## Branes, graphs and singularities

by
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M.Sc., Eötvös University (2004)

Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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#### Abstract

In this thesis, we study various aspects of string theory on geometric and nongeometric backgrounds in the presence of branes.

In the first part of the thesis, we study non-compact geometries. We introduce "brane tilings" which efficiently encode the gauge group, matter content and superpotential of various quiver gauge theories that arise as low-energy effective theories for D-branes probing singular non-compact Calabi-Yau spaces with toric symmetries. Brane tilings also offer a generalization of the AdS/CFT correspondence.

A technique is developed which enables one to quickly compute the toric vacuum moduli space of the quiver gauge theory. The equivalence of this procedure and the earlier approach that used gauged linear sigma models is explicitly shown. As an application of brane tilings, four dimensional quiver gauge theories are constructed that are AdS/CFT dual to infinite families of Sasaki-Einstein spaces. Various checks of the correspondence are performed.

We then develop a procedure that constructs the brane tiling for an arbitrary toric Calabi-Yau threefold. This solves a longstanding problem by computing superpotentials for these theories directly from the toric diagram of the singularity.

A different approach to the low-energy theory of D-branes uses exceptional collections of sheaves associated to the base of the threefold. We provide a dictionary that translates between the language of brane tilings and that of exceptional collections.

Geometric compactifications represent only a very small subclass of the landscape: the generic vacua are non-geometric. In the second part of the thesis, we study perturbative compactifications of string theory that rely on a fibration structure of the extra dimensions. Non-geometric spaces preserving $\mathcal{N}=1$ supersymmetry in four dimensions are obtained by using T-dualities as monodromies. Several examples are discussed, some of which admit an asymmetric orbifold description. We explore the possibility of twisted reductions where left-moving spacetime fermion number Wilson lines are turned on in the fiber.


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## Contents

I Introduction ..... 31
II Local geometry ..... 36
1 D-branes and quiver gauge theories ..... 37
2 Toric geometry ..... 41
3 Brane tilings ..... 45
3.0.1 Unification of quiver and superpotential data ..... 49
3.1 Dimer model technology ..... 53
3.2 An explicit correspondence between dimers and GLSMs ..... 57
3.2.1 A detailed example: the Suspended Pinch Point ..... 61
3.3 Massive fields ..... 63
3.4 Seiberg duality ..... 66
3.4.1 Seiberg duality as a transformation of the quiver ..... 66
3.4.2 Seiberg duality as a transformation of the brane tiling ..... 67
3.5 Partial resolution ..... 70
3.6 Different toric superpotentials for a given quiver ..... 72
3.7 Examples ..... 75
3.7.1 Del Pezzo 2 ..... 75
3.7.2 Pseudo del Pezzo 5 ..... 80
3.7.3 Tilings for infinite families of gauge theories ..... 82
4 Equivalence of algorithms ..... 93
4.1 Introduction ..... 93
4.2 Toric quivers and brane tilings ..... 94
4.2.1 Geometry of the tiling embedding from conformal invariance ..... 98
4.2.2 Height function ..... 99
4.3 Toric geometry from gauge theory ..... 101
4.4 The conjecture ..... 105
4.5 The proof ..... 107
4.5.1 Solving $\mathrm{F}-$ term conditions: gauge transformations and mag- netic coordinates ..... 108
4.5.2 The GLSM fields are perfect matchings ..... 111
4.5.3 Height changes as positions in a toric diagram ..... 115
4.6 Conclusions ..... 118
5 Infinite families of examples ..... 121
5.1 Introduction ..... 121
5.2 Quiver content from toric geometry ..... 124
5.2.1 General geometrical set-up ..... 124
5.2.2 Quantum numbers of fields ..... 127
5.3 The $L^{a, b, c}$ toric singularities ..... 136
5.3.1 The sub-family $L^{a, b, a}$ ..... 140
5.3.2 Quantum numbers of fields ..... 141
5.3.3 The geometry ..... 143
5.4 Superpotential and gauge groups ..... 149
5.4.1 The superpotential ..... 149
5.4.2 The gauge groups ..... 150
5.5 R-charges from $a$-maximization ..... 152
5.6 Constructing the gauge theories using brane tilings ..... 154
5.6.1 Seiberg duality and transformations of the tiling ..... 156
5.6.2 Explicit examples ..... 157
5.7 Conclusions ..... 165
5.8 Appendix: More examples ..... 166
5.8.1 Brane tiling and quiver for $L^{1,5,2}$ ..... 166
5.8.2 Brane tiling and quiver for $L^{1,7.3}$ ..... 167
6 Fast Inverse Algorithm ..... 169
6.1 Superconformal fixed point and R -charges ..... 169
6.2 Isoradial embeddings and R-charges ..... 171
6.3 Rhombus loops and zig-zag paths ..... 175
6.3.1 Inconsistent theories ..... 177
6.3.2 Conjecture of ( $p, q$ )-legs and rhombus loops ..... 181
6.3.3 Parameter space of a-maximization ..... 184
6.4 Fast Inverse Algorithm ..... 187
6.4.1 $\quad \mathcal{C}^{3} \quad(\mathcal{N}=4)$ ..... 187
6.4.2 Conifold ..... 190
6.4.3 $\quad L^{131}$ ..... 192
6.4.4 $\quad L^{152}$ ..... 194
6.5 Toric duality and Seiberg duality ..... 195
6.5.1 Seiberg duality in the hexagonal lattice with extra line ..... 197
7 Exceptional Collections ..... 201
7.1 Introduction ..... 201
7.2 Exceptional collections ..... 203
7.2.1 From Exceptional Collection to Periodic Quiver ..... 205
7.2.2 Line Bundles and Curvature Forms for Toric Surfaces ..... 208
7.2.3 Bundles on $\mathbb{P}^{2}$ ..... 212
7.2.4 Constructing the Quiver in General ..... 214
7.2.5 Vanishing Euler Character ..... 217
7.3 Compatibility ..... 218
7.3.1 Beilinson quivers and internal matchings ..... 219
7.3.2 Line bundles from tiling: The $\Psi$-map ..... 222
7.3.3 Reconstructing the quiver ..... 228
7.4 Conclusions ..... 238
III Global geometry ..... 244
8 Semi-flat spaces ..... 245
8.1 Introduction ..... 245
8.2 Semi-flat limit ..... 249
8.2.1 One dimension ..... 249
8.2.2 Two dimensions ..... 250
8.2.3 Three dimensions ..... 256
8.2.4 Flat vertices ..... 261
9 Non-geometric spaces ..... 265
9.1 Stringy monodromies ..... 265
9.1.1 Reduction to seven dimensions ..... 266
9.1.2 The perturbative duality group ..... 269
9.1.3 Embedding $S L(2)^{2}$ in $S L(4)$ ..... 272
9.1.4 U-duality and $G_{2}$ manifolds ..... 276
9.2 Compactifications with $\mathrm{D}_{\mathbf{4}}$ singularities ..... 278
9.2.1 Modified $K 3 \times T^{2}$ ..... 278
9.2.2 Non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ ..... 280
9.2.3 Asymmetric orbifolds ..... 281
9.2.4 Joyce manifolds ..... 284
9.2.5 Dualities between models ..... 290
9.2.6 U-duality and affine monodromies ..... 292
9.3 Compactifications with $\mathbf{E}_{\mathbf{n}}$ singularities ..... 293
9.3.1 Orbifold limits of $K 3$ ..... 293
9.3.2 Example: $T^{6} / \mathbb{Z}_{3}$ ..... 295
9.3.3 Example: $T^{6} / \Delta_{12}$ ..... 297
9.3.4 Example: $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$ ..... 297
9.3.5 Example: $T^{6} / \Delta_{24}$ ..... 298
9.3.6 Non-geometric modifications ..... 299
9.4 Chiral Scherk-Schwarz reduction ..... 301
9.4.1 One dimension ..... 301
9.4.2 Two dimensions ..... 302
9.5 Conclusions ..... 306
9.6 Appendix: Flat-torus reduction of type IIA to seven dimensions ..... 308
9.7 Appendix: Semi-flat vs. exact solutions ..... 310
9.8 Appendix: The Hanany-Witten effect from the semiflat approximation ..... 315
9.9 Appendix: Type IIA on $\mathbf{T}^{\mathbf{5}} / \mathbb{Z}_{\mathbf{2}}$ and Type IIB on $\mathbf{S}^{\mathbf{1}} \times \mathbf{K} \mathbf{3}$ ..... 318
9.10 Appendix: List of asymmetric orbifolds ..... 322
9.11 Appendix: Spectrum of $\mathrm{T}^{\mathbf{6}} / \mathbb{Z}_{\mathbf{2}} \times \mathbb{Z}_{\mathbf{2}}$ and the two-plaquette model ..... 325
9.12 Appendix: Spectrum of the one-plaquette model ..... 331
9.13 Appendix: Spectra of Joyce orbifolds ..... 334
9.14 Appendix: Polyhedron patterns ..... 338

## List of Figures

1-1 D3-branes probing the transverse geometry. ..... 38
1-2 Quiver of $\mathbf{d P}_{1}$. The theory contains four $U(N)$ gauge groups labeled by the nodes of the quiver. The arrows label bifundamental fields transforming in the (anti-)fundamental representation of the groups at the endpoints. ..... 39
1-3 $\mathbf{d P}_{1}$ Beilinson quiver. ..... 40
2-1 The cone for the variety. The coordinates of the spanning vectors are integers. The endpoints are coplanar following from the Calabi-Yau condition. ..... 42
2-2 The toric diagram for the conifold which can also be described by the equation $z_{1} z_{2}=z_{3} z_{4}$ with $z_{i} \in \mathbb{C}$. The normal vectors are also shown. ..... 44
2-3 The toric diagram for the del Pezzo 1 surface is shown. ..... 44
2-4 The toric diagram for $L^{1,7,3}$ which is part of the recently discovered series of $L^{a b c}$ metrics $([61,60])$. ..... 44
3-1 A finite region in the infinite brane tiling and quiver diagram for Model I of $\mathbf{d P}_{3}$. We indicate the correspondence between: gauge groups $\leftrightarrow$ faces, bifundamental fields $\leftrightarrow$ edges and superpotential terms $\leftrightarrow$ nodes. 50
3-2 The logical flowchart. ..... 51

3-3 The quiver gauge theory associated to one of the toric phases of the cone over $\mathbf{F}_{0}$. In the upper right the quiver and superpotential (3.0.6) are combined into the periodic quiver defined on $T^{2}$. The terms in the superpotential bound the faces of the periodic quiver, and the signs are indicated and have the dual-bipartite property that all adjacent faces have opposite sign. To get the bottom picture, we take the planar dual graph and indicate the bipartite property of this graph by coloring the vertices alternately. The dashed lines indicate edges of the graph that are duplicated by the periodicity of the torus. This defines the brane tiling associated to this $\mathcal{N}=1$ gauge theory.

3-4 a) Brane tiling for Model I of $\mathrm{dP}_{3}$ with flux lines indicated in red.
b) Unit cell for Model I of $\mathrm{dP}_{3}$. We show the edges connecting to
images of the fundamental nodes in green. We also indicate the signs
associated to each edge as well as the powers of $w$ and $z$ corresponding
to crossing flux lines. ..... 55
3-5 Toric diagram for Model I of $\mathrm{dP}_{3}$ derived from the characteristic poly- nomial in (3.1.10). ..... 56
3-6 F-term equations from the brane tiling perspective. ..... 60
3-7 Quiver diagram for the SPP. ..... 61
3-8 Brane tiling for the SPP. ..... 61
3-9 Toric diagram for the SPP. We indicate the perfect matchings corre-sponding to each node in the toric diagram.62
3-10 Perfect matchings for the SPP. We indicate the slopes $\left(h_{w}, h_{z}\right)$, whichallow the identification of the corresponding node in the toric diagramas shown in Figure 3-9.63

3-11 Integrating out a massive field corresponds to collapsing the two vertices adjacent to a bivalent vertex into a single vertex of higher valence. 64
3-12 The action of Seiberg duality on a periodic quiver to produce anothertoric phase of the quiver. Also marked are the signs of superpotentialterms, showing that the new terms (faces) are consistent with the pre-existing 2-coloring of the global graph.683-13 Seiberg duality acting on a brane tiling to produce another toric phase.This is the planar dual to the operation depicted in Figure 3-12. When-ever 2 -valent nodes are generated by this transformation, the corre-sponding massive fields can be integrated out as explained in Section3.3. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 69
3-14 The operation of Seiberg duality on a phase of $\mathbf{F}_{0}$. ..... 69
3-15 Removing the edge from between faces 5 and 6 Higgses Model I of $\mathrm{dP}_{3}$ (top) to one of the two toric phases of $\mathbf{d P}_{2}$ (bottom). ..... 71
3-16 The $\mathbf{d P}_{2}$ tiling (top) can be taken to either $\mathbf{d P}_{1}$ (bottom left) or $\mathbf{F}_{\mathbf{0}}$ (bottom right), depending on which edge gets removed. In the $\mathbf{F}_{0}$ tiling, one should collapse the edge between regions 2 and 4 to a point; this corresponds to the bifundamentals on the diagonal of the quiver. ..... 72
3-17 Quiver diagram admitting two toric superpotentials. ..... 73
3-18 Brane tiling corresponding to the quiver diagram in Figure 3-17 and the superpotential in (3.6.28). ..... 74
3-19 Brane tiling corresponding to the quiver diagram in Figure $3-17$ and the superpotential in (3.6.29). ..... 74
3-20 Toric diagram for the quiver in Figure 3-17 and superpotentials $W_{A}$ and $W_{B}$ ..... 75
3-21 Brane tiling for Model I of $\mathbf{d P} \mathbf{P}_{2}$. ..... 76
3-22 Brane tiling for Model II of $\mathbf{d P}_{2}$. ..... 76
3-23 Brane tiling for Model III of $\mathbf{d P}_{3}$. ..... 77
3-24 Brane tiling for Model IV of $\mathbf{d P}_{3}$. ..... 78
3-25 Brane tilings for the four toric phases of $P d P_{5}$. ..... 80
3-26 Toric diagrams with multiplicition lor the four toric phases of $P d P_{5}$. We observe that the GLSM multiplicities are the same for Models I and III. ..... 82
3-27 Brane tiling for $Y^{3,3}$ ..... 83
3-28 Brane tiling for $Y^{3,2}$. The impurit $:$ is the blue area. ..... 83
3-29 Brane tiling for $Y^{3,1}$. ..... 84
3-30 Toric diagram of a phase of $Y^{5,3}$ ..... 86
3-31 Toric diagram of a phase of $Y^{3,0}$ with three single impurities. ..... 87
3-32 Dualizing face 3 in $Y^{3,1}$ with two single impurities. In resulting tiling, we indicate the double impurity in pink. ..... 88
3-33 The double impurity in $Y^{3,1}$ ..... 88
3-34 Toric diagram for $Y^{3,1}$ in the double impurity phase ..... 89
3-35 Brane tilings for $Y^{3, q}$. ..... 90
3-36 A brane tiling for $X^{3,1}$. ..... 90
4-1 Quiver diagram for Model II of $d P_{2}$. ..... 95
4-2 Periodic quiver for Model II of $d P_{2}$. We show several fundamental cells ..... 95
4-3 Two plaquettes are equal once the F -term equation for the common field is imposed. ..... 96
4-4 Brane tiling for Model II of $d P_{2}$. ..... 98
4-5 (a) The dimers in the a perfect matching $M$ are shown in cyan. (b)The dimers in the reference perfect matching $M_{0}$ are shown in red. (c)The height function, whose level curves are given by $M-M_{0}$. . . . 99
4-6 Relevant matrices in the Forward Algorithm ..... 105
4-7 Contours defining $\tilde{v}_{x}$ and $\tilde{v}_{y}$. ..... 111
4-8 Sets of edges $E_{x}$ and $E_{y}$ that enter the computation of $\left(h_{x}, h_{y}\right)$. ..... 116
4-9 Perfect matchings and their slopes for Model II of $d P_{2}$. ..... 119
5-1 A four-faceted polyhedral cone in $\mathbb{R}^{3}$. ..... 125
5-2 Toric diagram for the $L^{a, b, c}$ geometries. ..... 138
5-3 (p,q)-web for the $L^{a, b, c}$ theories. ..... 139
5-4 a) Toric diagram and b) ( $p, q$ ) web for the $L^{a, b, a}$ sub-family. ..... 141
$5-5$ The four building blocks for the construction of brane tilings for $L^{a, b, c}$. ..... 155
5-6 Seiberg duality on a self-dual node that does not change the hexagon content. ..... 157
5-7 Seiberg duality on a self-dual node under which $\left(n_{A}, n_{B}, n_{C}, n_{D}\right) \rightarrow$ $\left(n_{A}+1, n_{B}-2, n_{C}, n_{D}+1\right)$. ..... 157
5-8 Brane tiling for $L^{2,6,3}$. ..... 158
5-9 Quiver diagram for $L^{2,6,3}$. ..... 159
5-10 Toric diagram for $L^{2,6,3}$ determined using the characteristic polynomial of the Kasteleyn matrix for the tiling in Figure 5-8. ..... 160
5-11 Brane tiling for a Seiberg dual phase of $L^{2,6,3}$. ..... 160
5-12 Brane tiling for $L^{2,6,4}$. ..... 161
5-13 Brane tiling for $L^{2,6,4}$ ..... 161
5-14 Brane tiling for $L^{a, b, a}$. ..... 164
5-15 Quiver diagram for $L^{a, b, a}$. It consists of $2 a \mathrm{C}$ nodes and ( $b-a$ ) A nodes. The last node is connected to the first one by a bidirectional arrow. ..... 164
5-16 Brane tiling for $L^{1,5,2}$ ..... 166
5-17 Quiver diagram for $L^{1,5,2}$. ..... 166
5-18 Toric diagram for $L^{1,5,2}$. ..... 167
5-19 Brane tiling for $L^{1,7,3}$. ..... 167
5-20 Quiver diagram for $L^{1,7,3}$. ..... 168
5-21 Toric diagram for $L^{1,7,3}$. ..... 168
6-1 Isoradially embedded part of an arbitrary brane tiling (in green). ..... 172
6-2 (i) Circumcircles around the faces (in black), (ii) and the corresponding rhombus lattice (in red). ..... 172
6-3 A rhombus in the lattice. The green line is an edge in the branetiling, the magenta line is the corresponding bifundamental field in theperiodic quiver.173
6-4 (i) Rhombus path in the rhombus lattice. (ii) Equivalmint zig zag path in the brane tiling. We will use blue lines to depict rhombus loops schematically. The edges which are crossed by the blue line in (i) are all parallel. Their orientation can be described by an angle, the so-called rhombus loop angle.

6-5 Tilting along the horizontal $R$ rhombus loop. The rhombus loop angle $\alpha$ changes during the Dehn-twist. Here we have chosen $\alpha=0$ to be the vertical direction $(\mid)$, hence $\alpha=\pi / 4$ corresponds to the skew edges

$$
(/)
$$176

6-6 Hirzebruch zero brane tiling. ..... 178
6-7 (i) Hirzebruch zero toric diagram (ii) un-Higgsed Hirzebruch. The area remains the same, external multiplicities appear ..... 178

6-8 (i) Hirzebruch zero toric diagram (ii) un-Higgsed Hirzebruch. The area increases by $1 / 2$ corresponding to the new face in the brane tiling.179

6-9 (i) Hirzebruch zero inconsistently un-Higgsed. (ii) Consistent unHiggsing.179

6-10 (i) Inconsistently un-Higgsed Hirzebruch. The rhombus loops are indicated with the blue lines. The zig-zag paths contain the edges that are crossed by the blue paths. The following rhombus loops are obtained: $A:(0,-1) \quad B:(-2,2) \quad C:(2,-1)$. Here $(a, b)$ denotes the homology class of the path.
(ii) Consistently un-Higgsed F0. The rhombus loops reproduce the $(p, q)$-legs of the toric diagram (Figures 6,7$): A:(0,-1) \quad B:(0,1)$ $C:(-2,1) \quad D:(1,-1) \quad E:(1,0) \ldots \ldots . . . . . . . . . . . . .$.

6-11 The subgraph connects to the rest of the tiling through its four nodes in the corner. No consistent brane tiling can contain this subgraph, because it results in collapsing rhombi and vanishing $R$-charges.

6-12 Toric diagram (i) and ( $p, q$ ) web (ii) for del Pezzo 2. The charges of the external branes are shown. According to the conjecture, these correspond to the homology classes of the rhombus loops in the brane tiling.

6-13 Toric diagram for the SPP. We have drawn the blue $(p, q)$-leg between the nodes $(1,1)$ and $(2,0)$. The zig-zag path corresponding to the leg is shown in Figure 19.

6-14 The six periodic perfect matchings of SPP [94]. The green edges are contained in the matching, the dashed lines are the other edges of the tiling. The $(s, t)$ numbers are the corresponding points in the toric diagram (Figure 17)

6-15 The $(1,1)$ and $(2,0)$ perfect matchings on top of each other. We see the emerging $(1,1)$ homology zig-zag loop which corresponds to the blue $(p, q)$-leg in Figure 17.

6-16 $P d P_{4}$ model I brane tiling with a $(1,0,1,0,1,0,0) \mathcal{N}=2$ fractional brane [98]. The bounding rhombus loops ( $A$ and $B$ zig-zag paths) are shown in blue

6-17 Assigning angles $(\theta)$ to the rhombus loops. The figure shows two intersecting blue rhombus paths. There is a single rhombus and a green bifundamental edge at the intersection of these paths. This bifundamental has an R -charge that is proportional to the angle $\theta$ of the rhombus. This angle is just the difference of the rhombus loop angles $\alpha$ and $\beta$ assigned to the two rhombus paths: $R \pi=\theta=|\alpha-\beta|$ (or $\pi-|\alpha-\beta|$ depending on the orientation).187
6-18 $\mathcal{C}^{3}$ toric diagram ..... 188

6-19 Rhombus loop diagram of $\mathcal{C}^{3}$. The blue rhombus loops are the D6branes. At the intersection points we get massless fields. The dark faces are terms in the superpotential, the light faces are the gauge groups. These correspond respectively to nodes and faces in the brane tiling. The rhombi are shown in red, the brane tiling edges are green.
6-20 From the rhombi to the brane tiling. We glue the rhombi together that arise at the intersections of rhombus loops (Figure 27). We glue the edges that are connected by the rhombus loops. Each rhombus has a green tiling edge in it, from which we obtain the entire (hexagonal) brane tiling. ..... 189
6-21 Conifold toric diagram ..... 190
6-22 Conifold rhombus loops and brane tiling ..... 190
6-23 Four fundamental cells of the conifold rhombus loop diagram. If we consider these cells as one big fundamental cell then we gain the rhom- bus loop diagram of the $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold of the conifold. ..... 191
6-24 $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold of the conifold. ..... 191
6-25 $L^{131}$ toric diagram ..... 192
6-26 $L^{131}$ rhombus loops and brane tiling ..... 192
6-27 (i) $L^{131}$ brane tiling (ii) and the corresponding quiver. ..... 193
6-28 Toric diagram of $L^{152}$ ..... 194
6-29 $L^{152}$ brane tiling from the rhombus loops ..... 195
$6-302 \times 2$ fundamental cells of the rhombus loop diagram of $L^{152}$. The brane tiling is shown in green. ..... 196
6-31 (i) $L^{152}$ brane tiling (ii) and the corresponding quiver ..... 197
6-32 The elementary Picard-Lefschetz-Yang-Baxter transformation. ..... 197
6-33 The Yang-Baxter-Reidemeister transformation on the rhombus lattice. Star-triangle ..... 198
6-34 (i) Four hexagon with one extra line. (ii) Seiberg dualizing the red square $(F)$. The extra edge in the upper hexagon $(B \& F)$ gets into the lower one $(F \& C)$. ..... 198
6-35 Seiberg duality in the hexagonal tiling with extra edge. The brane tiling is shown in green, the (deformed) rhombus lattice is in red, the relevant rhombus loops are in blue. ..... 199
6-36 Seiberg duality in the level of the rhombus loops. ..... 199

7-1 Four unit cells of the $\mathbb{P}^{2}$ periodic quiver for basepoint $\left(x_{1}, x_{2}\right)=(3 / 4,-1 / 2) .214$
7-2 The periodic Beilinson quiver for $\mathbf{d} \mathbf{P}_{1}$ with fundamental cell. . . . . . 216
7-3 The eight periodic perfect matchings of $\mathbf{d} \mathbf{P}_{1}$. The green edges are contained in the matching. The dashed lines are the edges left in the tiling. The $(s, t)$ numbers are the corresponding points in the toric diagram

7-4 Allowed face paths (i.e. paths in the Beilinson quiver) go always uphill. The height function increases by one at the line constituted of the black perfect matching and the green reference matching. The red path cannot cross the green edges (they are not in the Beilinson quiver). Hence when crossing the contour line, the red path has to cross a black edge. Crossing the black edge increases the value of the height function. 222

7-5 Gradient vectors in the toric diagram. The coordinates of the blue $\left(s_{i}, t_{i}\right)$ vectors give the monodromy of the height function of the perfect matching sitting at their endpoints. The red $(x, y)$ arrow is the gradient vector of the hypothetical nontrivial loop.

7-6 An integer function over the external nodes determines a divisor and therefore a sheaf of sections of the corresponding line bundle. The numbers in the figure denote $\mathcal{O}\left(D_{1}+D_{3}+2 D_{4}\right)$.

7-7 The $\Psi$-map.
7-8 The reference paths are allowed paths to each face. They start from face 1 and don't cross the edges of the green internal matching; hence they are paths in the Beilinson quiver.225
7-9 The three divisors computed from the paths to the faces. ..... 226

7-10 Face 4 can be assigned with either the red or the yellow allowed path. The resulting Weil divisors are shown on the right-hand side. We see that they differ by a linear function, i.e. they are equivalent.226

T-11 Determining the $S_{2,4}$ matrix element. In this case $E_{4} \otimes E_{2}^{*}=(1,0,1,1)-$ (1.0. 0,0$)=(0,0,1,1)$. The figure shows the lattice of the $\Delta_{2,4}$ polygon anl its bounding inequalities. The red lattice points inside $\Delta_{2,4}$ can be identified with adjacent fundamental cells in the brane tiling.

7-12 The figure shows the allowed paths that start on face 2 and end on face 4. The endpoints of these paths are in different fundamental cells which are in one-to-one correspondence with the lattice points inside $\Delta_{2.4}$ that has been used to compute $\operatorname{dim} \operatorname{Hom}\left(E_{2}, E_{4}\right)$.

7-13 The F-flatness equation for the $X$ bifundamental field is $C B A=V U$. This states the equivalence of the two green paths in the figure.

7-14 Homotopic paths are equivalent. The left-hand side of the figure shows two paths represented schematically by green lines. The tiling is colored black and the underlying rhombus lattice is shown by dotted lines. The pink area surrounded by the two paths is also shown separately.

7-15 Two homotopic paths that pass around the pink area. Each boundary node $\left(A_{1}, \ldots, A_{n}, B_{1}, \ldots, B_{m}\right)$ has at least one rhombus edge which ensures that the area cannot be reduced by F-terms.

7-16 The straight rhombus path in the area contains rhombi $r_{0}, \ldots, r_{n}$. The existence of this series of rhombi constrains $A B_{1}$ to be parallel to $B_{m} B .232$

7-17 The embedding of the dual cone in the tiling torus.233
7-18 Inequivalent $A \rightarrow B$ homotopic paths in an inconsistent tiling. ..... 234

7-19 Homotopic paths with different R -charge are not equivalent. After applying the F -term equation for $A_{3} B_{1}$, the long path (solid green line) gets transformed to the short path (dashed line) plus a small loop around the $A_{1}$ node in the tiling.

7-20 The figure schematically depicts three allowed green paths from $A$ to $B$. The shading indicates one of the height coordinates. The height increases in the direction of the small arrows. The allowed paths can only cross the dashed lines in this direction, and thus we obtain a bounding inequality for $\Delta_{A B}$. The remaining edges can be determined by means of the other heights.
7-21 $Y^{3,2}$ quiver. ..... 240
7-22 $Y^{3,2}$ perfect matchings $\left(1^{s t} \ldots 9^{\text {th }}\right)$. ..... 241
7-23 $Y^{3,2}$ perfect matchings $\left(10^{t h} \ldots 18^{t h}\right)$. ..... 242
7-24 $Y^{3,2}$ Beilinson quiver. Bifundamentals in internal matching 7 are omitted. 242
7-25 $Y^{3,2}$ tiling. The purple lines indicate the chosen paths that are used to compute the exceptional collections. The paths start on face 1 and connect it to the other faces. ..... 243
7-26 A set of reference paths for $Y^{3,2}$. ..... 243

8-1 A possible fundamental domain (gray area) for the action of the $S L(2, \mathbb{Z})$ modular group on the upper half-plane. The upper-half plane parametrizes the possible values of $\tau$ (or $\rho$ ): the moduli of a two-torus. The gray domain can be folded into an $S^{2}$ with three special points (the two orbifold points: $\tau_{\mathbb{Z}_{6}}=e^{2 \pi i / 6}$ and $\tau_{\mathbb{Z}_{4}}=i$, and the decompactification point: $\tau \rightarrow i \infty$ ).

8-2 Base of the $T^{4} / \mathbb{Z}_{2}$ orbifold. The $\mathbb{Z}_{2}$ action inverts the base coordinates and has four fixed points denoted by red stars. They have $180^{\circ}$ deficit angle. As the arrows show, one has to fold the diagram and this gives an $S^{2}$

8-3 Flat $S^{2}$ base constructed from four triangles: base of $K 3$ in the $\mathbb{Z}_{2}$ orbifold limit.

8-4 Singularities in the base of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. The big dashed cube is the original $T^{3}$ base. The orbifold group generates the singular lines as depicted in the figure. The red dots show the intersection points of these edges.

8-5 (i) Rhombic dodecahedron: fundamental domain for the base of $T^{6} / \mathbb{Z}_{2} \times$ $\mathbb{Z}_{2}$. Six pyramids are glued on top of the faces of a cube. Neighboring pyramid triangles give rhombi since the vertices are coplanar (e.g. $A B C D$ ). (ii) The $S^{3}$ base can be constructed by identifying triangles as shown by the arrows. After gluing, the deficit angle around cube edges is $180^{\circ}$ which is appropriate for a $D_{4}$ singularity. The dihedral angles of the dashed lines are $120^{\circ}$ and since three of them are glued together, there is no deficit angle. The tips of the pyramids get identified and the space finally becomes an $S^{3}$.

8-6 The base of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ is homeomorphic to a three-ball with an $S^{2}$
boundary which has to be folded as shown in the figure.

8-7 Monodromies for the edges. . . . . . . . . . . . . . . . . . . . . . . . 260
8-8 The solid angle at the apex is determined by the dihedral angles be-
tween the planes. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 261
8-9 Flat vertex. $A, B$ and $C$ are singular edges. $C$ is pointing towards the reader. The dashed lines must be glued together to account for the deficit angle around $C$.

8-10 Junction condition for monodromies. The red loop around $A$ can be
smoothly deformed into two loops around $B$ and $C$. . . . . . . . . . 262

9-1 Almost non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ spaces. Monodromies are modified along the red loops. We refer to the models as one-plaquette, twoplaquette, "L", "U" and "X", respectively. . . . . . . . . . . . . . . . 281

9-2 Simple non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. . . . . . . . . . . . . . . . . . . . 282

9-3 (i) Fundamental domain of the base after modding by $\gamma$ : half of a rhombic dodecahedron. The arrows show how the faces are identified. (ii) Schematic picture indicating the structure of the degenerations.

9-4 (i) Half of the fundamental domain after modding by $\gamma$. (ii) Schematic picture.

9-5 The base of $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ where the generators of $\mathbb{Z}_{2}$ 's include coordinate shifts. Four non-intersecting $D_{4}$ strings (dashed green lines in the middle of hexagons) curve the space into an $S^{3}$. See the figure in Appendix 9.13 for a pattern that can be cut out

9-6 (i) The base can be constructed by gluing the truncated tetrahedron (dashed lines) to itself along with a small tetrahedron. It is easy to check that the $D_{4}$ strings (solid lines) have $180^{\circ}$ deficit angle whereas the dashed lines are non-singular. (ii) Schematic picture. The truncated tetrahedron example can roughly be understood as four linked rings of $D_{4}$ singularities. All of the rings are penetrated by two other rings which curve the space into a cylinder as they have tension 12 . This forces the string to come back to itself.

9-7 (i) The base of the $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ Joyce orbifold. There are six strings located on the faces of a cube. These faces are folded up which generates the $180^{\circ}$ deficit angles. (ii) Schematic picture. The degenerations form three rings of $D_{4}$ singularities. . . . . . . . . . . . . . . . . . . . . . . 289

9-8 Monodromies of the one-shift Joyce orbifold. . . . . . . . . . . . . . . 290
9-9 The base of the $T^{4} / \mathbb{Z}_{3}$ orbifold contains three $E_{6}$ singularities. . . 294
9-10 The base of the $T^{4} / \mathbb{Z}_{4}$ orbifold contains two $E_{7}$ and one $D_{4}$ singularities.

9-11 The base of the $T^{4} / \mathbb{Z}_{6}$ orbifold contains $E_{8}, E_{6}$ and $D_{4}$ singularities.
The three black dots denote one non-singular point.

$$
\begin{aligned}
& \text { 9-12 The base of } T^{6} / \mathbb{Z}_{3} \text {. The green line shows the } E_{6} \text { singularity. Six trian- } \\
& \text { gles bound the domain. Two triangles touching the singular green line } \\
& \text { are identified by folding. Two triangles should be identified according } \\
& \text { to the orientation given by the arrows. The remaining two triangles } \\
& \text { are identified in a similar fashion. . . . . . . . . . . . . . . . . . . . . } 296
\end{aligned}
$$

9-13 (i) The base of $T^{6} / \Delta_{12}$. The red and green lines indicate $E_{6}, D_{4}$ sin
gularities, respectively. The other edges are non-singular. The solid
green cube indicates the $D_{4}$ singularities of the original $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ orb
ifold. (ii) Schematic picture describing the topology of the singular
lines. See Appendix 9.14 for building this polyhedron at home.

9-14 (i) The base of $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$. The red and green lines indicate $E_{7}$,
$D_{4}$ singularities, respectively. The other edges are non-singular. The
solid green cube indicates the $D_{4}$ singularities of the original $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$
orbifold. (ii) Schematic picture describing the topology of the singular
lines.

9-15 (i) The base of $T^{6} / \Delta_{24}$. The cyan, red and green lines indicate $E_{7}$,
$E_{6}$ and $D_{4}$ singularities, respectively. (ii) Schematic picture describing
the topology of the singular lines.

9-16 Non-geometric $T^{6} / \Delta_{12}$. The red lines indicate extra $(-1)^{F_{L}}$ factors. . 301

9-17 Non-geometric $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$.

9-18 Non-geometric $T^{6} / \Delta_{24}$.

9-19 Fundamental domain (gray area) for the action of the $\Gamma_{0}(2)$ on the upper half-plane.

9-20 (i) Comparing semi-flat and exact metrics for around degenerating fibers. The base is 3d, parametrized by the periodic $x$ coordinate and the complex $z$-plane. The red line / red dots indicate where the $S^{1}$ fiber vanishes. Translational invariance of the semi-flat solution is replaced by periodicity of the exact metric in the $x$ direction. (ii) The same (exact) metric from a different viewpoint. The horizontal direction in the torus fiber is the $x$ coordinate. The torus pinches at the degeneration point (red dot) in the 2d base. Topologically, the singular fiber is an $S^{2}$ with two points glued together. This replaces the degenerating $\tau_{2} \rightarrow \infty$ torus of the semi-flat solution.
9-21 Hanany-Witten brane creation mechanism. $A$ and $B$ are the monodromies of the NS5- and D6-branes, respectively. As the two branes pass through each other, a new brane appears with a monodromy around the green circle (see right-hand side). This monodromy can be easily computed in the original configuration (left-hand side) where the green path was a deformed loop around the two branes. The result is $A B A^{-1} B^{-1}$ which is simply the monodromy of a D4-brane.
9-22 Determining the monodromy around the green loop by means of 2 d branch cuts. ..... 317
9-23 $T^{5} / \mathbb{Z}_{2}$ as a fibration over $S^{2}$. The geometric $T^{3}$ fiber gets promoted to $T^{5}$ by adding $x^{10}$ and the M -theory circle $x^{11}$. The monodromy $\mathcal{M}$ then acts on this $T^{5}$. ..... 319
9-24 Almost non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ which is also a Joyce orbifold. It is T-dual to $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. ..... 322
9-25 This modified $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ is also a Joyce manifold. ..... 323
9-26 Truncated tetrahedron: the shifted $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. ..... 338
9-27 Rhombic dodecahedron: fundamental domain of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. ..... 339
9-28 The $T^{6} / \Delta_{12}$ fundamental domain. ..... 340

## List of Tables

3.1 Dictionary for translating between brane tiling, string theory and gauge
theory objects. ..... 47
5.1 Charge assignments for the six different types of fields present in the general quiver diagram for $L^{a, b, c}$. ..... 143
8.1 Examples for composite vertices. ..... 263
9.1 The two basic representations of the duality group ..... 269
9.2 Some well-known examples for the Gibbons-Hawking ansatz. ..... 313
9.3 Branes in the Hanany-Witten setup. 456 are the base, 789 are the fiber coordinates. ..... 315
9.4 Massless $\mathcal{N}=2$ multiplets in four dimensions (Weyl fermions and real scalars). ..... 325
9.5 Untwisted NS and R sectors. In the R sector, only the spins of compact complex dimensions are indicated. The remaining one is determined by the GSO projection as indicated by the "gso" label. This depends on whether it's the left or right R sector ..... 326
9.6 Untwisted sectors. The signs show the matter GSO projection (which, due to the superghost contributions, differ from the full GSO in the NS sectors). ..... 326
9.7 $(\alpha \beta)$-twisted NS and R sectors. The other twisted sectors are analo- gous. The dots indicate half-integer moded oscillators which generate massive states. ..... 327
9.8 Each twisted sector gives an $\mathcal{N}=2$ vector multiplet. ..... 327
9.9 Massless $\mathcal{N}=1$ multiplets in four dimensions (Weyl fermions and real scalars). ..... 328
9.10 Signs of projections in various twisted sectors. ..... 329
9.11 Assignment of signs for discrete torsion. ..... 329
$9.12 \beta$-twisted NS and R sectors. ..... 329
$9.13(\alpha \beta)$-twisted NS and R sectors. ..... 331
9.14 Assignment of phases for the twisted sectors (columns). Dots indicate signs that do not affect the spectrum calculation. The group elements that are not listed here have no non-trivial fixed loci. ..... 332
9.15 Untwisted closed sectors. ..... 332
$9.16 \alpha$-twisted sector: a chiral multiplet. ..... 333
9.17 Twisted sector: a vector and a chiral multiplet. ..... 333
9.18 Twisted sectors that include $(-1)^{F_{L}}$ : two chiral multiplets. The left- moving GSO projections are modified compared to the usual twisted sectors. ..... 333
9.19 Twisted sectors for $\left(b_{1}, b_{2}, c_{1}, c_{3}, c_{5}\right)=\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0\right)$ give a vector and three chiral multiplets. ..... 335

## Part I

## Introduction

Our current understanding of Nature is centered around two theories: general relativity which describes gravity, and quantum field theory which describes the strong and electroweak interactions and various low-energy phenomena. Naive attempts to unify these two theories lead to insurmountable difficulties. Moreover, a unified theory would ideally explain recent cosmological observations such as the acceleration of the universe, dark matter and cosmic inflation. This presents challenges to our understanding that must be addressed by new ideas.

As of today, the best candidate framework for unification is string theory. The best understood solutions of the theory are, however, ten dimensional. The extra six dimensions may be compactified. The various deformations of the compact space can lead to light moduli fields. Fields with flat potentials typically modify the gravitational law in a way which is experimentally ruled out. More generic states of string theory also contain field strengths and heavy solitonic objects, branes, which can generate a potential for the moduli fields. It is in fact possible to fix all the moduli in particular examples. There is expected to exist a large set of consistent string theory vacua which is referred to as the 'landscape'. It is important to study the properties of these vacua through examples and determine possible correlations between their features. It is conceivable that this way one can obtain predictions for low-energy physics and constrain the set of effective field theories.

After compactifying, the structure of the extra dimensions governs the particle content and interactions of the four--dimensional effective field theory. Much of the work in this thesis has focused on this interesting correspondence. In many cases, the basic features of the field theory depend only on the local structure of the extra dimensions. In this introduction, we briefly explain the results which fully solve this correspondence for the case of 'toric' geometries. The tools that we developed can also be used to generalize the recent discoveries for M2-branes in M-theory. Finally, we describe results in global compactifications, in particular, in constructing $4 \mathrm{~d} \mathcal{N}=1$ perturbative non-geometric backgrounds. Global issues are of importance since certain mechanisms (e.g. inflation) can depend on the structure of the entire compact space.

## Local geometries and singularities

A popular scenario for phenomenology describes visible particles as excitations of three-dimensional branes. In order to generate the chiral particle content of the Standard Model in Type II string theory, the compact space must contain singularities (or intersecting branes in a dual picture) where the branes are placed. Although general relativity breaks down in the presence of such singularities, string theory can still be well-defined [73, 71]. The data for specifying the low-energy effective theory on the brane include the superpotential and the quiver which encodes the gauge groups and the particle content. These features depend only on the local region near the singularity.

The existing methods for analyzing the correspondence were computationally prohibitive for most singularities. For Calabi-Yau geometries with toric symmetries, we introduced "brane tilings" which efficiently solved this problem in a graphical way [94]. This paper is the basis of Chapter 3. The tiling can be interpreted as a physical configuration of branes. It encodes the gauge group, matter content and superpotential of the gauge theory. Brane tilings give the largest class of $\mathcal{N}=1$ quiver gauge theories yet studied and they offer a generalization of the $\mathrm{AdS}_{5} / \mathrm{CFT}_{4}$ correspondence to infinite sets of non-spherical horizons.

The technique we developed also enabled one to quickly compute the toric vacuum moduli space of the quiver gauge theory. In [100], we explicitly proved the equivalence between this procedure and the earlier approach that used gauged linear sigma models [232]. This is summarized in Chapter 4.

As an application of brane tilings, in [97] we found the four dimensional quiver gauge theories that are AdS/CFT dual to the recently discovered $L^{a, b, c}$ families of 5 d Sasaki-Einstein metrics. Chapter 5 is based on this paper. We perform various checks of the correspondence, such as volume calculations on the string side which match the $R$-charges on the gauge theory side which are determined by a-maximization.

In [132], we developed a procedure that constructs the brane tiling for any toric Calabi-Yau threefold. This is summarized in Chapter 6. The algorithm solved a
longstanding problem by computing superpotentials for these theories directly from the toric diagram of the singularity. The rules for the consistency of tilings were also determined. In general, the correspondence between field theories and geometries is not one-to-one: various field theories can have the same moduli space. This ambiguity manifests itself as Seiberg duality which was further elucidated by the results.

Brane tilings give a simple pictorial way to determine the low energy gauge theory on a stack of D3-branes probing a toric singularity. Another more abstract approach to this problem uses so-called exceptional collections of sheaves associated to the base of the threefold. Although this method is not restricted to the toric case, it is considerably more complicated. In [125] we provided a dictionary that translates between these two languages. These results are described in Chapter 7.

In order to gain a better understanding of the field theory / geometry correspondence, in [51] we discussed in detail the problem of counting BPS gauge invariant operators in the chiral ring of quiver gauge theories. These operators are dual to generalized giant gravitons, i.e. D3-branes wrapped on generically nontrivial threecycles on the gravity side. We found an intriguing relation between a certain decomposition of the generating function and the discretized Kähler moduli space of the Calabi-Yau space.

In [95] we developed techniques for orientifolding toric Calabi-Yau singularities. With these new tools, one recovers many orientifolded theories known so far. Furthermore, new orientifolds of non-orbifold toric singularities were obtained. One particular application of the results is the construction of models which feature dynamical supersymmetry breaking as well as the computation of instanton induced superpotential terms.

As discovered recently, Chern-Simons-matter theories play a role in M-theory [210, 20, 21, 122]. In particular, they are conjectured to describe the $2+1$ dimensional low-energy theory living on M2-branes. Understanding the physics of these branes will be a further important step towards understanding M-theory and nonperturbative strings. Brane tilings proved to be efficient tools for studying a subset of
$\mathcal{N}=2$ Chern-Simons-matter theories [135]. In [133], we described a techuique which computes the three dimensional toric diagram of the non-compact modnli space of a single probe brane. As a byproduct, one obtained new examples for the $\mathrm{AdS}_{4} / \mathrm{CFT}_{3}$ correspondence. These examples may be useful for the study of $2+1$ dimensional condensed matter systems.

## Global non-geometric compactifications

The study of geometric compactifications is possible due to the abundance of available mathematical tools. Such vacua, however, represent only a very small subclass of the landscape: the generic vacua are non-geometric. The classification of such theories seems prohibitively difficult and therefore simple tractable examples are valuable. A first step can be made in a controlled environment using perturbative string dualities to build non-geometric spaces.

The second part of this thesis (Chapters 8-9) focuses on implementing these ideas to obtain four-dimensional $\mathcal{N}=1$ compactifications [229]. At the 'large complex structure point' in the moduli space, Calabi-Yau spaces can be approximated by torus fibrations $[113,220]$. Instead of using only geometric $S L(n)$ transformations to glue the torus fibers, one wishes to use the whole T -duality (or more ambitiously, U-duality) group [138]. The non-geometric spaces that we obtain this way have a nice geometric representation. The construction is dual to $G_{2}$ compactifications of M-theory and has asymmetric orbifold limits. It also allows for new ways to stabilize the moduli fields. In particular, it is useful in eliminating the modulus that is related to the overall size of the compact manifold which otherwise poses an intrinsic difficulty for ordinary (geometric) string compactifications. We also give a simple explanation for the Hanany-Witten brane-creation mechanism [134] and for the equivalence of the $T^{5} / \mathbb{Z}_{2}$ Type IIA orientifold and Type IIB on $S^{1} \times K 3[234,67]$.

## Part II

## Local geometry

## Chapter 1

## D-branes and quiver gauge theories

String theory contains a wide variety of extended objects. In addition to $1+1$ dimensional strings, it also contains branes, which are higher dimensional analogs of two-dimensional membranes. There exist various types of branes ${ }^{1}$ : NS5-branes, $\mathrm{D} p$-branes (in Type II string theory) and also M2- and M5-branes (in M-theory). In this first part of the thesis, we will focus on the physics of $D$-branes and how local features in the geometry affect their dynamics.

In perturbative string theory, D -branes are submanifolds in spacetime where strings can end. The effective action for a D-brane is given by the Dirac-BornInfeld action coupled by a Wess-Zumino term to other spacetime fields [182]. One can consider a limit where the length scale of the strings vanishes and the massless modes on the D -brane decouple from the tower of massive open string modes and other modes arising from closed strings in the bulk of spacetime. For a single D-brane in flat space, the low-energy limit corresponds to the dimensional reduction of the ten-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory with $U(1)$ gauge group.

Placing D-branes in curved background geometries offers an immediate generalization to the flat space configuration. The following question arises naturally: for a given geometry, what field theory governs the low-energy dynamics of the D-branes?

A possible approach to study this question, which is particularly interesting due to its relationship with different branches of geometry, is to use D-branes to probe

[^0]a singularity in the geometry. The geometry of the singularity then determines the anmunt of supersymmetry, the gauge group structure, the matter content and the suj»rpotential interactions on the worldvolume of the D -branes.

The richest of such examples which are both tractable and non-trivial, are given by the $4 d \mathcal{N}=1$ gauge theories that arise on a stack of D3-branes probing a singular Calabi- Yau 3-fold. This scenario is depicted in Figure 1-1. The background is a product of $(3+1)$-dimensional Minkowski space and a six-dimensional Calabi-Yau space. The D3-branes are filling the Minkowski factor. Their position in the extra six dimensions is given by a point in the Calabi-Yau manifold. If this is a smooth point in the Calabi-Yau geometry, we obtain $\mathcal{N}=4$ SYM on the D-branes. However, if the point is a singular point, the low-energy theory of the D -branes will be more interesting.


Figure 1-1: D3-branes probing the transverse geometry.

This setup also provides generalizations of the celebrated AdS/CFT correspondence $[189,118,236,6]$. The AdS/CFT conjecture states that the large $N$ 't Hooft limit of $\mathcal{N}=4 S U(N)$ super Yang-Mills is equivalent to type IIB string theory on $A d S \times S^{5}$ with $N$ units of Ramond-Ramond 5 -form flux on the $S^{5}$. The $\mathcal{N}=4$ gauge theory in question arises as the worldvolume theory of a stack of $N$ D3-branes in flat ten dimensional space. Since its original formulation, the AdS/CFT correspondence has been extended to and checked in a variety of more realistic, less supersymmetric situations. The worldvolume theory of D3-branes over a singular Calabi-Yau threefold is an $\mathcal{N}=1$ "quiver" gauge theory [73, 71]. The structure of the gauge theory
reflects the properties of the singular manifold. When the Calabi-Yau is a metric cone over an $X_{5}$ Sasaki-Einstein manifold, the corresponding dual is type IIB string theory on $A d S_{5} \times X_{5}$.

The matter content of the quiver gauge theory is neatly summarized in the quiver graph [73] which also generalizes the familiar Dynkin diagrams. Each node in the quiver (see e.g. Figure 1-2) may carry an index, $N_{i}$, for the $i^{\text {th }}$ node and denotes a $U\left(N_{i}\right)$ gauge group. The edges (arrows) label the chiral bifundamental multiplets. These fields transform in the fundamental representation of $U\left(N_{i}\right)$ and in the antifundamental of $U\left(N_{j}\right)$ where $i$ and $j$ represent the nodes in the quiver that are the head and tail of the corresponding arrow.


Figure 1-2: Quiver of $\mathrm{dP}_{1}$. The theory contains four $U(N)$ gauge groups labeled by the nodes of the quiver. The arrows label bifundamental fields transforming in the (anti-)fundamental representation of the groups at the endpoints.

In order for the gauge theory to be gauge anomaly free, for each gauge group, the number of chiral fermions in the fundamental representation must equal the number in the antifundamental representation. This anomaly cancellation constraint means that for a fixed node in the quiver, the number of incoming and outgoing arrows are the same.

In order to write down the Lagrangian of the quiver gauge theory, we further need to give the superpotential, which is a polynomial in gauge invariant operators. For
example, for $\mathbf{d P}_{1}$ the superpotential is ${ }^{2}$

$$
\begin{equation*}
W=\epsilon_{\alpha \beta} U_{1}^{\alpha} V^{\beta} Y_{1}-\epsilon_{\alpha \beta} U_{2}^{\alpha} Y_{2} V^{\beta}-\epsilon_{\alpha \beta} U_{1}^{\alpha} Y_{3} U_{2}^{\beta} Z \tag{1.0.1}
\end{equation*}
$$

By deleting certain arrows in the quiver, one obtains another graph, the so-called Beilinson quiver. This type of quiver will be important in Chapter 7. In this quiver there exists an ordering of the nodes such that there are no arrows pointing backwards (for an example see Figure 1-3). Generically, there are many Beilinson quivers corresponding to a given quiver. These quivers can be thought of as subquivers that contain no oriented loops.


Figure 1-3: $\mathbf{d P}_{1}$ Beilinson quiver.

[^1]
## Chapter 2

## Toric geometry

In this chapter, we give a brief introduction to toric geometry, focussing on features that are relevant for this thesis. In particular, we will concentrate on singular noncompact toric varieties $Y$ whose Calabi-Yau metric is a cone over an $X_{5}$ SasakiEinstein manifold. For more detailed discussions, we refer the reader to [185, 102, 45]

Toric non-compact Calabi-Yau spaces are a particularly simple, yet extremely rich, subset in the space of Calabi-Yau threefolds. Their simplicity resides in that they are defined by a relatively small amount of combinatorial data. This will allow us to extract the data of the quiver gauge theory that arises on D3-branes probing such toric spaces without knowing the metric explicitly. This is very important since Calabi-Yau metrics are rarely known in general.

In order to use toric methods, we restrict the class of possible spaces to toric ones, i.e. we assume that the isometry group of $Y$ contains a 3 -torus. The variety then can be defined by a "strongly convex rational polyhedral cone" $\sigma$ on the integer lattice $N$ (see Figure 2-1). Such a cone has the origin of the lattice as its apex and it is bounded by a finite set of hyperplanes (this is the "polyhedral" property). The edges of the cone are spanned by lattice vectors $\left\{v_{r}\right\}$. We also assume this set of vectors is minimal in the sense that removing any vector in the definition changes the cone.

The lattice $N$ is three dimensional so that we obtain a (complex) 3d space. Let $M=\operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice with pairing denoted by $\langle\cdot, \cdot\rangle$. The dual cone $\sigma^{v}$ is the set of vectors that are nonnegative on $\sigma$.


Figure 2-1: The cone for the variety. The coordinates of the spanning vectors are integers. The endpoints are coplanar following from the Calabi-Yau condition.

A collection of cones $\Sigma$ is a "fan", if each face of a cone is also present in $\Sigma$. Moreover, the intersection of two cones in $\Sigma$ is a bounding face of each. Let us denote the one-dimensional cones in $\Sigma$ by $v_{i}(i=1 \ldots k)$. We can associate to each $v_{i}$ an homogeneous coordinate $z_{i} \in \mathbb{C}$ defining a $\mathbb{C}^{k}$ space.

From this space we first remove an $S$ set,

$$
\begin{equation*}
S=\bigcup_{I}\left\{\left(z_{1}, \ldots, z_{k}\right) \mid z_{i}=0 \text { for all } i \in I\right\} \tag{2.0.1}
\end{equation*}
$$

where $I \subset\{1, \ldots, k\}$ labels the sets for which the $\left\{v_{i} \mid i \in I\right\}$ vectors are not in the same cone in the fan.

After subtracting this set, we can define the toric variety as a quotient,

$$
\begin{equation*}
Y=\frac{\mathbb{C}^{k} \backslash S}{G \times F} \tag{2.0.2}
\end{equation*}
$$

where $G$ is $\left(\mathbb{C}^{*}\right)^{k-3}$ and $F$ is a finite Abelian group. This finite Abelian group arises when the $v_{i}$ vectors generate only a sublattice $N^{\prime}$ of $N$. The quotient by $G$ is given as follows. We define $Q_{a}^{i}$ integer vectors by the relation,

$$
\begin{equation*}
\sum_{i=1}^{k} Q_{a}^{i} v_{i}=0 \tag{2.0.3}
\end{equation*}
$$

The quotient is given by identifying points with the following equivalence relations,

$$
\begin{equation*}
\left(z_{1}, \ldots, z_{k}\right) \sim\left(\lambda^{Q_{a}^{1}} z_{1}, \ldots, \lambda^{Q_{a}^{k}} z_{k}\right) \tag{2.0.4}
\end{equation*}
$$

where $\lambda \in \mathbb{C}^{*}$. There are $k-3$ independent relations and thus the toric variety is three dimensional.

A simple two-dimensional example is given by $\mathbb{C P}^{2}$. The fan is given by the following three vectors,

$$
\begin{equation*}
v_{1}=(1,0) \quad v_{2}=(0,1) \quad v_{3}=(-1,-1) . \tag{2.0.5}
\end{equation*}
$$

The $S$ subset is simply the origin, and the toric space is given by,

$$
\begin{equation*}
\frac{\mathbb{C}^{3} \backslash\{0\}}{\mathbb{C}^{*}} \tag{2.0.6}
\end{equation*}
$$

The action of $\mathbb{C}^{*}$ can be easily determined since

$$
\begin{equation*}
1 \times v_{1}+1 \times v_{2}+1 \times v_{3}=(0,0) \tag{2.0.7}
\end{equation*}
$$

therefore $Q^{i}=(1,1,1)$ and thus the equivalence relation is $\left(z_{1}, z_{2}, z_{3}\right) \sim \lambda\left(z_{1}, z_{2}, z_{3}\right)$. We see that this is just the usual definition of $\mathbb{C P}^{2}$.

For each spanning vector $v_{r}$ there is a corresponding $D_{r}$ (Weil) divisor in the toric variety. Principal divisors are of the form

$$
\begin{equation*}
\sum_{r}\left\langle m, v_{r}\right\rangle D_{r}, \tag{2.0.8}
\end{equation*}
$$

for $m \in M$. The Calabi-Yau condition states that $c_{1}(Y)=0$, i.e. the canonical class is trivial

$$
\begin{equation*}
K=-\sum_{r} D_{r}=-\sum_{r}\left\langle m, v_{r}\right\rangle D_{r} . \tag{2.0.9}
\end{equation*}
$$

The last equality implies that the endpoints of the $\left\{v_{r}\right\}$ vectors are coplanar, so with an appropriate $S L(3, \mathbb{Z})$ transformation a convex integer polygon in two dimensions can be obtained (see e.g. Figure 2-4). We will refer to this polygon as the toric diagram of the singularity [191, 190, 97]. Weil divisors can be specified as integer functions over the external lattice points of the toric diagram. Principal divisors are
simply linear functions: the canonical class is a constant function.


Figure 2-2: The toric diagram for the conifold which can also be described br the equation $z_{1} z_{2}=z_{3} z_{4}$ with $z_{i} \in \mathbb{C}$. The normal vectors are also shown.


Figure 2-3: The toric diagram for the del Pezzo 1 surface is shown.


Figure 2-4: The toric diagram for $L^{1,7,3}$ which is part of the recently discovered series of $L^{a b c}$ metrics ([61, 60]).

## Chapter 3

## Brane tilings

In this Chapter, we introduce the concept of brane tilings. They can be thought of as configurations of NS5- and D5-branes that generalize the brane box [131] and brane diamond [5] constructions and are dual to gauge theories on D3-branes transverse to arbitrary toric singularities. From now on, we proceed assuming that the dual geometry is toric and introduce the relevant brane configurations. The reason for the requirement that the corresponding singularities are toric will become clear in this and subsequent sections.

In our construction, the NS5-brane extends in the 0123 directions and wraps a holomorphic curve embedded in the 4567 directions (the 46 directions are taken to be compact). D5-branes span the 012346 directions and stretch inside the holes in the NS5 skeleton like soap bubbles. The D5-branes are bounded by NS5-branes in the 46 directions, leading to a $3+1$ dimensional theory in their world-volume at low energies. The branes break supersymmetry to $1 / 8$ of the original value, leading to 4 supercharges, i.e. $\mathcal{N}=1$ in four dimensions. In principle, there can be a different number of D5-branes $N_{I}$ in each stack. This would lead to a product gauge group $\prod_{I} S U\left(N_{I}\right)$. Strings stretching between D5-branes in a given stack give rise to the gauge bosons of $S U\left(N_{I}\right)$ while strings connecting D5-branes in adjacent stacks $I$ and $J$ correspond to states in the bifundamental of $S U\left(N_{I}\right) \times S U\left(N_{J}\right)$. We will restrict ourselves to the case $N_{I}=N$ for all $I$. Theories satisfying this restriction on the ranks were dubbed toric phases in [80], We should emphasize though, that there
are quivers that are dual to toric geometries but that do not satisfy this condition.
It is worthwhile here to note a few properties of NS5-branes that are relevant for this construction. As is well-known, an NS5-brane backreacts on its surrounding spacetime to create a throat geometry. When we have two sets of D5-branes ending on different sides of the NS5-brane, the throat separates the two sets of branes. The D-branes may then only interact via fundamental strings stretching between them; these are the bifundamentals in the quiver gauge theory. Initially it might seem like there are two conjugate bifundamentals which pair up to form hypermultiplets, but in this case, where the NS5-brane wraps a holomorphic curve, the orientation of the NS5-brane projects one of these out of the massless spectrum [76]. Thus the resulting quiver theory will generically have arrows pointing in only one direction (it is easy to get quivers with bidirectional arrows as well, but these will instead come from strings stretching across different NS5-branes rather than both orientations across the same NS5-brane).

The important physics is captured by drawing the brane tiling in the 46 plane. The NS5-branes wrap a holomorphic curve, the real section of which is a graph $G$ in the 46 plane, which we will later show must be bipartite. A graph is bipartite when its nodes can be colored in white and black, such that edges only connect black nodes to white nodes and vice versa. By construction, $G$ is $\mathbb{Z}^{2}$-periodic under translations in the 46 plane since these directions are taken to be compact. We will see in the next section that the existence of $G$ is associated to the duality between quiver gauge theories and dimer models.

Given a brane tiling, it is straightforward to derive its associated quiver gauge theory. The brane tiling encodes both the quiver diagram and the superpotential, which can be constructed according to the dictionary given in Table 3.1 (see the following section). Conversely, we can use this set of rules to construct a brane tiling from a given quiver with a superpotential. In the following section we will make this correspondence precise.

Several interesting consequences follow naturally from this simple set of rules. Some of them are well known, while others are new. The fact that the graphs under

| Brane tiling | String theory | Gauge theory |
| :--- | :--- | :--- |
| $2 n$-sided face | D5-branes | Gauge group with $n$ flavors |
| Edge between two <br> polygons I and J | String stretched between D5- <br> branes through NS5 brane. | Bifundamental chiral multiplet <br> between gauge groups I and J; <br> We orient the arrow such that <br> the white node is to the right. |
| k-valent vertex | Region where $k$ strings <br> interact locally. | Interaction between $k$ chiral <br> multiplets, i.e. order $k$ term in <br> the superpotential. The signs for <br> the superpotential terms are <br> assigned such that white and <br> black nodes correspond to plus <br> and minus signs respectively. |

Table 3.1: Dictionary for translating between brane tiling, string theory and gauge theory objects.
consideration are bipartite implies that each edge has a black and a white endpoint. Edges correspond to bifundamental fields while nodes indicate superpotential terms, with their sign determined by the color of the node. Thus, we conclude that each bifundamental field appears exactly twice in the superpotential, once with a plus and once with a minus sign. We refer to this as the toric condition and it follows from the underlying geometry being an affine toric variety [80].

The total number of nodes inside a unit cell is even (there are equal numbers of black and white nodes). Thus, we conclude that the total number of terms in the superpotential of a quiver theory for a toric singularity is even. Although this condition is reminiscent of the toric condition, it is different. It is comforting to see that it is satisfied by all the examples in the literature (orbifolds, del Pezzos, $\mathbf{F}_{0}$, pseudo-del Pezzos, SPP, $Y^{p, q}, X^{p, q}$, etc).

Bidirectional arrows and even adjoint fields in the quiver can be simply implemented in this construction, by suitably choosing the adjacency of polygons. We will present an example containing both situations in section 3.2.1.

Let us define

| Brane tiling | Gauge theory |
| :--- | :--- |
| $F:$ number of faces | $N_{9}:$ number of gauge groups |
| $E:$ number of edges | $N_{f}:$ number of fields |
| $N:$ number of nodes | $N_{11}:$ number of superpotential terms |

According to the dictionary above, $F=N_{g}, E=N_{f}$ and $N=N_{W}$. Applying Euler's formula to a unit cell in the graph, we see that $F+N-E=2 g-2=0$ (where we have used that the graph lives on the torus), which translates into the following identity for quiver theories ${ }^{1}$ :

$$
\begin{equation*}
N_{g}+N_{U}-N_{f}=0 \tag{3.0.1}
\end{equation*}
$$

The geometric intuition we gain when using brane tilings make the derivation of this remarkable identity straightforward.

It is interesting to point out here that the Euler formula has another interpretation. Let us assign an R-charge to each bifundamental field in the quiver, i.e. to each edge in the brane tiling. At the IR superconformal fixed point, we know that each term in the superpotential must satisfy

$$
\begin{equation*}
\sum_{i \in \text { edges around node }} R_{i}=2 \quad \text { for each node } \tag{3.0.2}
\end{equation*}
$$

where the sum is over all edges surrounding a given node. We can sum over all the nodes in the tiling, each of which corresponds to a superpotential term, to get $\sum_{\text {edges,nodes }} R=2 N$. Additionally, the beta function for each gauge coupling must vanish,

$$
\begin{equation*}
2+\sum_{i \in \text { edges around face }}\left(R_{i}-1\right)=0 \quad \text { for each face } \tag{3.0.3}
\end{equation*}
$$

where the sum is over all edges surrounding a given face. But we can now sum this

[^2]over all the faces in the tiling to get $2 F+2 N-2 E=0$, where we have used the fact that the double sum hits every edge twice, and (6.1.1). The sums $\sum_{\text {edges,nodes }} R$ and $\sum_{\text {edges,faces }} R$ are equal because each double sum has the R-charge of each bifundamental contributing twice. Thus we see that the requirements that the superpotential have $R(W)=2$ and the beta functions vanish (i.e. that the theory is superconformal in the IR) imply that the Euler characteristic of the tiling is zero. This condition is the analog of a similar condition for superconformal quivers discussed in [155, 27]. Conversely we see that, in the case in which the ranks of all gauge groups are equal, the construction of tilings over Riemann surfaces different from a torus leads to nonconformal gauge theories.

Let us illustrate the concepts introduced in this section with a simple example, one of the toric phases of $\mathrm{dP}_{3}$, denoted Model I in [80]. Its corresponding quiver diagram is presented in Figure 3-1 and its superpotential is

$$
\begin{align*}
W & =X_{12} X_{23} X_{34} X_{45} X_{56} X_{61}-\left(X_{23} X_{35} X_{56} X_{62}+X_{13} X_{34} X_{46} X_{61}+X_{12} X_{24} X_{45} X_{51}\right) \\
& +\left(X_{13} X_{35} X_{51}+X_{24} X_{46} X_{62}\right) \tag{3.0.4}
\end{align*}
$$

The quiver diagram has 6 gauge groups and 12 bifundamental fields. Hence, the brane configuration will have 6 faces and 12 edges in a unit cell. The superpotential (3.0.4) has 1 order six, 3 quartic and 2 cubic terms. According to (3.0.1) we thus have 16 -valent, 34 -valent and 23 -valent nodes. The final brane tiling is shown in Figure 3-1.

### 3.0.1 Unification of quiver and superpotential data

An $\mathcal{N}=1$ quiver gauge theory is described by the following data: a directed graph representing the gauge groups and matter content, and a set of closed paths on the graph representing the gauge invariant interactions in the superpotential. An equivalent way to characterise this data is to view it as defining a CW-complex; in other words, we may take the superpotential terms to define the 2 -dimensional faces


Figure 3-1: A finite region in the infinite brane tiling and quiver diagram for Model I of $\mathbf{d P}_{3}$. We indicate the correspondence between: gauge groups $\leftrightarrow$ faces, bifundamental fields $\leftrightarrow$ edges and superpotential terms $\leftrightarrow$ nodes.
of the complex bounded by a given set of edges and vertices (the 1 -skeleton and 0 -skeleton of the complex). Thus, the quiver and superpotential may be combined into a single object, a planar tiling of a 2-dimensional (possibly singular) space. Toric quiver theories, as we will see, are defined by planar tilings of the 2 -dimensional torus.

This is a key observation. Given the presentation of the quiver data (quiver graph and superpotential) as a planar graph tiling the torus, the bipartite graph appearing in the dimer model (the brane construction of the previous section) is nothing but the planar dual of this graph! Moreover, as we have argued, this dual presentation of the quiver data is physical, in that it appears directly in string theory as a way to construct the 3+1-dimensional quiver gauge theory in terms of intersecting NS5 and D5-branes. The logical flow of these ideas is shown in Figure 3-2.

Let us see how the properties of the brane tiling arise from those of the quiver theory. We will show that we can think of the superpotential and quiver together as a tiling of a two-dimensional surface, where bifundamentals are edges, superpotential terms are faces, and gauge groups are nodes. We refer to this as the "periodic quiver" representation. The toric condition, which states that each matter field appears in precisely two superpotential terms of opposite sign, means that the faces all glue


Figure 3-2: The logical flowchart.
together in pairs along the common edges. Since every field is represented exactly twice in the superpotential, this tiling has no boundaries. Thus, the quiver and its superpotential may be combined to give a tiling of a Riemann surface without boundary; this periodic quiver gives a discretization of the torus. Since the Euler characteristic of the quiver is zero for toric theories (as discussed in the previous section), the quiver and superpotential data are equivalent to a planar tiling of the two-dimensional torus. See Figure 5 of [82] for an early example of a periodic quiver.

This tiling has additional structure. The toric condition implies that adjacent faces of the tiling may be labelled with opposite signs according to the sign of the corresponding term in the superpotential. Thus, under the planar duality the vertices of the dual graph may be labelled with opposite signs; this is the bipartite property of the dimer model. Since the periodic quiver is defined on the torus, the dual bipartite graph also lives on the torus.

Anomaly cancellation of the quiver gauge theory is represented by the balancing of all incoming and outgoing arrows at every node of the quiver. In the dual graph, bipartiteness means that the edges carry a natural orientation (e.g. from black to white). This induces an orientation for the dual edges, which transition between adjacent faces of the brane tiling (vertices of the planar quiver). For example, these dual arrows point in a direction such that, looking at an arrow from its tail to its head, the black node is to the left and the white node is to the right (this is just a convention
and the opposite choice is equivalent by charge conjugation). Arrows around a face in $G$ alternate between incoming and outcoming arrows of the quiver; this is how anomaly cancellation is manifested in the brane tiling picture. Alternatively, we can say that arrows "circulate" clockwise around white nodes and counterclockwise around black nodes.


Figure 3-3: The quiver gauge theory associated to one of the toric phases of the cone over $\mathbf{F}_{0}$. In the upper right the quiver and superpotential (3.0.6) are combined into the periodic quiver defined on $T^{2}$. The terms in the superpotential bound the faces of the periodic quiver, and the signs are indicated and have the dual-bipartite property that all adjacent faces have opposite sign. To get the bottom picture, we take the planar dual graph and indicate the bipartite property of this graph by coloring the vertices alternately. The dashed lines indicate edges of the graph that are duplicated by the periodicity of the torus. This defines the brane tiling associated to this $\mathcal{N}=1$ gauge theory.

Figure 3-3 shows an example of the periodic quiver construction for the quiver gauge theory associated to one of the toric phases of the Calabi-Yau cone over $\mathrm{F}_{0}$. The superpotential for this theory is [83]

$$
\begin{align*}
W & =X_{1} X_{10} X_{8}-X_{3} X_{10} X_{7}-X_{2} X_{8} X_{9}-X_{1} X_{6} X_{12}  \tag{3.0.5}\\
& +X_{3} X_{6} X_{11}+X_{4} X_{7} X_{9}+X_{2} X_{12} X_{5}-X_{4} X_{11} X_{5}
\end{align*}
$$

### 3.1 Dimer model technology

Given a bipartite graph, a problem of interest to physicists and mathematicians is to count the number of perfect matchings of the graph. A perfect matching of a bipartite graph is a subset of edges ("dimers") such that every vertex in the graph is an endpoint of precisely one edge in the set. A dimer model is the statistical mechanics of such a system, i.e. of random perfect matchings of the graph with assigned edge weights. As discussed in the previous section, we are interested in dimer models associated to doubly-periodic graphs, i.e. graphs defined on the torus $T^{2}$. We will now review some basic properties of dimers; for additional review, see [128, 168].

