Applying SCET to Parton Showers

by

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Abstract

In this thesis we study corrections to parton showers in the context of soft collinear effective theory (SCET). Monte Carlo event generators like Pythia or Herwig are heavily used by experimentalists to simulate events and they are indispensable tools to make exclusive theoretical predictions. They are based on a leading log parton shower algorithm that allows to resum the dominant contributions in the soft and collinear radiation. In this work we construct a framework to classify corrections to the parton shower that can be used to systematically improve event generators. We formulate parton showers as a standard matching procedure between a tower of soft collinear effective field theories called SCET_i. We find two different kinds of corrections: hard-scattering corrections and jet-structure corrections. To relate these different effective field theories we make use of an important symmetry of SCET, called reparametrization invariance. In order to systematically study this symmetry, we construct operators that are invariant under reparametrization and we use them to find a minimal basis of operators that are homogeneous in the power counting. Complete basis of operators are constructed for pure glue operators for deep inelasting scattering at twist-4, for production of two and three jets from e^+e^- and for production of two jets via gluon fusion.

Thesis Supervisor: Iain W. Stewart
Title: Associate Professor of Physics

To my wife Chiara

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Chapter 1

Introduction

1.1 QCD

Quantum Chromodynamics (QCD) is the theory of the strong interaction which is one of the four fundamental forces. It describes the interactions between quarks and gluons and how they bind together to form particles, called hadrons, such as the proton, neutron and pion. It is also one of the building blocks of the Standard Model (SM) of particles physics. QCD emerged in the 1970s and since then its predictions have been verified by a huge number of experiments. With the Large Hadron Collider (LHC) at CERN, we will probe nature at energies never experimentally reached before. The discovery of the Higgs boson, the last piece of the SM that has not been detected yet, and possibly of physics beyond the Standard Model are the major goals of the LHC. Detecting a signal at the LHC of Higgs particle or of new physics, is not easy because of the huge background of events that depend on SM processes. For this reason it is extremely important to improve our prediction of QCD events as much as possible.

One of the key predictions of QCD is asymptotic freedom [57]. This means that the coupling constant, α_s, becomes large at low energy and small at high energy. This makes it possible to use at high energy "fixed order perturbative perturbation theory", where we can expand QCD in powers of the small α_s. When we study a physical event, in general, several energy scales are involved. This poses two main problems to perturbative QCD. First it is fundamental to disentangle the physics that happens at high energies, where we can use perturbation theory, from the physics at lower energies that is non-perturbative. This is achieved using "factorization theorems" [37]. Second, even in the regime where α_s is small, in the perturbative expansion, α_s often appears multiplied by factor of $\log(\mu_1/\mu_2)$, where μ_1 and μ_2 are two scales that are present in the process. In general a generic observable ($\mathcal{O}$) has the following schematic expansion in perturbation theory

$$\begin{aligned}\mathcal{O} = 1 + \alpha_s L^2 + \alpha_s^2 L^4 + \alpha_s^3 L^6 + \cdots \\ \alpha_s L + \alpha_s^2 L^3 + \alpha_s^3 L^5 + \cdots \\ \alpha_s + \alpha_s^2 L^2 + \alpha_s^3 L^4 + \cdots, \\ \vdots\end{aligned} \tag{1.1}$$

where 1 denotes the tree level result and $L = \log(\mu_1/\mu_2)$. Even if $\alpha_s \ll 1$, for $\mu_1 \gg \mu_2$ we may have $\alpha_s \log^2(\mu_1/\mu_2) \sim 1$, and the perturbative expansion breaks down. In this case the large logarithms need to be resummed. Resumming the first row of Eq. (1.1) is called leading logarithmic (LL) resumation, resumming the second row of Eq. (1.1) is called next-to-leading logarithmic (NLL) resumation and so on. In this thesis we study large logarithms that arise in collinear and soft emissions using an effective field theory approach.

1.2 EFT

In nature different phenomena happen at different length, time, or energy scales. The idea behind effective field theory (EFT) is that the physics at a lower scale should not depend on the details of the physics at a higher scale [76, 88]. For example at the length scale of everyday life, $\sim 10^0$ meter (m), classical mechanics is a very successful theory, but we know that at the scale of atoms, $\sim 10^{-10}\,m$, classical mechanics breaks down and the theory that describes the motion of objects is quantum mechanics. We can consider classical mechanics as an EFT of quantum mechanics at length scales of

$\sim 10^0\, m$. Another example is Newtonian gravity. Planetary motion (orbital radius $\sim 10^6 m$) can be well explained using Newtonian gravity but we know that the more fundamental theory that describes the gravitational force is general relativity (GR). We can recover Newtonian gravity from a perturbative expansion of GR in the ratio $\phi = G_N M/(c^2 R)$, where M is the mass of the sun and R is the typical orbital radius. Newtonian gravity is an EFT of GR in the limit of small ϕ. An EFT is an approximate theory of an underlying more fundamental theory, that includes only the appropriate degrees of freedom to describe physical phenomena occurring at a chosen length scale. The appropriate parameters are those that are at the same scale as the physical quantities we are interested in studying, and the rest of the parameters are either too small or too large for the description. In some cases an EFT can be obtained from the underlyine theory by doing a perturbative expansion in the small parameters or in the inverse of the large parameters. Using an EFT makes it possible to do calculations, and to make predictions, that would be much harder to do using the underlying theory. For example it is possible in principle to study phenomena like the motion of a bullet using quantum mechanics, but it is in practice computationally prohibitive.

There are many EFTs for QCD; each one is suitable to describe phenomena at a particular scale with particular degrees of freedom. Some examples are: chiral perturbation theory [97], heavy quark effective field theory [79] and non relativistic QCD [88]. In this work will use soft collinear effective field theory (SCET) [8, 10, 14, 18]. SCET is the theory that describes soft and collinear particles that we define in the next section. We give an introduction of SCET in chapter 2.

1.3 Parton Showers

The LHC is a proton-proton collider. In a typical proton-proton collision, two partons, quarks or gluons, are extracted from the protons and interact in a hard collision. The collision may produce leptons, such as electrons, muons and neutrinos, and hadrons. There are at least two energetic scales involved: Q, the energy at which the protons interact, and Λ_{QCD} the scale where perturbation theory is not valid anymore. Q is called a hard scale and in an energetic collision it is high enough that we can use perturbation theory. Λ_{QCD} is called the hadronization scale because it is the energy scale governing how quarks and gluons bind together to form hadrons. Using the fundamental theory of QCD, fixed order perturbative calculations have been implemented only at next-to-leading order (NLO), α_s^2, in many cases and next-next-to-leading order (NNLO), α_s^3, in few cases. At tree level, the number of Feynman diagrams to calculate grows factorially with the number of partons in the final state. These calculations have only been performed for a relatively low number of external particles. Even tree-level expressions have only been worked out for $\mathcal{O}(10)$ particles [55, 69, 75, 83]. At one-loop the frontier is four external particles processes [26, 27, 67] and at two loops is three [2, 3, 30]. These limitation show that a direct computation of QCD processes with many external particles is not currently feasible. However, high energy colliders produce events with thousands of particles in each event. Besides, there are regions of phase space in which high-order terms are enhanced and cannot be neglected. If Q is a hard scale in the process then, in these regions, the amplitude gets enhanced so that its coefficient is $(\alpha_s \ln^2(Q/q))^m$, where $q \ll Q$ refers to a small scale that is induced by the choice of observables.

Instead of studying a process for a precise prediction to some order in perturbation theory, a different approach is to seek an approximate result where we capture the dominant contributions taking into account such enhanced terms at all orders. Enhanced higher-order regions come from kinematic configurations where the relevant QCD matrix element becomes large. In particular, this is associated with emissions of soft or collinear particles. To be more precise, let us consider a particle i with momentum q_i coming from some hard scattering that happens at energy scale Q, branching into two particles j and k with momenta q_j and q_k, see Fig. 1-1-(A). Because of the internal propagator, the amplitude (A) of this process is proportional to $1/q_i^2$ and in the limit where the mass of the particles j and k is zero, we have

$$A \sim \frac{1}{q_i^2} = \frac{1}{2E_j E_k\,(1-\cos\theta)} \tag{1.2}$$

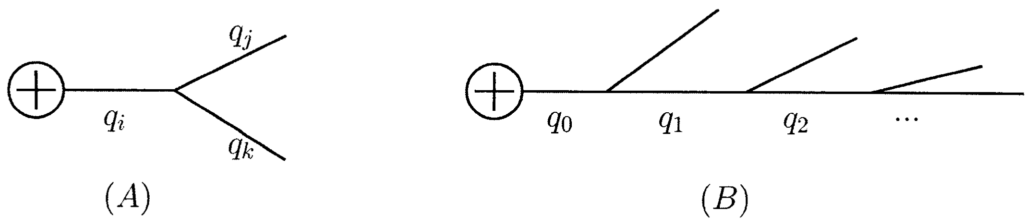

Figure 1-1: (A): branching of particle i to particles jk; (B): strongly ordered limit where $q_0^2 \gg q_1^2 \gg q_2^2 \gg \ldots$, this implies that each emission is more collinear than the previous one.

where E_j and E_k are the energies of the particles j and k, and θ is the angle between $\vec{q}_j$ and $\vec{q}_k$. From Eq. (1.2) we see that the amplitude gets enhanced when E_j or $E_k \to 0$, or $\theta \to 0$. In the first case the particles emitted are called soft because of the low energy, while in the second case they are called collinear because the particles are close to each other. A key characteristic is that in the collinear limit the cross section factorizes

$$\mathrm{d}\sigma_{X+jk} = dz\frac{dq_i}{q_i^2}P_{i\to jk}\mathrm{d}\sigma_{X+i}, \tag{1.3}$$

where $\mathrm{d}\sigma_{X+i}$ is the cross section to produce the particle i inside some larger process X, $\mathrm{d}\sigma_{X+jk}$ is the cross section to produce the particles jk, and P is called "splitting function" and represents the probability of the particle i to split into particle jk and can be calculated perturbatively in α_s. As an example, for a quark splitting to a quark and a gluon ($q \to qg$) after averaging and summing over spins, at leading order in α_s, this is

$$P^{(0)}_{q\to qg} = \frac{\alpha_s}{2\pi}C_F\frac{1+z^2}{1-z}, \tag{1.4}$$

where z is the momentum fraction of the emitted quark with respect to the parent quark and $C_F = 4/3$. If we integrated Eq. (1.3) over q_i^2, we would get a large logarithm $\log(Q/q)$. We can extend this argument to the emission of an arbitrary number of particles (q_0, q_1, q_2, ...) and we get large logarithms in the "strongly ordered" region

where

$$q_0^2 \gg q_1^2 \gg q_2^2 \gg \dots . \tag{1.5}$$

Condition Eq. (1.5) is equivalent to $q_{0\perp} \gg q_{1\perp} \gg q_{2\perp} \gg \dots$ where $\perp$ refers to the perpendicular component of the momentum with respect to the momentum of the mother particle. This means that in the strongly ordered limit, each emission is more collinear than the previous one, as depicted in Fig. 1-1-(B).

We saw that $P^{(0)}$ gives the probability for a parton to split. Using the Altarelli-Parisi equation it is possible to prove that the "Sudakov Factor", $\Delta(q^2, q_0^2)$, gives the probability of a parton to evolve from q_0^2 to q^2 without branching [44], where

$$\Delta(q^2, q_0^2) = \exp\left[-\int_{q_0^2}^{q^2} \frac{dq'^2}{q'^2} \int dx \frac{\alpha_s}{2\pi} P_{jk}^{(0)}(x)\right]. \tag{1.6}$$

We can use the splitting function and the Sudakov factor to construct an algorithm that branches a parton i into two partons jk and then iterates this process to produce an arbitrary number of partons in the final state. This process is called a parton shower and can be implemented in a Monte Carlo simulation as an event generator. Whereas a fixed order calculation is based on the perturbative expansion of α_s, the parton showers is defined in the soft-collinear limit and uses a probabilistic Markov chain of $1 \to 2$ particle splittings to recursively generate partons. In this way we resum the LL contributions by systematically treating real parton radiation.

To study an event in a hadron collider both fixed order calculations and parton shower event generators are used. We generate an event in three phases [40]. First we select the hard process at the parton level with a probability proportional to its production cross section, calculated using standard fixed order perturbation theory. Second, the produced partons, which are taken to be highly off-shell at the hard scale Q, radiate additional partons using an event generator and evolve down until their off-shellness reaches the hadronization scale Λ_{QCD}. Finally, all the partons hadronize using a confinement model. We illustrate these three phases in Fig. 1-2 for

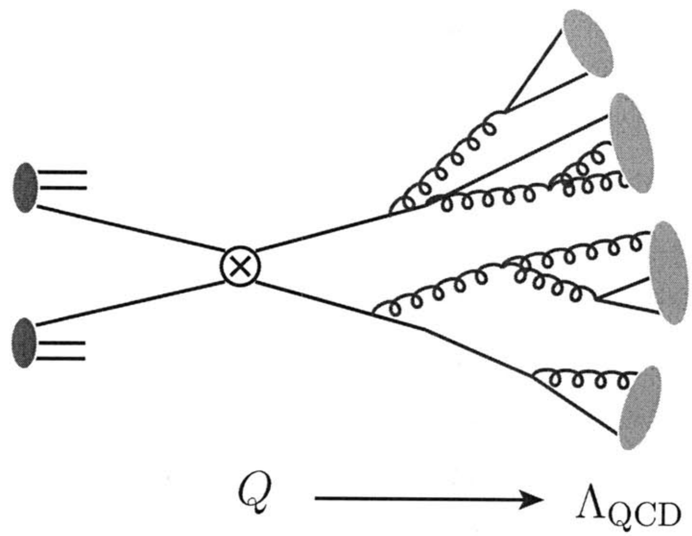

Figure 1-2: $p\bar{p}$ collision with final shower: two partons taken from the protons (red blobs) interact at scale Q. parton showers is used to emit radiations from Q down to the scale Λ_{QCD} where the partons hadronize (blue blobs). There is also radiation from the initial partons and from the remnants of the protons (not shown).

a $p\bar{p}$ collision. Two partons taken from the protons (red blobs) interact at scale Q producing two partons. A parton shower event generator is used to emit radiation from Q down to the scale Λ_{QCD} where the partons hadronize (blue blobs). There is also radiation from the initial partons and from the remnants (not shown); these are also described using a parton showers approach. Monte Carlo event generators like Pythia [91, 92] or Herwig [5, 38] are heavily used by experimentalists to simulate these types of events and have proved indispensable for making exclusive theoretical predictions.

There have been several improvements to LL parton showers such as MC@NLO [52], or CKKW [31], but a systematic way to resum NLL is missing in the literature and there is not even a clear method to catalog all the necessary corrections. The main problem to include NLL corrections is that we have to take into account emissions that are not strongly ordered, where $q_i^2 \gg q_{i+i}^2$ and where the factorization formula Eq. (1.3) is no longer valid. This means that we have to consider interference between different amplitudes as well as spin and color correlations.

In this thesis we set up a rigorous framework to study corrections to parton showers using SCET to pave the way for an implementation of a NLL parton shower algorithm. SCET is an appropriate effective theory for studying parton showers because it is designed to reproduce exactly the limit of soft and collinear particles. Moreover, SCET is organized in an expansion of a power counting parameter that makes it possible to classify all corrections to a known order in this parameter. The first work on parton showers using SCET was Ref. [17], where the authors proved how the splitting function and the Sudakov factor emerge naturally in SCET. They reproduced the LL parton showers using SCET but they introduce choices and approximations at several points which makes prohibitive to calculate corrections.

In our work, we describe the parton shower using a tower of independent but related effective field theories that we call SCET_i. Each SCET_i is a soft-collinear effective field theory. The difference between SCET_i and SCET_{i+1} is that SCET_{i+1} describes collinear particles with virtuality that is much smaller than the virtuality of a collinear particle in SCET_i, that is if q_{i+1} is a generic collinear particle in SCET_{i+1} and q_i a generic collinear particle in SCET_i we have that $q_{i+1}^2 \ll q_i^2$. We describe each emission in the strongly ordered region using a different SCET_i. Even if we have many EFTs, we will use only a single power counting parameter that we call λ. The LO operator in λ for n partons describes n emissions in the strongly ordered region. To resum NLL we need to calculate the NLO operators in λ as well as correction in α_s. In this work we will focus on corrections of the power counting, but in our framework we can also calculate corrections in α_s. We find two different kinds of corrections at NLO in λ: hard-scattering corrections and jet-structure corrections. The hard-scattering corrections depend on the hard-scattering process being investigated. The jet-structure corrections are independent from what happens at the scale Q, hence they are universal in the sense that they are the same for each process we want to study. The SCET_i picture, besides defining a clear method to calculate NLO corrections, has another important advantage in that we have factorization between emissions also at NLO. This characteristic is a crucial ingredient that gives hope for a future construction of a NLL parton shower algorithm.

In order to relate these different SCET_i we will make use of an important symmetry of SCET, called reparametrization invariance (RPI). Part of this thesis is devoted to study RPI. Symmetries are fundamental tools in all fields of physics. Knowing that a quantity is invariant under a set of transformations allows us to make some predictions. For example, if we have a quantity that depends only on two vectors, p^μ and q^μ, and we know it is a Lorentz scalar, we can immediately say that it can be only a function of q^2, p^2 and $p \cdot q$. Similarly, knowing that SCET is invariant under RPI, we are allowed to express all quantities in SCET using a complete a set of RPI invariant objects. In this thesis we construct operators that are invariant under reparametrization. Using RPI operators will turn out to be very powerful method to find a minimal basis that is homogeneous in the power counting in particular for processes with multiple jets.

1.4 Outline

In chapter 2 we give a brief review of SCET. After describing the main ingredients of SCET, we define RPI in this context. In chapter 3 we construct a set of operators that are invariant under reparametrization. It is based on Ref. [80]. We can use this set to reduce the number of operators in SCET. We construct a minimum basis of pure glue operators for DIS at twist-4, for production of two and three jets from e^+e^-, and for production of 2 jets from gluon fusion. Chapter 4 is dedicated to parton showers and is based on Ref. [21]. We formulate the shower emissions as a standard matching procedure between different SCET_i, namely $\mathrm{SCET}_i \to \mathrm{SCET}_{i+1}$. We use this formulation to classify and compute various corrections to the shower. Conclusions are given in chapter 5. We leave many of the technical details to the appendices.

