\widehat{v}_i < v_i$. Suppose by contradiction that $x_i(\widehat{v}_i, v_{-i}) = 1$. By Lemma 1, $p_i(v) = 0$. By individual rationality, $p_i(\widehat{v}_i, v_{-i}) \leq \widehat{v}_i < v_i$. Thus, $v_i \cdot x_i(\widehat{v}_i, v_{-i}) - p_i(\widehat{v}_i, v_{-i}) > 0 = v_i \cdot x_i(v) - p_i(v)$. This contradicts strategy-proofness. Therefore, $x_i(\widehat{v}_i, v_{-i}) = 0$, and by Lemma 1, $p_i(\widehat{v}_i, v_{-i}) = 0$. \square

Lemma 4 *Let f be a rule satisfying strategy-proofness, anonymity in welfare, and individual rationality.*

- (a) *Let $v \in V, i \in N, \widehat{v}_i \in V_i, v_0 > 0$, and $N' \subseteq N$ be such that $x_i(v) = 1, v_i < v_0 = \widehat{v}_i$, and $v_j = v_0$ for all $j \in N'$. Then, $x_i(\widehat{v}_i, v_{-i}) = x_j(\widehat{v}_i, v_{-i}) = 1$ for all $j \in N'$.*
- (b) *Let $v \in V, i \in N, \widehat{v}_i \in V_i, v_0 \geq 0$, and $N' \subseteq N$ be such that $x_i(v) = 0, v_i > v_0 = \widehat{v}_i$, and $v_j = v_0$ for all $j \in N'$. Then, $x_i(\widehat{v}_i, v_{-i}) = x_j(\widehat{v}_i, v_{-i}) = 0$ for all $j \in N'$.*

Proof of a: By Lemma 3(a), since $x_i(v) = 1$ and $v_i < \widehat{v}_i$, we have $x_i(\widehat{v}_i, v_{-i}) = 1$ and $p_i(\widehat{v}_i, v_{-i}) = p_i(v) \leq v_i < v_0$. Thus, since $v_j = v_0 = \widehat{v}_i$ for all $j \in N'$, it follows from anonymity in welfare that $v_j \cdot x_j(\widehat{v}_i, v_{-i}) - p_j(\widehat{v}_i, v_{-i}) = \widehat{v}_i \cdot x_i(\widehat{v}_i, v_{-i}) - p_i(\widehat{v}_i, v_{-i}) \geq v_0 - p_i(v) > 0$ for all $j \in N'$. Therefore, by Lemma 1, $x_j(\widehat{v}_i, v_{-i}) = 1$ for all $j \in N'$. \square

Proof of b: By Lemma 1, since $x_i(v) = 0$, we have $p_i(v) = 0$. By Lemma 3(b), since $x_i(v) = 0$, and $\widehat{v}_i < v_i$, we have $x_i(\widehat{v}_i, v_{-i}) = 0$, and so by Lemma 1, $p_i(\widehat{v}_i, v_{-i}) = 0$. Thus, since $v_j = v_0 = \widehat{v}_i$ for all $j \in N'$, it follows from anonymity in welfare that $v_j \cdot x_j(\widehat{v}_i, v_{-i}) - p_j(\widehat{v}_i, v_{-i}) = \widehat{v}_i \cdot x_i(\widehat{v}_i, v_{-i}) - p_i(\widehat{v}_i, v_{-i}) = 0$ for all $j \in N'$. Therefore, for all $j \in N'$, if $x_j(\widehat{v}_i, v_{-i}) = 1, p_j(\widehat{v}_i, v_{-i}) = v_0$.

Since $x_i(\widehat{v}_i, v_{-i}) = 0$, suppose that there is $j \in N'$ such that $x_j(\widehat{v}_i, v_{-i}) = 1$. Let $\widehat{v}_j = v_i$. Then, since $v_i > v_0 = v_j$, Lemma 3(a) implies that $x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1, p_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = p_j(\widehat{v}_i, v_{-i}) = v_0 < v_i = \widehat{v}_j$, and $\widehat{v}_j \cdot x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) - p_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = \widehat{v}_j - v_0 > 0$. Since $v_i \cdot x_i(v) - p_i(v) = 0, \widehat{v}_i = v_0 = v_j$, and $\widehat{v}_j = v_i$, this contradicts anonymity in welfare. Hence, for all $j \in N', x_j(\widehat{v}_i, v_{-i}) = 0$. \square

Now we prove the Proposition.