Many important properties of the dimer model are governed by the Kasteleyn matrix $K(z, w)$, a weighted, signed adjacency matrix of the graph with (in our conventions) the rows indexed by the white nodes, and the columns indexed by the black nodes. It is constructed as follows:

To each edge in the graph, multiply the edge weight by $\pm 1$ so that around every face of the graph the product of the edge weights over edges bounding the face has the following sign

$$
\operatorname{sign}\left(\prod_{i} e_{i}\right)=\left\{\begin{array}{lll}
+1 & \text { if }(\# \text { edges })=2 & \bmod 4  \tag{3.1.6}\\
-1 & \text { if }(\# \text { edges })=0 & \bmod 4
\end{array}\right.
$$

It is always possible to arrange this [165].
The coloring of vertices in the graph induces an orientation to the edges, for example the orientation "black" to "white". This orientation corresponds to the orientation of the chiral multiplets of the quiver theory, as discussed in the previous section. Now construct paths $\gamma_{w}, \gamma_{z}$ in the dual graph (i.e. the periodic quiver) that
wind once around the $(0,1)$ and $(1,0)$ cycles of the torus, respectively. We will refer to these fundamental paths as flux lines. In terms of the periodic quiver, the paths $\gamma$ pick out a subset of the chiral multiplets whose product is gauge-invariant and forms a closed path that winds around one of the fundamental cycles of the torus. For every such edge (chiral multiplet) in $G$ crossed by $\gamma$, multiply the edge weight by a factor of $w$ or $1 / w$ (respectively $z, 1 / z$ ) according to the relative orientation of the edges in $G$ crossed by $\gamma$.

The adjacency matrix of the graph $G$ weighted by the above factors is the Kasteleyn matrix $K(z, w)$ of the graph. The determinant of this matrix $P(z, w)=\operatorname{det} K$ is a Laurent polynomial (i.e. negative powers may appear) called the characteristic polynomial of the dimer model

$$
\begin{equation*}
P(z, w)=\sum_{i, j} c_{i j} z^{i} w^{j} \tag{3.1.7}
\end{equation*}
$$

This polynomial provides the link between dimer models and toric geometry [128].
Given an arbitrary "reference" matching $M_{0}$ on the graph, for any matching $M$ the difference $M-M_{0}$ defines a set of closed curves on the graph in $T^{2}$. This in turn defines a height function on the faces of the graph: when a path in the dual graph crosses the curve, the height is increased or decreased by 1 according to the orientation of the crossing. A different choice of reference matching $M_{0}$ shifts the height function by a constant. Thus, only differences in height are physically significant.

In terms of the height function, the characteristic polynomial takes the following form:

$$
\begin{equation*}
P(z, w)=z^{h_{x 0}} w^{h_{y 0}} \sum c_{h_{x}, h_{y}}(-1)^{h_{x}+h_{y}+h_{x} h_{y}} z^{h_{x}} w^{h_{y}} \tag{3.1.8}
\end{equation*}
$$

where $c_{h_{x}, h_{y}}$ are integer coefficients that count the number of paths on the graph with height change ( $h_{x}, h_{y}$ ) around the two fundamental cycles of the torus.

The overall normalization of $P(z, w)$ is not physically meaningful: since the graph does not come with a prescribed embedding into the torus (only a choice of periodicity), the paths $\gamma_{z, w}$ winding around the primitive cycles of the torus may be taken
to (russ any edges en route. Different choices of paths $\gamma$ multiply the characteristic polrumial by an overall power $z^{i} w^{j}$, and by an appropriate choice of path $P(z, w)$ can always be normalized to contain only non-negative powers of $z$ and $w$.

The Newton polygon $N(P)$ is a convex polygon in $\mathbb{Z}^{2}$ generated by the set of integer exponents of the monomials in $P$. In [128], it was conjectured that the Newton polygon can be interpreted as the toric diagram associated to the moduli space of the quiver gauge theory, which by assumption is a non-compact toric Calabi-Yau 3fold. In the following section, we will prove that the perfect matchings of the dimer model are in 1-1 correspondence with the fields of the gauged linear sigma model that describes the probed toric geometry.

Let us illustrate how the computation of the Kasteleyn matrix and the toric diagram works for the case of Model I of $\mathbf{d P}_{3}$. The brane configuration is shown in Figure $3-4 \mathrm{a}$. The corresponding unit cell is presented in Figure 3-4b. As expected, it contains one valence 6 , three valence 4 and two valence 3 nodes. It also contains twelve edges, corresponding to the twelve bifundamental fields in the quiver.

b)


Figure 3-4: a) Brane tiling for Model $I$ of $\mathbf{d P}_{3}$ with flux lines indicated in red. b) Unit cell for Model I of $\mathbf{d P}_{3}$. We show the edges connecting to images of the fundamental nodes in green. We also indicate the signs associated to each edge as well as the powers of $w$ and $z$ corresponding to crossing flux lines.

From the unit cell, we derive the following Kasteleyn matrix

$$
K=\left(\begin{array}{c|ccc} 
& 2 & 4 & 6  \tag{3.1.9}\\
\hline 1 & 1+w & 1-z w & 1+z \\
3 & 1 & -1 & -w^{-1} \\
5 & -z^{-1} & -1 & 1
\end{array}\right)
$$

We observe that is has twelve monomials, associated to the twelve bifundamental fields. This matrix leads to the characteristic polynomial

$$
\begin{equation*}
P(z, w)=w^{-1} z^{-1}-z^{-1}-w^{-1}-6-w-z+w z \tag{3.1.10}
\end{equation*}
$$

The toric data corresponding to this gauge theory can be read from this polynomial, and is shown in Figure 3-5.


Figure 3-5: Toric diagram for Model I of $\mathrm{dP}_{3}$ derived from the characteristic polynomial in (3.1.10).

The Kasteleyn matrix is a square matrix whose size is equal to half the total number of points in the unit cell. Thus, for a given toric quiver $K$ is a $N_{W} / 2 \times$ $N_{W} / 2$ matrix. This is remarkable, since this size can be very modest even for very complicated gauge theories. The simplicity of computing the toric data using this procedure should be contrasted with the difficulty of the Forward Algorithm.

This procedure has a profound impact on the study of quiver theories for arbitrary toric singularities. Given a candidate quiver theory for D3-branes over some geometry, instead of running the lengthy Forward Algorithm, one simply constructs the associated brane tiling using the rules of Section 4.2 and computes the corresponding
characteristic polynomial. We can thus refer to the determination of toric data from brane tilings as the Fast Forward Algorithm ${ }^{2}$. This simplification will become clear when we present explicit results for infinite families of arbitrarily large quivers in Sections 3.7.3.

### 3.2 An explicit correspondence between dimers and GLSMs

In the previous section we have argued that the characteristic polynomial encodes the toric data of the probed geometry. We now explore the reason for this connection, establishing a correspondence between fields in the gauged linear sigma model description of the singularity and perfect matchings in the brane tiling.

Given a toric Calabi-Yau 3-fold, the principles of determining the gauge theory on the world-volume of a stack of D3-brane probes are well established. Conversely, the determination of the toric data of the singularity from the gauge theory is also clear. This procedure has been algorithmized in [82] and dubbed the Forward Algorithm. Nevertheless, although a general prescription exists, its applicability beyond the simplest cases is limited due to the computational complexity of the algorithm.

Let us review the main ideas underlying the Forward Algorithm (for a detailed description and explicit examples, we refer the reader to [82]). The starting point is a quiver with $r S U(N)$ gauge groups and bifundamentals $X_{i}, i=1, \ldots, m$, together with a superpotential. The toric data that describes the probed geometry is computed using the following steps:

- Use F-term equations to express all bifundamental fields $X_{i}$ in terms of $r+2$ independent variables $v_{j}$. The $v_{j}$ 's can be simply equal to a subset of the bifundamentals. The connection between these variables and the original bifundamental fields is encoded in an $m \times(r+2)$ matrix $K$ (this matrix should not be confused with the Kasteleyn matrix; which of them we are talking about

[^3]will be clear from the context), such that
\[

$$
\begin{equation*}
X_{i}=\prod_{j} v_{j}^{K_{i j}}, \quad i=1,2, \ldots, m, \quad j=1,2, \ldots, r+2 \tag{3.2.11}
\end{equation*}
$$

\]

Since the F-term equations take the form of a monomial equated to another monomial, it is clear that generically $K_{i j}$ has negative entries (i.e. negative powers of the $v_{j}$ can appear in the expressions for the $X_{i}$ ).

- In order to avoid the use of negative powers, a new set of variables $p_{\alpha}, \alpha=$ $1, \ldots, c$, is introduced. The number $c$ is not known a priori in this approach, and must be determined as part of the algorithm. We will later see that it corresponds to the number of perfect matchings of $G$, the periodic bipartite graph dual to the quiver.
- The reduction of the $c p_{\alpha}$ 's to the $r+2$ independent variables $v_{i}$ is achieved by introducing a $U(1)^{c-(r+2)}$ gauge group. The action of this group is encoded in a $(c-r-2) \times c$ charge matrix $Q$.
- The original $U(1)^{r-1}$ action (one of the $r U(1)$ 's is redundant) determining the D-terms is recast in terms of the $p_{\alpha}$ by means of a $(r-1) \times c$ charge matrix $Q_{D}$.
- $Q$ and $Q_{D}$ are combined in the total matrix of charges $Q_{t}$. The $U(1)$ actions of the symplectic quotient defining the toric variety correspond to a basis of linear relations among the vectors in the toric diagram. Thus, the toric diagram corresponds to the columns in a matrix $G_{t}$ such that $G_{t}=\left(\operatorname{ker} Q_{t}\right)^{T}$.

At this stage, it is important to stress some points. The main difficulty in the Forward Algorithm is the computation of $T$, which is used to map the intermediate variables $v_{i}$ to the GLSM fields $p_{\alpha}$. Its determination involves the computation of a dual cone, consisting of vectors such that $\vec{K} \cdot \vec{T} \geq 0$. The number of operations involved grows drastically with the "size" (i.e. the number of nodes and bifundamental fields) of the quiver. The computation becomes prohibitive even for quivers of moderate complexity. Thus, one is forced to appeal to alternative approaches such
as (un-)Higgsing [81]. Perhaps the most dramatic examples of this limitation are provided by recently discovered infinite families of gauge theories for the $Y^{p, q}$ [26] and $X^{p, q}[127]$ singularities. The methods presented in this section will enable us to treat such geometries. This also represents a significant improvement over the brute force methods of [128], since the relevant brane tiling may essentially be written down directly from the data of the quiver theory.

It is natural to ask whether the possibility of associating dimer configurations to a gauge theory, made possible due to the introduction of brane tilings, can be exploited to find a natural set of variables playing the role of the $p_{\alpha}$ 's, overcoming the main intricacies of the Forward Algorithm. This is indeed the case, and we now elaborate on the details of the dimer/GLSM correspondence. The fact that the GLSM multiplicities are counted by the $c_{i j}$ coefficients in the characteristic polynomial provides some motivation for the correspondence.

We denote the perfect matchings as $\tilde{p}_{\alpha}$. Every perfect matching corresponds to a collection of edges in the tiling. Hence, we can define a natural product between an edge $e_{i}$, corresponding to a bifundamental field $X_{i}$, and a perfect matching $\tilde{p}_{\alpha}$

$$
<e_{i}, \tilde{p}_{\alpha}>=\left\{\begin{array}{l}
1 \text { if } e_{i} \in \tilde{p}_{\alpha}  \tag{3.2.12}\\
0 \text { if } e_{i} \notin \tilde{p}_{\alpha}
\end{array}\right.
$$

Given this product, we propose the following mapping between bifundamental fields and the perfect matching variables $\tilde{p}_{\alpha}$

$$
\begin{equation*}
X_{i}=\prod_{\alpha} \tilde{p}_{\alpha}^{\left\langle e_{i}, \tilde{p}_{\alpha}\right\rangle} \tag{3.2.13}
\end{equation*}
$$

According to (3.2.12), the $X_{i}$ involve only possitive powers of the $\tilde{p}_{\alpha}$. We will now show that F-term equations are trivially satisfied when the bifundamental fields are expressed in terms of perfect matchings variables according to (3.2.13). For any given bifundamental field $X_{0}$, we have

$$
\begin{equation*}
W=X_{0} P_{1}\left(X_{i}\right)-X_{0} P_{2}\left(X_{i}\right)+\ldots \tag{3.2.14}
\end{equation*}
$$

where we have singled out the two terms in the superpotential that involve $X_{0} . P_{1}\left(X_{i}\right)$ and $P_{2}\left(X_{1}\right)$ represent products of bifundamental fields. The F-term equation associated to $X_{11}$ becomes

$$
\begin{equation*}
\partial_{X_{0}} W=0 \quad \Leftrightarrow \quad P_{1}\left(X_{i}\right)=P_{2}\left(X_{i}\right) \tag{3.2.15}
\end{equation*}
$$

This condition has a simple interpretation in terms of the bipartite graph, as shown in Figure 3-6.


Figure 3-6: F-term equations from the brane tiling perspective.

After excluding the edge associated to $X_{0}$, the product of edges connected to node 1 has to be equal to the product of edges connected to node 2 . In terms of perfect matchings, (3.2.15) becomes

$$
\begin{equation*}
\prod_{i \in P_{1}} \prod_{\alpha} \tilde{p}_{\alpha}^{\left.<e_{i}, \tilde{p}_{\alpha}\right\rangle}=\prod_{i \in P_{2}} \prod_{\alpha} \tilde{p}_{\alpha}^{\left.<e_{i}, \tilde{p}_{\alpha}\right\rangle} \tag{3.2.16}
\end{equation*}
$$

Every time that a given $\tilde{p}_{\alpha}$ appears on the L.H.S. of (3.2.16), it has to appear on the R.H.S. Here is where the fact that the $\tilde{p}_{\alpha}$ 's are perfect matchings becomes important: since nodes 1 and 2 are separated exactly by one edge (the one corresponding to $X_{0}$ ) every time a perfect matching contains any of the edges in $P_{1}$, it contains one of the edges in $P_{2}$. This is necessary for the $p_{\alpha}$ to be a perfect matching (nodes 1 and 2 have to be covered exactly once). Thus, perfect matchings are the appropriate choice of variables that satisfy F-term conditions automatically. We conclude that the perfect matchings can be identified with the GLSM fields $\tilde{p}_{\alpha}=p_{\alpha}$. Then, the matrix that maps the bifundamental fields to the GLSM fields is

$$
\begin{equation*}
(K T)_{i \alpha}=<e_{i}, \tilde{p}_{\alpha}> \tag{3.2.17}
\end{equation*}
$$

### 3.2.1 A detailed example: the Suspended Pinch Point

Let us illustrate the simplifications achieved by identifying GLSM fields with perfect matchings with an explicit example. To do so, we choose the Suspended Pinch Point (SPP) [196]. The SPP has a quiver shown in Figure 3-7 with superpotential

$$
\begin{equation*}
W=X_{21} X_{12} X_{23} X_{32}-X_{32} X_{23} X_{31} X_{13}+X_{13} X_{31} X_{11}-X_{12} X_{21} X_{11} \tag{3.2.18}
\end{equation*}
$$



Figure 3-7: Quiver diagram for the SPP.

It is interesting to see how our methods apply to this example, since it has both adjoint fields and bidirectional arrows. Figure 3-8 shows the brane tiling for the SPP. The adjoint field in the quiver corresponds to an edge between two faces in the tiling representing the first gauge group. The Kasteleyn matrix is


Figure 3-8: Brane tiling for the SPP.

$$
K=\left(\begin{array}{c|cc} 
& 2 & 4  \tag{3.2.19}\\
\hline 1 & 1+w^{-1} & z+w^{-1} z \\
3 & 1 & 1+w^{-1}
\end{array}\right)
$$

from which we determine the characteristic polynomial

$$
\begin{equation*}
P(z, w)=w^{-2}+2 w^{-1}+1-w^{-1} z-z \tag{3.2.20}
\end{equation*}
$$

From it, we construct the toric diagram shown in Figure 3-9.


Figure 3-9: Toric diagram for the SPP. We indicate the perfect matchings corresponding to each node in the toric diagram.

There are six perfect matchings of the SPP tiling. We show them in Figure 3-10. Setting a reference perfect matching, we can compute the slope ( $h_{w}, h_{z}$ ) for each of them, i.e. the height change when moving around the two fundamental cycles of the torus.

Using (3.2.12) and (3.2.17), it is straightforward to determine the $K T$ matrix.

$$
K T=\left(\begin{array}{c|cccccc} 
& p_{1} & p_{2} & p_{3} & p_{4} & p_{5} & p_{6}  \tag{3.2.21}\\
\hline X_{11} & 0 & 0 & 0 & 0 & 1 & 1 \\
X_{12} & 1 & 1 & 0 & 0 & 0 & 0 \\
X_{21} & 0 & 0 & 1 & 1 & 0 & 0 \\
X_{31} & 1 & 0 & 1 & 0 & 0 & 0 \\
X_{13} & 0 & 1 & 0 & 1 & 0 & 0 \\
X_{23} & 0 & 0 & 0 & 0 & 1 & 0 \\
X_{32} & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

This agrees with the computation of this matrix done in Section 3.2 of [82].


Figure 3-10: Perfect matchings for the SPP. We indicate the slopes ( $h_{w}, h_{z}$ ), which allow the identification of the corresponding node in the toric diagram as shown in Figure 3-9.

### 3.3 Massive fields

By definition, massive fields appear in the superpotential as quadratic terms. Therefore they appear in the brane tiling as bivalent vertices. In the IR limit of the gauge theory, these massive fields become non-dynamical and should be integrated out using their equations of motion. We now show how to perform this procedure on the brane tiling and Kasteleyn matrix.

By performing a suitable relabelling of fields, one can always write the superpotential as follows (up to an overall minus sign if the quadratic term comes with opposite sign):

$$
\begin{equation*}
W\left(\Phi_{i}\right)=\Phi_{1} \Phi_{2}-\Phi_{1} P_{1}\left(\Phi_{i}\right)-\Phi_{2} P_{2}\left(\Phi_{i}\right)+\ldots \tag{3.3.22}
\end{equation*}
$$

where the omitted terms do not involve $\Phi_{1}, \Phi_{2}$, and $P_{1}, P_{2}$ are products of two disjoint subsets of the remaining $\Phi$ that do not include $\Phi_{1}$ or $\Phi_{2}$. This structure
follows from the toric condition, which specifies that each field appears exactly twice in the superpotential. with terms of opposite signs.

Integrating out $\Phi_{1}$ and $\Phi_{2}$ by their equations of motion gives

$$
\begin{equation*}
W(\Phi)=-P_{1}\left(\Phi_{i}\right) P_{2}\left(\Phi_{i}\right)+\ldots \tag{3.3.23}
\end{equation*}
$$

This operation takes the form shown in Figure 3-11 and collapses two nodes separated by a bivalent node of the opposite color into a single node of valence equal to the sum of the valences of the original nodes.


Figure 3-11: Integrating out a massive field corresponds to collapsing the two vertices adjacent to a bivalent vertex into a single vertex of higher valence.

The operation of integrating out a massive field can also be implemented in terms of the Kasteleyn matrix. From this perspective, it is simply row or column reduction of the matrix on rows or columns with two non-zero entries (or a single entry containing two summands, if both neighboring vertices to the bivalent vertex are identified in the graph). In the example of figure 3-11, if the bivalent white node has label 1 and the adjacent black nodes are $1^{\prime}$ and $2^{\prime}$ (this can always be arranged by a reordering of rows or columns, with the corresponding action of $(-1)$ to preserve the determinant), the Kasteleyn matrix (or its transpose) has the following structure:

$$
K=\left(\begin{array}{ccccc}
v_{1}^{(1)} & v_{1}^{(2)} & 0 & \ldots & 0  \tag{3.3.24}\\
v_{2}^{(1)} & v_{2}^{(2)} & & & \\
\vdots & \vdots & & \star & \\
v_{n}^{(1)} & v_{n}^{(2)} & & &
\end{array}\right)
$$

where $v^{(1)}$ and $v^{(2)}$ index the adjacent nodes to $1^{\prime}$ and $2^{\prime}$, i.e. contain $\operatorname{deg}\left(P_{1,2}(\Phi)\right)+1$ non-zero entries.

Performing elementary column operations ${ }^{3}$, the matrix can be brought to the following form ${ }^{4}$ :

$$
K=\left(\begin{array}{ccccc}
1 & 0 & 0 & \ldots & 0  \tag{3.3.25}\\
v_{2}^{(1)} / v_{1}^{(1)} & v_{2}^{(2)} v_{1}^{(1)}-v_{2}^{(1)} v_{1}^{(2)} & & \\
\vdots & \vdots & & \star & \\
v_{n}^{(1)} / v_{1}^{(1)} & v_{n}^{(2)} v_{1}^{(1)}-v_{n}^{(1)} v_{1}^{(2)} & &
\end{array}\right)
$$

and therefore can be reduced in rank without changing the determinant, by deleting the first row and column, giving the reduced Kasteleyn matrix

$$
K=\left(\begin{array}{cccc}
v_{2}^{(2)} v_{1}^{(1)}-v_{2}^{(1)} v_{1}^{(2)} & \star & \star & \ldots  \tag{3.3.26}\\
\vdots & & \star & \star \\
\ldots \\
v_{n}^{(2)} v_{1}^{(1)}-v_{n}^{(1)} v_{1}^{(2)} & \star & \star & \ldots
\end{array}\right)
$$

corresponding to the graph with bivalent vertex deleted.

[^4]
### 3.4 Seiberg duality

### 3.4.1 Seiberg duality as a transformation of the quiver

We now discuss how one can understand Seiberg duality from the perspective of the brane tilings. To motivate our construction, let us first recall what happens to a quiver theory when performing Seiberg duality at a single node. This was first done for orbifold quivers in [208]. Recall first that since Seiberg duality takes a given gauge group $S U\left(N_{c}\right)$ with $N_{f}$ fundamentals and $N_{f}$ anti-fundamentals to $S U\left(N_{f}-N_{c}\right)$, if we want the dual quiver to remain in a toric phase, we are only allowed to dualize on nodes with $N_{f}=2 N_{c}$. Dualizing on such a node (call it $I$ ) is straightforward, and is done as follows:

- To decouple the dynamics of node $I$ from the rest of the theory, the gauge couplings of the other gauge groups and superpotential should be scaled to zero. The fields corresponding to edges in the quiver that are not adjacent to $I$ decouple, and the edges between $I$ and other nodes reduce from bifundamental matter to fundamental matter transforming under a global flavor symmetry group. This reduces the theory to the SQCD-like theory with $2 N_{c}$ flavors and additional gauge singlets, to which Seiberg duality may be applied.
- Next, reverse the direction of all arrows entering or exiting the dualized node. This is because Seiberg duality requires that the dual quarks transform in the conjugate flavor representations to the originals, and the other end of each bifundamental transforms under a gauge group which acts as an effective flavor symmetry group. Because we want to describe our theory with a quiver, we perform charge conjugation on the dualized node to get back bifundamentals. This is exactly the same as reversing the arrows in the quiver.
- Next, draw in $N_{f}$ bifundamentals which correspond to composite (mesonic) operators that are singlets at the dualized node $I$ and carry flavor indices in the pairs nodes connected to $I$. This is just the usual $Q_{i} \widetilde{Q}^{j} \rightarrow M_{i}^{j}$ "electric quark $\rightarrow$ meson" map of Seiberg duality, but since each flavor group becomes
gauged in the full quiver theory. the Seiberg mesons are promoted to fields in the bifundamental representation of the gauge groups.
- In the superpotential, replace any romposite singlet operators with the new mesons, and write down new terms corresponding to any new triangles formed by the operators above. It is possible that this will make some fields massive (e.g. if a cubic term becomes quadratic), in which case the appropriate fields should then be integrated out.


### 3.4.2 Seiberg duality as a transformation of the brane tiling

By writing the action of Seiberg duality in the periodic quiver picture, one may derive the corresponding transformation on the dual brane tiling. This operation may be encoded in a transformation on the Kasteleyn matrix of the graph, and the recursive application of Seiberg duality may be implemented by computer to traverse the Seiberg duality tree [93, 92] and enumerate all toric phases ${ }^{5}$.

Consider a node in the periodic quiver. For the toric phases of the quiver all nodes in the quiver correspond to gauge groups of equal rank. If the node has 2 incoming arrows (and therefore 2 outgoing arrows by anomaly cancellation, for a total of 4 arrows), then for this gauge group $N_{f}=2 N_{c}$, and Seiberg duality maps

$$
\begin{equation*}
N_{c} \mapsto \hat{N}_{c}=N_{f}-N_{c}=N_{c} \tag{3.4.27}
\end{equation*}
$$

so after the duality the theory remains in a toric phase.
At such a node $V$, a generic quiver can be represented as in Figure 3-12. The 4 faces $F_{i}$ adjacent to $V$ share an edge with their adjacent faces, and contain some number of additional edges.

The neighboring vertices to $V$ are not necessarily all distinct (they may be identi-

[^5]

Figure 3-12: The action of Seiberg duality on a periodic quiver to produce another toric phase of the quiver. Also marked are the signs of superpotential terms, showing that the new terms (faces) are consistent with the pre-existing 2 -coloring of the global graph.
fied by the periodicity of the torus on which the quiver lives). However by the periodic quiver construction, if there are multiple fields in the quiver connecting the same two vertices, these appear as distinct edges in the periodic quiver.

Note that the new mesons can only appear between adjacent vertices in the planar quiver, because the edges connecting opposing vertices do not have a compatible orientation, so they cannot form a holomorphic, gauge-invariant combination. There are indeed 4 such arrows that can be drawn on the quiver corresponding to the $2 \times 2=4$ Seiberg mesons.

It is easy to translate this operation to the dual brane tiling. Gauge groups with $N_{f}=2 N_{c}$ correspond to quadrilaterals in the tiling. Performing Seiberg duality on such a face corresponds to the operation depicted in Figure 3-13 ${ }^{6}$.

Note that this operation (and the dual operation on the quiver) are local operations on the graph, in that they only affect a face and its neighbors, and the global structure of the graph is unaffected ${ }^{7}$.

As a simple example, consider $\mathbf{F}_{0}$. This is a $\mathbb{Z}_{2}$ orbifold of the conifold, and as such one can simply take the two-cell fundamental domain of the conifold and

[^6]

Figure 3-13: Seiberg duality acting on a brane tiling to produce another toric phase. This is the planar dual to the operation depicted in Figure 3-12. Whenever 2-valent nodes are generated by this transformation, the corresponding massive fields can be integrated out as explained in Section 3.3.
double its area (with an appropriate choice of periodicity) to get the $\mathbf{F}_{0}$ fundamental domain; this phase of this theory is given by a square graph with four different cells. In Figure 3-14, we have drawn this phase of the theory as well as the phase obtained by dualizing on face 1 . The blue dotted lines are the lines of magnetic flux delineating the fundamental region, which do not change during Seiberg duality. It is straightforward to see that these regions give the correct Kasteleyn matrices, and reproduce the known multiplicities of sigma model fields [128].


Figure 3-14: The operation of Seiberg duality on a phase of $\mathbf{F}_{0}$.

It is useful to see how this action of Seiberg duality can be understood from the brane perspective. Since the area of each cell (volume of the D-brane) is related to the gauge coupling of the corresponding group [130], one would expect that Seiberg duality could be viewed as a cell shrinking and then growing with the opposite orientation, e.g. as branes move through one another. It is possible to see this from

Figure 3-14: we can simply take the NST-branes at the sides of region 1 and pull them through one another. In doing this we generate the diagonal lines. Since we are in a toric phase with $N_{f}=2 N_{c}$, the ranks of the gauge groups do not change in this crossing operation and no new branes are created.

### 3.5 Partial resolution

Many of the first known examples of gauge theories dual to toric geometries were described by embedding them in orbifolds [82, 83, 24]. For example, partial resolutions of $\mathbb{C}^{3} / \mathbb{Z}_{3} \times \mathbb{Z}_{3}$ give the first three del Pezzo theories and $\mathbf{F}_{0}$, among others. Partially resolving the orbifold singularity corresponds to turning on Fayet-Iliopoulos terms in the dual gauge theory, which by the D-flatness conditions gives vacuum expectation values to bifundamental fields. These vevs then reduce the rank of the gauge group via the Higgs mechanism. From the standpoint of the toric diagram, this is simply removing an external point. Doing so decreases the area of the toric diagram, and consequently decreases the number of gauge groups in the dual superconformal theory.

It is straightforward to see how Higgsing operates from the perspective of the dimer models. We give a non-zero vev to a bifundamental field, which reduces the two gauge group factors under which the bifundamental is charged to the diagonal combination. Hence, Higgsing is nothing more than the removal of an edge from the fundamental region of the graph, which causes two faces of the graph to become one.

This method was used in [128] to obtain the bipartite graphs corresponding to an arbitrary toric singularity, but the algorithm presented was computationally expensive since it was unknown how to identify the desired Higgsing in the quiver side. Using the duality between the quivers and brane tilings, it is straightforward to identify the edge of the bipartite graph to be removed that corresponds to the Higgsing of any given field in the quiver. Thus, the relations between quiver theories under Higgsing may be easily followed on the dual brane tiling, avoiding any computational difficulties.

Let us begin with Model I of $\mathrm{dP}_{3}$, since we have already studied this tiling in a previous section. Since this model is perfectly symmetric and contains only single
bifundamental field between any two gauge groups, giving a vev to any field should result in the same theory. This theory is $\mathbf{d} \mathbf{P}_{2}$, which has five nodes in its quiver. One can easily check that removing any edge from this tiling for $\mathbf{d P}_{3}$ results in the expected gauge theory. Figure 3-15 illustrates this process: removing the edge between regions 5 and 6 is equivalent to removing the bifundamental between the corresponding nodes.


Figure 3-15: Removing the edge from between faces 5 and 6 Higgses Model I of $\mathrm{dP}_{3}$ (top) to one of the two toric phases of $\mathbf{d P}_{2}$ (bottom).

The example of taking Model I of $\mathbf{d P}_{3}$ to one of the two toric phases of $\mathbf{d P}_{2}$ (called Model II in [80]) is particularly simple, since no fields acquire a mass when $X_{56}$ gets a vev. It is not any more difficult to see what happens when bifundamentals do become massive, as we can see by considering the $\mathbf{d} \mathbf{P}_{2}$ example. We know that the $\mathbf{d P}_{2}$ theory can be Higgsed to either $\mathbf{d} \mathbf{P}_{1}$ or $\mathbf{F}_{0}$; this corresponds to giving a vev to $X_{34}$ (or equivalently $X_{12}$ by the symmetry of the quiver) or $X_{23}$, respectively. In the brane tiling, we delete the edge between regions 2 and 3 of the tiling. This puts an isolated node between the two regions. As per our discussion in Section 3.3, we then simply collapse those two edges to a point, which corresponds to integrating out the fields $X_{35}$ and $X_{52}$. See Figure 3-16.


Figure 3-16: The $\mathbf{d P}_{2}$ tiling (top) can be taken to either $\mathbf{d P}_{1}$ (bottom left) or $\mathbf{F}_{0}$ (bottom right), depending on which edge gets removed. In the $\mathbf{F}_{0}$ tiling, one should collapse the edge between regions 2 and 4 to a point; this corresponds to the bifundamentals on the diagonal of the quiver.

We expect from string theory that we may embed any toric quiver in an appropriately large Abelian orbifold theory of the form $\mathbb{C}^{3} / \mathbb{Z}_{m} \times \mathbb{Z}_{n}$. The tiling for $\mathbb{C}^{3} / \mathbb{Z}_{m} \times \mathbb{Z}_{n}$ is hexagonal, so one expects that we can reach any tiling by removing edges from hexagons. This is indeed the case, as noted by [169] and used extensively in [128].

### 3.6 Different toric superpotentials for a given quiver

Dimer methods can be used to tackle another interesting problem. Given a quiver diagram, it is sometimes possible to construct more than one consistent toric superpotential. Constructing the corresponding tilings shows immediately how these theories differ and enables a straightforward computation of their toric data.

Let us consider a concrete example, given by the quiver diagram shown in Figure 317. This is the quiver for Model II of $\mathbf{d P}_{3}$ [80]. This quiver has 6 gauge groups and 14 bifundamental fields. From (3.0.1), we see that the number of superpotential terms is $N_{W}=14-6=8$. There are two possible toric superpotentials consistent with the node symmetry group of the quiver. They have been considered in [80] and [88] and


Figure 3-17: Quiver diagram admitting two toric superpotentials.
are

$$
\begin{align*}
W_{A} & =\left[X_{12} X_{26} X_{61}-X_{12} X_{25} X_{51}+X_{36} X_{64} X_{43}-X_{35} X_{54} X_{43}\right] \\
& +\left[-X_{61} X_{13} X_{36}+X_{51} Y_{13} X_{35}\right]+\left[-X_{26} X_{64} X_{41} Y_{13} X_{32}+X_{25} X_{54} X_{41} X_{13} X_{32}\right] \tag{3.6.28}
\end{align*}
$$

$$
\begin{align*}
W_{B} & =Y_{13} X_{36} X_{61}+X_{13} X_{35} X_{51}-X_{61} X_{12} X_{26}-X_{43} X_{35} X_{54} \\
& +X_{12} X_{25} X_{54} X_{41}+X_{26} X_{64} X_{43} X_{32}-X_{25} X_{51} Y_{13} X_{32}-X_{64} X_{41} X_{13} X_{36} \tag{3.6.29}
\end{align*}
$$

$W_{A}$ corresponds to a brane tiling with six valence- 3 and two valence- 5 nodes. This brane tiling is shown in Figure 3-18. For $W_{B}$ the brane tiling has four valence-3 and four valence-4 nodes and it is shown in Figure 3-19. The Kasteleyn matrices for these tilings are


Figure 3-18: Brane tiling corresponding to the quiver diagram in Figure 3-17 and the superpotential in (3.6.28).


Figure 3-19: Brane tiling corresponding to the quiver diagram in Figure 3-17 and the superpotential in (3.6.29).

$$
K_{A}=\left(\begin{array}{c|cccc} 
& 2 & 4 & 6 & 8  \tag{3.6.30}\\
\hline 1 & 1 & z^{-1} & 0 & w \\
3 & -1 & 1 & 1 & 0 \\
5 & z+w^{-1} & w^{-1} & 1 & -z \\
7 & 1 & 0 & 1 & 1
\end{array}\right) \quad K_{B}=\left(\begin{array}{c|cccc} 
& 2 & 4 & 6 & 8 \\
\hline 1 & 1 & 0 & w^{-1} & w^{-1} \\
3 & z & 1 & 0 & w^{-1} \\
5 & 1 & 1 & -1 & 1 \\
7 & z & -1 & z & 1
\end{array}\right)
$$

The corresponding characteristic polynomials are

$$
\begin{array}{r}
P_{A}(z, w)=w^{-1} z^{-1}+z^{-1}-w^{-1}+7-w+z+w z \\
P_{B}(z, w)=-w^{-2}-2 w^{-1}-1-w^{-2} z+7 w^{-1} z-z-w^{-1} z^{2} \tag{3.6.32}
\end{array}
$$

From (3.6.31) and (3.6.32), we extract the toric diagrams shown in Figure 3-20.


Figure 3-20: Toric diagram for the quiver in Figure 3-17 and superpotentials $W_{A}$ and $W_{B}$

Thus we see that $W_{A}$ leads to Model II of $\mathrm{dP}_{3}$ (the multiplicities of GLSM fields are in agreement with the ones derived in [80]) while $W_{B}$ leads to a non-generic blow-up of $\mathbb{C P}^{2}$ at three points, denoted $P d P_{3 b}$ in [88].

### 3.7 Examples

Here we present the brane tiling configurations for several interesting gauge theories. Many of them can be obtained using the Seiberg duality and partial resolution ideas discussed in previous sections. When doing so, we generate data on GLSM multiplicities for all these models.

### 3.7.1 Del Pezzo 2

There are two toric phases for $\mathbf{d P}_{2}$. Their corresponding quivers and superpotentials can be found in [80]. We now construct their corresponding brane tilings.

## Model I

The brane tiling for this model is shown in Figure 3-21. The Kasteleyn matrix is

$$
K=\left(\begin{array}{cccc}
1 & w^{-1} & w^{-1} z^{-1} & 1  \tag{3.7.33}\\
1 & 1 & -z^{-1} & 0 \\
0 & 1 & -1 & -w \\
z & 0 & 1 & 1
\end{array}\right)
$$



Figure 3-21: Brane tiling for Model I of $\mathrm{dP}_{2}$.
leading to

$$
\begin{equation*}
P(z, w)=w^{-1} z^{-1}+z^{-1}+w^{-1}-6+w+z \tag{3.7.34}
\end{equation*}
$$

## Model II

The tiling for this model was obtained in Section 3.5 by means of partial resolution. We show it again in Figure 3-22. The corresponding Kasteleyn matrix is


Figure 3-22: Brane tiling for Model II of $\mathbf{d} \mathbf{P}_{2}$.

$$
K=\left(\begin{array}{ccc}
1-z^{-1} & w & 1  \tag{3.7.35}\\
1 & 1 & z \\
-1+w^{-1} z^{-1} & 1 & 1
\end{array}\right)
$$

which leads to the following characteristic polynomial

$$
\begin{equation*}
P(z, w)=w^{-1} z^{-1}-z^{-1}+5-w-z-w z \tag{3.7.36}
\end{equation*}
$$

From (3.7.34) and (3.7.36) we can determine the toric diagrams along with the GLSM multiplicities, which are in agreement with the results in [80].

## Del Pezzo 3

There are four toric phases for $\mathbf{d P}_{3}$. We refer the reader to [80] for their quivers and superpotentials. We have already presented the tiling for Model I in Figure 3-1. Its Kasteleyn matrix and characteristic polynomial are written in (3.1.9) and (3.1.10). Figure 3-18 shows the tiling for Model II. Its Kasteleyn matrix and characteristic polynomial are presented in (3.6.30) and (3.6.31). We now proceed with the construction of the brane tilings for Models III and IV.

## Model III

We show the brane tiling in Figure 3-23. The Kasteleyn matrix is


Figure 3-23: Brane tiling for Model III of $\mathbf{d P}_{3}$.

$$
K=\left(\begin{array}{c|cccc} 
& 2 & 4 & 6 & 8  \tag{3.7.37}\\
\hline 1 & 1 & w^{-1} & w^{-1} z^{-1} & 1 \\
3 & 1 & 1 & -z^{-1} & 0 \\
5 & w z & 1 & -1 & -w \\
7 & z & 0 & 1 & 1
\end{array}\right)
$$

from which we compute the determinant

$$
\begin{equation*}
P(z, w)=w^{-1} z^{-1}+z^{-1}-w^{-1}-8+w+z+w z \tag{3.7.38}
\end{equation*}
$$

This corresponds to the toric diagram of $\mathbf{d} \mathbf{P}_{3}$ with multiplicity 8 for the central point. This result agrees with the Forward Algorithm computations in [80].

## Model IV

Figure 3-24 shows the brane tiling for this theory.


Figure 3-24: Brane tiling for Model IV of $\mathbf{d P}_{3}$.

The Kasteleyn matrix is given by

$$
K=\left(\begin{array}{c|cccccc} 
& 2 & 4 & 6 & 8 & 10 & 12  \tag{3.7.39}\\
\hline 1 & 1 & 0 & 0 & -w z & 0 & -1 \\
3 & 1 & 1 & 0 & 0 & z & 0 \\
5 & 0 & -1 & 1 & 0 & 0 & -w^{-1} \\
7 & w^{-1} z^{-1} & 0 & 1 & 1 & 0 & 0 \\
9 & 0 & -z^{-1} & 0 & -1 & 1 & 0 \\
11 & 0 & 0 & w & 0 & 1 & 1
\end{array}\right)
$$

and the characteristic polynomial is

$$
\begin{equation*}
P(z, w)=-w^{-1} z^{-1}-z^{-1}-w^{-1}+11-w-z-w z \tag{3.7.40}
\end{equation*}
$$

Once again, this corresponds to the toric diagram of $\mathbf{d P}_{3}$. In this case, the multiplicity of the central point is 11 , in agreement with the computations in [80].

### 3.7.2 Pseudo del Pezzo 5

We now consider a complex cone over non-generic, toric blow-up of $\mathbb{C P}^{2}$ at five points. The geometry corresponds to a $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold of the conifold and was dubbed $P d P_{5}$ in [81], where the corresponding gauge theories were studied. There are four toric phases for this geometry. We refer the reader to [81] for the quivers and superpotentials. The brane tilings for these four phases are shown in Figure 3-25


Model I


Model III


Model II


Model IV

Figure 3-25: Brane tilings for the four toric phases of $P d P_{5}$.

$$
\begin{align*}
& K_{I}=\left(\begin{array}{c|cccc} 
& 2 & 4 & 6 & 8 \\
\hline 1 & -1 & -w & 1 & u^{\prime} z \\
3 & -1 & -1 & z & 1 \\
5 & z^{-1} & w & 1 & w \\
7 & 1 & z^{-1} & 1 & 1
\end{array}\right) \quad K_{I I}=\left(\begin{array}{c|cccccc} 
& 2 & 4 & 6 & 8 & 10 & 12 \\
\hline 1 & -1 & 0 & -w & 0 & 0 & w z \\
3 & -1 & 1 & 0 & 0 & 1 & 0 \\
5 & 0 & -1 & -1 & z & 0 & 1 \\
7 & 0 & z^{-1} & w & 1 & 0 & w \\
9 & 1 & 0 & 0 & 1 & 1 & 0 \\
11 & 0 & 0 & z^{-1} & 0 & -1 & 1
\end{array}\right)  \tag{3.7.41}\\
& K_{I I I}=\left(\begin{array}{c|cccccc} 
& 2 & 4 & 6 & 8 & 10 & 12 \\
\hline 1 & 1 & 0 & w & 0 & 0 & -w z \\
3 & 1 & 1 & 0 & -z & -1 & 0 \\
5 & 0 & -1 & 1 & 0 & 0 & -1 \\
7 & 0 & 0 & -w & -1 & 0 & -w \\
9 & -1 & -z^{-1} & 0 & -1 & -1 & 0 \\
11 & 0 & 0 & -z^{-1} & 0 & 1 & -1
\end{array}\right)  \tag{3.7.42}\\
& K_{I V}=\left(\begin{array}{c|cccccccc} 
& 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \\
\hline 1 & 1 & 0 & 0 & w & 0 & 0 & w z & 0 \\
3 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 1 & 1 & 0 & z & 0 & 0 & 0 \\
7 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\
9 & 0 & z^{-1} & 0 & 0 & -1 & 0 & 0 & w \\
11 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
13 & 0 & 0 & 0 & z^{-1} & 0 & 1 & -1 & 0 \\
15 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1
\end{array}\right) \tag{3.7.43}
\end{align*}
$$

From these matrices, we compute the corresponding characteristic polynomials

$$
\left.\begin{array}{rl}
P_{I}(z, w)= & P_{I I I}\left(z, u^{\prime}\right)
\end{array}=z^{-2}+2 z^{-1}+2 w z^{-1}+1-12 w+w^{2}+2 u z+2 w^{2} z+w^{2} z^{2}\right) ~=z^{-2}+2 z^{-1}+2 w z^{-1}+1-14 w+w^{2}+2 u z+2 w^{2} z+w^{2} z^{2} .
$$

Remarkably, although $K_{I}$ and $K_{I I I}$ are different matrices with different dimensions, their characteristic polynomials turn out to be identical. This is a counterexample to the conjecture that GLSM multiplicities are in one to one correspondence with the dual phases of the gauge theory. Different phases can indeed lead to exactly the same multiplicities. We present the toric diagrams with multiplicities in Figure 3-26.


Figure 3-26: Toric diagrams with multiplicities for the four toric phases of $P d P_{5}$. We observe that the GLSM multiplicities are the same for Models I and III.

### 3.7.3 Tilings for infinite families of gauge theories

One of the problems for which dimer methods show their full power is in the determination of dual geometries for infinite families of gauge theories. Infinite sets of quiver theories have recently been constructed in [26] and [127]. On one hand, we have already discussed that the application of the Forward Algorithm to large quivers becomes computationally prohibitive. In addition, it is obviously impossible to apply the Forward Algorithm to an infinite number of theories. Hence, the determination of gauge theories dual to an infinite number of geometries usually involve indirect evidence such as: (un)higgsing, global symmetries, computation of R-charges and central charges and comparison to volumes in the underlying geometry $[26,127]$.

## $Y^{p, q}$ tilings

Let us discuss now how the $Y^{p . q}$ theories [26] appear in the brane tiling picture. A simple way to construct the $Y^{p, q}$ s is to start with $Y^{p, q=p}$ and decrease $q$ by introducing "impurities" into the quiver [29]. This procedure can be similarly carried out with tilings. Since $Y^{p, p}$ is the base of the orbifold $\mathbb{C}^{3} / \mathbb{Z}_{2 p}$, it corresponds to the hexagonal graph with a fundamental cell containing $2 \times p$ hexagons. This is shown in Figure 3-27 for $Y^{3,3}$.


Figure 3-27: Brane tiling for $Y^{3,3}$

Let us put now a single impurity into the tiling. The impurity covers four hexagons, and is indicated in blue in Figure 3-28. Two disjoint single impurities


Figure 3-28: Brane tiling for $Y^{3,2}$. The impurity is the blue area.
can be generated by adding an identical shaded region into the tiling, separated from the first one by some hexagonal faces. For $Y^{3,1}$ this is not possible because the fundamental cell consists of only six hexagons, whereas two separated single impurities would cover eight of them. Instead, we can consider the case in which the two impurities are adjacent. This corresponds to a similar impurity graph, which is shown in Figure 3-29.


Figure 3-29: Brane tiling for $Y^{3,1}$.
One can continue adding impurities and discover the simplicity of the Kasteleyn matrix for $Y^{p, q}$. It contains elements only in the diagonal and its neighbors and in the corners. It can be written down immediately, without actually drawing the corresponding brane tiling. One starts with the following $2 p \times 2 p$ Kasteleyn matrix for $Y^{p, p}$.

$$
K=\left(\begin{array}{cccccccc}
1 & 1 & 0 & 0 & \ldots \ldots & 0 & z^{-1}  \tag{3.7.45}\\
1 & w & 1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 1 & 1 & 1 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 1 & w & 1 & 0 & \ldots & 0 \\
& \vdots & & & & \ddots & & \\
0 & \ldots & \ldots & 0 & 1 & 1 & 1 \\
z & 0 & \ldots & \ldots & 0 & 1 & w
\end{array}\right)
$$

We see that the elements around the diagonal consist of the alternating "codons":

$$
\begin{align*}
& A_{1}:=(1,1,1)  \tag{3.7.46}\\
& A_{2}:=(1, w, 1) \tag{3.7.47}
\end{align*}
$$

We define three other codons

$$
\begin{align*}
S & :=(1, w, w)  \tag{3.7.48}\\
I & :=(1,-1+w, w)  \tag{3.7.49}\\
E & :=(1,-1+w, 1) \tag{3.7.50}
\end{align*}
$$

Placing impurities into the quiver means changing the $A_{1}, A_{2}, A_{1}, A_{2}, \ldots$ sequence in the matrix. For $n$ single impurities we get $Y^{p, p-n}$ and the Kasteleyn matrix gets smaller, it is now a $(2 p-n) \times(2 p-n)$ matrix. We change the sequence of the codons as the following. In the second row we put $S$ (start codon), then $n-1$ times the $I$ (iteration codon), and we close it with $E$ (end codon). Then we continue the series with $A_{1}, A_{2}, A_{1}, A_{2}, \ldots$ until the end of the matrix. As an example, we present the Kasteleyn matrix for $Y^{5,3}$ (i.e. $n=2$ )

$$
K=\left(\begin{array}{c|cccccccc}
A_{1} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & z^{-1}  \tag{3.7.51}\\
S & 1 & w & w & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 1 & -1+w & w & 0 & 0 & 0 & 0 \\
E & 0 & 0 & 1 & -1+w & 1 & 0 & 0 & 0 \\
A_{1} & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
A_{2} & 0 & 0 & 0 & 0 & 1 & w & 1 & 0 \\
A_{1} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
A_{2} & z & 0 & 0 & 0 & 0 & 0 & 1 & w
\end{array}\right)
$$

The determinant of the Kasteleyn matrix is then

$$
\begin{equation*}
P(w, z)=-1+16 w-41 w^{2}+33 w^{3}-10 w^{4}+w^{5}-z^{-1}-w^{2} z \tag{3.7.52}
\end{equation*}
$$

and the toric diagam (with GLSM multiplicities) is given in Figure 3-30.


Figure 3-30: Toric diagram of a phase of $Y^{5,3}$

We note that the above rules for constructing the Kasteleyn matrix produce the toric diagrams for all $Y^{p, q}$ with $p>q>0$. To check this, we can see that the correct monomials appear in the determinant. First, the only powers of $z$ that appear in $\operatorname{det} K$ are $-1,0$, and 1 . Terms of the form $z^{0} w^{k}$ appear for all $k=0, \ldots, p$; these come from the diagonal. Second, there is a term $z^{-1} w^{0}$ that comes from the lower offdiagonal. Finally, the term $z w^{n}$, where $n=p-q$ is the number of single impurities, comes from the upper off-diagonal and gets contributions from only the $S$ and $I$ codons. Thus, we have shown that the moduli spaces of the $Y^{p, q}$ quivers reproduce the correct toric geometries.

For $Y^{p, 0}$ the matrix gets too small and there is not enough space for $A_{1}$ and $A_{2}$. The Kasteleyn matrix consists of only $I$ codons:

$$
K=\left(\begin{array}{ccccccc}
-1+w & w & 0 & 0 & \ldots \ldots \ldots \ldots \ldots & z^{-1}  \tag{3.7.53}\\
1 & -1+w & w & 0 & \ldots \ldots \ldots \ldots \ldots & 0 \\
0 & 1 & -1+w & w & 0 & \ldots \ldots \ldots \ldots & 0 \\
0 & 0 & 1 & -1+w & w & 0 & \ldots \\
& \vdots & & & & \ddots & \\
0 & \ldots \ldots \ldots \ldots \ldots \ldots \ldots & 0 & 1 & -1+w & w \\
0 & 0 & \ldots \ldots \ldots \ldots \ldots \ldots & 0 & 1 & -1+w
\end{array}\right)
$$

For example, the Kasteleyn matrix of $Y^{3.0}$ is:

$$
K=\left(\begin{array}{ccc}
-1+w & w & z^{-1}  \tag{3.7.54}\\
1 & -1+w & w \\
w z & 1 & -1+w
\end{array}\right)
$$

with determinant

$$
\begin{equation*}
P(w, z)=-1+6 w-6 w^{2}+w^{3}+z^{-1}+w^{3} z \tag{3.7.55}
\end{equation*}
$$

and the toric diagram of Figure 3-31.


Figure 3-31: Toric diagram of a phase of $Y^{3,0}$ with three single impurities.

These Kasteleyn matrices give the toric diagrams of a certain phase of the theories (the one with only single impurities, all of them together). Other phases can be obtained by performing Seiberg duality transformations. As discussed in section 3.4 this may be efficiently implemented on a computer and used to enumerate the toric phases of the theory, together with the duality graph showing the interconnections between phases.

## $Y^{3,1}$ with double impurity

In [29], it was shown that all toric phases of $Y^{p, q}$ theories can be constructed by adding single and double impurities to the $\mathbb{C}^{3} / \mathbb{Z}_{2 p}$ quiver. Double impurities arise when Seiberg duality makes two single impurities "collide". As an example of Seiberg duality, we now study the double impurity phase of $Y^{3,1}$. This phase can be obtained by dualizing face 3 (see Figure 3-32). The resulting graph can be deformed to the more symmetric form which is shown in Figure 3-33. The determinant of the Kasteleyn matrix again gives the $P(w, z)$ polynomial, from which we get the toric diagram
(Figure 3-34).


Figure 3-32: Dualizing face 3 in $Y^{3,1}$ with two single impurities. In resulting tiling, we indicate the double impurity in pink.


Figure 3-33: The double impurity in $Y^{3,1}$

$$
\begin{gather*}
K=\left(\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & z^{-1} \\
w^{-1} & 1 & w^{-1} & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & w^{-1} & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \\
z^{-1} & 0 & 0 & 0 & w & 1
\end{array}\right)  \tag{3.7.56}\\
P(w, z)=-7-w^{-2}+9 w^{-1}+w+w^{-1} z^{-1}+z w^{-1} \tag{3.7.57}
\end{gather*}
$$



Figure 3-34: Toric diagram for $Y^{3.1}$ in the double impurity phase

All the multiplicity results in this and previous section agree with the ones derived using the Forward Algorithm.

## $X^{p, q}$ tilings

We now describe the brane tilings for the $X^{p, q}$ spaces constructed in [127]. Recall that these spaces are defined by the property that an $X^{p, q}$ theory can be Higgsed to both $Y^{p, q}$ and $Y^{p, q-1}$. Constructing the brane tilings for the $X^{p, q}$ is quite straightforward, but it will be convenient for our purposes to use a slightly modified (but entirely equivalent) description of the $Y^{p, q}$ spaces from the one used in the previous section.

We use the following description of $Y^{p, q}$, with $p-q$ single impurities. For this tiling, we need $2(p-q)$ quadrilaterals and $2 q$ hexagons. We build the quadrilaterals by starting with a hexagonal grid, and drawing lines through the center of a given hexagon, connecting opposite vertices. This divides the hexagon into two quadrilaterals. A given $Y^{p, q}$ is then given by placing these hexagons and divided hexagons along a single diagonal such that the divided hexagons are separated from each other by an even number of non-divided hexagons; this is simply the requirement that the single impurities be separated from each other by an odd number of doublets. For examples of this construction, see Figure 3-35.

Constructing the $X^{p, q}$ brane tilings is now straightforward. We give the example of $X^{3,1}$ below; the other $X^{p, q}$ tilings work similarly. To build the tilings, simply insert diagonal lines in hexagons such that removing the line from the $X^{p, q}$ tiling gives the $Y^{p, q}$ to which it descends. This diagonal line should always share a node with one of the horizontal lines subdividing a hexagon in two; this is what allows one to blow down to $Y^{p, q-1}$ as well as $Y^{p, q}$. In Figure 3-36, one may remove the line between

$\mathbf{Y}^{3.0}$

$r^{3.1}$

$\mathrm{Y}^{3,2}$

Figure 3-35: Brane tilings for $Y^{3, q}$.
regions 6 and 7 or the line between regions 5 and 6 to yield $Y^{3,1}$ and $Y^{3,0}$, respectively.


Figure 3-36: A brane tiling for $X^{3,1}$.

The Kasteleyn matrix for this tiling is

$$
K=\left(\begin{array}{c|cccc} 
& 2 & 4 & 6 & 8  \tag{3.7.58}\\
\hline 1 & 1+w^{-1} & 1 & 0 & z \\
3 & 1 & -1-w & 1 & 0 \\
5 & 0 & 1 & w^{-1} & 1 \\
7 & z^{-1} & 1 & 1 & -1
\end{array}\right)
$$

which has determinant $\operatorname{det} K=7 w^{-1}+w^{-2}+8+w-z^{-1}-z+w^{-1} z$. This yields the proper toric diagram and multiplicities for this phase of $X^{3,1}$.

## Chapter 4

## Equivalence of algorithms

### 4.1 Introduction

In the previous Chapter, we introduced the brane tiling approach to quiver theories on D-branes over toric singularities. We have seen a striking correspondence between the perfect matching partition function and the toric diagram of the underlying geometry. In addition, a specific correspondence between GLSM fields and perfect matchings was conjectured, noticing also how perfect matchings are natural variables to solve F term equations. This correspondence, which we call Mathematical Dimer Conjecture, leads to impressive simplifications in the study of branes on toric singularities and lies at the core of the breakthrough of the dimer ideas. The main result of this Chapter is the proof of the Mathematical Dimer Conjecture.

We devote Section 4.2 to a discussion of toric quivers and brane tilings. Section 4.3 presents the gauged linear sigma model (GLSM) approach for computing toric moduli spaces of toric gauge theories. In Section 4.4 we present the conjecture of [94], splitting it into the Mathematical and Physical dimer conjectures. Finally, we prove the Mathematical Dimer Conjecture in Section 4.5. We illustrate all discussions with the relatively non-trivial example of a quiver theory for D3-branes probing a complex cone over the second del Pezzo surface.

### 4.2 Toric quivers and brane tilings

We consider the $\mathcal{N}=1$ superconformal gauge theories that live on the worldvolume of a stack of $N$ D3-branes probing a non-compact toric Calabi-Yau 3-fold. For every singularity, the gauge theory on the D3-branes is not unique. In fact, we have an infinite number of gauge theories connected by Seiberg duality [211, 24, 86, 53, 93] that flow to the same universality class in the infrared limit. Every gauge theory is specified by a gauge group and a matter content, which are encoded in a quiver diagram, and a superpotential. We will concentrate on a particular subset of this infinite set of dual theories, denoted toric phases. A toric phase is defined as a phase in which the gauge group is $\Pi S U(N)$, i.e. the ranks of all gauge group factors are the same. Non-toric phases are obtained by Seiberg duality on a node for which the number of flavors is larger than twice the number of colors. The fact that the probed geometry is an affine toric variety constraints the possible structure of the superpotential. It has to be such that all F -term equations are of the form "monomial = monomial". This constraint is dubbed the toric condition [80] and can be rephrased by saying that every field in the quiver must appear exactly in two terms of the superpotential, with both terms having opposite signs. In addition, all superpotential coefficients can be normalized to 1 by rescaling the fields.

Figure 4-1 shows one toric phase for the complex cone over $d P_{2}[82,80]$, usually referred to as Model II. The corresponding superpotential is given by

$$
\begin{align*}
W & =\left[X_{34} X_{45} X_{53}\right]-\left[X_{53} Y_{31} X_{15}+X_{34} X_{42} Y_{23}\right]  \tag{4.2.1}\\
& +\left[Y_{23} X_{31} X_{15} X_{52}+X_{42} X_{23} Y_{31} X_{14}\right]-\left[X_{23} X_{31} X_{14} X_{45} X_{52}\right]
\end{align*}
$$

where we have grouped terms to make a $\mathbb{Z}_{2}$ global symmetry that acts by interchanging nodes $1 \leftrightarrow 2$ and $4 \leftrightarrow 5$ and charge conjugating all the fields manifest. We will use this example to illustrate all our discussions.

In [94], it was realized that all the information in the quiver diagram and the superpotential of a toric phase can be encapsulated in a single object: the periodic quiver. A periodic quiver is a planar quiver drawn on the surface of a 2 -torus


5
Figure 4-1: Quiver diagram for Model II of $d P_{2}$.
(equivalently, a doubly periodic infinite quiver on the plane) s. t. every plaquette corresponds to a term in the superpotential. The sign of the superpotential terms is given by the orientation of the corresponding plaquettes, which alternates between clockwise and counterclockwise. The toric condition is automatically incorporated in the periodic quiver, since every field appears precisely in two neighboring plaquettes with opposite orientation.

It has been conjectured that any quiver corresponding to D3-branes probing noncompact, toric Calabi-Yau threefolds can be embedded in a $T^{2}$ [94]. Furthermore, the two cycles around the $T^{2}$ have been identified with the non-R symmetry $U(1)$ isometries [31]. In Section 4.2.1, we discuss how conformal invariance restricts the possible embeddings of the periodic quiver. Figure 4-2 shows the periodic quiver for our $d P_{2}$ example.


Figure 4-2: Periodic quiver for Model II of $d P_{2}$. We show several fundamental cells.

Our working hypothesis will be that we consider gauge theories that are described by periodic quivers on $T^{2}$. For this class of theories, we will show that the GLSM
determination of the moduli space can be translated into a dimer problem.
The superpotential can be written schematically as

$$
\begin{equation*}
W=\sum_{\mu} \pm W_{\mu} \tag{4.2.2}
\end{equation*}
$$

where every superpotential term $W_{\mu}$ is a gauge invariant mesonic operator with Rcharge equal to 2 and neutral under the $U(1) \times U(1)$ flavor symmetry ${ }^{1}$. We have explicitly indicated the sign of each term, which satisfy the toric condition.

In toric quivers, F -term equations can be used to show that all these operators are equivalent in the chiral ring. The toric condition implies that every field $X_{i}$ appears (linearly) in exactly two superpotential terms. Let us call them $W_{1}$ and $-W_{2}$ (according to the toric condition both contributions have opposite signs). Then

$$
\begin{equation*}
0=X \partial_{X} W=X \partial_{X}\left(W_{1}-W_{2}\right)=W_{1}-W_{2} \tag{4.2.3}
\end{equation*}
$$

This becomes very intuitive from the perspective of the periodic quiver (see Figure 4-3), where one can show that any two adjacent plaquettes are equal by using the F -term relation for the common field. Iterating this process we see that, once F-term equations are taken into account, all superpotential terms are identical. This idea has already been used in [31].


Figure 4-3: Two plaquettes are equal once the F-term equation for the common field is imposed.

In [94], an alternative representation of the gauge theory, dubbed brane tiling was introduced. The brane tiling is constructed by dualizing the periodic quiver

[^7]graph: Nodes, arrows and plaquettes of the periodic quiver are replaced by faces, transverse lines and nodes, respectively.

The resulting tiling is a bipartite graph. This means that it is possible to assign nodes two colors (by convention we choose black and white) such that white nodes are only connected to black nodes and viceversa. The coloring of nodes is in one-toone correspondence with the orientation of plaquettes in the periodic quiver (hence the sign of superpotential terms). Edges in the tiling carry a natural orientation (for example from white to black nodes), which corresponds to the orientation of arrows in the periodic quiver.

We can translate among periodic quiver, brane tiling and gauge theory concepts using the following dictionary

| Periodic quiver | Brane tiling | Gauge theory |
| :---: | :---: | :---: |
| node | face | $S U(N)$ gauge group |
| arrow | edge | bifundamental (or adjoint) |
| plaquette | node | superpotential term |

We denote $F, E$ and $N$ the number of faces, edges and nodes in the tiling. They correspond to the number of gauge groups, chiral multiplets and superpotential terms in the gauge theory.

For a comprehensive description of brane tilings we refer the reader to [94]. Figure 4-4 shows the brane tiling for the $d P_{2}$ example under consideration, obtained by dualizing the periodic quiver in Figure 4-2

In analogy to the chemical terminology, every edge in the tiling is denoted a dimer. A perfect matching is a collection of edges (dimers) such that every node in the tiling is the endpoint of exactly one edge in the set. For later reference, we list all perfect matchings for the $d P_{2}$ brane tiling in the Appendix. Perfect matchings play a fundamental role in our forthcoming discussion.


Figure 4-4: Brane tiling for Model II of $d P_{2}$.

### 4.2.1 Geometry of the tiling embedding from conformal invariance

In the previous section we stated that we will focus on tilings of a two dimensional torus. Since the gauge theories under consideration have a finite number of gauge groups, fields and superpotential terms, it is natural to represent them by a tiling of a compact Riemann surface $\Sigma$. But, is any $\Sigma$ a valid option? Why do we choose a $T^{2}$ ? Interestingly, as we discuss in this section, the gauge theory actually constraints the geometry of $\Sigma$.

Conformal invariance at the IR fixed point requires the beta functions for all superpotential and gauge couplings to be zero. For superpotential couplings this implies that

$$
\begin{equation*}
\sum_{\substack{i \in \text { edges } \\ \text { around node }}} R_{i}=2 \quad \text { for every node } \tag{4.2.4}
\end{equation*}
$$

while vanishing of gauge coupling beta functions corresponds to

$$
\begin{equation*}
2+\sum_{\substack{i \in \text { edges } \\ \text { around face }}}\left(R_{i}-1\right)=0 \quad \text { for every face } \tag{4.2.5}
\end{equation*}
$$

Adding (4.2.5) over all faces and using (4.2.4) we conclude that

$$
\begin{equation*}
F+N-E=\chi(\Sigma)=0 \tag{4.2.6}
\end{equation*}
$$

Hence, conformal invariance implies that the Euler characteristic of $\Sigma$ has to be zero.

### 4.2.2 Height function

Given a perfect matching $M$, it is possible to define an integer-valued height function $h$ over the brane tiling $[168,169]$. In order to do so we fix a reference perfect matching $M_{0}$ and a face $f_{0}$. The difference $M-M_{0}$ defines a set of closed curves over the tiling. The minus sign flips the orientation of bifundamentals associated with the edges of $M_{0}$, giving the resulting closed curves a definite orientation. The height function jumps by $\pm 1$ when crossing a curve, where the sign is given by the orientation of the crossing. The height for $f_{0}$ is set to be zero. Notice that the difference of the height functions of two matchings is well-defined independently of $M_{0}$.

The slope of a perfect matching is defined as the height change $\left(h_{x}, h_{y}\right)$ when moving from one unit cell to the next one along the two fundamental directions. Changing $M_{0}$ amounts to a constant shift $\left(h_{x 0}, h_{y 0}\right)$ in the slopes of all perfect matchings.

We exemplify the concepts presented in this section with $d P_{2}$. Figure 4-5 shows a perfect matching, a reference perfect matching and the corresponding height function. In this case, we see that the slope is $\left(h_{x}, h_{y}\right)=(-1,0)$.


Figure 4-5: (a) The dimers in the a perfect matching $M$ are shown in cyan. (b) The dimers in the reference perfect matching $M_{0}$ are shown in red. (c) The height function, whose level curves are given by $M-M_{0}$.

There is an equivalent way to define slopes, that later will turn out to be useful. To every perfect matching we can associate a unit flow on its edges. directed from white to black nodes. The slope then corresponds to the net flux between adjacent fundamental regions in the $x$ and $y$ directions. The Appendix gives the slopes for all perfect matchings of Model II of $d P_{2}$. We will come back to the interpretation of matchings as unit flows in Section 4.5.2.

It is straightforward to count the number of perfect matchings with a given slope [168, 169]. In order to do so, we first introduce the Kasteleyn matrix of the tiling $K(x, y)$. It is a weighted, signed, $(N / 2) \times(N / 2)$ adjacency matrix defined as follows. In our convention, the rows of $K(x, y)$ are indexed by white nodes and its columns by black nodes. We associate a $\pm 1$ weight to every edge $e_{i}$ in the tiling such that when we multiply the weights around every face we have

$$
\operatorname{sign}\left(\prod e_{i}\right)= \begin{cases}+1 \text { if }(\# \text { edges })=2 & \bmod 4  \tag{4.2.7}\\ -1 \text { if }(\# \text { edges })=0 & \bmod 4\end{cases}
$$

Next we take two fundamental paths $\mathcal{C}_{x}$ and $\mathcal{C}_{y}$ in the graph dual to the brane tiling winding once around the $(1,0)$ and $(0,1)$ cycles of the 2 -torus. These paths are conventionally denoted flux lines and can be visualized as the boundaries of the fundamental region. The weight of every edge in the tiling that is crossed by $\mathcal{C}_{x}$ is then multiplied by $x$ or $x^{-1}$ depending on the orientation of the crossing. Respectively, edges crossed by $\mathcal{C}_{y}$ are multiplied by $y$ or $y^{-1}$.

The determinant of the Kasteleyn matrix $P(x, y)=\operatorname{det} K(x, y)$ is a Laurent polynomial, the so-called characteristic polynomial of the dimer model. It has the following general form

$$
\begin{equation*}
P(x, y)=x^{h_{x 0}} y^{h_{y 0}} \sum c_{h_{x}, h_{y}} x^{h_{x}} y^{h_{y}} \tag{4.2.8}
\end{equation*}
$$

$P(x, y)$ is the partition function of perfect matchings on the brane tiling, in the sense that the integer coefficients $\left|c_{h_{x}, h_{y}}\right|$ count the number of perfect matchings with slope $\left(h_{x}, h_{y}\right)$ [169].

In our example, we have

$$
K=\left(\begin{array}{ccc}
1-x^{-1} & y & 1  \tag{4.2.9}\\
1 & 1 & x \\
-1+x^{-1} y^{-1} & 1 & 1
\end{array}\right)
$$

Then

$$
\begin{equation*}
P(x, y)=x^{-1} y^{-1}-x^{-1}+5-x-y-x y \tag{4.2.10}
\end{equation*}
$$

which gives the following counting of perfect matchings

| slope | \# matchings |
| :---: | :---: |
| $(-1,-1)$ | 1 |
| $(-1,0)$ | 1 |
| $(0,0)$ | 5 |
| $(1,0)$ | 1 |
| $(0,1)$ | 1 |
| $(1,1)$ | 1 |

that is in precise agreement with the direct counting in the Appendix.

### 4.3 Toric geometry from gauge theory

We now review the procedure for computing the moduli space of a given toric quiver (i.e. quiver plus toric superpotential). For $N \mathrm{D} 3$-brane probes, the moduli space along the mesonic branch corresponds to the symmetric product of $N$ copies of the probed geometry. This procedure has been algorithmized in [82] and dubbed the Forward Algorithm. It involves the following steps:

- Use F-flatness equations to express the fields in the quiver (which transform in bifundamental or adjoint representations) $X_{i}, i=1, \ldots, E$ in terms of $F+2$ independent variables $v_{j}$. Although the $v_{j}$ 's can be taken to be a subset of the $X_{i}$ fields, other choices are also possible. For example, as we will discuss later, dimers pick other combinations which turn out to be more natural. The final answer does not depend on this choice. Since for toric quivers the F-term
equations are of the form monomial $=$ monomial , each $X_{i}$ is given by a product of $v_{j}$ 's to appropriate powers. This can be encoded in an $E \times(F+2)$ matrix $K$ according to

$$
\begin{equation*}
X_{i}=\prod v_{j}^{K_{i j}}, \quad i=1, \ldots, E, \quad j=1, \ldots, F+2 \tag{4.3.11}
\end{equation*}
$$

The $X_{i}$ can involve negative powers of the $v_{j}$ 's, i.e. $K_{i j}$ may have negative entries. The row vectors $\vec{K}_{i}$ of $K$ span a cone $M_{+}$in $\mathbb{R}^{F+2}$, corresponding to non-negative linear combinations of them.

- Next, to get rid of the negative powers, we introduce new variables $p_{\alpha}, \alpha=$ $1, \ldots, N_{\sigma}$. In order to do so, we compute the cone $N_{+}$dual to $M_{+} . N_{+}$is spanned by vectors $\vec{T}_{\alpha}$, such that $\vec{K}_{i} \cdot \vec{T}_{\alpha} \geq 0$. These vectors can be organized as the columns of an $(F+2) \times N_{\sigma}$ integer matrix $T$ such that $K \cdot T \geq 0$ for all entries. The dimension of the dual cone $N_{\sigma}$ is not known a priori and is determined by explicitly computing $N_{+}$. The intermediate and original variables $v_{j}$ and $X_{i}$ are expressed in terms of the $p_{\alpha}$ as follow

$$
\begin{equation*}
v_{j}=\prod_{\alpha} p_{\alpha}^{T_{j \alpha}} \quad X_{i}=\prod_{\alpha} p_{\alpha}^{\sum_{j} K_{i j} T_{j \alpha}} \tag{4.3.12}
\end{equation*}
$$

The amount of operations required to compute $N_{\sigma}$ grows with the size of the gauge theory. This growth becomes prohibitive when trying to apply the Forward Algorithm to gauge theories with large quivers. Later, we will explain how this difficulty is circumvented by the dimer model.