Chapter 2

Soft Collinear Effective Field Theory

2.1 Introduction to SCET

Soft Collinear Effective Theory (SCET) is an effective field theory of QCD that describes the interactions of collinear and soft particles [8, 10, 14, 18]. The momentum p of any particle can be decomposed along two light-cone vectors, n and $\bar{n}$, with $n^2 = 0$, $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$, as

$$p^\mu = n{\cdot}p\frac{\bar{n}^\mu}{2} + \bar{p}\frac{n^\mu}{2} + p_\perp^\mu\,, \tag{2.1}$$

where $\bar{p} = \bar{n} \cdot p$ and the particle's invariant mass is $p^2 = p^+\bar{p} + p_\perp^2$. We use a Minkowskian notation for $p_\perp^2 = -\vec{p}_\perp^{\,2}$, where $\vec{p}_\perp$ is Euclidean. SCET's degrees of freedom include collinear and soft fields. [1]

A particle is collinear to a direction n if its momentum scales as:

$$(n \cdot p,\, \bar{p},\, p_\perp) \sim (\lambda^2,\, 1,\, \lambda)\,\bar{p}\,, \tag{2.2}$$

[1]Our primary interest here is the perturbative structure of jets, so we use what in literature is often called $\mathrm{SCET_I}$ theories with collinear and ultrasoft modes. For simplicitly we will always use the phrase soft in place of ultrasoft.

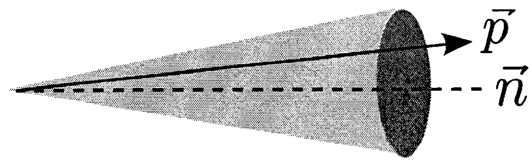

Figure 2-1: In SCET, a particle with momentum p is collinear to the direction n if $\vec{p}$ is inside a cone centered in $\vec{n}$ and of angle λ.

where $\bar{p} \sim Q$ is the hard scale, and $\lambda \ll 1$ is the power counting parameter of the SCET. Pictorially, we can think of a particle collinear to the direction n as having its three-momentum $\vec{p}$ is inside a cone centered on $\vec{n}$ and of angle λ, see Fig. (2-1). A particle is soft if

$$(n \cdot p,\, \bar{p},\, p_\perp) \sim (\lambda^2,\, \lambda^2,\, \lambda^2)\, Q\,. \tag{2.3}$$

Both Eqs. (2.2) and (2.3) imply $p^2 \lesssim Q\lambda^2$.

We obtain SCET from QCD by expanding in powers of λ, integrating out the hard modes and dividing the quark and gluon fields into separate soft and collinear modes. The soft fields are $\xi_s(x)$ and $A_s(x)$ where $\partial^\mu\, \xi_s(x) \sim \partial^\mu\, A_s(x) \sim \lambda^2$. For the collinear case, we introduce a momentum-space lattice and we give each field an index-label describing which lattice vector it is collinear to. To divide the QCD fields in this way, we split the momentum of a collinear particle into a "large" part $\tilde{p}^\mu$ and a residual one $k^\mu \sim \lambda^2$

$$p^\mu = \tilde{p}^\mu + k^\mu\,, \quad \text{where} \;\; \tilde{p}^\mu \equiv n\!\cdot\! p\frac{n^\mu}{2} + p_\perp^\mu\,. \tag{2.4}$$

We can remove the large momenta $\tilde{p}$ from the fermion field by the Fourier transform

$$\psi(x) = \sum_{\tilde{p},\, n} e^{-i\tilde{p}\cdot x}\, \psi_{n,\tilde{p}}\,. \tag{2.5}$$

For a collinear particle along n, $\partial^\mu\, \psi_{n,\tilde{p}}(x) \sim \lambda^2$. The four component field, $\psi_{n,\tilde{p}}$, has two large components $\xi_{n,\tilde{p}}$, and two small components $\xi_{\bar{n},\tilde{p}}$, that are defined using the following Dirac-space projectors:

$$\psi_{n,\tilde{p}} = \frac{\not{n}\not{\bar{n}}}{4}\psi_{n,\tilde{p}} + \frac{\not{\bar{n}}\not{n}}{4}\psi_{n,\tilde{p}} \equiv \xi_{n,\tilde{p}} + \xi_{\bar{n},\tilde{p}}. \tag{2.6}$$

These satisfy the relations

$$\begin{aligned} \frac{\not{n}\not{\bar{n}}}{4}\,\xi_{n,\tilde{p}} &= \xi_{n,\tilde{p}}\,, \quad \not{n}\,\xi_{n,\tilde{p}} = 0\,, \\ \frac{\not{\bar{n}}\not{n}}{4}\,\xi_{\bar{n},\tilde{p}} &= \xi_{\bar{n},\tilde{p}}\,, \quad \not{\bar{n}}\,\xi_{\bar{n},\tilde{p}} = 0\,. \end{aligned} \tag{2.7}$$

We can also similarly define a collinear gluon field $A^{\mu}_{n,\tilde{q}}(x)$ whose fluctuations are characterized by the scale of its residual momenta, $q^2 \sim \lambda^2$. Pictorially, we can think of $\xi_{n,\tilde{p}}$ and $A^{\mu}_{n,\tilde{q}}(x)$ as fields that create a particle whose three-momentum lies inside a cone with opening angle $\sim \lambda$ about the three-direction $\vec{n}$. We define $\mathcal{P}^{\mu}_{n}$ as the momentum operator that picks up the large component of the momentum: $\mathcal{P}^{\mu}_{n}\,\xi_{n,\tilde{p}}(x) = \tilde{p}^{\mu}\,\xi_{n,\tilde{p}}(x)$. Collinear fields always appear with a sum over $\tilde{p}$, so it is useful to redefine the fields as

$$\xi_n = \sum_{\tilde{p}} e^{i\tilde{p}\cdot x}\xi_{n,\tilde{p}}\,, \quad A^{\mu}_{n} = \sum_{\tilde{q}} e^{i\tilde{q}\cdot x} A^{\mu}_{n,\tilde{q}}\,. \tag{2.8}$$

The leading order SCET Lagrangian is

$$\mathcal{L}^{(0)}_{\text{SCET}} = \mathcal{L}^{(0)}_{s} + \sum_{n\in\{n_i\}} \mathcal{L}^{(0)}_{n} \tag{2.9}$$

where $\mathcal{L}^{(0)}_{n}$ has only collinear interactions among particles collinear to the same n. Particles collinear to different directions can interact either by the exchange of soft modes in $\mathcal{L}^{(0)}_{s}$, or from their coupling to other sectors in external operators. Two collinear sectors in SCET, n_1 and n_2, are distinct if [9]

$$n_1\!\cdot n_2 \gg \lambda^2\,, \tag{2.10}$$

so any particle is collinear in at most one direction within a given SCET.

Thus the key defining concepts of an SCET-theory are its hard-scale Q, its collinear sectors $\{[n_i]\}$, and its power counting parameter λ which governs the importance of operators and size of the collinear sectors.

We define collinear covariant derivatives as

$$i\bar{n}\cdot D_n = \overline{\mathcal{P}}_n + g\bar{n}\cdot A_{n,p}\,, \quad iD_n^{\perp\mu} = \mathcal{P}^\mu_{n\perp} + gA^{\perp\mu}_{n,p}\,, \quad in\cdot D_n = in\cdot\partial + gn\cdot A_{n,p}\,, \tag{2.11}$$

where $\mathcal{P}_n$ is the momentum operator that picks up the large component of the momentum, that is $\mathcal{P}_n\xi_{\tilde{p},n} = \tilde{p}\,\xi_{p,n}$. When integrating out hard offshell fluctuations and constructing gauge invariant structures in SCET, it is necessary to include collinear Wilson lines, W_n, defined by [14]

$$W_n(x) \;=\; \Big[\sum_{\text{perms}} \exp\Big(\frac{-g}{\overline{\mathcal{P}}_n}\,\bar{n}\cdot A_{n,p}(x)\,\Big)\Big]\,. \tag{2.12}$$

The collinear fields $A^\mu_{n,p}$ are defined with the zero-bin procedure [78]. To couple soft degrees of freedom we define an soft covariant derivative

$$iD^\mu_s = i\partial^\mu + gA^\mu_s\,, \tag{2.13}$$

that can act on collinear fields. At lowest order the coupling to n-collinear fields involves $n{\cdot}D_s$ and can be removed from the Lagrangian by the BPS field redefinition [14]

$$\xi_{n,p}(x) \to Y_n(x)\xi_{n,p}(x)\,, \qquad A_{n,q}(x) \to Y_n(x)A_{n,q}(x)Y_n^\dagger(x)\,, \tag{2.14}$$

with the soft Wilson line

$$Y_n(x^\mu) = \mathrm{P}\exp\Big(i\,g\int_{-\infty}^0 ds\;n\cdot A_s(x^\mu + sn^\mu)\Big)\,. \tag{2.15}$$

This field redefinition allows us to organize power corrections as gauge invariant products of collinear and soft fields as we discuss in the next subsection.

The SCET collinear lagrangian, $\mathcal{L}_n$ is derived from the QCD lagrangian by integrating out the field, $\xi_{\bar{n}}$. At LO, we have [10, 18]

$$\mathcal{L}_n^{(0)} = \bar{\xi}_n\big(in\!\cdot\!\partial + g\,n\!\cdot\! A_n + i\not{D}_{n\perp} W_n \frac{1}{\bar{\mathcal{P}}_n} W_n^\dagger i\not{D}_{n\perp}\big)\frac{\bar{\not{n}}}{2}\xi_n\,, \tag{2.16}$$

where we intrinsically sum over the large, label momenta, $\tilde{p}$, as well as the collinear index, n, which we have kept explicit as a label. The LO collinear Lagrangian for gluons is given in [14].

Operators are formed from products of the above fields, and the power counting for an operator is determined by adding up contributions from its constituents. The power counting for the fields and derivatives in SCET is

$$\begin{aligned}
&\xi_n \sim \lambda, \qquad (n\!\cdot\! A_n, \bar{n}\!\cdot\! A_n, A_n^\perp) \sim (\lambda^2, 1, \lambda)\,, \qquad q_s \sim \lambda^3, \qquad A_s \sim \lambda^2\,,\\
&i\partial^\mu \sim \lambda^2\,, \qquad (in\!\cdot\!\partial, \bar{n}\!\cdot\!\mathcal{P}, \mathcal{P}_{n\perp}) \sim (\lambda^2, 1, \lambda)\,, \qquad W_n \sim Y_n \sim \lambda^0\,.
\end{aligned} \tag{2.17}$$

2.1.1 Gauge Invariant Field Products

To build operators in SCET we want to use structures which are gauge invariant and homogeneous in the power counting. For SCET a convenient set of gauge invariant structures are:

$$\chi_n \equiv W_n^\dagger \xi_n, \qquad \mathcal{D}_n^\mu \equiv W_n^\dagger D_n^\mu W_n\,, \tag{2.18}$$

together with the $\mathcal{P}_n^\mu$ label momentum operator and derivative operator $i\partial^\mu$ acting on these structures. The collinear fields in Eq. (2.18) are the ones obtained after the field redefinition in Eq. (2.14). It is convenient to be able to switch the collinear derivatives multiplied by Wilson lines for gauge invariant field strengths, for which we use

$$\begin{aligned}
i\mathcal{D}_n^{\perp\mu} &= \mathcal{P}_{n\perp}^\mu + g\mathcal{B}_{n\perp}^\mu\,, \qquad i\overleftarrow{\mathcal{D}}_n^{\perp\mu} = -\mathcal{P}_{n\perp}^{\dagger\mu} - g\mathcal{B}_{n\perp}^\mu\,,\\
in\!\cdot\!\mathcal{D}_n &= in\!\cdot\!\partial + gn\!\cdot\!\mathcal{B}_n\,, \qquad in\!\cdot\!\overleftarrow{\mathcal{D}}_n = in\!\cdot\!\overleftarrow{\partial} - gn\!\cdot\!\mathcal{B}_n\,,
\end{aligned} \tag{2.19}$$

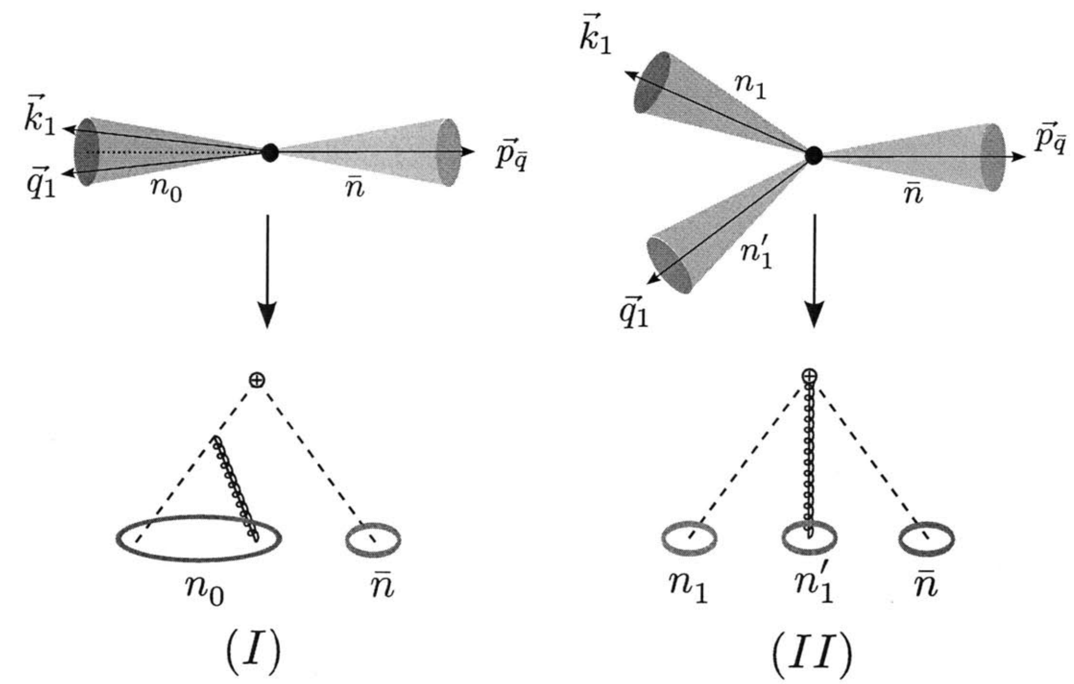

Figure 2-2: Final state with a quark (q_1), antiquark ($p_{\bar{q}}$), and gluon (k_1). Different kinematic configurations are described by different SCET operators. In **(I)**, the quark and the gluon are collinear to the direction n_0, represented by their sharing a cone. In **(II)**, the vectors q_1 and k_1 are too far apart to be collinear. The Feynman diagrams below show that collinear fields can come from the Lagrangian vertices, but non-collinear ones arise from higher-multiplicity operators.

and note that $\bar{n}\cdot\mathcal{D}_n = \bar{\mathcal{P}}_n$. Here the field strength tensors are

$$g\mathcal{B}^{\mu}_{n\perp} \equiv \left[\frac{1}{\overline{\mathcal{P}}_n}[i\bar{n}\cdot\mathcal{D}_n, i\mathcal{D}^{\perp\mu}_n]\right], \qquad gn\cdot\mathcal{B}_n \equiv \left[\frac{1}{\overline{\mathcal{P}}_n}[i\bar{n}\cdot\mathcal{D}_n, in\cdot\mathcal{D}_n]\right], \tag{2.20}$$

where the label operators and derivatives act only on fields inside the outer square brackets, and $g\mathcal{B}^{\mu}_{n\perp}$ and $gn\cdot\mathcal{B}_n$ are Hermitian.

We can construct gauge invariant operators using the fields defined above. Since the collinear fields carry a label referring to a specific light-cone vector, these operators describe particles in a specific region of phase space. SCET therefore distinguishes situations with the same number particle but different kinematics using different operators. For example, one can take an amplitude for three external particles: a quark (q_1), a gluon (k_1) and an antiquark ($p_{\bar{q}}$). We can consider two different configurations that we call $|q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle$ and $|q_{n_1} g_{n_1'} \bar{q}_{\bar{n}}\rangle$. In the first, shown in Fig. 2-2(I), the quark and the gluon are n_0-collinear, and the antiquark is collinear to a different direction, $\bar{n}$. Here the amplitude is described by operators with two distinct directions, say

$$\bar{\chi}_{n_0}\Gamma\chi_{\bar{n}} \sim \lambda^2\,, \quad \bar{\chi}_{n_0} g\mathcal{B}^{\mu}_{n_0\perp}\Gamma'\chi_{\bar{n}} \sim \lambda^3\,, \tag{2.21}$$

(where the form of the Dirac structures Γ and Γ' are not central to our discussion here). The first operator in (2.21) can emit $\bar{n}\cdot A_{n_0}$ gluons from the Wilson line in χ_{n_0} but requires a Lagrangian insertion to emit an $A^{\perp}_{n_0}$ gluon. Schematically the amplitude for a transverse gluon has contributions

$$A^I = \int d^4x\, \langle 0|T\,\mathcal{L}^{(0)}_{n_0}(x)\,\bar{\chi}_{n_0}\Gamma\chi_{\bar{n}}(0)|q_{n_0}g_{n_0}\bar{q}_{\bar{n}}\rangle + \langle 0|\bar{\chi}_{n_0}g\mathcal{B}^{\mu}_{n_0\perp}\Gamma'\chi_{\bar{n}}(0)|q_{n_0}g_{n_0}\bar{q}_{\bar{n}}\rangle\,. \tag{2.22}$$

In Fig. 2-2(II) each of the particles is collinear in a distinct direction, so no cone of size $\sim\lambda$ fits two of the momenta. In this case, the amplitude can only come from an operator with three distinct directions, such as $\bar{\chi}_{n_1}\mathcal{B}^{\mu}_{n_1'\perp}\Gamma'\chi_{\bar{n}}$:

$$A^{II} = \langle 0|\bar{\chi}_{n_1}g\mathcal{B}^{\mu}_{n_1'\perp}\Gamma''\chi_{\bar{n}}|q_{n_1}g_{n_1'}\bar{q}_{\bar{n}}\rangle\,. \tag{2.23}$$

2.2 Reparametrization invariance

When a set of fields have their largest momentum component in a light-like or time-like direction then the structure of operators built from these fields is constrained by reparametrization invariance. This invariance appears due to the ambiguity in the decomposition of momenta in terms of basis vectors and in terms of large and small components, in other words reparametrization constraints arise because the decomposition in Eq. (2.1) is not unique. We can shift n by a small amount and still have a suitable basis vector for the particle. We also have a large amount of freedom in the choice of $\bar{n}$. For each $\{n, \bar{n}\}$ pair the most general set of RPI transformations which preserves the relations $n^2 = 0$, $\bar{n}^2 = 0$, and $n\cdot\bar{n} = 2$ are

$$
\text{(I)}\begin{cases} n_\mu \to n_\mu + \Delta^\perp_\mu \\ \bar{n}_\mu \to \bar{n}_\mu \end{cases} \qquad \text{(II)}\begin{cases} n_\mu \to n_\mu \\ \bar{n}_\mu \to \bar{n}_\mu + \varepsilon^\perp_\mu \end{cases} \qquad \text{(III)}\begin{cases} n_\mu \to (1+\alpha)\, n_\mu \\ \bar{n}_\mu \to (1-\alpha)\, \bar{n}_\mu \end{cases} \tag{2.24}
$$

where the five infinitesimal parameters are $\{\Delta^\perp_\mu, \epsilon^\perp_\mu, \alpha\}$, and satisfy $\bar{n}\cdot\epsilon^\perp = n\cdot\epsilon^\perp = \bar{n}\cdot\Delta^\perp = n\cdot\Delta^\perp = 0$. The transformations (I), (II) and (III) in Eqs. (2.24) are called RPI-I, RPI-II and RPI-III. To ensure that n provides an equivalent physical description of the collinear direction for these particles requires the power counting $\{\Delta^\perp_\mu, \epsilon^\perp_\mu, \alpha\} \sim \{\lambda^1, \lambda^0, \lambda^0\}$ [77]. Thus n can only be shifted by a small amount, while parametrically large values of α and $\varepsilon^\perp_\mu$ are allowed. This is because the vector n has physical meaning, $\vec{n}$ is the direction where most of the momentum is allocated, that is the direction $\vec{p}$ is inside a cone centered on $\vec{n}$ with on opening angle $\sim \lambda$. The RPI-I transformations moves n inside this collinear cone. $\bar{n}$ does not carry any real physical meaning and it is only needed to decompose the momentum in (2.1). The collinear sectors $\{n_i\}$ in SCET are really equivalence classes of null vectors, $\{[n_i]\}$, where an equivalence class $[n_j]$ is defined as

$$
[n_j] = \{n \in [n_j] |\, n\cdot n_j \lesssim \lambda^2\}\,. \tag{2.25}
$$

The class $[n_j]$ consists of all light-like vectors connected to n_j^μ by a type-I RPI transformation $n_j^\mu \to n_j^\mu + \Delta^\mu_{n_j\perp}$.