Let f be a rule satisfying strategy-proofness, anonymity in welfare, and individual rationality. Let $v \in V$. We show that $f(v)$ is efficient for v . Without loss of generality, assume $v_1 \geq v_2 \geq \dots \geq v_m = \dots = v_{n'} > v_{n'+1} \geq v_{n'+2} \geq \dots \geq v_n$,¹⁴ and let $\bar{v} = v_m$. Note that for all $i \in N, v_i > \bar{v}$ implies $i < m$, and that $v_j < \bar{v}$ implies $j > n'$. Therefore, $f(v)$ is efficient for v if and only if

- (A) for all $i \in N, v_i > \bar{v}$ implies $x_i(v) = 1$, and
- (B) for all $i \in N, x_i(v) = 1$ implies $i \leq n'$.

Thus, if $f(v)$ is not efficient for v , then (A) or (B) is violated.

¹⁴ n' may be equal to m or n . In the case where $n' = m, v_1 \geq v_2 \geq \dots \geq v_m > v_{m+1} \geq v_{m+2} \geq \dots \geq v_n$. In the case where $n' = n, v_1 \geq v_2 \geq \dots \geq v_m = v_{m+1} = \dots = v_n$.

If (B) is violated, that is, if there is an agent $i \in N$ such that $x_i(v) = 1$ and $i > n'$, then we raise his valuation to $\widehat{v}_i = \bar{v}$, and by Lemma 4(a), we obtain that

$$x_i(\widehat{v}_i, v_{-i}) = x_m(\widehat{v}_i, v_{-i}) = x_{m+1}(\widehat{v}_i, v_{-i}) = \dots = x_{n'}(\widehat{v}_i, v_{-i}) = 1.$$

Since $i > n'$ and there are only m units of the goods, it follows that there is an agent $j < m$ such that $x_j(\widehat{v}_i, v_{-i}) = 0$. $j < m$ implies $v_j \geq \bar{v}$. If $v_j = \bar{v}$, Lemma 4(a) implies $x_j(\widehat{v}_i, v_{-i}) = 1$, contradicting $x_j(\widehat{v}_i, v_{-i}) = 0$. Thus, $v_j > \bar{v}$. For $j \in N$, (A) is violated. Thus, we assume that (A) is violated and derive a contradiction.

Since (A) is violated, there is an agent i such that $v_i > \bar{v}$ and $x_i(v) = 0$. $v_i > \bar{v}$ implies $i < m$. We lower his valuation to $\widehat{v}_i = \bar{v}$, and by Lemma 4(b), we obtain

$$x_i(\widehat{v}_i, v_{-i}) = x_m(\widehat{v}_i, v_{-i}) = x_{m+1}(\widehat{v}_i, v_{-i}) = \dots = x_{n'}(\widehat{v}_i, v_{-i}) = 0.$$

Since $i < m$ and the supply is m , there is an agent j such that $j > n'$, and $x_j(\widehat{v}_i, v_{-i}) = 1$. $j > n'$ implies $v_j < \bar{v}$.

We raise the valuation of j to $\widehat{v}_j = \bar{v}$, and by Lemma 4(a), we obtain

$$\begin{aligned} x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) &= x_i(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_m(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) \\ &= x_{m+1}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = \dots = x_{n'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1. \end{aligned}$$

Since there are only m units of the goods, it follows that there is an agent $i' < m$ such that $i' \neq i$, and $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$. $i' < m$ implies $v_{i'} \geq \bar{v}$. If $v_{i'} = \bar{v}$, Lemma 4(a) implies $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1$, contradicting $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$. Thus, $v_{i'} > \bar{v}$.

Again, we lower the valuation of i' to $\widehat{v}_{i'} = \bar{v}$ and by Lemma 4(b), we obtain

$$\begin{aligned} x_{i'}(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) &= x_i(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = x_j(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) \\ &= x_m(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = x_{m+1}(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = \dots = x_{n'}(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = 0. \end{aligned}$$

In this manner, we can continue to apply Lemma 4(a) and 4(b) alternately.¹⁵ As the result of the application of Lemma 4(a) and 4(b), respectively, we find an agent with a valuation below \bar{v} who is awarded an object, and an agent with a valuation above \bar{v} who is not awarded an object. We apply again Lemma 4(b) and 4(a), respectively, to these two cases. The application of Lemma 4(a) will alternate the outcome of all agents with the valuation of \bar{v} from not having an object allocated to having an object allocated. On the other hand, the application of Lemma 4(b) will alternate the outcome of them from having an object allocated to not having an object allocated. Notice that at each step there is one more agent with the valuation of \bar{v} so that eventually there will be at least $m + 1$ agents with valuation \bar{v} . This is a contradiction, since it is not possible to allocate more than m objects. This will happen at the latest at m -times application. But depending on the number of agents who started with the valuation of \bar{v} , this could occur at an earlier step. Hence, *efficiency* must hold.

We have completed the proof of the Proposition.

¹⁵ Serizawa (2006), which is the previous version of this article, explains the details of this argument in the form of induction.

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