- A convenient way to encode the relations among the $N_{\sigma}$ variables $p_{\alpha}$ and the original $F+2 v_{j}$ is by obtaining them as D -terms of an appropriately chosen $U(1)^{N_{\sigma}-(F+2)}$ gauge group. Its action is given by an $\left(N_{\sigma}-F-2\right) \times N_{\sigma}$ charge matrix $Q_{F}$ (where the subindex $F$ indicates that $Q_{F}$ contains all the information about F -term equations). Gauge invariance of the $v_{j}$ 's under the new gauge
group gives rise to the desired relations. Hence, $Q_{F}$ is such that

$$
\begin{equation*}
T \cdot Q_{F}^{T}=0 \tag{4.3.13}
\end{equation*}
$$

- The charges of fields under the $F$ gauge groups of the quiver are summarized by the $F \times E$ incidence matrix $d$. It is defined as $d_{l i}=\delta_{l, \text { head }\left(X_{i}\right)}-\delta_{l, \text { tail }\left(X_{i}\right)}$. Every column associated to a bifundamental field contains a 1 and a -1 and the rest of the entries are 0 's. Adjoint fields are represented in quiver language by arrows starting from and ending at the same node. Hence, the corresponding columns have all 0 's. It is clear that one of the rows of $d$ is redundant. Thus, we define the matrix $(F-1) \times E$ matrix $\Delta$, which is obtained from $d$ by deleting one of its rows. For our example, we have

$$
\Delta=\left[\begin{array}{c|ccccccccccc} 
& X_{14} & X_{31} & X_{15} & Y_{31} & X_{23} & X_{52} & Y_{23} & X_{42} & X_{34} & X_{53} & X_{45}  \tag{4.3.14}\\
\hline 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\
3 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 1 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1
\end{array}\right]
$$

The $F-1$ independent $D$-term equations of the original theory are implemented by adding a $U(1)^{F-1}$ gauge symmetry to the GLSM. The charges of the $p_{\alpha}$ under these symmetries is given by an $(F-1) \times N_{\sigma}$ matrix $Q_{D}$ which can be determined in two steps. First, we construct an $(F-1) \times(F+2)$ matrix $V$ that translates the charges of the $X_{i}$ 's to those of the $v_{j}$ 's. Thus,

$$
\begin{equation*}
V \cdot K^{T}=\Delta \tag{4.3.15}
\end{equation*}
$$

Next, we find an $(F+2) \times N_{\sigma}$ matrix $U$ that transform the charges of $v_{j}$ 's into those of the $p_{\alpha}$ 's

$$
\begin{equation*}
U \cdot T^{T}=\operatorname{Id}_{(F+2) \times(F+2)} \tag{4.3.16}
\end{equation*}
$$

Finally, we have

$$
\begin{equation*}
Q_{D}=V \cdot U \tag{4.3.17}
\end{equation*}
$$

$Q_{D}$ and $Q_{F}$ are cont,ined into a single $\left(N_{\sigma}-2\right) \times N_{\sigma}$ charge matrix $Q$

$$
\begin{equation*}
Q=\binom{Q_{D}}{Q_{F}} \tag{4.3.18}
\end{equation*}
$$

The construction we outlined can interpreted as a Witten's two dimensional gauged linear sigma model (GLSM) of $N_{\sigma}$ chiral fields $p_{\alpha}$ and $U(1)^{N_{\sigma}-3}$ gauge group with charges given by $Q$.

- The $U(1)$ charges defined above are exactly those that appear in the construction of a toric variety as a symplectic quotient. In toric geometry it is standard to encode the charge matrix by means of a toric diagram.

$$
\begin{equation*}
G=(\operatorname{Ker}(Q))^{T} \tag{4.3.19}
\end{equation*}
$$

One of the rows in $G$ can be set to have all entries equal to 1 by an appropriate $S L(3, \mathbb{Z})$ transformation. This is the Calabi-Yau condition and amounts to the fact that the sum of the charges of all the $p_{\alpha}$ under any of the $U(1)$ gauge symmetries is zero. Effectively, we are left with a two dimensional toric diagram. Every GLSM field $p_{\alpha}$ corresponds to a point in the toric diagram, which is a vector $\vec{v}_{\alpha}$ in $\mathbb{Z}^{3} . Q$ is given by linear relations of the form

$$
\begin{equation*}
\sum_{i=1}^{n} Q_{a}^{\alpha} \vec{v}_{\alpha}=0 \tag{4.3.20}
\end{equation*}
$$

satisfied by the $\vec{v}_{\alpha}$ 's.

Figure 4-6 summarizes the relevant matrices in the Forward Algorithm.


Figure 4-6: Relevant matrices in the Forward Algorithm.

### 4.4 The conjecture

Having introduced all necessary concepts, we are ready to study the conjecture of [94]. It is convenient to divide the conjecture into two parts, to which we refer as the Mathematical and the Physical Dimer Conjectures.

## Mathematical dimer conjecture

The mathematical dimer conjecture states that there is a one-to-one correspondence between fields $p_{\alpha}$ in the gauged linear sigma model construction of the toric moduli space of the given toric gauge theory and perfect matchings in the brane tiling dual to the toric quiver. Here, when we refer to a toric gauge theory we mean a gauge theory whose quiver can be drawn on a surface of a 2 -torus, s. t. the plaquettes give the terms in the superpotential (see discussion in Section 4.2.1). Furthermore, according to the conjecture, the toric diagram is the Newton polygon of the characteristic polynomial (i.e. the set of integer exponents of monomials [169]) which, as we have already discussed, is the set of height function monodromies ("slopes") of the perfect matchings.

## Physical dimer conjecture

The physical dimer conjecture identifies dimers and tilings with physical objects. According to the conjecture, the brane tiling is interpreted as a physical brane configuration. It consists of an NS5-brane extended in the 0123 directions that wraps an holomorphic curve in 4567 . The 5 and 6 directions are periodically identified giving rise to the 2-torus. D5-branes extend in 012346, suspended within the "holes" of the NS5-brane in the 46 torus. Every stack of D5-branes gives rise to a gauge group. Strings crossing every NS5-brane segment and connecting two D5-brane stacks correspond to chiral multiplets transforming in the bifundamental representation of the corresponding gauge groups. Gauge invariant superpotential terms are produced by the coupling of massless string states at the nodes of the NS5-brane configuration. This configuration is conjectured to be related to the D3-branes over the singularity by two T-dualities. The suspended D5-branes are dual to the probe D3-branes and the NS5-brane structure is dual to the singular geometry.

The correspondence between dimers and a physical brane system could be more subtle and might differ from the one suggested by the physical dimer conjecture. However, the validity of the mathematical dimer conjecture, which is the main subject of this chapter, is completely independent of how tilings are realized in terms of branes ${ }^{2}$.

Having introduced the conjectures of [94], we devote the rest of the chapter to proving the mathematical dimer conjecture.

[^8]
### 4.5 The proof

In this section we prove the Mathematical Dimer Conjecture. As we said before, we prove it for toric gauge theories whose quivers (and hence their brane tilings) are embedded in a two-torus. A considerable amount of evidence supporting its validity has been accumulated in the literature. This includes:

- Construction of the correct toric diagram for the moduli space of gauge theories for an infinite number of singularities. This number is infinite thanks to the determination of the tilings for the $Y^{p, q}[94]$ and $L^{a, b, c}$ manifolds $[97,52]$.
- Precise agreement between the number of perfect matchings and the multiplicity of GLSM fields in toric diagrams for various models [97].
- Derivation of Seiberg dual theories by transformations of the tilings preserving the Newton polygon of the characteristic polynomial [97, 132].
- In [97], it was shown that given a simple proposal to express quiver fields in terms of perfect matchings, F-term conditions are straightforwardly satisfied. This proposal will be derived as part of our proof.
- The geometry of brane tilings has been investigated in [87]. The results of this paper show how tilings appear in the description of toric gauge theories by explicitly deriving them from the mirror geometry but do not prove the correspondence between perfect matchings and GLSM fields.

Our computations with dimers will closely follow those of the Forward Algorithm. It is important to keep in mind that some of the steps (or intermediate matrices) are naturally skipped by the inherent simplifications of the dimer approach. In order to avoid confusion we will use tilded variables at some stages of the proof. In the end, we will show that they can be identified with the untilded ones of the Forward Algorithm.

### 4.5.1 Solving $\mathbf{F}$-term conditions: gauge transformations and magnetic coordinates

The tiling is bipartite, therefore each edge has a natural orientation from its white vertex to its black vertex. Any weight function $\varepsilon(e)$ on the edges defines a 1 -form, satisfying $\varepsilon(-e)=-\varepsilon(e)$, where $-e$ is the edge with opposite direction [169]. We denote the linear space of 1 -forms on the tiling by $\Omega^{1}$. Analogously, the functions on nodes and faces define $0-$ and 2 -forms in $\Omega^{0}$ and $\Omega^{2}$. The three spaces are related by differentials

$$
\begin{equation*}
0 \rightarrow \Omega^{0} \xrightarrow{d} \Omega^{1} \xrightarrow{d} \Omega^{2} \rightarrow 0 \tag{4.5.21}
\end{equation*}
$$

We can now define gauge transformations on the tiling, whose action on the 1 -forms is given by [169]

$$
\begin{equation*}
\varepsilon^{\prime}\left(e_{i}\right)=\varepsilon\left(e_{i}\right)+d f \quad f \in \Omega^{0} \tag{4.5.22}
\end{equation*}
$$

That is

$$
\begin{equation*}
\varepsilon^{\prime}\left(e_{i}\right)=\varepsilon\left(e_{i}\right)+f\left(\mathrm{~b}_{i}\right)-f\left(\mathrm{w}_{i}\right) \tag{4.5.23}
\end{equation*}
$$

with $\mathrm{b}_{i}$ and $\mathrm{w}_{i}$ the black and white nodes at the endpoints of edge $e_{i}$. These gauge transformations of the tiling should not be confused with the gauge symmetries of the quiver theory. We are confident that the distinction between both types of gauge transformations will be clear from the context in which we use them. Given a closed path on the tiling

$$
\begin{equation*}
\gamma=\left\{\mathrm{w}_{0}, \mathrm{~b}_{0}, \mathrm{w}_{1}, \mathrm{~b}_{1}, \ldots, \mathrm{~b}_{k-1}, \mathrm{w}_{k}\right\} \quad \mathrm{w}_{k}=\mathrm{w}_{0} \tag{4.5.24}
\end{equation*}
$$

we define the magnetic flux through $\gamma$ as

$$
\begin{equation*}
B(\gamma)=\int_{\gamma} \varepsilon=\sum_{i=1}^{k-1}\left[\varepsilon\left(\mathrm{w}_{i}, \mathrm{~b}_{i}\right)-\varepsilon\left(\mathrm{w}_{i+1}, \mathrm{~b}_{i}\right)\right] \tag{4.5.25}
\end{equation*}
$$

Magnetic fluxes are clearly gauge invariant. The brane tiling is embedded in a two dimensional torus. Hence, gauge inequivalent classes of 1 -forms are parameterized by $\mathbb{R}^{F-1} \oplus \mathbb{R}^{2}$. The first term corresponds to $d \varepsilon \in \Omega^{2}$, a function on the faces of the tiling subject to the condition $\sum d \varepsilon=0$. We can specify the $\mathbb{R}^{F-1}$ part by the magnetic fluxes $B_{z}(j)(j=1, \ldots, F-1)$ through the $\gamma_{i}$ contours around $F-1$ faces. The remaining two parameters $\left(B_{x}, B_{y}\right)$ correspond to fluxes around two non-trivial cycles $\left(\gamma_{x}, \gamma_{y}\right)$ winding around the torus.

Gauge transformations are of particular interest because taking $\varepsilon$ to be the energy function they do not modify the energy difference between perfect matchings. Hence, the probability distribution of perfect matchings is invariant under gauge transformations.

In this section, we will exploit gauge transformations with a different goal, namely to provide a convenient set of variables (mostly in $\Omega^{2}$ ) that solve the F -term equations. For this purpose, we define the complex 1-form

$$
\begin{equation*}
\varepsilon\left(e_{i}\right)=\ln X_{i} \Rightarrow \quad \text { under gauge transformations: } X_{i}^{\prime}=X_{i} e^{f\left(\mathrm{~b}_{i}\right)-f\left(\mathbf{w}_{i}\right)} \tag{4.5.26}
\end{equation*}
$$

In this context, we refer to the $X_{i}$ 's as weights ${ }^{3}$.
Using (4.5.26), we can define new variables associated to closed paths

$$
\begin{equation*}
\tilde{v}(\gamma)=e^{\int_{\gamma} \varepsilon}=\prod_{i=1}^{k-1} \frac{X\left(\mathrm{w}_{i}, \mathrm{~b}_{i}\right)}{X\left(\mathrm{w}_{i+1}, \mathrm{~b}_{i}\right)} \tag{4.5.27}
\end{equation*}
$$

where the product runs over the contour $\gamma$. Then $\left\{\tilde{v}_{j} \equiv \tilde{v}\left(\gamma_{j}\right), \tilde{v}_{x}, \tilde{v}_{y}\right\}$ provides a parametrization of inequivalent gauge classes.

We define a convenient basis of 0 -forms $F^{(\mu)}, \mu=1, \ldots, N$,

$$
F^{(\mu)}\left\{\begin{array}{l}
f_{\mu}=1  \tag{4.5.28}\\
f_{\nu}=0 \text { for } \nu \neq \mu
\end{array}\right.
$$

[^9]Their virtue is that superpotential terms transform simply under the corresponding gauge transformations. Taking the gauge transformation for $\alpha_{\mu} F^{(\mu)}$, with $\alpha_{\mu}$ a complex coefficient, we get

$$
\begin{equation*}
W_{\mu}^{\prime}=W_{\mu} e^{\operatorname{sign}(\mu) v_{\mu} \alpha_{\mu}} \tag{4.5.29}
\end{equation*}
$$

where $v_{\mu}$ is the valence of node $\mu$ (i.e. the order of the associated superpotential term $W_{\mu}$ ) and following (4.5.26) $\operatorname{sign}(\mu)$ is 1 for black nodes and -1 for white nodes.

As discussed in Section 4.2, solving F-term conditions corresponds to setting all the $W_{\mu}$ 's equal. Given arbitrary values of the $W_{\mu}$, it is possible to set them equal to $W_{1}$ by the basic gauge transformations of (4.5.29) with

$$
\begin{equation*}
\alpha_{\mu}=\frac{\operatorname{sign}(\mu)}{v_{\mu}} \frac{\ln W_{1}}{\ln W_{\mu}} \tag{4.5.30}
\end{equation*}
$$

In other words, solving F -term equations corresponds in this language to partially fixing the gauge. Each gauge choice can be labeled by the common value of $W_{\mu}=W_{1}{ }^{4}$. Equivalently, one can label gauge choices using the more symmetric variable $\mathcal{V}$ defined as

$$
\begin{equation*}
\mathcal{V}=W_{1}^{N}=\prod_{\mu=1}^{N} W_{\mu}=\prod_{i=1}^{E} X_{i}^{2} \tag{4.5.31}
\end{equation*}
$$

We denote $\mathcal{V}$, the $\tilde{v}_{j}$ 's, $\tilde{v}_{x}$ and $\tilde{v}_{y}$ the flux variables.
We have just seen that on each gauge orbit there exists a unique solution to F term equations for every value of $\mathcal{V}$. Hence, we conclude that solutions to $F$-flatness equations are parametrized by the $F+2$ flux variables: the value of $\mathcal{V}$ indicating a partial gauge fixing, along with the variables $\tilde{v}_{j}(j=1, \ldots, F-1), \tilde{v}_{x}$ and $\tilde{v}_{y}$ parametrizing gauge equivalence classes. It is now clear that these fluxes can be identified with the $v_{j}(j=1, \ldots, F+2)$ variables of the Forward Algorithm.

With this identification, it is straightforward to write down a left inverse matrix for $K$, which we call $K_{L}^{-1}$. This is an $(F+2) \times E$ matrix such that $K_{L}^{-1} K=\operatorname{Id}_{(F+2) \times(F+2)}$.

For our $d P_{2}$ example, we have

[^10]\[

K_{L}^{-1}=\left[$$
\begin{array}{c|ccccccccccc} 
& X_{14} & X_{31} & X_{15} & Y_{31} & X_{23} & X_{52} & Y_{23} & X_{42} & X_{34} & X_{53} & X_{45}  \tag{4.5.32}\\
\hline \tilde{v}_{1} & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{v}_{2} & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\
\tilde{v}_{3} & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & -1 & 0 \\
\tilde{v}_{4} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\
\hline \tilde{v}_{x} & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\tilde{v}_{y} & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\hline \mathcal{V} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}
$$\right]
\]

for which we have taken the $\gamma_{i}$ loops to run clockwise around faces, and $\gamma_{x}$ and $\gamma_{y}$ are the two non-trivial cycles shown in Figure 4-7, i.e.

$$
\begin{align*}
& \tilde{v}_{x}=X_{14}^{-1} X_{42} X_{23}^{-1} Y_{31}  \tag{4.5.33}\\
& \tilde{v}_{y}=X_{53} X_{31}^{-1} X_{14} X_{45}^{-1}
\end{align*}
$$

With this choice of contours, it is clear that the first $F-1$ rows of $K_{L}^{-1}$ are equal to $-\Delta$ (see (4.3.14)). There are other paths equivalent to $\gamma_{x}$ and $\gamma_{y}$ that are obtained by deforming them using F -term equations.


Figure 4-7: Contours defining $\tilde{v}_{x}$ and $\tilde{v}_{y}$.

The matrix $K$ converts magnetic variables into weight variables. We do not determine $K$ explicitly in this section as it is not necessary for our discussion. As explained in Section 4.3, the vectors $\vec{n}_{i}$ corresponding to rows in $K(i=1, \ldots, E)$ span a cone $\mathcal{S}$ in $\mathbb{R}^{F+2}$.

### 4.5.2 The GLSM fields are perfect matchings

In the previous section we discussed at length how the $\mathrm{F}-$ flatness conditions can be satisfied in terms of the $\tilde{v}_{i}$ magnetic fluxes that are in one-to-one correspondence
with the variables $v_{j}$ according to (4.5.27). The relation between these variables and the original $X_{i}$ fields are encoded in the matrix $K$, whose rows span the cone $M_{+}$in $\mathbb{R}^{F+2}$. The Forward Algorithm proceeds by computing the cone dual to $M_{+}$:

$$
\begin{equation*}
N_{+}=\left\{x \in \mathbb{R}^{F+2} \mid\left\langle\vec{K}_{i}, x\right\rangle \geq 0 \text { for } i=0, \ldots, E\right\} \tag{4.5.34}
\end{equation*}
$$

There are $N_{\sigma}$ spanning vectors for this dual cone $N_{+}$. These $N_{\sigma}$ vectors define the columns $\vec{T}_{j}$ of the matrix $T$ and they are in one-to-one correspondence with the homogeneous $p_{\alpha}$ GLSM coordinates.

We would like to understand the computation of the dual cone in terms of tiling techniques. In order to do so, we introduce a slightly different viewpoint that will prove to be useful.

An arbitrary real weight system on the edges can be interpreted as a white-toblack flow ${ }^{5}$ [169]. The (possibly negative) strength of the flow from white to black node along an edge $e_{i}$ is given by the corresponding real weight $c_{i}$. The real weights considered in this section are not to be confused with the complex weights given by $X_{i}$ that we have discussed earlier.

A flow is nonnegative if it has a nonnegative strength on all edges of the tiling ( $c_{i}>0$ for all $e_{i}$ ). The flows are typically not divergence free, therefore there can be sinks and sources at the vertices. The net flux coming out of a given white node or into a black node is denoted the vorticity of the node.

For each point in flux space $x \in \mathbb{R}^{F+2}$ we define a real flow on the tiling whose strength at the $i^{\text {th }}$ edge is given by $\sum_{j} K_{i j} x_{j}$. Hence the points inside $N_{+}$correspond to nonnegative flows in this picture.

We want to find the spanning vectors $\vec{T}_{\alpha}$ of the dual cone $N_{+} \in \mathbb{R}^{N_{F}+2}$. Following our discussion in Section 4.5.1, we can rescale the vectors $\vec{T}_{\alpha}$ by a positive real number using the gauge transformations of the dimer model. Thus we can set their vorticity to one. Therefore, we can focus on the hyperplane $H \subset \mathbb{R}^{N_{F}+2}$ such there is a unit source residing at every white vertex and a unit sink at the black ones. The flows

[^11]associated with this hyperplane are called unit flows.

The vectors $\vec{T}_{\alpha}$ span the cone $N_{+}$, hence they also span the intersection $H \cap N_{+}$ in flux space. From the previous discussion, we know that this intersection is linearly mapped by $K_{i j}$ to nonnegative unit flows $P \subset \mathbb{R}^{E}$ in flow space. It is well-known in the literature that the set of nonnegative unit flows is a convex polytope in the flow space and that perfect matchings are vertices of this polytope (Perfect Matching Polytope Theorem, [75]). Their preimages are the spanning vectors $\vec{T}_{\alpha}$ in flux space. For $\vec{T}_{\alpha}$, the flow on the $i^{\text {th }}$ edge is given by $\sum_{j} K_{i j}\left(\vec{T}_{\alpha}\right)_{j}=\sum_{j} K_{i j} T_{j \alpha}$. We conclude that there is a one-to-one correspondence between the GLSM fields in the Forward Algorithm and perfect matchings.

Perfect matchings are naturally represented as unit flows, hence they immediately determine $K T$. By introducing the following "product" between perfect matchings and edges in the tiling

$$
\left\langle e_{i}, p_{\alpha}\right\rangle=\left\{\begin{array}{l}
1 \text { if } e_{i} \in p_{\alpha}  \tag{4.5.35}\\
0 \text { if } e_{i} \notin p_{\alpha}
\end{array}\right.
$$

the matrix $K T$ is simply

$$
\begin{equation*}
(K T)_{i \alpha}=\left\langle e_{i}, p_{\alpha}\right\rangle \tag{4.5.36}
\end{equation*}
$$

The correspondence between GLSM fields and perfect matchings and the computation of $K T$ in terms of perfect matchings that we derived in this section was originally proposed in [94].

Using (4.5.36) for $d P_{2}$, we have

$$
K T^{T}=\left[\begin{array}{c|ccccccccccc} 
& X_{14} & X_{31} & X_{15} & Y_{31} & X_{2,3} & X_{6,2} & Y_{23} & X_{42} & X_{34} & X_{53} & X_{45}  \tag{4.5.37}\\
\hline p_{1} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
p_{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
p_{3} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
p_{4} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
p_{5} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
p_{6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
p_{7} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
p_{8} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
p_{9} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
p_{10} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
$$

As we discussed in the previous section, the left inverse of $K$, which we called $K_{L}^{-1}$, arises naturally using dimer methods. Then, it is straightforward to write down

$$
\begin{equation*}
 \tag{4.5.38}
\end{equation*}
$$

Notice that the fact that $T$ may have negative entries is not a problem. The important point is that $(K T)_{i \alpha} \geq 0$. In fact we can give a straightforward definition of $T$ in terms of the tiling, similar to (4.5.36). In order to do so, we take into account the edges $e_{i}$ in the curves $\gamma_{j}$ that define the magnetic fluxes (similarly, all $e_{i}$ 's are included for $\mathcal{V}$ ). The $\gamma_{j}$ 's have an orientation and then the fields $X_{i}$ associated to
edges $\epsilon_{i}$ appear with a $\pm 1$ power that we denote $\operatorname{sign}\left(e_{i}\right)$. Combining these ideas, we get

$$
\begin{equation*}
T_{j \alpha}=\sum_{e_{i} \in \gamma_{j}} \operatorname{sign}\left(e_{i}\right)\left\langle e_{i}, p_{\alpha}\right\rangle \tag{4.5.40}
\end{equation*}
$$

### 4.5.3 Height changes as positions in a toric diagram

So far we have shown that GLSM fields are perfect matchings. This is half of the proof of the Mathematical Dimer Conjecture, which in addition states that the height changes $\left(h_{x}, h_{y}\right)$ of a given perfect matching should be interpreted as the position in the toric diagram of the corresponding GLSM field.

Let us define the following $3 \times N_{\sigma}$ matrix

$$
G_{h}=\left(\begin{array}{c}
h_{x}  \tag{4.5.41}\\
h_{y} \\
1
\end{array}\right)
$$

The non-trivial piece of $G_{h}$ is given by $\left(h_{x}, h_{y}\right)$. We have included a third row with value 1 for all perfect matchings that plays the role of the trivial coordinate of the toric diagram.

Our goal is to prove that $G_{h}$ defines the GLSM charge matrix $Q$ through the vanishing linear relations among its columns, and thus can be identified with $G$ in (4.3.19). I.e. we want to show that

$$
\begin{align*}
Q G_{h}^{T}=0 \quad \Leftrightarrow \quad Q_{F} G_{h}^{T} & =0  \tag{4.5.42}\\
\text { and } Q_{D} G_{h}^{T} & =0
\end{align*}
$$

For the third row of $G_{h},(4.5 .42)$ means that the trace over perfect matchings of any given GLSM $U(1)$ charge vanishes. It is straightforward to see that this condition is always satisfied. Thus, from now on we concentrate on the $\left(h_{x}, h_{y}\right)$ piece of $G_{h}$.

Let us first show that $Q_{F} G_{h}^{T}=0$. From (4.5.43), we have

$$
\begin{equation*}
T Q_{F}^{T}=0 \tag{4.5.43}
\end{equation*}
$$

Hence. it is sufficient to prove that $h_{x}$ and $h_{y}$ are given by linear combinations of the rows of $T$. It is straightforward not only to show that this is the case but also to identify the precise form of these linear combinations. The key ideas are the interpretation of height changes as horizontal and vertical net flows as discussed in Section 4.2 .2 and that $K T$ is computed as the "overlap" of perfect matchings and edges (4.5.36). With this in mind, we can express the height changes as

$$
\begin{align*}
& h_{x}\left(p_{\alpha}\right)=\sum_{j}\left(\sum_{e_{i} \in E_{x}} \operatorname{sign}^{x}\left(e_{i}\right) K_{i j}\right) T_{j \alpha}  \tag{4.5.44}\\
& h_{y}\left(p_{\alpha}\right)=\sum_{j}\left(\sum_{e_{i} \in E_{y}} \operatorname{sign}^{y}\left(e_{i}\right) K_{i j}\right) T_{j \alpha} \tag{4.5.45}
\end{align*}
$$

where $E_{x}$ and $E_{y}$ denote the set of edges crossing the horizontal and vertical boundaries of the unit cell (i.e. the flux lines $\mathcal{C}_{x}$ and $\mathcal{C}_{y}$ ), and $\operatorname{sign}^{x}\left(e_{i}\right)$ and $\operatorname{sign}^{y}\left(e_{i}\right)$ indicate the direction of the crossing. For illustration, let us consider our $d P_{2}$ example, for which

$$
\begin{array}{ll}
E_{x}=\left\{X_{52}, X_{53}, Y_{23}\right\} & \operatorname{sign}^{x}\left(e_{i}\right)=\{-1,1,-1\}  \tag{4.5.46}\\
E_{y}=\left\{X_{23}, Y_{23}\right\} & \operatorname{sign}^{y}\left(e_{i}\right)=\{1,-1\}
\end{array}
$$

Figure 4-8 shows $E_{x}$ and $E_{y}$ in the tiling.

$\mathrm{E}_{\mathrm{x}}$


Ey
Figure 4-8: Sets of edges $E_{x}$ and $E_{y}$ that enter the computation of $\left(h_{x}, h_{y}\right)$.
Using (4.5.43), the fact that $\left(h_{x}, h_{y}\right)$ is given by the linear combinations constructed in (4.5.44) and (4.5.45) implies that

$$
\begin{equation*}
Q_{F} G_{h}^{T}=0 \tag{4.5.47}
\end{equation*}
$$

as we want. The missing part of the proof is to show that $Q_{D} G_{h}^{T}=0$. This can be
done as follows

$$
\begin{align*}
\left(Q_{D} G_{h}^{T}\right)_{l x} & =\sum_{e_{i} \in E_{x}} \operatorname{sign}^{x}\left(e_{i}\right)\left(V U T^{T} K^{T}\right)_{l i}=\sum_{e_{i} \in E_{r}} \operatorname{sign}^{x}\left(\epsilon_{i}\right)\left(V K^{T}\right)_{l i} \\
& =\sum_{e_{i} \in E_{x}} \operatorname{sign}^{x}\left(e_{i}\right) \Delta_{l i}=0 \tag{4.5.48}
\end{align*}
$$

In the first equality we have used (4.5.44) and (4.3.17). In the second one, we used (4.3.16). In the third one, we used (4.3.15). The last step uses the following reasoning. Every face $l$ of a tiling $(l=1, \ldots, F)$ is crossed by $\mathcal{C}_{x}$ over an even number of edges ${ }^{6}$. Typically, as in the $d P_{2}$ example we are considering, this intersection number is 0 or 2 , but larger values are also possible. Every edge intersected by $\mathcal{C}_{x}$ corresponds to a field $X_{i}$ in $E_{x}$ that transforms either in the fundamental ( $\Delta_{l i}=1$ ) or antifundamental $\left(\Delta_{l i}=-1\right)$ representation of the $S U(N)$ gauge group associated with face $l^{7}$. Let us consider two edges in $e_{i}$ and $e_{j}$ in $E_{x}$ that are consecutive as we move around face $l$. Then, $\Delta_{l i} / \Delta_{l j}=1$ or -1 provided $e_{i}$ and $e_{j}$ are separated by and odd or even number of edges, respectively. Conversely, $\operatorname{sign}^{x}\left(e_{i}\right) / \operatorname{sign}^{x}\left(e_{j}\right)=1$ or -1 if they are separated by and even or odd number of edges. Hence, we have that $\operatorname{sign}^{x}\left(e_{i}\right) \Delta_{l i} / \operatorname{sign}^{x}\left(e_{j}\right) \Delta_{l j}=-1$, and thus $\sum_{e_{i} \in E_{x}} \operatorname{sign}^{x}\left(e_{i}\right) \Delta_{l i}=0$.

With identical reasoning, it follows that

$$
\begin{equation*}
\left(Q_{D} G_{h}^{T}\right)_{l y}=\sum_{e_{i} \in E_{y}} \operatorname{sign}^{y}\left(e_{i}\right) \Delta_{l i}=0 \tag{4.5.49}
\end{equation*}
$$

From (4.5.48) and (4.5.49), we conclude that

$$
\begin{equation*}
Q_{D} G_{h}^{T}=0 \tag{4.5.50}
\end{equation*}
$$

[^12]Hence. we have $Q G_{h}^{T}=0$ and we can identify

$$
\begin{equation*}
G_{h} \equiv G \tag{4.5.51}
\end{equation*}
$$

We have shown that the slopes of the perfect matchings are the positions of the corresponding GLSM fields in the toric diagram, completing our proof of the Mathematical Dimer Conjecture.

Before closing this section we notice an interesting result that was possible due the use of dimers. Equations (4.5.44) and (4.5.45) give the positions of GLSM fields in the toric diagram directly as linear combinations of rows of $K T$. Nothing like these expressions was clear from the Forward Algorithm and shows, once again, how dimers manage to pick the natural variables for computing the moduli space.

### 4.6 Conclusions

In this chapter we have proved the Mathematical Dimer Conjecture. That is, we have explicitly shown that there is a one-to-one mapping between the GLSM fields that realize the moduli space of a toric quiver and perfect matchings in the brane tiling dual to the periodic quiver. We have also demonstrated that the position of each GLSM field in the toric diagram is given by the slope of the corresponding perfect matching.

We have witnessed how dimers often provide an intuitive interpretation of otherwise obscure steps in the computation of the moduli space. An example of this type is that F -term equations can be easily solved using gauge transformations of weights as shown in Section 4.5.1. This leads to the magnetic flux variables and $\mathcal{V}$ as natural intermediate variables of the Forward Algorithm.

## Perfect matchings for $d P_{2}$

Figure 4-9 presents the ten perfect matchings for Model II of $d P_{2}$ and their slopes.


Figure 4-9: Perfect matchings and their slopes for Model II of $d P_{2}$.

## Chapter 5

## Infinite families of examples

### 5.1 Introduction

At low energies the theory on the D3-brane is expected to flow to a superconformal fixed point. The AdS/CFT correspondence [189, 118, 236] connects the strong coupling regime of such gauge theories with supergravity in a mildly curved geometry. For the case of D3-branes placed at the tips of Calabi-Yau cones over five-dimensional geometries $Y_{5}$, the gravity dual is of the form $A d S_{5} \times Y_{5}$, where $Y_{5}$ is a Sasaki-Einstein manifold [167, 174, 2, 196]. There has been considerable progress in this subject recently: for a long time, there was only one non-trivial Sasaki-Einstein five-manifold, $T^{1,1}$, where the metric was known. Thanks to recent progress, we now have an infinite family of explicit metrics which, when non-singular and simply-connected, have topology $S^{2} \times S^{3}$. The most general such family is specified by 3 positive integers $a, b, c$, with the metrics denoted $L^{a, b, c}[61,192]^{1}$. When $a=p-q, b=p+q, c=p$ these reduce to the $Y^{p, q}$ family of metrics, which have an enhanced $S U(2)$ isometry [107, 106, 191]. Aided by the toric description in [191], the entire infinite family of gauge theories dual to these metrics was constructed in [26]. These theories have subsequently been analyzed in considerable detail $[142,29,202,28,187,37,99,54,48,32,98,31,42]$. There has also been progress on the non-conformal extensions of these theories (and

[^13]others) both from the supergravity [142, 48] and gauge theory sides [99. 54. 32, 98, 42]. These extensions exhibit many interesting features, such as cascades [173] and dynamical supersymmetry breaking. In addition to the $Y^{p, q}$ spaces, there are also several other interesting infinite families of geometries which have been studied recently: the $X^{p, q}$ spaces [127], deformations of geometries with $U(1) \times U(1)$ isometry [187], and deformations of geometries with $U(1)^{3}$ isometry [10].

Another key ingredient in obtaining the gauge theories dual to singular CalabiYau manifolds is the principle of $a$-maximization [154], which permits the determination of exact R -charges of superconformal field theories. Recall that all $d=4$ $\mathcal{N}=1$ gauge theories possess a $U(1)_{R}$ symmetry which is part of the superconformal group $S U(2,2 \mid 1)$. If this superconformal R-symmetry is correctly identified, many properties of the gauge theory may be determined. $a$-maximization [154] is a simple procedure - maximizing a cubic function - that allows one to identify the R -symmetry from among the set of global symmetries of any given gauge theory. Plugging the superconformal R-charges into this cubic function gives exactly the central charge $a$ of the SCFT [14, 12, 140]. Although here we will focus on superconformal theories with known geometric duals, $a$-maximization is a general procedure which applies to any $\mathcal{N}=1 d=4$ superconformal field theory, and has been studied in this context in a number of recent works, with much emphasis on its utility for proving the $a$-theorem [178, 156, 177, 59, 22, 179, 23].

In the case that the gauge theory has a geometric dual, one can use the AdS/CFT correspondence to compute the volume of the dual Sasaki-Einstein manifold, as well as the volumes of certain supersymmetric 3-dimensional submanifolds, from the R charges. For example, remarkable agreement was found for these two computations in the case of the $Y^{p, q}$ singularities [41,26]. Moreover, a general geometric procedure that allows one to compute the volume of any toric Sasaki-Einstein manifold, as well as its toric supersymmetric submanifolds, was then given in [190]. In [190] it was shown that one can determine the Reeb vector field, which is dual to the R-symmetry, of any toric Sasaki-Einstein manifold by minimizing a function $Z$ that depends only on the toric data that defines the singularity. For example, the volumes of the $Y^{p, q}$ manifolds
are easily reproduced this way. Remarkably, one can also compute the volumes of manifolds for which the metric is not known explicitly. In all cases agreement has been found between the geometric and field theoretic calculations. This was therefore interpreted as a geometric dual of $a$-maximization in [190], although to date there is no general proof that the two extremal problems, within the class of superconformal gauge theories dual to toric Sasaki-Einstein manifolds, are in fact equivalent.

In this chapter, we will use this recent progress in geometry, field theory, and dimer models to obtain a lot of information about gauge theories dual to general toric Calabi-Yau cones. Our geometrical knowledge will specify many requirements of the gauge theory, and we describe how one can read off gauge theory quantities rather straightforwardly from the geometry. As a particular example of our methods, we construct the gauge theories dual to the recently discovered $L^{a, b, c}$ geometries. We will realize the geometrically derived requirements by using the brane tiling approach. Since the $L^{a, b, c}$ spaces are substantially more complicated than the $Y^{p, q}$ 's, we will not give a closed form expression for the gauge theory. We will, however, specify all the necessary building blocks for the brane tiling, and discuss how these building blocks are related to quantities derived from the geometry.

The plan of the chapter is as follows: In Section 2, we discuss how to read gauge theory data from a given toric geometry. In particular, we give a detailed prescription for computing the quantum numbers (e.g. baryon charges, flavor charges, and R -charges) and multiplicities for the different fields in the gauge theory. Section 3 applies these results to the $L^{a, b, c}$ spaces. We derive the toric diagram for a general $L^{a, b, c}$ geometry, and briefly review the metrics $[61,192]$ for these theories. We compute the volumes of the supersymmetric 3 -cycles in these Sasaki-Einstein spaces, and discuss the constraints these put on the gauge theories. In Section 4 we discuss how our geometrical computations constrain the superpotential, and describe how one may always find a phase of the gauge theory with at most only three different types of interactions. In Section 5, we prove that $a$-maximization reduces to the same equations required by the geometry for computing R-charges and central charges. Thus we show that $a$-maximization and the geometric computation agree. In Section 6,
we construct the gauge theories dual to the $L^{a, b, c}$ spaces by using the brane tiling perspective, and give several examples of interesting theories. In particular, we describe a particularly simple infinite subclass of theories, the $L^{a, b, a}$ theories. for which we can simply specify the toric data and brane tiling. We check via $Z$-minimization and $a$-maximization that all volumes and dimensions reproduce the results expected from AdS/CFT. Finally, in the Appendix, we give some more interesting examples which use our construction.

### 5.2 Quiver content from toric geometry

In this section we explain how one can extract a considerable amount of information about the gauge theories on D3-branes probing toric Calabi-Yau singularities using simple geometric methods. In particular, we show that there is always a distinguished set of fields whose multiplicities, baryon charges, and flavor charges can be computed straightforwardly using the toric data.

### 5.2.1 General geometrical set-up

Let us first review the basic geometrical set-up. For more details, the reader is referred to [190]. Let $(X, \omega)$ be a toric Calabi-Yau cone of complex dimension $n$, where $\omega$ is the Kähler form on $X$. In particular $X=C(Y) \cong \mathbb{R}^{+} \times Y$ has an isometry group containing an $n$-torus, $T^{n}$. A conical metric on $X$ which is both Ricci-flat and Kähler then gives a Sasaki-Einstein metric on the base of the cone, $Y$. The moment map for the torus action exhibits $X$ as a Lagrangian $T^{n}$ fibration over a strictly convex rational polyhedral cone $\mathcal{C} \subset \mathbb{R}^{n}$. This is a subset of $\mathbb{R}^{n}$ of the form

$$
\begin{equation*}
\mathcal{C}=\left\{y \in \mathbb{R}^{n} \mid\left(y, v_{A}\right) \geq 0, A=1, \ldots, D\right\} \tag{5.2.1}
\end{equation*}
$$

Thus $\mathcal{C}$ is made by intersecting $D$ hyperplanes through the origin in order to make a convex polyhedral cone. Here $y \in \mathbb{R}^{n}$ are coordinates on $\mathbb{R}^{n}$ and $v_{A}$ are the inward pointing normal vectors to the $D$ hyperplanes, or facets, that define the polyhedral
cone. The normals are rational and hence one can normalise them to be primitive ${ }^{2}$ elements of the lattice $\mathbb{Z}^{n}$. We also assume this set of vectors is minimal in the sense that removing any vector $v_{A}$ in the definition (5.2.1) changes $\mathcal{C}$. The condition that $\mathcal{C}$ be strictly convex is simply the condition that it is a cone over a convex polytope.


Figure 5-1: A four-faceted polyhedral cone in $\mathbb{R}^{3}$.

The condition that $X$ is Calabi-Yau, $c_{1}(X)=0$, implies that the vectors $v_{A}$ may, by an appropriate $S L(n ; \mathbb{Z})$ transformation of the torus, be all written as $v_{A}=\left(1, w_{A}\right)$. In particular, in complex dimension $n=3$ we may therefore represent any toric Calabi-Yau cone by a convex lattice polytope in $\mathbb{Z}^{2}$, where the vertices are simply the vectors $w_{A}$. This is usually called the toric diagram.

From the vectors $v_{A}$ one can reconstruct $X$ as a Kähler quotient or, more physically, as the classical vacuum moduli space of a gauged linear sigma model (GLSM). To explain this, denote by $\Lambda \subset \mathbb{Z}^{n}$ the span of the normals $\left\{v_{A}\right\}$ over $\mathbb{Z}$. This is a lattice of maximal rank since $\mathcal{C}$ is strictly convex. Consider the linear map

$$
\begin{align*}
& A: \quad \mathbb{R}^{D} \rightarrow \mathbb{R}^{n}  \tag{5.2.2}\\
&  \tag{5.2.3}\\
& e_{A} \mapsto v_{A}
\end{align*}
$$

which maps each standard orthonormal basis vector $e_{A}$ of $\mathbb{R}^{D}$ to the vector $v_{A}$. This induces a map of tori

$$
\begin{equation*}
T^{D} \cong \mathbb{R}^{D} / \mathbb{Z}^{D} \rightarrow \mathbb{R}^{n} / \Lambda \tag{5.2.4}
\end{equation*}
$$

[^14]In general the kernel of this map is $\mathcal{A} \cong T^{D-n} \times \Gamma$ where $\Gamma$ is a finite abelian group. Then $X$ is given by the Kälıler quotient

$$
\begin{equation*}
X=\mathbb{C}^{D} / / \mathcal{A} \tag{5.2.5}
\end{equation*}
$$

Recall we may write this more explicitly as follows. The torus $T^{D-n} \subset T^{D}$ is specified by a charge matrix $Q_{I}^{A}$ with integer coefficients, $I=1, \ldots, D-n$, and we define

$$
\begin{equation*}
\mathcal{K} \equiv\left\{\left.\left(Z_{1}, \ldots, Z_{D}\right) \in \mathbb{C}^{D}\left|\sum_{A} Q_{I}^{A}\right| Z_{A}\right|^{2}=0\right\} \subset \mathbb{C}^{D} \tag{5.2.6}
\end{equation*}
$$

where $Z_{A}$ denote complex coordinates on $\mathbb{C}^{D}$. In GLSM language, $\mathcal{K}$ is simply the space of solutions to the D term equations. Dividing out by gauge transformations gives the quotient

$$
\begin{equation*}
X=\mathcal{K} / T^{D-n} \times \Gamma \tag{5.2.7}
\end{equation*}
$$

We also denote by $L$ the link of $\mathcal{K}$ with the sphere $S^{2 D-1} \subset \mathbb{C}^{D}$. We then have a fibration

$$
\begin{equation*}
\mathcal{A} \hookrightarrow L \rightarrow Y \tag{5.2.8}
\end{equation*}
$$

where $Y$ is the Sasakian manifold which is the base of the cone $X=C(Y)$. For a general set of vectors $v_{A}$, the space $Y$ will not be smooth. In fact typically one has orbifold singularities. $Y$ is smooth if and only if the polyhedral cone is good [183], although we will not enter into the general details of this here - see, for example, [190].

Finally in this subsection we note some topological properties of $Y$, in the case that $Y$ is a smooth manifold. In [184] it is shown that $L$ has trivial homotopy groups in dimensions 0,1 and 2 . From the long exact homotopy sequence for the fibration
(5.2.8) one concludes that [184]

$$
\begin{align*}
& \pi_{1}(Y) \cong \pi_{0}(\mathcal{A}) \cong \Gamma \cong \mathbb{Z}^{n} / \Lambda  \tag{5.2.9}\\
& \pi_{2}(Y) \cong \pi_{1}(\mathcal{A}) \cong \mathbb{Z}^{D-n} \tag{5.2.10}
\end{align*}
$$

In particular, $Y$ is simply-connected if and only if the $\left\{v_{A}\right\}$ span $\mathbb{Z}^{n}$ over $\mathbb{Z}$. In fact we will assume this throughout in the following any finite quotient of a toric singularity will correspond to an orbifold of the corresponding gauge theory, and this process is well-understood by now.

From now on we also restrict to the physical case of complex dimension $n=3$. Moreover, throughout this section we assume that the Sasaki-Einstein manifold $Y$ is smooth. The reason for this assumption is firstly to simplify the geometrical and topological analysis, and secondly because the physics in the case that $Y$ is an orbifold which is not a global quotient of a smooth manifold is not well-understood. However, as we shall see later, one can apparently relax this assumption with the results essentially going through without modification. The various cohomology groups that we introduce would then need replacing by their appropriate orbifold versions.

### 5.2.2 Quantum numbers of fields

In this subsection we explain how one can deduce the quantum numbers for a certain distinguished set of fields in any toric quiver gauge theory. Recall that, quite generally, $N$ D3-branes placed at a toric Calabi-Yau singularity have an AdS/CFT dual that may be described by a toric quiver gauge theory. In particular, the matter content is specified by giving the number of gauge groups, $N_{g}$, and number of fields $N_{f}$, together with the charge assignments of the fields. In fact these fields are always bifundamentals (or adjoints). This means that the matter content may be neatly summarised by a quiver diagram.

We may describe the toric singularity as a convex lattice polytope in $\mathbb{Z}^{2}$ or by giving the GLSM charges, as described in the previous section. By setting each complex
coordinate $Z_{A}=0, A=1, \ldots, D$, one obtains a toric divisor $D_{A}$ in the Calabi-Yau cone. This is also a cone, with $D_{A}=C^{\prime}\left(\Sigma_{A}\right)$ where $\Sigma_{A}$ is a 3-dimensional supersymmetric submanifold of $Y$. Thus in particular wrapping a D3-brane over $\Sigma_{A}$ gives rise to a BPS state, which via the AdS/CFT correspondence is conjectured to be dual to a dibaryonic operator in the dual gauge theory. We claim that there is always a distinguished subset of the fields, for any toric quiver gauge theory, which are associated to these dibaryonic states. To explain this, recall that given any bifundamental field $X$, one can construct the dibaryonic operator

$$
\begin{equation*}
\mathcal{B}[X]=\varepsilon^{\alpha_{1} \ldots \alpha_{N}} X_{\alpha_{1}}^{\beta_{1}} \ldots X_{\alpha_{N}}^{\beta_{N}} \varepsilon_{\beta_{1} \ldots \beta_{N}} \tag{5.2.11}
\end{equation*}
$$

using the epsilon tensors of the corresponding two $S U(N)$ gauge groups. This is dual to a D3-brane wrapped on a supersymmetric submanifold, for example one of the $\Sigma_{A}$. In fact to each toric divisor $\Sigma_{A}$ let us associate a bifundamental field $X_{A}$ whose corresponding dibaryonic operator (5.2.11) is dual to a D3-brane wrapped on $\Sigma_{A}$. These fields in fact have multiplicities, as we explain momentarily. In particular each field in such a multiplet has the same baryon charge, flavor charge, and R-charge.

## Multiplicities

Recall that $D_{A}=\left\{Z_{A}=0\right\}=C\left(\Sigma_{A}\right)$ where $\Sigma_{A}$ is a 3-submanifold of $Y$. To each such submanifold we associated a bifundamental field $X_{A}$. As we now explain, these fields have multiplicities given by the simple formula

$$
\begin{equation*}
m_{A}=\left|\left(v_{A-1}, v_{A}, v_{A+1}\right)\right| \tag{5.2.12}
\end{equation*}
$$

where we have defined the cyclic identification $v_{D+1}=v_{1}$, and we list the normal vectors $v_{A}$ in order around the polyhedral cone, or equivalently, the toric diagram. Here $(\cdot, \cdot, \cdot)$ denotes a $3 \times 3$ determinant, as in [190].

In fact, when $Y$ is smooth, each $\Sigma_{A}$ is a Lens space $L\left(n_{1}, n_{2}\right)$ for appropriate $n_{1}$ and $n_{2}$. To see this, note that each $\Sigma_{A}$ is a principle $T^{2}$ fibration over an interval,
say $[0,1]$. By an $S L(2 ; \mathbb{Z})$ transformation one can always arrange that at 0 the $(1,0)-$ cycle collapses, and at 1 the $\left(n_{1}, n_{2}\right)$ cycle collapses. It is well-known that this can be equivalently described as the quotient of $S^{3} \subset \mathbb{C}^{2}$ by the $\mathbb{Z}_{n_{1}}$ action

$$
\begin{equation*}
\left(z_{1}, z_{2}\right) \rightarrow\left(z_{1} \omega_{n_{1}}, z_{2} \omega_{n_{1}}^{n_{2}}\right) \tag{5.2.13}
\end{equation*}
$$

where $\operatorname{hcf}\left(n_{1}, n_{2}\right)=1$ and $\omega_{n_{1}}$ denotes an $n_{1}$ th root of unity. These spaces have a rich history, and even the classification of homeomorphism types is rather involved. We shall only need to know that $\pi_{1}\left(L\left(n_{1}, n_{2}\right)\right) \cong \mathbb{Z}_{n_{1}}$, which is immediate from the second definition above.

Consider now wrapping a D3-brane over some smooth $\Sigma$, where $\pi_{1}(\Sigma)=\mathbb{Z}_{m}$. As we just explained, when $\Sigma$ is toric, it is necessarily some Lens space $L\left(m, n_{2}\right)$. In fact, when the order of the fundamental group is greater than one, there is not a single such D3-brane, but in fact $m$ D3-branes. The reason is that, for each $m$, we can turn on a flat line bundle for the $U(1)$ gauge field on the D 3 -brane worldvolume. Indeed, recall that line bundles on $\Sigma$ are classified topologically by $H^{2}(\Sigma ; \mathbb{Z}) \cong H_{1}(\Sigma ; \mathbb{Z}) \cong \pi_{1}(\Sigma)$, where the last relation follows for abelian fundamental group. A torsion line bundle always admits a flat connection, which has zero energy. Since these D3-brane states have different charge - namely torsion D1-brane charge - they must correspond to different operators in the gauge theory. However, as will become clear, these operators all have the same baryon charge, flavor charge and $R$-charge. We thus learn that the multiplicity of the bifundamental field $X_{A}$ associated with $D_{A}$ is given by $m$.

It remains then to relate $m$ to the formula (5.2.12). Without loss of generality, pick a facet $A$, and suppose that the normal vector is $v_{A}=(1,0,0)$. The facet is itself a polyhedral cone in the $\mathbb{R}^{2}$ plane transverse to this vector. To obtain the normals that define this cone we simply project $v_{A-1}, v_{A+1}$ onto the plane. Again, by a special linear transformation we may take these 2 -vectors to be $(0,1),\left(n_{1},-n_{2}\right)$, respectively, for some integers $n_{1}$ and $n_{2}$. One can then verify that this toric diagram indeed corresponds to the cone over $L\left(n_{1}, n_{2}\right)$, as defined above. By direct calculation
we now see that

$$
\begin{equation*}
\mid\left(v_{A-1}, v_{A}, v_{A-1}\left|=\left|(0.1) \times\left(n_{1},-n_{2}\right)\right|=n_{1}\right.\right. \tag{5.2.14}
\end{equation*}
$$

which is the order of $\pi_{1}\left(\Sigma_{A}\right)$. The determinant is independent of the choice of basis we have made, and thus this relation is true in general, thus proving the formula (5.2.12). One can verify this formula in a large number of examples where the gauge theories are already known.

## Baryon charges

In this subsection we explain how one can deduce the baryonic charges of the fields $X_{A}$. Recall that, in general, the toric Sasaki Einstein manifold $Y$ arises from a quotient by a torus

$$
\begin{equation*}
T^{D-3} \hookrightarrow L \rightarrow Y \tag{5.2.15}
\end{equation*}
$$

This fibration can be thought of as $D-3$ circle fibrations over $Y$ with total space $L$. Equivalently we can think of these as complex line bundles $\mathcal{M}_{I}$. Let $C_{I}, I=$ $1, \ldots, D-3$, denote the Poincaré duals of the first Chern classes of these bundles. Thus they are classes in $H_{3}(Y ; \mathbb{Z})$. Recall from (5.2.9) that $\pi_{2}(Y) \cong \mathbb{Z}^{D-3}$ when $Y$ is smooth. Provided $Y$ is also simply-connected ${ }^{3}$ one can use the Hurewicz isomorphism, Poincaré duality and the universal coefficients theorem to deduce that

$$
\begin{equation*}
H_{3}(Y: \mathbb{Z}) \cong \mathbb{Z}^{D-3} \tag{5.2.16}
\end{equation*}
$$

In particular note that the number of independent 3 -cycles is just $D-3$. A fairly straightforward calculation ${ }^{4}$ in algebraic topology shows that the classes $C_{I}$ above actually generate the homology group $H_{3}(Y ; \mathbb{Z}) \cong \mathbb{Z}^{D-3}$. Thus $\left\{C_{I}\right\}$ form a basis of 3-cycles on $Y$.

[^15]In Type IIB supergravity one can Kaluza-Klein reduce the Ramond-Ramond four-form potential $C_{4}$ to obtain $D-3$ gauge fields $A_{I}$ in the $A d S_{5}$ space:

$$
\begin{equation*}
C_{4}=\sum_{I=1}^{D-3} A_{I} \wedge \mathcal{H}_{I} \tag{5.2.17}
\end{equation*}
$$

Here $\mathcal{H}_{I}$ is a harmonic 3 -form on $Y$ that is Poincare dual to the 3 -cycle $C_{I}$. In the superconformal gauge theory, which recall may be thought of as living on the conformal boundary of $A d S_{5}$, these become $D-3$ global $U(1)$ symmetries

$$
\begin{equation*}
U(1)_{B}^{D-3} . \tag{5.2.18}
\end{equation*}
$$

These are baryonic symmetries precisely because the D3-brane is charged under $C_{4}$ and a D3-brane wrapped over a supersymmetric submanifold of $Y$ is interpreted as a dibaryonic state in the gauge theory. Indeed, the $\Sigma_{A}$ are precisely such a set of submanifolds.

Again, a fairly standard calculation in toric geometry then shows that topologically

$$
\begin{equation*}
\left[\Sigma_{A}\right]=\sum_{I=1}^{D-3} Q_{I}^{A} C_{I} \in H_{3}(Y ; \mathbb{Z}) \tag{5.2.19}
\end{equation*}
$$

This perhaps requires a little explanation. Each GLSM field $Z_{A}, A=1, \ldots, D$, can be viewed as a section of a complex line bundle $\mathcal{L}_{A}$ over $Y$. They are necessarily sections of line bundles, rather than functions, because the fields $Z_{A}$ are charged under the torus $T^{D-3}$. Now $Z_{A}=0$ is the zero section of the line bundle associated to $Z_{A}$, and by definition this cuts out the submanifold $\Sigma_{A}$ on $Y$. Moreover, the first Chern class of this line bundle is then Poincaré dual to $\left[\Sigma_{A}\right]$. Recall that the charge matrix $Q$ specifies the embedding of the torus $T^{D-3}$ in $T^{D}$, which then acts on the fields/coordinates $Z_{A}$; the element $Q_{I}^{A}$ specifies the charge of $Z_{A}$, which is a section of $\mathcal{L}_{A}$, under the circle $\mathcal{M}_{I}$. This means that the two sets of line bundles are related
by

$$
\begin{equation*}
\mathcal{L}_{A}=\bigotimes_{I=1}^{D-3} \mathcal{M}_{I}^{Q_{I}^{A}} \tag{5.2.20}
\end{equation*}
$$

Taking the first Chern class of this relation and applying Poincaré duality then proves (5.2.19).

It follows that the baryon charges of the fields $X_{A}$ are given precisely by the matrix $Q$ that enters in defining the GLSM. Thus if $B_{I}\left[X_{A}\right]$ denotes the baryon charge of $X_{A}$ under the $I$ th copy of $U(1)$ in (5.2.18) we have

$$
\begin{equation*}
B_{I}\left[X_{A}\right]=Q_{I}^{A} \tag{5.2.21}
\end{equation*}
$$

Note that from the Calabi-Yau condition the charges of the linear sigma model sum to zero

$$
\begin{equation*}
\sum_{A} B_{I}\left[X_{A}\right]=\sum_{A} Q_{I}^{A}=0 \quad I=1, \ldots, D-3 \tag{5.2.22}
\end{equation*}
$$

Moreover, the statement that

$$
\begin{equation*}
\sum_{A} v_{A}^{i}\left[\Sigma_{A}\right]=0 \tag{5.2.23}
\end{equation*}
$$

may then be interpreted as saying that, for each $i$, one can construct a state in the gauge theory of zero baryon charge by using $v_{A}^{i}$ copies of the field $X_{A}$, for each $A$.

## Flavor charges

In this subsection we explain how one can compute the flavor charges of the $X_{A}$. Recall that the horizon Sasaki-Einstein manifolds have at least a $U(1)^{3}$ isometry since they are toric. By definition a flavor symmetry in the gauge theory is a non- R symmetry - that is, the supercharges are left invariant under such a symmetry. The geometric dual of this statement is that the Killing spinor $\psi$ on the Sasaki-Einstein manifold $Y$ is left invariant by the corresponding isometry. Thus a Killing vector field
$V_{F}$ is dual to a flavor symmetry in the gauge theory if and only if

$$
\begin{equation*}
\mathcal{L}_{V_{F}} \psi=0 \tag{5.2.24}
\end{equation*}
$$

where $\psi$ is a Killing spinor on $Y$. In fact there is always precisely a $U(1)^{2}$ subgroup of $U(1)^{3}$ that satisfies this condition. This can be shown by considering the holomorphic $(3,0)$ form of the corresponding Calabi-Yau cone [190]. It is well known that this is constructed from the Killing spinors as a bilinear

$$
\begin{equation*}
\Omega=\psi^{c} \Gamma_{(3)} \psi, \tag{5.2.25}
\end{equation*}
$$

where $\Gamma_{(3)}$ is the totally antisymmetrised product of 3 gamma matrices in Cliff $(6,0)$. In particular, in the basis in which the normal vectors of the polyhedral cone $\mathcal{C}$ are of the form $v_{A}=\left(1, w_{A}\right)$, the Lie algebra elements $(0,1,0),(0,0,1)$ generate the group $U(1)_{F}^{2}$ of flavor isometries. Note that, for $Y^{p, q}$, one of these $U(1)_{F}$ symmetries is enhanced to an $S U(2)$ flavor symmetry. However, $U(1)_{F}^{2}$ is the generic case.

We would like to determine the charges of the fields $X_{A}$ under $U(1)_{F}^{2}$. In fact in the gauge theory this symmetry group is far from unique - one is always free to mix any flavor symmetry with part of the baryonic symmetry group $U(1)_{B}^{D-3}$. The baryonic symmetries are distinguished by the fact that mesons in the gauge theory, for example constructed from closed loops in a quiver gauge theory, should have zero baryonic charge. Thus the flavor symmetry group is unique only up to mixing with baryonic symmetries, and of course mixing with each other.

This mixing ambiguity has a beautiful geometric interpretation. Recall that the Calabi-Yau cone $X$ is constructed as a symplectic quotient

$$
\begin{equation*}
X=\mathbb{C}^{D} / / T^{D-3} \tag{5.2.26}
\end{equation*}
$$

where the torus $T^{D-3} \subset T^{D}$ is defined by the kernel of the map

$$
\begin{align*}
A: & \mathbb{R}^{D} \rightarrow \mathbb{R}^{3}  \tag{5.2.27}\\
& e_{A} \mapsto v_{A} \tag{5.2.28}
\end{align*}
$$

More precisely the kernel of $A$ is generated by the matrix $Q_{I}^{A}$, which in turn defines a sublattice $\Upsilon$ of $\mathbb{Z}^{D}$ of rank $D-3$. The torus is then $T^{D-3}=\mathbb{R}^{D-3} / \Upsilon$. We may also consider the quotient $\mathbb{Z}^{D} / \Upsilon$. The map induced from $A$ then maps this quotient space isomorphically onto $\mathbb{Z}^{3}$ and the corresponding torus $T^{3}=T^{D} / T^{D-3}$ is then precisely the torus isometry of $X$.

Let us pick two elements $\alpha_{1}, \alpha_{2}$ of $\mathbb{Z}^{D}$ that map to the basis vectors $(0,1,0),(0,0,1)$ under $A$. From the last paragraph these are defined only up to elements of the lattice $\Upsilon$, and thus may be considered as elements of the quotient $\mathbb{Z}^{D} / \Upsilon$. Geometrically, $\alpha_{1}, \alpha_{2}$ define circle subgroups of $T^{D}$ that descend to the two $U(1)$ flavor isometries generated by $(0,1,0)$ and $(0,0,1)$. The charges of the complex coordinates $Z_{A}$ on $\mathbb{C}^{D}$ are then simply $\alpha_{1}^{A}, \alpha_{2}^{A}$ for each $A=1, \ldots, D$. However, as discussed in the last subsection, the $Z_{A}$ descend to complex line bundles on $Y$ whose Poincaré duals are precisely the submanifolds $\Sigma_{A}$. Thus the flavor charges of $X_{A}$ may be identified with $\alpha_{1}^{A}, \alpha_{2}^{A}$. Moreover, by construction, each $\alpha$ was unique only up to addition by some element in the lattice $\Upsilon$ generated by $Q_{I}^{A}$. But as we just saw in the previous subsection, this is precisely the set of baryon charges in the gauge theory. We thus see that the ambiguity in the choice of flavor symmetries in the gauge theory is in 1-1 correspondence with the ambiguity in choosing $\alpha_{1}, \alpha_{2}$.

## R-charges

The R-charges were treated in reference [190], so we will be brief here. Let us begin by emphasising that all the quantities computed so far can be extracted in a simple way from the toric data, or equivalently from the charges of the gauged linear sigma model, without the need of an explicit metric. In [190], it was shown that the total volume of any toric Sasaki-Einstein manifold, as well as the volumes of its supersymmetric toric
submanifolds, can be computed by solving a simple extremal problem which is defined in terms of the polyhedral cone $\mathcal{C}$. This toric data is encoded in a function $Z$, which depends on a "trial" Reeb vector living in $\mathbb{R}^{3}$. Minimizing $Z$ determines the Reeb vector for the Sasaki-Einstein metric on $Y$ uniquely, and as a result one can compute the volumes of the $\Sigma_{A}$. This is a geometric analogue of $a$-maximization [154]. Indeed, recall that the volumes are related to the R -charges of the corresponding fields $X_{A}$ by the simple formula

$$
\begin{equation*}
R\left[X_{A}\right]=\frac{\pi}{3} \frac{\operatorname{vol}\left(\Sigma_{A}\right)}{\operatorname{vol}(Y)} . \tag{5.2.29}
\end{equation*}
$$

This formula has been used in many AdS/CFT calculations to compare the R-charges of dibaryons with their corresponding 3 -manifolds [34, 155, 146, 147].

Moreover, in [190] a general formula relating the volume of supersymmetric submanifolds to the total volume of the toric Sasaki-Einstein manifold was given. This reads

$$
\begin{equation*}
\pi \sum_{A=1}^{D} \operatorname{vol}\left(\Sigma_{A}\right)=6 \operatorname{vol}(Y) \tag{5.2.30}
\end{equation*}
$$

Then the physical interpretation of (5.2.30) is that the R -charges of the bifundamental fields $X_{A}$ sum to 2 :

$$
\begin{equation*}
\sum_{A} R\left[X_{A}\right]=2 \tag{5.2.31}
\end{equation*}
$$

This is related to the fact that each term in the superpotential is necessarily the sum

$$
\begin{equation*}
\sum_{A=1}^{D} \Sigma_{A} \tag{5.2.32}
\end{equation*}
$$

and the superpotential has R -charge 2 by definition. We shall discuss this further in Section 4.

### 5.3 The $L^{a, b, c}$ toric singularities

In the remainder of this chapter, we will be interested in the specific GLSM with charges

$$
\begin{equation*}
Q=(a,-c, b,-d) \tag{5.3.33}
\end{equation*}
$$

where of course $d=a+b-c$ in order to satisfy the Calabi-Yau condition. We will define this singularity to be $L^{a, b, c}$. The reason we choose this family is two-fold: firstly, the Sasaki-Einstein metrics are known explicitly in this case $[61,192]$ and, secondly, this family is sufficiently simple that we will be able to give a general prescription for constructing the gauge theories.

Let us begin by noting that this is essentially the most general GLSM with four charges, and hence the most general toric quiver gauge theories with a single $U(1)_{B}$ symmetry, up to orbifolding. Indeed, provided all the charges are non-zero, either two have the same sign or else three have the same sign. The latter are in fact just orbifolds of $S^{5}$, and this case where all but one of the charges have the same sign is slightly degenerate. Specifically, the charges ( $e, f, g,-e-f-g$ ) describe the orbifold of $S^{5} \subset \mathbb{C}^{3}$ by $\mathbb{Z}_{e+f+g}$ with weights $(e, f, g)$. The polyhedral cones therefore have three facets, and not four, or equivalently the $(p, q)$ web has 3 external legs. By our general analysis there is therefore no $U(1)$ baryonic symmetry, as expected. Indeed, note that setting $Z_{4}=0$ does not give a divisor in this case, since the remaining charges are all positive and there is no solultion to the remaining D-terms. The Sasaki-Einstein metrics are just the quotients of the round metric on $S^{5}$ and these theories are therefore not particularly interesting. In the case that one of the charges is zero, we instead obtain $\mathcal{N}=2$ orbifolds of $S^{5}$, which are also well-studied.

We are therefore left with the case that two charges have the same sign. In (5.3.33) we therefore take all integers to be positive. Without loss of generality we may of course take $0<a \leq b$. Also, by swapping $c$ and $d$ if necessary, we can always arrange that $c \leq b$. By definition $\operatorname{hcf}(a, b, c, d)=1$ in order that the $U(1)$ action specified by (5.3.33) is effective, and it then follows that any three integers are
coprime. The explicit Sasaki-Einstein metrics on the horizons of these singularities were constructed in [61]. The toric description above was then given in [192]. The manifolds were named $L^{p, q, r}$ in reference [61] but, following [192]. we have renamed these $L^{\text {t.b.c }}$ in order to avoid confusion with $Y^{p, q}$. Indeed, notice that these spaces reduce to, $Y^{p, q}$ when $c=d=p$, and then $a=p-q, b=p+q$. In particular there is an enhanced $S U(2)$ symmetry in the metric in this limit. It is straightforward to determine when the space $Y=L^{a, b, c}$ is non-singular: each of the pair $a, b$ must be coprime to each of $c, d$. This condition is necessary to avoid codimension four orbifold singularities on $Y$. To see this, consider setting $Z_{1}=Z_{4}=0$. If $b$ and $c$ had a common factor $h$, then the circle action specified by (5.3.33) would factor through a cyclic group $\mathbb{Z}_{h}$ of order $h$, and this would descend to a local orbifold group on the quotient space. In fact it is simple to see that this subspace is just an $S^{1}$ family of $\mathbb{Z}_{h}$ orbifold singularities. All such singularities arise in this way. When $Y=L^{a, b, c}$ is non-singular it follows from the last section that $\pi_{2}(Y) \cong \mathbb{Z}$ and hence $H_{2}(Y ; \mathbb{Z}) \cong \mathbb{Z}$. By Smale's theorem $Y$ is therefore diffeomorphic to $S^{2} \times S^{3}$. In particular there is one 3 -cycle and hence one $U(1)_{B}$ for these theories.

The toric diagram can be described by an appropriate set of four vectors $v_{A}=$ $\left(1, w_{A}\right)$. We take the following set

$$
\begin{equation*}
w_{1}=[1,0] \quad w_{2}=[a k, b] \quad w_{3}=[-a l, c] \quad w_{4}=[0,0] \tag{5.3.34}
\end{equation*}
$$

where $k$ and $l$ are two integers satisfying

$$
\begin{equation*}
c k+b l=1 \tag{5.3.35}
\end{equation*}
$$

and we have assumed for simplicity of exposition that $\operatorname{hcf}(b, c)=1$. This toric diagram is depicted in Figure 5-2.

The solution to the above equation always exists by Euclid's algorithm. Moreover, there is a countable infinity of solutions to this equation, where one shifts $k$ and $l$ by $-t b$ and $t c$, respectively, for any integer $t$. However, it is simple to check that


Figure 5-2: Toric diagram for the $L^{a, b, c}$ geometries.
different solutions are related by the $S L(2 ; \mathbb{Z})$ transformation

$$
\left(\begin{array}{cc}
1 & -t a  \tag{5.3.36}\\
0 & 1
\end{array}\right)
$$

acting on the $w_{A}$, as must be the case of course. The kernel of the linear map (5.2.2) is then generated by the charge vector $Q$ in (5.3.33).

It is now simple to see that the toric diagram for $L^{a, b, c}$ always admits a triangulation with $a+b$ triangles. It is well known that this gives the number of gauge groups $N_{g}$ in the gauge theory. To see this one uses the fact that the area of the toric diagram is the Euler number of the (any) completely resolved Calabi-Yau $\tilde{X}$ obtained by toric crepant resolution, and then for toric manifolds this is the dimension of the even cohomology of $\tilde{X}$. Now on 0,2 and 4 -cycles in $\tilde{X}$ one can wrap space-filling D3, D5 and D7-branes, respectively, and these then form a basis of fractional branes. The gauge groups may then be viewed as the gauge groups on these fractional branes. By varying the Kähler moduli of $\tilde{X}$ one can blow down to the conical singularity $X$. The holomorphic part of the gauge theory is independent of the Kähler moduli, which is why the matter content of the superconformal gauge theory can be computed at large volume in this way. To summarise, we have

$$
\begin{equation*}
N_{g}=a+b \tag{5.3.37}
\end{equation*}
$$

Note that, different from $Y^{p, q}$, the number of gauge groups for $L^{a, b, c}$ can be odd.

We may now draw the $(p, q)$ web $[8,7,185]$. Recall that this is simply the graphtheoreric dual to the toric diagram, or, completely equivalently, is the projection of the polyhedral cone $\mathcal{C}$ onto the plane with normal vector ( $1,0,0$ ). The external legs of the ( $p, q$ ) web are easily computed to be

$$
\begin{align*}
& \left(p_{1}, q_{1}\right)=(-c,-a l) \\
& \left(p_{2}, q_{2}\right)=(c-b, a(k+l))  \tag{5.3.38}\\
& \left(p_{3}, q_{3}\right)=(b,-a k+1) \\
& \left(p_{4}, q_{4}\right)=(0,-1)
\end{align*}
$$

This ( $\mathrm{p}, \mathrm{q}$ )-web is pictured in Figure 5-3. Using this information we can compute the


Figure 5-3: $(\mathrm{p}, \mathrm{q})$-web for the $L^{a, b, c}$ theories.
total number of fields in the gauge theory. Specifically we have

$$
N_{f}=\frac{1}{2} \sum_{i, j \in \operatorname{legs}}^{4}\left|\operatorname{det}\left(\begin{array}{cc}
p_{i} & q_{i}  \tag{5.3.39}\\
p_{j} & q_{j}
\end{array}\right)\right|
$$

This formula comes from computing intersection numbers of 3 -cycles in the mirror geometry [126]. In fact the four adjacent legs each contribute $a, b, c, d$ fields, which are simply the GLSM charges, up to sign. The two cross terms then contribute $c-a$ and $b-c$ fields, giving

$$
\begin{equation*}
N_{f}=a+3 b \tag{5.3.40}
\end{equation*}
$$

To summarise this section so far, the gauge theory for $L^{a, b, c}$ has $N_{g}=a+b$ gauge
groups, ant $\boldsymbol{V}_{\boldsymbol{j}}=a+3 b$ fields in total. In Section 3.2 we will determine the multiplicities of the fields, as well as their baryon and flavor charges, using the results of the previous section.