The type-III boost simply ensures that $(\#Nn_i)-(\#N\bar{n}_i)-(\#Dn_i)+(\#D\bar{n}_i) = 0$ for each i, where $(\#Nn_i)$ counts the number of n_i factors in the numerator of an operator, $(\#D\bar{n}_i)$ counts the numbers of $\bar{n}_i$ factors in the denominator, etc. With three collinear directions an example of a type-III RPI invariant parameter is

$$
\frac{n_1\cdot\bar{n}_2\ \bar{n}_1\cdot\bar{n}_3}{\bar{n}_2\cdot\bar{n}_3}\,. \tag{2.26}
$$

The type-I and type-II transformations of collinear objects are more interesting and are summarized in Table 2.1, which we take from Ref. [77]. Since the factors induced

Type (I)			Type (II)		
n	$\to$	$n+\Delta^{\perp}$	n	$\to$	n
$\bar{n}$	$\to$	$\bar{n}$	$\bar{n}$	$\to$	$\bar{n}+\varepsilon^{\perp}$
$n\cdot D_n$	$\to$	$n\cdot D_n+\Delta^{\perp}\cdot D_n^{\perp}$	$n\cdot D_n$	$\to$	$n\cdot D_n$
$D^{\mu}_{n\perp}$	$\to$	$D^{\mu}_{n\perp}-\frac{\Delta^{\mu}_{\perp}}{2}\,\bar{n}\cdot D_n-\frac{\bar{n}^{\mu}}{2}\Delta^{\perp}\cdot D_{n\perp}$	$D^{\mu}_{n\perp}$	$\to$	$D^{\mu}_{n\perp}-\frac{\varepsilon^{\mu}_{\perp}}{2}\,n\cdot D_n-\frac{n^{\mu}}{2}\,\varepsilon^{\perp}\cdot D_n^{\perp}$
$\bar{n}\cdot D_n$	$\to$	$\bar{n}\cdot D_n$	$\bar{n}\cdot D_n$	$\to$	$\bar{n}\cdot D_n+\varepsilon^{\perp}\cdot D_n^{\perp}$
ξ_n	$\to$	$\left(1+\frac{1}{4}\,\Delta\!\!\!/^{\perp}\bar{n}\!\!\!/\right)\xi_n$	ξ_n	$\to$	$\left(1+\frac{1}{2}\,\varepsilon\!\!/^{\perp}\frac{1}{\bar{n}\cdot D_n}\,D\!\!\!\!/_n^{\perp}\right)\xi_n$
W	$\to$	W	W	$\to$	$\left[\left(1-\frac{1}{\bar{n}\cdot D_n}\,\epsilon^{\perp}\cdot D_n^{\perp}\right)W\right]$

Table 2.1: Summary of infinitesimal type I and II transformations from Ref. [77]. With multiple collinear directions these transformations exist for each $\{n_i, \bar{n}_i\}$ pair.

by these transformations occur at different orders in λ, demanding overall invariance of a physical process provides connections between the Wilson coefficients of operators at different orders in the expansion.

When we couple collinear and soft particles there is another ambiguity, associated with the decomposition of a collinear momentum into large and small pieces. If the total momentum P^{μ} of a collinear particle is decomposed into the sum of a large collinear p^{μ} and a small soft momentum k^{μ}:

$$P^{\mu} \;=\; p^{\mu}+k^{\mu}=\frac{n^{\mu}}{2}\bar{n}\cdot(p+k)+\frac{\bar{n}^{\mu}}{2}n\cdot k+(p_{\perp}+k_{\perp})^{\mu}, \tag{2.27}$$

then operators must be invariant under a transformation that takes $\bar{n}\cdot p\to\bar{n}\cdot p+\bar{n}\cdot\ell$, $p^{\mu}_{\perp}\to p^{\mu}_{\perp}+\ell^{\mu}_{\perp}$, $\bar{n}\cdot k\to\bar{n}\cdot k-\bar{n}\cdot\ell$, and $k^{\mu}_{\perp}\to k^{\mu}_{\perp}-\ell^{\mu}_{\perp}$. To construct invariant objects that have nice gauge transformation properties we use the combined covariant derivatives [15, 25],

$$iD^{\mu}_{n\perp}+W_n iD^{\mu}_{s\perp}W_n^{\dagger}\,, \qquad\qquad i\bar{n}\cdot D_n+W_n i\bar{n}\cdot D_s W_n^{\dagger}\,. \tag{2.28}$$

This can be implemented by taking

$$i\mathcal{D}^{\mu}_{n\perp}\to i\mathrm{D}^{\perp\mu}_{\text{full}}=i\mathcal{D}^{\mu}_{n\perp}+iD_{\mathrm{s}}^{\perp\mu}\,,\qquad \overline{\mathcal{P}}_n\to i\bar{n}\cdot\mathrm{D}_{\text{full}}=\overline{\mathcal{P}}_n+i\bar{n}\cdot D_{\mathrm{s}}\,, \tag{2.29}$$

and then expanding in λ. The results in Eq. (2.29) give powerful relations as they relate the coefficients of operators involving collinear fields to those involving soft fields. These relations are quite easy to derive order by order in λ. Note that reparametrization constraints associated with transformation of the soft Wilson line Y_n are automatically enforced by the other constraints.[2]

[2]For example, prior to the field redefinition only the combination $in\cdot \mathrm{D} = in\cdot\partial + gn\cdot A_s + gn\cdot A_n$ appears acting on collinear fields. A type-I transformation connects this to a $\mathcal{D}_n^\perp$, and Eq. (2.29) then connects this to the same $iD_s^\perp$ that one would find by direct transformation of Y_n.

Chapter 3

Reparametrization Invariant Collinear Operators

3.1 Introduction

To study a process using SCET, the standard procedure is to take the QCD current, $J_{\rm QCD}$, underlying that event and to expand it in terms of SCET operators using an operator expansion:

$$J_{\rm QCD} = \sum_i C_i O_i \,, \tag{3.1}$$

where C_i are the Wilson coefficients describing the physics at the hard scale, and O_i are the SCET operators that reproduce the infra-red (IR) behavior.[1] The process of calculating the Wilson coefficients is called matching. All the operators in SCET have a power counting in λ, and the OPE is organized as an expansion in λ. In order to fully reproduce $J_{\rm QCD}$, we have to match it to an infinite tower of SCET operators with higher and higher power counting, but at a given power of λ, the number of operators is finite, and we only match $J_{\rm QCD}$ to SCET operators up to a fixed order in λ. To construct the expansion (3.1), the standard procedure is to build

[1] We will see in the next paragraph that the product of Wilson coefficients and operators in Eq. (3.1) is actually a convolution.

a gauge invariant basis of operators with a definite power counting, using the gauge invariant fields defined in Section (2.1.1). We call leading order (LO) operators, $O_i^{\rm LO}$, the operators in (3.1) with the lowest power counting, next-to-leading-order (NLO) operators, $O_i^{\rm NLO}$, the operators with the next higher power counting and so forth. Thus we can write (3.1) as

$$J_{\rm QCD} = \sum_i C_i^{\rm LO} O_i^{\rm LO} + \sum_i C_i^{\rm NLO} O_i^{\rm NLO} + \dots\,. \tag{3.2}$$

SCET is invariant under reparametrization invariance, thus we have

$$\delta_{\rm RPI}\Big(\sum_i C_i O_i\Big) = 0\,, \tag{3.3}$$

where with $\delta_{\rm RPI}$ we indicate the set of transformations in table (2.1). We can solve the Eqs. (3.3) order by order in λ and find relations among Wilson coefficients, and because the transformations in (2.1) occur at different order, Eqs. (3.3) allow us to relate Wilson coefficients at different order. In other words, we use RPI transformations to reduce the basis of operators that we need for the matching. Because the reparametrization invariant transformations depend on the collinear direction n, if we have operators with different directions, we have a different set of transformations for each n. Thus when dealing with operators with multiple directions, solving Eqs. (3.3) becomes hard, if not prohibitive.

In this chapter we will construct RPI operators $\mathbf{Q}^i$, which are reparametrization invariant, that is $\delta_{\rm RPI}(\mathbf{Q}^i) = \mathbf{Q}^i$. The results of the chapter were presented in Ref. [80]. The operators $\mathbf{Q}^i$ are made of reparametrization invariant fermion fields Ψ_n, and gluon fields $\mathcal{G}^{\mu\nu}$, that we call superfields. The superfields are made gauge invariant using a reparametrization invariant Wilson line $\mathcal{W}_n$ that is the generalization of the usual W_n. These objects do not have a definite power counting order, in particular we will know the order in the λ-expansion where they start, but they will contain terms at all higher orders as well. We build a basis with these RPI and gauge invariant objects, which is made minimal using equations of motion and kinematic constraints as discussed below in Section 3.4. Each element of this basis is assigned a Wilson coefficient, and then the elements are expanded to find the final basis with elements of a definite power counting. In this way we immediately obtain relations between Wilson coefficients of operators at different orders. Once we expand and check for redundancy, the number of independent Wilson coefficients is equal to the number of independent RPI operators in the reduced basis. We will apply RPI operators to construct the minimal basis of operator for several processes.

In hard-scattering processes, DIS provides a familiar context where the construction of a minimal operator basis requires judicial use of the quark and gluon equations of motion, and an invariance under reparametrizations of a light-like direction [41, 42, 62, 63, 86], for a review see [64]. The invariance under reparametrizations becomes more valuable at higher orders in the expansion, being particularly constraining on the basis of twist-4 operators derived in Refs. [41, 42, 62, 63]. We derive RPI constraints for collinear operators in DIS and compare to these classic results as a test of our setup. For DIS the minimization of the basis of RPI operators is quite similar to the reduction of operators in Ref. [63]. On the other hand the basis of SCET operators are comprised entirely of analogs of "good" quark and gluon fields, namely a two-component quark field χ_n and just two components of the gluon field strength, $\mathcal{B}^{\mu}_{n\perp}$. These objects both incorporate Wilson lines, and for these operators it is easier to find a minimal basis. The RPI relations provide Lorentz invariance connections between the Wilson coefficients in this basis. These constraints carry a process independence, they depend on the type of operators being considered, but not on the precise process in which they will be used. It should be emphasized that when matrix elements are considered for a particular process, a further reduction in the number of independent hadronic functions becomes possible. For twist-4 quark operators in DIS this type of further reduction was discussed in detail in Ref. [42] and for inclusive B-decays in. [95], but this type of reduction is not our focus here.

Our construction is general enough that it applies not just to DIS like processes, but to operators with multiple collinear directions, which are useful for processes with multiple hadrons and jets. These operator bases provide a starting point for deriving appropriate factorization theorems for different processes. The invariant operator procedure becomes more and more efficient as the number of directions grows.

The outline of this chapter is as follows. In subsection 3.2 we study the convolution between Wilson coefficient and operator. We divide hard interactions into two categories, those with an external hard leptonic reference vector q^μ, and those where the hard interaction is between strongly interacting particles. Since most SCET applications focus on the former case, we address some of the additional notational complications that occur for the latter. A set of RPI invariant collinear objects is constructed in subsection 3.3, followed by a summary of identities that can be used to reduce the operator basis in subsection 3.4. The inclusion of mass effects is considered in subsection 3.5, and the expansion of the RPI objects is carried out in subsection 3.6. Applications for constructing operators are considered in subsection 3.7. In subsection 3.7.1 we verify that our approach provides a simple way to reproduce the known RPI result for the chiral-even scalar current given in Ref. [58]. In subsection 3.7.2 we construct a general basis of field structures involving up to four active quark or gluon operators, and with up to four distinct collinear directions. In subsection 3.7.3 we consider the special case of quark operators for DIS at twist-4 with one collinear direction, and compare with the literature. In subsection 3.7.4 we derive a basis of operators for pure gluon scattering in DIS up to twist-4. Finally we apply the formalism to jet production. In subsection 3.7.5 we demonstrate that very little information is gained about the operator basis describing $e^+e^- \to 2\,\text{jets}$. In subsection 3.7.6 we show that RPI turns out to be quite powerful for constraining the $e^+e^- \to 3\,\text{jet}$ operators. Finally we show that RPI is also useful for two jet production from gluon-fusion, $gg \to q\bar{q}$, and we construct a basis of operators for this process in subsection 3.7.7. Conclusions are given in subsection 3.8.

3.2 Convolutions

In the presence of collinear fields, a hard interaction can introduce convolutions in variables ω_i between the perturbatively calculable Wilson coefficient $C(Q^2, \omega_i)$ and the matrix element of the collinear operators. In this case the amplitude, cross-section, or decay rate has the form

$$A = \int [d\omega_1 \cdots d\omega_k]\, C(Q^2, \omega_i)\, \langle O(\omega_i) \rangle \,. \tag{3.4}$$

The convolutions occur because a component of the hard momentum and of one or more collinear momenta are $\mathcal{O}(\lambda^0)$. The exchange of momentum between the hard and collinear components yields a convolution in variables ω_i, where the number of such variables is constrained by gauge invariance and by momentum conservation in the matrix element. A gauge invariant momentum from the collinear fields can be picked out by a delta function acting on one of the collinear objects in Eq. (2.18), such as $[\delta(\omega - \bar{n}\cdot\mathcal{P}_n)\chi_n]$, and traditionally in SCET a subscript notation is used for these products,

$$\begin{aligned} \chi_{n,\omega} &\equiv \Big[\, \delta\big(\omega - \overline{\mathcal{P}}_n\big)\chi_n \Big] \,, & (i\mathcal{D}^\mu_{n\perp})_\omega &\equiv \Big[i\mathcal{D}^\mu_{n\perp}\, \delta\big(\omega - \overline{\mathcal{P}}^\dagger_n\big) \Big] \,, \\ (g\mathcal{B}^\mu_{n\perp})_\omega &\equiv \Big[g\mathcal{B}^\mu_{n\perp}\, \delta\big(\omega - \overline{\mathcal{P}}^\dagger_n\big) \Big] \,, & (gn\cdot\mathcal{B}_n)_\omega &\equiv \Big[gn\cdot\mathcal{B}_n\, \delta\big(\omega - \overline{\mathcal{P}}^\dagger_n\big) \Big] \,. \end{aligned} \tag{3.5}$$

We will refer to these as homogeneous objects since they have a definite order in λ, and call the operators build from these objects homogeneous operators. As an example we have the bilinear scalar operator,

$$O(\omega_1, \omega_2) = \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega_2} \,. \tag{3.6}$$

When we consider RPI it will be convenient to use different δ functions and convolution variables $\hat{\omega}$, that are type-III invariant. Essentially each $\overline{\mathcal{P}}_n = \bar{n}\cdot\mathcal{P}_n$ must be multiplied by a scalar transforming as n under RPI type-III. There are two cases to consider:

i) situations where there is a reference vector q^μ for the hard interaction, $|q^2| = Q^2 \gg \Lambda_{\rm QCD}$, which is external to the QCD dynamics,

ii) situations where the hard interactions are purely from strongly interacting particles.

Case i) applies to examples such as DIS where q^μ is the momentum transfer from the virtual photon, or $e^+e^- \to$ jets where q^μ is the four momentum of the e^+e^- pair. Here we can use $n\cdot q \sim \lambda^0$ to make the δ-function type-III invariant for n-collinear fields. Since $Q^2 \gg \Delta\Lambda_{\rm QCD} \gg \Lambda^2_{\rm QCD}$ we know that $n\cdot q \gg n\cdot p$, where p is the momentum of a collinear particle in the jet. Thus we use a variable $\hat\omega$ with mass dimension two, and will find δ-functions of the form[2]

$$\delta\big(\hat\omega - n\cdot q\,\overline{\mathcal{P}}_n\big)\,. \tag{3.7}$$

We also introduce a subscript notation with hatted variables,

$$\begin{aligned} \chi_{n,\hat\omega} &\equiv \Big[\,\delta\big(\hat\omega - n\cdot q\,\overline{\mathcal{P}}_n\big)\chi_n\Big]\,, & (i\mathcal{D}^\mu_{n\perp})_{\hat\omega} &\equiv \Big[i\mathcal{D}^\mu_{n\perp}\,\delta\big(\hat\omega - n\cdot q\,\overline{\mathcal{P}}^\dagger_n\big)\Big]\,, \\ (g\mathcal{B}^\mu_{n\perp})_{\hat\omega} &\equiv \Big[g\mathcal{B}^\mu_{n\perp}\,\delta\big(\hat\omega - n\cdot q\,\overline{\mathcal{P}}^\dagger_n\big)\Big]\,, & (gn\cdot\mathcal{B}_n)_{\hat\omega} &\equiv \Big[gn\cdot\mathcal{B}_n\,\delta\big(\hat\omega - n\cdot q\,\overline{\mathcal{P}}^\dagger_n\big)\Big]\,. \end{aligned} \tag{3.8}$$

Since $\delta(\hat\omega - n\cdot q\overline{\mathcal{P}}_n) \sim \lambda^0$, it is leading order in the power counting. Furthermore, we have $\delta(\hat\omega - n\cdot q\,\overline{\mathcal{P}}) = \delta(\hat\omega/n\cdot q - \overline{\mathcal{P}})/n\cdot q$, so identifying $\hat\omega = n\cdot q\,\omega$ there is no real change to the structure of Eq. (3.4). An operator built out of the components given in Eq. (3.8) has multiple labels, $O(\hat\omega_1, \hat\omega_2, \ldots)$, and the Wilson coefficient for the operator will be a function of the same parameters, $C(\hat\omega_1, \hat\omega_2, \ldots)$, yielding Eq. (3.4) with $\hat\omega$'s replacing ω's.