### 5.3.1 The sub-family $L^{a, b, a}$

The observant reader will have noticed that the charges in (5.3.38) are not always primitive. In fact this is a consequence of orbifold singularities in the Sasaki-Einstein space. In such singular cases one can have some number of lattice points, say $m-1$, on the edges of the toric diagram, and then the corresponding leg of the $(p, q)$ web in (5.3.38) is not a primitive vector. One should then really write the primitive vector, and associate to that leg the label, or multiplicity, $m$. Each leg of the $(p, q)$ web corresponds to a circle on $Y$ which is a locus of singular points if $m>1$, where $m$ gives the order of the orbifold group. Nevertheless, the charges (5.3.38) as written above give the correct numbers of fields.

Rather than explain this point in generality, it is easier to give an example. Here we consider the family $L^{a, b, a}$, which are always singular if one of $a$ or $b$ is greater than 1. In fact by the $S L(2 ; \mathbb{Z})$ transformation

$$
\left(\begin{array}{ll}
1 & l  \tag{5.3.41}\\
0 & 1
\end{array}\right)
$$

one maps the toric diagram to an isosceles trapezoid as shown in Figure 5-4.a.
Notice that there are $a-1$ lattice points on one external edge, and $b-1$ lattice points on the opposite edge. This is indicative of the singular nature of these spaces. Correspondingly, the ( $p, q$ ) web has non-primitive charges (or else one can assign positive labels $b$ and $a$ to primitive charges). Indeed, the leg with label $a$ is just the submanifold obtained by setting $Z_{3}=Z_{4}=0$. On $Y$ the D -terms, modulo the $U(1)$ gauge transformation, just give a circle $S^{1}$. However, the $U(1)$ group factors through $a$ times, due to the charges of $Z_{1}$ and $Z_{2}$ being both equal to $a$. This means that the $S^{1}$ is a locus of $\mathbb{Z}_{a}$ orbifold singularities. Obviously, similar remarks apply to


Figure 5-4: a) Toric diagram and b) $(p, q)$ web for the $L^{a, b, a}$ sub-family.
$Z_{1}=Z_{2}=0$. The singular nature of these spaces will also show up in the gauge theory: certain types of fields will be absent, and there will be adjoints, as well as bifundamentals. The $L^{a, b, a}$ family will be revisited in Section 5.6 .2 , where we will construct their associated brane tilings, gauge theories and compare the computations performed in the field theories with those in the dual supergravity backgrounds.

### 5.3.2 Quantum numbers of fields

Let us denote the distinguished fields as

$$
\begin{equation*}
X_{1}=Y \quad X_{2}=U_{1} \quad X_{3}=Z \quad X_{4}=U_{2} \tag{5.3.42}
\end{equation*}
$$

In the limit $c=d=p$ we have that $L^{a, b, c}$ reduces to a $Y^{p, q}$. Specifically, $b=p+q$ and $a=p-q$. Then this notation for the fields coincides with that of reference [26]. In particular, the $U_{i}$ become a doublet under the $S U(2)$ isometry/flavor symmetry in this limit.

The multiplicities of the fields can be read off from the results of the last section:

$$
\begin{equation*}
\operatorname{mult}[Y]=b \quad \operatorname{mult}\left[U_{1}\right]=d \quad \operatorname{mult}[Z]=a \quad \operatorname{mult}\left[U_{2}\right]=c \tag{5.3.43}
\end{equation*}
$$

This accounts for $2(a+b)$ fields, which means that there are $b-a$ fields missing. The
$(p, q)$ web suggests that there are two more fields $V_{1}$ and $V_{2}$ with multiplicities

$$
\begin{equation*}
\operatorname{mult}\left[V_{1}\right]=c-a \quad \operatorname{mult}\left[V_{2}\right]=b-c \tag{5.3.44}
\end{equation*}
$$

Indeed, this also reproduces a $Y^{p, q}$ theory in the limit $c=d$, where the fields $V_{i}$ again become an $S U(2)$ doublet.

It is now simple to work out which toric divisors these additional fields are associated to. As will be explained later, each divisor must appear precisely $b$ times in the list of fields. Roughly, this is because there are necessarily $a+3 b-(a+b)=2 b$ terms in the superpotential, and every field must appear precisely twice by the quiver toric condition [80]. From this we deduce that we may view the remaining fields $V_{1}, V_{2}$ as "composites" - more precisely, we identify them with unions of adjacent toric divisors $D_{i} \cup D_{j}$ in the Calabi-Yau, or equivalently in terms of supersymmetric 3 -submanifolds in the Sasaki-Einstein space:

$$
\begin{array}{ll}
V_{1}: & \Sigma_{3} \cup \Sigma_{4}  \tag{5.3.45}\\
& \\
V_{2}: & \Sigma_{2} \cup \Sigma_{3} .
\end{array}
$$

We may now compute the baryon and flavor charges of all the fields. The charges for the fields $V_{1}, V_{2}$ can be read off from their relation to the divisors $\Sigma_{A}$ above. We summarise the various quantum numbers in Table 5.1.

Notice that the $S L(2 ; \mathbb{Z})$ transformation (5.3.36) that shifts $k$ and $l$ is equivalent to redefining the flavor symmetry

$$
\begin{equation*}
U(1)_{F_{2}} \rightarrow U(1)_{F_{2}}-t U(1)_{B}+t a U(1)_{F_{1}} . \tag{5.3.47}
\end{equation*}
$$

Note also that each toric divisor appears precisely $b$ times in the table. This fact automatically ensures that the linear traces vanish

$$
\begin{equation*}
\operatorname{Tr} U(1)_{B}=0 \quad \text { and } \quad \operatorname{Tr} U(1)_{F_{1}}=\operatorname{Tr} U(1)_{F_{2}}=0 \tag{5.3.48}
\end{equation*}
$$

| Field | SUSY sumanifold | number | $U(1)_{B}$ | $U(1)_{F_{1}}$ | $U(1)_{F_{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $Y$ | $\Sigma_{1}$ | $b$ | $a$ | 1 | 0 |
| $U_{1}$ | $\Sigma_{2}$ | $d$ | $-c$ | 0 | $l$ |
| $Z$ | $\Sigma_{3}$ | $a$ | $b$ | 0 | $k$ |
| $U_{2}$ | $\Sigma_{1}$ | $c$ | $-d$ | -1 | $-k-l$ |
| $V_{1}$ | $\Sigma_{3} \cup \Sigma_{1}$ | $b-c$ | $c-a$ | -1 | $-l$ |
| $V_{2}$ | $\Sigma_{2} \cup \Sigma_{3}$ | $c-a$ | $b-c$ | 0 | $k+l$ |

Table 5.1: Charge assignments for the six different types of fields present in the general quiver diagram for $L^{a, b, c}$.
as must be the case. As a non trivial check of these assignments, one can compute that the cubic baryonic trace vanishes as well

$$
\begin{equation*}
\operatorname{Tr} U(1)_{B}^{3}=b a^{3}-d c^{3}+a b^{3}-c d^{3}+(b-c)(c-a)^{3}+(c-a)(b-c)^{3}=0 \tag{5.3.49}
\end{equation*}
$$

### 5.3.3 The geometry

In this subsection we summarise some aspects of the geometry of the toric SasakiEinstein manifolds $L^{a, b, c}$. First, we recall the metrics [61], and how these are associated to the toric singularities discussed earlier [192]. We also discuss supersymmetric submanifolds, compute their volumes, and use these results to extract the R-charges of the dual field theory.

The local metrics were given in [61] in the form

$$
\begin{align*}
\mathrm{d} s^{2}= & \frac{\rho^{2} \mathrm{~d} x^{2}}{4 \Delta_{x}}+\frac{\rho^{2} \mathrm{~d} \theta^{2}}{\Delta_{\theta}}+\frac{\Delta_{x}}{\rho^{2}}\left(\frac{\sin ^{2} \theta}{\alpha} \mathrm{~d} \varphi+\frac{\cos ^{2} \theta}{\beta} \mathrm{~d} \psi\right)^{2}  \tag{5.3.50}\\
& +\frac{\Delta_{\theta} \sin ^{2} \theta \cos ^{2} \theta}{\rho^{2}}\left(\frac{\alpha-x}{\alpha} \mathrm{~d} \varphi-\frac{\beta-x}{\beta} \mathrm{~d} \psi\right)^{2}+(\mathrm{d} \tau+\sigma)^{2} \tag{5.3.51}
\end{align*}
$$

where

$$
\begin{align*}
\sigma & =\frac{\alpha-x}{\alpha} \sin ^{2} \theta \mathrm{~d} \varphi+\frac{\beta-x}{\beta} \cos ^{2} \theta \mathrm{~d} \psi  \tag{5.3.52}\\
\Delta_{x} & =x(\alpha-x)(\beta-x)-\mu, \quad \rho^{2}=\Delta_{\theta}-x  \tag{5.3.53}\\
\Delta_{\theta} & =\alpha \cos ^{2} \theta+\beta \sin ^{2} \theta \tag{5.3.54}
\end{align*}
$$

Here $\alpha, \beta, \mu$ are a priori arbitrary constants. These local metrics are Sasaki-Einstein which can be equivalently stated by saying that the metric cone $\mathrm{d} r^{2}+r^{2} \mathrm{~d} s^{2}$ is Ricciflat and Kähler, or that the four-dimensional part of the metric (suppressing the $\tau$ direction) is a local Kähler-Einstein metric of positive curvature. These local metrics were also found in [192]. The coordinates in (5.3.50) have the following ranges: $0 \leq \theta \leq \pi / 2,0 \leq \varphi \leq 2 \pi, 0 \leq \psi \leq 2 \pi$, and $x_{1} \leq x \leq x_{2}$, where $x_{1}, x_{2}$ are the smallest two roots of the cubic polynomial $\Delta_{x}$. The coordinate $\tau$, which parameterises the orbits of the Reeb Killing vector $\partial / \partial \tau$ is generically non-periodic. In particular, generically the orbits of the Reeb vector field do not close, implying that the SasakiEinstein manifolds are in general irregular.

The metrics are clearly toric, meaning that there is a $U(1)^{3}$ contained in the isometry group. Three commuting Killing vectors are simply given by $\partial / \partial \psi, \partial / \partial \psi, \partial / \partial \tau$. The global properties of the spaces are then conveniently described in terms of those linear combinations of the vector fields that vanish over real codimension two fixed point sets. This will correspond to toric divisors in the Calabi-Yau cone - see e.g. [191]. It is shown in [61] that there are precisely four such vector fields, and in particular these are $\partial / \partial \varphi$ and $\partial / \partial \psi$, vanishing on $\theta=0$ and $\theta=\pi / 2$ respectively, and two additional vectors

$$
\begin{equation*}
\ell_{i}=a_{i} \frac{\partial}{\partial \varphi}+b_{i} \frac{\partial}{\partial \psi}+c_{i} \frac{\partial}{\partial \tau} \quad i=1,2 \tag{5.3.55}
\end{equation*}
$$

which vanish over $x=x_{1}$ and $x=x_{2}$, respectively. The constants are given by [61]

$$
\begin{gather*}
a_{i}=\frac{\alpha c_{i}}{x_{i}-\alpha}, \quad b_{i}=\frac{\beta c_{i}}{x_{i}-\beta}  \tag{5.3.56}\\
c_{i}=\frac{\left(\alpha-x_{i}\right)\left(\beta-x_{i}\right)}{2(\alpha+\beta) x_{i}-\alpha \beta-3 x_{i}^{2}} \tag{5.3.57}
\end{gather*}
$$

In order that the corresponding space is globally well-defined, there must be a linear relation between the four Killing vector fields

$$
\begin{equation*}
a \ell_{1}+b \ell_{2}+c \frac{\partial}{\partial \varphi}+d \frac{\partial}{\partial \psi}=0 \tag{5.3.58}
\end{equation*}
$$

where $(a, b, c, d)$ are relatively prime integers. It is shown in [61] that for appropriately chosen coefficients $a_{i}, b_{i}, c_{i}$ there are then countably infinite families of complete Sasaki-Einstein manifolds.

The fact that there are four Killing vector fields that vanish on codimension 2 submanifolds implies that the image of the Calabi--Yau cone under the moment map for the $T^{3}$ action is a four faceted polyhedral cone in $\mathbb{R}^{3}[191]$. Using the linear relation among the vectors (5.3.58) one can show that the normal vectors to this polyhedral cone satisfy the relation

$$
\begin{equation*}
a v_{1}-c v_{2}+b v_{3}-(a+b-c) v_{4}=0 \tag{5.3.59}
\end{equation*}
$$

where $v_{A}, A=1,2,3,4$ are the primitive vectors in $\mathbb{R}^{3}$ that define the cone. Note that we have listed the vectors according to the order of the facets of the polyhedral cone. As explained in [192], it follows that, for $a, b, c$ relatively prime, the Sasaki-Einstein manifolds arise from the symplectic quotient

$$
\begin{equation*}
\mathbb{C}^{4} / /(a,-c, b,-a-b+c) \tag{5.3.60}
\end{equation*}
$$

which is precisely the gauged linear sigma model considered in the previous subsection.

The volume of the Sasaki Einstein manifolds/orbifolds is given by $[61]$

$$
\begin{equation*}
\operatorname{vol}(Y)=\frac{\pi^{2}}{2 k \alpha \beta}\left(x_{2}-x_{1}\right)\left(\alpha+\beta-x_{1}-x_{2}\right) \Delta \tau \tag{5.3.61}
\end{equation*}
$$

where here $k=\operatorname{gcd}(a, b)$ and

$$
\begin{equation*}
\Delta \tau=\frac{2 \pi k\left|c_{1}\right|}{b} \tag{5.3.62}
\end{equation*}
$$

This can also be written as

$$
\begin{equation*}
\operatorname{vol}(Y)=\frac{\pi^{3}(a+b)^{3}}{8 a b c d} W \tag{5.3.63}
\end{equation*}
$$

where $W$ is a root of certain quartic polynomial given in [61]. This shows that the central charges of the dual conformal field theory will be generically quartic irrational.

In order to compute the R -charges from the metric, we need to know the volumes of the four supersymmetric 3 -submanifolds $\Sigma_{A}$. These volumes were not given in [61] but it is straightforward to compute them. We obtain

$$
\begin{array}{ll}
\operatorname{vol}\left(\Sigma_{1}\right)=\frac{\pi}{k}\left|\frac{c_{1}}{a_{1} b_{1}}\right| \Delta \tau & \operatorname{vol}\left(\Sigma_{2}\right)=\frac{\pi}{k \beta}\left(x_{2}-x_{1}\right) \Delta \tau \\
\operatorname{vol}\left(\Sigma_{3}\right)=\frac{\pi}{k}\left|\frac{c_{2}}{a_{2} b_{2}}\right| \Delta \tau & \operatorname{vol}\left(\Sigma_{4}\right)=\frac{\pi}{k \alpha}\left(x_{2}-x_{1}\right) \Delta \tau \tag{5.3.64}
\end{array}
$$

We can now complete the charge assignments of all the fields in the quiver by giving their R -charges purely from the geometry. The charges of the distinguished fields $Y, U_{1}, Z, U_{2}$ are obtained from the geometry using the formula

$$
\begin{equation*}
R\left[X_{A}\right]=\frac{\pi}{3} \frac{\operatorname{vol}\left(\Sigma_{A}\right)}{\operatorname{vol}(Y)} \tag{5.3.66}
\end{equation*}
$$

while those of the $V_{1}, V_{2}$ fields are simply deduced from (5.3.46). In particular

$$
\begin{equation*}
R\left[V_{1}\right]=R[Z]+R\left[U_{2}\right] \quad R\left[V_{2}\right]=R[Z]+R\left[U_{1}\right] . \tag{5.3.67}
\end{equation*}
$$

It will be convenient to note that the constants $\alpha, \beta, \mu$ appearing in $\Delta_{x}$ are related
to its roots as follows

$$
\begin{align*}
\mu & =x_{1} x_{2} x_{3}  \tag{5.3.68}\\
\alpha+\beta & =x_{1}+x_{2}+x_{3}  \tag{5.3.69}\\
\alpha \beta & =x_{1} x_{2}+x_{1} x_{3}+x_{2} x_{3} \tag{5.3.70}
\end{align*}
$$

where $x_{3}$ is the third root of the cubic, and $x_{3} \geq x_{2} \geq x_{1} \geq 0$. Using the volumes in (5.3.65), we then obtain the following set of R -charges

$$
\begin{array}{ll}
R[Y]=\frac{2}{3 x_{3}}\left(x_{3}-x_{1}\right) & R\left[U_{1}\right]=\frac{2 \alpha}{3 x_{3}} \\
R[Z]=\frac{2}{3 x_{3}}\left(x_{3}-x_{2}\right) & R\left[U_{2}\right]=\frac{2 \beta}{3 x_{3}} \tag{5.3.72}
\end{array}
$$

To obtain explicit expressions, one should now write the constants $x_{i}, \alpha, \beta$ in terms of the integers $a, b, c$. This can be done, using the equations (9) in [61]. We have

$$
\begin{align*}
\frac{x_{1}\left(x_{3}-x_{1}\right)}{x_{2}\left(x_{3}-x_{2}\right)} & =\frac{a}{b}  \tag{5.3.73}\\
\frac{\alpha\left(x_{3}-\alpha\right)}{\beta\left(x_{3}-\beta\right)} & =\frac{c}{d} \tag{5.3.74}
\end{align*}
$$

Notice that $\alpha=\beta$ implies $c=d$, as claimed in [61]. Combining these two equations with (5.3.70), one obtains a complicated system of quartic polynomials, which in principle can be solved. However, we will proceed differently. Our aim is simply to show that the resulting R -charges will match with the $a$-maximization computation in the field theory. Therefore, using the relations above, we can write down a system of
equations involving the R -charges and the integers $a, b, c, d$. We obtain the following:

$$
\begin{align*}
\frac{R[Y](2-3 R[Y])}{R[Z](2-3 R[Z])} & =\frac{a}{b} \\
\frac{R\left[U_{1}\right]\left(2-3 R\left[U_{1}\right]\right)}{R\left[U_{2}\right]\left(2-3 R\left[U_{2}\right]\right)} & =\frac{c}{d} \\
\frac{3}{4}\left(R\left[U_{1}\right] R\left[U_{2}\right]-R[Y] R[Z]\right)+R[Z]+R[Y] & =1 \\
R[Y]+R\left[U_{1}\right]+R[Z]+R\left[U_{2}\right] & =2 . \tag{5.3.75}
\end{align*}
$$

With the aid of a computer program, one can check that the solutions to this system are given in terms of roots of various quartic polynomials involving $a, b, c$. For the case of $L^{a, b, a}$ the polynomials reduce to quadratics and the R -charges can be given in closed form. These in fact match precisely with the values that we will compute later using $a$-maximization, as well as $Z$-minimization. Therefore we won't record them here.

In the general case, instead of giving the charges in terms of unwieldy quartic roots, we can more elegantly show that the system (5.3.75) can be recast into an equivalent form which is obtained from $a$-maximization. In order to do so, we can use the last equation to solve for $R\left[U_{1}\right]$. Expressing the first three equations in terms of $R\left[U_{2}\right]=x, R[Y]=y$ and $R[z]=z$, we have

$$
\begin{align*}
b(2-3 y)+a z(3 z-2) & =0 \\
c(x+y-2)(3 x+3 y-4)-(a+b-c)(x-z)(3 x-3 z-2) & =0  \tag{5.3.76}\\
3 x^{2}-4 y+2(z+2)+x(3 y-3 z-6) & =0
\end{align*}
$$

Interestingly, the third equation does not involve any of the parameters. For later comparison with the results coming from $a$-maximization, it is important to find a way to reduce this system of three coupled quadratic equations in three variables to a standard form. The simplest way of doing so is to 'solve' for one of the variables $x$, $y$ or $z$ and two of the parameters. A particularly simple choice is to solve for $y, a$ and $b$. The simplicity follows from the fact that it is possible to use the third equation to solve for $y$ and the parameters then appear linearly in all the equations. Doing this,
we obtain

$$
\begin{align*}
a & =\frac{c(3 x-2)(3 x(x-z-2)+2(2+z))}{(3 x-4)^{2}(x-z)} \\
b & =\frac{c z(3 z-2)}{(3 x-3 z-2)(x-z)} \\
y & =\frac{-2(2+z)-3 x(x-z-2)}{(3 x-4)} \tag{5.3.77}
\end{align*}
$$

This system of equations is equivalent to the original one, and is the one we will compare with the results of $a$-maximization.

Of course, one could also compute these R -charges using $Z$-minimization [190]. The algebra encountered in tackling the minimization problem is rather involved, but it is straightforward to check agreement of explicit results on a case by case basis.

### 5.4 Superpotential and gauge groups

In the previous sections we have already described how rather generally one can obtain the number of gauge groups, and the field content of a quiver whose vacuum moduli space should reproduce the given toric variety. In particular, we have listed the multiplicities of every field and their complete charge assignments, namely their baryonic, flavor, and R-charges. In the following we go further and predict the form of the superpotential as well as the nature of the gauge groups, that is, the types of nodes appearing in the quivers.

### 5.4.1 The superpotential

First, we recall that in [94] a general formula was derived relating the number of gauge groups $N_{g}$, the number of fields $N_{f}$, and the number of terms in the superpotential $N_{W}$. This follows from applying Euler's formula to a brane tiling that lives on the surface of a 2-torus, and reads

$$
\begin{equation*}
N_{W}=N_{f}-N_{g} . \tag{5.4.78}
\end{equation*}
$$

Using this we find that the number of superpotential terms for $L^{a, b, c}$ is $N_{W}=2 b$. Now we use the fact each term in the superpotential $W$ must be

$$
\begin{equation*}
\cup_{A=1}^{4} \Sigma_{A} \tag{5.4.79}
\end{equation*}
$$

In fact this is just the canonical class of $X$ - a standard result in toric geometry. One can justify the above form as follows. Each term in $W$ is a product of fields, and each field is associated to a union of toric divisors. The superpotential has R -charge 2, and is uncharged under the baryonic and flavor symmetries. This is true, using the results of Section 2 and (5.4.79).

A quick inspection of Table 5.1 then allows us to identify three types of monomials that may appear in the superpotential

$$
\begin{equation*}
W_{q}=\operatorname{Tr} Y U_{1} Z U_{2} \quad W_{c_{1}}=\operatorname{Tr} Y U_{1} V_{1} \quad W_{c_{2}}=\operatorname{Tr} Y U_{2} V_{2} \tag{5.4.80}
\end{equation*}
$$

Furthermore, their number is uniquely fixed by the mutiplicities of the fields, and the fact that $N_{W}=2 b$. The schematic form of the superpotential for a general $L^{a, b, c}$ quiver theory is then

$$
\begin{equation*}
W=2\left[a W_{q}+(b-c) W_{c_{1}}+(c-a) W_{c_{2}}\right] \tag{5.4.81}
\end{equation*}
$$

In the language of dimer models, this is telling us the types of vertices in the brane tilings [94]. In particular, in each fundamental domain of the tiling we must have $2 a$ four-valent vertices, $2(b-c)$ three-valent vertices of type 1 , and $2(c-a)$ three-valent vertices of type 2 .

### 5.4.2 The gauge groups

Finally, we discuss the nature of the $N_{g}=a+b$ gauge groups of the gauge theory, i.e. we determine the types of nodes in the quiver. This information, together with the above, will be used to construct the brane tilings. First, we will identify the allowed types of nodes, and then we will determine the number of times each node appears
in the quiver.
The allowed types of nodes can be deduced by requiring that at any given node

1. the total baryonic and flavor charge is zero: $\quad \sum_{i \in \text { node }} U(1)_{i}=0$
2. the beta function vanishes: $\quad \sum_{i \in \text { node }}\left(R_{i}-1\right)+2=0$
3. there are an even number of legs.

These requirements are physically rather obvious. The first property is satisfied if we construct a node out of products (and powers) of the building blocks of the superpotential (5.4.80). Moreover, using (5.2.31), this also guarantees that the total R -charge at the node is even.

Imposing these three requirements turns out to be rather restrictive, and we obtain four different types of nodes that we list below:

$$
\begin{array}{ll}
A: U_{1} Y V_{1} \cdot U_{1} Y V_{1} & B: V_{2} Y V_{1} \cdot U_{1} Y U_{2} \\
D: U_{2} Y V_{2} \cdot U_{2} Y V_{2} & C: U_{1} Y U_{2} Z \tag{5.4.83}
\end{array}
$$

Next, we determine the number of times each node appears in the quiver. Denote these numbers $n_{A}, n_{B}, n_{D}, 2 n_{C}$ respectively. Taking into account the multiplicities of all the fields imposes six linear relations. However, it turns out that these do not uniquely fix the number of different nodes. We have

$$
\begin{align*}
n_{C} & =a  \tag{5.4.84}\\
n_{B}+2 n_{A} & =2(b-c)  \tag{5.4.85}\\
n_{B}+2 n_{D} & =2(c-a) . \tag{5.4.86}
\end{align*}
$$

Although the number of fields and schematic form of superpotential terms are fixed by the geometry, the number of $A, B$ and $D$ nodes are not. We can then have
different types of quivers that are nevert heless described by the same toric singularity. This suggests that the theories with different types of nodes are related by Seiberg dualities. We will show that this is the case in Section 6.

It is interesting to see what happens for the $L^{a . b, a}$ geometries discussed in Section 5.3.1. In this case $c-a=0$ (the case that $b-c=0$ is symmetric with this) and the theory has some peculiar properties. Recall that this corresponds to a linear sigma model with charges $(a,-a, b,-b)$. These theories have no $V_{2}$ fields, while the $b-a$ $V_{1}$ fields have zero baryonic charge, and must therefore be adjoints. Moreover, from (5.4.86), we see that $n_{B}=n_{D}=0$, so that there aren't any $B$ and $D$ type nodes, while there are $b-a A$-type nodes. In terms of tilings, these theories are then just constructed out of $C$-type quadrilaterals and $A$-type hexagons. We will consider in detail these models in Section 5.6.2.

Finally, we note that the general conclusions derived for $L^{a, b, c}$ quivers in Sections 5.4.1 and 5.4.2 are based on the underlying assumption that we are dealing with a generic theory (i.e. one in which the R-charges of different types of fields are not degenerate). It is always possible to find at least one generic phase for a given $L^{a, b, c}$, and thus the results discussed so far apply. Non-generic phases can be generated by Seiberg duality transformations. In these cases, new types of superpotential interactions and quiver nodes may emerge, as well as new types of fields. This was for instance the case for the toric phases of the $Y^{p, q}$ theories [29].

### 5.5 R-charges from $a$-maximization

A remarkable check of the AdS/CFT correspondence consists of matching the gauge theory computation of R -charges and central charge with the corresponding calculations of volumes of the dual Sasaki-Einstein manifold and supersymmetric submanifolds on the gravity side. This is perhaps the most convincing evidence that the dual field theory is the correct one. Since explicit expressions for the Sasaki-Einstein metrics are available, it is natural to attempt such a check. Actually, the volumes of toric manifolds and supersymmetric submanifolds thereof can also be computed from
the toric data [190], without using a metric. This gives a third independent check that everything is indeed consistent.

Here we will calculate the R -charges and central charge $a$ for an arbitrary $L^{a, b, c}$ quiver gauge theory using a-maximization. From the field theory point of view, initially, there are six different R-charges, corresponding to the six types of bifundamental fields $U_{1}, U_{2}, V_{1}, V_{2}, Y$ and $Z$. Since the field theories are superconformal, these R -charges are such that the beta functions for the gauge and superpotential couplings vanish. Using the constraints (5.4.80) and (5.4.83) it is possible to see that these conditions always leave us with a three-dimensional space of possible R -charges. This is in precise agreement with the fact that the non-R abelian global symmetry is $U(1)^{3} \simeq U(1)_{F}^{2} \times U(1)_{B}$. It is convenient to adopt the parametrization of R -charges of Section 5.3.3:

$$
\begin{array}{ll}
R\left[U_{1}\right]=x-z & R\left[U_{2}\right]=2-x-y \\
R\left[V_{1}\right]=2-x-y+z & R\left[V_{2}\right]=x  \tag{5.5.87}\\
R[Y]=y & R[Z]=z
\end{array}
$$

This guarantees that all beta functions vanish. Using the multiplicities in Table 5.1, we can check that $\operatorname{tr} R(x, y, z)=0$. This is expected, since this trace is proportional to the sum of all the beta functions. In addition, the trial $a$ central charge can be written as

$$
\begin{align*}
\operatorname{tr} R^{3}(x, y, z) & =\frac{1}{3}[a(9(2-x)(x-z) z-2)+b(9 x y(2-x-y)-2) \\
& +9(b-c) y z(2 x+y-z-2)] \tag{5.5.88}
\end{align*}
$$

The R -charges are determined by maximising (5.5.88) with respect to $x, y$ and $z$.

This corresponds to the following equations

$$
\begin{align*}
& \partial_{x} \operatorname{tr} R^{3}(x, y, z)=0=-3 b y(2 x+y-2 z-2)+3 z(a(2-2 x+z-2 c y)) \\
& \partial_{y} \operatorname{tr} R^{3}(x, y, z)=0=-3 b(x-z)(x+2 y-z-2)-3 c z(2 x+2 y-z-2) . \\
& \partial_{z} \operatorname{tr} R^{3}(x, y, z)=0=-3 a(x-2)(x-2 z)+3(b-c) y(2 x+y-2 z-2) \tag{5.5.89}
\end{align*}
$$

It is straightforward to show that this system of equations is equivalent to (5.3.77). In fact, proceeding as in Section 5.3.3, we reduce (5.5.89) to an equivalent system by 'solving' for $y, a$ and $b$. In this case, there are three solutions, although only one of them does not produce zero R -charges for some of the fields, and indeed corresponds to the local maximum of (5.5.88). This solution corresponds to the following system of equations

$$
\begin{align*}
a & =\frac{c(3 x-2)(3 x(x-z-2)+2(2+z))}{(3 x-4)^{2}(x-z)} \\
b & =\frac{c z(3 z-2)}{(3 x-3 z-2)(x-z)} \\
y & =\frac{-2(2+z)-3 x(x-z-2)}{(3 x-4)} \tag{5.5.90}
\end{align*}
$$

which is identical to (5.3.77). We conclude that, for the entire $L^{a, b, c}$ family, the gauge theory computation of R -charges and central charge using a-maximization agrees precisely with the values determined using geometric methods on the gravity side of the AdS/CFT correspondence.

### 5.6 Constructing the gauge theories using brane tilings

In Sections 5.3 and 5.4 we derived detailed information regarding the gauge theory on D3-branes transverse to the cone over an arbitrary $L^{a, b, c}$ space. Table 5.1 gives the types of fields along with their multiplicities and global $U(1)$ charges, (5.4.80) presents the possible superpotential interactions and (5.4.83) and (5.4.86) give the
types of nodes in the quiver along with some constraints on their multiplicities.
This information is sufficient for constructing the corresponding gauge theories. We have used it in Section 5.5 to prove perfect agreement between the geometric and gauge theory computations of R-charges and central charges. Nevertheless, it is usually a formidable task to combine all these pieces of information to generate the gauge theory. In this section we introduce a simple set of rules for the construction of the gauge theories for the $L^{a, b, c}$ geometries. In particular, our goal is to find a simple procedure in the spirit of the 'impurity idea' of $[26,29]$.

Our approach uses the concept of a brane tiling, which was introduced in [94], following the discovery of the connection between toric geometry and dimer models of [128]. Brane tilings encode both the quiver diagram and the superpotential of gauge theories on D -branes probing toric singularities. Because of this simplicity, they provide the most suitable language for describing complicated gauge theories associated with toric geometries. We refer the reader to [94] for a detailed explanation of brane tilings and their relation to dimer models.

All the conditions of Sections 5.3 and 5.4 can be encoded in the properties of four elementary building blocks. These blocks are shown in Figure 5-5. We denote them $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D , following the corresponding labeling of gauge groups in (5.4.83). It is important to note that a C hexagon contains two nodes of type C .

(A)

(B)

(C)

(D)

Figure 5-5: The four building blocks for the construction of brane tilings for $L^{a, b, c}$.
Every edge in the elementary hexagons is associated with a particular type of field. These edge labels fully determine the way in which hexagons can be glued together along their edges to form a periodic tiling. The quiver diagram and superpotential can then be read off from the resulting tiling using the results of [94]. The elementary hexagons automatically incorporate the three superpotential interactions of (5.4.80). The number of $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D hexagons is $n_{A}, n_{B}, n_{C}$ and $n_{D}$, respectively. Taking
their values as given by (5.4.86), the multiplicities in Table 5.1 are reproduced.
Following the discussion in Section 5.4, the number of A, B, C and D hexagons is not fixed for a given $L^{a, b, c}$ geometry. There is a one parameter space of solutions to (5.4.86), which we can take to be indexed by $n_{D}$. It is possible to go from one solution of (5.4.86) to another one by decreasing the number of B hexagons by two and introducing one A and one D, i.e. $\left(n_{A}, n_{B}, n_{C}, n_{D}\right) \rightarrow\left(n_{A}+1, n_{B}-2, n_{C}, n_{D}+1\right)$. We show in the next section that this freedom in the number of each type of hexagon is associated with Seiberg duality.

### 5.6.1 Seiberg duality and transformations of the tiling

We now study Seiberg duality [211] transformations that produce 'toric quivers' ${ }^{5}$. We can go from one toric quiver to another one by applying Seiberg duality to the so-called self-dual nodes. These are nodes for which the number of flavors is twice the number of colours, thus ensuring that the rank of the dual gauge group does not change after Seiberg duality. Such nodes are represented by squares in the brane tiling [94]. Hence, for $L^{a, b, c}$ theories, we only have to consider dualizing C nodes. Seiberg duality on a self-dual node corresponds to a local transformation of the brane tiling [94]. This is important, since it means that we can focus on the subtilings surrounding the nodes of interest in order to analyse the possible behavior of the tiling.

We will focus on cases in which the tiling that results from dualizing a self-dual node can also be described in terms of $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D hexagons. There are some cases in which Seiberg duality generates tilings that are not constructed using the elementary building blocks of Figure 5-5. In these cases, the assumption of the six types of fields being non-degenerate does not hold. We present an example of this non-generic case below, corresponding to $L^{2,6,3}$.

Leaving aside non-generic cases, we see that we only need to consider two pos-

[^16]sibilities. They are presented in Figures 5-6 and 5-7. Figure 5-6 shows the case in which dualization of the central square does not change the number of hexagons of each type in the tiling. The new tiling is identical to the original one up to a shift. Figure 5-7 shows a situation in which dualization of the central square removes two type B hexagons and adds an A and a D .


Figure 5-6: Seiberg duality on a self-dual node that does not change the hexagon content.


Figure 5-7: Seiberg duality on a self-dual node under which $\left(n_{A}, n_{B}, n_{C}, n_{D}\right) \rightarrow$ $\left(n_{A}+1, n_{B}-2, n_{C}, n_{D}+1\right)$.

The operations discussed above leave the labels of the edges on the boundary of the sub-tiling invariant ${ }^{6}$. Hence, the types of hexagons outside the sub-tilings are not modified. The above discussion answers the question of how to interpret the different solutions of (5.4.86): they just describe Seiberg dual theories.

### 5.6.2 Explicit examples

Having presented the rules for constructing tilings for a given $L^{a, b, c}$, we now illustrate their application with several examples. We first consider $L^{2,6,3}$, which is interesting since it has eight gauge groups and involves A, B and C elementary hexagons. We also discuss how its tiling is transformed under the action of Seiberg duality. We then

[^17]present tilings for $L^{2,6,4}$, showing how D hexagons are generated by Seiberg duality. Finally, we classify all sub-families whose brane tilings can be constructed using only two types of elementary hexagons. These theories are particularly simple and it is straightforward to match the geometric and gauge theory computations of R-charges and central charges explicitly. We analyse the $L^{a, b, a}$ sub-family in detail, and present other interesting examples in the appendix.

## Gauge theory for $L^{2,6,3}$

Let us construct the brane tiling for $L^{2,6,3}$. We consider the $\left(n_{A}, n_{B}, n_{C}, n_{D}\right)=$ $(2,2,2,0)$ solution to (5.4.86). Hence, we have two A, two B and two C hexagons. Using the gluing rules given by the edge labeling in Figure 5-5, it is straightforward to construct the brane tiling shown in Figure 5-8.


Figure 5-8: Brane tiling for $L^{2,6,3}$.

From the tiling we determine the quiver diagram shown in Figure 5-9. The multiplicities of each type of field are in agreement with the values in Table 2. In addition, we can also read off the corresponding superpotential


Figure 5-9: Quiver diagram for $L^{2,6,3}$.

$$
\begin{align*}
W & =Y_{31} U_{12}^{(1)} V_{23}^{(1)}-Y_{42} V_{23}^{(1)} U_{34}^{(1)}+Y_{42} V_{21}^{(2)} U_{12}^{(2)}+Y_{85} U_{53}^{(1)} V_{38}^{(1)} \\
& +Y_{17} U_{78}^{(1)} V_{81}^{(1)}-Y_{63} V_{38}^{(1)} U_{86}^{(1)}-Y_{28} V_{81}^{(1)} U_{12}^{(1)}-Y_{17} U_{72}^{(2)} V_{21}^{(2)} \\
& +Z_{45} U_{56}^{(2)} Y_{63} U_{34}^{(1)}-Z_{45} U_{53}^{(1)} Y_{31} U_{14}^{(2)}-Z_{67} U_{78}^{(1)} Y_{85} U_{56}^{(2)}+Z_{67} U_{72}^{(2)} Y_{28} U_{86}^{(1)} \tag{5.6.91}
\end{align*}
$$

where for simplicity we have indicated the type of $U$ and $V$ fields with a superscript and have used subscripts for the gauge groups under which the bifundamental fields are charged.

Having the brane tiling for a gauge theory at hand makes the derivation of its moduli space straightforward. The corresponding toric diagram is determined from the characteristic polynomial of the Kasteleyn matrix of the tiling [128, 94]. In this case, we obtain the toric diagram shown in Figure 5-10. This is an additional check of our construction.

Let us now consider how Seiberg duality on self-dual nodes acts on this tiling. Dualization of nodes 4 or 7 corresponds to the situation in Figure 5-6. The resulting tiling is identical to the original one up to an upward or downward shift, respectively. The situation is different when we dualize node 5 or 6 . In these cases, Seiberg duality 'splits apart' the two squares corresponding to the C nodes forming C hexagons.


Figure 5-10: Toric diagram for $L^{2,6,3}$ determined using the characteristic polynomial of the Kasteleyn matrix for the tiling in Figure 5-8.

Figure $5-11$ shows the tiling after dualizing node 5 . This tiling seems to violate the classification of possible gauge groups given in Section 5.4.80. In particular, some of the hexagons would have at least one edge corresponding to a Z field. As we discussed in Section 5.4.2, this is not a contradiction, but just indicates that we are in a non-generic situation in which some of the six types of fields are degenerate.


Figure 5-11: Brane tiling for a Seiberg dual phase of $L^{2,6,3}$.

## Generating D hexagons by Seiberg duality: $L^{2,6,4}$

We now construct brane tilings for $L^{2,6,4}$. This geometry is actually a $\mathbb{Z}_{2}$ orbifold of $L^{1,3,2}$. This example illustrates how $D$ hexagons are generated by Seiberg duality. We
start with the $\left(n_{A}, n_{B}, n_{C}, n_{D}\right)=(0,4,2,0)$ solution to (5.4.86), whose corresponding tiling is shown in Figure 5-12.


Figure 5-12: Brane tiling for $L^{2,6,4}$.

We see that all self-dual nodes are of the form presented in Figure 5-7. Seiberg duality on node 4 leads to a tiling with $\left(n_{A}, n_{B}, n_{C}, n_{D}\right)=(1,2,2,1)$, which we show in Figure 5-13.


Figure 5-13: Brane tiling for $L^{2,6,4}$.

## The $L^{a, b, a}$ sub-family

It is possible to use brane tilings to identify infinite sub-families of the $L^{a, b, c}$ theories whose study is considerably simpler than the generic case. In particular, classifying the geometries whose corresponding tilings can be constructed using only two different types of hexagons is straightforward. We now proceed with such a classification.

Let us first consider those models that do not involve C type hexagons. These tilings consist entirely of 'pure' hexagons and thus correspond to orbifolds [128, 94]. We have already discussed them in Section 5.3, where we mentioned the case in which $a$, and thus $n_{C}$, is equal to zero. The orbifold action is determined by the choice of a fundamental cell (equivalently, by the choice of labeling of faces in the tiling).

We now focus on theories for which one of the two types of hexagons is of type C. There are only three possibilities of this form:

$$
\begin{array}{lcc} 
& \text { Hexagon types } & \text { Sub-family } \\
n_{B}=n_{D}=0 & \mathrm{~A} \text { and C } & L^{a, b . a} \\
n_{A}=n_{D}=0 & \mathrm{~B} \text { and C } & L^{a, b, \frac{a+b}{2}}=Y^{\frac{a+b}{2}, \frac{a-b}{2}} \\
n_{A}=n_{B}=0 & \mathrm{C} \text { and D } & L^{a, b . b}
\end{array}
$$

It is interesting to see that the $Y^{p, q}$ theories emerge naturally from this classification of simple models. In addition to orbifolds and $Y^{p, q}$ 's, the only new family is that of $L^{a, b, a}$. The $L^{a, b, b}$ family is equivalent to the latter by a trivial reordering of the GLSM charges, which in the gauge theory exchanges $U_{1} \leftrightarrow U_{2}$ and $V_{1} \leftrightarrow V_{2}$.

Let us study the gauge theories for the $L^{a, b, a}$ manifolds. These theories were first studied in [224] using Type IIA configurations of relatively rotated NS5-branes and D4-branes. The simplest example of this family is the SPP theory [196], which in our notation is $L^{1,2,1}$, and has GLSM charges $(1,-1,2,-2)$. The brane tiling for this theory was constructed in [94] and indeed uses one $A$ and one $C$ building block. We will see that it is possible to construct the entire family of gauge theories. We have already shown in Section 5.5 that the computation of R -charges and central charge using a-maximization agrees with the geometric calculation for an arbitrary $L^{a, b, c}$. We now compute these values explicitly for this sub-family and show agreement with the results derived using the metric [61] and the toric diagram [190]. These types of checks have already been performed for another infinite sub-family of the $L^{a, b, c}$ geometries, namely the $Y^{p, q}$ manifolds, in [26] and [190].

Let us first compute the volume of $L^{a, b, a}$ from the metric. The quartic equation in [61] from which the value of $W$ entering (5.3.63) is determined becomes

$$
\begin{equation*}
W^{2}\left(\frac{1024 a^{2}(a-b)^{2} b^{2}}{(a+b)^{6}}+\frac{64(2 a-b)(2 b-a) W}{(a+b)^{2}}-27 W^{2}\right)=0 \tag{5.6.93}
\end{equation*}
$$

Taking the positive solution to this equation, we obtain

$$
\begin{equation*}
\operatorname{vol}\left(L^{a, b, a}\right)=\frac{4 \pi^{3}}{27 a^{2} b^{2}}\left[(2 b-a)(2 a-b)(a+b)+2\left(a^{2}-a b+b^{2}\right)^{3 / 2}\right] \tag{5.6.94}
\end{equation*}
$$

There is an alternative geometric approach to computing the volume of $L^{a, b, a}$ which uses the toric diagram instead of the metric: $Z$-minimization. This method for calculating the volume of the base of a toric cone from its toric diagram was introduced in [190]. For 3-complex dimensional cones, it corresponds to the minimization of a two variable function $Z[y, t]$. The toric diagram for $L^{a, b, a}$, as we presented in Section 5.3.1, has vertices

$$
\begin{equation*}
[0,0] \quad[1,0] \quad[1, b] \quad[0, a] \tag{5.6.95}
\end{equation*}
$$

We then have

$$
\begin{equation*}
Z[y, t]=3 \frac{y(b-a)+3 a}{t(y-3) y(t-y(b-a)-3 a)} . \tag{5.6.96}
\end{equation*}
$$

The values of $t$ and $y$ that minimize $Z[y, t]$ are

$$
\begin{equation*}
t_{\min }=\frac{1}{2}(a+b+w) \quad y_{\min }=\frac{2 a-b-w}{a-b} \tag{5.6.97}
\end{equation*}
$$

where

$$
\begin{equation*}
w=\sqrt{a^{2}-a b+b^{2}} \tag{5.6.98}
\end{equation*}
$$

Computing $\operatorname{vol}(Y)=\pi^{3} Z_{\min } / 3$, we recover (5.6.94), which was determined using the metric. We now show how this result is reproduced by a gauge theory computation. The unique solution to (5.4.86) for the case of $L^{a, b, a}$ is $\left(n_{A}, n_{B}, n_{C}, n_{D}\right)=(b-a, 0, a, 0)$, so the brane tiling consists of $(b-a) \mathrm{A}$ and $a \mathrm{C}$ hexagons. This tiling is shown in Figure 5-14. First, we note that these theories are non-chiral. Figure 5-15 shows their quiver diagram.

Their superpotential can be easily read from the tiling in Figure 5-14. These models do not have $V_{2}$ fields. Nevertheless, the parametrization of R-charges given in (5.5.87) is applicable to this case. Using it, we have


Figure 5-14: Brane tiling for $L^{a, b, a}$.


Figure 5-15: Quiver diagram for $L^{a, b, a}$. It consists of $2 a \mathrm{C}$ nodes and ( $b-a$ ) A nodes. The last node is connected to the first one by a bidirectional arrow.

$$
\begin{equation*}
\operatorname{tr} R^{3}(x, y, z)=\frac{1}{3}[b(9 y(z-x)(x+y-z-2)-2)+a(9(2-x-y)(x+y-z) z-2)] . \tag{5.6.99}
\end{equation*}
$$

Maximising (5.6.99), we obtain the R -charges

$$
\begin{array}{ll}
R\left[U_{1}\right]=\frac{1}{3} \frac{b-2 a+w}{b-a} & R\left[U_{2}\right]=\frac{1}{3} \frac{2 b-a-w}{b-a} \\
R\left[V_{1}\right]=\frac{2}{3} \frac{2 b-a-w}{b-a} & R[Y]=\frac{1}{3} \frac{b-2 a+w}{b-a} . \\
R[Z]=\frac{1}{3} \frac{2 b-a-w}{b-a} &
\end{array}
$$

The central charge $a$ is then

$$
\begin{equation*}
a\left(L^{a b a}\right)=\frac{27 a^{2} b^{2}}{16}-\left[(2 b-a)(2 a-b)(a+b)+2\left(a^{2}-a b+b^{2}\right)^{3 / 2}\right]^{-1} \tag{5.6.101}
\end{equation*}
$$

which reproduces (5.6.94) on using $a=\pi^{3} / 4 \operatorname{vol}(Y)$.

### 5.7 Conclusions

The main result of this chapter is the development of a combination of techniques which allow one to extract the data defining a (superconformal) quiver gauge theory purely from toric and Sasaki-Einstein geometry. We have shown that the brane tiling method provides a rather powerful organizing principle for these theories, which generically have very intricate quivers. We emphasise that, in the spirit of [190], our results do not rely on knowledge of explicit metrics, and are therefore applicable in principle to an arbitrary toric singularity. It is nevertheless interesting, for a variety of reasons, to know the corresponding Sasaki-Einstein metrics in explicit form.

For illustrating these general principles, we have discussed an infinite family of toric singularities denoted $L^{a, b, c}$. These generalise the $Y^{p, q}$ family, which have been the subject of much attention; the corresponding $L^{a, b, c}$ Sasaki-Einstein metrics have been recently constructed in [61] (see also [192]). The main input into constructing these theories came from the geometrical data, which strongly restricts the allowed gauge theories. Subsequently, the brane tiling technique provides a very elegant way of organizing the data of the gauge theory. In constrast to the quivers and superpotentials, which are very complicated to write down in general, it is comparatively easy to describe the building blocks of the brane tiling associated to any given $L^{a, b, c}$. We have computed the exact R -charges of the entire family using three different methods and found perfect agreement of the results, thus confirming the validity of our construction.

### 5.8 Appendix: More examples

In this appendix, we include additional examples that illustrate the simplicity of our approach to the construction of brane tilings and gauge theories.

### 5.8.1 Brane tiling and quiver for $L^{1,5,2}$

Figure 5-16 shows a brane tiling for $L^{1,5,2}$ with $\left(n_{A}, n_{B}, n_{C}, n_{D}\right)=(2,2,1,0)$.


Figure 5-16: Brane tiling for $L^{1,5,2}$.
The corresponding quiver diagram is shown Figure 5-17.


Figure 5-17: Quiver diagram for $L^{1,5,2}$.
The toric diagram computed from the tiling according to the prescription in [128, 94] is presented in Figure 5-18.


Figure 5-18: Toric diagram for $L^{1,5,2}$.

### 5.8.2 Brane tiling and quiver for $L^{1,7,3}$

The brane tiling for $L^{1,7,3}$ corresponding to $\left(n_{A}, n_{B}, n_{C}, n_{D}\right)=(2,4,1,0)$ is presented in Figure 5-19.


Figure 5-19: Brane tiling for $L^{1,7,3}$.

Figure 5-20 shows the quiver diagram for this phase.
The toric diagram is given in Figure 5-21.


Figure 5-20: Quiver diagram for $L^{1,7,3}$.


Figure 5-21: Toric diagram for $L^{1,7,3}$.

## Chapter 6

## Fast Inverse Algorithm

In this chapter, we describe a simple "Fast Inverse" algorithm that computes brane tilings for a given generic toric singular Calabi-Yau threefold. This therefore gives AdS/CFT dual quiver gauge theories for D3-branes probing the given noncompact manifold. We study the parameter space of a-maximization; this study is made possible by identifying the R-charges of bifundamental fields as angles in the brane tiling. We also study Seiberg duality from a new perspective.

### 6.1 Superconformal fixed point and R -charges

In this section we are going to find a new connection between R -charges of fundamental fields and some basic properties of the tiling configuration.

The quiver gauge theories described by the brane tilings are expected to flow at low energies to a superconformal fixed point. The global symmetry group of the theory contains the $U(1) \mathrm{R}$-symmetry. The Sasaki-Einstein manifolds have a canonical Killing vector field called the Reeb vector. This is dual to the R-symmetry of the quiver gauge theory.

It has been shown in [154] that the superconformal R-charges can be determined by a-maximization: the R -symmetry is the $U(1)$ symmetry, which maximizes the combination of 't Hooft anomalies $a(R) \equiv\left(9 \operatorname{Tr} R^{3}-3 \operatorname{Tr} R\right) / 32$. The maximal value of $a$ is then suggested to be the central charge of the superconformal theory (for
details see [ 14,13$]$ ).

The R-charges are related by the AdS/CFT correspondence to volumes of supersymmetric submanifolds in the dual Sasaki-Einstein manifold. Recently, it has been shown [190] that these volumes can be extracted from the toric data of the Calabi-Yau singularity without knowing the metric explicitly. The $\mathrm{R}-\mathrm{charges}$ can be obtained by minimizing a function $Z$ that depends only on the toric data of the singularity and the trial Reeb vector. This method is called the geometric dual to a-maximization.

Let us assign an R -charge to each bifundamental field in the brane tiling. At the IR superconformal fixed point, each term in the superpotential satisfies

$$
\begin{equation*}
\sum_{i \in e d g e s} R_{i}=2 \quad \text { for eannd node } \quad \text { each node } \tag{6.1.1}
\end{equation*}
$$

where the sum is over all edges surrounding a given node. The (numerator of the) NSVZ beta function for each gauge coupling vanishes, which leads to the following equation:

$$
\begin{equation*}
\sum_{i \in \text { edges around face }}\left(1-R_{i}\right)=2 \quad \text { for each face } \tag{6.1.2}
\end{equation*}
$$

where the sum is over all edges surrounding a given face. These constraints will get a nice geometric interpretation in section 6.2.

Let $F$ denote the number of faces, $E$ the number of edges and $V$ the number of vertices in the brane tiling. By summing equation (6.1.1) over the nodes, we get $2 \sum_{\text {edges }} R_{i}=2 V$. Using this and summing equation (6.1.2) over all the faces in the tiling we arrive at the Euler formula for a torus:

$$
\begin{equation*}
F-E+V=0 \tag{6.1.3}
\end{equation*}
$$

This is a non-trivial statement about the quiver theory which was first observed in [129] and derived in [94].

In our case the linear 't Hooft anomaly vanishes [27]: $\operatorname{Tr} R=\sum$ beta functions $=$

0 , so we have to maximize the following function:

$$
\begin{equation*}
a\left(R_{i}\right)=\frac{9}{32} \sum_{i}\left(R_{i}-1\right)^{3} \tag{6.1.4}
\end{equation*}
$$

The computation of a-maximization for a given quiver gauge theory has by now turned into a standard procedure for solving for supersymmetric gauge theories. Furthermore, it serves as good probe for consistency checks on quiver theories. Indeed, while there are many theories for which this procedure leads to nice and impressive results, it turns out that there is a large class of quiver gauge theories for which a straightforward application of a-maximization gives rise to negative or zero R charges. This obviously indicates some sign of inconsistency. Such theories were termed in [19] as having tachyons, in [84] as fractional Seiberg duals and in [144] as mutations. All these examples share the same property of having negative R -charges.

### 6.2 Isoradial embeddings and R-charges

Let us consider again the constraints for the R -charges (section 6.1):

$$
\begin{array}{cc}
\sum_{i \in e d g e s} R_{i}=2 & \text { for each node } \\
2+\sum_{i \in \text { edges around face }}\left(R_{i}-1\right)=0 & \text { for each face } \tag{6.2.6}
\end{array}
$$

After multiplying both equations by $\pi$ and rearranging the second one, we arrive at

$$
\begin{gather*}
\sum_{i \in \text { edges around node }}\left(\pi R_{i}\right)=2 \pi  \tag{6.2.7}\\
\sum_{i \in \text { edges around face }}\left(\pi R_{i}\right)=(\# \text { edges }-2) \pi \tag{6.2.8}
\end{gather*} \quad \text { for each node } \quad \text { each face }
$$

Now, if we think of $\pi R_{i}$ as an angle, then we see that the first equation is just the statement that the angles around a node sum up to $2 \pi$, whereas the second equation tells us that the sum of the internal angles in a polygon is $(\# e d g e s-2) \pi$.


Figure 6-1: Isoradially embedded part of an arbitrary brane tiling (in green).

Where are these angles in the brane tiling? To show this, we need the notion of isoradial embedding [74, 194, 168]. So far the brane tiling was only a graph for us, we could freely move around its nodes without causing self-intersection. The isoradial embedding is an embedding of the tiling graph into the plane, where the nodes of each face are on a circle of unit radius. (The edges of the tiling are straight lines). The square lattice for the conifold provides a trivial example (Figure 4 (i)), where the unit circles are just the circumcircles of the squares in the tiling. The squares are of same size so the circumcircles will have the same radius which can be chosen to be one.


Figure 6-2: (i) Circumcircles around the faces (in black), (ii) and the corresponding rhombus lattice (in red).

To demonstrate a non-trivial example, Figure 6-1 shows a small part of a brane tiling ${ }^{1}$. This tiling graph is isoradially embedded into the plane. This can be seen in

[^18]Figure 6-2 (i), where the black circles are the circumcircles of unit radius of the faces in the tiling. The nodes of the brane tiling are sitting at the intersection points of the circles.

Once we have the tiling isoradially embedded, we can immediately draw the corresponding rhombus lattice ${ }^{2}$ (Figure 6-2 (ii) shows the rhombus lattice in red), which can be obtained by simply connecting the center of the circles with the nodes of the face in the brane tiling. The rhombi (a.k.a. "diamonds" in [170]) in this lattice have edges of unit length. This is guaranteed by the equality of the radii of the circles. We see that by isoradially embedding our original tiling we gain a lattice of rhombi. The bifundamental fields of the quiver theory (i. e. edges in the brane tiling) are in one-to-one correspondence with the rhombi of this rhombus lattice.


Figure 6-3: A rhombus in the lattice. The green line is an edge in the brane tiling, the magenta line is the corresponding bifundamental field in the periodic quiver.

Let us study now a single rhombus that is shown in Figure 6-3. The green bifundamental edge ( $A C$ in Figure 6-3) is just one of the diagonals of the rhombus. If instead of the green lines we draw the flipped magenta ones ( $B D$ in Figure 6-3) into the rhombus lattice, then we obtain the dual graph to the tiling, the periodic quiver (which is also isoradially embedded). We immediately see that on the level of the rhombus lattice, the quiver and the brane tiling are on the same footing.

In the figure, $\theta_{i}$ denotes the $D C B$ and $B A D$ angles in the rhombus. The shape of the rhombus is characterized by this single angle. We are now in the position to
${ }^{2}$ Also known as quad-graph or diamond lattice.
visualize the $R$-charges if we set

$$
\begin{equation*}
\theta_{i} \equiv \pi R_{i} \tag{6.2.9}
\end{equation*}
$$

We see that the condition for vanishing beta function to superpotential terms, equation (6.2.7) says that the angles around a node in the brane tiling sum up to $2 \pi$, whereas the condition for vanishing beta function to gauge groups, equation (6.2.8) is equivalent to the statement that the sum of the internal angles of each face in the tiling is $(\# e d g e s-2) \pi$. This is certainly true for a flat torus.

It is not a priori clear that an arbitrary brane tiling graph can be isoradially embedded into the plane. If the exact R -charges are strictly greater than zero and less than one, then they provide a good embedding of the rhombus lattice, hence an isoradial tiling. If some $R_{i}=0$ (or 1 ), then $\theta_{i}=0$ (or $\pi$ ), that is the corresponding rhombus becomes degenerate.

The results of this section is that we identified the R -charges of the bifundamental fields with certain angles in the brane tiling. For any periodic embedding of the rhombus lattice of the brane tiling into the plane the trial R -charges (defined by the $\theta_{i}$ angles in the rhombi) automatically satisfy the equations (6.2.7) and (6.2.8), and vice versa, the set of exact R -charges of the quiver gauge theory gives a good rhombus lattice and thereby an isoradial embedding of the brane tiling.

Finally, let us transform equation (6.1.4) into the following form using the angles in Figure 6-3:

$$
\begin{equation*}
a=-\frac{9}{32 \pi^{3}} \sum_{i} \alpha_{i}^{3} \tag{6.2.10}
\end{equation*}
$$

Here we used the fact that $\alpha_{i}=\pi-\theta_{i}=\pi\left(1-R_{i}\right)$. The parameter space of the different possible embeddings of the rhombus lattice is nothing, but the manifold over which one has to do a-maximization. This space will be investigated in the next section.

### 6.3 Rhombus loops and zig-zag paths

In the last section we introduced a very special type of embedding of the tiling, the so called isoradial embedding. This has been used to visualize the R -charges of the bifundamental fields. In this section we go further and develop new mathematical concepts that will allow us to study the moduli space of isoradial embeddings that is the parameter space of a-maximization.


Figure 6-4: (i) Rhombus path in the rhombus lattice. (ii) Equivalent zig-zag path in the brane tiling. We will use blue lines to depict rhombus loops schematically. The edges which are crossed by the blue line in (i) are all parallel. Their orientation can be described by an angle, the so-called rhombus loop angle.

The most important new concept that we will continuously use in the present chapter is the notion of the rhombus path (a.k.a. "train track" [170]). A rhombus path is defined in the rhombus lattice as a path on rhombi which "does not turn", i. e. after entering to a rhombus on one edge, we are exiting on the opposite side (see Figure 6-4). We can assume that the rhombus path is extended to its maximal size, which means that in a rhombus lattice on the surface of $\mathbb{T}^{2}$ (or, equivalently, in the periodic rhombus lattice) it is a closed loop, the rhombus loop. The rhombus loops will be of great importance in the Fast Inverse Algorithm in section 6.4.

The rhombus edges we are crossing while going along the rhombus loop are all parallel. Their direction, which can be parametrized by a characteristic angle, the rhombus loop angle ( $\alpha$ and $\beta$ in Figure 6-17). This angle can be changed by tilting the rhombus loop as in Figure 6-5.

In [170] it was shown that: (i) No rhombus path crosses itself (or it is periodic), and (ii) two distinct rhombus paths cross each other at most once. These conditions are
not always true in our case, because we allow the existence of degenerate rhombi. Two-valence nodes also result in collapsing rhombi, they have to be integrated out before drawing the rhombus lattice.


Figure 6-5: Tilting along the horizontal $R$ rhombus loop. The rhombus loop angle $\alpha$ changes during the Dehn-twist. Here we have chosen $\alpha=0$ to be the vertical direction $(\mid)$, hence $\alpha=\pi / 4$ corresponds to the skew edges (/).

If the R-charge of a bifundamental field is one, the corresponding rhombus collapses $\left(\theta_{i}=\pi\right)$. This happens for example in the square-octagon phase of the zeroth Hirzebruch surface. We can also squash the rhombus in the perpendicular direction, if we set the R -charge equal to zero. As opposed to the $R_{i}=1$ situation, this case is not allowed, it leads to the so-called tachyonic quivers.

If we color the edges in the brane tiling corresponding to the rhombus loop (the blue lines in Figure 6-4), we get the so-called zig-zag path [168]. This is a path in the tiling which turns maximally left at a node, then maximally right at the next node, then again left, and so on. An example is presented in Figure 6-4. The first picture shows the rhombus path, the second one is the corresponding zig-zag path in the brane tiling. See Figure 6-15 for another example in SPP. Here the blue zig-zag path is periodic.

The zig-zag paths and the rhombus loops are equivalent, the only difference is that they refer to the same path in different lattices. Henceforth we will use both terms depending on the context. At first, it might be non-trivial to understand why there are exactly two zig-zag paths going through each tiling edge. This is best seen in the rhombus lattice where these two paths are the two "perpendicular" rhombus loops that are crossing the corresponding rhombus.

The zig-zag paths in the tiling are in one-to-one correspondence with zig-zag paths in the periodic quiver. These paths in the quiver are oriented loops hence there are gauge-invariant trace operators that can be constructed by multiplying the
bifundamentals one after the other along the path. Such an operator is called the zig-zag operator.

### 6.3.1 Inconsistent theories

In the previous sections we reviewed the construction of brane tilings, visualized R charges as certain angles in the tiling and introduced the new concept of zig-zag paths. One may now imagine that for any arbitrary bipartite tiling there exists a corresponding quiver theory. Unfortunately, this is not the case and there exist some bipartite graphs which do not give meaningful quiver theories. So far in the literature there was no other restriction on consistent tilings, than bipartiteness. In this chapter we are going to give a simple constraint that has to be satisfied by every consistent brane tiling.

One can construct the $Y_{6}$ Calabi-Yau manifold as a Kähler quotient [175] that is as the (classical) vacuum moduli space of a gauged linear sigma model (GLSM) [232, 71]. The "Fast Forward Algorithm" ([94], see also [128]) computes the toric diagram of the singular Calabi-Yau from the brane tiling. The algorithm also gives the multiplicities of the GLSM fields, these appear in the toric diagram. It is possible that from a given tiling the Fast Forward Algorithm produces a toric diagram, whose area is smaller than what we expect from the number of the corresponding gauge theory. This is a good sign of inconsistency of the theory. Then, typically, a-maximization gives zero R-charges for some of the bifundamental fields ${ }^{3}$. For such theories, we also get GLSM field multiplicities in the corners of the toric diagram (see Figure 6-7).

Partial resolutions of the singularity correspond to turning on Fayet-Iliopoulos terms in the supersymmetric gauge theory side and leads to Higgsing in the quiver gauge theory. The FI terms govern the size of the blow-ups. The effective theory at scales smaller than the expectation value of the Higgsed field can be described by the Higgsed quiver and superpotential [82, 81]. Here we consider the inverse of this process, the so-called un-Higgsing. In the level of brane tiling this can be implemented by adding a new edge to the graph. This edge divides a face into two

[^19]

Figure 6-6: Hirzebruch zero brane tiling.
faces, therefore the number of gauge groups increases by one, the number of bifundamental fields increases by one while the number of terms in the superpotential remain the same. Alas, not all possible un-Higgsings of the theory are consistent, in fact, it is a non-trivial problem to determine the allowed un-Higgsings for a given brane tiling.

To demonstrate consistent and inconsistent un-Higgsing, we consider the Hirzebruch zero $(F 0)$ surface. $F 0$ has two toric phases that are connected by Seiberg duality. The brane tiling for one of the phases is the square lattice. We will study the other phase that is the square-octagon lattice which is depicted in Figure 6-6. We consider two possible un-Higgsings of the theory that are shown in Figure 6-9. The new edge (dashed line) is dividing the original face 4 into two faces $4 \& 5$. The first un-Higgsing (i) leads to an inconsistent theory. By means of the Fast Forward Algorithm we can compute its toric diagram with the multiplicities of the GLSM fields. The results are shown in Figure 6-7. We see that during the un-Higgsing the area of the diagram remained the same, meanwhile an external multiplicity (the 3 in the corner) appeared.


Figure 6-7: (i) Hirzebruch zero toric diagram (ii) un-Higgsed Hirzebruch. The area remains the same, external multiplicities appear.


Figure 6-8: (i) Hirzebruch zero toric diagram (ii) un-Higgsed Hirzebruch. The area increases by $1 / 2$ corresponding to the new face in the brane tiling.

We now consider another un-Higgsing that adds the line with a different orientation (see Figure 6-9 (ii)). This theory is consistent. The corresponding toric diagram is shown in Figure 6-8.


Figure 6-9: (i) Hirzebruch zero inconsistently un-Higgsed. (ii) Consistent unHiggsing.

In the Fast Forward Algorithm it is a priori unclear why these small changes in the tiling lead at one time to a consistent and at another time to an inconsistent theory. Having discussed the main mathematical concepts that we need, we can now understand what causes the inconsistency.

Figure 6-10 shows the rhombus loops ${ }^{4}$ for the two different un-Higgsing of $F 0$. The blue lines are crossing edges which are the edges of the corresponding zig-zag paths. The pictures show the rhombus loops only inside the fundamental cell. For the inconsistent tiling (i) we obtain only three rhombus loops, it does not reproduce the $(p, q)$-legs of the toric diagram which we obtained by the Fast Forward Algorithm

[^20]

Figure 6-10: (i) Inconsistently un-Higgsed Hirzebruch. The rhombus loops are indicated with the blue lines. The zig-zag paths contain the edges that are crossed by the blue paths. The following rhombus loops are obtained: $A:(0,-1) \quad B:(-2,2)$ $C:(2,-1)$. Here $(a, b)$ denotes the homology class of the path.
(ii) Consistently un-Higgsed F0. The rhombus loops reproduce the $(p, q)$-legs of the toric diagram (Figures 6, 7): $A:(0,-1) B:(0,1) C:(-2,1) D:(1,-1) E:(1,0)$.
(Figure 6-7). On the other hand, the zig-zag paths of the consistent tiling (ii) give the legs properly (Figure 6-8).

In the first tiling the edge between face 4 and 5 is at the intersection point of the $B$ loop with itself. The corresponding rhombus in such cases is always degenerate, because all the four edges of the rhombus must be parallel, therefore the first tiling is inconsistent. We can state this in general: Self-intersecting zig-zag paths lead to inconsistent brane tilings.


Figure 6-11: The subgraph connects to the rest of the tiling through its four nodes in the corner. No consistent brane tiling can contain this subgraph, because it results in collapsing rhombi and vanishing R -charges.

This example demonstrated how zig-zag paths can be used to determine whether the tiling is consistent or not. Besides these computations, the rhombus loop tech-
nique enables us to generate simple rules that must be satisfied by any consistent tiling. This might help in the construction of such tilings. An example is presented in Figure 6-11. This subtiling cannot be part of any consistent tiling. It is clear from the corresponding (degenerate) rhombus lattice that there are zero R -charges as the reader may check. The inconsistency can be also seen by performing Seiberg duality on face 1 that creates a face with only two edges.

### 6.3.2 Conjecture of $(p, q)$-legs and rhombus loops

In the previous section we investigated rhombus loops in the rhombus lattice and equivalent zig-zag paths in the brane tiling. We have seen that one can use these paths to decide whether the tiling is a priori consistent or not (i. e. before doing a-maximization). In the followings we make an observation which will enable us to develop the Fast Inverse Algorithm in section 6.4.

To state the conjecture we introduce the notion of $(p, q)$-webs. $(p, q)$-webs were introduced in [7] to study five dimensional gauge theories with 8 supercharges (i. e. $\mathcal{N}=1$ ). The $(p, q)$-web describes a configuration of 5 -branes in Type IIB string theory. These webs might be interpreted as "dual graphs" to toric diagrams as it was noticed in [8]. This observation has been proven in [185]. An example is shown in Figure 6-12. The geometry can be described by a $\mathbb{T}^{2}$ fibration over the web. A circle in the fibre degenerates at each line of the diagram and at the nodes the whole fibre collapses. The lines of the web have rational slopes denoted by two integers: $\left(p_{i}, q_{i}\right)$. These are the $(p, q)$ charges of the branes. A D5-brane is assigned a $(1,0)$ charge whereas the NS5-brane carries $(0,1)$ charge. These two type of branes correspond to horizontal and vertical lines in the web. At each node we have three branes intersecting each other and their charges must sum up to zero:

$$
\begin{equation*}
\sum_{i} p_{i}=0 \quad \sum_{i} q_{i}=0 \tag{6.3.11}
\end{equation*}
$$

In the followings we will use $(p, q)$-legs. These are the external lines in the $(p, q)-$ web and they extend to infinity. Their direction is perpendicular to the corresponding
rdge of the (dual) toric diagram.


Figure 6-12: Toric diagram (i) and ( $p, q$ )-web (ii) for del Pezzo 2. The charges of the external branes are shown. According to the conjecture, these correspond to the homology classes of the rhombus loops in the brane tiling.

An important observation is that for each rhombus loop of homology class $(p, q)$ there is a corresponding $(p, q)-\operatorname{leg}$ in the toric diagram. We will heavily use this in section 6.4. The conjecture has been checked for many consistent brane tilings. Inconsistent tilings tipically do not satisfy this criterion. By reading off the zig-zag paths from the tiling we might arrive at the toric data faster than by the usual Kasteleyn matrix process $[94,128]$. We simply need to draw all the zig-zag paths (each edge has two of them) and from their homology classes the ( $p, q$ )-legs are obtained. These legs uniquely determine the toric diagram of the Calabi-Yau cone.

Another observation ${ }^{5}$ is that we can generate zig-zag paths by means of perfect matchings. A perfect matching is a subgraph of the tiling which contains all the nodes and each node has valence one $[169,168]$. This means that a perfect matching is a set of dimers (edges in the brane tiling) that are separated, i. e. they don't touch each other, furthermore they cover all the nodes. Therefore, we have altogether $V / 2$ dimers in each perfect matching, where $V$ denotes the number of nodes in the tiling. To demonstrate this, we have drawn the periodic perfect matchings for the Suspended Pinch Point (Figure 6-14) whose toric diagram is shown in Figure 6-13.