For processes in case ii) there is no analog of the external q^μ. Examples here include $pp \to$ jets, or any other hard process that does not involve external leptons or photons. The key difference with case i) is that here the hard interaction must involve two or more collinear directions, so we are guaranteed that there are scalar products $n_i\cdot n_j \sim \lambda^0$. For this type of reaction the type-III invariant δ-functions which are convoluted with Wilson coefficients always involve large momenta for two different

[2] For B-decays these type-III invariant δ-functions were used in Ref. [85], with $q^\mu \simeq m_b v^\mu$, $\delta(\hat\omega - n\cdot q\,\overline{\mathcal{P}}_n) = \delta(\hat\omega - m_b n\cdot v\,\overline{\mathcal{P}}_n) = 1/m_b\,\delta(\hat\omega' - n\cdot v\,\overline{\mathcal{P}}_n)$, where $\hat\omega = m_b\hat\omega'$. This form of invariant δ-function was also quite useful for analyzing the factorization theorem for $e^+e^- \to J/\psi X$ in Ref. [49].

collinear directions,

$$\Delta_{ij} \equiv \delta\Big(\hat{\omega}_{ij} - \frac{1}{2} n_i \cdot n_j \, \overline{\mathcal{P}}_{n_i} \overline{\mathcal{P}}_{n_j}\Big) \,. \tag{3.9}$$

Here $\overline{\mathcal{P}}_{n_i}$ acts on a gauge invariant block of n_i-collinear fields, and $\overline{\mathcal{P}}_{n_j}$ acts on a block of n_j-collinear fields. Since this δ-operator does not act on a single block of collinear fields we will not use a subscript notation like Eq. (3.8) for $\hat{\omega}_{ij}$. In this case the structure of the factorization theorem between operators and Wilson coefficients is a bit different than in Eq. (3.4). For example, consider an operator with collinear objects for four directions, where the convolution is

$$\int \Big[\prod_{ij} d\hat{\omega}_{ij}\Big] \, C(\hat{\omega}_{ij}) \Big[\prod_{km} \Delta_{km}\Big] \, \bar{\chi}_{n_1} (g\mathcal{B}^{\perp}_{n_3})(g\mathcal{B}^{\perp}_{n_4}) \chi_{n_2} \,. \tag{3.10}$$

Here the products are over the six unique pairs ij with $i \neq j$, and $\overline{\mathcal{P}}_{n_i}$ in the Δ_{km} acts on the n_i-collinear field(s). The convolutions in Eq. (3.10) can be manipulated into the form of Eq. (3.4) by inserting four factors of $1 = \int d\omega_i \, \delta(\omega_i - \overline{\mathcal{P}}_{n_i})$, writing $\hat{\delta}_{ij} = \delta(\hat{\omega}_{ij} - n_i \cdot n_j \, \omega_i \omega_j / 2)$ and carrying out the integrals over the six $\hat{\omega}_{ij}$'s to give

$$\int \big[d\omega_1 \cdots d\omega_4\big] \, C\big(n_i \cdot n_j \, \omega_i \omega_j\big) \bar{\chi}_{n_1,\omega_1} (g\mathcal{B}^{\perp}_{n_3,\omega_3})(g\mathcal{B}^{\perp}_{n_4,\omega_4}) \chi_{n_2,\omega_2} \,. \tag{3.11}$$

Here the RPI-III transformation of the measure cancels against that of the δ-functions in the operator, and RPI has constrained the Wilson coefficients to only depend on invariant products $n_1 \cdot n_2 \omega_1 \omega_2$, $n_1 \cdot n_3 \omega_1 \omega_3$, etc.

Due to the simplicity of the soft-collinear coupling at leading order in SCET a further factorization of the EFT matrix element can be made into collinear pieces J, and soft pieces S at each order in the power counting:

$$\langle O(\omega_i) \rangle = \int dk_j \; J(\omega_i, k_j) \; S(k_j) \,. \tag{3.12}$$

However it is the factorization in Eq. (3.4) that will be central to our discussion of reparametrization invariant operators.

3.3 Construction of RPI and Gauge Invariant objects

We now construct reparametrization invariant objects in SCET whose leading terms give the fields in Eq. (2.17). These are then generalized to objects that are simultaneously RPI and gauge invariant whose leading terms give the objects in Eqs. (2.18,3.8). For simplicity only collinear objects are considered in this section. Pulling out the large phases from the collinear quark field and gluon field strength, and decomposing the full theory field into independent collinear sectors we have at tree level,

$$\psi(x) = \sum_n e^{-ix\cdot\mathcal{P}_n}\psi_n(x)\,, \qquad G^{\mu\nu}(x) = \sum_n e^{-ix\cdot\mathcal{P}_n}G_n^{\mu\nu}(x)\,. \tag{3.13}$$

Full Lorentz invariance act on the fields $\psi(x)$ and $G^{\mu\nu}(x)$, but the RPI transformations that we are interested acts independently on each collinear sector labeled by n. Two sectors i, j are independent if $n_i\cdot n_j \gg \lambda^2$, and the sums in Eq. (3.13) are really over equivalence classes, $\{n\}$, where a class consists of vectors related by RPI. From the discussion in section 2.2 the n-reparametrization invariant collinear quark and field strength are easy to identify

$$\psi_n \;\equiv\; \left(1+\frac{1}{\bar n\cdot D_n}\not{D}_n^{\perp}\frac{\bar{\not{n}}}{2}\right)\xi_n\,, \qquad\qquad ig\,G^n_{\mu\nu}\equiv\left[iD^n_\mu, iD^n_\nu\right]\,. \tag{3.14}$$

Under the transformations in Table 2.1 for $\{n,\bar n\}$, the quark field ψ_n remains invariant [77], while the gluon tensor is invariant because the vector D_n^μ is invariant. To make the fields in Eq. (3.14) invariant under the additional reparametrization transformations that link collinear and soft derivatives we replace $in\cdot D_n \to in\cdot D_n + gn\cdot A_s$, $iD^\mu_{n\perp} \to iD^\mu_{n\perp} + W_n iD^\mu_{s\perp}W_n^\dagger$, and $i\bar n\cdot D_n \to i\bar n\cdot D_n + W_n i\bar n\cdot D_s W_n^\dagger$. After this replacement the decoupling field redefinitions in Eq. (2.14) can be made. In Eq. (3.14) $\not{n}\xi_n = 0$, and the term in ψ_n with a $\perp$-covariant derivative corresponds to the two components of the full fermion field that are small when $p_\perp/\bar n\cdot p \ll 1$. Since $\not{n}\psi_n \neq 0$, the ψ_n field does not provide a definite power counting for operators. For example,

$\bar{\psi}_n \not{\bar{n}} \psi_n \sim \lambda^0$ whereas $\bar{\psi}_n \not{n} \psi_n \sim \lambda^4$.

We also need reparametrization invariant δ-functions whose expansions reproduce Eqs. (3.7) and (3.9) at lowest order. For example, these are needed to construct an RPI operator which when expanded gives $\bar{\chi}_{n,\omega_1} \not{\bar{n}} \chi_{n,\omega_2}$ at lowest order. For situations where there is an external hard vector q^μ the invariant δ-function is

$$\hat{\Delta}_i \equiv \delta(\hat{\omega} - 2q \cdot i\partial_{n_i}) = \delta(\hat{\omega} - n_i{\cdot}q\, \overline{\mathcal{P}}_{n_i}) + \dots\,, \tag{3.15}$$

where as described in section 2.1.1, q^μ is a parameter specific to the kinematics of the process being studied. Notice that $\delta(\hat{\omega} - 2q \cdot i\partial_n)$ starts at $\mathcal{O}(\lambda^0)$, is RPI, and is gauge invariant when acting on singlet operators. Here

$$i\partial_n^\mu \equiv \frac{n^\mu}{2} \overline{\mathcal{P}}_n + \mathcal{P}_{n\perp}^\mu + \frac{\bar{n}^\mu}{2} in{\cdot}\partial_n\,, \tag{3.16}$$

and functions of $i\partial_n^\mu \sim (\lambda^2, 1, \lambda)$ can be expanded in powers of λ. Note that $\overline{\mathcal{P}}_n$ and $\mathcal{P}_{n\perp}^\mu$ are only non-zero when they act on n-collinear fields. It is useful to extend this property to the full $i\partial_n^\mu$, which we can do by distributing an $i\partial^\mu$ derivative across all fields that it acts on, writing for example $i\partial^\mu \bar{\psi}_{n_1} \psi_{n_2} = (i\partial_{n_1}^\mu \bar{\psi}_{n_1})\psi_{n_2} + \bar{\psi}_{n_1}(i\partial_{n_2}^\mu \psi_{n_2})$. In some hard processes there is more than one external hard vector, and a natural question arises as to whether q^μ provides a unique choice for this construction. For example, in DVCS, $\gamma^* p \to \gamma^{(*)} p'$ we have the momentum q^μ of the incoming γ^* and the momentum q'^μ of the outgoing $\gamma^{(*)}$. In appendix A we show that as long as $q_\perp - q'_\perp \sim \lambda$ or smaller, the choice q suffices, since for the purpose of constructing a basis of operators it is equivalent to the choice of any linear combination of q and q'. On the other hand, for situations where there is no external hard vector q^μ, the appropriate RPI δ-function is

$$\hat{\Delta}_{ij} \equiv \delta(\hat{\omega}_{ij} - 2i\partial_{n_i} \cdot i\partial_{n_j}) = \delta\Big(\hat{\omega}_{ij} - \frac{1}{2} n_i \cdot n_j\, \overline{\mathcal{P}}_{n_i} \overline{\mathcal{P}}_{n_j}\Big) + \dots\,. \tag{3.17}$$

This δ-function operator acts on two independent collinear directions. In general we must include in an operator a set of $\hat{\Delta}_i$ and $\hat{\Delta}_{ij}$ which are linearly independent. Once we expand, the first term in the series for $\hat{\Delta}_{ij}$ is not independent of the first term from $\hat{\Delta}_i$, so the δ-function shown on the RHS of Eq. (3.17) can always be eliminated, as we did in Eq. (3.11).

We will also make use of a reparametrization invariant Wilson line, $\mathcal{W}_n$, which has the same gauge transformation properties as W_n,

$$\mathcal{W}_n = W_n \, e^{-iR_n} \,. \tag{3.18}$$

Here the operator R_n starts with a term at $\mathcal{O}(\lambda)$ and is built of n-collinear gluon fields,

$$R_n = R_n\big[\overline{\mathcal{P}}_n \,, \mathcal{P}^\mu_{n\perp} \,, g\mathcal{B}^\mu_{n\perp} \,, t^\mu\big] \,, \tag{3.19}$$

where the vector t^μ is either q^μ or $i\partial^\mu_{n'}$ with $n \cdot n' \sim \lambda^0$. Furthermore, R_n is Hermitian, dimensionless, and collinear gauge invariant. We leave the explicit construction of R_n to section 3.6 below, and for the remainder of this section take these properties as given.

Under collinear gauge transformations, ψ_n and $\mathcal{W}_n$ transform the same way as ξ_n and W_n, and $G^{\mu\nu}_n$ transforms as a nonabelian field strength. Thus using $\mathcal{W}_n$ we can form analogs of the results in Eq. (3.8) that are simultaneously RPI and gauge invariant, namely the superfields

$$\Psi_n \equiv \mathcal{W}^\dagger_n \psi_n \,, \qquad \mathcal{G}^{\mu\nu}_n \equiv \mathcal{W}^\dagger_n G^{\mu\nu}_n \mathcal{W}_n \,. \tag{3.20}$$

For cases with an external q^μ we also introduce a subscript notation,

$$\Psi_{n,\hat{\omega}} \equiv \big[\delta(\hat{\omega} - 2q\cdot i\partial_n)\Psi_n\big] \,, \qquad \mathcal{G}^{\mu\nu}_{n,\hat{\omega}} \equiv \Big[\mathcal{G}^{\mu\nu}_n \, \delta(\hat{\omega} + 2q\cdot i \overleftarrow{\partial}_n)\Big] \,. \tag{3.21}$$

Operators built out of the superfields Ψ_n and $\mathcal{G}^{\mu\nu}_n$ are simultaneously RPI and gauge invariant. They are not homogeneous in the power counting, but the superfields reduce to the objects in Eq. (3.8) at lowest order in the λ expansion. For example, the superfield for the fermion

$$\Psi_n = e^{iR_n}\, W_n^\dagger \left(1+\frac{1}{i\bar{n}\cdot D_n} i\not{D}_n^\perp \frac{\not{\bar{n}}}{2}\right)\xi_n = e^{iR_n[i\partial_n,\mathcal{B}_n]}\left(1+\frac{1}{\overline{\mathcal{P}}_n} i\not{\mathcal{D}}_n^\perp \frac{\not{\bar{n}}}{2}\right)\chi_n$$
$$= \chi_n + \ldots . \qquad (3.22)$$

Similarly, $(g^\perp_{\nu\nu'}\bar{n}_\mu) ig(\mathcal{G}_n^{\mu\nu'}) = \overline{\mathcal{P}}_n\, g\mathcal{B}^\perp_{n\nu} + \ldots$. Thus to form a RPI version of the bilinear fermion operator $O(\omega_1,\omega_2)$ in Eq. (3.6) we simply take

$$\mathbf{Q}(\hat{\omega}_1,\hat{\omega}_2) = \bar{\Psi}_{n,\hat{\omega}_1}\, \not{q}\, \Psi_{n,\hat{\omega}_2}\,, \qquad (3.23)$$

and note that expanding in λ gives $\mathbf{Q}(\hat{\omega}_1,\hat{\omega}_2) = (n\cdot q)^{-1}\, O(\omega_1,\omega_2) + \ldots$.

We will also need the equations of motion for the RPI quark and gauge superfields in Eq. (3.20). The n-collinear Lagrangian for the quark field is [10]

$$\mathcal{L}_{qn} = \bar{\xi}_n \left(in\cdot D_n + i\not{D}_n^\perp \frac{1}{i\bar{n}\cdot D_n} i\not{D}_n^\perp\right)\frac{\not{\bar{n}}}{2}\xi_n\,, \qquad (3.24)$$

We can write Eq. (3.24) in terms of ψ_n as a simple Dirac Lagrangian

$$\mathcal{L}_{qn} = \bar{\psi}_n\, i\not{D}_n\, \psi_n\,, \qquad (3.25)$$

The equation of motion for ψ_n is a simple Dirac equation $\not{D}_n\,\psi_n = 0$. Using $\mathcal{W}_n\mathcal{W}_n^\dagger = 1$, we can write $\mathcal{W}_n^\dagger i\not{D}_n\,\mathcal{W}_n\mathcal{W}_n^\dagger\psi_n = 0$, and thus obtain the equation of motion for Ψ_n

$$\hat{\not{\mathcal{D}}}_n\,\Psi_n = 0\,. \qquad (3.26)$$

Here $\hat{\mathcal{D}}_n^\mu$ is the RPI and gauge invariant derivative

$$\hat{\mathcal{D}}_n^\mu \equiv \mathcal{W}_n^\dagger D_n^\mu\, \mathcal{W}_n = e^{iR_n}\, \mathcal{D}_n^\mu\, e^{-iR_n}\,. \qquad (3.27)$$

For the gluon field we have the equation of motion $[iD_\nu^n, G_n^{\mu\nu}] = igT^A\sum_f \bar{\psi}_n^f T^A\gamma^\mu\psi_n^f$, and for the superfield

$$i\hat{\mathcal{D}}^n_\nu \mathcal{G}^{\nu\mu}_n = [i\hat{\mathcal{D}}^n_\nu, \mathcal{G}^{\nu\mu}_n] = -ig\, T^A \bar{\Psi}^f_n T^A \gamma^\mu \Psi^f_n\,. \tag{3.28}$$

Note that $ig\mathcal{G}^{\mu\nu}_n = [i\hat{\mathcal{D}}^\mu_n, i\hat{\mathcal{D}}^\nu_n]$.

3.4 Reducing the Operator Basis

In general there are three steps that one can consider to reduce the perturbative and nonperturbative information in the EFT to its minimal form:

a) Find a minimal basis of homogeneous operators and of RPI operators that suffice at the desired order in λ. The homogeneous operators can be written entirely in terms of χ_n, $\mathcal{B}^\mu_{n\perp}$, and $\mathcal{P}^\mu_\perp$.

b) Compare the homogeneous and RPI basis to determine which perturbative Wilson coefficients are fixed by RPI.

c) Consider the decomposition of matrix elements of operators in the homogeneous basis, and derive further relations between the resulting non-perturbative functions.

Generically the relation between the operator basis looks like

$$\sum_{n_i}\sum_{\ell}\int [\prod_j d\hat{\omega}_j]\, \hat{C}_\ell(\hat{\omega}_j)\, [\mathbf{Q}_\ell(\hat{\omega}_j)] = \sum_{n_i}\sum_{\ell}\int [\prod_j d\omega_j]\, C_\ell(\omega_j)\, [O_\ell(\omega_j)] + \dots\,, \tag{3.29}$$

where $\mathbf{Q}_\ell(\hat{\omega}_j)$ are RPI operators and $O_\ell(\omega_j)$ are homogeneous operators, and the ellipse denotes higher order terms in the power expansion. In general our focus in this article is to carry out b) which is still largely process independent. For the most part we give no discussion of item c), which obviously must be considered process by process. In order to consider b) we must first determine a) which is the focus of this Section. We will discuss the equations of motion and other relations that allow a reduction in the basis of operators at each order in λ.

First we consider the gauge invariant objects with homogeneous power counting. We would like to demonstrate that all operators can be reduced to a form that only involves the basic building blocks χ_n, $g\mathcal{B}^\mu_{n\perp}$, and $\mathcal{P}^\mu_\perp$. All other homogeneous objects can be reduced to these. For example, one might think that the objects $g\mathcal{B}^{\mu\nu}_{\perp\perp} \equiv [1/\overline{\mathcal{P}}\, W^\dagger [iD^\mu_{n\perp}, iD^\nu_{n\perp}] W]$ and $g\mathcal{B}^\mu_{\perp 2} \equiv [1/\overline{\mathcal{P}}\, W^\dagger [iD^\mu_{n\perp}, in\cdot D_n] W]$ are independent. However they are related to the building blocks by

$$
\begin{aligned}
g\mathcal{B}^{\mu\nu}_{\perp\perp} &= \frac{1}{\overline{\mathcal{P}}}\mathcal{P}^\mu_\perp (g\mathcal{B}^\nu_\perp) - \frac{1}{\overline{\mathcal{P}}}\mathcal{P}^\nu_\perp (g\mathcal{B}^\mu_\perp) + \frac{1}{\overline{\mathcal{P}}}\left[g\mathcal{B}^\mu_\perp, g\mathcal{B}^\nu_\perp\right], \\
g\mathcal{B}^\mu_{\perp 2} &= \frac{1}{\overline{\mathcal{P}}}\mathcal{P}^\mu_\perp (gn\cdot\mathcal{B}) - \frac{1}{\overline{\mathcal{P}}} in\cdot\partial_n (g\mathcal{B}^\mu_\perp) + \frac{1}{\overline{\mathcal{P}}}\left[g\mathcal{B}^\mu_\perp, gn\cdot\mathcal{B}\right],
\end{aligned}
\tag{3.30}
$$

where we will see below that $n\cdot\mathcal{B}$ and $in\cdot\partial_n\mathcal{B}^\mu_\perp$ can also be reduced using the gluon equation of motion. For χ_n the equation of motion is

$$
in\cdot\partial_n\chi_n = -(gn\cdot\mathcal{B}_n)\chi_n - i\not{\mathcal{D}}^\perp_n \frac{1}{\overline{\mathcal{P}}_n} i\not{\mathcal{D}}^\perp_n\, \chi_n\,,
\tag{3.31}
$$

which allows us to eliminate $in\partial_n$ derivatives on χ_n. To obtain the equations of motion for the gluon objects we consider $-g^2 T^A \sum_f \bar\psi^f_n W_n T^A W^\dagger_n \gamma^\mu \psi^f_n = [i\mathcal{D}^n_\nu, [i\mathcal{D}^\mu_n, i\mathcal{D}^\nu_n]]$. Expanding in λ and multiplying on the right with $\delta(\omega - \overline{\mathcal{P}}^\dagger_n)$ gives three equations

$$
\omega\,(gn\cdot\mathcal{B})_\omega = 2\mathcal{P}^\perp_\nu (g\mathcal{B}^\nu_\perp)_\omega + \frac{2\omega'}{\omega}\left[(g\mathcal{B}^\nu_\perp)_{\omega-\omega'}, (g\mathcal{B}^\perp_\nu)_{\omega'}\right] - \frac{2}{\omega}\, g^2 T^A \sum_f \left[\bar\chi^f_n\, T^A \not{\bar n}\chi^f_n\right]_\omega,
$$

$$
\begin{aligned}
\omega\left[in\cdot\partial_n g\mathcal{B}^\mu_\perp\right]_\omega &= -\left[\mathcal{P}^\perp_\nu [g\mathcal{B}^\mu_\perp, g\mathcal{B}^\nu_\perp]\right]_\omega - \left[g\mathcal{B}^\perp_\nu, [\mathcal{P}^{[\mu}_\perp g\mathcal{B}^{\nu]}_\perp]\right]_\omega - \left[g\mathcal{B}^\perp_\nu, [g\mathcal{B}^\mu_\perp, g\mathcal{B}^\nu_\perp]\right]_\omega \\
&+ \frac{\omega}{2}\left[\mathcal{P}^\mu_\perp gn\cdot\mathcal{B}\right]_\omega - \left[\mathcal{P}^\perp_\nu \mathcal{P}^{[\mu}_\perp\, g\mathcal{B}^{\nu]}_\perp\right]_\omega + \frac{\omega}{2}\left[g\mathcal{B}^\mu_\perp, gn\cdot\mathcal{B}\right]_\omega - \frac{\omega'}{2}\left[(gn\cdot\mathcal{B})_{\omega-\omega'}, (g\mathcal{B}^\mu_\perp)_{\omega'}\right] \\
&- g^2 T^A \sum_f \left[\bar\chi^f_n T^A \gamma^\mu_\perp \frac{1}{\overline{\mathcal{P}}}(\not{\mathcal{P}}_\perp + g\not{\mathcal{B}}_\perp)\frac{\not{\bar n}}{2}\chi^f_n\right]_\omega g^2 T^A \sum_f \left[\bar\chi^f_n \frac{\not{\bar n}}{2}(\not{\mathcal{P}}^\dagger_\perp + g\not{\mathcal{B}}_\perp)\frac{1}{\overline{\mathcal{P}}^\dagger} T^A \gamma^\mu_\perp \chi^f_n\right]_\omega,
\end{aligned}
$$

$$
\begin{aligned}
g^2 T^A \sum_f & \left[\bar\chi^f_n (\not{\mathcal{P}}^\dagger_\perp + g\not{\mathcal{B}}_\perp)\frac{1}{\overline{\mathcal{P}}^\dagger} T^A \frac{1}{\overline{\mathcal{P}}}(\not{\mathcal{P}}_\perp + g\not{\mathcal{B}}_\perp)\not{\bar n}\chi^f_n\right]_\omega \\
&= \frac{\omega}{2}\left[in\cdot\partial_n gn\cdot\mathcal{B}\right]_\omega - \left[(\mathcal{P}_\perp)^2 gn\cdot\mathcal{B}\right]_\omega - \left[\mathcal{P}^\perp_\nu [g\mathcal{B}^\nu_\perp, gn\cdot\mathcal{B}]\right]_\omega - \left[g\mathcal{B}_{\perp\nu}, [\mathcal{P}^\nu_\perp gn\cdot\mathcal{B}]\right]_\omega \\
&+ \left[g\mathcal{B}_{\perp\nu}, [in\cdot\partial_n g\mathcal{B}^\nu_\perp]\right]_\omega - \left[g\mathcal{B}_{\perp\nu}, [g\mathcal{B}^\nu_\perp, gn\cdot\mathcal{B}]\right]_\omega + \left[in\cdot\partial_n \mathcal{P}^\perp_\nu g\mathcal{B}^\nu_\perp\right]_\omega
\end{aligned}
$$

$$+\frac{\omega'}{2}\big[(gn\cdot\mathcal{B})_{\omega-\omega'},(gn\cdot\mathcal{B})_{\omega'}\big]\,. \tag{3.32}$$

Here we sum over the color A, over the flavors f, and integrate over the repeated index ω'. In our analysis the first two equations will be used to eliminate $gn\cdot\mathcal{B}_n$ and $in\cdot\partial_n\, g\mathcal{B}_\perp^\mu$ respectively. The last relation only becomes relevant at higher orders than those we consider here. The above relations imply that when building a homogeneous basis of operators we do not need to consider the objects

$$in\cdot\partial_n\chi_n\,,\qquad n\cdot\mathcal{B}_n\,,\qquad in\cdot\partial_n\mathcal{B}^\mu_{n\perp}\,,\qquad \mathcal{B}^{\mu\nu}_{\perp\perp}\,,\qquad \mathcal{B}^\mu_{\perp 2}\,. \tag{3.33}$$

Next we derive relations that can be used to reduce RPI operators to a minimal form. Given the definition in Eq. (3.27), we can write $i\hat{\mathcal{D}}_n^\mu = i\partial_n^\mu + [i\hat{\mathcal{D}}_n^\mu]$, and it is straightforward using Eq. (3.50) below to prove that

$$[q\cdot i\partial_n i\hat{\mathcal{D}}_n^\nu] = q_\mu ig\mathcal{G}_n^{\mu\nu}\,, \tag{3.34}$$

and hence that $q_\mu[\hat{\mathcal{D}}_n^\mu] = 0$. (The results here and below apply equally well for $t = q$ and $t = i\partial_{n'}$ with $n\cdot n' \sim \lambda^0$. For simplicity we use the notation with $t = q$.) Eq. (3.34) can be used to rewrite the quark superfields equation of motion in Eq. (3.26) as

$$i\partial\!\!\!/_n\Psi_n = -\Big[\frac{1}{q\cdot i\partial_n}\, q_\mu\gamma_\nu ig\mathcal{G}_n^{\mu\nu}\Big]\Psi_n\,, \tag{3.35}$$

Since $q\cdot i\partial_n\,\delta(\hat\omega - 2q\cdot i\partial_n) = \frac{1}{2}\hat\omega\,\delta(\hat\omega - 2q\cdot i\partial_n)$ we also have the result

$$q\cdot i\partial_n\Psi_{n,\hat\omega} = \frac{\hat\omega}{2}\,\Psi_{n,\hat\omega}\,. \tag{3.36}$$

In a similar way, $q\cdot i\partial_n\mathcal{G}^{\mu\nu}_{n,\hat\omega} = (-\hat\omega/2)\mathcal{G}^{\mu\nu}_{n,\hat\omega}$. The collinear gluon equation of motion for $\mathcal{G}_n^{\mu\nu}$ in Eq. (3.28) can be rewritten as

$$[i\partial^n_\nu\mathcal{G}_n^{\nu\mu}] = -ig\,T^A\bar\Psi^f_n T^A\gamma^\mu\Psi^f_n + \Big[\Big[\frac{1}{q\cdot i\partial_n}q_\alpha\mathcal{G}^\alpha_{n\,\nu}\Big]\,,\,ig\mathcal{G}_n^{\mu\nu}\Big]\,. \tag{3.37}$$

The quark and gluon operators will have $\hat{\omega}$ subscripts, $\Psi_{n,\hat{\omega}}$ and $\mathcal{G}_{n,\hat{\omega}}$, so only the equations of motion in Eqs. (3.35,3.37) should be used to remove derivatives since the $i\partial_n$ derivatives commute with the presence of the δ-function denoted by the $\hat{\omega}$ subscript. The QCD Bianchi identity, $D_\mu G_{\nu\sigma} + D_\nu G_{\sigma\mu} + D_\sigma G_{\mu\nu} = 0$, also gives a relation for $\mathcal{G}_n^{\mu\nu}$, namely $\hat{\mathcal{D}}_n^\mu \mathcal{G}_n^{\nu\sigma} + \hat{\mathcal{D}}_n^\nu \mathcal{G}_n^{\sigma\mu} + \hat{\mathcal{D}}_n^\sigma \mathcal{G}_n^{\mu\nu} = 0$. Rearranging it gives the following relation

$$i\partial_n^\alpha \mathcal{G}_n^{\mu\nu} = q_\beta \bigg\{ \bigg[\Big[\frac{ig}{q\cdot i\partial_n} \mathcal{G}_n^{\alpha\beta} \Big], \mathcal{G}_n^{\mu\nu} \bigg] - \bigg[\Big[\frac{ig}{q\cdot i\partial_n} \mathcal{G}_n^{\beta\mu} \Big], \mathcal{G}_n^{\nu\alpha} \bigg] + \bigg[\Big[\frac{ig}{q\cdot i\partial_n} \mathcal{G}_n^{\beta\nu} \Big], \mathcal{G}_n^{\mu\alpha} \bigg] \bigg\} - i\partial_n^{[\mu} \mathcal{G}_n^{\nu]\alpha} \,, \tag{3.38}$$

which implies that $i\partial_n^\alpha \mathcal{G}_n^{\mu\nu}$, $i\partial_n^\mu \mathcal{G}_n^{\nu\alpha}$, and $i\partial_n^\nu \mathcal{G}_n^{\alpha\mu}$ are not all independent. Closing Eq. (3.38) with γ^μ allows us to remove $i\partial\!\!\!/_n \mathcal{G}_n^{\mu\nu}$, which is how we will choose to use this identity in quark operators. An analog of the Bianchi identity does not occur for the building block $g\mathcal{B}_n^\mu$ in homogeneous operators; it easy to verify that when expanded in λ, Eq. (3.38) is trivially satisfied. Eqs. (3.35–3.38) are the RPI equivalent of the results in Eqs. (3.31,3.32), and can be used to reduce the RPI operator basis.

The above results imply that when building an RPI operator basis we do not need to consider the objects

$$i\partial\!\!\!/_n \Psi_n \,, \qquad q\cdot i\partial_n \Psi_{n,\hat{\omega}} \,, \qquad [i\partial_\nu^n \mathcal{G}_n^{\nu\mu}] \,, \qquad i\partial\!\!\!/_n \mathcal{G}_n^{\mu\nu} \,, \qquad [q\cdot i\partial_n\, \mathcal{G}_{n,\hat{\omega}}^{\nu\mu}] \,. \tag{3.39}$$

This list is not exhaustive. By manipulating operators in specific situations further structures can be eliminated using a combination of the above identities. For example, for in Sections 3.7.3 and 3.7.4 below we will see that $q_\mu \mathcal{G}_n^{\mu\nu} i\partial_\nu^n$, with the $i\partial_\nu^n$ acts on a n-collinear quark or gluon field, can be eliminated.

In principle one can just count the number of RPI operators and compare to the number of operators in a homogeneous operator basis with definite power counting to determine whether there are any RPI constraints on the Wilson coefficients. The key issue here is that of linear independence, even if one has the the same number of operators in the RPI and homogeneous basis, it could be that two RPI operators constrain the same linear combination of operators in the homogeneous basis.

3.5 Extension to Massive Collinear Fields

Massive collinear quarks in SCET were first studied in Refs. [74, 89]. After the field redefinition in Eq. (2.14) they have the LO Lagrangian

$$\mathcal{L}_{qn} = \bar{\xi}_n \left[in \cdot D_n + (i\not{D}_n^{\perp} - m)\frac{1}{i\bar{n} \cdot D_n}(i\not{D}_n^{\perp} + m)\right] \frac{\not{\bar{n}}}{2}\xi_n \,. \tag{3.40}$$

The appropriate RPI transformations with massive quarks were determined in Ref. [34]. The only change is in the type-II transformation of the fermion field, where one has to add a mass dependent term:

$$\xi_n \xrightarrow{\text{II}} \left[1 + \frac{\not{\epsilon}^{\perp}}{2}\frac{1}{\bar{n} \cdot iD_n}(i\not{D}_{n\perp} - m)\right] \xi_n. \tag{3.41}$$

Under this transformation the Lagrangian in Eq. (2.14) falls into two invariant parts, one fixed by the leading order kinetic term and one whose coefficient encodes the choice of mass scheme. Note that the RPI transformation itself is not modified by the presence of a mass term, the transformation of $\bar{n}$ is still exactly as in Eq. (2.24).

We can now build an analog of the RPI superfield for a massive collinear quark. The reparametrization invariant quark field is

$$\psi_n = \left(1 + \frac{1}{\bar{n} \cdot iD_n}(i\not{D}_n^{\perp} + m)\frac{\not{\bar{n}}}{2}\right) \xi_n. \tag{3.42}$$

This leads to the modified RPI superfield for a massive collinear quark

$$\Psi_n = e^{iR_n}\left[1 + \frac{1}{\overline{\mathcal{P}}_n}(i\not{\mathcal{D}}_n^{\perp} + m)\frac{\not{\bar{n}}}{2}\right] \chi_n \,. \tag{3.43}$$

This result is included for completeness. Our focus in the remainder of the chapter will be on massless collinear quark fields.

3.6 Determination of R_n and Expansion of Ψ_n and $\mathcal{G}_n^{\mu\nu}$

In this Section we derive an expression for R_n appearing in the RPI Wilson line, and then expand the invariant objects Ψ_n, $\mathcal{G}_n^{\mu\nu}$, $\delta(\hat{\omega} - 2q \cdot i\partial_n)$, and $\delta(\hat{\omega}_{12} - 2i\partial_{n_1} \cdot i\partial_{n_2})$. We can define the collinear Wilson line W_n by the equation:

$$[(\bar{n} \cdot D_n)W_n] = 0\,. \tag{3.44}$$

We define the RPI $\mathcal{W}_n$ generalizing (3.44) to a covariant derivative D_n along a (non light-like) direction t as:

$$[(t \cdot D_n)\mathcal{W}_n] = 0\,, \tag{3.45}$$

where t is such that $n \cdot t \sim \lambda^0$. This implies the momentum space representation:

$$\mathcal{W}_n = \left[\sum_{\text{perms}} \exp\left(\frac{-g}{(t \cdot i\partial_n)} t \cdot A_n\right)\right]. \tag{3.46}$$

We would like to find R_n such that $\mathcal{W}_n = W_n e^{-iR_n}$. Thus e^{-iR_n} is the operator that rotates W_n from the light-like direction n to the direction t. $\mathcal{W}_n$ is reparametrization invariant to the choice of the basis vector n, which labels the n-collinear fields A_n^μ, since such reparametrizations cannot change the fact that $n \cdot t \sim \lambda^0$. Recall that the subscript n on $\mathcal{W}_n$ labels the equivalence class $\{n\}$ of vectors that are related by type-I and type-III RPI transformations. For any t such that $n \cdot t \sim \lambda^0$ we have

$$\frac{1}{t \cdot i\partial_n} t \cdot A_n = \frac{1}{\overline{\mathcal{P}}_n} \bar{n} \cdot A_n + \dots\,, \tag{3.47}$$

and thus

$$\mathcal{W}_n = W_n + \dots\,, \tag{3.48}$$

where the ellipses represent power suppressed terms. In Eq. (3.47) the $n \cdot t$'s in the numerator and denominator cancel out in the leading term, leaving a t independent result.

For situations where we have an external hard vector q^μ, we can simply take $t^\mu = q^\mu$ and use the corresponding $\mathcal{W}_n$ as the RPI invariant Wilson line.

For situations where there is no external q^μ, the choice for t^μ in $\mathcal{W}_n$ is less obvious since the only available RPI vectors are operators themselves, $i\partial^\mu_{n'}$, where n' is a distinct collinear direction from n. In this situation, any choice $t^\mu = i\partial^\mu_{n'}$ satisfying $n \cdot t = n \cdot n' \,\overline{\mathcal{P}}_{n'} + \ldots \sim \lambda^0 + \ldots$ is equally good, and the existence of the hard interaction guarantees that such an n' exists. In this case $\mathcal{W}_n$ still yields W_n at lowest order, and hence only behaves like an operator in the n' direction through terms in the power corrections, namely the ellipsis in Eq. (3.48). In these ellipse terms the $i\partial_{n'}$'s appear linearly order by order. Since the derivative $i\partial_{n'}$ does not act on n-collinear fields it behaves just like an external vector q as far as manipulations related to the n-collinear fields are concerned.

In the remainder of this Section we adopt the notation $t = q$, even though the algebra applies equally well to both cases mentioned above, with the substitution $q \to t = i\partial_{n'}$ in appropriate places. The only complication for the case $t = i\partial_{n'}$ is that the dot product $n \cdot i\partial_{n'}$ must be expanded using

$$\begin{aligned} 2i\partial_n \!\cdot\! i\partial_{n'} = {} & \frac{n\!\cdot\! n'}{2}\,\overline{\mathcal{P}}_n \overline{\mathcal{P}}_{n'} + n'\!\cdot\! i\partial_{n\perp}\overline{\mathcal{P}}_{n'} + n\!\cdot\! i\partial_{n'\perp}\overline{\mathcal{P}}_n + 2i\partial_{n\perp}\!\cdot\! i\partial_{n'\perp} + \frac{\bar{n}\cdot n'}{2}\, n\!\cdot\! i\partial_n\,\overline{\mathcal{P}}_{n'} \\ & + \frac{\bar{n}'\cdot n}{2}\, n'\!\cdot\! i\partial_{n'}\overline{\mathcal{P}}_n + \bar{n}\!\cdot\! i\partial^\perp_{n'}\, n\!\cdot\! i\partial_n + \bar{n}'\!\cdot\! i\partial^\perp_n\, n'\!\cdot\! i\partial_{n'} + \frac{\bar{n}\cdot\bar{n}'}{2}\, n\!\cdot\! i\partial_n\, n'\!\cdot\! i\partial_{n'}\,, \end{aligned} \tag{3.49}$$

where the first term is $\sim \lambda^0$, the next two $\sim \lambda$, the following three are $\sim \lambda^2$, then the next two are $\sim \lambda^3$, and the last one is $\sim \lambda^4$.