It can be easily checked by the reader that if we put two perfect matchings $A$ and $B$ on top of each other (this is denoted by $A+B$ ), then we obtain loops and separate edges which we neglect. Let us fix a reference perfect matching $R$. Now for each

[^21]

Figure 6-13: Toric diagram for the SPP. We have drawn the blue $(p, q)$-leg between the nodes $(1,1)$ and $(2,0)$. The zig-zag path corresponding to the leg is shown in Figure 19.
matching $A_{i}$ we can define an integer height function. The loops of $R+A_{i}$ denote the change in the height as in an ordinary map. The height function is a well-defined function on the infinite periodic tiling, but on the 2-torus it has monodromy that is described by two integers: $(s, t)$. These numbers are the change in the height as we go along the two non-trivial cycles of the torus of the brane tiling. Such pairs are assigned to every perfect matching. For SPP these vectors are shown in Figure 6-14. (Here we used the first perfect matching as a reference matching.) These pairs are coordinates of points in the toric diagram, in fact, the toric diagram is the (convex) set of all such points. The change in the reference matching merely translates the toric diagram.

Now if we choose two adjacent points in the toric diagram then there are perfect matchings corresponding to them whose superposition is (experimentally) a zig-zag path. We demonstrate an example for SPP. The two neighboring matchings have $(1,1)$ and $(2,0)$ coordinates in the toric diagram. Their superposition is shown in Figure 6-15. The emerging non-trivial blue cycle (zig-zag path) has homology ( 1,1 ) which precisely corresponds to the blue $(1,1)$ leg in Figure $6-13$ which is sitting between the two adjacent points.

For further informations on perfect matchings and the dimer model the reader should refer to $[94,128,169,168]$.

In a recent paper [98], fractional branes were studied in the context of brane tilings. The fractional brane is a D5-brane wrapped on a 2 -cycle that vanishes at the tip of the cone. Adding $M$ fractional branes changes the rank of the $S U(N)$ gauge groups of the quiver. For deformation branes some of the ranks increase by $M$. One can shade these tiles as shown in Figure 6-16. Zig-zag paths naturally show up as


Figure 6-14: The six periodic perfect matchings of SPP [94]. The green edges are contained in the matching, the dashed lines are the other edges of the tiling. The $(s, t)$ numbers are the corresponding points in the toric diagram (Figure 17).
boundaries of these shaded areas.

### 6.3.3 Parameter space of a-maximization

We have defined the rhombus loop angle that is assigned to a rhombus path. This angle gives the relative orientation of the parallel edges in the path. We have seen that we can tilt the rhombi in a rhombus path by changing its rhombus loop angle (Figure 6-5). In fact, we can parametrize the entire space of different embeddings of the rhombus lattice (i. e. the isoradial embeddings of the brane tiling) by these rhombus loop angles [170]. At the intersection point of two rhombus paths, we find a single rhombus, whose angles $(\theta$ and $\pi-\theta)$ are determined by the difference of the rhombus loop angles of the paths (Figure 6-17), because they fix the orientation of the edges of the rhombus. This angle $\theta$ is proportional to the R -charge of the field sitting in the rhombus as we have seen in section 6.2.


Figure 6-15: The $(1,1)$ and $(2,0)$ perfect matchings on top of each other. We see the emerging (1,1) homology zig-zag loop which corresponds to the blue ( $p, q$ )-leg in Figure 17.

This means that we can parametrize the convex polyhedron space [170] of trial R -charges by the set of rhombus loop angles. The number of such loops is $d$, which is equal to the number of the edges of the toric diagram according to our conjecture in section 6.3.2. One of the rhombus loop angles can be set to zero by a global rotation of the rhombus lattice. This reduces the dimension of the parameter space to $d-1$. In Figure 6-17 this has already been done, because the $\alpha$ angle is zero (the parallel edges in the corresponding rhombus path are horizontal).

Let us see how can we identify the $d-1$ different parameters in the quiver gauge theory: In the superconformal quiver gauge theory the R -symmetry can mix with every anomaly-free global $U(1)$ symmetry that commutes with charge conjugation.

The global baryonic $U(1)$ 's are gauge symmetries in the gravity dual picture. $H_{3}\left(X_{5}, \mathbb{Z}\right)=\mathbb{Z}^{d-3}$ (see [96]), i. e. the number of independent 3 -cycles in the $X_{5}$ Sasaki-Einstein manifold is $d-3$, hence the Kaluza-Klein reduction of the RamondRamond 4-form gives $d-3$ different gauge fields in $A d S_{5}$. These local symmetries correspond in the dual quiver theory to global baryonic $U(1)$ 's.

Tilting the lattice along a rhombus loop means that the R -symmetry is mixing with a certain $U(1)$ charge. The bifundamentals along the loop have +1 and -1 charges alternatingly under this $U(1)$ and all the other fields have zero charges. The baryonic $U(1)$ 's are linear combinations of these charges.

We identified $d-3$ degrees of freedom as the mixing of the R -charge with the baryonic charges. The two remaining charges correspond to the mixing with the two flavor $U(1)$ charges. These are dual to the Abelian part of the isometry group of


Figure 6-16: $P d P_{4}$ model I brane tiling with a $(1,0,1,0,1,0,0) \mathcal{N}=2$ fractional brane [98]. The bounding rhombus loops ( $A$ and $B$ zig-zag paths) are shown in blue.
the Sasaki-Einstein manifold which is mixing with the Reeb vector in the sense of Zminimization (see [190] for details). The corresponding tiltings are roughly speaking Dehn-twists along the two nontrivial $(1,0)$ and $(0,1)$ cycles.

One can compute the number of possibly different R -charges for the quiver theory in the following way. Let us fix two rhombus loops (zig-zag paths). It is clear that whenever they cross one another, they produce a bifundamental field with the same R -charge. This follows from the fact that the rhombus loop angles of the two loops fully determine the orientation of the rhombus edges, i. e. they fix the R-charge of the field. Therefore we can get different charges only from different rhombus loop intersection points. We can count the number of different possible R -charges. Out of the $d$ loops we are choosing two in all possible ways:

$$
\begin{equation*}
\binom{d}{2}=\frac{d(d-1)}{2} \tag{6.3.12}
\end{equation*}
$$

which gives the maximum possible number of different R -charges of the quiver theory.


Figure 6-17: Assigning angles $(\theta)$ to the rhombus loops. The figure shows two intersecting blue rhombus paths. There is a single rhombus and a green bifundamental edge at the intersection of these paths. This bifundamental has an R-charge that is proportional to the angle $\theta$ of the rhombus. This angle is just the difference of the rhombus loop angles $\alpha$ and $\beta$ assigned to the two rhombus paths: $R \pi=\theta=|\alpha-\beta|$ (or $\pi-|\alpha-\beta|$ depending on the orientation).

### 6.4 Fast Inverse Algorithm

The above discussed techniques based on the isoradial embeddings, rhombus loops and zig-zag paths allow us to develop the Fast Inverse Algorithm, which constructs the brane tiling from arbitrary toric diagrams. The brane tiling encodes the quiver (dual graph), the superpotential data (nodes), hence uniquely describes the quiver gauge theory. Therefore, by means of the Fast Inverse Algorithm we are able to compute an AdS/CFT dual to any toric singularity. (The algorithm is somewhat complicated by the fact that the resulting theory is highly non-unique. This phenomenon will be investigated in section 6.5.)

In the following, we describe the algorithm by presenting examples.

### 6.4.1 $\quad \mathcal{C}^{3} \quad(\mathcal{N}=4)$

In Figure 6-18 the toric diagram of the flat $\mathcal{C}^{3}$ Calabi-Yau manifold is shown. The polygon has three edges, hence three ( $p, q$ )-legs (the blue arrows) with homology classes $(-1,0),(0,-1)$ and $(1,1)$. These correspond to the three rhombus loops in
the rhombus lattice. or to the zig-zag paths in the tiling.


Figure 6-18: $\mathcal{C}^{3}$ toric diagram

We now draw these $(p, q)$ cycles in the fundamental cell (see the blue lines in Figure 6-19) this is the rhombus loop diagram. In the language of the rhombus lattice, at each intersection point we have a single rhombus, which is shown in red. Each rhombus comes with a single bifundamental edge in the brane tiling, these edges are shown in green. To obtain the rhombus lattice, we have to glue these rhombi together (in a periodic fashion), so that along the blue lines we get rhombus paths (Figure 6-20). Once we have the (red) rhombus lattice it is trivial to obtain the (green) brane tiling which encodes the quiver gauge theory.


Figure 6-19: Rhombus loop diagram of $\mathcal{C}^{3}$. The blue rhombus loops are the D6branes. At the intersection points we get massless fields. The dark faces are terms in the superpotential, the light faces are the gauge groups. These correspond respectively to nodes and faces in the brane tiling. The rhombi are shown in red, the brane tiling edges are green.

We shaded some of the faces in the rhombus loop diagram. From the algorithm it is clear that these correspond to the (black or white) nodes, whereas the light faces correspond to the faces in the brane tiling (see also Figure 6-26 where the green brane
tiling is drawn directly on top of the rhombus loop diagram of $L^{131}$ ). We see that the rhombus loop diagram treats the gauge groups and the terms in the superpotential on equal footing.


Figure 6-20: From the rhombi to the brane tiling. We glue the rhombi together that arise at the intersections of rhombus loops (Figure 27). We glue the edges that are connected by the rhombus loops. Each rhombus has a green tiling edge in it, from which we obtain the entire (hexagonal) brane tiling.

For this simple example, we have only three rhombi, i. e. three fields, which turn out to be adjoints, because we have only one gauge group in the tiling. In Figure 6-20 we recover the hexagonal lattice of $\mathcal{N}=4$.

### 6.4.2 Conifold

We now turn to the conifold and will see how we can reproduce the well-known square lattice brane tiling for this theory (Figure 4). The toric diagram (Figure (i-21) has four legs, these cycles can be seen in the rhombus loop diagram in Figure 6-22.


Figure 6-21: Conifold toric diagram


Figure 6-22: Conifold rhombus loops and brane tiling

We can actually skip the rhombus lattice step and draw the brane tiling immediately in the rhombus loop diagram. The emerging green square tiling is better seen in Figure 6-23 where we have drawn a $2 \times 2$ block of adjacent fundamental cells. The square lattice tiling reproduces the superpotential of $[174,196]$.

Refining the integer lattice of the toric diagram means orbifoldizing the singular Calabi-Yau manifold. The resulting toric diagram has more $(p, q)$-legs as seen in Figure 6-24. Clearly, we can realize orbifolding by increasing the size of the fundamental cell of the rhombus loop diagram to $n \times m$ times the size of the original cell. This


Figure 6-23: Four fundamental cells of the conifold rhombus loop diagram. If we consider these cells as one big fundamental cell then we gain the rhombus loop diagram of the $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold of the conifold.
means orbifolding the space by $\mathbb{Z}_{n} \times \mathbb{Z}_{m}$. The action is generated by

$$
\begin{array}{ll}
\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(\lambda z_{1}, z_{2}, \lambda^{-1} z_{3}\right), & \lambda^{m}=1 \\
\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(z_{1}, \omega z_{2}, \omega^{-1} z_{3}\right), & \omega^{n}=1 \tag{6.4.14}
\end{array}
$$

Multiplying the unit cell of the rhombus loop diagram is the same as increasing the size of the fundamental cell in the brane tiling therefore it justifies the observations made in [128].


Figure 6-24: $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold of the conifold.

### 6.4.3 $\quad L^{131}$

As a more complex example, we generate brane tiling for $L^{131}$ which denotes one of the recently discovered 5d Sasaki-Einstein metrics ([61], see also [60]). The space is topologically $S^{2} \times S^{3}$. The toric diagram (Figure 6-25) has six legs, one possible rhombus loop diagram for them is shown in Figure 6-26. We notice that by moving the blue loops around, we may get a different tiling. This important phenomenon is toric duality and will be investigated in section 6.5.


Figure 6-25: $L^{131}$ toric diagram


Figure 6-26: $L^{131}$ rhombus loops and brane tiling

We can immediately draw the green tiling edges in the rhombus loop diagram, the final brane tiling can be seen in Figure 6-27 (i). From the tiling we trivially obtain the quiver (the "compactified" dual graph to the tiling, Figure 6-27 (ii)) and the following
superpotential:

$$
\begin{align*}
W & =X_{11} X_{12} X_{21}+X_{22} X_{23} X_{32}+X_{43} X_{34} X_{41} X_{14}  \tag{6.4.15}\\
& -X_{21} X_{12} X_{22}-X_{32} X_{23} X_{34} X_{43}-X_{11} X_{14} X_{41} \tag{6.4.16}
\end{align*}
$$

The fundamental cell in the tiling is denoted by a red box, this is the same as the fundamental cell of the rhombus loop diagram.

Closed oriented loops in the rhombus loop diagram (Figure 6-26) have a corresponding gauge invariant trace operator which is the product of the bifundamentals (at the intersection points) along the loop. These operators give a subset of all possible gauge invariant operators. Superpotential terms are trivial examples, these are small loops around the dark faces in the diagram. Another example is provided by the zig-zag operator, for which the above mentioned oriented loop is just one of the rhombus loops.


Figure 6-27: (i) $L^{131}$ brane tiling (ii) and the corresponding quiver.

If there are no degenerate rhombi, then we can use the results of [170] and count the number of bifundamental fields directly from the toric diagram. This can be done by summing up the intersection numbers as in [126]. The number of fields coming from the crossing ( $p_{1}, q_{1}$ ) and ( $p_{2}, q_{2}$ ) rhombus loops is simply

$$
\begin{equation*}
\#\left(S_{i} \cdot S_{j}\right)=\#\left(C_{i} \cdot C_{j}\right)=\left|p_{1} q_{2}-p_{2} q_{1}\right| \tag{6.4.17}
\end{equation*}
$$

Hirzebruch zero has two phases. one of them is non-degenerate (i. e. there are no degenerate rhombi). The formula gives the right value (eight) for the number of fields. The other phase has $R_{i}=1$ for some of the bifundamentals, so the above formula can't be used.

### 6.4.4 $L^{152}$

Our last example ${ }^{6}$ is $L^{152}$, its toric diagram is in Figure 6-28.


Figure 6-28: Toric diagram of $L^{152}$

The drawing of the rhombus loop diagram (Figure 6-29) is more involved than in the previous cases. To obtain an anomaly free tiling, one has to make sure that every other face (the light areas) has an even number of bounding rhombus loops (i. e. in the tiling the the corresponding face has even number of edges). To decide which face is dark and which one is light, we recall that the dark superpotential faces are distinguished by the fact that the rhombus loops are oriented around them. The gauge invariant trace operators built up from these small oriented loops are present in the superpotential, the order of the operator is given by the number of bounding rhombus loops of the dark face (this can be arbitrary).

Again, to see the tiling emerging out of the rhombus loop diagram, we have drawn more fundamental cells next to each other (Figure 6-30). The dark faces get black and white nodes, the edges of the tiling are stretching between them.

Finally, Figure 6-31 shows the resulting brane tiling and quiver. Six gauge groups are present in the theory.

[^22]

Figure 6-29: $L^{152}$ brane tiling from the rhombus loops

We can check the resulting tiling by computing the characteristic polynomial (6.4.19) of the dimer model by means of the determinant of the Kasteleyn matrix (6.4.18) (for details see [94]). The Newton polygon reproduces our starting point, the toric diagram of $L^{152}$ (Figure 6-28) therefore justifies our computation.

$$
\begin{gather*}
K=\left(\begin{array}{ccccc}
1 & 1 & -1 & w^{-1} & 0 \\
w & 1 & 0 & 0 & z \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & w & 1 & 1 \\
z^{-1} & 0 & 1 & 0 & 1
\end{array}\right)  \tag{6.4.18}\\
P(w, z) \equiv \operatorname{det}(K)=6-6 w+w^{2}+z^{-1}+w^{-1} z^{-1}+z \tag{6.4.19}
\end{gather*}
$$

### 6.5 Toric duality and Seiberg duality

We have seen ambiguities while constructing the brane tilings for a given singularity. The non-uniqueness manifests itself through the fact that we can freely move the rhombus loops which certainly changes the tiling and therefore the quiver gauge theory. Some of the resulting tiling might not be bipartite. Out of the bipartite tilings we are also only interested in the consistent ones. These "phases" of the


Figure 6-30: $2 \times 2$ fundamental cells of the rhombus loop diagram of $L^{152}$. The brane tiling is shown in green.
theory are believed to be Seiberg-dual to each other [24, 86, 53, 33].

The simplest transformation is when we move a single rhombus loop across an intersection point as in Figure 6-32. This is the Yang-Baxter transformation. We can build up a generic transformation from such elementary steps. The YangBaxter move changes the rhombus lattice locally which is shown in Figure 6-33.

On the other hand, the brane tiling (and the periodic quiver) has been changed globally. Apart from the local change in the rhombus lattice, we are forced to "flip" the tiling edges in the rhombi (Figure 6-3), i. e. the periodic quiver and the brane tiling get interchanged. The periodic quiver is usually non-bipartite (the only exception is the square lattice), therefore the resulting tiling is non-bipartite. However, one can perform more such Yang-Baxter transformations so that the final brane tiling is anomaly-free. Then, by definition, the resulting theory is toric dual to the original one. We provide an example in the followings.


Figure 6-31: (i) $L^{152}$ brane tiling (ii) and the corresponding quiver


Figure 6-32: The elementary Picard-Lefschetz-Yang-Baxter transformation.

### 6.5.1 Seiberg duality in the hexagonal lattice with extra line

Let us consider an arbitrary brane tiling with a subtiling shown in Figure 6-34 (i). This setup has been used in [96]. If we dualize group $F$, the extra edge moves into the neighboring hexagon.

What happened to the rhombus loops during this dualization? We can see that immediately, if we draw the (red) rhombus lattice (Figure 6-35). The relevant rhombus loops $(A, B, C, D)$ are shown in blue as usual. Only these loops are affected by the transformation.

Figure 6-34 shows the rhombus loops only. In this picture we see how Seiberg duality can be realized on the level of rhombus loops. It can be easily checked that the transformation contains four elementary Yang-Baxter steps. For another brane


Figure 6-33: The Yang-Baxter-Reidemeister transformation on the rhombus lattice. Star-triangle


Figure 6-34: (i) Four hexagon with one extra line. (ii) Seiberg dualizing the red square $(F)$. The extra edge in the upper hexagon $(B \& F)$ gets into the lower one $(F \& C)$.
realization of Seiberg duality see [134, 78].
With this knowledge, a thorough study of the possible moves of the rhombus loops (and the inconsistencies of the tiling) should reveal whether or not Seiberg duality is equivalent to toric duality.


Figure 6-35: Seiberg duality in the hexagonal tiling with extra edge. The brane tiling is shown in green, the (deformed) rhombus lattice is in red, the relevant rhombus loops are in blue.


Figure 6-36: Seiberg duality in the level of the rhombus loops.

## Chapter 7

## Exceptional Collections

### 7.1 Introduction

Determining the low energy gauge theory on a stack of D-branes probing a CalabiYau singularity is an important, interesting, and in general unsolved problem. These D-brane constructions can be used to build flux vacua in string theory, and they play an important role in the AdS/CFT correspondence, where they yield a geometric understanding of strongly coupled gauge theories. While much progress has been made in understanding orbifold, toric, and other simple Calabi-Yau singularities, the general case remains elusive.

Two of the most powerful techniques for unearthing these gauge theories are the brane tiling method pioneered by [128, 94, 97] and exceptional collections first mentioned in the AdS/CFT context in [53]. The relation between these two methods has up to this point remained obscure. In this chapter, we show how to translate one language into the other.

One of the best features of the brane tiling method is the ease with which the superpotential of the quiver gauge theory can be extracted. A brane tiling is a bipartite tiling of the torus $T^{2}$, and the superpotential terms are just the nodes of this tiling with coefficient $\pm 1$ given by the coloring of the node. No other method of relating gauge theory to geometric singularity has as yet produced such a simple way of extracting the superpotential.

For the brane tiling method to work, one starts with a toric Calabi-Yau three-fold sing ularity. The toric condition means that $Y$ possesses three $U(1)$ isometries. There are countably many interesting toric Calabi-Yau singularities, but the toric condition is a substantial restriction on $Y$. By using brane tilings, older algorithms ([82, 83]) get vastly simplified and reinterpreted.

For the exceptional collection method to work, one needs to be able to resolve partially the Calabi-Yau singularity by blowing up a complex surface - the exceptional collection lives on this surface. There are many both toric and non-toric Calabi-Yau singularities which can be resolved in this manner. The exceptional collection method was in large part developed to study some simple non-toric singularities, the non-toric del Pezzos [231].

While the superpotential can be extracted from an exceptional collection, the process is more abstract and less intuitive than for the brane tiling. In the exceptional collection case, deriving the superpotential requires working with $A$-infinity algebras $[18,17]$.

The exceptional collection method as applied to deriving quiver gauge theories rests on relatively firm mathematical and physical foundations [144, 19, 145, 38, 147]. From the perspective of the topological B-model, the objects in the collection can be understood as a nice basis of D-branes and the maps between the objects as massless open strings.

By providing a translation between the brane tiling and the exceptional collection, we put the brane tiling, along with its easy superpotential calculation, on a firmer mathematical and physical footing. Our results fall short of a general proof that the brane tiling method is equivalent to exceptional collections for toric Calabi-Yau singularities. Instead, we provide a well motivated conjecture of the way this map will work which we can prove example by example. By relating the tiling to exceptional collections which are topological B-model objects, our approach is complementary to that of [87].

In order for our translation between the brane tiling and the exceptional collection to work, we henceforth restrict to toric Calabi-Yau threefold singularities which can
be partially resolved by blowing up a complex surface.
The next section contains a brief review of the exceptional collection method and a map from the exceptional collection to the brane tiling. We argue that the periodic quiver which is the dual graph of the brane tiling can be constructed from a consideration of Wilson lines.

We then proceed in the other direction, mapping the brane tiling onto an exceptional collection. The cornerstone of this mapping is the realization that internal perfect matchings are in one-to-one correspondence with exceptional collections of line bundles.

### 7.2 Exceptional collections

Exceptional collections provide a powerful tool for deriving the low energy gauge theory description of a stack of D-branes probing a Calabi--Yau singularity. Given a Calabi-Yau cone $Y$, a stack of D -branes at the singularity will fragment into a set of fractional branes from which the gauge theory is easily deduced. These fractional branes are best described as objects in $D^{b}(Y)$, the derived category of coherent sheaves on $Y$. Exceptional collections provide a way of finding a good set of fractional branes and avoiding a direct confrontation with $D^{b}(Y) .{ }^{1}$

If $Y$ can be partially resolved by blowing up a possibly singular complex surface $V$, instead of looking for fractional branes on $Y$, we look for an exceptional collection of sheaves on $V$. There is then a simple procedure for converting this collection into a good set of fractional branes $[144,19]$, and in fact the gauge theory can often be deduced directly from the exceptional collection.

An exceptional collection of sheaves $\mathcal{E}=\left(E_{1}, E_{2}, \ldots, E_{n}\right)$ is an ordered set of sheaves which satisfy the following special properties:

1. Each $E_{i}$ is exceptional:

$$
\operatorname{Ext}^{q}\left(E_{i}, E_{i}\right)=0 \text { for } q>0 \text { and } \operatorname{Ext}^{0}\left(E_{i}, E_{i}\right)=\operatorname{Hom}\left(E_{i}, E_{i}\right)=\mathbb{C}
$$

[^23]
## 2. Ext ${ }^{\prime \prime}\left(E_{i}, E_{j}\right)=0$ for $i>j$ and $\forall q$.

In these notes, we will be most interested in the case where the collection is strongly exceptional. in which case $\operatorname{Ext}^{q}\left(E_{i}, E_{j}\right)=0$ for $i<j$ and $q>0$. For smooth toric surfaces. the collection must be strong to generate a physical quiver gauge theory [144, 19]. and the same is true for singular surfaces as well. ${ }^{2}$

For the most part, our sheaves can be thought of as line bundles, and line bundles are easy to describe in a toric context. ${ }^{3}$ For each ray $v_{r}$ in the fan, there is a toric Weil divisor $D_{r}$. The line bundles can then be expressed as $\mathcal{O}\left(\sum_{r} a_{r} D_{r}\right)$ for $a_{r} \in \mathbb{Z}$. One very special line bundle is the anti-canonical bundle:

$$
\begin{equation*}
\mathcal{O}(-K)=\mathcal{O}\left(\sum_{r} D_{r}\right) \tag{7.2.1}
\end{equation*}
$$

As we said earlier, the Calabi-Yau cone is the total space of the canonical bundle over our surface. The fact that our fan defines a convex polygon means that $K$ is negative.

Given a strongly exceptional collection $\mathcal{E}$, the quiver gauge theory can be constructed from the inverse collection $\mathcal{E}^{\vee}$. The members of $\mathcal{E}^{\vee}$ are no longer sheaves but objects in $D^{b}(V)$. Lifting these objects to $Y$ yields the fractional branes. At the level of D-brane charges, the inverse collection can be constructed from the Euler character on $V, \chi\left(E_{i}, E_{j}^{\vee}\right)=\delta_{i j}$. As a set of objects in $D^{b}(V), \mathcal{E}^{\vee}$ is constructed via a braiding operation called mutation described in detail in [144]. The inverse collection is also exceptional although no longer strongly exceptional. The Euler character $\chi\left(E_{i}^{\vee}, E_{j}^{\vee}\right)$ can be interpreted as the number of arrows in the quiver from node $i$ to node $j$ minus the number of arrows from node $j$ to node $i[147,19]$. This matrix is sometimes referred to as the antisymmetric part of the adjacency matrix. More precisely, the Euler character tells us the net number of $\operatorname{Hom}_{D^{b}(Y)}^{1}\left(E_{i}^{\vee}, E_{j}^{\vee}\right)$ maps in the Calabi-Yau between the fractional branes. For each of these maps, we have a massless open string which translates into a bifundamental field in the quiver gauge

[^24]theory.
It is often convenient to write down an intermediate quiver, the so-called Beilinson quiver, which lives on $V$ instead of $Y$. This quiver contains arrows corresponding only to the negative entries of $\chi\left(E_{i}^{\vee}, E_{j}^{\vee}\right)$, or more precisely maps in $\operatorname{Ext}^{1}\left(E_{i}^{\vee}, E_{j}^{\vee}\right)$. The Beilinson quiver algebra can be thought of as
\[

$$
\begin{equation*}
\oplus_{i, j} \operatorname{Hom}\left(E_{i}, E_{j}\right), \tag{7.2.2}
\end{equation*}
$$

\]

but the quiver contains arrows only for the generators of this algebra which are encoded simply in $\mathcal{E}^{\vee}$. Because $V$ is compact, the Beilinson quiver contains no oriented loops.

### 7.2.1 From Exceptional Collection to Periodic Quiver

In this section we assume that we have a compact toric surface $V$ with positive anticanonical class and a strongly exceptional collection of line bundles $\mathcal{E}$ on $V$. We would like to construct from this data a periodic quiver. In particular, we will write the Beilinson quiver on a torus.

Any toric surface can be described by a fan by which we mean a collection of at least three vectors $v_{r}, r=1, \ldots, n$ on an integer lattice $\mathbb{Z}^{2}$. That the surface is compact means that the polygon defined by the endpoints of the vectors $v_{r}$ includes the origin. That the anti-canonical class of this surface is positive means that the polygon is convex. (We would like to allow $V$ to have quotient singularities.)

One way of understanding $V$ is as a quotient of $\mathbb{C}^{n}$. Given $n$ vectors in $\mathbb{Z}^{2}$, we expect that there will be $n-2$ linearly independent relations between the $v_{r}$, which we write as

$$
\begin{equation*}
\sum_{r} Q_{a r} v_{r}=0 \tag{7.2.3}
\end{equation*}
$$

where $a=1, \ldots, n-2$ and $Q_{a r} \in \mathbb{Z}$. Geometrically, we quotient

$$
\begin{equation*}
\frac{\mathbb{C}^{n}-F_{\Delta}}{\left(\mathbb{C}^{*}\right)^{n-2}} \tag{7.2.4}
\end{equation*}
$$

where the action of the $\left(\mathbb{C}^{*}\right)^{n-2}$ is given by the $Q_{a r}$. The set $F_{\Delta}$ is a small set of points inside $\mathbb{C}^{n}$ which we need to remove to have a well defined quotient.

As an example, consider $\mathbb{P}^{2}$ for which the fan is $v_{1}=(1,0), v_{2}=(0,1)$, and $v_{3}=(-1,-1)$. There is just one relation which we write as $Q=(1,1,1)$. This quotient construction is nothing but the usual equivalence relation of the homogenous coordinates on $\mathbb{P}^{2}$, namely $\left(X_{1}, X_{2}, X_{3}\right) \sim\left(\lambda X_{1}, \lambda X_{2}, \lambda X_{3}\right)$ for $\lambda \in \mathbb{C}^{*}$. $F_{\Delta}$ is the origin $(0,0,0)$ of $\mathbb{C}^{3}$.

For arbitrary $V$, we can think of $X \in \mathbb{C}^{n}$ as generalized homogenous coordinates. The $n-2$ equivalence relations (7.2.3) leave a two complex dimensional space which is $V$ itself:

$$
\begin{equation*}
\left(X_{1}, X_{2}, \ldots, X_{n}\right) \sim\left(\lambda^{Q_{a 1}} X_{1}, \lambda^{Q_{a 2}} X_{2}, \ldots, \lambda^{Q_{a n}} X_{n}\right) \tag{7.2.5}
\end{equation*}
$$

This two complex dimensional space $V$ is a fiber bundle $\pi: V \rightarrow B$ where $B$ is a real two dimensional surface and the fibers are real, two dimensional tori. More simply put, the fibers are coordinatized by the phase angles of the complex coordinates on $V$. First, we characterize this torus in greater detail.

Given the $n-2$ vectors $Q_{a}$ and using the standard inner product on $\mathbb{Z}^{n}$, we find two additional vectors $q_{1}$ and $q_{2}$ such that $q_{i} \cdot Q_{a}=0$ and $q_{1}$ and $q_{2}$ are linearly independent. A canonical set of $q_{i}$ are the $v_{r}$ reinterpreted as two $n$ dimensional vectors rather than $n$ two dimensional vectors: we could set $q_{1 r}=v_{r, 1}$ and $q_{2 r}=v_{r, 2}$. These $q_{i}$ can be used to measure relative positions on the real two torus. Given the homogenous coordinates $\left(X_{1}, X_{2}, \ldots, X_{n}\right)$, we define the two torus coordinates to be

$$
\begin{equation*}
\left(\theta_{1}, \theta_{2}\right)=\left(\sum_{r} q_{1 r} \operatorname{Arg} X_{r}, \sum_{r} q_{2 r} \operatorname{Arg} X_{r}\right) \tag{7.2.6}
\end{equation*}
$$

Notice that if we shift $X_{r}$ by $\lambda^{Q_{a r}},\left(\theta_{1}, \theta_{2}\right)$ remains invariant because $q_{i} \cdot Q_{a}=0$.
Our D-branes are line bundles on $V$, and thus we can think of them as Euclidean D 4 -branes filling all of $V$. If we perform fiberwise T -duality twice on the two torus, we should find D2-branes localized at points on the torus. The open strings will then connect these points together. The periodic Beilinson quiver is nothing but this web of D2-branes and open strings.

We will characterize this web using the original line bundle (or D4 brane) description. The notation $\mathcal{O}(I)$ indicates a D4-brane with a dissolved D 2 brane; this dissolved D2-brane has the same charges as a D2-brane wrapping the divisor $D \subset V$. We can describe this dissolved D2-brane as magnetic flux. Because the line bundle is holomorphic, the field strength components $F_{i j}=0=F_{\bar{\imath} \jmath}$ vanish, and locally the field strength takes the form

$$
\begin{equation*}
F=i \partial_{i} \bar{\partial}_{\bar{\jmath}}\left(f+f^{*}\right) d y^{i} \wedge d \bar{y}^{\bar{\jmath}} \tag{7.2.7}
\end{equation*}
$$

where $A_{j}=-i \partial_{j} f, A_{\bar{\jmath}}=i \bar{\partial}_{\bar{\jmath}} f^{*}$ and $f$ is some function of the coordinate patch. By a gauge choice, we may take the imaginary part of $f$ to vanish.

In a toric variety, the phase angle directions $\theta_{i}$ are isometries, and the field strength $F$ describing the D 2 -brane should not depend on the $\theta_{i}$. Because our variety is toric, we can choose a complex structure such that $y^{j}=\ln r_{j}+i \theta_{j}=\rho_{j}+i \theta_{j}$. In this coordinate system, the field strength becomes

$$
\begin{equation*}
F=\left(\frac{\partial^{2} f}{\partial \rho_{i} \partial \rho_{j}}+\frac{\partial^{2} f}{\partial \theta_{i} \partial \theta_{j}}\right) d \rho_{i} \wedge d \theta_{j}+\frac{\partial^{2} f}{\partial \theta_{i} \partial \rho_{j}}\left(d \rho_{i} \wedge d \rho_{j}+d \theta_{i} \wedge d \theta_{j}\right) \tag{7.2.8}
\end{equation*}
$$

In order for $F$ to be independent of $\theta_{i}, f$ must take a very special form. In particular, $f=g(r)+C_{i \bar{\jmath}} y^{i} \bar{y}^{\bar{\jmath}}$ where the second term leads to a constant field strength. We will assume this second term in $f$ vanishes in which case the vector potential takes the very simple form

$$
\begin{equation*}
A=\frac{\partial f}{\partial \rho_{i}} d \theta_{i} \tag{7.2.9}
\end{equation*}
$$

At this point, we fix a point $\left(r_{1}, r_{2}\right) \in B$ and look at the $T^{2}$ fiber, where we recognize a Wilson line. Locally on the $T^{2}, A=w_{j} d \theta_{j}$ is pure gauge; $A=i d \ln \Lambda$ where $\Lambda=\exp \left(-i w_{j} \theta_{j}\right)$. However, globally, $\Lambda$ does not respect the periodicity conditions. We have a distinct set of Wilson lines for $0 \leq w_{j}<1$, with $\left(w_{1}, w_{2}\right) \sim\left(w_{1}+n, w_{2}+m\right)$ for $n$ and $m$ integers. This set of Wilson lines lives on a dual torus we will call $\tilde{T}^{2}$.

Given a collection of line bundles, we can calculate the value of the Wilson line for each such bundle and plot that point $\left(w_{1}, w_{2}\right)$ on our $\tilde{T}^{2}$ of length and height one.

This plot gives us the nodes of the periodic Beilinson quiver.
The strings between the D4-branes come from the generators of the Beilinson quiver algebra and as such are maps of the form $\operatorname{Hom}\left(E_{i}, E_{j}\right)$. Since the branes are line bundles, we may write $E_{i}=\mathcal{O}(D), E_{j}=\mathcal{O}\left(D^{\prime}\right)$, and $\operatorname{Hom}\left(E_{i}, E_{j}\right)=H^{0}\left(V \cdot \mathcal{O}\left(D^{\prime}-\right.\right.$ $D))$. We expect, given a generating element in $\operatorname{Hom}\left(E_{i}, E_{j}\right)$, to find a corresponding string between $\mathcal{O}(D)$ and $\mathcal{O}\left(D^{\prime}\right)$. Moreover, $\mathcal{O}(D)$ and $\mathcal{O}\left(D^{\prime}\right)$ should be separated by a vector on the torus given by the value of the Wilson line for $\mathcal{O}\left(D^{\prime}-D\right)$.

From the derived category point of view on $Y$, we know how to compute the masses of these open strings $[144,19,16]$, and the answer depends on being able to understand instanton corrections as we move in the Kähler moduli space of $Y$. From the point of view of the complex surface $V$ and the Wilson line discussion, our intuition is that a string stretching between two of these D 4 -branes will have a mass proportional to the distance between the corresponding points on $\tilde{T}^{2}$ [204]. As we change the base point, the Wilson lines will all move around. Our naive expectation is that for massless strings, there is a particular choice of base point for which the Wilson line corresponding to $\mathcal{O}\left(D^{\prime}-D\right)$ vanishes. It would be interesting to understand these masses better from the Wilson line point of view.

### 7.2.2 Line Bundles and Curvature Forms for Toric Surfaces

In the previous section, we sketched a procedure for converting a set of line bundles on a toric variety into a periodic quiver, but we did not explain why the construction would respect the periodicity of the torus. For example, take two linearly equivalent divisors $D$ and $D^{\prime}$. The corresponding line bundles $\mathcal{O}(D)$ and $\mathcal{O}\left(D^{\prime}\right)$ correspond to the same D-brane. Why then are the Wilson lines for $\mathcal{O}(D)$ and $\mathcal{O}\left(D^{\prime}\right)$ the same? In this section, we will attempt to answer this question and elucidate the structure of the corresponding vector potentials.

Given a line bundle, $\mathcal{O}(D)$, and a particular choice of Kähler metric on a toric variety, one can construct an explicit coordinate dependent expression for a representative of $c_{1}(D) \in H^{2}(V, \mathbb{Z})$. These representatives were first worked out by [119] (for a readable and more recent account see [1]). This representative of $\mathcal{O}(D)$ is holomor-
phic, i.e. locally of the form $i \partial \bar{\partial} f$. Also, it is independent of the angular coordinates $\theta_{i}$ and so takes the form (7.2.9) discussed previously.

These representatives have a number of disadvantages. In most cases, these representatives do not satisfy the remaining equation of motion $g^{i \bar{\jmath}} F_{i \bar{\jmath}}=\mu$. Here, $\mu$ is a constant often called the slope. Moreover, they depend on a particular canonical choice of Kähler metric which is usually not the one of physical interest. Typically, we would be more interested in a metric which is compatible with a Ricci flat metric on the cone over $V .{ }^{4}$ Despite these disadvantages, we use these explicit representatives for they form a useful beginning from which to argue more general results.

We have thus far been working with complex coordinates $\rho+i \theta$, but these representatives are most easily expressed in symplectic coordinates on $V, x+i \theta$. The phase angles $\theta_{i}$ remain the same in both the complex and symplectic system. For the $x$, we define a polytope

$$
\begin{equation*}
\Delta=\left\{x \in \mathbb{R}^{2}:\left\langle x, v_{r}\right\rangle \geq-1 \forall r\right\} \tag{7.2.10}
\end{equation*}
$$

The symplectic form is then $\omega=\sum_{i} d x_{i} \wedge d \theta_{i}$.
In these symplectic coordinates, the Kähler metric and complex structure depend on a potential function $g(x)$. Define

$$
\begin{equation*}
g_{i j}=\frac{\partial^{2} g(x)}{\partial x_{i} \partial x_{j}} \tag{7.2.11}
\end{equation*}
$$

The line element becomes

$$
\begin{equation*}
d s^{2}=g_{i j} d x_{i} d x_{j}+g^{i j} d \theta_{i} d \theta_{j} \tag{7.2.12}
\end{equation*}
$$

where $g^{i j}$ is the inverse of $g_{i j}$ and summation on the indices is implied. The symplectic coordinates are related to the complex ones by a Legendre transformation, $\rho=\partial g / \partial x$.

[^25]The representatives of $H^{2}(V, \mathbb{Z})$ depend on a particular choice of $g$,

$$
\begin{equation*}
g_{c a n}=\frac{1}{2} \sum_{r} \ell_{r} \log \ell_{r}, \tag{7.2.13}
\end{equation*}
$$

where we have defined

$$
\begin{equation*}
\ell_{r}=\left\langle x, v_{r}\right\rangle+1 \tag{7.2.14}
\end{equation*}
$$

In the case of projective space, this metric is physically interesting: it's Einstein and is thus compatible with a Ricci flat metric on the cone over $V$. In general $g_{c a n}$ will produce a metric which is physically uninteresting albeit simple. A general Kähler metric is related to $g_{c a n}$ in a smooth way:

$$
\begin{equation*}
g=g_{c a n}+h \tag{7.2.15}
\end{equation*}
$$

where $h$ is a smooth function on $\Delta$.

We have seen already that a holomorphic vector bundle has a curvature form which may be written as $2 i \partial \bar{\partial} f(\rho)$ for some locally defined function of $f$. In symplectic coordinates, this two-form becomes

$$
\begin{equation*}
2 i \partial \bar{\partial} f=\sum_{j, k} \frac{\partial}{\partial x_{j}}\left(g^{k l} \frac{\partial f}{\partial x_{l}}\right) d x_{j} \wedge d \theta_{k} \tag{7.2.16}
\end{equation*}
$$

For the canonical choice of metric, we take the vector potential corresponding to $\mathcal{O}\left(D_{r}\right)$ to be

$$
\begin{equation*}
A_{r}=\frac{1}{2}\left(g_{c a n}\right)^{k l} \frac{\partial \log \ell_{r}}{\partial x_{l}} d \theta_{k} \tag{7.2.17}
\end{equation*}
$$

This $A_{r}$ yields a curvature two-form which represents the class $c_{1}\left(D_{r}\right)$ but is in general not harmonic. Note that $A_{r}$ is only well defined away from the side $\ell_{r}=0$.

Using (7.2.17), we will prove a result about the $A_{r}$ and then argue that the same result must hold more generally for non-canonical metrics and $A_{r}$ which do satisfy
the equations of motion. The result is that

$$
\begin{equation*}
\sum_{r} v_{r, i} A_{r}=d \theta_{i} \tag{7.2.18}
\end{equation*}
$$

or in other words, this particular combination of the $A_{r}$ is pure gauge. The result follows simply from noting that

$$
\begin{equation*}
\left(g_{c a n}\right)_{i j}=\sum_{r} \frac{v_{r, i} v_{r, j}}{2 \ell_{r}} \tag{7.2.19}
\end{equation*}
$$

More generally, because every divisor $D=\sum_{r} a_{r} D_{r}$ can be expressed as a sum of primitive Weil divisors, we expect there to be a basis of primitive vector potentials $A_{r}, r=1, \ldots, n$ such that $A_{D}=\sum_{r} a_{r} A_{r}$. We have now chosen the $A_{r}$ to satisfy the equations of motion, but they should be related to the canonical $A_{r}$ in a smooth way. We say two divisors $D$ and $D^{\prime}$ are linearly equivalent when they have the same $Q$ charges, $\sum_{r} Q_{a r}\left(a_{r}-a_{r}^{\prime}\right)=0$. All such linear equivalence relations are generated by the $q_{i}$. If $D$ and $D^{\prime}$ are linearly equivalent, then $\mathcal{O}\left(D-D^{\prime}\right) \sim \mathcal{O}$. But $\mathcal{O}$ corresponds to a single D4-brane with no dissolved D2-brane charge. The associated field strength must vanish, and it must be that

$$
\begin{equation*}
\sum_{r} q_{i r} A_{r} \tag{7.2.20}
\end{equation*}
$$

is pure gauge for $i=1$ and 2 .

We can deduce more from the statement that (7.2.20) is pure gauge. A gauge transformation $A \rightarrow A+i d \ln \Lambda$ must respect the periodicity of the torus. Since the $A_{r}$ take the form $f(r) d \theta$, the gauge transformation $\ln \Lambda$ which annihilates (7.2.20) must depend only linearly on $\theta$ and not at all on $x$. The only choice is $\Lambda=\exp (\operatorname{in} \theta)$, from which we conclude that

$$
\begin{equation*}
\sum_{r} q_{i r} A_{r}=n_{i 1} d \theta_{1}+n_{i 2} d \theta_{2} \tag{7.2.21}
\end{equation*}
$$

for integers $n_{i j}$. The $v_{r}$ and our Wilson line torus are only defined up to an $S L_{2}(\mathbb{Z})$
transformation so we choose

$$
\begin{equation*}
\sum_{r} q_{1 r} A_{r}=d \theta_{1} ; \quad \sum_{r} q_{2 r} A_{r}=d \theta_{2} \tag{7.2.22}
\end{equation*}
$$

recovering the canonical result (7.2.18) in a more general context. This reasoning answers the question posed earlier about why for linearly equivalent $D$ and $D^{\prime}, \mathcal{O}(D)$ and $\mathcal{O}\left(D^{\prime}\right)$ give the same Wilson line.

Before moving on, we study the vanishing of the generating set $A_{r}$ because of a possible relation to massless open strings. We wish to show that the $A_{r}$ will vanish at corners of $\Delta$ where $A_{r}$ is well defined. For this demonstration, we rely on a result of Abreu [1] that

$$
\begin{equation*}
\operatorname{det}\left(g_{i j}\right)=\left[\delta(x) \prod_{r=1}^{n} \ell_{r}(x)\right]^{-1} \tag{7.2.23}
\end{equation*}
$$

where $\delta$ is a smooth function on $\Delta$. Since we are on a surface, at a corner of $\Delta$, the determinant of $g^{i j}$ involves a double zero, and it is straightforward to show that $g^{i j}$ must vanish. Since $g^{i j}$ vanishes, from (7.2.16) we see that $A_{r}$ will vanish as well unless the corner is associated with the vanishing of $\ell_{r}$.

### 7.2.3 Bundles on $\mathbb{P}^{2}$

To illustrate these ideas concretely, we present them for $\mathbb{P}^{2}$. There are three Weil divisors $D_{1}, D_{2}$, and $D_{3}$ on $\mathbb{P}^{2}$ corresponding to the three rays of the fan $v_{1}=$ $(1,0), v_{2}=(0,1)$, and $v_{3}=(-1,-1)$. From (7.2.17), the vector potentials for the corresponding three line bundles, which in this case satisfy the equations of motion, are

$$
\begin{align*}
& A_{1}=-\frac{1}{3}\left(x_{1}-2\right) d \theta_{1}-\frac{1}{3}\left(1+x_{2}\right) d \theta_{2},  \tag{7.2.24}\\
& A_{2}=-\frac{1}{3}\left(1+x_{1}\right) d \theta_{1}-\frac{1}{3}\left(x_{2}-2\right) d \theta_{2},  \tag{7.2.25}\\
& A_{3}=-\frac{1}{3}\left(1+x_{1}\right) d \theta_{1}-\frac{1}{3}\left(1+x_{2}\right) d \theta_{2}, \tag{7.2.26}
\end{align*}
$$

where the $x_{i}$ lie inside the triangle defined by $x_{1}>-1, x_{2}>-1$ and $x_{1}+x_{2}<1$. These $A_{r}$ are all gauge equivalent to each other, which is expected since the corresponding divisors are all linearly equivalent. The gauge transformation takes the form $A \rightarrow$ $A+d \lambda$ where $\lambda=n_{1} \theta_{1}+n_{2} \theta_{2}$ and $n_{i}$ is an integer. The Wilson line corresponding to the $A_{r}$ will not change because the gauge transformation respects the periodicity of this square torus of height and length one. Thus we see that $\mathcal{O}\left(D_{1}\right), \mathcal{O}\left(D_{2}\right)$ and $\mathcal{O}\left(D_{3}\right)$ appear as the same point on $\tilde{T}^{2}$. Indeed, for any line bundle of the form $\mathcal{O}\left(a D_{1}+b D_{2}+c D_{3}\right)$, the point on the torus will depend only on $a+b+c$. Any line bundle of the form $\mathcal{O}\left(a D_{1}+b D_{2}+c D_{3}\right)$ can equivalently be written as $\mathcal{O}(a+b+c)$.

We can take the vector potential corresponding to $\mathcal{O}(n)$ to be

$$
\begin{equation*}
-\frac{n}{3}\left(1+x_{1}\right) d \theta_{1}-\frac{n}{3}\left(1+x_{2}\right) d \theta_{2} \tag{7.2.27}
\end{equation*}
$$

Thus, given the exceptional collection $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$, we should plot points at $p_{1}=$ $(0,0), p_{2}=\left(-1-x_{1},-1-x_{2}\right) / 3$ and $p_{3}=2\left(-1-x_{1},-1-x_{2}\right) / 3$ or their translates on $\tilde{T}^{2}$. These three points correspond to the D -branes.

To connect these three D -branes with open strings, we return to the $A_{i}$ (7.2.24)(7.2.26). Between $\mathcal{O}$ and $\mathcal{O}(1)$ or between $\mathcal{O}(1)$ and $\mathcal{O}(2)$, there are three possible paths corresponding to $D_{1}, D_{2}$, and $D_{3}$. The path corresponding to $D_{i}$ is defined by the Wilson line associated to $A_{i}$. Instead of thinking of the Wilson line as a point on the torus, we now think of it as a vector that joins two points. The resulting Beilinson quiver for $\mathbb{P}^{2}$ is shown in Figure 7-1. We do not need to draw in additional arrows corresponding to maps between $\mathcal{O}$ and $\mathcal{O}(2)$. All the requisite maps can be formed by joining together the arrows already drawn.

These vectors corresponding to the $D_{i}$ shrink to zero size at special base points on the polytope $\Delta$. In particular, the string corresponding to $D_{1}$ shrinks to zero at $(2,-1), D_{2}$ shrinks to zero at $(-1,2)$, and $D_{3}$ shrinks to zero at $(-1,-1)$.

One startling feature of this Beilinson quiver is that the arrows will never cross, no matter what our choice of basepoint $\left(x_{1}, x_{2}\right)$. As the ( $x_{1}, x_{2}$ ) moves to the boundaries of $\Delta$, arrows may become parallel and the three points may touch, but the arrows


Figure 7-1: Four unit cells of the $\mathbb{P}^{2}$ periodic quiver for basepoint $\left(x_{1}, x_{2}\right)=$ (3/4, -1/2).
never cross.

### 7.2.4 Constructing the Quiver in General

Given a set of generating field strengths for the $\mathcal{O}\left(D_{r}\right)$, we can construct a family of periodic quivers from an exceptional collection. A particular quiver in the family will depend on the choice of basepoint $\left(x_{1}, x_{2}\right) \in \Delta$. If the metric is of physical interest, e.g. it lifts to a Ricci flat metric on the cone and provides a starting point for AdS/CFT constructions, and the field strengths satisfy the equations of motion, we expect this periodic quiver to be the quiver of physical interest. Thus, the quiver we described for $\mathbb{P}^{2}$ should be the "correct" quiver. Unfortunately, we in general do not have explicit expressions for the metric and the field strengths, only the canonical representatives detailed above.

In the absence of physical data, we will work with the canonical metric and hope that the resulting quiver is topologically if not geometrically accurate. Because we only expect topological data, we will fix a particularly convenient choice of basepoint in $\Delta:\left(x_{1}, x_{2}\right)=(0,0)$. In this case, the vector potential becomes

$$
\begin{equation*}
A_{r}=\frac{1}{2} g^{k l} v_{r, l} d \theta_{k} \tag{7.2.28}
\end{equation*}
$$

From this vector potential, we see that a general line bundle of the form $\mathcal{O}\left(\sum_{r} a_{r} D_{r}\right)$
will be plotted on the torus with coordinates

$$
\begin{equation*}
\left(\sum_{r} q_{1 r} a_{r}, \sum_{r} q_{2 r} a_{r}\right) \tag{7.2.29}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{i r}=\frac{1}{2} g^{i l} v_{r, l} . \tag{7.2.30}
\end{equation*}
$$

These two $q_{i r}$ are orthogonal to the $Q_{a}$ and are in fact the same as the $q_{i}$ discussed previously. Because $g^{k l}$ is complicated and we are after only topological information, let us rescale the $q_{i r}$ and the associated torus by a $g_{k l} \in G L_{2}(\mathbb{R})$ transformation, choosing $q_{j r}=v_{r, j}$ as before.

The procedure for constructing the quiver is very simple. Given a strongly exceptional collection of line bundles $\mathcal{E}=\left(E_{1}, E_{2}, \ldots, E_{n}\right)$, take $E_{j}=\mathcal{O}\left(\sum_{r} a_{r} D_{r}\right)$ and $E_{k}=\mathcal{O}\left(\sum_{r} b_{r} D_{r}\right)$. The homomorphisms from $E_{j}$ to $E_{k}$ are generated by the global sections of $\mathcal{O}\left(\sum_{r}\left(b_{r}-a_{r}\right) D_{r}\right)$. Start with the monomial

$$
\begin{equation*}
\prod_{r} X_{r}^{b_{r}-a_{r}} \tag{7.2.31}
\end{equation*}
$$

This monomial has charges $\sum_{r} Q_{a r}\left(b_{r}-a_{r}\right)$. To be a global section, $b_{r}-a_{r} \geq 0$ for all $r$ (or there will be a pole). However, there may be more than one such monomial with this charge. Construct all such monomials. Call the set of such monomials $M_{j k}$. For each $m \in M_{j k}$, where $m=\prod_{r} X_{r}^{c_{r}}$, we compute

$$
\begin{equation*}
\left(\varphi_{1}, \varphi_{2}\right)=\left(\sum_{i} q_{1 r} c_{r}, \sum_{i} q_{2 r} c_{r}\right) \tag{7.2.32}
\end{equation*}
$$

This vector $\left(\varphi_{1}, \varphi_{2}\right)$ is the relative position of nodes $j$ and $k$ on $\tilde{T}^{2}$. Fixing the position of $E_{1}$, we now have specified the location of all the nodes of the quiver.

Instead of a $\tilde{T}^{2}$ of length and height one as before, because of the rescaling, the period vectors of this torus are the $q_{i}$. If we take two points of the quiver separated by $a q_{1}+b q_{2}$, in the language of line bundles, we have $\mathcal{O}(D)$ and $\mathcal{O}\left(D+\sum_{r}\left(a q_{1 r}+b q_{2 r}\right) D_{r}\right)$. However, since the $q_{i}$ are orthogonal to the $Q_{i}, D$ and $D+\sum_{r}\left(a q_{1 r}+b q_{2 r}\right) D_{r}$ have
the same $Q$ charges and are linearly equivalent as divisors. In other words, these two points are the same.

Starting with the set $M_{k, k+1}$, we draw an arrow from node $k$ to node $k+1$ for each $m \in M_{k, k+1}$. We repeat this procedure for line bundles of the form $E_{k}$ and $E_{k+2}$. There is an additional complication now. It may happen that the monomial $m=m_{1} m_{2}$ where $m_{1}$ joins nodes $E_{k}$ with $E_{k+1}$ and $m_{2}$ joins nodes $E_{k+1}$ and $E_{k+2}$. If such is the case, then we do not add an arrow corresponding to $m$. The entries of $\chi\left(E_{i}^{\vee}, E_{j}^{\vee}\right)$ let us know how many arrows we should be writing down. Recursively, we consider $E_{k}$ and $E_{k+i}$ and continue until all the arrows in the Beilinson quiver are drawn.

Take $\mathbf{d} \mathbf{P}_{1}$ to illustrate these ideas. A fan is $v_{1}=(0,1), v_{2}=(1,1), v_{3}=(0,-1)$, and $v_{4}=(-1,0)$ from which we choose

$$
q=\left(\begin{array}{cccc}
0 & -1 & 0 & 1  \tag{7.2.33}\\
-1 & -1 & 1 & 0
\end{array}\right)
$$

An exceptional collection on $\mathrm{dP}_{1}$ is $\mathcal{O}, \mathcal{O}\left(D_{1}\right), \mathcal{O}\left(D_{4}+D_{1}\right), \mathcal{O}\left(D_{4}+D_{1}+D_{3}\right)$. Using the procedure described above, we find the Beilinson quiver, figure 7-2.


Figure 7-2: The periodic Beilinson quiver for $\mathbf{d} \mathbf{P}_{\mathbf{1}}$ with fundamental cell.
For example, consider the paths between $\mathcal{O}\left(D_{4}+D_{1}\right)$ and $\mathcal{O}\left(D_{4}+D_{1}+D_{3}\right)$. We
look for all monomials with the $Q$ charges of $D_{3}$. in other words $x_{3}, x_{1} x_{4}$, and $x_{1} x_{2}$. These three monomials have torus charges $q,(0,1),(1,-1)$, and $(-1,-2)$ respectively. () $)_{1}$ our torus, node 4 is indeed at relative positions $(0,1),(1,-1)$, and $(-1,-2)$ to node 3 with corresponding arrows drawn in.

### 7.2.5 Vanishing Euler Character

We can argue that the Euler character of the torus (to be distinguished from the Euler character of the exceptional collection) must vanish and so the most obvious obstruction to writing the quiver on a torus is eliminated. (Of course, we don't have an arbitrary collection of lines, vertices, and faces, but have instead completely specified the connectivity, and it remains unclear that the pattern of connectivity will be compatible with a torus structure.) Given exceptional collections $\mathcal{E}$ and $\mathcal{E}^{\vee}$, in terms of charges, we can decompose any sheaf $F$ into the $E_{j}^{\vee}$ or the $E_{j}$ :

$$
\begin{equation*}
\operatorname{ch}(F)=\sum_{j} \chi\left(E_{j}, F\right) \operatorname{ch}\left(E_{j}^{\vee}\right) ; \quad \operatorname{ch}(F)=\sum_{j} \chi\left(F, E_{j}^{\vee}\right) \operatorname{ch}\left(E_{j}\right) . \tag{7.2.34}
\end{equation*}
$$

We are interested in quivers that come from a stack of D3-branes, which look like a point in $V$. Thus, for a skyscraper sheaf

$$
\begin{equation*}
\operatorname{ch}\left(\mathcal{O}_{p t}\right)=\sum_{j} \chi\left(E_{j}, \mathcal{O}_{p t}\right) \operatorname{ch}\left(E_{j}^{\vee}\right)=\sum_{i, j} \chi\left(E_{j}, \mathcal{O}_{p t}\right) \chi\left(E_{j}^{\vee}, E_{i}^{\vee}\right) \operatorname{ch}\left(E_{i}\right) \tag{7.2.35}
\end{equation*}
$$

The rank component of the chern class of a skyscraper sheaf vanishes, and $\chi\left(E_{i}, \mathcal{O}_{p t}\right)=$ $\operatorname{rk}\left(E_{i}\right)$. Thus,

$$
\begin{equation*}
0=\sum_{i, j} \operatorname{rk}\left(E_{i}\right) \operatorname{rk}\left(E_{j}\right) \chi\left(E_{i}^{\vee}, E_{j}^{\vee}\right) \tag{7.2.36}
\end{equation*}
$$

For these toric exceptional collections, we find exceptional collections of line bundles where the ranks are all one. Thus, the sum over the entries of the Euler character must vanish. But this sum has a different interpretation. The sum over the diagonal entries is the number of gauge groups. The sum over the negative entries is the number of arrows in the Beilinson quiver, and the sum over the off-diagonal positive
entries is the number of relations:

$$
\begin{equation*}
\sum_{i, j} \chi\left(E_{i}^{\vee}, E_{j}^{\vee}\right)=\text { gauge groups }- \text { arrows }+ \text { relations } \tag{7.2.37}
\end{equation*}
$$

Now for these toric quivers, we know that each relation corresponds to two superpotential terms. Moreover, when we lift to the Calabi-Yau quiver, each relation also becomes an additional arrow. Thus, for the Calabi-Yau quiver

$$
\begin{equation*}
\text { gauge groups }- \text { arrows }+ \text { superpotential terms }=0 \tag{7.2.38}
\end{equation*}
$$

which is exactly the condition that the Euler character of the torus vanish because for each gauge group we have a node, for each arrow an edge, and each superpotential term a face in the quiver. ${ }^{5}$ Moving back to the Beilinson quiver now consists of removing a set of arrows, which cannot change the Euler character of the graph. This demonstration of vanishing Euler character is complementary to but distinct from a similar observation in [94] where the authors use R-charge constraints to prove that the Euler character of the brane tiling vanishes.

### 7.3 Compatibility

Having established that one can derive periodic quivers from exceptional collections, we now study the possibility of generating such collections by means of brane tilings. In this section we define a map that assigns line bundles to paths in the quiver. This map can be used to compute an exceptional collection on a complex surface that shrinks to zero size at the singularity. The exceptionality can be checked on a case-by-case basis. Given these bundles, one can reconstruct the quiver based on mathematically rigorous procedures [53, 231, 18, 17, 144, 19, 145, 143]. By reinterpreting paths and perfect matchings in the tiling language, we explicitly prove that this construction gives back our original quiver.

[^26]

Figure 7-3: The eight periodic perfect matchings of $\mathbf{d P}_{1}$. The green edges are contained in the matching. The dashed lines are the edges left in the tiling. The $(s, t)$ numbers are the corresponding points in the toric diagram.

### 7.3.1 Beilinson quivers and internal matchings

For the exceptional collection technique to be useful when applied to toric CalabiYau manifolds, we need the toric diagram to contain at least one internal point. This restriction means that our manifold can be partially resolved by blowing up a 4-cycle. Let us consider the tiling for this Calabi-Yau which can be most efficiently constructed
by the Fast Inverse Algorithm [132, 87]. Let us also fix a reference internal matching $P M_{11}$ that resides at one of the internal points of the toric diagram. We can set the origin at this point.

If we remove those bifundamentals from the quiver that are contained in $P M_{0}$, then we obtain another smaller quiver. We will show that this subquiver contains no oriented loops and therefore has the right properties to be a Beilinson quiver for the relevant 4 -cycle. ${ }^{6}$ For an example see Figure 1-3. This Beilinson quiver is generated by deleting the bifundamentals that are contained in the $4^{\text {th }}$ perfect matching of Figure 7-3. Recall that the Beilinson quiver was defined at the beginning of Section 7.2 from an exceptional collection. Here, we define an intermediate notion

Definition 7.3.1.1. We define a pre-Beilinson quiver to be a connected subquiver of the gauge theory quiver that contains no oriented loops and all the nodes of the original.

Let us summarize some additional terminology we use in the following.

Definition 7.3.1.2. An oriented path is a path in the quiver that respects the direction of the arrows.

Definition 7.3.1.3. Paths in the quiver that also exist in a Beilinson (or preBeilinson) quiver are called allowed paths.

We say that a path crosses an edge in the tiling if the path contains the corresponding arrow in the quiver. Paths that exist in the Beilinson quiver will not intersect the edges of $P M_{0}$. It is easy to see that F-terms transform allowed paths to allowed paths. Closed paths may wind around the tiling torus, and the winding can be characterized by the homology class of the loop $(p, q)$. The $(0,0)$ loops are called trivial loops. By definition, the length of an oriented path is the R -charge of the corresponding operator. Paths can be related by F-term transformations, but these transformations will not change the total R -charge associated to a path. The height functions of the external matchings with respect to $P M_{0}$ are called height coordinates.

[^27]Lemma 7.3.1.4. In a consistent tiling, an internal perfect matching determines a pre-Beilinson quiver by removing those bifundamentals from the quiver that are contained in the matching.

Proof. Removing bifundamentals from the gauge theory quiver that are contained in $P M_{0}$ does not remove nodes and does not create disconnected pieces. The nontrivial part of the proof involves the oriented loops.
(i) First we show that trivial allowed loops cannot exist. Such a loop would contain at least one edge $e$. By crossing this edge in the tiling, some of the height functions would increase by one. The increase happens exactly when the corresponding perfect matchings contain $e$. Allowed paths will never go "downhill" on the graph of any height function, because then they would have to cross an edge in $P M_{0}$ which is not allowed (the edge is not present in the pre-Beilinson quiver). See Figure 7-4 for the schematic picture. The increase of the height function is "irreversible", i.e. the function is monotone along an allowed path; hence we have arrived at a contradiction.

For this argument to hold one has to show that $e$ is contained in at least one perfect matching. We can suppose this, since otherwise we can omit this edge from the tiling and still get the same toric diagram which questions the consistency of the original tiling.
(ii) We also need to show that there are no non-trivial loops in the pre-Beilinson quiver. These non-trivial loops wrap the torus cycles. Suppose that there exists such a loop. This oriented loop is a face path on the brane tiling with homology class $(x, y) \in \mathbb{Z}^{2}$ as in Figure 7-5. Let us take an arbitrary external matching $P M_{i}$ at $\left(s_{i}, t_{i}\right)$. We can compute the height function assigned to this matching with respect to $P M_{0}$.

The height function should not decrease along the path. As an immediate consequence, the scalar product $\left(s_{i}, t_{i}\right) \cdot(x, y)$ must be nonnegative. On the other hand, the set of vectors $\left\{\left(s_{i}, t_{i}\right)\right\}$ span the whole 2 d space with positive coefficients, and thus at least one of these vectors has negative scalar product with $(x, y)$. This is a contradiction; therefore the pre-Beilinson quiver doesn't contain non-trivial loops.


Figure 7-4: Allowed face paths (i.e. paths in the Beilinson quiver) go always uphill. The height function increases by one at the line constituted of the black perfect matching and the green reference matching. The red path cannot cross the green edges (they are not in the Beilinson quiver). Hence when crossing the contour line, the red path has to cross a black edge. Crossing the black edge increases the value of the height function.


Figure 7-5: Gradient vectors in the toric diagram. The coordinates of the blue $\left(s_{i}, t_{i}\right)$ vectors give the monodromy of the height function of the perfect matching sitting at their endpoints. The red $(x, y)$ arrow is the gradient vector of the hypothetical nontrivial loop.

### 7.3.2 Line bundles from tiling: The $\Psi$-map

In the last section we saw that a candidate Beilinson quiver could be created from an internal perfect matching. In this section we continue by defining a map $\Psi$ that assigns a divisor to an allowed path by using external perfect matchings. We conjecture that these divisors give exceptional collections of line bundles which we will use to reconstruct the Beilinson quiver.

A Weil divisor can be represented by an integer function over the external vertices of the toric diagram polygon (see Figure 7-6). We call two such integer functions equivalent if they differ by a linear function $f(x, y)=x m+y n$ which defines a principal divisor. (Here $x$ and $y$ are coordinates on the plane of the polygon.)

Let us fix an arbitrary oriented path $P$. Then, $\Psi(P)$ gives a divisor, i.e. an integer


Figure 7-6: An integer function over the external nodes determines a divisor and therefore a sheaf of sections of the corresponding line bundle. The numbers in the figure denote $\mathcal{O}\left(D_{1}+D_{3}+2 D_{4}\right)$.
function over the external nodes. We define this map by using the matchings of the tiling. For each external node $v_{r}$, there is a corresponding unique perfect matching ${ }^{7}$ $P M_{r}$. We assign to the divisor $D_{r}$ the integer $\Psi_{r}(P)$ that is the number of edges in $P M_{r}$ which are crossed by the path $P$. In Figure 7-7 we see an example.


Figure 7-7: The $\Psi$-map.

The left hand side shows the brane tiling for $\mathrm{dP}_{1}$. The red path $P$ crosses two edges; hence it labels the operator $X_{13} \cdot X_{34}^{(1)}$. There is a corresponding oriented $1 \rightarrow 3 \rightarrow 4$ path in the quiver as in Figure 1-2. We have chosen the 4 th matching from Figure 7-3 as the green reference matching. To show how to compute $\Psi_{8}(P)$, we have drawn the 8th matching of Figure 7-3 (in blue). The shading of the faces indicates the height function of this matching that has $(0,1)$ monodromy. The red

[^28]path crosses one blue edge in the matching (namely $X_{34}^{(1)}$ ); hence $\Psi_{8}(P)=1$. One can compute the other integer "intersection numbers" with the help of the other external perfect matchings. The resulting numbers are indicated in red. These numbers define a Weil divisor on the base of the threefold. The numbers can also be interpreted as the increase in the height coordinates as we go along the path $P$. If the path is an allowed path (Definition 7.3.1.3) starting at face $A$ and ending at $B$, then $\Psi_{r}$ is simply the $h_{r}(B)-h_{r}(A)$ difference in the height function that corresponds to the $r^{\text {th }}$ external node. $\Psi$ is a well-defined function on the paths of the quiver. In fact, it does not depend on the choice of the reference perfect matching (modulo linear equivalence).

The $\Psi$-map can be extended to unoriented paths, i.e. paths that do not respect the arrow direction in the quiver. When crossing an edge in $P M_{i}$ in the reverse direction, we subtract one instead of adding one in computing $\Psi_{r}(P)$.

Let $C_{i}$ denote the Abelian group of chains in the periodic quiver. Here the quiver is understood as a discretization of the 2-torus. This is the free group generated by the edges in the quiver with integer coefficients. The elements of $C_{1}$ take the following form

$$
\begin{equation*}
P=\sum_{i} c_{i} X_{i} \quad\left(c_{i} \in \mathbb{Z}\right) \tag{7.3.39}
\end{equation*}
$$

where $X_{i}$ denotes the $i^{\text {th }}$ edge. We denote the cycles in $C_{i}$ by $Z_{i}$ and the boundaries by $B_{i}$. Elements of $B_{1}$ are built out of trivial loops. $\Psi$ can be extended in a straightforward way to be defined on $C_{1}$

$$
\begin{equation*}
\Psi_{r}=\sum_{j} c_{P_{j}} \tag{7.3.40}
\end{equation*}
$$

where $\left\{p_{j}\right\}$ is the list of edges in the $r^{\text {th }}$ external matching. In the following, we will study the properties of this extended $\Psi$-map.

For an elementary loop around a node in the tiling, the image of $\Psi$ is a constant function (the anticanonical class $K$ ). Since all the perfect matchings cover this node, each matching is intersected by the loop precisely once; hence $\Psi_{r}=1$ for all $r$. This
coincides with the observations made in [97]. In fact, one can easily prove that the entire $B_{1}$ subgroup is mapped to constant functions.

Gauge invariant mesonic operators can be constructed from arbitrary oriented loops ${ }^{8}$. These are the elements of $Z_{1}$. For these loops $\Psi$ assigns non-negative affine functions on the toric diagram parametrized by three integers. These functions are points in the dual cone.

We will now use $\Psi$ to compute a collection of line bundles. We choose an internal reference matching which determines a Beilinson quiver and therefore an ordering of the faces in the tiling. Without losing generality, we relabel the groups such that there are no arrows from node $i$ to $j$ if $i>j$.

Let us fix an allowed path $P_{i}$ for each face in the tiling (for $\mathrm{dP}_{1}$ see Figure 7-8). We will call $\left\{P_{i}\right\}$ the set of reference paths. We choose these paths such that they start on face 1 and end on the specific face. This is possible because the Beilinson quiver is connected. Then, $\Psi$ maps each of these paths to a Weil divisor (see Figure 7-9 for the image). These divisors determine a collection of line bundles.


Figure 7-8: The reference paths are allowed paths to each face. They start from face 1 and don't cross the edges of the green internal matching; hence they are paths in the Beilinson quiver.

There is a general freedom in the choice of these paths. The terminal faces can also be chosen from different fundamental cells. We demonstrate this ambiguity in Figure 7-10. Let us pick two different paths that end on the same faces but in different fundamental cells. Recall that $\Psi$ maps closed loops to linear functions;

[^29]hence the difference of the resulting divisors is linear, which mecms that they are in fact equivalent. Note that $\Psi$ gives the same set of integers for uperators (paths) related by F - term equations.


Figure 7-9: The three divisors computed from the paths to the faces.


Figure 7-10: Face 4 can be assigned with either the red or the yellow allowed path. The resulting Weil divisors are shown on the right-hand side. We see that they differ by a linear function, i.e. they are equivalent.

After determining the divisors that correspond to the $P_{i}$ paths, we are ready to write down an exceptional collection. We introduce the notation

$$
\begin{equation*}
\mathcal{O}\left(\sum_{r} a_{r} D_{r}\right) \equiv\left(a_{1}, a_{2}, \ldots, a_{n}\right) \tag{7.3.41}
\end{equation*}
$$

We assign the line bundle of the divisor $\Psi\left(P_{i}\right)$ to the $i^{\text {th }}$ face. The integer numbers sitting at the external nodes are the $a_{i}$ coefficients. For the first face we assign $(0,0, \ldots, 0)$. In our $\mathbf{d P}_{1}$ example from Figure 7-9 we obtain the following collection:

$$
\begin{equation*}
(0,0,0,0),(1,0,0,0),(0,0,1,0),(1,0,1,1) \tag{7.3.42}
\end{equation*}
$$

which is exactly the collection discussed in section 7.2.4.

Another example for the $Y^{3,2}$ theory is presented in the Appendix.
Before moving on, we would like to point out that the $\Psi$-map efficiently computes the divisors that correspond to dibaryons. In order to obtain the divisor for the bifundamental $X$, we simply compute $\Psi(X)$. For $\mathbf{d} \mathbf{P}_{1}$ we get the following list

| field | divisor |
| :---: | :---: |
| $X_{12}$ | $(1,0,0,0)$ |
| $X_{23}^{(1)}, X_{23}^{(2)}$ | $(0,1,0,0) \cong(0,0,0,1)$ |
| $X_{41}^{(1)}, X_{41}^{(2)}$ | $(0,1,0,0) \cong(0,0,0,1)$ |
| $X_{42}$ | $(0,0,1,0)$ |
| $X_{13}$ | $(0,0,1,0)$ |
| $X_{34}^{(1)}, X_{34}^{(2)}, X_{34}^{(3)}$ | $(0,0,1,0) \cong(1,1,0,0) \cong(1,0,0,1)$ |

in precise agreement with section 5.1 of [147]. The linear equivalence relations $\cong$ are easily established. Let us show that $(0,0,1,0) \cong(1,0,0,1)$. The difference divisor $(1,0,0,1)-(0,0,1,0)=(1,0,-1,1)$, shown on the right hand side of Figure $7-10$, has a $\Psi$ map of the form $\Psi=y-x$. In other words $(1,0,-1,1)$ is a principal divisor and the linear equivalence follows.

In this section we defined the linear $\Psi$-map that computes the divisors corresponding to the bifundamental fields. This map can be used explicitly to write down a collection of line bundles for the singularity. Unfortunately, we are lacking a general proof that the generated collections are always exceptional. Strong exceptionality may be checked on a case-by-case basis.

### 7.3.3 Reconstructing the quiver

In section 7.3 .2 we introduced the general method, the $\Psi$-map, that computes a collection of line bundles that is presumably strongly exceptional. Given such a collection, we can use rigorous methods to construct the quiver of the gauge theory. In this section we prove that the quiver obtained this way matches with the dual graph of the tiling which was our starting point. ${ }^{9}$

Let us denote the exceptional collection by $\left\{E_{i}\right\}$. We define the matrix

$$
\begin{equation*}
S_{i j}=\operatorname{dim} \operatorname{Hom}\left(E_{i}, E_{j}\right) \tag{7.3.43}
\end{equation*}
$$

The matrix elements in $S$ tell the number of ways of getting from node $i$ to node $j$ in the Beilinson quiver, taking the relations into account. The inverse of this matrix gives the quiver directly up to bidirectional arrows. The nonzero elements of $S_{i j}^{-1}$ $(i<j)$ are the number of arrows from $j$ to $i$ minus the number of arrows from $i$ to $j$ in the quiver.

Since we are dealing with line bundles on toric manifolds, the computation of $\operatorname{dim} \operatorname{Hom}\left(E_{i}, E_{j}\right)$ gets vastly simplified [102]. This dimension is equal to the number of global sections of the bundle $E_{j} \otimes E_{i}^{*}$, which we denote by $\mathcal{O}\left(\sum_{r} a_{r} D_{r}\right)$. Then, the dimension is obtained by counting the lattice points inside the polygon

$$
\begin{equation*}
\Delta_{i j}=\left\{u \in \mathbb{R}^{2}: u \cdot v_{r} \leq a_{r} \text { for all } r\right\} \tag{7.3.44}
\end{equation*}
$$

where $v_{r} \in \mathbb{Z}^{2}$ is the position of the $r$ th external node in the toric diagram. See the left-hand side of Figure 7-11 for an example.

In section 7.3 .2 we computed the (7.3.42) exceptional collection for $\mathbf{d P}_{1}$. Using the above described method, the $S$ matrix and its inverse are determined

[^30]\[

S=\left($$
\begin{array}{cccc}
1 & 1 & 3 & 6  \tag{7.3.45}\\
0 & 1 & 2 & 5 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{array}
$$\right) \quad S^{-1}=\left($$
\begin{array}{cccc}
1 & -1 & -1 & 2 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{array}
$$\right)
\]

We see that $S^{-1}$ gives precisely the quiver in Figure 1-2.
In the following, we will show that this lattice point counting method of determining the number of paths from node $i$ to node $j$ in the quiver is identical to the same computation on the brane tiling. Since the number of paths essentially encodes the quiver via $S$ and $S^{-1}$, we are proving that the collection of line bundles encodes the quiver of the original brane tiling.



Brane tiling cells

Figure 7-11: Determining the $S_{2,4}$ matrix element. In this case $E_{4} \otimes E_{2}^{*}=$ $(1,0,1,1)-(1,0,0,0)=(0,0,1,1)$. The figure shows the lattice of the $\Delta_{2,4}$ polygon and its bounding inequalities. The red lattice points inside $\Delta_{2,4}$ can be identified with adjacent fundamental cells in the brane tiling.

The key observation is that the lattice of $\Delta_{i j}$ can be identified with the lattice of fundamental cells of the brane tiling. ${ }^{10}$ This is shown in Figure 7-11. In particular, we will assign the lattice points to the $j^{\text {th }}$ faces in the cells. The simple counting of lattice points also counts the inequivalent allowed paths from face $i$ to face $j$. There can be many such paths, but their number is finite, since no loops are allowed. The lattice points in $\Delta_{i j}$ are in one-to-one correspondence with adjacent fundamental cells that contain the final $j$ faces where these paths end. In Figure $7-12$ these are

[^31]

Figure 7-12: The figure shows the allowed paths that start on face 2 and end on face 4. The endpoints of these paths are in different fundamental cells which are in one-to-one correspondence with the lattice points inside $\Delta_{2,4}$ that has been used to compute dim $\operatorname{Hom}\left(E_{2}, E_{4}\right)$.
the five faces marked in yellow. We see that to one of these faces there are two allowed paths leading. This shouldn't trouble us, since these are equivalent paths related by the $U_{2}^{1} V^{2}=U_{2}^{2} V^{1} \mathrm{~F}$-term equation for the $Y_{2}$ bifundamental field that separates face 2 and face 4 . In fact, it turns out that a general feature of consistent tilings is that homotopic paths of the same length (measured by the R-charge of the corresponding trace operator ${ }^{11}$ ) are F -term equivalent. In the following, we will prove this statement.

Lemma 7.3.3.1. In a consistent tiling, paths of the same length are $F$-term equivalent iff they are homotopic.

Proof. F-flatness equations are local transformations of the paths (Figure 7-13); hence they transform homotopic paths into one another. Applying such a transformation to the path does not change the R -charge of the corresponding operator. We need to show that two homotopic paths are equivalent.

As an illustration, Figure 7-14 shows two such paths in a square lattice that can be deformed into one another by F-terms. The rhombi they surround are also shown separately in the right-hand side of the figure. This area has two bounding lines: $A A_{1} A_{2} A_{3} B$ and $A B_{1} B_{2} B_{3} B$. On the boundary we find two kinds of rhombus nodes alternating: Every other node is also a node of the tiling $\left(A_{1}, A_{3}, B_{1}, B_{3}\right)$. We call
${ }^{11}$ In fact, any trial R -charge can be used to measure the length.


Figure 7-13: The F -flatness equation for the $X$ bifundamental field is $C B A=V U$. This states the equivalence of the two green paths in the figure.
these odd nodes. The remaining even nodes $\left(A, A_{2}, B, B_{2}\right)$ are only vertices in the rhombus lattice.

We can start deforming path 1 by using the F -term equation for the tiling edge $A_{3} B_{3}$. We also see that using the F -term equation for $A_{1} B_{1}$ is not possible because path 1 does not contain $A_{1} B_{3}$. At the level of the rhombus lattice the difference of the two nodes $A_{3}$ and $A_{1}$ can be quickly seen: There is no red rhombus lattice edge in the pink area that connects $A_{3}$ to another node, whereas $A_{1}$ has one edge, namely $A_{1} B_{2}$. To summarize, the area between two paths can be reduced by F-terms where the boundary nodes don't have rhombus edges.


Figure 7-14: Homotopic paths are equivalent. The left-hand side of the figure shows two paths represented schematically by green lines. The tiling is colored black and the underlying rhombus lattice is shown by dotted lines. The pink area surrounded by the two paths is also shown separately.

Let us consider two homotopic paths that start and end on the same two faces. For simplicity, we assume that the paths are not intersecting. We also assume that the area between the two paths has been completely reduced, i.e. there are no more

F-terms that we can use to decrease it. This is equivalent to requiring that the orld nodes along the boundary have at least one rhombus edge going to the interior of the area. One can check that by construction the even nodes always have at least ont rhombus edge. (In the previous example, such nodes were $A_{2}$ and $B_{2}$.) The reduced area can be schematically drawn as in Figure 7-15.


Figure 7-15: Two homotopic paths that pass around the pink area. Each boundary node $\left(A_{1}, \ldots, A_{n}, B_{1}, \ldots, B_{m}\right)$ has at least one rhombus edge which ensures that the area cannot be reduced by F -terms.

If we suppose that there is precisely one red rhombus edge at each $A_{i}$ and $B_{j}$ node and there are no edges at $A$ and $B$, then we recognize a straight rhombus path built out of the $r_{i}(i=0,1,2, \ldots, n)$ rhombi. These are located at the boundary next to path 1 (see Figure 7-16).


Figure 7-16: The straight rhombus path in the area contains rhombi $r_{0}, \ldots, r_{n}$. The existence of this series of rhombi constrains $A B_{1}$ to be parallel to $B_{m} B$.

This rhombus path corresponds to a zig-zag path in the tiling. The opposite edges of the rhombi are parallel; hence $A B_{1}$ is parallel to $B_{m} B$. The same argument applies for the rhombi on the other side of the area; hence $A A_{1}$ is parallel to $A_{n} B$. As a consequence, some of the rhombi in the area must be degenerate (here $r_{0}$ and $r_{n}$ ), i.e. the R -charges of the corresponding fields are zero or negative and the tiling
is inconsistent. Here we used that there is one rhombus edge for each node.
Extra rhombus edges joining to $A_{i} . B_{j}$ or to the endpoints $A$ or $B$ can't be used to restore the consistency of the tiling since they make the rhombi even more degenerate. This can also be seen by looking at the sum of internal angles of the $A, A_{1}, \ldots, A_{n}, B, B_{m}, \ldots, B_{1}, A$ pink polygon. This polygon has $n+m+2$ vertices, hence the sum of angles should be $(n+m) \pi$. Every rhombus next to the boundary contributes $\pi$ to the sum, except for the rhombi at $A$ and $B$ whose contribution can be bigger. If there are extra rhombus edges at a particular node, then we also get contribution from those rhombi that touch this node but they don't have a common edge with the boundary polygon. Since there are at least $n+m$ rhombi, the total sum of angles is greater than $(n+m) \pi$; hence the polygon must be degenerate.

As an immediate corollary, the lemma proves the following observation of [31]

Corollary 7.3.3.2. The structure of the chiral ring is naturally encoded in the nontrivial cycles of the tiling torus. In particular, the dual cone can be "embedded" in the infinite tiling [172].

The embedding is sketched in Figure 7-17.


Figure 7-17: The embedding of the dual cone in the tiling torus.

One can assign gauge invariant mesonic operators to each of the monomials in the dual cone. For the $A, B, D$ monomials we assigned three green paths that are schematically shown in the right-hand side of the figure. They start and end on the
same square in the tiling. Keeping these endpoints and the lengths fixed, they can be freely deformed due to Lemma 7.3.3.1.

Then, the endpoints of the paths in the lattice of fundamental cells can be identified with the projection of the monomials onto the red tiling plane. To reach the bulk of the cone (here the monomials $B$ and $C$ ), the path has to contain loops, e.g. small loops around a tiling node. For instance, the tip of the cone and $C$ are projected to the same point; therefore the corresponding path to $C$ must be a trivial loop. It can be chosen to be the appropriate power of any term in the superpotential.

Corollary 7.3.3.3. For the consistency of the tiling a necessary condition is that homotopic paths of the same length are F-term equivalent. ${ }^{12}$


Figure 7-18: Inequivalent $A \rightarrow B$ homotopic paths in an inconsistent tiling.

If the tiling is inconsistent, it might be possible to construct two inequivalent paths surrounding the "inconsistency". An example is shown in Figure 7-18 where the tiling contains the subgraph of Figure 15 in [132]. We recognize the two rhombus paths and the corresponding tiling zig-zags along the boundary of the pink area. Since no F-terms can be used, the paths are inequivalent.

After proving the lemma and investigating some of its corollaries, let us turn back to the original problem. We want to show that the matrix element $S_{i j}$ gives the number of inequivalent paths from $i$ to $j$. In order to prove this, we need to show that for each $u$ lattice point in $\Delta_{i j}$, we have a unique allowed path in the tiling starting on

[^32]the $i^{\text {th }}$ face and ending on the $j^{\text {th }}$ one. These $j^{\text {th }}$ faces are in different fundamental cells that are in one-to-one correspondence with the $u$ lattice points.

The previous lemma ensures that we have a single path for each cell. To see this, we need to prove that allowed homotopic paths have the same length. Suppose that there exist two homotopic paths of different lengths. Using F-term equations, we can deform the longer path to the shorter one as in Figure 7-19. Thus, we end up with loops around tiling nodes which are evidently not allowed, since these loops intersect $P M_{0}$. Recalling that F-terms transform allowed paths to allowed paths, we arrive at a contradiction. This means that in a consistent tiling homotopic allowed paths always have the same $R$-charge and are equivalent.


Figure 7-19: Homotopic paths with different R -charge are not equivalent. After applying the F -term equation for $A_{3} B_{1}$, the long path (solid green line) gets transformed to the short path (dashed line) plus a small loop around the $A_{1}$ node in the tiling.

Having proved that from the $i^{\text {th }}$ face of a fixed fundamental cell there exists at most one inequivalent path to the $j^{\text {th }}$ face of any cell, we also need to show that these cells where the paths can end are in one-to-one correspondence with the $u$ lattice points. In order to do so, we reinterpret the (7.3.44) bounding inequalities of $\Delta_{i j}$.

In the definition of $\Delta_{i j}$, we have a $u \cdot v_{r} \leq a_{r}$ constraint for each external node of the toric diagram. For a given path, $u$ is interpreted as the integer vector defined on the lattice of fundamental cells giving the distance of the cells wherein the $i^{\text {th }}$ and $j^{\text {th }}$ faces reside. In the tiling language, $v_{r}$ is the monodromy of the height of the $r^{\text {th }}$ external perfect matching. Thus, the scalar product gives the increase in the $r^{\text {th }}$ height coordinate. Hence, the $a_{r}$ variables should be interpreted as height differences.

In fact, this is exactly how we computed them with the $\Psi$-map in section 7.3.2.
Figure 7-20 illustrates the correspondence schematically. The figure shows three inequivalent allowed paths that connect face $A$ to different $B$ faces. The shading indicates the $r^{\text {th }}$ height function. The height changes along the edges in the superposition of the corresponding matching and $P M_{0}$. This level set is represented by purple dashed lines.


Figure 7-20: The figure schematically depicts three allowed green paths from $A$ to $B$. The shading indicates one of the height coordinates. The height increases in the direction of the small arrows. The allowed paths can only cross the dashed lines in this direction, and thus we obtain a bounding inequality for $\Delta_{A B}$. The remaining edges can be determined by means of the other heights.

The right-hand side of Figure 7-20 shows the lattice of $\Delta_{A B}$ along with a green bounding line. The lattice points are in one-to-one correspondence with the red fundamental cells on the left-hand side. In particular, we assign them to the $B$ faces sitting in the cells. We set the origin at the middle point which is assigned to the upper left $B$ face in the tiling.

How does the green constraint come about? From previous discussions in section 7.3.1 we know that allowed paths can only go uphill on the height function. For example, in Figure 7-20 the paths can cross the dashed lines in the direction of the small arrows; therefore we can't reach the $B$ face in the lower right corner. This face corresponds to the excluded point on the right-hand side shown by the dotted arrow.

Using the above interpretation of $u$, we can immediately write down a necessary (and sufficient) condition for the allowed paths. In our schematic example, we have $v_{r}=(1,-1)$ which is the average "gradient vector" of the height function. Naively, the constraint translates to the following inequality for the allowed paths

$$
\begin{equation*}
u \cdot v_{r} \leq 0 \tag{7.3.46}
\end{equation*}
$$

This is not quite right, because the paths start from $A$ not $B$. One can take this into account by adding the difference in their height coordinates to the right-hand side

$$
\begin{equation*}
u \cdot v_{r} \leq d_{r} \tag{7.3.47}
\end{equation*}
$$

By using the $P_{A}$ and $P_{B}$ reference paths that connect the first node of the Beilinson quiver to $A$ and $B$, one can see this difference is given by $d_{r}=\Psi_{r}\left(P_{B}\right)-\Psi_{r}\left(P_{A}\right){ }^{13}$ Let us denote the $i^{\text {th }}$ line bundle in the exceptional collection by $\left(a_{1}^{i}, a_{2}^{i}, \ldots, a_{n}^{i}\right)$. Recalling from section 7.3.2 how we have determined the collection, we obtain $d_{r}=a_{r}^{B}-a_{r}^{A}$. Our final expression is then

$$
\begin{equation*}
u \cdot v_{r} \leq a_{r}^{B}-a_{r}^{A} \tag{7.3.48}
\end{equation*}
$$

which is precisely the inequality in the definition of $\Delta_{A B}$ !
We can write down the remaining inequalities for the constraints coming from the other height functions in exactly the same way. Thus, we obtain the boundaries of $\Delta_{A B}$.

We have seen that the inequalities are equivalent to the fact that allowed paths can't go downhill on any of the height functions of the external matchings. This completes the correspondence between the lattice points of $\Delta$ and the allowed paths, and thus proves that $S_{i j}$ indeed counts the inequivalent paths in the tiling.

Let us summarize the main results of this section. Given a consistent brane tiling, we can compute a $\mathcal{B}$ Beilinson quiver and an $\left\{E_{i}\right\}$ collection of line bundles by means

[^33]of an internal matching and the $\Psi$-map. ${ }^{14}$ One may check on a case-by-case basis that this collection is exceptional.

In this section we have proved that the "true" Beilinson quiver of the gauge theory living in the worldvolume of the D3-branes is the same as $\mathcal{B}$, the original quiver which is obtained directly from the tiling. In particular, we proved that the number of inequivalent paths between two nodes are the same.

As a byproduct, we obtained that homotopic paths with the same R -charge are F-term equivalent. Thus, we could clarify the relation of the brane tiling to the dual cone by a projection of the lattice points of the cone onto the tiling plane. This gave an explicit correspondence between monomials and paths.

### 7.4 Conclusions

Brane tilings can be deceptively simple. With a few strokes of a pen, all of the data of a $\mathcal{N}=1$ supersymmetric quiver gauge theory - the matter fields, the gauge groups, the superpotential - are captured. Given these simple pictures, theorems should be easy to prove, but we have often found otherwise. In the following paragraphs, we outline our successes but also the work that remains to be done to prove our dictionary between brane tilings and exceptional collections.

In section 7.2 , we provided a recipe that will convert any exceptional collection of line bundles into a periodic quiver and motivated the recipe using Wilson lines and a little mirror symmetry. In the cases we looked at, this periodic quiver was the graph theoretic dual of a brane tiling. Thinking of the periodic quiver as a triangulation of a surface, we proved that the Euler character vanished. Since the exceptional collection specifies the connectivity of all the vertices, edges, and faces, a vanishing Euler character is not necessarily enough to ensure the quiver can be written on a torus.

[^34]In section 7.3 , we provided a recipe that will convert any brane tiling into a collection of line bundles. Two key observations underlie this recipe. The first is that internal perfect matchings of the tilling are in one to one correspondence with Beilinson quivers and hence with exceptional collections. The second is that external perfect matchings are in one-to-one correspondence with the generating Weil divisors $D_{r}$ and can be used to convert paths in the brane tiling into sums of divisors $\sum a_{r} D_{r}$ via the $\Psi$-map.

We left the word exceptional out of the first sentence of the preceding paragraph on purpose. On a case by case basis, we can verify the collections are exceptional, using for example the techniques described in [145]. However, proving that the collection is exceptional in general is difficult. There is a paper by Altmann and Hille [11] who prove strong exceptionality for quivers without relations (no superpotential) using Kodaira vanishing. The Kodaira vanishing theorem and certain generalizations are a powerful way of proving strong exceptionality. Given a line bundle $\mathcal{O}(D)$ corresponding to an ample divisor $D$, then

$$
\begin{equation*}
\operatorname{dim} H^{q}(X, \mathcal{O}(D \otimes K))=0, \text { for any } q>0 \tag{7.4.49}
\end{equation*}
$$

Unfortunately, for us, even in relatively simple exceptional collections, one finds a $D$ which is not ample even though these higher cohomology groups vanish. To see the vanishing, one must rely on techniques specific to the complex surface $V$ in question.

We hope the future brings new progress on both these fronts.

## Appendix

To demonstrate the computation of exceptional collections with the $\Psi$-map of section 7.3.2, we give another example. This is the $Y^{3,2}$ theory, whose quiver is shown in Figure 7-21.

The brane tiling of this geometry and the 18 perfect matchings are given in Figure 7-22 and Figure 7-23. In the upper left corner of the figures the toric diagram is


Figure 7-21: $Y^{3,2}$ quiver.
shown with a red dot giving the position of the matching. For reference matching we pick the $7^{\text {th }}$ matching of Figure 7-22. Deleting the corresponding arrows in the quiver gives the Beilinson quiver (Figure 7-24). We need to fix allowed reference paths in the tiling that connect the first node of the Beilinson quiver to all the other nodes. The chosen paths are shown in Figure 7-25.


Figure 7-22: $Y^{3,2}$ perfect matchings $\left(1^{s t} \ldots 9^{t h}\right)$.




Figure 7-23: $Y^{3,2}$ perfect matchings $\left(10^{t h} \ldots 18^{t h}\right)$.


Figure 7-24: $Y^{3,2}$ Beilinson quiver. Bifundamentals in internal matching 7 are omitted.


Figure 7-25: $Y^{3,2}$ tiling. The purple lines indicate the chosen paths that are used to compute the exceptional collections. The paths start on face 1 and connect it to the other faces.

$1 \rightarrow 2$
$1 \rightarrow 2 \rightarrow 4$
$1 \rightarrow 2 \rightarrow 4 \rightarrow 5$
$1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$
$1 \rightarrow 3$
Figure 7-26: A set of reference paths for $Y^{3,2}$.

From the intersection number of the paths and the external perfect matchings we can immediately derive the following collection:

$$
\begin{equation*}
(0,0,0,0),(1,0,0,0),(0,0,1,0),(1,1,0,0),(1,1,1,0),(1,1,2,0) \tag{7.4.50}
\end{equation*}
$$

## Part III

## Global geometry

## Chapter 8

## Semi-flat spaces

### 8.1 Introduction

A great deal of progress has been made in the study of string compactification using the ten-dimensional supergravity approximation (for a review, see [72]). However, it has become clear that certain interesting physical features of our world are difficult (if not impossible) to realize when this description is valid. Examples which come to mind include a period of slow-roll inflation [141, 68, 114], certain models of dynamical supersymmetry breaking [89], chiral matter with stabilized moduli [44] and parametrically-small perturbatively-stabilized extra dimensions [72]. This strongly motivates attempts to find descriptions of moduli-stabilized string vacua which transcend the simple geometric description.

One approach to vacua outside the domain of validity of 10 d supergravity is to rely only on the 4 d gravity description, as in e.g. [217, 219]. This can be combined with insight into the microscopic ingredients to give a description of much more generic candidate string vacua. A drawback of this approach is that it is difficult to control systematically the interactions between the ingredients. Another promising direction is heterotic constructions, which do not require RR flux and hence are more amenable to a worldsheet treatment [4, 3]. However, stabilization of the dilaton in these constructions requires non-perturbative physics.

A third technique, which is at an earlier state of development, was implemented
in [138]. anl was inspired by [113, 226]. The idea is to build a compactification out of locally ten-dimensional geometric descriptions, glued together by transition functions which inchude large gauge transformations, such as stringy dualities. This technique is uniquely adapted to construct examples with no global geometric description. In this chapter. we build on the work of [138] to give $4 \mathrm{~d} \mathcal{N}=1$ examples.

In [138]. early examples of vacua were constructed involving such 'non-geometric fluxes'. These examples were constructed by compactifying string theory on a flat $n$-torus, and allowing the moduli of this torus to vary over some base manifold. The description of these spaces where the torus fiber is flat is called the semi-flat approximation [220]. Allowing the torus to degenerate at real codimension two on the base reduces the construction of interesting spaces to a Riemann-Hilbert problem; the relevant data is in the monodromy of the torus around the degenerations [113]. Generalizing this monodromy group to include not just modular transformations of the torus, but more general discrete gauge symmetries of string theory (generally known as string dualities) allows the construction of vacua of string theory which have no global geometric description [138]. The examples studied in detail in [138] had two-torus fibers, which allowed the use of complex geometry.

A natural explanation of mirror symmetry is provided by the conjecture [220] that any CY has a description as a three-torus $\left(T^{3}\right)$ fibration, over a 3-manifold base. In the large complex structure limit, the locus in the base where the torus degenerates is a trivalent graph; the data of the CY is encoded in the monodromies experienced by the fibers in circumnavigating this graph. Further, the edges of the graph carry energy and create a deficit angle - in this description a compact CY is a self-gravitating cosmic string network whose back-reaction compactifies the space around itself. In this chapter, our goal is to use this description of ordinary CY manifolds to construct non-geometric vacua, again by enlarging the monodromy group. We find a number of interesting new examples of non-geometric vacua with $4 \mathrm{~d} \mathcal{N}=1$ supersymmetry. In a limit, they have an exact CFT description as asymmetric orbifolds, and hence can be considered 'blowups' thereof. We study the spectrum, particularly the massless scalars, and develop some insight into how these vacua fit into the web of known
constructions.
We emphasize at the outset two limitations of our analysis. First, the examples constructed so far are special cases which have arbitrarily-weakly-coupled perturbative descriptions and (therefore) unfixed moduli. Our goal is to use them to develop the semiflat techniques in a controllable context. Generalizations with nonzero RR fluxes are naturally incorporated by further enlarging the monodromy group to include large RR gauge transformations, as in F-theory [226]. There one can hope that all moduli will be lifted. This is the next step once we have reliable tools for understanding such vacua using the fibration description.

The second limitation is that we have not yet learned to describe configurations where the base of the $T^{3}$-fibration is not flat away from the degeneration locus. The examples of SYZ fibrations we construct (analogous to F-theory at constant coupling [66]) all involve composite degenerations which we do not know how to resolve. The set of rules we find for fitting these composite degenerations into compact examples will be a useful guide to the more difficult general case.

A number of intriguing observations arise in the course of our analysis. One can "geometrize" these non-geometric compactifications by realizing the action of the T-duality group as a geometric action on a $T^{4}$ fiber. The semi-flat metric on the fiber contains the original metric and the Hodge dual of the B-field. Hence, we are led to study seven-manifolds $\mathcal{X}_{7}$ which are $T^{4}$ fibrations over a 3 d base. They can be embedded into flat $T^{4}$ compactifications of $M$-theory down to seven dimensions where the reduced theory has an $S L(5)$ U-symmetry. U-duality then suggests that $\mathcal{X}_{7}$ may be a $G_{2}$ manifold since the non-geometric Type IIA configuration can be rotated into a purely geometric solution of maximal supergravity in seven dimensions. Whether or not these solutions can in general be lifted to eleven dimensions is a question for further investigation. In this chapter, we study explicit examples of $G_{2}$ (and Calabi-Yau) manifolds and show that they do provide perturbative non-geometric solutions to Type IIA in ten dimensions through this correspondence. The spectrum of these spaces can be computed by noticing that they admit an asymmetric orbifold description, and it matches that computed from M -theory when a comparison is
possible.
The chapter is organized as follows. In the next section we review the semiflat approximation to geometric compactification in various dimensions. We describe in detail the semiflat decomposition of an orbifold limit of a Calabi-Yau threefold: this will be used as a starting point for nongeometric generalizations in section 9.2. In section 9.1 we describe the effective field theory for type II strings on a Hat $T^{3}$. We show that the special class of field configurations which participate in $T^{3}$-fibrations with perturbative monodromies can alternatively be described in terms of geometric $T^{4}$-fibrations. We explain the U-duality map which relates these constructions to M-theory on $T^{4}$-fibered $G_{2}$-manifolds. In sections 9.2 and 9.3 we put this information together to construct nongeometric compactifications. In section 9.4 we consider generalizations where the fiber theory involves discrete Wilson lines. Hidden after the conclusions are many appendices. Appendix 9.6 gives more detail of the reduction on $T^{3}$. The purpose of Appendices 9.7-9.9 is to build confidence in and intuition about the semiflat approximation: Appendix 9.7 is a check on the relationship between the semiflat approximation and the exact solution which it approximates; Appendix 9.8 is a derivation of the Hanany-Witten brane-creation effect using the semiflat limit; Appendix 9.9 derives a known duality using the semiflat description. In Appendix 9.10 we record asymmetric orbifold descriptions of the nongeometric constructions of section 9.2. In Appendices 9.11 through 9.13, we study in detail the massless spectra of many of our constructions, and compare to the spectra of M-theory on the corresponding $G_{2}$-manifolds when we can. Appendix 9.14 contains templates to help the reader to build these models at home.

### 8.2 Semi-flat limit

Since we want to construct non-geometric spaces by means of T-duality, we exhibit the spaces as torus fibrations. We need isometries in the fiber directions in which the dualities act. Hence, we wish to study manifolds in a semi-flat limit where the fields do not depend on the fiber coordinates. This is the realm of the SYZ conjecture [220]. Mirror symmetry of Calabi-Yau manifolds implies that they have a special Lagrangian $T^{n}$ fibration. Branes can be wrapped on the fibers in a supersymmetric way and their moduli space is the mirror Calabi-Yau. At tree level, this moduli space is a semi-flat fibration, i.e. the metric has a $U(1)^{n}$ isometry along the fiber. However, there are world-sheet instanton corrections to this tree-level metric. Such corrections are suppressed (away from singular fibers) in the large volume limit. The mirror Calabi-Yau is then in the large complex structure limit. In this limit the metric is semi-flat and mirror symmetry boils down to T-duality along the fiber directions ${ }^{1}$.

As a warm-up, we will now briefly review the one-complex-dimensional case of a torus, and the two-dimensional case of stringy cosmic strings [113]. These sections may be skipped by experts. In Section 8.2.3, we construct a fibration for a three-dimensional orbifold that will in later sections be modified to a non-geometric compactification.

### 8.2.1 One dimension

The simplest example is the flat two-torus. Its complex structure is given by modding out the complex plane by a lattice generated by 1 and $\tau=\tau_{1}+i \tau_{2} \in \mathbb{C}$ (with $\tau_{2}>0$ ). The Kähler structure is $\rho=b+i V / 2$ where $b=\int_{T^{2}} B$ and $V$ the area of the torus (again, $V>0$ ).

There is an $S L(2, \mathbb{Z})_{\tau}$ group acting on the complex modulus $\tau$. This is a redundancy in defining the lattice. The group action is generated by $\tau \mapsto \tau+1$ and $\tau \mapsto-1 / \tau$. Another $S L(2, \mathbb{Z})_{\rho}$ group acts on $\rho$. This is generated by the shift in

[^35]the B-field $b \mapsto b+1$ and a double T-duality combined with a $90^{\circ}$ rotation that is $\rho \mapsto-1 / \rho$. The fundamental domain for the moduli is shown in Figure 8-1.

The torus can naturally be regarded as a semi-flat circle fibration over a circle. For special Lagrangian fibers, we choose the real slices in the complex plane. In the $\tau_{2} \rightarrow \infty$ large complex structure limit, these fibers are small compared to the base $S^{1}$ which is along the imaginary axis.

Mirror symmetry exchanges the complex structure $\tau$ with the Kähler structure $\rho$. This boils down to T-duality along the fiber direction according to the Buscher rules [49, 50]. It maps the large complex structure into large Kähler structure that is $\rho_{2}=V \rightarrow \infty$.


Figure 8-1: A possible fundamental domain (gray area) for the action of the $S L(2, \mathbb{Z})$ modular group on the upper half-plane. The upper-half plane parametrizes the possible values of $\tau$ (or $\rho$ ): the moduli of a two-torus. The gray domain can be folded into an $S^{2}$ with three special points (the two orbifold points: $\tau_{\mathbb{Z}_{6}}=e^{2 \pi i / 6}$ and $\tau_{\mathbb{Z}_{4}}=i$, and the decompactification point: $\tau \rightarrow i \infty)$.

### 8.2.2 Two dimensions

In order to construct semi-flat fibrations in two dimensions, let us consider the dynamics first. Type IIA on a flat two-torus can be described by the effective action in Einstein frame

$$
\begin{equation*}
S=\int d^{8} x \sqrt{g}\left(R+\frac{\partial_{\mu} \tau \partial^{\mu} \bar{\tau}}{\tau_{2}^{2}}+\frac{\partial_{\mu} \rho \partial^{\mu} \bar{\rho}}{\rho_{2}^{2}}\right) \tag{8.2.1}
\end{equation*}
$$

where $\tau$ is the complex structure of the torns, and $\rho=b+i V / 2$ is the Kähler modulus as described earlier. The action is invariant under the $S L(2, \mathbb{Z})_{\tau} \times S L(2, \mathbb{Z})_{\rho}$ perturbative duality group, which acts on $\tau$ and $\rho$ by fractional linear transformations.

Variation with respect to $\tau$ gives

$$
\begin{equation*}
\partial \bar{\partial} \tau+\frac{2 \dot{\partial} \tau \bar{\partial} \tau}{\bar{\tau}-\tau}=0 \tag{8.2.2}
\end{equation*}
$$

and $\rho$ obeys the same equation. Stringy cosmic string solutions to the EOM can be obtained by choosing a complex coordinate $z$ on two of the remaining eight dimensions, and taking $\tau(z)$ a holomorphic section of an $S L(2, \mathbb{Z})$ bundle. Such solutions are not modified by considering the following ansatz for the metric around the string ${ }^{2}$

$$
\begin{equation*}
d s^{2}=d s_{\mathrm{Mink}}^{2}+e^{v(z, \bar{z})} d z d \bar{z}+d s_{\mathrm{fiber}}^{2} \tag{8.2.3}
\end{equation*}
$$

where

$$
d s_{\text {fiber }}^{2}=\frac{1}{\tau_{2}}\left(\begin{array}{cc}
1 & \tau_{1}  \tag{8.2.4}\\
\tau_{1} & |\tau|^{2}
\end{array}\right)
$$

The Einstein equation is the Poisson equation,

$$
\begin{equation*}
\partial \bar{\partial} \psi=\partial \bar{\partial} \log \tau_{2} \tag{8.2.5}
\end{equation*}
$$

Far away from the strings, the metric of the base goes like [113]

$$
\begin{equation*}
d s_{2 D}^{2} \sim\left|z^{-N / 12} d z\right|^{2} \tag{8.2.6}
\end{equation*}
$$

where $N$ is the number of strings. This can be coordinate transformed by $\tilde{z}=z^{1-N / 12}$ to a flat metric with $2 \pi N / 12$ deficit angle.

Solutions and orbifold points. One could in principle write down solutions by

[^36]means of the $j$-function,
\[

$$
\begin{equation*}
j(\tau)=\eta(\tau)^{-24}\left(\theta_{1}^{8}(\tau)+\theta_{2}^{8}(\tau)+\theta_{3}^{8}(\tau)\right)^{3} \tag{8.2.7}
\end{equation*}
$$

\]

which maps the $\tau_{\mathbb{Z}_{6}}=e^{2 \pi i / 6}$ and $\tau_{\mathbb{Z}_{4}}=i$ orbifold points to 0 and 1 , respectively. The $\tau_{2} \rightarrow \infty$ degeneration point gets mapped to $j \rightarrow \infty$. A simple solution would then be

$$
\begin{equation*}
j(\tau)=\frac{1}{z-z_{0}}+j_{0} \tag{8.2.8}
\end{equation*}
$$

At infinity, the shape of the fiber is constant, i.e. $\tau_{\infty}=j^{-1}\left(j_{0}\right)$ and thus this non-compact solution may be glued to any other solution with constant $\tau$ at infinity. However, since $\tau$ covers the entire fundamental domain once, there will be two points in the base where $\tau(z)=\tau_{\mathbb{Z}_{6}}$ or $\tau_{\mathbb{Z}_{4}}$. Over these points, the fiber is an orbifold of the two-torus. These singular points cannot be resolved in a Ricci-flat way and we can't use this solution for superstrings.

There is, however, a six-string solution which evades this problem [113]. It is possible to collect six strings together in a way that $\tau$ approaches a constant value at infinity. $\tau$ can be given implicitly by e.g.

$$
\begin{equation*}
y^{2}=x(x-1)(x-2)(x-z) \tag{8.2.9}
\end{equation*}
$$

There are no orbifold points now because $\tau$ can be written as a holomorphic function over the base. The above equation describes three double degenerations, that is, three strings of tension twice the basic unit. In the limit when the strings are on top of one another, we obtain what is known (according to the Kodaira classification) as a $D_{4}$ singularity with deficit angle $180^{\circ}$.

The monodromy of the fiber around this singularity is described by

$$
\mathcal{M}_{D_{4}}=\left(\begin{array}{rr}
-1 & 0  \tag{8.2.10}\\
0 & -1
\end{array}\right)
$$

acting on $\binom{\omega_{1}}{\omega_{2}}$ with $\tau \equiv \frac{\omega_{1}}{\omega_{2}}$. This monodromy decomposes into that of six elementary
strings which are mutually non-local ${ }^{3}$.

This can be generalized to more than six strings using the Weierstrass equation

$$
\begin{equation*}
y^{2}=x^{3}+f(z) x+g(z) \tag{8.2.11}
\end{equation*}
$$

The modular parameter of the torus is determined by

$$
\begin{equation*}
j(\tau(z))=\frac{4 f^{3}}{4 f^{3}+27 g^{2}} \tag{8.2.12}
\end{equation*}
$$

Whenever the numerator vanishes, $\tau=\tau_{\mathbb{Z}_{6}}$ and we are at an orbifold point. We see however that it is a triple root of $f^{3}$ and no orbifolding of the fiber is necessary. The same applies for the $\mathbb{Z}_{4}$ points. The strings are located where $\tau_{2} \rightarrow \infty$ that is where the modular discriminant $\Delta \equiv 4 f^{3}+27 g^{2}$ vanishes. Note that the monodromy of the fibers around a smooth point is automatically the identity in such a construction.

Kodaira classification. Degenerations of elliptic fibrations have been classified according to their monodromy by Kodaira. For convenience, we summarize the result in the following table [39]:

[^37]| ord(f) | ord(g) | ord $(\Delta)$ | monodromy | singularity |
| :---: | :---: | :---: | :---: | :---: |
| $\geq 0$ | $\geq 0$ | 0 | $\left(\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right)$ | none |
| 0 | 0 | $n$ | $\left(\begin{array}{cc}1 & n \\ 0 & 1\end{array}\right)$ | $A_{n-1}$ |
| $\geq 1$ | 1 | 2 | $\left(\begin{array}{cc}1 & 1 \\ -1 & 0\end{array}\right)$ | none |
| 1 | $\geq 2$ | 3 | $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ | $A_{1}$ |
| $\geq 2$ | 2 | 4 | $\left(\begin{array}{cc}0 & 1 \\ -1 & -1\end{array}\right)$ | $A_{2}$ |
| 2 | $\geq 3$ | $n+6$ | $\left(\begin{array}{cc}-1 & -n \\ 0 & -1\end{array}\right)$ | $D_{n+4}$ |
| $\geq 2$ | 3 | 8 | $\left(\begin{array}{cc}-1 & -n \\ 0 & -1\end{array}\right)$ | $D_{n+4}$ |
| $\geq 3$ | $\geq 5$ | 9 | $\left(\begin{array}{cc}-1 & -1 \\ 1 & 0\end{array}\right)$ | $E_{6}$ |
| $\geq 4$ | 5 | 10 | $\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$ | $E_{7}$ |
| $\left(\begin{array}{cc}0 & -1 \\ 1 & 1\end{array}\right)$ | $E_{8}$ |  |  |  |

Constructing K3. One can construct a compact example where the fiber experiences $24 A_{0}$ degenerations. In the Weierstrass description $(2.11)$, this means that $f$ has degree $8, g$ has degree 12 , and $\Delta$ has degree 24 . This is the semi-flat description of a $K 3$ manifold. In a certain limit where we group the strings into four composite $D_{4}$ singularities, the base is flat and the total space becomes $T^{4} / \mathbb{Z}_{2}$. The base can be obtained by gluing four flat triangles as seen in Figure 8-3. At each $D_{4}$ degeneration, the base has $180^{\circ}$ deficit angle which adds up to $4 \pi$ and closes the space into a flat sphere with the curvature concentrated at four points.


Figure 8-2: Base of the $T^{4} / \mathbb{Z}_{2}$ orbifold. The $\mathbb{Z}_{2}$ action inverts the base coordinates and has four fixed points denoted by red stars. They have $180^{\circ}$ deficit angle. As the arrows show, one has to fold the diagram and this gives an $S^{2}$.

As we have seen, in two dimensions the Weierstrass equation solves the problem of orbifold points. In higher dimensions, we don't have this tool but we can still try to glue patches of spaces in order to get compact solutions. Gluing is especially easy if the base is flat. However, generically this is not the case. Having a look at the Einstein equation (8.2.2), we see that a flat base can be obtained if $\tau(z)$ is constant. This happens in the case of $D_{4}$ and $E_{n}$ singularities. Our discussion in this chapter will (unfortunately) be restricted to these singularities.

The cosmic string metric is singular in the above semi-flat description. It must be slightly modified in order to get a smooth Calabi-Yau metric for the total space. This will be discussed in Appendix 9.7.


Figure 8-3: Flat $S^{2}$ base constructed from four triangles: base of $K 3$ in the $\mathbb{Z}_{2}$ orbifold limit.

### 8.2.3 Three dimensions

In two dimensions, the only smooth compact Calabi-Yau is the $K 3$ surface. In three dimensions, there are many different spaces and therefore the situation is much more complicated. The SYZ conjecture [220] says that every Calabi-Yau threefold which has a geometric mirror, is a special Lagrangian $T^{3}$ fibration with possibly degenerate fibers at some points. For the generic case, the base is an $S^{3}$. Without the special Lagrangian condition, the conjecture has been well understood in the context of topological mirror symmetry $[115,223]$. There, the degeneration loci form a (real) codimension two subset in the base. A graph $\Gamma$ is formed by edges and trivalent vertices. The fiber suffers from monodromy around the edges. This is specified by a homomorphism

$$
\begin{equation*}
M: \pi_{1}\left(S^{3} \backslash \Gamma\right) \longrightarrow S L(3, \mathbb{Z}) \tag{8.2.13}
\end{equation*}
$$

There are two types of vertices which contribute $\pm 1$ to the Euler character of the total space ${ }^{4}$. At the vertices, the topological junction condition relates the monodromies of the edges.

One of the most studied non-trivial Calabi-Yau spaces is the quintic in $\mathbb{P}^{4}$. However, even the topological description of this example is fairly complicated [115]. The topological construction contains $250+50$ vertices and 450 edges in the $S^{3}$ base.

Constructing not only topological, but special Lagrangian SYZ fibrations is a much harder task. In fact, it is expected that away from the semi-flat limit, the real codi-

[^38]mension two singular loci in the base get promoted to codimension one singularities, i.e. surfaces in three dimensions. These were termed ribbon graphs [157] and their description remains elusive.

A compact orbifold example. In the following, we will describe the singular $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold in the SYZ fibration picture. One starts with $T^{6}$ that is a product of three tori with complex coordinates $z_{i}$. Without discrete torsion, the orbifold action is generated by the geometric transformations,

$$
\begin{align*}
& \alpha:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(-z_{1},-z_{2}, z_{3}\right)  \tag{8.2.14}\\
& \beta:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(-z_{1}, z_{2},-z_{3}\right) \tag{8.2.15}
\end{align*}
$$

These transformations have unit determinant and thus the resulting space may be resolved into a smooth Calabi-Yau manifold.

In order to obtain a fibration structure, we need to specify the base and the fibers. For the base coordinates, we choose $x_{i} \equiv \operatorname{Re}\left(z_{i}\right)$ and for the fibers $y_{i} \equiv \operatorname{Im}\left(z_{i}\right)$. Under the orbifold action, fibers are transformed into fibers and they don't mix with the base ${ }^{5}$.


Figure 8-4: Singularities in the base of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. The big dashed cube is the original $T^{3}$ base. The orbifold group generates the singular lines as depicted in the figure. The red dots show the intersection points of these edges.

[^39]Degeneration loci in the base. The base originally is a $T^{3}$. What happens after orbifolding? If we fix, for instance. the $x_{3}$ coordinate, then the orbifold action locally reduces to $\alpha$ (since the other two non-trivial group elements change $x_{3}$ ). This means that we simply obtain four fixed points in this slice of the base. This is exactly analogous to the $T^{4} / \mathbb{Z}_{2}$ example. The fixed points correspond to $D_{4}$ singularities with a deficit angle of $180^{\circ}$. As we change $x_{3}$, we obtain four parallel edges in the base. By keeping instead $x_{1}$ or $x_{2}$ fixed, we get perpendicular lines corresponding to conjugate $D_{4} \mathrm{~S}$ whose monodromies act on another $T^{2}$ in the $T^{3}$ fiber. Altogether, we get $3 \times 4$ lines of degeneration as depicted in Figure 8-4. These edges meet at (half-)integer points in the $T^{3}$ base.

Some parts of the base have been identified by the orbifold group. We can take this into account by a folding procedure which we have already seen for $T^{4} / \mathbb{Z}_{2}$. The degeneration loci are the edges of a cube. The volume of this cube is $\frac{1}{8}$ of the volume of the original $T^{3}$. The base can be obtained by gluing six pyramids on top of the faces (see Figure 8-5). The top vertices of these pyramids are the reflection of the center of the cube on the faces and thus the total volume is twice that of the cube. This polyhedron is a Catalan solid ${ }^{6}$ : the rhombic dodecahedron. (Note that one can also construct the same base by gluing two separate cubes together along their faces.)

In order to have a compact space, we finally glue the faces of the pyramids to neighboring faces (see the right-hand side of Figure 8-5). This is analogous to the case of $T^{4} / \mathbb{Z}_{2}$ where triangles were glued along their edges (Figure 8-2).

[^40]

Figure 8-5: (i) Rhombic dodecahedron: fundamental domain for the base of $T^{6} / \mathbb{Z}_{2} \times$ $\mathbb{Z}_{2}$. Six pyramids are glued on top of the faces of a cube. Neighboring pyramid triangles give rhombi since the vertices are coplanar (e.g. $A B C D$ ). (ii) The $S^{3}$ base can be constructed by identifying triangles as shown by the arrows. After gluing, the deficit angle around cube edges is $180^{\circ}$ which is appropriate for a $D_{4}$ singularity. The dihedral angles of the dashed lines are $120^{\circ}$ and since three of them are glued together, there is no deficit angle. The tips of the pyramids get identified and the space finally becomes an $S^{3}$.

The topology of the base. The base is an $S^{3}$ which can be seen as follows ${ }^{7}$. First fold the three rombi $A B C D, A F G D$ and $A B E F$, and the corresponding three on the other side of the fundamental domain. Then, we are still left with six rhombi that we need to fold. It is not hard to see that the problem is topologically the same as having a $B^{3}$ ball with boundary $S^{2}$. Twelve triangles cover the $S^{2}$ and we need to glue them together as depicted in Figure 8-6. This operation is the same as taking the $S^{2}$ and identifying its points by an $x \mapsto-x$ flip. This on the other hand, exhibits the space as an $S^{1}$ fibration over $D^{2}$. The fiber vanishes at the boundary of the disk. This is further equivalent to an $S^{2}$ fibration over an interval where the fiber vanishes at both endpoints. This space is simply an $S^{3}$. The degeneration loci are on the $S^{2}$ equator of this $S^{3}$ base and form the edges of the cube.

Edges and vertices. The monodromies of the edges are shown in Figure 8-7. The

[^41]

Figure 8-6: The base of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ is homeomorphic to a three-ball with an $S^{2}$ boundary which has to be folded as shown in the figure.
letters on the degeneration edges denote the following $S L(3)$ monodromies:

$$
x=\left(\begin{array}{rrr}
1 & 0 & 0  \tag{8.2.16}\\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right) \quad y=\left(\begin{array}{rrr}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right) \quad z=\left(\begin{array}{rrr}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$



Figure 8-7: Monodromies for the edges.

This orbifold example contained $D_{4}$ strings. These are composite edges made out of six "mutually non-local" elementary edges. The edges have $180^{\circ}$ deficit angle around them which is $6 \times \frac{\pi}{6}$ where $\frac{\pi}{6}$ is the deficit angle of the elementary string.

Note that the base is flat. This made it possible to easily glue the fundamental cell to itself yielding a compact space. Since the edges around any vertex meet in a symmetric way, the cancellation of forces is automatic.

There are other spaces that one can describe using $D_{4}$ edges and the above mentioned composite vertices. Some examples are presented in Section 9.2. The strategy is to make a compact space by gluing polyhedra like the above described cubes, then
make sure that the dihedral deficit angles are appropriate for the $D_{4}$ singularity.

### 8.2.4 Flat vertices

Codimension two degeneration loci meet at vertices in the base. In the generic case, these are trivalent vertices of elementary strings. Such strings have $30^{\circ}$ deficit angle around them measured at infinity. This creates a solid deficit angle around the vertex.

In some cases when composite singularities meet, the base is flat and the vertex is easier to understand. In particular, the total deficit angle arises already in the vicinity of the strings. An example was given in Section 8.2 .3 where composite vertices arise from the "collision" of three $D_{4}$ singularities (see Figure 8-5). The singular edges have a deficit angle $\pi$. The vertex can be constructed by taking an octant of three dimensional space and gluing another octant to it along the boundary walls. The curvature is then concentrated in the axes. The solid angle can be computed as twice the solid angle of an octant. This gives $\pi$ (or a deficit solid angle of $3 \pi$ ).


Figure 8-8: The solid angle at the apex is determined by the dihedral angles between the planes.

In the general (flat) case, a composite vertex may be described by gluing two identical cones (the analogs of octants). Such a cone is shown in Figure 8-8. Note that the solid angle spanned by three vectors is given by the formula

$$
\begin{equation*}
\theta=\alpha+\beta+\gamma-\pi \tag{8.2.17}
\end{equation*}
$$

where $\alpha, \beta$ and $\gamma$ are the dihedral angles at the edges. This can be used to compute
the solid angle around a composite vertex.


Figure 8-9: Flat vertex. $A, B$ and $C$ are singular edges. $C$ is pointing towards the reader. The dashed lines must be glued together to account for the deficit angle around $C$.

The singular edges have a tension which is proportional to the deficit angle around them. This leads to the problem of force balance. In Figure 8-9, a flat vertex is shown. The two solid lines $(A$ and $B)$ are degeneration loci. The third edge $(C)$ is pointing towards the reader as indicated by the arrow head. The deficit angle around $C$ is shown by the shaded area. In the weak tension limit (where we rescale the deficit angles by a small number), one condition for force balance is that these edges are in a plane. (Otherwise, energy could be decreased by moving the vertex.) This can be generalized for almost flat spaces by ensuring that $\alpha+\beta=\gamma$. This is automatic when we construct the neighborhood of a vertex by gluing two identical cones ${ }^{8}$.


Figure 8-10: Junction condition for monodromies. The red loop around $A$ can be smoothly deformed into two loops around $B$ and $C$.

Another problem to be solved is related to the fiber monodromies. These can be described by matrices $A, B$ and $C$ (see Figure 8-10). The loop around one of the

[^42]edges (say $A$ ) can be smoothly deformed into the union of the other two $(B, C)$. This gives the monodromy condition ${ }^{9} A B C=1$.

Some composite strings can be easier described than elementary ones because the base metric can be flat around them. Such singularities are $D_{4}, E_{6}, E_{7}$ and $E_{8}$ with deficit angles $\pi, 4 \pi / 3,3 \pi / 2$ and $5 \pi / 3$, respectively [113]. Vertices where composite lines meet can also be easily found by studying flat $\mathbb{C}^{3}$ orbifolds. Here we list some of the vertices that will later arise in the examples.

| orbifold group | colliding singularities | solid angle |
| :---: | :---: | :---: |
| $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ | $D_{4}-D_{4}-D_{4}$ | $\pi$ |
| $\mathbb{Z}_{2} \times \mathbb{Z}_{4}$ | $D_{4}-D_{4}-E_{7}$ | $\pi / 2$ |
| $\Delta_{12}$ | $D_{4}-E_{6}-E_{6}$ | $\pi / 3$ |
| $\Delta_{24}$ | $D_{4}-E_{6}-E_{7}$ | $\pi / 6$ |

Table 8.1: Examples for composite vertices.

We have already seen the $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ vertex in Section 8.2.3. If the vertex is located at the origin, then the strings are stretched along the coordinate axes,

$$
\begin{equation*}
D_{4}^{(1)}:(1,0,0) \quad D_{4}^{(2)}:(0,1,0) \quad D_{4}^{(3)}:(0,0,1) \tag{8.2.18}
\end{equation*}
$$

The second example is generated by

$$
\begin{aligned}
& \alpha:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(-z_{3}, z_{2}, z_{1}\right) \\
& \beta:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(z_{1},-z_{2},-z_{3}\right)
\end{aligned}
$$

It contains different colliding singularities. Their directions are given by

$$
\begin{equation*}
D_{4}^{(1)}:(1,0,0) \quad D_{4}^{(2)}:(1,0,1) \quad E_{7}:(0,1,0) \tag{8.2.19}
\end{equation*}
$$

[^43]The $\Delta_{12}$ group has $\left(\mathbb{Z}_{2}\right)^{2}$ and $\mathbb{Z}_{3}$ subgroups. It is generated by

$$
\begin{aligned}
\alpha:\left(z_{1}, z_{2}, z_{3}\right) & \mapsto\left(z_{2}, z_{3}, z_{1}\right) \\
\beta:\left(z_{1}, z_{2}, z_{3}\right) & \mapsto\left(-z_{1},-z_{2}, z_{3}\right)
\end{aligned}
$$

The strings directions are

$$
\begin{equation*}
D_{4}:(1,0,0) \quad E_{6}^{(1)}:(1,1,1) \quad E_{6}^{(2)}:(1,1,-1) \tag{8.2.20}
\end{equation*}
$$

The last example is generated by combining $\mathbb{Z}_{3}$ and $\mathbb{Z}_{4}$ generators,

$$
\begin{aligned}
\alpha:\left(z_{1}, z_{2}, z_{3}\right) & \mapsto\left(z_{2}, z_{3}, z_{1}\right) \\
\beta:\left(z_{1}, z_{2}, z_{3}\right) & \mapsto\left(-z_{2}, z_{1}, z_{3}\right)
\end{aligned}
$$

which generate the $\Delta_{24}$ group. The direction of the strings are the following,

$$
\begin{equation*}
D_{4}:(1,1,0) \quad E_{6}:(1,1,1) \quad E_{7}:(1,0,0) \tag{8.2.21}
\end{equation*}
$$

This is not an exhaustive list; a thorough study based on the finite subgroups of $S U(3)$ [79] would be interesting.

## Chapter 9

## Non-geometric spaces

### 9.1 Stringy monodromies

In this section, we wish to extend the discussion by including the full perturbative duality group of type II string theory on $T^{3}$ in the possible set of monodromies. We will find that this duality group can be interpreted as the geometric duality group of an auxiliary $T^{4}$. The extra circle is to be distinguished from the M-theory circle but it is related to it by a U-duality transformation.

For simplicity, the Ramond-Ramond field strengths will be turned off. This allows us to use perturbative dualities only. However, in moduli stabilization these fields play an important role. In fact, in the Appendices 9.8 and 9.9, we use U-duality [153] monodromies which act on RR-fields in order to describe two familiar phenomena.

From the worldsheet point of view, string compactifications are expected to be typically non-geometric, since the 2d CFT does not necessarily have a geometric target space. Even though we construct our examples directly based on intuition from supergravity, they will have a worldsheet description as modular invariant asymmetric orbifolds.

For other related works on non-geometric spaces, see $[176,138,164,148,117,110$, $162,121,111,180,64,151,90,152,217,218,25]$ and references therein.

In the following, we study the perturbative duality group in Type IIA string theory compactified on a flat three-torus. We gain intuition by studying the reduced

7d Lagrangian of the supergravity approximation. Finally we discuss how U-duality relates non-geometric compactifications to $G_{2}$ manifolds in M-theory which will be fruitful when constructing examples in the next section.

### 9.1.1 Reduction to seven dimensions

Action and symmetries. Let us consider the bosonic sector of (massless) 10d Type IIA supergravity,

$$
\begin{equation*}
S_{\mathrm{IIA}}=S_{\mathrm{NS}}+S_{\mathrm{R}}+S_{\mathrm{CS}} \tag{9.1.1}
\end{equation*}
$$

where

$$
\begin{gather*}
S_{\mathrm{NS}}=\frac{1}{2 \kappa_{10}^{2}} \int d^{10} x \sqrt{-g} e^{-2 \varphi}\left(R+4 \partial_{\mu} \varphi \partial^{\mu} \varphi-\frac{1}{2}\left|H_{3}\right|^{2}\right)  \tag{9.1.2}\\
S_{\mathrm{R}}=-\frac{1}{4 \kappa_{10}^{2}} \int d^{10} x \sqrt{-g}\left(\left|F_{2}\right|^{2}+\left|\tilde{F}_{4}\right|^{2}\right) \tag{9.1.3}
\end{gather*}
$$

and the Chern-Simons term is

$$
\begin{equation*}
S_{\mathrm{CS}}=-\frac{1}{4 \kappa_{10}^{2}} \int B \wedge F_{4} \wedge F_{4} \tag{9.1.4}
\end{equation*}
$$

with $\tilde{F}_{4}=d A_{3}-A_{1} \wedge d B$ and $\kappa_{10}^{2}=\kappa_{11}^{2} / 2 \pi R$.
First we set the RR fields to zero ${ }^{1}$. This truncates the theory to the NS part which is identical to the IIB $S_{\text {NS }}$ action. Compactifying Type IIA on a flat $T^{3}$ yields the perturbative T-duality group $S O(3,3, \mathbb{Z})$ which acts on the coset $S O(3,3, \mathbb{R}) / S O(3)^{2}$.

The equivalences of Lie algebras

$$
\begin{gather*}
\mathfrak{s o}(3,3) \cong \mathfrak{s l}(4)  \tag{9.1.5}\\
\mathfrak{s o}(3) \oplus \mathfrak{s o}(3) \cong \mathfrak{s u}(2) \oplus \mathfrak{s u}(2) \cong \mathfrak{s o}(4) \tag{9.1.6}
\end{gather*}
$$

enable us to realize the T-duality group as an $S L(4, \mathbb{Z})$ action on $S L(4, \mathbb{R}) / S O(4)$. This latter space is simply the moduli space of a flat $T^{4}$ with constant volume. Therefore, we can think of the T-duality group as the mapping class group of an auxiliary

[^44]four-torus of unit volume. What is the metric on this $T^{4}$ in terms of the data of the $T^{3}$ ? To answer this question, we have to study the Lagrangian.

Reduction to seven dimensions. One obtains the following terms after reduction on $T^{3}$ [188] (see Appendix 9.6 for more details and notation)

$$
\begin{equation*}
S=\int d x \sqrt{-g} e^{-\varphi} \mathcal{L} \tag{9.1.7}
\end{equation*}
$$

with $\mathcal{L}=\mathcal{L}_{1}+\mathcal{L}_{2}+\mathcal{L}_{3}+\mathcal{L}_{4}$ and

$$
\begin{align*}
\mathcal{L}_{1} & =R+\partial_{\mu} \varphi \partial^{\mu} \varphi  \tag{9.1.8}\\
\mathcal{L}_{2} & =\frac{1}{4}\left(\partial_{\mu} G_{\alpha \beta} \partial^{\mu} G^{\alpha \beta}-G^{\alpha \beta} G^{\gamma \delta} \partial_{\mu} B_{\alpha \gamma} \partial^{\mu} B_{\beta \delta}\right)  \tag{9.1.9}\\
\mathcal{L}_{3} & =-\frac{1}{4} g^{\mu \rho} g^{\nu \lambda}\left(G_{\alpha \beta} F_{\mu \nu}^{(1) \alpha} F_{\rho \lambda}^{(1) \beta}+G^{\alpha \beta} H_{\mu \nu \alpha} H_{\rho \lambda \beta}\right)  \tag{9.1.10}\\
\mathcal{L}_{4} & =-\frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} \tag{9.1.11}
\end{align*}
$$

The relation of these fields and the ten dimensional fields are presented in Appendix 9.6. In order to see the $S O(d, d, \mathbb{Z})$ symmetry, one introduces the symmetric positive definite $2 d \times 2 d$ matrix

$$
M=\left(\begin{array}{cc}
G^{-1} & G^{-1} B  \tag{9.1.12}\\
B G^{-1} & G-B G^{-1} B
\end{array}\right) \in S O(3,3)
$$

The kinetic terms $\mathcal{L}_{2}$ can be written as the $\sigma$-model Lagrangian

$$
\begin{equation*}
\mathcal{L}_{2}=\frac{1}{8} \operatorname{Tr}\left(\partial_{\mu} M^{-1} \partial^{\mu} M\right) \tag{9.1.13}
\end{equation*}
$$

The other terms in the Lagrangian are also invariant under $S O(3,3)$.

The SL(4) duality symmetry and "N-theory". Let us now put the bosonic action in a manifestly $S L(4)$ invariant form (see [47]). Rewrite $\mathcal{L}_{2}$ as

$$
\begin{equation*}
\mathcal{L}_{2}=\frac{1}{8} \operatorname{Tr}\left(\partial_{\mu} M^{-1} \partial^{\mu} M\right)=\frac{1}{4} \operatorname{Tr}\left(\partial_{\mu} N^{-1} \partial^{\mu} N\right) \tag{9.1.14}
\end{equation*}
$$

where we introduced the symmetric $S L(4)$ matrix $^{2}$

$$
\begin{gather*}
N_{4 \times 1}=(\operatorname{det} G)^{-1 / 2}\left(\begin{array}{cc}
G & G \vec{b} \\
\vec{b}^{T} G & \operatorname{det} G+\vec{b}^{T} G \vec{b}
\end{array}\right)  \tag{9.1.15}\\
B_{i j}=\varepsilon_{i j k} b_{k} \quad b_{i}=\frac{1}{2} \varepsilon_{i j k} B_{j k} . \tag{9.1.16}
\end{gather*}
$$

The equality of the Lagrangians can be checked by lengthy algebraic manipulations (or a computer algebra software). We included the Hodge-dualized B-field in the metric as a Kaluza-Klein vector. The inverse of $N$ is

$$
N^{-1}=(\operatorname{det} G)^{-1 / 2}\left(\begin{array}{cc}
(\operatorname{det} G) G^{-1}+\vec{b}^{T} b & -\vec{b}  \tag{9.1.17}\\
-\vec{b}^{T} & 1
\end{array}\right)
$$

Keeping $N$ symmetric, the Lagrangian is invariant under the global transformation,

$$
\begin{equation*}
N(x) \mapsto U^{T} N(x) U, \quad \text { with } U \in S L(4) \tag{9.1.18}
\end{equation*}
$$

A useful device for interpreting $N$ is the following. Note that we would get the exact same bosonic terms of $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$, if we were to reduce an eleven dimensional classical theory to seven dimensions. This theory is given by the Einstein-Hilbert action plus a scalar, the "11d dilaton" ${ }^{3}$

$$
\begin{equation*}
S=\int d^{11} x \sqrt{-\tilde{g}} e^{-\varphi}\left(R(\tilde{g})+\partial_{\mu} \varphi \partial^{\mu} \varphi\right) \tag{9.1.19}
\end{equation*}
$$

This Lagrangian contains no B-field. The description in terms of (9.1.19) is only useful when $\mathcal{L}_{3}$ and $\mathcal{L}_{4}$ vanish. This means that $F_{\mu \nu}^{(1) \alpha}=H_{\mu \nu \alpha}=H_{\mu \nu \rho}=0$. Since the size of $T^{4}$ is constant, its dimensions are not treated on the same footing as the three geometric fiber dimensions. It is similar to the situation in F-theory [226], where the area of the $T^{2}$ is fixed and the Kähler modulus of the torus is not a dynamical

[^45]parameter.
We have seen that the matrix $N$ can be interpreted as a semi-flat metric on a $T^{4}$ torus fiber. Part of this torus is the original $T^{3}$ fiber and the overall volume is set to one. The T-duality group $S O(3,3, \mathbb{Z})$ acts on $T^{4}$ in a geometric way. This means that we can hope to study non-geometric compactifications by studying purely geometric ones in higher dimension.

### 9.1.2 The perturbative duality group

In the previous section, we have transformed the coset space $S O(3,3) / S O(3)^{2}$ into $S L(4) / S O(4)$ via Eq. (9.1.15). We also would like to see how the discrete T-duality group $S O(3,3, \mathbb{Z})$ maps to $S L(4, \mathbb{Z})$. We will denote the $S O(3,3)$ matrices by $\mathcal{Q}$, and the $S L(4)$ matrices by $\mathcal{W}$.

| SO(3,3) | SL(4) | $\operatorname{dim}$ | examples |
| :---: | :---: | :---: | :---: |
| spinor | fundamental | 4 | RR fields |
| fundamental | antisym. tensor | 6 | momenta \& winding |

Table 9.1: The two basic representations of the duality group.

Generators of $\mathbf{S O}(3,3, \mathbb{Z})$. It was shown in [209] that the following $S O(3,3, \mathbb{Z})$ elements generate the whole group
$\mathcal{Q}_{1}(n)=\left(\begin{array}{c|c}\mathbb{1}_{3 \times 3} & n \\ \hline 0 & \mathbb{1}_{3 \times 3}\end{array}\right) \quad \mathcal{Q}_{2}(R)=\left(\begin{array}{c|c}R & 0 \\ \hline 0 & \left(R^{-1}\right)^{T}\end{array}\right) \quad \mathcal{Q}_{3}=\left(\begin{array}{ccc|ccc}0 & & & 1 & \\ & 0 & & \\ & & 1 & \\ & & 1 & & & 0 \\ \hline 1 & & & 0 & \\ & 1 & & & 0 & \\ & & 0 & & 1\end{array}\right)$
where $n^{T}=-n$, $\operatorname{det} R= \pm 1$. The first two matrices correspond to a change of basis of the compactification lattice. The last matrix is T-duality along the $x^{7}-x^{8}$ coordinates. Instead of using $\mathcal{Q}_{3}$ directly, we combine double T-duality with a $90^{\circ}$
rotation. This gives the $S O(3,3)$ matrix

$$
\widetilde{\mathcal{Q}}_{3}=\left(\begin{array}{cc|cc} 
& & & -1  \tag{9.1.21}\\
& & 1 & \\
& & & \\
\hline & 1 & & \\
\hline-1 & & & \\
\hline & & & \\
& & &
\end{array}\right)
$$

Generators of $\mathrm{SL}(4, Z)$. In the Appendix of [46], it was shown that the above matrices have an integral $4 \times 4$ spinor representation and in fact generate the entire $S L(4, \mathbb{Z})$. We now list the spinor representations corresponding to these generators ${ }^{4}$.

- $\mathcal{Q}_{1}(n)$ is mapped to matrices

$$
\mathcal{W}_{1}(n)=\left(\begin{array}{cc|cc}
1 & & & n_{23}  \tag{9.1.22}\\
& 1 & & n_{31} \\
\hline & 1 & n_{12} \\
& & 1
\end{array}\right)
$$

These are the generators corresponding to " T " transformations of various $S L(2)$ subgroups.

- $\widetilde{\mathcal{Q}}_{3}$ is mapped to

$$
\begin{equation*}
\widetilde{\mathcal{W}}_{3}=\left(\right) \tag{9.1.23}
\end{equation*}
$$

This corresponds to a modular " S " transformation. Note that $\left(\widetilde{\mathcal{W}}_{3}\right)^{2} \neq \mathbb{1}$.

[^46]- When $\operatorname{det} R=+1$, the matrix $\mathcal{Q}_{2}(R)$ is mapped to the $S L(4, \mathbb{Z})$ matrix

$$
\mathcal{W}_{2}(R)=\left(\begin{array}{ll}
R & 0  \tag{9.1.24}\\
0 & 1
\end{array}\right)
$$

For symmetric $R$ matrices it coincides with the prescription of Eq. (9.1.15).

- The $\operatorname{det} R=-1$ case is more subtle. Even though Type IIA string theory is parity invariant, in the microscopic description reflecting an odd number of coordinates does not give a symmetry by itself. Since this transformation flips the spinor representations $16 \leftrightarrow 16^{\prime}$, it must be accompanied by an internal symmetry $\Omega$ which changes the orientation of the world-sheet and thus exchanges the left-moving and right-moving spinors.
$S O(3,3)$ has maximal subgroup $S(O(3) \times O(3))$ and hence has two connected components [117]. Inversion of an odd number of coordinates is not in the identity component. $S L(4, \mathbb{Z})$ is the double cover of the connected component of $S O(3,3, \mathbb{Z})$ only. We must allow for $\operatorname{det} \mathcal{W}= \pm 1$ to obtain $\operatorname{Spin}(3,3, \mathbb{Z})$, the double cover of the full $S O(3,3, \mathbb{Z})$. Then, the reflections of the $x^{7}, x^{8}$ or $x^{9}$ coordinates have the following representations ${ }^{5}$

$$
\begin{equation*}
\mathcal{W}_{I_{7}}=\operatorname{diag}(-1,1,1,1) \quad \mathcal{W}_{I_{8}}=\operatorname{diag}(1,-1,1,1) \quad \mathcal{W}_{I_{9}}=\operatorname{diag}(1,1,-1,1) \tag{9.1.25}
\end{equation*}
$$

Upon restriction to $G L(3) \subset S O(3,3)$, the $\operatorname{Spin}(3,3)$ group is a trivial covering.

Ramond-Ramond fields transform in the spinor representation of the T-duality group ${ }^{6}$. Therefore they form fundamental $S L(4)$ multiplets. We can check the above representation for the coordinate reflections. Reflection of say $x^{7}$

[^47]combined with a flip of the three-form field gives
\[

$$
\begin{equation*}
\left(C_{7}, C_{8}, C_{9}, C_{789}\right) \mapsto\left(-C_{7}, C_{8}, C_{9}, C_{789}\right) \tag{9.1.26}
\end{equation*}
$$

\]

which is precisely the action of $\mathcal{W}_{I_{7}}$.