Adopting $t = q$, Eq. (3.45) can be used to prove that

$$(q \cdot iD_n) = \mathcal{W}_n \,(q \cdot i\partial_n)\, \mathcal{W}_n^\dagger\,. \tag{3.50}$$

To calculate iR_n we exploit Eq. (3.50) and calculate iR_n order by order in λ. Substituting Eq. (3.18) into Eq. (3.50) we find

$$(q \cdot i\mathcal{D}_n) = e^{-iR_n} \, (q \cdot i\partial_n) \, e^{iR_n} \, . \tag{3.51}$$

Because of the Hermicity of $i\mathcal{D}_n^\mu$ and $i\partial_n^\mu$, R_n is Hermitian. Applying the Hadamard formula to Eq. (3.51) we obtain

$$(q \cdot i\mathcal{D}_n) = (q \cdot i\partial_n) + \sum_{j=1}^{\infty} \frac{1}{j!} \{\{(q \cdot i\partial_n), (iR_n)^j \}\} \, , \tag{3.52}$$

where $\{\{A, B\}\} = [A, B]$ and

$$\{\{A, B^j\}\} = \{\{[A, B], B^{j-1}\}\} = [[\cdots [A, \overbrace{B,]B,] \ldots ,]B]^{j} \, . \tag{3.53}$$

Expanding R_n in terms with $R_n^{(k)} \sim \lambda^k$ we can expand all the objects in Eq. (3.52) in λ and solve the resulting equations order by order for $R_n^{(k)}$. Thus we write

$$\begin{aligned} iR_n &= \sum_{k=1}^{\infty} iR_n^{(k)} \, , \\ (q \cdot i\mathcal{D}_n) &= \frac{n \cdot q}{2} \overline{\mathcal{P}}_n + (q_\perp \cdot \mathcal{P}_{n\perp}) + (q_\perp \cdot g\mathcal{B}_{n\perp}) + \frac{\bar{n} \cdot q}{2} (n \cdot i\partial_n) + \frac{\bar{n} \cdot q}{2} (gn \!\cdot\! \mathcal{B}_n) \, , \\ (q \cdot i\partial_n) &= \frac{n \cdot q}{2} \overline{\mathcal{P}}_n + (q_\perp \cdot \mathcal{P}_{n\perp}) + \frac{\bar{n} \cdot q}{2} (n \cdot i\partial_n) \, . \end{aligned} \tag{3.54}$$

$(q \cdot i\partial_n)$ is a derivative operator, so when it acts in a commutator with $(g\mathcal{B}_n^\mu)$ we have

$$[(q \cdot i\partial_n), (g\mathcal{B}_n^\mu)] = \big[q \cdot i\partial_n \, (g\mathcal{B}_n^\mu) \big] \, , \tag{3.55}$$

where the last set of square brackets means that the derivative acts only inside. Substituting Eq. (3.54) into (3.52) we can solve for $iR_n^{(k)}$. The first two terms are

$$iR_n^{(1)} = \Big[\frac{2}{n \!\cdot\! q \, \overline{\mathcal{P}}_n} \, q_\perp \!\cdot\! (g\mathcal{B}_n^\perp) \Big] \, , \tag{3.56}$$

$$iR_n^{(2)} = \Big[\frac{1}{n\!\cdot\! q\,\overline{\mathcal{P}}_n}\,(\bar{n}\!\cdot\! q)\,(gn\!\cdot\!\mathcal{B}_n)\Big] - \Big[\frac{4q_\perp\!\cdot\!\mathcal{P}_{n\perp}}{(n\!\cdot\! q\,\overline{\mathcal{P}}_n)^2}\,q_\perp\!\cdot\!(g\mathcal{B}_n^\perp)\Big]$$
$$+\Big[\frac{2}{n\!\cdot\! q\,\overline{\mathcal{P}}_n}\Big[\Big[\frac{1}{n\!\cdot\! q\,\overline{\mathcal{P}}_n}\,q_\perp\!\cdot\!(g\mathcal{B}_n^\perp)\Big],\,q_\perp\!\cdot\!(g\mathcal{B}_n^\perp)\Big]\Big].$$

The $n\cdot\mathcal{B}_n$ term should be further reduced with the equation of motion in Eq. (3.32) to terms involving χ_n and $\mathcal{B}^\mu_{n\perp}$. In terms of the $iR_n^{(k)}$ we can determine the λ expansion of the invariant Wilson line

$$\mathcal{W}_n = \sum_{k=0}^{\infty} \mathcal{W}_n^{(k)}\,. \tag{3.57}$$

Using the definition in Eq. (3.18) the first few terms are

$$\mathcal{W}_n^{(0)} = W_n\,, \qquad \mathcal{W}_n^{(1)} = -W_n(iR_n^{(1)})\,, \qquad \mathcal{W}_n^{(2)} = \Big[\frac{1}{2}(iR_n^{(1)})^2 - (iR_n^{(2)})\Big]\,. \tag{3.58}$$

The expansion of the invariant Wilson line is therefore

$$\mathcal{W}_n = W_n - W_n(iR_n^{(1)}) + W_n\Big[\frac{1}{2}(iR_n^{(1)})^2 - (iR_n^{(2)})\Big] + \ldots\,. \tag{3.59}$$

Using these $R_n^{(k)}$'s and Table 2.1 it is simple to check explicitly that $\mathcal{W}_n$ is RPI up to order $O(\lambda^3)$. Note that we did not assign a suppression for $q_\perp$ anywhere above (ie, we took $q_\perp \sim \lambda^0$). Taking $q_\perp \sim \lambda$ causes further suppression of some of the terms in Eq. (3.56). For cases where $q_\perp = 0$ the expansion of $\mathcal{W}_n$ starts at $O(\lambda^2)$.

We will also need the λ expansion of the invariant δ-functions, $\delta(\hat{\omega} - 2q\cdot i\partial_n)$ and $\delta(\hat{\omega}_{12} - 2i\partial_{n_1}\cdot i\partial_{n_2})$. For the former we have

$$\begin{aligned}\delta(\hat{\omega} - 2q\cdot i\partial_n) &= \delta(\hat{\omega} - n\!\cdot\! q\,\overline{\mathcal{P}}_n - 2q_\perp\!\cdot\!\mathcal{P}_{n\perp} - \bar{n}\!\cdot\! q\, in\!\cdot\!\partial_n) \\ &= \frac{1}{n\!\cdot\! q}\Big[\Big(1 + \sum_{k=1}^{\infty} p_n^{(k)}\Big)\delta(\omega - \overline{\mathcal{P}}_n)\Big]\,,\end{aligned} \tag{3.60}$$

where the first two terms are

$$p_n^{(1)} = -\frac{2q_\perp\!\cdot\!\mathcal{P}_{n\perp}}{n\!\cdot\! q}\frac{d}{d\omega}\,, \qquad p_n^{(2)} = 2\Big(\frac{q_\perp\!\cdot\!\mathcal{P}_{n\perp}}{n\!\cdot\! q}\Big)^2\frac{d^2}{d\omega^2} - \frac{\bar{n}\!\cdot\! q}{n\!\cdot\! q}(in\!\cdot\!\partial_n)\frac{d}{d\omega}\,. \tag{3.61}$$

When combining the operator with the Wilson coefficient $C(\omega_i)$ we can integrate by parts to move these derivatives onto the $C(\omega_i)$ and leave a simple δ-function in the operator. For the δ-function with two collinear directions we have

$$
\begin{aligned}
&\delta(\hat\omega_{12} - 2i\partial_{n_1}\!\cdot i\partial_{n_2}) \qquad (3.62)\\
&= \delta\Big(\hat\omega_{12} - \frac{n_1\!\cdot n_2}{2}\overline{\mathcal{P}}_{n_1}\overline{\mathcal{P}}_{n_2} - \overline{\mathcal{P}}_{n_1} n_1\!\cdot i\partial_{n_2\perp} - \overline{\mathcal{P}}_{n_2} n_2\!\cdot i\partial_{n_1\perp} - 2i\partial_{n_1\perp}\!\cdot i\partial_{n_2\perp} - \frac{\bar n_1\!\cdot n_2}{2}\, n_1\!\cdot i\partial_{n_1}\overline{\mathcal{P}}_{n_2}\\
&\quad - \frac{\bar n_2\!\cdot n_1}{2}\, n_2\!\cdot i\partial_{n_2}\overline{\mathcal{P}}_{n_1} - n_1\!\cdot i\partial_{n_1}\bar n_1\!\cdot i\partial_{n_2\perp} - n_2\!\cdot i\partial_{n_2}\bar n_2\!\cdot i\partial_{n_1\perp} - \frac{\bar n_1\!\cdot \bar n_2}{2} n_1\!\cdot i\partial_{n_1} n_2\!\cdot i\partial_{n_2}\Big)\\
&= \Big[\Big(1 + \sum_{k=1}^{\infty} p^{(k)}_{n_1 n_2}\Big)\delta\Big(\hat\omega_{12} - \frac{n_1\!\cdot n_2}{2}\overline{\mathcal{P}}_{n_1}\overline{\mathcal{P}}_{n_2}\Big)\Big]\,,
\end{aligned}
$$

where the first two terms are

$$
\begin{aligned}
p^{(1)}_{n_1 n_2} &= -\Big[\overline{\mathcal{P}}_{n_1} n_1\!\cdot i\partial_{n_2\perp} + \overline{\mathcal{P}}_{n_2} n_2\!\cdot i\partial_{n_1\perp}\Big]\frac{d}{d\omega}\,,\\
p^{(2)}_{n_1 n_2} &= \Big[\overline{\mathcal{P}}_{n_1} n_1\!\cdot i\partial_{n_2\perp} + \overline{\mathcal{P}}_{n_2} n_2\!\cdot i\partial_{n_1\perp}\Big]^2\frac{d^2}{d\omega^2}\\
&\quad - \Big[2i\partial_{n_1\perp}\!\cdot i\partial_{n_2\perp} + \frac{\bar n_1\!\cdot n_2}{2}\, n_1\!\cdot i\partial_{n_1}\overline{\mathcal{P}}_{n_2} + \frac{\bar n_2\!\cdot n_1}{2}\, n_2\!\cdot i\partial_{n_2}\overline{\mathcal{P}}_{n_1}\Big]\frac{d}{d\omega}\,. \qquad (3.63)
\end{aligned}
$$

All terms with $n\!\cdot i\partial_n$ in Eqs. (3.61) and (3.63) will be further reduced by the equations of motion in Eqs. (3.31) and (3.32) when they appear in operators.

Finally we expand the superfields in Eq. (3.20) in λ, writing

$$
\Psi_{n,\hat\omega} = \sum_{k=1}^{\infty}\Psi^{(k)}_{n,\hat\omega}\,, \qquad \mathcal{G}^{\mu\nu}_{n,\hat\omega} = \sum_{k=1}^{\infty}\mathcal{G}^{(k)\mu\nu}_{n,\hat\omega}\,, \qquad (3.64)
$$

where $\Psi^{(k)}_{n,\hat\omega} \sim \lambda^k$ and $\mathcal{G}^{(k)\mu\nu}_{n,\hat\omega} \sim \lambda^k$. The expansion of the quark superfield is straightforward, the first few orders are

$$
\begin{aligned}
\Psi^{(1)}_{n,\hat\omega} &= \frac{1}{n\!\cdot q}\,\chi_{n,\omega}\,, \qquad (3.65)\\
\Psi^{(2)}_{n,\hat\omega} &= \frac{1}{n\!\cdot q}\Big(\frac{1}{\omega} i\mathcal{D\!\!\!/}^{\perp}_{n,\omega_a-\omega}\frac{\bar{n\!\!\!/}}{2}\chi_{n,\omega_a} + iR^{(1)}_{n,\omega_a-\omega}\chi_{n,\omega_a} + \big[p^{(1)}_n\chi_{n,\omega}\big]\Big)\,,\\
\Psi^{(3)}_{n,\hat\omega} &= \frac{1}{n\!\cdot q}\Big(iR^{(2)}_{n,\omega_a-\omega}\chi_{n,\omega_a} + \big[p^{(2)}_n\chi_{n,\omega}\big] + \frac{1}{\omega_a+\omega_b}\, iR^{(1)}_{n,\omega_a-\omega_b-\omega}\, i\mathcal{D\!\!\!/}^{\perp}_{n,\omega_b}\frac{\bar{n\!\!\!/}}{2}\chi_{n,\omega_a}
\end{aligned}
$$

$$+\left[p_n^{(1)}\frac{1}{\omega}\,i\mathcal{P}\!\!\!/^{\perp}_{n,\omega_a-\omega}\frac{\bar{n}\!\!\!/}{2}\chi_{n,\omega_a}\right]+\left[p_n^{(1)}iR^{(1)}_{n,\omega_a-\omega}\chi_{n,\omega_a}\right]+\frac{1}{2}iR^{(1)}_{n,\omega_a-\omega_b-\omega}iR^{(1)}_{n,\omega_b}\chi_{n,\omega_a}\Big).$$

Here there is an implicit integration over the repeated indices ω_a and ω_b. For the gluon superfield first it is useful to expand $W^\dagger G_{\mu\nu}W$:

$$\begin{aligned}
W^\dagger igG_{\mu\nu}W &= \frac{n_\mu}{2}\left[i\bar{n}\cdot\mathcal{D},i\mathcal{D}_{\perp\nu}\right]-\frac{n_\nu}{2}\left[i\bar{n}\cdot\mathcal{D},i\mathcal{D}_{\perp\mu}\right]+\left[i\mathcal{D}_{\perp\mu},i\mathcal{D}_{\perp\nu}\right]+\frac{\bar{n}_\mu}{2}\frac{n_\nu}{2}\left[in\cdot\mathcal{D},i\bar{n}\cdot\mathcal{D}\right] \\
&\quad+\frac{n_\mu}{2}\frac{\bar{n}_\nu}{2}\left[i\bar{n}\cdot\mathcal{D},in\cdot\mathcal{D}\right]+\frac{\bar{n}_\mu}{2}\left[in\cdot\mathcal{D},i\mathcal{D}_{\perp\nu}\right]-\frac{\bar{n}_\nu}{2}\left[in\cdot\mathcal{D},i\mathcal{D}_{\perp\mu}\right] \\
&= \frac{n_\mu}{2}\left[\bar{\mathcal{P}}g\mathcal{B}_{\perp\nu}\right]-\frac{n_\nu}{2}\left[\bar{\mathcal{P}}g\mathcal{B}^{\perp}_{\mu}\right]+\left[\bar{\mathcal{P}}g\mathcal{B}^{\perp\perp}_{\mu\nu}\right]-\frac{\bar{n}_\mu n_\nu}{4}\left[\bar{\mathcal{P}}gn\cdot\mathcal{B}\right]+\frac{n_\mu\bar{n}_\nu}{4}\left[\bar{\mathcal{P}}gn\cdot\mathcal{B}\right] \\
&\quad-\frac{\bar{n}_\mu}{2}\left[\overline{\mathcal{P}}g\mathcal{B}^{\perp 2}_{\nu}\right]+\frac{\bar{n}_\nu}{2}\left[\overline{\mathcal{P}}g\mathcal{B}^{\perp 2}_{\mu}\right]\,, \qquad (3.66)
\end{aligned}$$

where $g\mathcal{B}^{\perp\perp}_{\mu\nu}$ and $g\mathcal{B}^{\perp 2}_{\mu}$ are given by the combinations of fields in Eq. (3.30). Using this result to determine the first few terms $\mathcal{G}^{(k)\mu\nu}_{n,\hat{\omega}}$ from expanding Eq. (3.20), we find

$$\begin{aligned}
ig\mathcal{G}^{(1)\mu\nu}_{n,\hat{\omega}} &= \frac{\omega}{2(n\cdot q)}\left[n^\nu(g\mathcal{B}^\mu_{n\perp})_\omega-n^\mu(g\mathcal{B}^\nu_{n\perp})_\omega\right]\,, \qquad (3.67)\\
ig\mathcal{G}^{(2)\mu\nu}_{n,\hat{\omega}} &= \frac{1}{n\cdot q}\Big\{\left[\mathcal{P}^\mu_\perp(g\mathcal{B}^\nu_{n\perp})_\omega\right]-\left[\mathcal{P}^\nu_\perp(g\mathcal{B}^\mu_{n\perp})_\omega\right]+\left[(g\mathcal{B}^\mu_{n\perp}),(g\mathcal{B}^\nu_{n\perp})\right]_\omega \\
&\quad+\frac{\omega}{4}(\bar{n}^\mu n^\nu-n^\mu\bar{n}^\nu)(gn\cdot\mathcal{B}_n)_\omega+\frac{\omega}{2}\left[iR^{(1)}_{n,\omega-\omega_a},n^\nu(g\mathcal{B}^\mu_{n\perp})_{\omega_a}-n^\mu(g\mathcal{B}^\nu_{n\perp})_{\omega_a}\right] \\
&\quad-\frac{1}{2}\left[n^\nu p^{(1)}_n\omega(g\mathcal{B}^\mu_{n\perp})_\omega-n^\mu p^{(1)}_n\omega(g\mathcal{B}^\nu_{n\perp})_\omega\right]\Big\}\,, \\
ig\mathcal{G}^{(3)\mu\nu}_{n,\hat{\omega}} &= \frac{-1}{2(n\cdot q)}\Big\{\left[(\bar{n}^\mu\mathcal{P}^\nu_\perp-\bar{n}^\nu\mathcal{P}^\mu_\perp)(gn\cdot\mathcal{B}_n)_\omega\right]-\left[in\cdot\partial_n\left(\bar{n}^\mu g\mathcal{B}^\nu_{\perp,\omega}-\bar{n}^\nu g\mathcal{B}^\mu_{\perp,\omega}\right)\right] \\
&\quad+\left[\bar{n}^\mu g\mathcal{B}^\nu_{\perp,\omega},gn\cdot\mathcal{B}_n\right]-\left[\bar{n}^\nu g\mathcal{B}^\mu_{\perp,\omega},gn\cdot\mathcal{B}_n\right]-\left[n^\nu\omega(g\mathcal{B}^\mu_{n\perp})_\omega p^{(2)\dagger}_n-n^\mu\omega(g\mathcal{B}^\nu_{n\perp})_\omega p^{(2)\dagger}_n\right] \\
&\quad+\ldots\Big\}\,,
\end{aligned}$$

where again there is an implicit integration over ω_a in terms where it appears. Here the ellipsis denotes terms in $\mathcal{G}^{(3)\mu\nu}_n$ with an $iR^{(1,2)}_n$ or $p^{(1)}_n$ which were not needed for our analysis. The $(gn\cdot\mathcal{B}_n)$ and $[in\cdot\partial_n\, g\mathcal{B}^\mu_\perp]$ terms are further reduced to $\mathcal{P}_\perp$'s, $(g\mathcal{B}^\mu_\perp)$'s, and χ_n's by using the equation of motion in Eq. (3.32). Finally, recall that the expansion coefficients in Eqs. (3.63,3.65,3.67) do not encode the RPI relations between collinear and soft fields which can be determined using Eq. (2.29).