### 9.1.3 Embedding $S L(2)^{2}$ in $S L(4)$

In order to get some intuition for the $S L(4)$ duality group that we discussed in the previous section, we first look at the simpler case of $T^{2}$ compactifications. In this section we describe how the T-duality group of $T^{2}$ compactifications can be embedded into the bigger $S L(4)$ group.

In eight dimensions, the duality group is $S L(2)_{\tau} \times S L(2)_{\rho}$ with the first factor acting on the $\tau$ complex structure of the torus and the second factor acting on $\rho=$ $b+i V / 2$ where $b=\int_{T^{2}} B$ and $V$ is the volume of $T^{2}$. If we consider a two dimensional base with complex coordinate $z$, then the equations of motion are satisfied if $\tau(z)$ and $\rho(z)$ are holomorphic sections of $S L(2, \mathbb{Z})$ bundles. Monodromies of $\tau$ around branch points points describe the geometric degenerations of the fibration. Monodromies of $\rho$, however, correspond to T-dualities and to the semi-flat description of NS5-branes. In particular, if there is a monodromy $\rho \mapsto \rho+1$ around a degeneration point in the base, then it implies $b \mapsto b+1$ which describes a unit magnetic charge for the B-field, i.e. an NS5-brane. The $\rho \mapsto-1 / \rho$ monodromy on the other hand is a double T-duality along the $T^{2}$ combined with a $90^{\circ}$ rotation.

Let us denote the two-torus coordinates by $x^{7,8}$. In order to embed this $S L(2) \times$ $S L(2)$ duality group into the $S L(4)$ of $T^{3}$ compactifications, we need to further compactify on a "spectator" circle of size $L$. We denote its coordinate by $x^{9}$. The metric on $T^{3}\left(x^{9}-x^{10}-x^{11}\right)$ is now

$$
G_{3 \times 3}=\left(\begin{array}{ll|l}
g_{11} & g_{12} &  \tag{9.1.27}\\
g_{21} & g_{22} & \\
\hline & & L^{2}
\end{array}\right)
$$

Then, one can construct the $4 \times 4$ metric on $T^{4}$ by the prescription of (9.1.15) which gives

$$
N=(\operatorname{det} g)^{-1 / 2}\left(\begin{array}{c|cc}
\frac{1}{L} g_{2 \times 2} & &  \tag{9.1.28}\\
& L & L b \\
& L b & L\left(\operatorname{det} g+b^{2}\right)
\end{array}\right) \equiv\left(\begin{array}{cc}
\frac{1}{L} \mathcal{T}_{2 \times 2} & \\
& L \mathcal{R}_{2 \times 2}
\end{array}\right)
$$

with

$$
\mathcal{T}=\frac{1}{\tau_{2}}\left(\begin{array}{cc}
1 & \tau_{1}  \tag{9.1.29}\\
\tau_{1} & |\tau|^{2}
\end{array}\right) \quad \mathcal{R}=\frac{1}{\rho_{2}}\left(\begin{array}{cc}
1 & \rho_{1} \\
\rho_{1} & |\rho|^{2}
\end{array}\right)
$$

The $\sigma$-model Lagrangian

$$
\begin{equation*}
\operatorname{Tr}\left(\partial_{\mu} N^{-1} \partial^{\mu} N\right)=-2\left(\frac{\partial_{\mu} \tau \partial^{\mu} \bar{\tau}}{\tau_{2}^{2}}+\frac{\partial_{\mu} \rho \partial^{\mu} \bar{\rho}}{\rho_{2}^{2}}\right) \tag{9.1.30}
\end{equation*}
$$

indeed gives the familiar kinetic terms for the torus moduli (in seven dimensions).
We have seen how the metric and the B-field parametrize the relevant subset of the $S L(4, \mathbb{R}) / S O(4)$ coset space. The generators of the $S L(2, \mathbb{Z}) \times S L(2, \mathbb{Z})$ duality group are also mapped to elements in $S L(4, \mathbb{Z})$. We now verify that these images in fact give the transformations that we expect.

## - Geometric transformations

These are simply generated by

$$
T=\left(\begin{array}{ll}
1 & 1  \tag{9.1.31}\\
0 & 1
\end{array}\right) \oplus \mathbb{1}_{2 \times 2} \quad \text { and } \quad S=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right) \oplus \mathbb{1}_{2 \times 2}
$$

They act on $g_{2 \times 2}$ by conjugation with the non-trivial $S L(2)$ part as expected. The determinant of $g$ stays the same. The first one is a Dehn-twist and the second one is a $90^{\circ}$ rotation.

- Non-geometric transformations

The generators

$$
T^{\prime}=\mathbb{1}_{2 \times 2} \oplus\left(\begin{array}{cc}
1 & 1  \tag{9.1.32}\\
0 & 1
\end{array}\right) \quad \text { and } \quad S^{\prime}=\mathbb{1}_{2 \times 2} \oplus\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)
$$

correspond respectively to the shift of the B-field and to a double T-duality on $x^{7,8}$ combined with a $90^{\circ}$ rotation. The latter one has the $S L(4)$ monodromy

$$
M=\left(\begin{array}{rr|rr}
1 & 0 & 0 & 0  \tag{9.1.33}\\
0 & 1 & 0 & 0 \\
\hline 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

This is basically an exchange of the $x^{9}-x^{10}$ coordinates and it transforms the $\mathcal{R}_{2 \times 2}$ submatrix of $N$ into its inverse

$$
\mathcal{R}^{-1}=(\operatorname{det} g)^{-1 / 2}\left(\begin{array}{cc}
\operatorname{det} g+b^{2} & -b  \tag{9.1.34}\\
-b & 1
\end{array}\right)
$$

After this double T-duality, the (geometric) metric on $T^{3}$ becomes

$$
G_{3 \times 3} \mapsto \widetilde{G}_{3 \times 3}=\left(\begin{array}{c|c}
\frac{1}{\operatorname{det} g+b^{2}} g_{2 \times 2} & 0  \tag{9.1.35}\\
\hline 0 & L^{2}
\end{array}\right)
$$

The B-field transforms as

$$
\begin{equation*}
b \mapsto \tilde{b}=-\frac{b}{\operatorname{det} g+b^{2}} \tag{9.1.36}
\end{equation*}
$$

The metric $g$ on $T^{2}$ changes, in particular if $b=0$, then the volume gets inverted. Since we exchanged the $x^{9}-x^{10}$ coordinates, one might have expected that this affects the metric on $x^{9}$. However, we see that it remains the same as it should since it was only a spectator circle.

- Left-moving spacetime fermion number: $(-1)^{F_{L}}$

This is a global transformation which inverts the sign of the Ramond-Ramond fields. It acts trivially on the vector representation of $S O(3,3)$ (which is the antisymmetric tensor of $S L(4)$ ). It will be important since T -duality squares to $(-1)^{F_{L}}$. In [138], its representation was determined,

$$
\mathcal{M}_{(-1)^{F_{L}}}=\left(\begin{array}{cc}
-1 & 0  \tag{9.1.37}\\
0 & -1
\end{array}\right) \oplus\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right) \in S L(2) \times S L(2)
$$

that is a $D_{4}$ monodromy combined with a $D_{4}^{\prime}$ (i.e. a conjugate $D_{4}$ ). This statement can be proven as follows. Let us define complex coordinates

$$
\begin{align*}
& z_{L}=x_{L}^{7}+i x_{L}^{8}  \tag{9.1.38}\\
& z_{R}=x_{R}^{7}+i x_{R}^{8} \tag{9.1.39}
\end{align*}
$$

where $x_{L}$ and $x_{R}$ are the left- and right-moving components of the bosonic coordinates. We denote a transformation

$$
\begin{equation*}
\left(z_{L}, z_{R}\right) \mapsto\left(e^{\theta_{L}} z_{L}, e^{\theta_{R}} z_{R}\right) \tag{9.1.40}
\end{equation*}
$$

by $\theta=\left(\theta_{L}, \theta_{R}\right)$. Then,

$$
\begin{equation*}
\theta_{D_{4}}=(-\pi,-\pi) \tag{9.1.41}
\end{equation*}
$$

as it is a reflection of the bosonic coordinates. Moreover, we can use $D_{4}^{\prime}=S^{2}$ where $S$ is a double T-duality with a $90^{\circ}$ rotation. We have

$$
\begin{equation*}
\theta_{S}=\underbrace{(-\pi, 0)}_{\text {double T-duality }}+\underbrace{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)}_{90^{\circ} \text { rotation }}=\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{9.1.42}
\end{equation*}
$$

from which we obtain

$$
\begin{equation*}
\theta_{D_{4}^{\prime}}=2 \times \theta_{S}=(-\pi, \pi) \tag{9.1.43}
\end{equation*}
$$

Finally,

$$
\begin{equation*}
\theta_{D_{4}+D_{4}^{\prime}}=\theta_{D_{4}}+\theta_{D_{4}^{\prime}}=(-2 \pi, 0) \tag{9.1.44}
\end{equation*}
$$

which acts trivially on the bosons. However, it inverts the sign of the spinors from left movers which is precisely the action of $(-1)^{F_{L}}$. Finally, it can be embedded into $S L(4)$ simply as

$$
\begin{equation*}
\mathcal{M}_{(-1)^{F_{L}}}=\operatorname{diag}(-1,-1,-1,-1) \tag{9.1.45}
\end{equation*}
$$

### 9.1.4 U-duality and $G_{2}$ manifolds

We have seen that upon compactifying Type IIA on $T^{3}$, a $T^{4}$ torus emerges. We will be eventually interested in compactifications to four dimensions. For vacua without fluxes and T-dualities, the total space of the $T^{3}$ fibration is a Calabi-Yau threefold. What can we say about the total space of the $T^{4}$ fibration?

Note that there is an analogous (more general) story in M-theory. Reducing eleven dimensional supergravity on a flat $T^{4}$ yields a Lagrangian that is symmetric under the $S L(5, \mathbb{R})$ U-duality group [153, 57, 216, 58]. By Hodge-dualizing the three-form $A_{I J K}=: \varepsilon_{I J K L} X^{L}(I, J, K, L=7,8,9,11)$, one can define a $5 \times 5$ matrix $^{7}$

$$
G^{-1}=\left(\begin{array}{c|c}
\omega g^{I J}+\frac{1}{\omega} X^{I} X^{J} & -\frac{1}{\omega} X^{I}  \tag{9.1.46}\\
\hline-\frac{1}{\omega} X^{I} & \frac{1}{\omega}
\end{array}\right)
$$

which contains the geometric metric $g$ on $T^{4}$ as well. We denote the dimensions ${ }^{8}$ by $x^{7}, x^{8}, x^{9}, x^{11}, x^{10}$, respectively. The bosonic kinetic terms can be written as a manifestly $S L(5)$ invariant $\sigma$-model in terms of this metric [58].

We can embed the $4 \times 4$ unit determinant matrix $N^{-1}$ (see Eq. 9.1.17) into the

[^48]$5 \times 5$ unit-determinant matrix $G^{-1}$ as follows
\[

G^{-1}=\left($$
\begin{array}{cc|c}
\delta g^{i j}+\frac{1}{\delta} b^{i} b^{j} & 0 & -\frac{1}{\delta} b^{i}  \tag{9.1.47}\\
0 & 1 & 0 \\
\hline-\frac{1}{\delta} b^{i} & 0 & \frac{1}{\bar{\delta}}
\end{array}
$$\right)
\]

with $\delta \equiv\left(\operatorname{det} g_{i j}\right)^{1 / 2}$. By setting $\omega:=\delta$, we arrive at the previous form of the metric. If we now perform a U-duality corresponding to the $x^{10}-x^{11}$ flip, then the solution is transformed into pure geometry in the 11d picture,

$$
G^{-1}=\left(\begin{array}{cc|c}
\delta g^{i j}+\frac{1}{\delta} b^{i} b^{j} & -\frac{1}{\delta} b^{i} &  \tag{9.1.48}\\
-\frac{1}{\delta} b^{i} & \frac{1}{\delta} & \\
\hline & 1
\end{array}\right) \equiv\left(\begin{array}{c|c}
g_{\text {new }}^{I J} & \\
\hline & 1
\end{array}\right)
$$

In 10d Type IIA language, this flip roughly corresponds to the exchange of the Ramond-Ramond one-form and the Hodge-dual of the B-field in the fiber directions.

In order to preserve minimal supersymmetry in four dimensions, one compactifies M-theory on a $G_{2}$ manifold. Semi-flat limits of $G_{2}$ manifolds are expected to exist by an SYZ-like argument [120]. Then, by the above U-duality in seven dimensions, a solution is obtained which is non-geometric from a 10 d point of view as shown in this diagram

"Oxidation" seems obscure in this context since we only have the 7d spacetime equations of motion. However, for the special case of $D_{4}$ singularities, we will be able to "lift the solutions" to 10d: they turn out to be asymmetric orbifolds, similar to some examples in [138].

### 9.2 Compactifications with $\mathrm{D}_{4}$ singularities

In the previous sections, we studied the semi-flat limit of various geometries which had a fibration structure. This corner of the moduli space is a natural playground for T-duality since isometries appear along the fiber directions. Almost everywhere the space locally looks like $\mathbb{R}^{n} \times T^{n}$ and the duality group can simply be studied by a torus reduction of the supergravity Lagrangian. The idea is then to glue patches of the base manifold by also including the T-duality group in the transition functions. Since the duality group is discrete, such deformations are "topological" and a priori cannot be achieved continuously. From the 10d point of view, the total space becomes nongeometric in general. In seven dimensions, the $S O(3,3, \mathbb{Z})$ group can be realized as the mapping class group of a $T^{4}$ of unit volume. This geometrizes the non-geometric space by going one real dimension higher. Considering such compactifications to four dimensions which preserve $\mathcal{N}=1$ supersymmetry, U-duality suggests that the total space of the geometrized internal non-geometric space is a $G_{2}$ manifold.

In this section, we use these ideas to build non-geometric compactifications. We deform geometric orbifold spaces by hand and also study particular examples of $G_{2}$ manifolds. These examples will only contain (conjugate) $D_{4}$ singularities. This allows for a constant arbitrary shape for the fiber and the base is also locally flat. Even though the examples are singular and supergravity breaks down at the orbifold points, we can embed the solutions into Type IIA string theory where they give consistent non-geometric vacua realized as modular invariant asymmetric orbifolds.

### 9.2.1 Modified $K 3 \times T^{2}$

Let us first consider $K 3$. The base of an elliptic fibration of $K 3$ is an $S^{2}$. At the $T^{4} / \mathbb{Z}_{2}$ orbifold point, there are four $D_{4}$ singularities in the base (see Figure 8-3). The purely geometric $D_{4}$ monodromies are

$$
\begin{equation*}
\mathcal{M}_{D_{4}}=\left(-\mathbb{1}_{2 \times 2}\right) \oplus \mathbb{1}_{2 \times 2} \in S L(2)_{\tau} \times S L(2)_{\rho} \tag{9.2.49}
\end{equation*}
$$

By changing the monodromies by hand, it is possible to construct non-geometric spaces. In [138], K3 was modified into the union of two half K3's which we denote by $\widetilde{K 3}$. This non-geometric space has two ordinary $D_{4}$ 's and two non-geometric $D_{4}^{\prime}$ singularities with monodromies

$$
\begin{equation*}
\mathcal{M}_{D_{4}^{\prime}}=\mathbb{1} \oplus(-\mathbb{1}) \tag{9.2.50}
\end{equation*}
$$

If we had changed one or three $D_{4} \mathrm{~S}$ into $D_{4}^{\prime}$, then the monodromy at infinity would not be trivial. In fact, it would be $\mathcal{M}_{D_{4}} \cdot \mathcal{M}_{D_{4}^{\prime}}=\mathcal{M}_{(-1)^{F_{L}}}$. This means that the $T^{2} \times T^{2}$ fiber is orbifolded everywhere in the base by the $\mathbb{Z}_{2}$ action which inverts the fiber coordinates. In principle, this could be interpreted as an overall orbifolding by $(-1)^{F_{L}}$ which moves us from Type IIA to IIB. However, it is not clear what should happen to the odd number of $D_{4}$ and $D_{4}^{\prime}$ singularities as they don't have a trivial monodromy at infinity in IIB either. Therefore, we do not consider such examples any further.

Let us now compactify further and consider $K 3 \times T^{2}$ or $\widetilde{K 3} \times T^{2}$. The base is $S^{2} \times S^{1}$ where the second factor is the base of the two-torus as described in Section 8.2.1. The relevant monodromies are embedded in the $S L(4)$ duality group as follows

$$
\begin{equation*}
\mathcal{M}_{D_{4}}=\operatorname{diag}(-1,-1,1,1) \quad \mathcal{M}_{D_{4}^{\prime}}=\operatorname{diag}(1,1,-1,-1) \tag{9.2.51}
\end{equation*}
$$

Since in lower dimension the duality group is larger, one can consider another $D_{4}$-like monodromy

$$
\begin{equation*}
\mathcal{M}_{D_{4}^{\prime \prime}}=\operatorname{diag}(1,-1,-1,1) \tag{9.2.52}
\end{equation*}
$$

which is not in the $S L(2) \times S L(2)$ subgroup of $S L(4)$, and thus it was not possible for the case of $T^{2}$ compactifications. In principle, we can have spaces with monodromies

$$
\begin{equation*}
\left(2 \times D_{4}\right)+\left(2 \times D_{4}^{\prime \prime}\right) \quad \text { or } \quad\left(2 \times D_{4}^{\prime}\right)+\left(2 \times D_{4}^{\prime \prime}\right) \tag{9.2.53}
\end{equation*}
$$

These are T-dual to each other by an $x^{7}-x^{10}$ flip. Thus, it is enough to consider the
first one which is geometric since the monodromies act only in the upper-left $S L(3)$ subsector of $S L(4)$. However, this space is not Calabi-Yau. Supersymmetry suggests that in the base. parallel lines ${ }^{9}$ of singularities should have the same monodromies (possibly up to a factor of $(-1)^{F_{L}}$ as in the case of $\widetilde{K 3} \times T^{2}$ ). This is not the case for this space. A way to explicitly see the absence of supersymmetry is to exhibit the total space as the $\left(\mathbb{R} \times T^{5}\right) /\langle\alpha, \beta\rangle$ orbifold,

$$
\begin{array}{ll}
\alpha: & \left(x, \theta_{1}, \theta_{2} \mid \theta_{3}, \theta_{4}, \theta_{5}\right) \mapsto\left(L-x, \theta_{1},-\theta_{2} \mid-\theta_{3},-\theta_{4}, \theta_{5}\right) \\
\beta: & \left(x, \theta_{1}, \theta_{2} \mid \theta_{3}, \theta_{4}, \theta_{5}\right) \mapsto\left(-x, \theta_{1},-\theta_{2} \mid \theta_{3},-\theta_{4},-\theta_{5}\right) \tag{9.2.55}
\end{array}
$$

Here $x, \theta_{1,2}$ are coordinates on the base and $\theta_{3,4,5}$ are coordinates on the fiber. $x$ is non-compact and $\theta_{i}$ are periodic. The orbifold group $\langle\alpha, \beta\rangle$ also contains the element

$$
\begin{equation*}
\alpha \beta:\left(x, \theta_{1}, \theta_{2} \mid \theta_{3}, \theta_{4}, \theta_{5}\right) \mapsto\left(x+L, \theta_{1}, \theta_{2} \mid-\theta_{3}, \theta_{4},-\theta_{5}\right) \tag{9.2.56}
\end{equation*}
$$

which breaks supersymmetry because it projects out the gravitini.
We see that by considering conjugate $D_{4}$ singularities, in the above reducible case we do not obtain any other supersymmetric examples than those already considered in [138] even if the duality group is extended. Hence, we move on to threefolds in the next section.

### 9.2.2 $\quad$ Non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$

Let us consider the orbifold $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ that we described in detail in Section 8.2.3. Figure 8-7 shows the monodromies of the singular edges. These monodromies have the following $S L(4)$ representations,

$$
\begin{equation*}
x=\operatorname{diag}(1,-1,-1,1) \quad y=\operatorname{diag}(-1,1,-1,1) \quad z=\operatorname{diag}(-1,-1,1,1) \tag{9.2.57}
\end{equation*}
$$

[^49]These are of course geometric since they only act on the first threr coordinates. How can we deform the orbifold into something non-geometric? There are three more $D_{4}$ type singularities that we can use. They have the following monodromies,

$$
\begin{equation*}
\bar{x} \equiv-x \quad \bar{y} \equiv-y \quad \bar{z} \equiv-z \tag{9.2.58}
\end{equation*}
$$

These all invert the $x^{10}$ coordinate. A simple modification of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ is possible by replacing the original monodromies by $\bar{x}, \bar{y}$ or $\bar{z}$. The junction condition says that an even number of negative signs should meet at each vertex. Therefore, consistent monodromy assignments are given by switching signs along loops. There are five theories obtained this way as shown in Figure 9-1. Since these simple spaces have a geometric total space at this orbifold point of their moduli space, we call them "almost non-geometric".


Figure 9-1: Almost non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ spaces. Monodromies are modified along the red loops. We refer to the models as one-plaquette, two-plaquette, "L", "U" and "X", respectively.

### 9.2.3 Asymmetric orbifolds

In the previous section, we changed the monodromies by hand and obtained "almost non-geometric" spaces. In particular, monodromies in the loops contained the extra
action of $(-1)^{F_{L}}$, which reverses the signs of all RR-charges,

$$
\begin{equation*}
x \cdot \mathcal{M}_{(-1)^{F_{L}}}=\bar{x} \quad y \cdot \mathcal{M}_{(-1)^{F_{L}}}=\bar{y} \quad z \cdot \mathcal{M}_{(-1)^{F_{L}}}=\bar{z} \tag{9.2.59}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{M}_{(-1)^{F_{L}}}=\operatorname{diag}(-1,-1,-1,-1) \tag{9.2.60}
\end{equation*}
$$

Hence, we can realize the non-geometric spaces of the previous section as asymmetric orbifolds $[198,197]$ (see also $[69,63,43,104,15,163]$ ). We consider the simple example of Figure 9-2: the one-plaquette model.


Figure 9-2: Simple non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

If we parametrize the $T^{6}$ torus by angles $\theta_{i}$, then the original $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold group action is generated by

$$
\begin{align*}
& \alpha:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2},-\theta_{3},-\theta_{4}, \theta_{5}, \theta_{6}\right)  \tag{9.2.61}\\
& \beta:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2}, \theta_{3}, \theta_{4},-\theta_{5},-\theta_{6}\right) \tag{9.2.62}
\end{align*}
$$

The base coordinates can be chosen to be $\left(\theta_{1}, \theta_{3}, \theta_{5}\right)$. The singular edges along these directions have monodromies $(x, y, z)$, respectively.

Now the example of Figure 9-2 has modified monodromies. In particular, edges on the top of the cube have monodromies which include $(-1)^{F_{L}}$. We use the same trick as in Section 9.2.1: let us choose the vertical $x_{5}$ coordinate to be non-compact
and then compactify it with an asymmetric action,

$$
\begin{array}{ll}
\alpha: & \left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2},-\theta_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \\
\beta_{1}: & \left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2}, \theta_{3}, \theta_{4},-x_{5},-\theta_{6}\right) \\
\beta_{2}: & \left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2}, \theta_{3}, \theta_{4}, L-x_{5},-\theta_{6}\right) \times(-1)^{F}(9.2 .65) \tag{9.2.65}
\end{array}
$$

This realizes the example as an asymmetric orbifold. The Type IIA spectrum is computed in Appendix 9.12. It has $\mathcal{N}=1$ supersymmetry with a gravity multiplet, 16 vector multiplets and 71 chiral multiplets.

The theory is consistent since decorating $D_{4}$ singularities with $(-1)^{F_{L}}$ does not destroy modular invariance. In the Green-Schwarz formalism, adding $(-1)^{F_{L}}$ changes the boundary conditions for the four complex left-moving fermionic coordinates as

$$
\begin{equation*}
D_{4}:(++--) \quad \longrightarrow \quad D_{4} \times(-1)^{F_{L}}:(--++) \tag{9.2.66}
\end{equation*}
$$

Hence, the energy of the twisted sector ground state does not change and thus levelmatching is satisfied [225]. In the RNS formalism, $(-1)^{F_{L}}$ does not act on the worldsheet fields and therefore the moding does not change. However, the left-moving GSO projection changes and various generalized discrete torsion signs show up in the twisted sectors as discussed in the Appendices. (See also related literature [15, 139].) For Abelian orbifolds, one-loop modular invariance implies higher loop modular invariance [225]. Here we are actually considering a non-Abelian orbifold ${ }^{10}$ for which level-matching is not sufficient for consistency. Further constraints may arise if a modular transformation takes a pair of commuting group elements $(g, h)$ into their own conjugacy class [101],

$$
\begin{equation*}
(g, h) \longrightarrow\left(g^{a} h^{b}, g^{c} h^{d}\right)=\left(p g p^{-1}, p h p^{-1}\right) \tag{9.2.67}
\end{equation*}
$$

where $a, b, c$ and $d$ are the elements of an $S L(2, \mathbb{Z})$ matrix. In this case, the path integral with boundary conditions $(g, h)$ and $\left(p g p^{-1}, p h p^{-1}\right)$ for the torus world-

[^50]sheet should give the same result. Since we only consider $D_{4}$ singularities, the twists of world-sheet fermion. by orbifold group elements do commute and thus noncommutativity can only cont from the action on the bosons. However, left-moving and right-moving bosons are treated symmetrically and thus we do not get any further constraints. Therefore, one expects this model to be modular invariant. Moreover, this theory has an alternative presentation as a $\left(\mathbb{Z}_{2}\right)^{3}$ Abelian orbifold of $T^{6}$ as we will see in Section 9.2.5.

The rest of the modified $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ spaces (Figure 9-1) have asymmetric orbifold descriptions as well. These are listed in Appendix 9.10. The modular invariance argument of the previous paragraph applies to these as well. Some of the models are dual to each other. This will be discussed in Section 9.2.5.

### 9.2.4 Joyce manifolds

In Section 9.1.4, we saw how a class of non-geometric spaces can be transformed into geometric M-theory compactifications by U-duality. Naturally, one can try to interpret existing $G_{2}$ spaces from the literature as "non-geometric" Type IIA string theory vacua.

Let us denote the coordinates on $\mathbb{R}^{7}$ (and $T^{7}$ ) by $x_{1}, x_{2}, x_{3}$ (base), $y_{1}, y_{2}, y_{3}, y_{4}$ (fiber). The exceptional group $G_{2}$ is the subgroup of $G L(7, \mathbb{R})$ which preserves the form

$$
\begin{aligned}
\varphi=d x_{1} \wedge d y_{1} \wedge & d y_{2}+d x_{2} \wedge d y_{1} \wedge d y_{3}+d x_{3} \wedge d y_{2} \wedge d y_{3}+d x_{2} \wedge d y_{2} \wedge d y_{4} \\
& -d x_{3} \wedge d y_{1} \wedge d y_{4}-d x_{1} \wedge d y_{3} \wedge d y_{4}-d x_{1} \wedge d x_{2} \wedge d x_{3}
\end{aligned}
$$

It also preserves the orientation and the Euclidean metric on $\mathbb{R}^{7}$ and so it is a subgroup of $S O(7)$. In this section, we consider particular compact examples. Joyce manifolds [158, 159] are (resolved) $T^{7} /\left(\mathbb{Z}_{2}\right)^{3}$ orbifolds which preserve the calibration.

We consider the following artion,

$$
\begin{aligned}
\alpha & :\left(x_{1}, x_{2}, x_{3} \mid y_{1} \cdot y_{2} \cdot y_{3}, y_{4}\right) \mapsto\left(x_{1},-x_{2},-x_{3} \mid y_{1}, y_{2},-y_{3},-y_{4}\right) \\
\beta & :\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2} \cdot y_{3}, y_{4}\right) \mapsto\left(-x_{1}, x_{2}, A_{1}-x_{3} \mid y_{1},-y_{2}, y_{3},-y_{4}\right) \\
\gamma & :\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(A_{2}-x_{1}, A_{3}-x_{2}, x_{3} \mid-y_{1}, y_{2}, y_{3},-y_{4}\right)
\end{aligned}
$$

where $A_{i} \in\left\{0, \frac{1}{2}\right\}$. Note that $\alpha^{2}=\beta^{2}=\gamma^{2}=1$ and $\alpha, \beta$ and $\gamma$ commute. Some of the choices of $\vec{A} \equiv\left(A_{1}, A_{2}, A_{3}\right)$ are equivalent to others by a change of coordinates. Only shifts for the base coordinates are included since fiber shifts can't be realized by a linear transformation. (We comment on this later in Section 9.2.6.) The blow-ups of these spaces are described in [159, 160].

These orbifolds can be interpreted as non-geometric Type II backgrounds as follows. The $T^{4}$ fiber coordinates are already chosen to be $\left\{y_{i}\right\}$. One needs to pick a direction for the extra $x^{10}$ circle. Theories that differ in this choice are T-dual to each other. Then, whenever a generator contains a minus sign for the $x^{10}$ circle, a $(-1)^{F_{L}}$ must be separated from its action. The geometric action is then given by inverting the fiber signs (and omitting the extra circle). For instance, if $y_{4}$ is the $x^{10}$ circle, then $\alpha$ will become

$$
\begin{equation*}
\alpha_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}\right) \mapsto\left(x_{1},-x_{2},-x_{3} \mid-y_{1},-y_{2}, y_{3}\right) \tag{9.2.68}
\end{equation*}
$$

and this geometric action will be accompanied by $(-1)^{F_{L}}$.
In the following, we list the spaces of different shifts and discuss their singularity structure.

- $\vec{A}=(0,0,0)$
(i) Let us first consider the $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold generated by only $\alpha$ and $\beta$. Then, by identifying $y^{4}$ with the extra $x^{10}$ coordinate, we obtain the model in Figure 9-24. This is U-dual to the pure geometry $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ by a $y_{1}-y_{4}$ flip.
(ii) Let us now include $\gamma$. This gives the most singular example of Joyce manifolds. The $x_{i}$ and $y_{i}$ coordinates parametrize the $S^{3}$ base and the $T^{4}$ fiber,
respectively. The $\left(\mathbb{Z}_{2}\right)^{3}$ orbifold group is equally well generated by $\langle\alpha, \beta, \alpha \beta \gamma\rangle$. It is important to note that the product $\alpha \beta \gamma$ does not act on the base coordinates. In principle, this could be interpreted as globally orbifolding ${ }^{11}$ by $(-1)^{F_{L}}$. However, this leads to problems similar to those in our earlier discussion in Section 9.2.1.

It is also easy to see that U-duality does not work in this case ${ }^{12}$. Compactifying M-theory on a $G_{2}$ manifold gives $\mathcal{N}=1$ supersymmetry in 4 d . However, the above configuration in Type II has $\mathcal{N}=2$ supersymmetry ${ }^{13}$, and therefore cannot be equivalent to the M -theory configuration. Thus we will not discuss this example any further.

- $\vec{A}=\left(0,0, \frac{1}{2}\right) \sim\left(0, \frac{1}{2}, 0\right) \sim\left(\frac{1}{2}, 0,0\right)$

The extra identification by $\gamma$ cuts the fundamental cell of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ in half. The resulting base is again an $S^{3}$ which can be constructed as shown in Figure 9-3. The non-geometric space has the same monodromies as the model in Figure 9-25 that we already constructed by directly modifying the monodromies of $T^{6} / \mathbb{Z}_{2} \times$ $\mathbb{Z}_{2}$.

- $\vec{A}=\left(0, \frac{1}{2}, \frac{1}{2}\right) \sim\left(\frac{1}{2}, 0, \frac{1}{2}\right) \sim\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Let us consider $\vec{A}=\left(0, \frac{1}{2}, \frac{1}{2}\right)$, as the others are equivalent by a coordinate transformation. The action of $\alpha$ and $\beta$ generate $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ as usual. The third $\mathbb{Z}_{2}$ is generated by $\gamma$. It has a fixed edge which goes through two parallel faces of the cube (see Figure 9-4). The base is again an $S^{3}$. (The proof of this statement goes roughly as that of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.)

- $\vec{A}=\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

[^51]

Figure 9-3: (i) Fundamental domain of the base after modding by $\gamma$ : half of a rhombic dodecahedron. The arrows show how the faces are identified. (ii) Schematic picture indicating the structure of the degenerations.
(i) Let us first omit the action of $\gamma$. This gives a somewhat simpler space with base depicted in Figure 9-5. It is the union of a truncated tetrahedron, plus a small tetrahedron. This base can be obtained as the intersection of fundamental domains of the two commuting $\mathbb{Z}_{2}$ actions. Both of these domains are $S^{1}$ times the square (with solid edges) depicted in Figure 8-2. The identification of the faces and the schematic structure of the degenerations are shown in Figure 9-6.
(ii) Let us now include $\gamma$ as well. The coordinate shifts in the $\mathbb{Z}_{2}$ actions make sure that the fixed edges do not intersect. The structure of the base is shown in Figure 9-7.


Figure 9-4: (i) Half of the fundamental domain after modding by $\gamma$. (ii) Schematic picture.


Figure 9-5: The base of $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ where the generators of $\mathbb{Z}_{2}$ 's include coordinate shifts. Four non-intersecting $D_{4}$ strings (dashed green lines in the middle of hexagons) curve the space into an $S^{3}$. See the figure in Appendix 9.13 for a pattern that can be cut out.


Figure 9-6: (i) The base can be constructed by gluing the truncated tetrahedron (dashed lines) to itself along with a small tetrahedron. It is easy to check that the $D_{4}$ strings (solid lines) have $180^{\circ}$ deficit angle whereas the dashed lines are nonsingular. (ii) Schematic picture. The truncated tetrahedron example can roughly be understood as four linked rings of $D_{4}$ singularities. All of the rings are penetrated by two other rings which curve the space into a cylinder as they have tension 12. This forces the string to come back to itself.


Figure 9-7: (i) The base of the $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ Joyce orbifold. There are six strings located on the faces of a cube. These faces are folded up which generates the $180^{\circ}$ deficit angles. (ii) Schematic picture. The degenerations form three rings of $D_{4}$ singularities.

### 9.2.5 Dualities between models

The two-plaquette model can be realized as $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ by the following orbifold action ${ }^{14}$,

$$
\begin{aligned}
& \alpha:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2},-\theta_{3},-\theta_{4}, \theta_{5}, \theta_{6}\right) \\
& \beta:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2}, \theta_{3}, \theta_{4},-\theta_{5},-\theta_{6}\right) \times(-1)^{F_{L}}
\end{aligned}
$$

Performing a single T -duality on $\theta_{6}$ turns $\beta$ into

$$
\begin{equation*}
\tilde{\beta}:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-\theta_{1},-\theta_{2}, \theta_{3}, \theta_{4},-\theta_{5},-\theta_{6}\right) \tag{9.2.69}
\end{equation*}
$$

and keeps $\alpha$ intact $^{15}$. We thus learn that Type IIA on the two-plaquette model is dual to Type IIB on $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. The details of the spectrum computation is presented in Appendix 9.11.


Figure 9-8: Monodromies of the one-shift Joyce orbifold.

Another duality is provided by considering the $\vec{A}=\left(\frac{1}{2}, 0,0\right)$ Joyce orbifold,

$$
\begin{aligned}
& \alpha:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{ccc|ccc}
x_{1}, & -x_{2}, & -x_{3} & \mid \quad y_{1}, & y_{2}, & -y_{3},
\end{array}-y_{4}\right) \\
& \beta:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(-x_{1}, \quad x_{2}, \left.\frac{1}{2}-x_{3} \right\rvert\, \quad y_{1},-y_{2}, \quad y_{3},-y_{4}\right) \\
& \gamma:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{lll|llll}
-x_{1}, & -x_{2}, & x_{3} & -y_{1}, & y_{2}, & y_{3}, & \left.-y_{4}\right)
\end{array}\right.
\end{aligned}
$$

[^52]The monodromies of the singularities in the base are shown in Figure 9-8 (see also Figure (9-3). The action of $\alpha$ and $\gamma$ creates the usual cubic structure and $\beta$ cuts the cube in half.

This $G_{2}$ orbifold can be interpreted as a Type IIA background in more than one way depending on which coordinate we choose for the $x^{10}$ (ircle. As discussed in the previous section, a minus sign in the $x^{10}$ direction is interpreted as $(-1)^{F_{L}}$ (this interpretation is accompanied by an inversion of fiber signs). From Figure 9-8 it is clear that $x^{10}=y_{2}$ or $y_{3}$ gives the one-plaquette model since in these cases $\beta$ or $\alpha$, respectively, will contain $(-1)^{F_{L}}$. On the other hand, choosing $x^{10}=y_{1}$ or $y_{4}$ gives model "U". Since relabeling $x^{10}$ is an element of the $S L(4)$ T-duality group, these backgrounds are T-dual to each other. The spectrum is computed in Appendix 9.12.

### 9.2.6 U-duality and affine monodromies

For usual orbifolds, it is known that the untwisted sector contains information about the singular space, whereas the twisted sectors describe resolutions (or deformations $[227,103])$ thereof. It is typically said that string theory "knows" about the nonsingular resolution and the number of the various particles are determined by the Hodge numbers. Here we can see this happening in a more general setup. In Mtheory, the number of $\mathcal{N}=1$ vector and chiral multiplets are respectively determined by the $b_{2}$ and $b_{3}$ Betti numbers of the $G_{2}$-manifold. When U-duality works, one should obtain the same massless spectrum from the asymmetric (non-geometric) orbifold of Type IIA.

Joyce $[158,159]$ computed Betti numbers for blown-up $T^{7} /\left(\mathbb{Z}_{2}\right)^{3}$ examples. These examples, however, contained $1 / 2$ shifts also in directions that were interpreted as fiber coordinates in the previous section ${ }^{16}$,

$$
\begin{array}{rlrr|rrrr}
\alpha: & \left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) & \mapsto\left(x_{1},\right. & -x_{2}, & -x_{3} \mid & y_{1}, & y_{2}, & -y_{3}, \\
\beta: & \left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) & \mapsto\left(-x_{1},\right. & x_{2}, & b_{2}-x_{3} \mid & y_{1}, & -y_{2}, & y_{3}, \\
\left.b_{1}-y_{4}\right) \\
\gamma: & \left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) & \mapsto\left(c_{5}-x_{1},\right. & c_{3}-x_{2}, & x_{3} \mid & -y_{1}, & y_{2}, & y_{3}, \\
\left.c_{1}-y_{4}\right)
\end{array}
$$

These shifts are recommended, otherwise one encounters "bad singularities" which can't easily be resolved. If interpreted as a fibration, the monodromies acting on $T^{4}$ are affine transformations which also include half-shifts for some of the fiber coordinates. Although these orbifolds can readily be interpreted as non-geometric backgrounds for Type IIA, the naive U-duality map does not necessarily work and the spectrum does not match with that of M-theory.

In Appendix 9.13, we discuss the cases of two Joyce manifolds, with two and three shifts $\left(b_{1}, b_{2}, c_{1}, c_{3}, c_{5}\right)=\left(0, \frac{1}{2}, \frac{1}{2}, 0,0,0\right)$ and $\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0\right)$. Naive U-duality works well for the three shift example and one obtains the same spectrum from the non-geometric compactification. However, the two shift example gives a different spectrum from what we expect from the Betti numbers of the $G_{2}$-manifold ${ }^{17}$. The

[^53]puzzle can simply be resolved by choosing a different (coassociative) fiber. Taking $\left\{x_{1}, x_{2}, y_{2}, y_{3}\right\}$ for fiber coordinates, the $\mathbb{Z}_{2}$ transformations have no shifts in these directions and the non-geometric Type IIA spectrum indeed matches the M-theory spectrum.

### 9.3 Compactifications with $\mathrm{E}_{\mathrm{n}}$ singularities

In this section, we list geometric orbifolds containing singularities other than $D_{4}$. Non-geometric modifications of these orbifolds may be done similarly to the previous section. For $D_{4}$ singularities, the constant shape of the fiber can be arbitrary. The main difference in the $E_{n}$ case is that the fiber shape is determined by the symmetry group. In practice, this means that in two dimensions $\tau=i$ or $\tau=e^{i \pi / 3}$.

### 9.3.1 Orbifold limits of $K 3$

Simple warm-up examples are provided by considering $T^{4} / \mathbb{Z}_{n}$ orbifolds. These have been analyzed from the F-theory point of view in [66].

The $\mathrm{T}^{4} / \mathbb{Z}_{\mathbf{3}}$ orbifold. The action of the generator of the orbifold group is given by

$$
\begin{equation*}
\alpha:\left(z_{1}, z_{2}\right) \mapsto\left(e^{i \pi / 3} z_{1}, e^{-i \pi / 3} z_{2}\right) \tag{9.3.70}
\end{equation*}
$$

which respects the torus identifications

$$
\begin{equation*}
z_{i} \sim z_{i}+1 \sim z_{i}+e^{i \pi / 3} \tag{9.3.71}
\end{equation*}
$$

The base is $T^{2} / \mathbb{Z}_{3}$ and can be parametrized by $z_{1}$. It contains three $E_{6}$ singularities
action for $(-1)^{F_{L}}$ will now include $1 / 2$ shifts in the fiber. In some cases, this ambiguity can be exploited to match the IIA and M-theory spectra.
of deficit angle $4 \pi / 3$. The monotiony around these are given by

$$
\mathcal{M}_{E_{\mathrm{ij}}}=(. \varsigma T)^{2}=\left(\begin{array}{cc}
-1 & -1  \tag{9.3.72}\\
1 & 0
\end{array}\right)
$$

A fundamental cell is shown in Figure 9-9.


Figure 9-9: The base of the $T^{4} / \mathbb{Z}_{3}$ orbifold contains three $E_{6}$ singularities.

The $T^{4} / \mathbb{Z}_{4}$ orbifold. The generator of $\mathbb{Z}_{4}$ is given by

$$
\begin{equation*}
\alpha:\left(z_{1}, z_{2}\right) \mapsto\left(i z_{1},-i z_{2}\right) \tag{9.3.73}
\end{equation*}
$$

with the torus identifications

$$
\begin{equation*}
z_{i} \sim z_{i}+1 \sim z_{i}+i \tag{9.3.74}
\end{equation*}
$$

The base is $T^{2} / \mathbb{Z}_{4}$. This orbifold contains two $E_{7}$ and one $D_{4}$ singularity. They have deficit angles $3 \pi / 2$ and $\pi$, respectively. The $E_{7}$ and $D_{4}$ monodromies are given by

$$
\mathcal{M}_{E_{7}}=S=\left(\begin{array}{cc}
0 & -1  \tag{9.3.75}\\
1 & 0
\end{array}\right) \quad \mathcal{M}_{D_{4}}=\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right)
$$

A fundamental cell is shown in Figure 9-10.

The $\mathrm{T}^{4} / \mathbb{Z}_{6}$ orbifold. The base is $T^{2} / \mathbb{Z}_{6}$. This orbifold contains $E_{8}, E_{6}$ and $D_{4}$


Figure 9-10: The base of the $T^{4} / \mathbb{Z}_{4}$ orbifold contains two $E_{7}$ and one $D_{4}$ singularities.
singularities. The $E_{8}$ monodromy is given by

$$
\mathcal{M}_{E_{8}}=S T=\left(\begin{array}{cc}
0 & -1  \tag{9.3.76}\\
1 & 1
\end{array}\right)
$$

A fundamental cell is shown in Figure 9-11.


Figure 9-11: The base of the $T^{4} / \mathbb{Z}_{6}$ orbifold contains $E_{8}, E_{6}$ and $D_{4}$ singularities. The three black dots denote one non-singular point.

### 9.3.2 Example: $T^{6} / \mathbb{Z}_{3}$

We continue by discussing three dimensional examples. The simplest one is $T^{6} / \mathbb{Z}_{3}$. This is created by orbifolding the square $T^{6}$ by cyclic permutations of (complex) coordinates

$$
\begin{equation*}
\alpha:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(z_{2}, z_{3}, z_{1}\right) \tag{9.3.77}
\end{equation*}
$$

Clearly, this action preserves the holomorphic volume form,

$$
\begin{equation*}
\Omega=d z_{1} \wedge d z_{2} \wedge d z_{3} \tag{9.3.78}
\end{equation*}
$$

and the Kähler form

$$
\begin{equation*}
\omega=\sum_{i} d z_{i} \wedge d \bar{z}_{i} \tag{9.3.79}
\end{equation*}
$$

Let us now choose the real parts of $z_{i}$ for the base coordinates. Before orbifolding, the base is a cube as shown in Figure 9-12. The fixed loci of $\alpha$ are at $z_{1}=z_{2}=z_{3}$ that is along a diagonal. The cube has a $\mathbb{Z}_{3}$ symmetry about this diagonal, and thus the orbifolding procedure respects the torus identifications.


Figure 9-12: The base of $T^{6} / \mathbb{Z}_{3}$. The green line shows the $E_{6}$ singularity. Six triangles bound the domain. Two triangles touching the singular green line are identified by folding. Two triangles should be identified according to the orientation given by the arrows. The remaining two triangles are identified in a similar fashion.

Since $\mathbb{Z}_{3} \subset S U(2)$, this example preserves $\mathcal{N}=4$ supersymmetry in four dimensions. By making the identifications of the bounding triangles, one can check that the only singularity is $E_{6}$. It is along the diagonal which gives a closed loop in the base. Since there are no other gravitating strings to curve the space, this is a good sign that the space factorizes. In particular, we do not expect it to be an $S^{3}$.

### 9.3.3 Example: $T^{6} / \Delta_{12}$

A more complicated example is gained brorbifolding $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ by the above described cyclic permutations. These permutations do not commute with the sign flips and together they give $\Delta_{12} \subset S U(3)$. This group has the faithful representation described by the following matrices (see [112], and also [35. 124, 85]) which act on the ( $z_{1}, z_{2}, z_{3}$ ) complex coordinates

$$
\left(\begin{array}{ccc}
(-1)^{p} & 0 & 0 \\
0 & (-1)^{q} & 0 \\
0 & 0 & (-1)^{p+q}
\end{array}\right) \quad\left(\begin{array}{ccc}
0 & 0 & (-1)^{p} \\
(-1)^{q} & 0 & 0 \\
0 & (-1)^{p+q} & 0
\end{array}\right) \quad\left(\begin{array}{ccc}
0 & (-1)^{p} & 0 \\
0 & 0 & (-1)^{q} \\
(-1)^{p+q} & 0 & 0
\end{array}\right)
$$

It can be generated by two elements,

$$
\begin{aligned}
& \alpha:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(z_{2}, z_{3}, z_{1}\right) \\
& \beta:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(-z_{1},-z_{2}, z_{3}\right)
\end{aligned}
$$

The fundamental domain is shown in Figure 9-13. There are two $E_{6}$ and four $D_{4}$ singularities in the base. They meet in $E_{6}-E_{6}-D_{4}$ and $D_{4}-D_{4}-D_{4}$ vertices. The solid angle around these vertices are $\pi / 3$ and $\pi$, respectively. The base is topologically an $S^{3}$.

### 9.3.4 $\quad$ Example: $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$

Another example is obtained from $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ by further orbifolding it by $\mathbb{Z}_{4}$. This is possible because the rhombic dodecahedron has fourfold symmetry axes. These are the axes of the green cube in Figure 8-5.

The resulting base is shown in Figure 9-14. There is one $E_{7}$ line which is topologically a circle. In contrast to the $T^{6} / \mathbb{Z}_{3}$ example, this happens because the other $D_{4}$ singularities curve the base and make this contractible loop a geodesic. The base only contains familiar $D_{4}-D_{4}-D_{4}$ vertices.


Figure 9-13: (i) The base of $T^{6} / \Delta_{12}$. The red and green lines indicate $E_{6}, D_{4}$ singularities, respectively. The other edges are non-singular. The solid green cube indicates the $D_{4}$ singularities of the original $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ orbifold. (ii) Schematic picture describing the topology of the singular lines. See Appendix 9.14 for building this polyhedron at home.

### 9.3.5 Example: $T^{6} / \Delta_{24}$

Our final example can be constructed by first taking $T^{6}$. Its base is a cube with opposite faces identified. We now place $E_{7}$ singularities on the twelve edges of the cube. We also add diagonal $E_{6}$ singularities as in Section 9.3.3. These are realized by the following matrices which act on $\left(z_{1}, z_{2}, z_{3}\right)$ complex coordinates

$$
\alpha_{E_{6}}=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{9.3.81}\\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right) \quad \beta_{E_{7}}=\left(\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

These generate the $\Delta_{24}$ group. Compared to $\Delta_{12}$, it also contains odd permutations of the coordinates. Since odd permutations come with an odd number of minus signs, the volume form is again invariant.

In Figure 9-15, the resulting base is shown. The green cube around the base is $1 / 8$ or the original base of $T^{6}$. The faces should be folded as indicated by the arrows. The rear faces touching $E_{6}$ should be also folded. This gives an $S^{3}$ with curvature


Figure 9-14: (i) The base of $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$. The red and green lines indicate $E_{7}, D_{4}$ singularities, respectively. The other edges are non-singular. The solid green cube indicates the $D_{4}$ singularities of the original $T^{6} /\left(\mathbb{Z}_{2}\right)^{2}$ orbifold. (ii) Schematic picture describing the topology of the singular lines.
concentrated in the singular lines (see the right-hand side of the figure). The base contains two types of composite vertices. One is an intersection of $E_{7}, E_{6}$ and $D_{4}$ edges. The other one comes from the collision of an $E_{7}$ and two $D_{4}$ singularities.

### 9.3.6 Non-geometric modifications

Having discussed the geometric structure of the fibrations with exceptional singularities, we can try to modify them into non-geometric spaces. Similarly to the examples in Section 9.2.2, closed loops of $D_{4}, E_{7}$ and $E_{8}$ singularities ${ }^{18}$ may be decorated with the action of $(-1)^{F_{L}}$. For example, $E_{8}=\left(\begin{array}{cc}0 & -1 \\ 1 & 1\end{array}\right) \oplus\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ with $(-1)^{F_{L}}$ has the same monodromy as a composite of $A_{2}=\left(\begin{array}{cc}0 & 1 \\ -1 & -1\end{array}\right)$ and a $D_{4}^{\prime}$ (which acts on the other $\left.T^{2} \subset T^{4}\right)$. The tension four $A_{2}$ and the tension six $D_{4}^{\prime}$ give the original deficit angle of the tension ten $E_{8}$ (see 8.2.2 for the Kodaira classification of singularities).

The simplest example is to add $(-1)^{F_{L}}$ to the $D_{4}$ and one of the $E_{7}$ singularities of the $T^{4} / \mathbb{Z}_{4}$ orbifold (Figure 9-10), or instead decorate both $E_{7}$ singularities. The $T^{4} / \mathbb{Z}_{6}$ orbifold (Figure 9-11) can similarly be modified by adding $(-1)^{F_{L}}$ to the $D_{4}$

[^54]

Figure 9-15: (i) The base of $T^{6} / \Delta_{24}$. The cyan, red and green lines indicate $E_{7}, E_{6}$ and $D_{4}$ singularities, respectively. (ii) Schematic picture describing the topology of the singular lines.
and the $E_{8}$ singularities. By performing a single T-duality in the fiber, the $T^{4} / \mathbb{Z}_{4}$ monodromies can be changed to act on $S L(2)_{\rho}$ instead of $S L(2)_{\tau}$. The resulting Type IIB theory has a $D_{4}^{\prime}=D_{4} \times(-1)^{F_{L}}$ and two $E_{7}^{\prime}$ singularities. The $E_{7}^{\prime}$ corresponds to a double T-duality and thus the background is globally non-geometric, even though it has a geometric dual.

Turning to the three dimensional examples, $(-1)^{F_{L}}$ can be added to the $D_{4}$ loop of $T^{6} / \Delta_{12}$ as shown in Figure 9-16. This is obtained by orbifolding the last example in Figure 9-1. The $D_{4}$ loops or the $E_{7}$ loop of $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$ can similarly be modified. An example is shown in Figure 9-17 where a single $D_{4}$ has been changed into $D_{4}^{\prime}$ corresponding to the first example of Figure 9-1. A single T-duality on the geometric $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$ gives Type IIB with a circle of $E_{7}^{\prime}$ and thus the dual background is non-geometric. $T^{6} / \Delta_{24}$ can similarly be modified (Figure 9-18).

These spaces can serve as perturbative string backgrounds. The consistency of these vacua, however, needs further investigation.


Figure 9-16: Non-geometric $T^{6} / \Delta_{12}$. The red lines indicate extra $(-1)^{F_{L}}$ factors.


Figure 9-17: Non-geometric $T^{6} /\left(\mathbb{Z}_{2}\right)^{2} \times \mathbb{Z}_{4}$.

### 9.4 Chiral Scherk-Schwarz reduction

In previous sections, we studied non-geometric spaces mainly by using a $(-1)^{F_{L}}$ monodromy around singular loci in the base. Another possibility is to have this transformation in the fiber as a Wilson line. Fields still do not depend on the fiber coordinates, and in this sense this is a (chiral) Scherk-Schwarz reduction.

### 9.4.1 One dimension

Let us consider Type IIA compactified on a circle with $(-1)^{F_{L}}$ Wilson line. This will be a one-dimensional fiber. The configuration breaks half of the supersymmetry keeping sixteen right-moving supercharges. In M-theory, $(-1)^{F_{L}}$ is described as reflection of $x^{11}$. Hence, the background lifts to M-theory as compactification on a Klein bottle [65].

An important feature of the background that one can try to exploit in the construction of non-geometric spaces is that T-duality on the circle takes Type IIA to


Figure 9-18: Non-geometric $T^{6} / \Delta_{24}$.

IIA (not IIB as usual) [123, 137, 9]. Although the duality switches between the $S O(8)$ spinor and conjugate spinor representations in the right-moving sector, it also exchanges the untwisted and twisted R/NS sectors [137]. Therefore, when the circle decompactifies, the two massless 10d gravitini have different chiralities and thus the theory is still Type IIA.

At self-dual radius, the bosonic string has additional massless states and one obtains the gauge group $S U(2) \times S U(2)$. In Type II strings, these extra states are destroyed by the GSO projection and one is left with $U(1) \times U(1)$ only. With the above Wilson line, however, an extended $S U(2) \times U(1)$ gauge symmetry is obtained. In the effective theory, T-duality is part of the $S U(2)$ gauge group and thus a T-duality monodromy can be regarded as a Wilson line.

A simple two-dimensional non-geometric space is obtained by compactifying on another base circle with a monodromy that is a T-duality on the fiber circle. The consistency of this model has to be further investigated.

### 9.4.2 Two dimensions

These ideas can be generalized by considering $T^{n}$ compactifications and turning on a $(-1)^{F_{L}}$ Wilson line. This still preserves half of the supersymmetry. In order to glue spaces, only those monodromies can be considered which preserve Wilson lines, that is the "spin structure" of the $T^{n}$ fiber. Therefore, the perturbative duality group will be a proper subgroup of $O(n, n, \mathbb{Z})$.

In the following, we consider the simplest examples where the base is taken to be
two dimensional and is parametrized by the complex coordinate $z$. The shape of the two-torus fiber is described by the $\tau$ complex parameter. We take the Wilson line ${ }^{19}$ to be along the real direction denoted by $x^{9}$. Then. along this coordinate axis, a single T-duality is possible. Applying the Buscher rules, this duality is mirror symmetry for the two-torus fiber.

Let us denote the components of an arbitrary $S L(2, \mathbb{Z})$ element $M$ by

$$
M=\left(\begin{array}{cc}
a & b  \tag{9.4.82}\\
c & d
\end{array}\right), \quad a d-b c=1
$$

Geometric transformations must preserve the $(-,+)$ spin structure. If $(x, y) \in \mathbb{Z}^{2}$ denotes the homotopy class of a one-cycle, then this constraint is equivalent to

$$
\begin{equation*}
(-1)^{x}=(-1)^{a x+b y} \tag{9.4.83}
\end{equation*}
$$

that is

$$
\begin{equation*}
(a-1) x+b y=0 \quad(\bmod 2) \tag{9.4.84}
\end{equation*}
$$

Since $y$ is arbitrary, $b$ must be even. Then, $\operatorname{det} M=1$ forces $a$ (and $d$ ) to be odd and the above equation is satisfied. Therefore, the geometric part of the duality group is the $\Gamma_{0}(2) \subset S L(2)$ congruence subgroup of index three. A maximal subgroup of it is $\Gamma(2)$ that contains matrices with even off-diagonal elements. $\Gamma_{0}(2)$ can be generated by $\Gamma(2)$ and the $T S T^{-1}$ transformation which exchanges two cycles in the fiber. Its fundamental domain is shown in Figure 9-19. The full duality group contains another copy of $\Gamma_{0}(2)$ for $\rho$, and a single T-duality along $x^{9}$.

A geometric $K 3$ fibration ${ }^{20}$ with such restricted transformations can be described

[^55]

Figure 9-19: Fundamental domain (gray area) for the action of the $\Gamma_{0}(2)$ on the upper half-plane.
by [36]

$$
\begin{equation*}
y^{2}+x^{4}+x^{2} w^{2} f_{4}(z)+w^{4} g_{8}(z)=0 \tag{9.4.85}
\end{equation*}
$$

where $(x, y, w) \in \mathbb{C} P_{1,2,1}^{2}$ and $f_{4}, g_{8}$ are holomorphic sections of degree 4 and 8 , respectively. The $j$-function is given by

$$
\begin{equation*}
j(\tau)=\frac{\left(f_{4}^{2}+12 g_{8}\right)^{3}}{108 g_{8}\left(-f_{4}^{2}+4 g_{8}\right)^{2}} \tag{9.4.86}
\end{equation*}
$$

The discriminant of the elliptic fibration vanishes generically at 16 points out of which 8 are double zeros. The moduli space is ten dimensional, in contrast to the 18 dimensional space of the cubic Weierstrass equation.

As explained in Appendix 9.7 of [36], the types of possible degenerations are $A_{n}$, $D_{n}$ and $E_{7}$. The $K 3$ geometry can reach the $T^{4} / \mathbb{Z}_{2}$ orbifold limit where four $D_{4}$ singularities close the base into an $S^{2}$. The orbifold is then generated by

$$
\begin{array}{rllll}
\alpha:\left(x_{1}, x_{2} \mid y_{1}, y_{2}\right) & \mapsto\left(\begin{array}{rrrr}
-x_{1}, & -x_{2} \mid & -y_{1}, & \left.-y_{2}\right) \\
\beta:\left(x_{1}, x_{2} \mid y_{1}, y_{2}\right) & \mapsto( & x_{1}, & x_{2} \mid
\end{array} y_{1},\right. & \left.\frac{1}{2}+y_{2}\right)
\end{array}
$$

with $\beta$ containing $(-1)^{F_{L}}$. This is the same theory as the asymmetric orbifold limit of the $12+12^{\prime}$ model of [138]. The anomaly free $\mathcal{N}=16 \mathrm{~d}$ spectrum contains a
supergravity multiplet, nine tensor multiplets, eight vector multiplets and twenty hypermultiplets. The strong coupling limit is M-theory on a $\mathbb{Z}_{2}$ orbifold ${ }^{21}$

$$
\begin{equation*}
\left(K 3 \times S^{1}\right) /\{\sigma \cdot(y \rightarrow-y)\} \tag{9.4.88}
\end{equation*}
$$

where $y \in[0,1)$ is the $S^{1}$ coordinate and $\sigma$ is an involution on $K 3$ that acts with eight fixed points. It preserves twelve of the harmonic $(1,1)$ forms and changes the sign of the other eight harmonic $(1,1)$ forms. The spectrum computation [213] matches that of the asymmetric orbifold.

The resolved $12+12^{\prime}$ model used a doubly elliptic Weierstrass fibration over an $S^{2}$ base,

$$
\begin{equation*}
y^{2}=x^{3}+p_{4}(z) x+q_{6}(z) \quad \tilde{y}^{2}=\tilde{x}^{3}+\tilde{p}_{4}(z) \tilde{x}+\tilde{q}_{6}(z) \tag{9.4.89}
\end{equation*}
$$

The constants in the polynomials give a 19 dimensional moduli space. In the above orbifold limit, the complex base coordinate includes $y_{2}$ (which has the Wilson line). The $\Gamma_{0}(2)$ construction resolves the orbifold in a different 'frame': it chooses a different set of base coordinates, namely $x_{1}$ and $x_{2}$. It presumably slices out a different subspace in the full moduli space of the model.

Finally, T-duality along the $x^{9}$ circle can also be considered. The $\tau(z)$ and $\rho(z)$ sections can be described by considering a doubly elliptic fibration over the base. In [138], the fiber tori were independent and thus $\tau(z)$ and $\rho(z)$ were unrelated. For the present configuration with a Wilson line, however, a single T-duality can exchange them and result in more complicated non-geometric spaces. The construction of such backgrounds is left for future work.

[^56]by utilizing the heterotic-Type II duality [233].

### 9.5 Conclusions

A perturbative vacuum of string theory is specified by a conformal field theory on the worldsheet. Only in special cases will the CFT have a geometric description. Such cases include flat space, Calabi-Yau and flux compactifications, which have been studied in great detail. The development of a more systematic understanding of the set of consistent string vacua will inevitably require the study of non-geometric compactifications.

String dualities allow for the construction of string vacua that are locally geometric but not necessarily manifolds globally. Using this idea, we have constructed nongeometric compactifications preserving $\mathcal{N}=1$ supersymmetry in four dimensions. In the two dimensional case, the Weierstrass equation with holomorphic coefficients solves the equation of motion and allows for sharing the $\mathbb{Z}_{4}$ and $\mathbb{Z}_{6}$ orbifold points which is necessary for $S U(2)$ holonomy. Since an appropriate generalization of the Weierstrass equation was not at our disposal, we were only able to describe such spaces at the asymmetric orbifold point in their moduli space. A strong motivation for departing the flat-base limit is that it presumably generates a non-trivial potential for the overall volume modulus. Note that for $D_{4}$ singularities, the size of the fiber is an arbitrary free parameter which (typically) runs to large volume.

Although our explicit examples were all orbifolds, in principle, it is possible to build non-orbifold examples by means of $D_{4}$ and $E_{n}$ singularities. Since the base in this case is flat, it could be obtained by gluing various polyhedra along their faces. By carefully choosing the dihedral angles of the building blocks, one can create the appropriate deficit angles for the edges. However, it is not easy to satisfy the constraints on monodromies coming from supersymmetry and the constructions quickly get complicated. A good step in this direction would be to find a good basis of building blocks which suffice even to reconstruct the orbifold examples. By the relation discussed in Section 9.1.4, such spaces would presumably give new examples of $G_{2}$ manifolds.

In Appendix 9.12, Type IIA string theory has been compactified in a non-geometric
way on the "one-shift" $T^{7} /\left(\mathbb{Z}_{2}\right)^{3}$ orbifold down to four dimensions. The massless spectrum is equivalent to that of the M-theory compactification on a particular resolution of this orbifold with $\left(b_{2}, b_{3}\right)=(16,71)$. The orbifold has, however, numerous other resolutions with very different Betti numbers [160]. It would be interesting to see whether these other resolutions arise in Type IIA by the introduction of discrete torsion (and possibly NS5-branes).

In the other direction, we have seen that a general $T^{3}$-fibration with $S O(3,3, \mathbb{Z})$ T-duality monodromies has a globally geometric M-theory dual. This is striking given the difficulty of describing such creatures from the string theory point of view. More generic constructions with $S L(5)$ monodromies presumably have no duality frame where they are globally geometric.

We have focussed on compactifications where the monodromy group was a subgroup of the perturbative duality group. There is no obstacle in principle to the extension of the monodromy group to include the full $S L(5)$ U-duality group ${ }^{22}$. In this manner one can extend these techniques to include in the compactification RamondRamond fields, D-branes and orientifolds, and presumably to find vacua with no massless scalars. In Appendices 9.8 and 9.9 we build confidence that such objects can be treated consistently in the semiflat approximation by rederiving from this viewpoint the Hanany-Witten brane-creation effect and the duality between M-theory on $T^{5} / \mathbb{Z}_{2}$ and type IIB on K3. Although we studied vacua of Type II string theory, the discussion can be applied to heterotic strings as well where the duality group $O(16+d, d)$ is much larger [90].

Another interesting direction is the study of leaving the large complex structure limit. Our special flat-base examples had a worldsheet description as modular invariant asymmetric orbifolds. However, in the generic case, this powerful tool is missing. Any available tools, such as the gauged linear sigma model [232], should be brought to bear on this problem.

In [113] it is proved for the $T^{2}$-fibered case that a solution in the semiflat approximation determines an exact solution. While the power of holomorphy is lacking in

[^57]the $T^{3}$-fibered case, the physical motivation for this statement [138] remains. The idea is that the violations of the semiflat approximation are localized in the base, and we have a microscopic description of the degenerations, as D-branes or NS-branes or as parts of well-understood CY manifolds or orbifolds or U-duality images of these things.

It is expected that the singular edges in the base transform into ribbon graphs as we move away from the semi-flat limit [157, 195]. It seems possible that one can construct local (in the base) invariants of the fibration which give 'NUT charges' [150]. These invariants, which are analogous to the number of seven-branes in the stringy cosmic strings construction, appear in the [217] mirror-symmetry-covariant superpotential.

### 9.6 Appendix: Flat-torus reduction of type IIA to seven dimensions

The following discussion is based on [188]. Let us consider the action for the massless NS-NS fields of type II strings (in any number of dimensions)

$$
\begin{equation*}
S=\int d x \int d y \sqrt{-\hat{g}} \cdot e^{-\hat{\varphi}}\left[R(\hat{g})+\partial_{\mu} \hat{\varphi} \partial^{\mu} \hat{\varphi}-\frac{1}{12} \hat{H}_{\mu \nu \rho} \hat{H}^{\mu \nu \rho}\right] . \tag{9.6.90}
\end{equation*}
$$

The $x$ coordinates label so-far-noncompact directions, and $y$ are coordinates on a $T^{d}$. We want to reduce the theory and eliminate the $y$ coordinates. Let $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ label the corresponding indices. Taking the following ansätze,

$$
\begin{gather*}
\hat{g}=:\left(\begin{array}{cc}
g_{\mu \nu}+A_{\mu}^{\gamma} A_{\nu \gamma} & A_{\mu \beta} \\
A_{\nu \alpha} & G_{\alpha \beta}
\end{array}\right)  \tag{9.6.91}\\
\varphi:=\hat{\varphi}-\frac{1}{2} \log \operatorname{det} G_{\alpha \beta} \quad F_{\mu \nu}^{(1) \alpha}:=\partial_{\mu} A_{\nu}^{\alpha}-\partial_{\nu} A_{\mu}^{\alpha}  \tag{9.6.92}\\
H_{\mu \nu \alpha}:=\hat{H}_{\mu \nu \alpha}-A_{\mu}^{\beta} \hat{H}_{\beta \nu \alpha}-A_{\nu}^{\beta} \hat{H}_{\mu \beta \alpha} \tag{9.6.93}
\end{gather*}
$$

one obtains the following terms after reduction

$$
\begin{equation*}
S=\int d x \sqrt{-g} e^{-\varphi} \mathcal{L} \tag{9.6.94}
\end{equation*}
$$

with $\mathcal{L}=\mathcal{L}_{1}+\mathcal{L}_{2}+\mathcal{L}_{3}+\mathcal{L}_{4}$ and

$$
\begin{align*}
\mathcal{L}_{1} & =R+\partial_{\mu} \varphi \partial^{\mu} \varphi  \tag{9.6.95}\\
\mathcal{L}_{2} & =\frac{1}{4}\left(\partial_{\mu} G_{\alpha \beta} \partial^{\mu} G^{\alpha \beta}-G^{\alpha \beta} G^{\gamma \delta} \partial_{\mu} B_{\alpha \gamma} \partial^{\mu} B_{\beta \delta}\right)  \tag{9.6.96}\\
\mathcal{L}_{3} & =-\frac{1}{4} g^{\mu \rho} g^{\nu \lambda}\left(G_{\alpha \beta} F_{\mu \nu}^{(1) \alpha} F_{\rho \lambda}^{(1) \beta}+G^{\alpha \beta} H_{\mu \nu \alpha} H_{\rho \lambda \beta}\right)  \tag{9.6.97}\\
\mathcal{L}_{4} & =-\frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} \tag{9.6.98}
\end{align*}
$$

In order to see the $S O(d, d, \mathbb{Z})$ symmetry, one introduces the $2 d \times 2 d$ matrix

$$
M=\left(\begin{array}{cc}
G^{-1} & G^{-1} B  \tag{9.6.99}\\
B G^{-1} & G-B G^{-1} B
\end{array}\right)
$$

This symmetric matrix is in $S O(d, d)$, that is

$$
M^{T} \eta M=\eta \quad \eta \equiv\left(\begin{array}{cc}
0 & 1_{2 \times 2}  \tag{9.6.100}\\
1_{2 \times 2} & 0
\end{array}\right)
$$

$M$ is positive definite which can be seen as follows. First notice that the above properties of $M$ imply that the eigenvalues are present with their reciprocals,

$$
\begin{equation*}
M \vec{v}=\lambda \vec{v} \quad \Longrightarrow \quad M(\eta \vec{v})=\lambda^{-1}(\eta \vec{v}) \tag{9.6.101}
\end{equation*}
$$

Let us now turn off the B-field. The eigenvalues of $M(B=0)$ are simply the eigenvalues of $G$ and the reciprocals: $\lambda_{i}$ and $1 / \lambda_{i}$, all positive. As we turn on the B-field, we do not expect any singularities in the eigenvalues since $M$ is quadratic in $B$. Therefore the eigenvalues remain positive.

Let us introduce

$$
\begin{equation*}
H_{\mu \nu \alpha}=\partial_{\mu} B_{\nu \alpha}-\partial_{\nu} B_{\mu \alpha}=: F_{\mu \nu \alpha}^{(2)} \tag{9.6.102}
\end{equation*}
$$

we can collect the field strength in the following $S O(d, d)$ vector

$$
\begin{equation*}
\mathcal{F}_{\mu \nu}^{i}:=\binom{F_{\mu \nu}^{(1) \alpha}}{F_{\mu \nu \beta}^{(2)}} \tag{9.6.103}
\end{equation*}
$$

With these ingredients, one can explicitly see the $S O(3,3)$ invariance of the Lagrangian. $\mathcal{L}_{1}$ is trivially invariant. The kinetic terms can be written as

$$
\begin{equation*}
\mathcal{L}_{2}=\frac{1}{8} \operatorname{Tr}\left(\partial_{\mu} M^{-1} \partial^{\mu} M\right) \tag{9.6.104}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\mathcal{L}_{3}=-\frac{1}{4} \mathcal{F}_{\mu \nu}^{i}\left(M^{-1}\right)_{i j} \mathcal{F}^{\mu \nu j} \tag{9.6.105}
\end{equation*}
$$

which is invariant. Since $H_{\mu \nu \rho}$ does not change under the duality group, $\mathcal{L}_{4}$ is also invariant.

### 9.7 Appendix: Semi-flat vs. exact solutions

In this section we compare the exact supergravity solutions to the semi-flat description. We study the approximation through the example of an NS5-brane. NS5-branes are parametrically heavier than D-branes ${ }^{23}$ and they curve spacetime even to zeroth approximation.

Semi-flat approximation. In order for the semi-flat machinery to work, we need to compactify the transverse space. The transverse space is now a two-torus fiber over a complex $z$-plane. The NS5-brane can be described by a $\rho \mapsto \rho+1$ monodromy around a singular point in the $z$-plane. This can be achieved by the following solution [113]

$$
\begin{equation*}
\rho(z) \sim \frac{1}{2 \pi i} \log (z) \tag{9.7.106}
\end{equation*}
$$

[^58]As we approach the brane $(|z| \rightarrow 0)$, the torus fiber decompactifies,

$$
\begin{equation*}
V_{\text {fiber }} \sim \rho_{2} \sim-\log |z| \tag{9.7.107}
\end{equation*}
$$

Since the eight dimensional dilaton can be set to constant [138], the ten dimensional dilaton is (Appendix 9.6) ${ }^{24}$

$$
\begin{equation*}
2 \varphi=\frac{1}{2} \log \operatorname{det} G_{\alpha \beta} \tag{9.7.108}
\end{equation*}
$$

that is

$$
\begin{equation*}
e^{2 \varphi}=V_{\text {fiber }} \tag{9.7.109}
\end{equation*}
$$

and the dilaton grows near the origin ${ }^{25}$.

Exact NS5 solution. If $x^{m}$ are transverse to the NS5-brane and $x^{\mu}$ are tangent to it, then the exact non-compact classical solution in the string frame is given by (see [203], page 183)

$$
\begin{align*}
G_{m n}=e^{2 \varphi} \delta_{m n} & G_{\mu \nu}=\eta_{\mu \nu}  \tag{9.7.110}\\
H_{m n p}=-\varepsilon_{m n p}^{q} \partial_{q} \varphi & e^{2 \varphi}=e^{2 \varphi(\infty)}+\frac{1}{2 \pi^{2} r^{2}} \tag{9.7.111}
\end{align*}
$$

where $r^{2}=\sum\left(x^{m}\right)^{2}$. The geometry has an infinite throat: the origin $x^{m}=0$ is at infinite distance and the angular $S^{3}$ approaches an asymptotic constant size. In the Einstein frame $\left(G_{\mu \nu}^{\text {Einstein }}=e^{-\varphi / 2} G_{\mu \nu}^{\text {string }}\right)$, the singularity is at finite distance. There is a growing dilaton in the throat and string perturbation theory eventually breaks down. At infinity, the metric asymptotes to flat space.

In order to derive the semi-flat solution from the exact one, we need to compactify the latter on a two-torus. In the covering space, this amounts to placing an infinite number of NS5-branes in a 2 d lattice $\Lambda$ in the 4 d transverse space. For sake of simplicity, we take this to be a square lattice. Since the branes are BPS, the solution

[^59]comes from simple superposition ${ }^{26}$,
\[

$$
\begin{equation*}
e^{2 \varphi(x)}=e^{2 \varphi(\infty)}+\frac{1}{2 \pi^{2}} \sum_{(n, m) \in \Lambda}\left(\frac{1}{\left|x-x_{n, m}\right|^{2}}-\frac{1}{\left|x_{n . m}\right|^{2}}\right) \tag{9.7.112}
\end{equation*}
$$

\]

where $x$ and $x_{n, m}$ are four-vectors, the latter one denoting the positions of the lattice points parametrized by two integers $n$ and $m$.

If we neglect distances smaller than the lattice spacing, then we obtain

$$
\begin{equation*}
e^{2 \varphi(z, \bar{z})}=e^{2 \varphi(\infty)}+\frac{1}{2 \pi^{2}} \sum_{n, m}\left(\frac{1}{z \bar{z}+n^{2}+m^{2}}-\frac{1}{n^{2}+m^{2}}\right) \tag{9.7.113}
\end{equation*}
$$

where we introduced a complex $z$ coordinate perpendicular to the 2 d lattice. This is the base coordinate. This expression can now be compared to the semi-flat solution. If we denote $r \equiv|z|$, then taking the derivative w.r.t $r$ gives for the semi-flat solution

$$
\begin{equation*}
\partial_{r} e^{2 \varphi_{\mathrm{semi} \mathrm{-fat}}}=\partial_{r} V_{\mathrm{fiber}} \sim-\frac{1}{r} \tag{9.7.114}
\end{equation*}
$$

For our expression,

$$
\begin{equation*}
\partial_{r} e^{2 \varphi(z, \bar{z})}=-\frac{1}{2 \pi^{2}} \sum_{n, m} \frac{2 r}{r^{2}+n^{2}+m^{2}} \approx-\int d u d v \frac{2 r}{r^{2}+u^{2}+v^{2}} \sim-\frac{1}{r} \tag{9.7.115}
\end{equation*}
$$

where we approximated the sum by an integral. Thus, we reproduced the semi-flat dilaton from the exact one.

Although the semi-flat approximation reproduces the qualitative features of the exact solution, due to the partially compactified transverse space, we have to deal with another problem. The function (9.7.106) is valid only close to the brane. A semi-flat solution that extends to the entire complex plane may be given by means of the $j$ function. This solution suffers from the problem of orbifold points as we described in Section 8.2.2 and it is not possible to describe a single NS5-brane consistently. The

[^60]exact solution avoids this problem by roughls speaking going to the limit very close to the brane (in the $z$-plane) compared to the size of the fiber. Hence, any possible orbifold points are pushed to an infinite distance (in the base) and thus are invisible.

| example | potential |
| :---: | :---: |
| flat space | $V(\vec{x})=\frac{1}{\|\vec{x}\|}$ |
| Taub-NUT | $I(\vec{x})=1+\frac{1}{\|\vec{x}\|}$ |
| Eguchi-Hanson | $V(\vec{x})=\frac{1}{\left\|\vec{x}-x_{1}\right\|}+\frac{1}{\left\|\vec{x}-\overrightarrow{x_{2}}\right\|}$ |

Table 9.2: Some well-known examples for the Gibbons-Hawking ansatz.

The Gibbons-Hawking ansatz. A perhaps more visual comparison of the semi-flat and exact metrics is possible through the ansatz ${ }^{27}$,

$$
\begin{equation*}
d s^{2}=V(\vec{x}) d \vec{x}^{2}+V(\vec{x})^{-1}(d t+\vec{A} \cdot d \vec{x})^{2} \tag{9.7.116}
\end{equation*}
$$

where $\vec{A}$ is given through $\vec{\nabla} \times \vec{A}=\vec{\nabla} V$. This defines a circle fibration over a three dimensional base parametrized by $\vec{x}$. A semi-flat solution for a degenerating fiber is given through

$$
\begin{equation*}
\tau(z) \sim \frac{1}{2 \pi i} \log (z) \tag{9.7.117}
\end{equation*}
$$

This corresponds to [199]

$$
\begin{gather*}
V=\tau_{2}=-\frac{1}{2 \pi} \log |z|  \tag{9.7.118}\\
A_{x}=\tau_{1}=\frac{1}{4 \pi i} \log (z / \bar{z}) \quad A_{z}=0 \quad A_{\bar{z}}=0 \tag{9.7.119}
\end{gather*}
$$

where the $3 \mathrm{~d} \vec{x}$ space has been decomposed into $(x, z, \bar{z})$. This gives a singular metric which is translationally invariant in the $x$ direction.

The exact non-singular hyper-Kähler metric is not translationally invariant, but periodic.

$$
\begin{equation*}
V=\frac{1}{4 \pi} \sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{(x-n)^{2}+z \bar{z}}}-\frac{1}{|n|}\right)+\text { const. } \tag{9.7.120}
\end{equation*}
$$

In Figure 9-20 (i), red dots indicate where this potential is singular. An $S^{3}$ (the

[^61]

Figure 9-20: (i) Comparing semi-flat and exact metrics for around degenerating fibers. The base is 3 d , parametrized by the periodic $x$ coordinate and the complex $z$-plane. The red line / red dots indicate where the $S^{1}$ fiber vanishes. Translational invariance of the semi-flat solution is replaced by periodicity of the exact metric in the $x$ direction. (ii) The same (exact) metric from a different viewpoint. The horizontal direction in the torus fiber is the $x$ coordinate. The torus pinches at the degeneration point (red dot) in the 2d base. Topologically, the singular fiber is an $S^{2}$ with two points glued together. This replaces the degenerating $\tau_{2} \rightarrow \infty$ torus of the semi-flat solution.
"throat") close to such a degeneration point can be seen as a Hopf-fibration with fiber $t$ above the $S^{2}$ surrounding the singularity. On the right-hand side of Figure 920, we see again the SYZ-like fibration with a two-torus fiber above the complex plane. The torus fiber degenerates into a "pillow" 28 above a codimension two locus.

It is possible to glue an approximate $K 3$ metric from 24 such patches. For details, see [116]. A more conventional way to obtain a smooth approximate metric is to start with the singular $T^{4} / \mathbb{Z}_{2}$ orbifold and blow up the 16 fixed points. This is possible by cutting out a small neighborhood (whose boundary is homeomorphic to $\mathbb{R P}^{3}$ ) around the fixed point and gluing there an Eguchi-Hanson space. This is the unique smooth hyper-Kähler metric which asymptotes to $\mathbb{C}^{2} / \mathbb{Z}_{2}$.

[^62]
### 9.8 Appendix: The Hanany-Witten effect from the semiflat approximation

The Hanany-Witten effect [134] is an interesting phenomenon of brane creation. It can be used to construct brane configurations that realize four dimensional $\mathcal{N}=1$ supersymmetric gauge theories [78] which exhibit Seiberg duality. This duality relates two different gauge theories which give the same infrared physics [211]. In string theory, it is realized in a very geometric way: as the branes move around, new branes appear which can change the rank of the gauge group in the 4 d theory ${ }^{29}$.

The Hanany-Witten setup co tains D4-branes stretched between NS5-branes in the presence of D6-branes. The D6, D4 and NS5 are magnetically charged under $C^{(1)}, C^{(3)}$ and $B$, respectively. The branes that we are going to use have the following orientations

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| NS5 | x | x | x | x | x |  |  |  |  | x |
| D6 | x | x | x | x |  | x |  | x | x |  |
| D4 | x | x | x | x |  |  | x |  |  |  |

Table 9.3: Branes in the Hanany-Witten setup. 456 are the base, 789 are the fiber coordinates.

Let us now consider an NS5-brane and a D6-brane as in the left-hand side of Figure 9-21. As the two branes pass though each other, a new brane is created. In order to verify that it is indeed a D4-brane, we need to determine its monodromy. Ramond-Ramond charges $\left(C_{7}, C_{8}, C_{9}, C_{789}\right)$ transform in the dual fundamental representation of $S L(4)$, which means that we have to use the transposed monodromy matrices.

[^63]

Figure 9-21: Hanany-Witten brane creation mechanism. $A$ and $B$ are the monodromies of the NS5- and D6-branes, respectively. As the two branes pass through each other, a new brane appears with a monodromy around the green circle (see righthand side). This monodromy can be easily computed in the original configuration (left-hand side) where the green path was a deformed loop around the two branes. The result is $A B A^{-1} B^{-1}$ which is simply the monodromy of a D 4 -brane.

The NS5 monodromy,

$$
A=\mathcal{M}_{\mathrm{NS} 5}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{9.8.121}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

and the D6 monodromy ${ }^{30}$

$$
\begin{equation*}
B: C_{9} \mapsto C_{9}+1 \tag{9.8.122}
\end{equation*}
$$

From these ingredients, we need to determine the monodromy along the green loop in the left-hand side of Figure 9-21. A moment's thought will convince the reader that it is the (group-theoretic) commutator, $A B A^{-1} B^{-1}$. This is best seen by choosing "branch cut planes" that start from the branes and studying how the green loop intersects these (see Figure 9-22).

Finally, the commutator can be computed

$$
\begin{equation*}
A B A^{-1} B^{-1}: C_{789} \mapsto C_{789}+1 \tag{9.8.123}
\end{equation*}
$$

[^64]

Figure 9-22: Determining the monodromy around the green loop by means of 2 d branch cuts.
which is the monodromy of a D4-brane ${ }^{31}$.

[^65]
### 9.9 Appendix: Type IIA on $\mathrm{T}^{5} / \mathbb{Z}_{2}$ and Type IIB on $\mathbf{S}^{1} \times \mathrm{K} 3$

Type IIA string theory on $T^{5} / \mathbb{Z}_{2}$ has been conjectured to be equivalent to Type IIB on $S^{1} \times K 3[234,67]$. We study this equivalence by means of the semi-flat machinery.

The IIA orientifold $T^{5} / \mathbb{Z}_{2}$ is generated by the action $\alpha \cdot \Omega$ that reverses the sign of all the circles ${ }^{32}$,

$$
\begin{equation*}
\alpha:\left(x^{5}, x^{6}, x^{7}, x^{8}, x^{9}\right) \mapsto\left(-x^{5},-x^{6},-x^{7},-x^{8},-x^{9}\right) \tag{9.9.124}
\end{equation*}
$$

and changes the parity of the world-sheet. There are 32 D 4 -branes located at the fixed points which cancel the RR tadpoles.

We cast the geometry in a fibration structure as follows. The 2d base coordinates will be $x^{5}$ and $x^{6}$. Since $\alpha$ inverts these coordinates, the base becomes $T^{2} / \mathbb{Z}_{2}$, i.e. a semi-flat $S^{2}$. Over this sphere there is a $T^{3}$ fiber parametrized by $x^{7}, x^{8}$ and $x^{9}$. The fiber degenerates at four points in the base. These singularities have a deficit angle of $180^{\circ}$ (tension six, like $D_{4}$ ). The monodromy around the singular points in the base is then

$$
\begin{equation*}
\mathcal{M}=R_{789} \cdot \Omega \tag{9.9.125}
\end{equation*}
$$

where $R_{i}$ is reflection of the i -th coordinate, $\Omega$ is the world-sheet parity transformation. This monodromy already includes the monodromies of eight D4-branes which cancel the RR-charges of the O4-plane. This is analogous to the orientifold limit of F-theory where the O7-plane monodromy is

$$
\mathcal{M}_{\mathrm{O} 7^{-}}=-T^{-4} \quad \text { with } \quad T=\mathcal{M}_{\mathrm{D} 7 \text {-brane }}=\left(\begin{array}{cc}
1 & 1  \tag{9.9.126}\\
0 & 1
\end{array}\right)
$$

which combines with the monodromy of the four D7-branes $\left(T^{4}\right)$ to cancel the RR-

[^66]charge. The final monodromy matrix is then diagonal.


Figure 9-23: $T^{5} / \mathbb{Z}_{2}$ as a fibration over $S^{2}$. The geometric $T^{3}$ fiber gets promoted to $T^{5}$ by adding $x^{10}$ and the M -theory circle $x^{11}$. The monodromy $\mathcal{M}$ then acts on this $T^{5}$.

What is $\mathcal{M}$ explicitly? Type IIB orientifolds at strong coupling can be described by F-theory. As already mentioned in Section 9.1.4, the torus fiber of F-theory is analogous to the $x^{11}-x^{10}$ coordinates in our case. Therefore, perturbative dualities are not sufficient for determining the monodromy around the orientifold fixed points and the $x^{11}$ coordinate should be included in the discussion. In the following, we determine the $5 \times 5$ U-duality monodromy matrices in the basis of $x^{7}-x^{8}-x^{9}-x^{11}-x^{10}$.

Reflection has an immediate interpretation in the vector representation of $S O(3,3)$. Since inversion of a coordinate exchanges the two spinors that have different chirality, only an even number of reflections give a symmetry of Type IIA. For instance, reflection of both $x^{7}$ and $x^{8}$ can be represented by

$$
\begin{equation*}
R_{78}=\operatorname{diag}(-1,-1,+1,-1,-1,+1) \in S O(3,3) \tag{9.9.127}
\end{equation*}
$$

which inverts the momentum and the winding as well. It has the spinor representation

$$
\begin{equation*}
R_{78}=\operatorname{diag}(-1,-1,+1,+1) \in S L(4) \tag{9.9.128}
\end{equation*}
$$

Odd number of reflections must be accompanied with other internal symmetries. However, we can still determine the corresponding monodromies, keeping in mind that we have to combine them with other symmetries. The relevant reflection is that
of $x^{7}-x^{8}-x^{9}$ which gives

$$
\begin{equation*}
R_{789}=\operatorname{diag}(-1,-1,-1,+1,-1) \tag{9.9.129}
\end{equation*}
$$

The inversion of $x^{10}$ comes about because the three coordinate reflections change the sign of the Levi-Civita pseudotensor that we used to convert the three-form field into a vector.

Let us consider the world-sheet parity transformation $\Omega$. In the Type IIA language, under this transformation the metric, the dilaton and the Ramond-Ramond one-form are even, whereas the B-field and the three-form are odd. In M-theory language this means that the three-form switches sign. From this, one can determine the $\Omega$ monodromy to be

$$
\begin{equation*}
\Omega= \pm \operatorname{diag}(+1,+1,+1,+1,-1) \tag{9.9.130}
\end{equation*}
$$

In order to fix the overall sign, let us consider an O6 orientifold plane. It is obtained by the action of $R_{78911}$ which reduces to $(-1)^{F_{L}} \Omega R_{789}$ in the IIA limit ([214], see also [228, 161]).

Since $(-1)^{F_{L}}$ changes the signs of the RR-fields, the corresponding monodromy is easy to determine, ${ }^{33}$

$$
\begin{equation*}
(-1)^{F_{L}}=\operatorname{diag}(-1,-1,-1,+1,-1) \tag{9.9.131}
\end{equation*}
$$

Since

$$
\begin{equation*}
R_{78911}=\operatorname{diag}(-1,-1,-1,-1,+1) \tag{9.9.132}
\end{equation*}
$$

we obtain ${ }^{34}$

$$
\begin{equation*}
\Omega=\operatorname{diag}(-1,-1,-1,-1,+1) \tag{9.9.133}
\end{equation*}
$$

From these ingredients we can now write down the explicit form of the monodromy

[^67]in the $T^{5} / \mathbb{Z}_{2}$ orientifold of Type IIA,
\[

$$
\begin{equation*}
\mathcal{M}=R_{789} \cdot \Omega=\operatorname{diag}(+1,+1,+1,-1,-1) \tag{9.9.134}
\end{equation*}
$$

\]

where we immediately recognize the monodromy of a (conjugate) $D_{4}$ singularity

$$
\begin{equation*}
\mathcal{M}=U \underbrace{\operatorname{diag}(-1,-1,+1,+1,+1)}_{R_{\bar{\gamma} \bar{\delta}}} U^{-1} \tag{9.9.135}
\end{equation*}
$$

Therefore by flipping $x^{7}-x^{11}$ and $x^{8}-x^{10}$, we obtain Type IIA on K3. ${ }^{35}$ The volume of the $T^{2}$ fiber is the inverse of the volume of the original $T^{3}$ fiber,

$$
\begin{equation*}
\operatorname{vol}\left(T^{2}\right)=R_{\overline{7}} R_{\overline{8}}=R_{11} R_{10}=R_{11} \frac{1}{R_{7} R_{8} R_{9} R_{11}}=\frac{1}{R_{7} R_{8} R_{9}}=\frac{1}{\operatorname{vol}\left(T^{3}\right)} \tag{9.9.136}
\end{equation*}
$$

Here we used the fact that the $T^{5}$ torus has unit volume in appropriate units.
Finally, by T-dualizing the spectator $x^{9}\left(R_{\tilde{9}}=1 / R_{9}\right)$, we arrive at the final equivalence,

$$
\begin{equation*}
\text { Type IIA on } T^{5} / \mathbb{Z}_{2} \text { orientifold } \cong \text { Type IIB on } K 3 \times S^{1} \tag{9.9.137}
\end{equation*}
$$

[^68]
### 9.10 Appendix: List of asymmetric orbifolds

In this Appendix, we list the asymmetric orbifoll actions that realize the almost non-geometric spaces of Section 9.2.2. The one-plaquette example was described in Section 9.2.3. We use the following trick [138]: some of the compact dimensions are "unfolded" and compactified back with an asymmetric action. The complexity of the model depends on how many dimensions have to be unfolded.

## Two-plaquette model



Figure 9-24: Almost non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ which is also a Joyce orbifold. It is T-dual to $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

$$
\begin{aligned}
\alpha:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) & \mapsto\left(-\theta_{1},-\theta_{2},-\theta_{3},-\theta_{4}, \theta_{5}, \theta_{6}\right) \\
\beta:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) & \mapsto\left(-\theta_{1},-\theta_{2}, \theta_{3}, \theta_{4},-\theta_{5},-\theta_{6}\right) \times(-1)^{F_{L}}
\end{aligned}
$$



Figure 9-25: This modified $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ is also a Joyce manifold.

$$
\begin{aligned}
\alpha:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) & \mapsto\left(-\theta_{1},-\theta_{2},-\theta_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \times(-1)^{F_{L}} \\
\gamma_{1}:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) & \mapsto\left(\theta_{1}, \theta_{2},-\theta_{3},-\theta_{4},-x_{5},-\theta_{6}\right) \times(-1)^{F_{L}} \\
\gamma_{2}:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) & \mapsto\left(\theta_{1}, \theta_{2},-\theta_{3},-\theta_{4}, L-x_{5},-\theta_{6}\right)
\end{aligned}
$$

## Model "L"



$$
\begin{aligned}
& \alpha_{1}:\left(x_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-x_{1},-\theta_{2},-\theta_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \times(-1)^{F_{L}} \\
& \alpha_{2}:\left(x_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(L-x_{1},-\theta_{2},-\theta_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \\
& \gamma_{1}:\left(x_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2},-\theta_{3},-\theta_{4},-x_{5},-\theta_{6}\right) \times(-1)^{F_{L}} \\
& \gamma_{2}:\left(x_{1}, \theta_{2}, \theta_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2},-\theta_{3},-\theta_{4}, L-x_{5},-\theta_{6}\right)
\end{aligned}
$$



Model "X"

$$
\begin{aligned}
& \text { el "X" } \\
& \alpha_{1}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-x_{1},-\theta_{2},-x_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \\
&
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{1}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-x_{1},-\theta_{2},\right. \\
& \alpha_{2}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(L-x_{1},-\theta_{2},-x_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \times(-1)^{F_{L}} \\
&\left.\theta_{1}\right) \mapsto\left(-x_{1},-\theta_{2}, L-x_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \times(-1)^{F_{L}}
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{2}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(L-x_{1},-\theta_{2}\right. \\
& \alpha_{3}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-x_{1},-\theta_{2}, L-x_{3},-\theta_{4}, x_{5}, \theta_{6}\right) \times(-1)^{F_{L}}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{3}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(-x_{1},-v_{2}, \theta_{3}, \theta_{3}\right) \\
\alpha_{4}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(L-x_{1},-\theta_{2}, L-x_{3},-\theta_{5},\right.
\end{aligned}
$$

$\gamma_{1}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2},-x_{3},-\theta_{4},-x_{5},-\theta_{6}\right)$
$\gamma_{2}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2}, L-x_{3},-\theta_{4},-x_{5},-\theta_{6}\right) \times(-1)^{F_{L}}$

$$
\begin{aligned}
& \gamma_{2}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2}, L-x_{3},\right. \\
& \gamma_{3}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2},-x_{3},-\theta_{4}, L-x_{5},-\theta_{6}\right) \times(-1)^{F_{L}} \\
& \hline
\end{aligned}
$$

$$
\begin{aligned}
& \gamma_{3}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto\left(x_{1}, \theta_{2}, L-x_{3},-\theta_{4}, L-x_{5},-\theta_{6}\right) \\
& \gamma_{4}:\left(x_{1}, \theta_{2}, x_{3}, \theta_{4}, x_{5}, \theta_{6}\right) \mapsto(
\end{aligned}
$$

### 9.11 Appendix: Spectrum of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and the two-plaquette model

For comparison and as a warm-up exercise, in this Appendix we review the details of the computation of massless spectra for $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}[227]$ and for its T-dual, the two-plaquette model. We will work in the RNS formalism. The compactification preserves $\mathcal{N}=2$ supersymmetry in four dimensions. The $\mathcal{N}=2$ multiplets are listed in the Table 9.4.

| hypermultiplet | 2 fermions, 4 scalars <br> vector multiplet <br> supergravity multiplet |
| :---: | :---: |
| vector, 2 fermions, 2 scalars <br> graviton, 2 gravitini, vector |  |

Table 9.4: Massless $\mathcal{N}=2$ multiplets in four dimensions (Weyl fermions and real scalars).

The massless spectrum of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$

## - untwisted sector

The spectrum contains the states that are invariant under the orbifold projection. Each of the orbifold group generators invert four spacetime coordinates. The action should also be specified on the fermions. We use the following convention ${ }^{36}$,

$$
\begin{equation*}
\alpha:(\pi,-\pi, 0) \quad \beta:(-\pi, 0, \pi) \quad \alpha \beta:(0,-\pi, \pi) \tag{9.11.138}
\end{equation*}
$$

The action of these elements on the states are summarized in the following table,

These states combine into 1 supergravity multiplet, 3 vector multiplets and 4 hypermultiplets according to Table 9.4.

[^69]| sector | state | $\alpha, \beta, \alpha \beta$ charge |
| :---: | :---: | :---: |
| $N S$ | $\psi_{-1 / 2}^{\mu}\|0 ; k\rangle$ | +++ |
|  | $\psi_{-1 / 2}^{4,5}\|0 ; k\rangle$ | --+ |
|  | $\psi_{-1 / 2}^{6,7}\|0 ; k\rangle$ | -+- |
|  | $\psi_{-1 / 2}^{8,9}\|0 ; k\rangle$ | +-- |
| $R$ | $\left.\left.\left.\right\|_{\text {gso }}+++\right\rangle,\left.\right\|_{\text {gso }}---\right\rangle$ | +++ |
|  | $\left.\left.\left.\right\|_{\text {gso }}++-\right\rangle,\left.\right\|_{\text {sso }}--+\right\rangle$ | +-- |
|  | $\left.\left.\left.\right\|_{\text {gso }}+-+\right\rangle,\left.\right\|_{\text {gso }}-+-\right\rangle$ | -+- |
|  | $\left.\left.\left.\right\|_{\text {gso }}+--\right\rangle,\left.\right\|_{\text {gso }}-++\right\rangle$ | --+ |

Table 9.5: Untwisted NS and $R$ sectors. In the $R$ sector, only the spins of compact complex dimensions are indicated. The remaining one is determined by the GSO projection as indicated by the "gso" label. This depends on whether it's the left or right $R$ sector.

| sector | fields |
| :---: | :---: |
| $N S_{-} / N S_{-}$ | $G_{\mu \nu}, B_{\mu \nu}$, dilaton, 12 real scalars |
| $N S_{-} / R_{+}$ | gravitino, 7 Weyl fermions |
| $R_{-} / N S_{-}$ | gravitino, 7 Weyl fermions |
| $R_{-} / R_{+}$ | 4 vectors, 4 cx. scalars |

Table 9.6: Untwisted sectors. The signs show the matter GSO projection (which, due to the superghost contributions, differ from the full GSO in the NS sectors).

## - twisted sectors

There are $16+16+16=48$ fixed tori under $\alpha, \beta$ and $\alpha \beta$. The zero-point energies vanish and both NS and R sectors have zero modes in the twisted and untwisted directions, respectively.

| sector | state | $\alpha, \beta, \alpha \beta$ charge |
| :---: | :---: | :---: |
| $N S$ | $\|\ldots++\rangle$ | $-i, i, 1$ |
|  | $\|\ldots--\rangle$ | $i,-i, 1$ |
| $R$ | $\left.\left.\right\|_{\text {gso }}+\ldots\right\rangle$ | $i,-i, 1$ |
|  | $\left.\left.\right\|_{\text {gso }}-\ldots\right\rangle$ | $-i, i, 1$ |

Table 9.7: $(\alpha \beta)$-twisted NS and R sectors. The other twisted sectors are analogous. The dots indicate half-integer moded oscillators which generate massive states.

| sector | fields |
| :---: | :---: |
| $N S_{+} / N S_{+}$ | cx. Scalar |
| $N S_{+} / R_{-}$ | Weyl fermion |
| $R_{+} / N S_{+}$ | Weyl fermion |
| $R_{+} / R_{-}$ | vector |

Table 9.8: Each twisted sector gives an $\mathcal{N}=2$ vector multiplet.
These states give 48 vector multiplets ${ }^{37}$.
Vertex operators are local with respect to the eight supercharges

$$
\begin{align*}
& e^{-\varphi / 2} e^{\frac{i}{2}\left(H_{0} \pm H_{1}\right) \pm \frac{i}{2}\left(H_{2}+H_{3}+H_{4}\right)}  \tag{9.11.139}\\
& e^{-\tilde{\varphi} / 2} e^{\frac{i}{2}\left(\tilde{H}_{0} \pm \tilde{H}_{1}\right) \pm \frac{i}{2}\left(\tilde{H}_{2}+\tilde{H}_{3}+\tilde{H}_{4}\right)} \tag{9.11.140}
\end{align*}
$$

In $\mathcal{N}=2$ language, this gives altogether one supergravity, 51 vector and 4 hypermultiplets. This is consistent with the expectations that Type IIA compactified on a Calabi-Yau should result in $h^{1,1}=51$ vector multiplets and $h^{2,1}+1=4$ hypermultiplets [227].

[^70]In $\mathcal{N}=1$ language, a hyper is two chirals, a vector is a vector+chiral, and the supergravity multiplet is a gravity + gravitino multiplet as seen from Table 9.9. Therefore, we obtain a gravity, a gravitino, 51 vector and 59 chiral multiplets.

| chiral multiplet | fermion, 2 scalars |
| :---: | :---: |
| vector multiplet | vector, fermion |
| gravitino multiplet | gravitino, vector |
| gravity multiplet | graviton, gravitino |

Table 9.9: Massless $\mathcal{N}=1$ multiplets in four dimensions (Weyl fermions and real scalars).

The massless spectrum of the two-plaquette model. The theory is described in Section 9.2.2 (see Figure 9-1 for the singularity structure). As discussed in Section 9.2.5, Type IIA on this background is T-dual to Type IIB on ordinary $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. As a further exercise, we compute the spectrum. The theory is defined as a $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ orbifold generated by $\alpha$ and $\beta$, similarly to the previous section. In this case, however, $\alpha$ includes the action of $(-1)^{F_{L}}$. In the RNS formalism, $(-1)^{F_{L}}$ does not act directly on the worldsheet fields. It changes the sign of the left-moving spin-fields and hence acts as charge conjugation on RR-fields. The GSO projection is switched in leftmoving sectors twisted by $\mathcal{R} \cdot(-1)^{F_{L}}$ compared to sectors twisted by $\mathcal{R}$ only. (Here $\mathcal{R}$ inverts four spacetime coordinates.) This can be deduced using the equivalence of the RNS and Green-Schwarz formalisms. In the latter description, $(-1)^{F_{L}}$ changes the sign of the $\theta^{a}$ left-moving world-sheet spinor fields in the light-cone gauge (for a review, see e.g. [62]).

We again need to impose GSO and orbifold invariance. In the twisted sectors, an ambiguity arises: one can keep even or odd states under the action of a certain $\mathbb{Z}_{2}$ generator. The choices are constrained by modular invariance. The signs are shown in Table 9.10.

The signs on the diagonal are directly related to the coloring of the edges (Figure 91). Perhaps the off-diagonal signs can be encoded in faces of the SYZ graph.

By the logic of Section 9.2.5, successive T-dualities attach $2 \times 2$ blocks of minus signs to Table 9.10 . T-duality on a $T^{3}$ then produces Table 9.11 which is precisely

|  | $\alpha$-t wisted | $\beta$-twisted | $\alpha \beta$-twisted |
| :---: | :---: | :---: | :---: |
| $P_{\alpha}$ | - | + | - |
| $P_{\beta}$ | + | + | + |
| $P_{\alpha \beta}$ | - | + | - |

Table 9.10: Signs of projections in various twisted sectors.
the choice of discrete torsion identified by [227] in the mirror of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ (see also [103, 70]).

|  | $\alpha$-twisted | $\beta$-twisted | $\alpha \beta$-twisted |
| :---: | :---: | :---: | :---: |
| $P_{\alpha}$ | + | - | - |
| $P_{\beta}$ | - | + | - |
| $P_{\alpha \beta}$ | - | - | + |

Table 9.11: Assignment of signs for discrete torsion.

- untwisted sector: Same result as untwisted $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.


## - $\beta$ twisted sectors

There are 16 fixed tori under $\beta$. The zero-point energies vanish and the GSO projection is the same as that of the $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ twisted sectors. In particular, we have $+/+$ in the NS/NS sector and $+/-$ in the $R / R$ sector. The orbifold projection preserves $\alpha$ even and $\beta$ even states. Due to $(-1)^{F_{L}}, \alpha$ and $\alpha \beta$ have an extra minus sign in the left R sector. The left and right states combine to give 16 hypermultiplets.


Table 9.12: $\beta$-twisted NS and R sectors.

- $\alpha$ and $\alpha \beta$ twisted sectors

There are $16+16$ fixed tori under the two group elements. The GSO projection is $+/-$ in the NS/NS sector and $-/-$ in the $R / R$ sector. The orbifold projection preserves $\alpha$ odd and $\beta$ even states, i.e. the twisted-sector vacuum has $\alpha$-charge $(-1)$. These twisted sectors give $16+16=32$ hypermultiplets.

| sector | state | $\alpha, \beta, \alpha \beta$ charge |
| :---: | :---: | :---: |
| $N S_{\text {left }}$ | $\|\ldots+-\rangle$ | $-i,-i,-1$ |
|  | $\|\ldots-+\rangle$ | $i, i,-1$ |
| $N S_{\text {right }}$ | $\|\ldots++\rangle$ | $-i, i, 1$ |
|  | $\|\ldots--\rangle$ | $i,-i, 1$ |
| $R_{\text {left }}$ | $\|-+\ldots\rangle$ | $-i,-i,-1$ |
|  | $\|+-\ldots\rangle$ | $i, i,-1$ |
| $R_{\text {right }}$ | $\|-+\ldots\rangle$ | $i,-i, 1$ |
|  | $\|+-\ldots\rangle$ | $-i, i, 1$ |

Table 9.13: $(\alpha \beta)$-twisted NS and R sectors.

Altogether we obtain a gravity multiplet, 52 hypermultiplets and 3 vector multiplets. Thus, the counting reproduces the massless spectrum of Type IIB on $T^{6} / \mathbb{Z}_{2} \times$ $\mathbb{Z}_{2}$.

### 9.12 Appendix: Spectrum of the one-plaquette model

The background is flat space divided by

$$
\begin{array}{llll}
\alpha:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) & \mapsto\left(-\theta_{1},-\theta_{2},-\theta_{3},-\theta_{4},\right. & +\theta_{5}, & \left.+\theta_{6}\right) \\
\beta_{1}:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) & \mapsto\left(-\theta_{1},-\theta_{2},+\theta_{3},+\theta_{4},\right. & -\theta_{5}, & \left.-\theta_{6}\right) \\
\beta_{2}:\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) & \mapsto & \left(-\theta_{1},-\theta_{2},+\theta_{3},+\theta_{4},\right. & \frac{1}{2}-\theta_{5}, \\
\hline & \left.-\theta_{6}\right) \times(-1)^{F_{L}}
\end{array}
$$

This gives $\mathcal{N}=1$ supersymmetry as we will see. Note that $\beta_{2} \cdot \beta_{1}$ defines a $(-1)^{F_{L}}$ Wilson-line for the $\theta_{5}$ base coordinate ${ }^{38}$. The signs of the projection in the twisted sectors we employ are given in Table 9.14. They are motivated by the logic of the previous example.

- untwisted sector

We need to carry out a projection on the invariant subspace. On the left NS states, $(-1)^{F_{L}}$ acts trivially. Therefore, the NS/NS and NS/R sectors are the same as those of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

[^71]|  | $\alpha$ | $\beta_{1}$ | $\beta_{2}$ | $\alpha \beta_{1}$ | $\alpha \beta_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $P_{\alpha}$ | + | + | + | + | + |
| $P_{\beta_{1}}$ | + | + | $\cdot$ | + | $\cdot$ |
| $P_{\beta_{2}}$ | + | $\cdot$ | - | $\cdot$ | - |
| $P_{\alpha \beta_{1}}$ | + | + | $\cdot$ | + | $\cdot$ |
| $P_{\alpha \beta_{2}}$ | + | $\cdot$ | - | $\cdot$ | - |

Table 9.14: Assignment of phases for the twisted sectors (columns). Dots indicate signs that do not affect the spectrum calculation. The group elements that are not listed here have no non-trivial fixed loci.

The R/NS and R/R sectors on the other hand do not contribute anything because there is no massless state invariant under both $\beta_{1}$ and $\beta_{2}$. In particular, this means that half of the gravitini are projected out compared to $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. We will see that no extra gravitini arise in the twisted sectors and hence only $\mathcal{N}=1$ supersymmetry is preserved in four dimensions. The fields combine into a gravity multiplet and seven chiral multiplets.

| sector | fields |
| :---: | :---: |
| $N S_{-} / N S_{-}$ | $G_{\mu \nu}, B_{\mu \nu}$, dilaton, 12 real scalars |
| $N S_{-} / R_{+}$ | gravitino, 7 Weyl fermions |
| $R_{-} / N S_{-}$ | - |
| $R_{-} / R_{+}$ | - |

Table 9.15: Untwisted closed sectors.

## - $\alpha$ twisted sectors

There are 16 fixed tori. Zero point energies vanish both in the NS and R sectors. These states give 16 chiral multiplets. The R/R and R/NS sectors do not contribute because no states are invariant under both $\beta_{1}$ and $\beta_{2}$.

- $\beta_{1}, \alpha \beta_{1}$ twisted sectors

There are $8+8$ invariant fixed tori, respectively, because $\beta_{2}$ permutes them in pairs. Zero point energies vanish both in the NS and R sectors. Each fixed locus gives an $\mathcal{N}=2$ vector multiplet, so in $\mathcal{N}=1$ language we obtain 16

| sector | fields |
| :---: | :---: |
| $N S_{+} / N S_{+}$ | cx. scalar |
| $N S_{+} / R_{-}$ | Weyl fermion |
| $R_{+} / N S_{+}$ | - |
| $R_{+} / R_{-}$ | - |

Table 9.16: $\alpha$-twisted sector: a chiral multiplet.
chiral multiplets and 16 vector multiplets.

| sector | fields |
| :---: | :---: |
| $N S_{+} / N S_{+}$ | cx. scalar |
| $N S_{+} / R_{-}$ | Weyl fermion |
| $R_{+} / N S_{+}$ | Weyl fermion |
| $R_{+} / R_{-}$ | vector |

Table 9.17: Twisted sector: a vector and a chiral multiplet.

## - $\beta_{2}, \alpha \beta_{2}$ twisted sectors

These $8+8$ sectors contain a twist by $(-1)^{F_{L}}$. The GSO projection is switched for all the left-moving states. Zero point energies still vanish as moding is not affected by $(-1)^{F_{L}}$. These states give 32 chiral multiplets.

| sector | fields |
| :---: | :---: |
| $N S_{-} / N S_{+}$ | cx. scalar |
| $N S_{-} / R_{-}$ | Weyl fermion |
| $R_{-} / N S_{+}$ | Weyl fermion |
| $R_{-} / R_{-}$ | cx. scalar |

Table 9.18: Twisted sectors that include $(-1)^{F_{L}}$ : two chiral multiplets. The leftmoving GSO projections are modified compared to the usual twisted sectors.

The other orbifold group elements have no fixed points. Vertex operators are local with respect to four right-moving supercharges,

$$
\begin{equation*}
e^{-\tilde{\varphi} / 2} e^{\frac{i}{2}\left(\tilde{H}_{0} \pm \tilde{H}_{1}\right) \pm \frac{i}{2}\left(\tilde{H}_{2}+\tilde{H}_{3}+\tilde{H}_{4}\right)} \tag{9.12.141}
\end{equation*}
$$

Finally, we obtain an $\mathcal{N}=1$ gravity multiplet, 16 vector multiplets and 71 chiral multiplets. It would be good to explicitly check modular invariance of the partition function for this example.

### 9.13 Appendix: Spectra of Joyce orbifolds

In this Appendix, we describe the spectra of two seven dimensional Joyce manifolds interpreted as non-geometric Type IIA compactifications down to four dimensions. The $T^{7} /\left(\mathbb{Z}_{2}\right)^{3}$ orbifolds are generated by the following involutions,

$$
\begin{aligned}
& \alpha:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(\quad x_{1}, \quad-x_{2}, \quad-x_{3} \left\lvert\, \begin{array}{lllll}
y_{1}, & y_{2}, & -y_{3}, & \left.-y_{4}\right)
\end{array}\right.\right. \\
& \beta:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(\quad-x_{1}, \quad x_{2}, b_{2}-x_{3} \mid \quad y_{1},-y_{2}, \quad y_{3}, \quad b_{1}-y_{4}\right) \\
& \gamma:\left(x_{1}, x_{2}, x_{3} \mid y_{1}, y_{2}, y_{3}, y_{4}\right) \mapsto\left(c_{5}-x_{1}, c_{3}-x_{2}, \quad x_{3} \mid-y_{1}, \quad y_{2}, \quad y_{3}, c_{1}-y_{4}\right)
\end{aligned}
$$

where $b_{1}, b_{2}, c_{1}, c_{3}, c_{5} \in\left\{0, \frac{1}{2}\right\}$ are constants. Note that $\alpha^{2}=\beta^{2}=\gamma^{2}=1$ and $\alpha, \beta$ and $\gamma$ commute. The action preserves the $G_{2}$-structure

$$
\begin{aligned}
\varphi=d x_{1} \wedge d y_{1} \wedge & d y_{2}+d x_{2} \wedge d y_{1} \wedge d y_{3}+d x_{3} \wedge d y_{2} \wedge d y_{3}+d x_{2} \wedge d y_{2} \wedge d y_{4} \\
& -d x_{3} \wedge d y_{1} \wedge d y_{4}-d x_{1} \wedge d y_{3} \wedge d y_{4}-d x_{1} \wedge d x_{2} \wedge d x_{3}
\end{aligned}
$$

The notation is the same as that of [159], but we reshuffled the coordinates to distinguish between base and fiber directions.

Example with three shifts: $\left(b_{1}, b_{2}, c_{1}, c_{3}, c_{5}\right)=\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0\right)$
This is Example 3 in [159]. The Betti numbers are computed to be $b^{2}=12, b^{3}=$ 43. Therefore when M-theory is compactified on this manifold, 12 vector multiplets and 43 chiral multiplets are obtained. In order to compute the U-dual Type IIA spectrum, we first need to choose the $x^{10}$ direction. This can be chosen to be $y_{1}$ since none of the $\mathbb{Z}_{2}$ actions contain a shift in this direction. However, $\gamma$ inverts $y_{1}$ and thus $(-1)^{F_{L}}$ must be separated from this transformation. This means that the geometric action on $T^{6}$ has inverted fiber coordinates for $\gamma$,

$$
\begin{aligned}
& \alpha_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\quad x_{1}, \quad-x_{2}, \quad-x_{3} \left\lvert\, \begin{array}{lll}
y_{2}, & -y_{3}, & \left.-y_{4}\right)
\end{array}\right.\right. \\
& \beta_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{cccccc}
-x_{1}, & x_{2}, & \left.\frac{1}{2}-x_{3} \right\rvert\, & -y_{2}, & y_{3}, & \left.-y_{4}\right)
\end{array}\right. \\
& \gamma_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{llllll}
-x_{1}, & \frac{1}{2}-x_{2}, & x_{3} & \mid-y_{2}, & -y_{3}, & \left.\frac{1}{2}+y_{4}\right)
\end{array} \times(-1)^{F_{L}}\right.
\end{aligned}
$$

The base and fiber coordinates nicely pair up. The orbifold group preserves the volume form of $T^{6}$ whose real part is obtained from $\varphi$. The untwisted sector contributes a gravity multiplet and seven chiral multiplets similarly to the non-geometric $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ in Appendix 9.11. Twisted sectors arise at the fixed $T^{2}$ tori of $\alpha_{0}, \beta_{0}$ and $\beta_{0} \gamma_{0}$. On the set of fixed loci for an element say $\alpha_{0}$, the other two group elements ( $\beta_{0}$ and $\beta_{0} \gamma_{0}$ ) act freely by permuting the tori. Therefore, we obtain $4+4+4$ two-tori each giving a vector and three chiral multiplets. This gives altogether 12 vector and 43 chiral multiplets which matches the U-dual M-theory result ${ }^{39}$.

| sector | fields |
| :---: | :---: |
| $N S / N S$ | 2 cx. scalars |
| $N S / R$ | 2 Weyl fermions |
| $R / N S$ | 2 Weyl fermions |
| $R / R$ | vector, cx. scalar |

Table 9.19: Twisted sectors for $\left(b_{1}, b_{2}, c_{1}, c_{3}, c_{5}\right)=\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0\right)$ give a vector and three chiral multiplets.

Example with two shifts: $\left(b_{1}, b_{2}, c_{1}, c_{3}, c_{5}\right)=\left(0, \frac{1}{2}, \frac{1}{2}, 0,0,0\right)$
This is Example 4 in [159]. The Betti numbers are $b^{2}=8+l, b^{3}=47-l$ where $l \in\{0, \ldots, 8\}$. The non-trivial elements with fixed loci are $\alpha, \beta$ and $\gamma$. These fix 48 copies of $T^{3}$. The group $\langle\beta, \gamma\rangle$ permutes the 16 three-tori fixed by $\alpha$, and $\langle\alpha, \gamma\rangle$ permutes the tori fixed by $\beta$. These give $4+4$ copies of $T^{3}$. However, the action of the element $\alpha \beta$ is trivial on the tori fixed by $\gamma$. Therefore, we obtain 8 copies of $T^{3} / \mathbb{Z}_{2}$. There are two topologically distinct ways to resolve each of these singularities and the choice of $l$ distinguishes between the various cases.

Similarly to the previous example, we can try to interpret this $G_{2}$ space as a Type IIA background

$$
\begin{aligned}
& \alpha_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{cccccc}
x_{1}, & -x_{2}, & -x_{3} & \mid & y_{2}, & -y_{3},
\end{array}-y_{4}\right) \\
& \beta_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{cccccc}
-x_{1}, & x_{2}, & 1 / 2-x_{3} & \mid & -y_{2}, & y_{3},
\end{array} \quad-y_{4}\right) \\
& \gamma_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\begin{array}{llllll}
-x_{1}, & -x_{2}, & x_{3} \quad \mid \quad-y_{2}, & -y_{3}, & \left.1 / 2+y_{4}\right) & \times(-1)^{F_{L}}, ~
\end{array}\right.
\end{aligned}
$$

where $\gamma_{0}$ also includes the action of $(-1)^{F_{L}}$. The untwisted sector again gives a gravity

[^72]multiplet and seven chiral multiplets. The non-trivial elements $\alpha_{0}, \beta_{0}, \alpha_{0} \gamma_{0}$ and $\beta_{0} \gamma_{0}$ give twisted sectors with massless fields. Taking into account the permutations by other group elements. $\alpha_{0}$ and $: 3_{0}$ give $4+4 T^{2}$ tori. These sectors each contribute a vector and three chiral multiplets.

Let us now consider the sectors that contain a twist by $(-1)^{F_{L}}$. As opposed to the seven dimensional interpretation where one obtained 8 copies of $T^{3} / \mathbb{Z}_{2}$, here the $16+$ 16 two-tori fixed by $\alpha_{0} \gamma_{0}$ and $\beta_{0} \gamma_{0}$ are permuted by the other group elements which gives 8 copies of $T^{2}$. Since each of them gives a vector and three chiral multiplets, the spectrum does not match that of the M-theory compactification.

What went wrong? Since the monodromies contained a $1 / 2$ shift, there is an ambiguity in the definition of $(-1)^{F_{L}}$. This can be seen by redefining the coordinates $\tilde{y} \equiv y+\frac{1}{4}$ which adds half-shifts for the fiber coordinates for the action of $(-1)^{F_{L}}$. The resulting monodromy is an affine transformation. We do not know how to fix this ambiguity in the general case.

By separating the $T^{7}$ coordinates into $x_{i}$ and $y_{j}$, we have chosen a coassociative four-cycle for the fiber (i.e. $\left.\varphi\right|_{\text {fiber }}=0$ ). The terms in the flat $G_{2}$-structure $\varphi$ basically tell us which coordinate triples can be chosen for base coordinates. Out of $\binom{7}{3}=35$ choices, there are precisely seven for which the $T^{4}$ fiber is coassociative. For some of the choices, however, an element of the $\left(\mathbb{Z}_{2}\right)^{3}$ group would be interpreted as an overall orbifolding by $(-1)^{F_{L}}$ in which case U-duality does not work. For example, if we choose $x_{1}, y_{1}$ and $y_{2}$ for the base coordinates, then $\alpha$ inverts the four fiber coordinates everywhere and the local model as $T^{4}$ over a base breaks down.

In the case of the three-shift example, the puzzle with the fiber shifts can be avoided if instead of $x_{i}$, we take $x_{3}, y_{1}$ and $y_{4}$ for base coordinates. Then, there will be no shifts in the fiber and we expect a perfect agreement with the M-theory spectrum. Picking $y_{3}$ for the $x^{10}$ coordinate, the generators have the following interpretation in Type IIA,

$$
\begin{aligned}
\alpha_{0}: & \left(x_{3}, y_{1}, y_{4} \mid x_{1}, x_{2}, y_{2}\right)
\end{aligned} \mapsto\left(\left.\begin{array}{rrr|rrr} 
& -x_{3}, & y_{1}, & -y_{4} & -x_{1}, & x_{2}, \\
\beta_{0} & \left.-y_{2}\right) & \times(-1)^{F_{L}} \\
\beta_{0}: & \left(x_{3}, y_{1}, y_{4} \mid x_{1}, x_{2}, y_{2}\right) & \mapsto( & \frac{1}{2}-x_{3}, & y_{1}, & -y_{4}
\end{array} \right\rvert\,-x_{1}, \quad x_{2}, \quad-y_{2}\right) .
$$

Twisted sectors come from $\alpha_{0}, \beta_{0}$ and $\gamma_{0}$. Similarly to the M-theory case, $\beta_{0}$ and $\gamma_{0}$ contributes $4+4$ fixed $T^{2}$. The $\alpha_{0}$-twisted sector gives $8 T^{2} / \mathbb{Z}_{2}$ and thus this Type IIA spectrum indeed reproduces the U-dual M-theory spectrum.

Another representation of the same model is possible by noticing that $\alpha \beta$ defines a $(-1)^{F_{L}}$ Wilson line for the $x_{3}$ fiber coordinate. Using this Wilson line, we are left with two generators,

$$
\begin{array}{rllll|rrr}
\alpha_{0}:\left(x_{2}, y_{2}, y_{4} \mid x_{1}, x_{3}, y_{1}\right) & \mapsto\left(\begin{array}{rrrrr}
x_{2}, & -y_{2} & -y_{4} & -x_{1}, & -x_{3}, \\
y_{1}
\end{array}\right) & \times(-1)^{F_{L}} \\
\gamma_{0}:\left(x_{2}, y_{2}, y_{4} \mid x_{1}, x_{3}, y_{1}\right) & \mapsto\left(\begin{array}{rrr|r}
-x_{2}, & y_{2}, & \frac{1}{2}-y_{4} & -x_{1}, \\
x_{3}, & -y_{1}
\end{array}\right)
\end{array}
$$

It is an example for chiral Scherk-Schwarz reduction (see Section 9.4).

Example with one shift: $\left(b_{1}, b_{2}, c_{1}, c_{3}, c_{5}\right)=\left(0, \frac{1}{2}, 0,0,0,0\right)$
For too few $1 / 2$ shifts, the orbifold has "bad singularities" (intersecting fixed loci) and the proper desingularization to a smooth $G_{2}$ holonomy manifold is more complicated [160]. These spaces can, however, still be embedded in string theory.

$$
\begin{aligned}
& \alpha_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(\quad x_{1},-x_{2}, \quad-x_{3} \mid \quad y_{2}, \quad-y_{3}, \quad-y_{4}\right) \\
& \beta_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(-x_{1}, \quad x_{2}, \quad \frac{1}{2}-x_{3} \left\lvert\, \begin{array}{llll}
2 & y_{3}, & \left.-y_{4}\right)
\end{array}\right.\right. \\
& \gamma_{0}:\left(x_{1}, x_{2}, x_{3} \mid y_{2}, y_{3}, y_{4}\right) \mapsto\left(-x_{1}, \quad-x_{2}, \quad x_{3} \left\lvert\, \begin{array}{llll} 
& -y_{2}, & \left.-y_{3}, \quad y_{4}\right) & \times(-1)^{F_{L}}, ~
\end{array}\right.\right.
\end{aligned}
$$

This background is dual to the one-plaquette model. Assuming that U-duality works, the spectrum calculation of Appendix 9.12 is a prediction for the Betti numbers of a resolution of this singular $G_{2}$ orbifold. Indeed, $b_{2}=16$ and $b_{3}=71$ is one of the possibilities as discussed in Section 12.5 in [160]. Some of the many remaining possibilities are presumably connected to this model by turning on discrete torsion [103].

### 9.14 Appendix: Polyhedron patterns



Figure 9-26: Truncated tetrahedron: the shifted $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.


Figure 9-27: Rhombic dodecahedron: fundamental domain of $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.


Figure 9-28: The $T^{6} / \Delta_{12}$ fundamental domain.

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[^0]:    ${ }^{1}$ The numbers indicate the (spatial) dimensionality of the branes.

[^1]:    ${ }^{2}$ In this usual schematic notation the coefficients and traces in front of the terms are not shown.

[^2]:    ${ }^{1}$ This identity was derived empirically with Barak Kol using the known examples. The brane tiling gives a proof for a generic $\mathcal{N}=1$ toric theory.

[^3]:    ${ }^{2}$ A name coined by Pavlos Kazakopoulous.

[^4]:    ${ }^{3}$ It is possible that some row and column operations produce a bipartite graph corresponding to a gauge theory with different matter content and interactions, but the same IR moduli space. It would be interesting to study the physical meaning of these operations in more detail.
    ${ }^{4}$ If the sets of vertices $v^{(1)}, v^{(2)}$ adjacent to vertices $1^{\prime}$ and $2^{\prime}$ (excluding the common neighbor 1 ) are not disjoint, then after integrating out there will be two or more edges between the same pairs of vertices. In such cases, these multiple edges may be replaced by a single edge carrying the sum of the weights of the individual edges, since this reproduces the correct counting of matchings of the graph. This is indeed what happens in the column reduction process, which may produce entries that are the sum or difference of two non-zero entries.

[^5]:    ${ }^{5}$ Assuming this graph is connected. In fact, this is not allways the case and it is possible for the toric phases to appear in disconnected (i.e. connected by non-toric phases) regions of the duality tree. A simple example of this situation is given by the duality tree of $\mathrm{dP}_{1}$. This tree is presented in [93], where the connected toric components where denoted "toric islands". In addition, it is interesting to see that if the theory is taken out of the conformal point by the addition of fractional branes, the cascading RG flow can actually "migrate" among these islands [?].

[^6]:    ${ }^{6}$ The extension to non-toric Seiberg dualities appears obvious on the periodic quiver, although there are subtleties involved in the precise operation on the Kasteleyn matrix.
    ${ }^{7}$ The transformation that we have identified with the action of Seiberg duality on the bipartite graph was discussed in [205], where it was referred to as "urban renewal". This work used a different assignment of weights in the transformed graph in order to keep the determinant (i.e. the GLSM field multiplicities, in our language) invariant across the operation. This is not what we want for Seiberg duality, which maps a toric diagram with one set of multiplicities to the same toric diagram with (in general) different multiplicities.

[^7]:    ${ }^{1}$ In some cases the $U(1)^{2}$ global symmetry can be enhanced. For example, for $Y^{p, q}$ theories the flavor symmetry is $S U(2) \times U(1)$ [26].

[^8]:    ${ }^{2}$ Recently, another physical description of the tiling has been developed in [87]. Using mirror symmetry, the D3-branes are mapped to a system of D6-branes that wraps a self-intersecting $T^{3}$ torus. The mirror geometry is a double fibration over the complex $W$ plane, one being the $W=u v$ torus fibration degenerating at the origin and another being the $W=P(w, z)$ fibration degenerating at some critical points. Here $P(w, z) \equiv \operatorname{det}$ (Kasteleyn) is the spectral curve with $(w, z)=\left(e^{s+i \theta}, e^{t+i \varphi}\right) \in\left(\mathbb{C}^{*}\right)^{2}$. The spectral curve can then be projected to the non-compact space $(s, t)$ which yields the amoeba whose spine is the pq-web of the toric diagram. Projection on the compact $(\theta, \varphi)$ coordinates gives the so-called alga of the curve. Its skeleton is the rhombus loop diagram that has been used to construct the brane tiling for a given toric diagram [132, 87]. This construction supports the D5-NS5 tiling proposal of [94], which appears when T-dualizing along the $S^{1}$ fibre in the $u v$ plane.

[^9]:    ${ }^{3}$ If we regard $-\varepsilon\left(e_{i}\right)$ as the energy of a link, the $X_{i}$ 's can be interpreted as complex valued Boltzmann weights.

[^10]:    ${ }^{4}$ We thank Alastair King for discussions on related ideas.

[^11]:    ${ }^{5}$ The flow space should not be confused with the flux space, which was introduced in the previous section and is $\mathbb{R}^{F+2}$.

[^12]:    ${ }^{6}$ Actually, a face of the tiling may be crossed by $\mathcal{C}_{x}$ over an odd number of edges. This happens when there are chiral multiplets transforming in the adjoint representation of the corresponding gauge group. Adjoint fields are represented in the tiling by edges such that the faces at both of its sides are identified (arrows beginning and ending at the same node in the dual quiver). For a field $X_{i}$ in the adjoint representation of the $l^{\text {th }}$ gauge group $\Delta_{l i}=0$ and thus the derivation of (4.5.48) still holds. The reader should keep in mind this subtlety.
    ${ }^{7}$ As we explained, it is straightforward to incorporate fields in the adjoint representation to the proof.

[^13]:    ${ }^{1}$ We have changed the notation to $L^{a, b, c}$ to avoid confusion with the $p$ and $q$ of $Y^{p, q}$. In our notation, $Y^{p, q}$ is $L^{p+q, p-q, p}$.

[^14]:    ${ }^{2} \mathrm{~A}$ vector $v \in \mathbb{Z}^{n}$ is primitive if it cannot be written as $m v^{\prime}$ with $v^{\prime} \in \mathbb{Z}^{n}$ and $\mathbb{Z} \ni m>1$.

[^15]:    ${ }^{3}$ Recall this is also one of our assumptions in this section.
    ${ }^{4}$ For example, one can use the Gysin sequence for each circle in turn.

[^16]:    ${ }^{5}$ This term was introduced in [82] and refers to quivers in which the ranks of all the gauge groups are equal. Toric quivers are a subset of the infinite set of Seiberg dual theories associated to a given toric singularity, i.e. it is possible to obtain quivers that are not toric on D -branes probing toric singularities.

[^17]:    ${ }^{6}$ There is a small subtlety in this argument: in some cases, identifications of faces due to the periodicity of the tiling can be such that the boundary of the sub-tiling is actually modified when performing Seiberg duality.

[^18]:    ${ }^{1}$ From now on, green lines will always denote edges in the brane tilings, red lines are edges of the

[^19]:    ${ }^{3}$ In [91] such tachyonic quivers were investigated in the context of $(p, q)$-webs.

[^20]:    ${ }^{4} \mathrm{We}$ can choose the direction of the rhombus loops so that they pass the black nodes on the left-hand side.

[^21]:    ${ }^{5}$ The results of the rest of the chapter will not depend on this observation.

[^22]:    ${ }^{6}$ The quiver gauge theory for $L^{a b c}$ has been constructed recently in $[96,30,52]$.

[^23]:    ${ }^{1}$ For earlier physics applications of exceptional collections to Landau Ginzburg models, see [193, 222, 109, 149, 237].

[^24]:    ${ }^{2}$ For a recent gauge theory interpretation of more general exceptional collections, see [230].
    ${ }^{3}$ For singular surfaces when $D$ is not a Cartier divisor, $\mathcal{O}(D)$ is actually not a line bundle but only a reflexive sheaf. Nevertheless, for simplicity, we will not emphasize this point further.

[^25]:    ${ }^{4}$ It may be that the metric compatible with a Ricci flat metric on the cone is not Kähler. For example, the metric on $\mathbf{d P}_{1}$ compatible with the $Y^{2,1}$ Sasaki-Einstein metric is not Kähler [191].

[^26]:    ${ }^{5}$ We would like to thank Aaron Bergman for this observation relating $\chi\left(E_{i}^{\vee}, E_{j}^{\vee}\right)$ to the Euler character of the torus.

[^27]:    ${ }^{6}$ We would like to thank Robert Karp for discussion about this point.

[^28]:    ${ }^{7} \mathrm{We}$ assume that the tiling is consistent and there are no "external multiplicities", i.e. there is a unique perfect matching corresponding to each external node of the toric diagram.

[^29]:    ${ }^{8}$ Related work on mesonic operators was recently done in [171, 201].

[^30]:    ${ }^{9}$ We will prove this for the non-periodic McKay quiver.

[^31]:    ${ }^{10}$ We thank Alastair King for related discussions.

[^32]:    ${ }^{12}$ An immediate question arises: Is this condition sufficient? Can consistency be defined as the equivalence of homotopic paths? We leave this question for future study.

[^33]:    ${ }^{13}$ In the example of Figure 7-20, the difference is $d_{r}=1-0=1$, i.e. there is one level line between $A$ and $B$.

[^34]:    ${ }^{14}$ For a specific Calabi-Yau, there are many equivalent Seiberg dual phases of the quiver theory $[83,86,80,33,84,92,29,144,132]$. Notice that the exceptional collection of section 7.3 .2 has the advantage that it gives back the right phase of the theory when computing the $S^{-1}$ quiver adjacency matrix.

[^35]:    ${ }^{1}$ It is best to think of the fiber as being very small compared to the size of the base. It is thought that in the large complex structure limit, the total space of the CY collapses to a metric space homeomorphic to $S^{n}$ which is the base of the fibration (see e.g. [116]).

[^36]:    ${ }^{2}$ By an appropriate coordinate transformation of the base coordinate, this metric can be recast into a symmetric $g \oplus g$ form (see $[220,186]$ ).

[^37]:    ${ }^{3}$ For explicit monodromies for the six strings, see [105].

[^38]:    ${ }^{4}$ These positive and negative vertices are also called type $(1,2) /$ type $(2,1)$ [115] or type III / type II [206] vertices by different authors. For an existence proof of metric on the vertex, see [186].

[^39]:    ${ }^{5}$ It is much harder in the general case to find a fibration that commutes with the group action.

[^40]:    ${ }^{6}$ Catalan solids are duals to Archimedean solids which are convex polyhedra composed of two or more types of regular polygons meeting in identical vertices. The dual of the rhombic dodecahedron is the cuboctahedron.

[^41]:    ${ }^{7}$ We thank A. Tomasiello for help in proving this.

[^42]:    ${ }^{8}$ In the weak tension limit, the two identical cones almost fill two half-spaces. The slopes of the edges are dictated by the tensions as in [215]. We leave the proof to the interested reader.

[^43]:    ${ }^{9}$ Since monodromy matrices do not generically commute, it is important to keep track of the branch cut planes.

[^44]:    ${ }^{1}$ This can be done consistently since the $(-1)^{F_{L}}$ symmetry forbids a tadpole for any RR field.

[^45]:    ${ }^{2}$ This matrix parametrizes the eight complex structure moduli, and one Kähler modulus of $T^{4}$.
    ${ }^{3}$ We will denote the extra dimension by $x^{10}$. This is not to be confused with the M-theory circle denoted by $x^{11}$.

[^46]:    ${ }^{4}$ Note that [46] uses a different basis for the spinors.

[^47]:    ${ }^{5}$ The only non-trivial element in the center of $S L(4)$ is $-\mathbb{1}$. This sign may be attached to all the group elements not in the identity component, giving an automorphism of $\operatorname{Spin}(3,3, \mathbb{Z})$.
    ${ }^{6}$ As discussed in [47], the fields that have simple transformation properties are $C^{(3)}=A^{(3)}+$ $A^{(1)} \wedge B$.

[^48]:    ${ }^{7}$ The relation to F-theory [226] can roughly be understood as follows. In the lower right corner of the $5 \times 5$ metric there is a $2 \times 2$ submatrix (with coordinates $x^{11,10}$ ). In the ten dimensional language, this matrix contains the dilaton and the three-form $X^{11} \sim C^{(3)}$ which is "mirror" to the $C^{(0)}$ axion in Type IIB. Roughly speaking, (conjugate) S-duality acts on this $T^{2} \subset T^{5}$.
    ${ }^{8}$ Note that $x^{10}$ and $x^{11}$ are switched. This is because we want to denote the extra M-theory dimension by $x^{11}$. We stick to this notation.

[^49]:    ${ }^{9}$ Parallelism makes sense in the context of $D_{4}$ singularities since the base has a flat metric.

[^50]:    ${ }^{10} \ldots$ since $x \mapsto-x$ and $x \mapsto L-x$ do not commute.

[^51]:    ${ }^{11}$ This interpretation would give $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ in Type IIB. This is mirror to Type IIA on $T^{6} / \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ with discrete torsion turned on [227].
    ${ }^{12}$ The general fiber in a Lagrangian fibration on any symplectic manifold is a torus. However, the general fiber for a coassociative $G_{2}$ fibration is expected to be $T^{4}$ or $K 3$ [181]. The adiabatic argument for U-duality only works for the $T^{4}$ case $[228,212]$ which must be taken into account when choosing the fiber coordinates.
    ${ }^{13}$ Although one of the gravitini is projected out by $(-1)^{F_{L}}$, it comes back in the twisted sector to give extended supersymmetry.

[^52]:    ${ }^{14}$ The action of $\alpha$ creates four parallel edges of the singular cube in the base. Then, $\beta$ and $\alpha \beta$ generate $4+4$ edges with $(-1)^{F_{L}}$. These give the two "red plaquettes" (see Figure 9-1).
    ${ }^{15}$ In the $T^{2}$ fiber language, the duality exchanges $\tau$ and $\rho$ and therefore takes a $D_{4}^{\prime}$ singularity into $D_{4}$.

[^53]:    ${ }^{16}$ The notation $b_{i}$ and $c_{i}$ is from [159]. These constants should not be confused with the Betti numbers.
    ${ }^{17} \mathrm{An}$ ambiguity is immediately discovered by noticing that a redefinition the fiber coordinates $\tilde{y} \equiv y+1 / 4$ changes the naive interpretation of $(-1)^{F_{L}}$ as $\operatorname{diag}(-1,-1,-1,-1)$. The new monodromy

[^54]:    ${ }^{18}$ Since the monodromy of $E_{6}$ is an order three modular transformation, adding a sign would make it order six.

[^55]:    ${ }^{19}$ The case of Wilson lines turned on for both fiber circles is the same since a modular $T$ transformation converts the $(-,-)$ spin structure into $(-,+)$.
    ${ }^{20}$ This $K 3$ fibration has been used in the literature ([36], see also [40]) to describe F-theory duals of 8d CHL strings [55, 56]. Nine dimensional CHL strings are defined by taking $E_{8} \times E_{8}$ heterotic strings and orbifolding by a $\mathbb{Z}_{2}$ action which shifts the ninth coordinate and interchanges the two $E_{8}$ factors. For a recent study of the moduli space of nine dimensional theories with sixteen supercharges, see [9].

[^56]:    ${ }^{21}$ This is to be compared with the CHL string in six dimensions which is dual [40] to M-theory on

    $$
    \begin{equation*}
    \left(K 3 \times S^{1}\right) /\{\sigma \cdot(y \rightarrow y+1 / 2)\} \tag{9.4.87}
    \end{equation*}
    $$

[^57]:    ${ }^{22}$ An early attempt to geometrize such examples was made in [176].

[^58]:    ${ }^{23} \mathrm{D}$-branes have a tension $T_{\mathrm{Dp} \text {-brane }}=1 / g_{s}\left(l_{s}\right)^{p+1}$ where $g_{s}$ is the string coupling and $l_{s}$ is the string length. On the other hand, NS5-branes have tension $T_{\text {NS5-brane }}=1 /\left(g_{s}\right)^{2}\left(l_{s}\right)^{6}$ which is much larger at weak coupling.

[^59]:    ${ }^{24}$ There is a slight change of variables compared to Appendix 9.6 which includes $\varphi \rightarrow 2 \varphi$.
    ${ }^{25}$ The Kähler potential for such 2 d semi-flat solutions can be computed and is given in [113, 186].

[^60]:    ${ }^{26}$ Note that the sum was made convergent by subtracting an infinite constant. For two dimensional lattices, this constant does not depend on $x$. This is the same trick that one uses in the definition of Weierstrass's elliptic function. Elliptic functions have been generalized to higher dimensions (see [207] and references therein, [136]).

[^61]:    ${ }^{27}$ Every 4d hyper-Kähler metric with a (triholomorphic) Killing vector can be written in this form [108].

[^62]:    ${ }^{28}$ Torus was the Latin word for a torus-shaped cushion.

[^63]:    ${ }^{29}$ See related works $[166,200,235,77,82,173,53,24,86,33,94,132]$ and references therein.

[^64]:    ${ }^{30}$ The D4- and D6-brane monodromies can be realized linearly if we include the $x^{11} \mathrm{M}$-theory circle in the discussion.

[^65]:    ${ }^{31} \mathrm{~A}$ similar observation about monodromies has recently been made by 't Hooft in [221]

[^66]:    ${ }^{32} \mathrm{An}$ Op-plane has charge $2^{p-5}$ and is generated by the action of $\begin{cases}R_{9-p} \Omega & \text { if } p=0,1(\bmod 4) \\ R_{9-p} \Omega(-1)^{F_{L}} & \text { if } p=2,3(\bmod 4) .\end{cases}$

[^67]:    ${ }^{33}$ This transformation effectively reflects the $x^{11}$ coordinate. In principle there could be an overall sign, but this can be fixed by remembering the $S L(4)$ representation of $(-1)^{F_{L}}$ which is simply $-\mathbb{1}_{4 \times 4}$.
    ${ }^{34}$ This reflects the well-known fact that conjugation by $S$-duality takes $(-1)^{F_{L}}$ to $\Omega$ in Type IIB.

[^68]:    ${ }^{35}$ For the weak coupling limit where Type IIA is defined, we need $R_{\widetilde{1} 1}=R_{7} \rightarrow 0$.

[^69]:    ${ }^{36}$ Other conventions give the same spectrum but preserve different supercharges.

[^70]:    ${ }^{37}$ If we take Type IIB instead, then we get complex scalars in the $R-R$ sectors which then combine into 48 hypermultiplets. Turning on discrete torsion has the same effect: in each twisted sector it changes the sign of projection for the other two non-trivial group elements of $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$. For example, in the ( $\alpha \beta$ )-twisted sector, the surviving $\mathrm{R}-\mathrm{R}$ states must be even under $\alpha \beta$ (as in the case without torsion), but odd under the $\alpha$ and $\beta$ transformations.

[^71]:    ${ }^{38}$ For generic circle radius, the resulting states are massive however. Massless states arise from the sector of zero momentum and winding. For further details, the reader is referred to [137].

[^72]:    ${ }^{39}$ This may be a coincidence since there was a half-shift in the fiber.