The above results can be used to expand the RPI basis of operators in terms of operators in the homogeneous basis as in Eq. (3.29).

3.7 Applications

3.7.1 Scalar Current

As a first example to show how the expansion of a RPI current works, we expand the scalar chiral-even bilinear currents (LL+RR), for processes with a hard external vector q^μ up to order λ^3. In the basis built from superfields there is only one current that satisfies these conditions

$$\bar{\Psi}_{n,\hat{\omega}_1} \not{q}\, \Psi_{n,\hat{\omega}_2} \,. \tag{3.68}$$

All the other possible currents (for example $\bar{\Psi}_{n,\hat{\omega}_1} \gamma_\mu q_\nu \mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3} \Psi_{n,\hat{\omega}_2}$) have expansions that start at $\mathcal{O}(\lambda^4)$ or beyond. To recover the basis with a homogeneous power counting, all we have to do is to expand (3.68) using Eq. (3.65),

$$\begin{aligned}
\bar{\Psi}_{n,\hat{\omega}_1} \not{q}\, \Psi_{n,\hat{\omega}_2} &= \frac{1}{(n\cdot q)} \bar{\chi}_{n,\omega_1} \frac{\not{\bar{n}}}{2} \chi_{n,\omega_2} \\
&+ \frac{1}{2\,\omega_1 (n\cdot q)^2} \bar{\chi}_{n,\omega_a} i \overleftarrow{\not{\mathcal{D}}}_{\perp \omega_1 - \omega_a} \not{\bar{n}}\, \not{q}_\perp \chi_{n,\omega_2} - \frac{1}{2\,\omega_2 (n\cdot q)^2} \bar{\chi}_{n,\omega_1} \not{q}_\perp \not{\bar{n}}\, i \not{\mathcal{D}}_{\perp \omega_a - \omega_2} \chi_{n,\omega_a} \\
&+ \frac{1}{(\omega_1 - \omega_a)(n\cdot q)^2} \bar{\chi}_{n,\omega_a} (q_\perp \cdot g\mathcal{B}_\perp)_{\omega_1 - \omega_a} \not{\bar{n}} \chi_{n,\omega_2} + \frac{1}{(\omega_2 - \omega_a)(n\cdot q)^2} \bar{\chi}_{n,\omega_1} \not{\bar{n}} (q_\perp \cdot g\mathcal{B}_\perp)_{\omega_a - \omega_2} \chi_{n,\omega_a} \\
&- \frac{1}{(n\cdot q)^2} \frac{\partial}{\partial \omega_1}\, \bar{\chi}_{n,\omega_1} \mathcal{P}_\perp^\dagger \cdot q_\perp \not{\bar{n}} \chi_{n,\omega_2} - \frac{1}{(n\cdot q)^2} \frac{\partial}{\partial \omega_2}\, \bar{\chi}_{n,\omega_1} \not{\bar{n}} \mathcal{P}_\perp \cdot q_\perp \chi_{n,\omega_2} \,.
\end{aligned} \tag{3.69}$$

Thus all the $\mathcal{O}(\lambda^3)$ terms (the twist-3 terms on the last three lines) are connected. Eq. (3.69) agrees with the original derivation of these constraints given in Eqs. (122-126) of Ref. [58]. The ease at which Eq. (3.69) was derived demonstrates the power of the invariant operator formalism. In this example there is only one supercurrent to $\mathcal{O}(\lambda^3)$, so all Wilson coefficients are connected to the coefficient of the leading operator

$\bar{\chi}_{n,\omega_1} \not{q} \chi_{n,\omega_2}$. Note that here all of the connected operators involve a $q_\perp$, which we have counted as $\mathcal{O}(\lambda^0)$. We will see below that for situations with two collinear directions, where in the end its natural to specialize to a frame where $q_\perp = 0$, the connections tend to appear at higher twist. For situations with three or more collinear directions RPI will provide useful constraints on the basis already at lowest order.

3.7.2 General Quark and Gluon Operators

In this Section we enumerate an operator basis for the general set of collinear quark and gluon operators up to $\mathcal{O}(\lambda^4)$. This basis is useful for many applications, and we keep our notation as general as possible. In particular we consider up to 4 distinct collinear directions (which for example could be used for $e^+e^- \to 4$jets, or $gg, qg, q\bar{q} \to$ 2 jets). We also discuss a basis both for the homogeneous operators with a definite power counting, and for the RPI operators.

For processes with a hard q^μ, the most general basis of homogeneous quark operators in SCET up to $\mathcal{O}(\lambda^4)$ is

$$
\begin{aligned}
O^{(0a)} &= \bar{\chi}_{n_1,\omega_1} \Gamma \chi_{n_2,\omega_2} , & O^{(1a)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_\alpha \mathcal{P}^{\dagger\alpha}_{n_1\perp} \chi_{n_2,\omega_2} , \\
O^{(1b)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_\alpha \mathcal{P}^{\alpha}_{n_2\perp} \chi_{n_2,\omega_2} , & O^{(1c)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_\alpha \big(ig\mathcal{B}^{\alpha}_{n_3\perp}\big)_{\omega_3} \chi_{n_2,\omega_2} , \\
O^{(2a)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\alpha\beta} \mathcal{P}^{\dagger\alpha}_{n_1\perp} \mathcal{P}^{\beta}_{n_2\perp} \chi_{n_2,\omega_2} , & O^{(2b)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\alpha\beta} \mathcal{P}^{\dagger\alpha}_{n_1\perp} \mathcal{P}^{\dagger\beta}_{n_1\perp} \chi_{n_2,\omega_2} , \\
O^{(2c)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\alpha\beta} \mathcal{P}^{\alpha}_{n_2\perp} \mathcal{P}^{\beta}_{n_2\perp} \chi_{n_2,\omega_2} , & O^{(2d)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\alpha\beta} \mathcal{P}^{\dagger\alpha}_{n_1\perp} \big(g\mathcal{B}^{\beta}_{n_3\perp}\big)_{\omega_3} \chi_{n_2,\omega_2} , \\
O^{(2e)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\beta\alpha} \big(g\mathcal{B}^{\beta}_{n_3\perp}\big)_{\omega_3} \mathcal{P}^{\perp\alpha}_{n_2} \chi_{n_2,\omega_2} , & O^{(2f)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\alpha\beta} \big[\mathcal{P}^{\perp\alpha}_{n_3} \big(g\mathcal{B}^{\perp\beta}_{n_3}\big)_{\omega_3}\big] \chi_{n_2,\omega_2} , \\
O^{(2g)} &= \bar{\chi}_{n_1,\omega_1} \Gamma_{\alpha\beta} \big(g\mathcal{B}^{\alpha}_{n_3\perp}\big)_{\omega_3} \big(g\mathcal{B}^{\beta}_{n_4\perp}\big)_{\omega_4} \chi_{n_2,\omega_2} , & O^{(2h)} &= \big(\bar{\chi}_{n_1,\omega_1} \Gamma_1 \chi_{n_2,\omega_2}\big) \big(\bar{\chi}_{n_3,\omega_3} \Gamma_2 \chi_{n_4,\omega_4}\big) .
\end{aligned}
\tag{3.70}
$$

If we need to specify the subscripts we write for example $O^{(2g)}(\omega_1, \omega_3, \omega_4, \omega_2)$, with the ω_i listed from left to right. Due to the equations of motion in Eqs. (3.31,3.32) we did not need to consider $in \cdot \partial_n \chi_n$ or $gn \cdot \mathcal{B}_n$. For each operator there may be a set of different Dirac, flavor, and color structures $\Gamma^j_{\alpha_1 \cdots \alpha_n}$ which depend on the particular phenomena being studied (including also two choices for color for the Γ_i

in the four-quark operators $O^{(2i)}$). In general for each independent $\Gamma^j_{\alpha_1\cdots\alpha_n}$ structure the operator has a Wilson coefficient that must be determined order by order in perturbation theory. We included in Eq. (3.70) the mixed quark and gluon operators. For pure gluon operators up $\mathcal{O}(\lambda^4)$ we have the homogeneous basis

$$\begin{aligned}
O^{(0b)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,, & O^{(1d)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{P}^{\dagger\alpha}_{n_1\perp}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,, \\
O^{(1e)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{P}^{\alpha}_{n_2\perp}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,, & O^{(1f)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,\mathcal{B}^{\perp\tau}_{n_3,\omega_3}\,, \\
O^{(2i)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{P}^{\dagger\alpha}_{n_1\perp}\mathcal{P}^{\dagger\beta}_{n_1\perp}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,, & O^{(2j)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{P}^{\dagger\alpha}_{n_1\perp}\mathcal{P}^{\beta}_{n_2\perp}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,, \\
O^{(2k)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{P}^{\alpha}_{n_2\perp}\mathcal{P}^{\beta}_{n_2\perp}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,, & O^{(2l)} &= \big[\mathcal{P}^{\alpha}_{n_1\perp}\mathcal{B}^{\perp\mu}_{n_1,\omega_1}\big]\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,\mathcal{B}^{\perp\tau}_{n_3,\omega_3}\,, \\
O^{(2m)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\big[\mathcal{P}^{\alpha}_{n_2\perp}\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\big]\,\mathcal{B}^{\perp\tau}_{n_3,\omega_3}\,, & O^{(2n)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,\big[\mathcal{P}^{\alpha}_{n_3\perp}\mathcal{B}^{\perp\tau}_{n_3,\omega_3}\big]\,, \\
O^{(2o)} &= \mathcal{B}^{\perp\mu}_{n_1,\omega_1}\,\mathcal{B}^{\perp\nu}_{n_2,\omega_2}\,\mathcal{B}^{\perp\sigma}_{n_3,\omega_3}\,\mathcal{B}^{\perp\tau}_{n_4,\omega_4}\,. & &
\end{aligned} \tag{3.71}$$

Here we do not need to consider operators with $gn\cdot\mathcal{B}_n$ and $gn\cdot\partial_n\mathcal{B}^{\mu}_{n\perp}$ because using the equations of motion in Eq. (3.32) they can be written in terms of the operators in Eq. (3.71), and are hence redundant.

To setup the computation of constraints on Wilson coefficients we also need to build an RPI basis of operators using the objects in Eq. (3.13) and $i\partial^{\mu}_{n}$. Because each operator will be RPI, its Wilson coefficient is truly independent of those for other operators in the basis. The RPI operators can then be expanded in terms of homogeneous operators made out of of gauge invariant objects, and doing so we obtain operators in the homogeneous basis with all the constraints coming from reparametrization invariance. The number of constraints on Wilson coefficients is equal to the number of homogeneous operators minus the number of RPI operators, once we have accounted for linear dependencies [46, 73].

Let's construct the RPI basis of operators which is the analog of those in Eqs. (3.70) and (3.71). The operators with no $i\partial^{\mu}_{n}$ derivatives are

$$\mathbf{Q}^{(0q)} = \bar{\Psi}_{n_1,\hat{\omega}_1}\Gamma\Psi_{n_2,\hat{\omega}_2}\,, \qquad \mathbf{Q}^{(0g)} = \mathcal{G}^{\mu\nu}_{n_1,\hat{\omega}_1}\,\mathcal{G}^{\sigma\tau}_{n_2,\hat{\omega}_2}\,, \tag{3.72}$$

where the basis of Dirac structures Γ, and contraction of indices $\mu\nu\sigma\tau$ in $\mathbf{Q}^{(0g)}$ depends on the kind of current we are studying. For cases without a q^μ the subscripts $\hat{\omega}_i$ are erased and RPI operators are multiplied by the $\hat{\Delta}_{ij}$ factors shown in Eq. (3.17). Recall that we do not have a good power counting in the RPI basis, this basis makes the RPI properties transparent but the power counting more tricky. When $\mathbf{Q}^{(aq)}$ and $\mathbf{Q}^{(ag)}$ are expanded in terms of operators that are homogeneous in the power counting, they contain a leading order term, so they are relevant operators to consider at LO. Of the RPI objects only $i\partial_n^\mu$ starts at leading order, so theoretically we can construct an infinite set of LO operators using $(i\partial_n)^k$ for any k. However, the structure of this operator provides additional constraints. In particular the $\mathcal{O}(\lambda^0)$ term is $i\partial_n^\mu = (n^\mu/2)\overline{\mathcal{P}}_n + \cdots$, and the collinear momentum $\overline{\mathcal{P}}_n$ acting on a n-collinear field such as χ_{n,ω_1} just gives a number, ω_1, which can be absorbed into the Wilson coefficient $C(\omega_1, \omega_2)$. For cases with a q^μ this implies that adding $i\partial_n^\mu$'s in a scalar operator (where all vector indices are contracted) most often gives an operator that differs from one we already have only at $\mathcal{O}(\lambda)$. For these scalar operators we can count $i\partial_n^\mu \sim \mathcal{O}(\lambda)$ when determining which RPI operators are required, and for simplicity we follow this counting in the remainder of this Section. If we have an operator with a free vector index μ, then this index can be carried by $i\partial_n^\mu = (n^\mu/2)\overline{\mathcal{P}}_n + \cdots$, and the partial derivative does count as $\mathcal{O}(\lambda^0)$.

The expansion of the RPI operators in Eq. (3.72) in terms of homogeneous operators up to $\mathcal{O}(\lambda^4)$ is

$$\begin{aligned}
\mathbf{Q}^{(0q)} &= \bar{\Psi}^{(1)}_{n_1,\hat{\omega}_1}\Gamma\Psi^{(1)}_{n_2,\hat{\omega}_2} + \bar{\Psi}^{(2)}_{n_1,\hat{\omega}_1}\Gamma\Psi^{(1)}_{n_2,\hat{\omega}_2} + \bar{\Psi}^{(1)}_{n_1,\hat{\omega}_1}\Gamma\Psi^{(2)}_{n_2,\hat{\omega}_2} \\
&\quad + \bar{\Psi}^{(2)}_{n_1,\hat{\omega}_1}\Gamma\Psi^{(2)}_{n_2,\hat{\omega}_2} + \bar{\Psi}^{(1)}_{n_1,\hat{\omega}_1}\Gamma\Psi^{(3)}_{n_2,\hat{\omega}_2} + \bar{\Psi}^{(3)}_{n_1,\hat{\omega}_1}\Gamma\Psi^{(1)}_{n_2,\hat{\omega}_2} + O(\lambda^5)\,, \\
\mathbf{Q}^{(0g)} &= \mathcal{G}^{(1)\mu\nu}_{n_1,\hat{\omega}_1}\mathcal{G}^{(1)\sigma\tau}_{n_2,\hat{\omega}_2} + \mathcal{G}^{(2)\mu\nu}_{n_1,\hat{\omega}_1}\mathcal{G}^{(1)\sigma\tau}_{n_2,\hat{\omega}_2} + \mathcal{G}^{(1)\mu\nu}_{n_1,\hat{\omega}_1}\mathcal{G}^{(2)\sigma\tau}_{n_2,\hat{\omega}_2} \\
&\quad + \mathcal{G}^{(2)\mu\nu}_{n_1,\hat{\omega}_1}\mathcal{G}^{(2)\sigma\tau}_{n_2,\hat{\omega}_2} + \mathcal{G}^{(3)\mu\nu}_{n_1,\hat{\omega}_1}\mathcal{G}^{(1)\sigma\tau}_{n_2,\hat{\omega}_2} + \mathcal{G}^{(1)\mu\nu}_{n_1,\hat{\omega}_1}\mathcal{G}^{(3)\sigma\tau}_{n_2,\hat{\omega}_2} + O(\lambda^5)\,,
\end{aligned} \tag{3.73}$$

where the $\Psi^{(k)}_{n,\hat{\omega}}$ and $\mathcal{G}^{(k)}_{n,\hat{\omega}}$ are given in Eqs. (3.65) and (3.67). To look for RPI relations the results of this expansion must be compared to power suppressed operators which also can generate $\mathcal{O}(\lambda^3)$ and $\mathcal{O}(\lambda^4)$ terms. Up to this order the power suppressed operators involving two or more quark fields are

$$
\begin{aligned}
\mathbf{Q}^{(1a)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_\alpha\, i\partial^\alpha_{n_2}\,\Psi_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(1b)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_\alpha\, i\overleftarrow{\partial}^\alpha_{n_1}\,\Psi_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(1c)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\beta\beta'}\,\mathcal{G}^{\beta\beta'}_{n_3,\hat\omega_3}\,\Psi_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(2a)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\alpha'}\, i\overleftarrow{\partial}^\alpha_{n_1}\, i\partial^{\alpha'}_{n_2}\,\Psi_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(2b)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\alpha'}\, i\overleftarrow{\partial}^\alpha_{n_1} i\overleftarrow{\partial}^{\alpha'}_{n_1}\,\Psi_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(2c)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\alpha'}\, i\partial^\alpha_{n_2}\, i\partial^{\alpha'}_{n_2}\,\Psi_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(2d)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\beta\beta'}\, i\overleftarrow{\partial}^\alpha_{n_1}\,\mathcal{G}^{\beta\beta'}_{n_3,\hat\omega_3}\Psi_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(2e)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\beta\beta'}\big[i\partial^\alpha_{n_3}\mathcal{G}^{\beta\beta'}_{n_3,\hat\omega_3}\big]\Psi_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(2f)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\beta\beta'}\,\mathcal{G}^{\beta\beta'}_{n_3,\hat\omega_3}\, i\partial^\alpha_{n_2}\,\Psi_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(2g)} &= \bar\Psi_{n_1,\hat\omega_1}\Gamma_{\alpha\beta\gamma\delta}\,\mathcal{G}^{\alpha\beta}_{n_3,\hat\omega_3}\,\mathcal{G}^{\gamma\delta}_{n_4,\hat\omega_4}\Psi_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(2h)} &= \big[\bar\Psi_{n_1,\hat\omega_1}\Gamma_1\Psi_{n_2,\hat\omega_2}\big]\,\big[\bar\Psi_{n_3,\hat\omega_3}\Gamma_2\Psi_{n_4,\hat\omega_4}\big]\,.
\end{aligned}
\tag{3.74}
$$

Again a minimal basis for Dirac structures Γ will depend on the process being studied and may differ between the various $\mathbf{Q}^{(ix)}$ operators. Such a basis will also in general differ from the one for the homogeneous operators in Eq. (3.70). We will adopt notation such as $\mathbf{Q}^{(2g)}(\hat\omega_1,\hat\omega_3,\hat\omega_4,\hat\omega_2)$ when we wish to specify these subscripts. For a field basis for the higher order operators with gluon fields (whose expansion starts at $\mathcal{O}(\lambda^3)$ or $\mathcal{O}(\lambda^4)$) we have

$$
\begin{aligned}
\mathbf{Q}^{(1d)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1} i\partial^\alpha_{n_2}\,\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(1e)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1} i\overleftarrow{\partial}^\alpha_{n_1}\,\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(1f)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1}\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\mathcal{G}^{\alpha\beta}_{n_3,\hat\omega_3}\,, &
\mathbf{Q}^{(2i)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1} i\partial^\alpha_{n_2}\, i\partial^\beta_{n_2}\,\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(2j)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1} i\overleftarrow{\partial}^\alpha_{n_1} i\overleftarrow{\partial}^\beta_{n_1}\,\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\,, &
\mathbf{Q}^{(2k)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1} i\overleftarrow{\partial}^\alpha_{n_1} i\partial^\beta_{n_2}\,\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\,, \\
\mathbf{Q}^{(2l)} &= [i\partial^\gamma_{n_1}\mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1}]\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\mathcal{G}^{\alpha\beta}_{n_3,\hat\omega_3}\,, &
\mathbf{Q}^{(2m)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1}[i\partial^\gamma_{n_2}\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}]\mathcal{G}^{\alpha\beta}_{n_3,\hat\omega_3}\,, \\
\mathbf{Q}^{(2n)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1}\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}[i\partial^\gamma_{n_3}\mathcal{G}^{\alpha\beta}_{n_3,\hat\omega_3}]\,, &
\mathbf{Q}^{(2o)} &= \mathcal{G}^{\mu\nu}_{n_1,\hat\omega_1}\mathcal{G}^{\sigma\tau}_{n_2,\hat\omega_2}\mathcal{G}^{\alpha\beta}_{n_3,\hat\omega_3}\mathcal{G}^{\gamma\delta}_{n_4,\hat\omega_4}\,.
\end{aligned}
\tag{3.75}
$$

We will include a basis of Dirac structures and expand the RPI operators in Eqs. (3.74) and (3.75) in terms of the homogeneous ones in several of the examples below, and consider whether there are non-trivial RPI relations on a case-by-case basis.

3.7.3 Deep Inelastic Scattering for Quarks at Twist-4

In this Section we consider spin-averaged DIS at twist-4. This provides a test of our technique of constructing a minimal basis, for an example where the basis is already well known [42, 62, 63]. We will see that RPI constrains the Wilson coefficients of the homogeneous collinear operators. Our analysis is really of scalar operators with one collinear direction, $q_\perp = 0$, with overall derivatives set to zero. DIS is the most popular application for these operators, so we frame our discussion in that language. For simplicity we consider the QCD electromagnetic current $J^\mu = \bar{q}\gamma^\mu q$ for one-flavor of quark. (We briefly discuss the generalization to non-singlet operators in a footnote.) The study of higher twist in DIS and related processes is an active field of research, for example [23, 28, 36, 53, 66, 87]. In the language of SCET, DIS was first studied in [9], whose notation we follow. The virtual photon has momentum transfer $q^2 = -Q^2$, and $x = Q^2/(2p \cdot q)$ is the Bjorken variable.

In the Breit frame the momentum of the virtual photon is $q^\mu = Q(\bar{n}^\mu - n^\mu)/2$, and the incoming proton momentum is $p^\mu = n^\mu \bar{n} \cdot p/2 + \bar{n}^\mu m_p^2/(2\bar{n} \cdot p)$ where m_p is the mass of the proton. Expanding in m_p/Q we have $\bar{n} \cdot p = Q/x - x m_p^2/Q + \ldots$. The energetic proton has a small invariant mass $p^2 = m_p^2 \sim \Lambda_{\rm QCD}^2$, and in the Breit frame it is described by collinear fields in the effective theory with a power counting in $\lambda = \Lambda_{\rm QCD}/Q$. It is convenient to pick this frame in order to be able to assign definite power counting to momentum components. What reparametrization invariance enforces is that all results are invariant to small perturbations about this frame, encoded by changes to the collinear reference vector n^μ. Since these changes are small we are free to use the same power counting when studying the RPI relations. There is a larger class of frame independence, which says for example that the same results would be found if we compare an analysis in the Breit-frame with an analysis made about the initial proton rest frame, but this set of "big" frame transformations does not encode non-trivial dynamic information that relates coefficients of operators at higher twist. All final results are of course entirely frame independent.

For spin-averaged DIS the hadronic tensor has the structure

$$T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(x, Q^2) + \left(p_\mu + \frac{q_\mu}{2x}\right)\left(p_\nu + \frac{q_\nu}{2x}\right) T_2(x, Q^2), \qquad (3.76)$$

where

$$T_{\mu\nu}(p,q) = \frac{1}{2}\sum_{\text{spin}} \langle p|\hat{T}_{\mu\nu}(q)|p\rangle, \qquad \hat{T}_{\mu\nu}(q) = i\int d^4z e^{iq\cdot z} \mathrm{T}[J_\mu(z), J_\nu(0)]\,. \qquad (3.77)$$

The scalar structure functions T_i can be projected out of $T_{\mu\nu}$ using

$$\begin{aligned} T_1(Q^2,x) &= -\frac{1}{2}\Big(g^{\mu\nu} - \frac{4x^2}{Q^2+4m_p^2x^2}\, p^\mu p^\nu\Big) T_{\mu\nu}\,, \\ T_2(Q^2,x) &= -\frac{2x^2}{Q^2+4m_p^2x^2}\Big(g^{\mu\nu} - \frac{12x^2}{Q^2+4m_p^2x^2}\, p^\mu p^\nu\Big) T_{\mu\nu}\,. \end{aligned} \qquad (3.78)$$

The expansion of T_1 and T_2 has been carried out up to twist-4 with the Wilson coefficients determined at tree level in Refs. [42, 62, 63]. To simplify our calculations we will make use of the fact that the projections in Eq. (3.78) commute with taking the proton matrix element, and hence can be applied directly to $\hat{T}_{\mu\nu}$ to give $\hat{T}_1$ and $\hat{T}_2$, where $\frac{1}{2}\sum_{\text{spin}}\langle p|\hat{T}_i|p\rangle = T_i(Q^2,x)$. Thus we consider the expansion of $\hat{T}_1$ and $\hat{T}_2$ in scalar chiral-even operators, by writing

$$\hat{T}_i = \sum_j \int [d\omega_k]\, C_j^{[i]}(\omega_k) O_j(\omega_k)\,. \qquad (3.79)$$

Here $[d\omega_k] = d\omega_1 \cdots d\omega_n$ is the integration measure over the independent parton momenta ω_k carried by the Wilson coefficients $C_j^{[i]}$ and the operators O_j. The superscript $[i]$ indicates that the Wilson coefficients for the two tensor structures will in general differ. We also consider a basis of RPI operators $\mathbf{Q}_j$ by writing

$$\hat{T}_i = \sum_j \int [d\hat{\omega}_k]\, \hat{C}_j^{[i]}(\hat{\omega}_k) \mathbf{Q}_j(\hat{\omega}_k)\,. \qquad (3.80)$$

Unlike the O_j's the $\mathbf{Q}_j$'s do not contain contributions of a definite order in the power counting. Using the RPI $\mathbf{Q}_j$ operators we can test if there are relations between the Wilson coefficients $C_j^{[i]}$ of the O_j's. A connection would mean, for example, that the one-loop coefficient for a twist-4 operator is determined by a coefficient at twist-2 at all orders in α_s.

We first write down a gauge invariant basis of chiral-even quark operators that are homogeneous in the power counting. This can be done using the general basis in Eq. (3.70) with all directions $n_i = n$. Furthermore, since the DIS matrix element is forward, we have $\langle p|[\mathcal{P}^\mu O]|p\rangle = 0$ for any operator O. Thus we are free to integrate $\perp$-label momentum operators by parts, and hence can ignore all terms with $\mathcal{P}_\perp^\dagger$'s in Eq. (3.70). (If we consider our analysis to be of the general scalar operators with one collinear direction, then this is the only simplification that we make which relies on the form of the final matrix element.) For simplicity we also drop the square-brackets from inside $O^{(2f)}$ in Eq. (3.70). A minimal basis of chiral-even parity-even Dirac structures between the n-collinear quark fields is easily constructed using the properties of the SCET χ_n fields. We have i) just $\{\bar{n}\!\!\!/\}$ when there are no vector indices on fields, ii) no elements at all when there is one vector index, and iii) just $\{\bar{n}\!\!\!/ g_\perp^{\mu\nu},\, i\epsilon_\perp^{\mu\nu}\bar{n}\!\!\!/\gamma_5\}$ or $\{\bar{n}\!\!\!/ g_\perp^{\mu\nu},\, \bar{n}\!\!\!/\gamma_\perp^\mu\gamma_\perp^\nu\}$ for two vector indices on fields. Here ii) is the standard fact that the spin-averaged case does not have twist-3 terms. (For polarized DIS it does not suffice to only consider the scalar operators.) For the four-quark operators we can have $\Gamma_1 \otimes \Gamma_2 = \{\bar{n}\!\!\!/ \otimes \bar{n}\!\!\!/,\, \bar{n}\!\!\!/\gamma_5 \otimes \bar{n}\!\!\!/\gamma_5\}$ and color structures $1 \otimes 1$ or $T^A \otimes T^A$. Thus the basis is

$$
\begin{aligned}
O_1 &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, \chi_{n,\omega_2}\,, &
O_2 &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, \mathcal{P}_\perp^2\, \chi_{n,\omega_2}\,, \qquad (3.81)\\
O_{3a} &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, (g\mathcal{B}_{n\perp}^\mu)_{\omega_3} \mathcal{P}_\mu^\perp \chi_{n,\omega_2}\,, &
O_{3b} &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, \mathcal{P}_\mu^\perp (g\mathcal{B}_{n\perp}^\mu)_{\omega_3} \chi_{n,\omega_2}\,,\\
O_{4a} &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, (g\mathcal{B}\!\!\!/_{n\perp})_{\omega_3} \mathcal{P}\!\!\!/_\perp \chi_{n,\omega_2}\,, &
O_{4b} &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, \mathcal{P}\!\!\!/_\perp (g\mathcal{B}\!\!\!/_{n\perp})_{\omega_3} \chi_{n,\omega_2}\,,\\
O_5 &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, (g\mathcal{B}\!\!\!/_{n\perp})_{\omega_3} (g\mathcal{B}\!\!\!/_{n\perp})_{\omega_4} \chi_{n,\omega_2}\,, &
O_6 &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, \mathrm{Tr}\big[(g\mathcal{B}\!\!\!/_{n\perp})_{\omega_3} (g\mathcal{B}\!\!\!/_{n\perp})_{\omega_4}\big] \chi_{n,\omega_2}\,,\\
O_7 &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, (g\mathcal{B}_{n\perp}^\mu)_{\omega_3} (g\mathcal{B}_\mu^{\perp n})_{\omega_4} \chi_{n,\omega_2}\,, &
O_8 &= \bar{\chi}_{n,\omega_1} \frac{\bar{n}\!\!\!/}{2}\, \mathrm{Tr}\big[(g\mathcal{B}_{n\perp}^\mu)_{\omega_3} (g\mathcal{B}_\mu^{\perp n})_{\omega_4}\big] \chi_{n,\omega_2}\,,
\end{aligned}
$$

$$O_9 = \left[\bar{\chi}_{n,\omega_1}\frac{\bar{n}\!\!\!/}{2}\chi_{n,\omega_2}\right]\left[\bar{\chi}_{n,\omega_3}\frac{\bar{n}\!\!\!/}{2}\chi_{n,\omega_4}\right], \qquad O_{10} = \left[\bar{\chi}_{n,\omega_1}\frac{\bar{n}\!\!\!/}{2}\gamma_5\chi_{n,\omega_2}\right]\left[\bar{\chi}_{n,\omega_3}\frac{\bar{n}\!\!\!/}{2}\gamma_5\chi_{n,\omega_4}\right],$$
$$O_{11} = \left[\bar{\chi}_{n,\omega_1}\frac{\bar{n}\!\!\!/}{2}\gamma_5 T^A\chi_{n,\omega_2}\right]\left[\bar{\chi}_{n,\omega_3}\frac{\bar{n}\!\!\!/}{2}\gamma_5 T^A\chi_{n,\omega_4}\right], \quad O_{12} = \left[\bar{\chi}_{n,\omega_1}\frac{\bar{n}\!\!\!/}{2}T^A\chi_{n,\omega_2}\right]\left[\bar{\chi}_{n,\omega_3}\frac{\bar{n}\!\!\!/}{2}T^A\chi_{n,\omega_4}\right].$$

Recall that in an operator like O_2 the position space analog of $\mathcal{P}_\perp^\mu$ is to translate all gluon and quark fields in χ_{n,ω_2} in $x_\perp$, differentiate twice with respect to $x_\perp^\mu$, and then set $x_\perp = 0$. The basis shown in Eq. (3.81) can be used to describe twist-4 effects in DIS at any order in α_s. Note that we have already discussed and taken into account the quark and gluon equations of motion in the general result in Eq. (3.70) and hence already in Eq. (3.81). For $O_{5,7}$ there are two color structures associated with the product of $\mathcal{B}_n$'s, but these are picked out by consider Wilson coefficients $C_{5,7}$ that are odd or even in the exchange $\omega_3 \leftrightarrow \omega_4$. The forward proton matrix element of these operators will be proportional to an overall δ-function, which is $\delta(\omega_1 - \omega_2)$ for $O_{1,2}$, $\delta(\omega_1 + \omega_3 - \omega_2)$ for $O_{3a,3b,4a,4b}$, $\delta(\omega_1 + \omega_3 + \omega_4 - \omega_2)$ for O_{5-8}, and $\delta(\omega_1 + \omega_3 - \omega_2 - \omega_4)$ for O_{9-12}.

Next we derive the analogous results for the RPI basis of chiral-even operators. From Eq. (3.77) the hadronic tensor operator $\hat{T}_{\mu\nu}$ depends on q^μ which we use as our reference vector. To construct this basis we cannot use n^μ or $\bar{n}^\mu$. Comparing Eqs. (3.77) and Eq. (3.78) we see that it suffices to construct a basis of scalar operators for the expansion of $g^{\mu\nu}\hat{T}_{\mu\nu}$ and $p^\mu p^\nu \hat{T}_{\mu\nu}$. The forward proton matrix element of the expansion of these operators then yields an expansion for the observables T_1 and T_2. Thus, for the scalar basis we allow any number of q's to appear, but only zero or two p's. This implies that at twist-2 there is only one RPI bilinear quark operator

$$\mathbf{Q}_1 = \bar{\Psi}_{n,\hat{\omega}_1} q\!\!\!/ \, \Psi_{n,\hat{\omega}_2} \,. \tag{3.82}$$

At twist-3 there are no scalar chiral-even RPI operators. The candidate operators $\bar{\Psi}_n i\partial\!\!\!/_n \Psi_n$ and $\bar{\Psi}_{n,,\hat{\omega}_1} q\!\!\!/ (q \cdot i\partial_n)\Psi_{n,,\hat{\omega}_2}$ are ruled out by the equations of motion in Eqs. (3.35) and (3.36). Another possible operator is $\bar{\Psi}_n p\!\!\!/ (p \cdot \partial_n)\Psi_n$ but it starts at twist-6, since $(p \cdot \partial_n) \sim O(\lambda^2)$, being suppressed either by an $n{\cdot}p$ or $n{\cdot}\partial_n$, and $p\!\!\!/$ adds another factor 2 to the power counting when it is squeezed between the n-collinear fermion fields χ_n. All the operators with $\mathcal{G}_n^{\mu\nu}$, like for example $\bar{\Psi}_n\gamma_\mu q_\nu \mathcal{G}_n^{\mu\nu}\Psi_n$, have expansions whose lowest term is twist-4 because the Dirac structure of the twist-3 component of this operator vanishes between the n-collinear fermion fields, since $\bar{\chi}_n \not{n}\ \chi_n = 0$. Thus the power suppressed terms start at twist-4 in agreement with the homogeneous basis in Eq. (3.81). Writing out the RPI operators different from zero at twist-4 and not connected by operator relations we have

$$\begin{aligned}
&\mathbf{Q}_2 = \bar{\Psi}_{n,\hat{\omega}_1}\gamma_\mu q_\nu \mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3}\Psi_{n,\hat{\omega}_2}\,, && \mathbf{Q}_3 = \bar{\Psi}_{n,\hat{\omega}_1}\not{q}\gamma_\mu\gamma_\nu ig\mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3}\Psi_{n,\hat{\omega}_2}\,, \qquad (3.83)\\
&\mathbf{Q}_4 = -g^2\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\,\gamma_\mu q_\nu \mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3}\gamma_\alpha q_\beta \mathcal{G}^{\alpha\beta}_{n,\hat{\omega}_4}\Psi_{n,\hat{\omega}_2}\,, && \mathbf{Q}_5 = -g^2\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\,\gamma_\mu q_\nu\gamma_\alpha q_\beta \mathrm{Tr}[\mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3}\mathcal{G}^{\alpha\beta}_{n,\hat{\omega}_4}]\Psi_{n,\hat{\omega}_2}\,,\\
&\mathbf{Q}_6 = -g^2\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\,q_\mu q_\nu(\mathcal{G}_{n,\hat{\omega}_3})^\mu{}_\alpha\mathcal{G}^{\alpha\nu}_{n,\hat{\omega}_4}\Psi_{n,\hat{\omega}_2}\,, && \mathbf{Q}_7 = -g^2\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\,q_\mu q_\nu \mathrm{Tr}[(\mathcal{G}_{n,\hat{\omega}_3})^\mu{}_\alpha\mathcal{G}^{\alpha\nu}_{n,\hat{\omega}_4}]\Psi_{n,\hat{\omega}_2}\,,\\
&\mathbf{Q}_8 = \big[\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\Psi_{n,\hat{\omega}_2}\big]\big[\bar{\Psi}_{n,\hat{\omega}_3}\not{q}\Psi_{n,\hat{\omega}_4}\big]\,, && \mathbf{Q}_9 = \big[\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\gamma_5\Psi_{n,\hat{\omega}_2}\big]\big[\bar{\Psi}_{n,\hat{\omega}_3}\not{q}\gamma_5\Psi_{n,\hat{\omega}_4}\big]\,,\\
&\mathbf{Q}_{10} = \big[\bar{\Psi}_{n,\hat{\omega}_1}\not{q}T^A\Psi_{n,\hat{\omega}_2}\big]\big[\bar{\Psi}_{n,\hat{\omega}_3}\not{q}T^A\Psi_{n,\hat{\omega}_4}\big]\,, && \mathbf{Q}_{11} = \big[\bar{\Psi}_{n,\hat{\omega}_1}\not{q}T^A\gamma_5\Psi_{n,\hat{\omega}_2}\big]\big[\bar{\Psi}_{n,\hat{\omega}_3}\not{q}T^A\gamma_5\Psi_{n,\hat{\omega}_4}\big]\,.
\end{aligned}$$

One can think of other possible operators at twist-4, but all of them are either ruled out by the equations of motion and operator relations, or start at higher twist. For example, there are not operators with both p^μ and $\mathcal{G}^{\mu\nu}_{\hat{\omega}_3}$, like $\bar{\Psi}_{n,\hat{\omega}_1}\gamma_\mu p_\nu \mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3}\Psi_{n,\hat{\omega}_2}$, because they all start at higher twist. We have integrated by parts making all derivatives act to the right, since here our interest is in forward matrix elements, and we removed $i\partial_\nu\mathcal{G}^{\mu\nu}_{n,\hat{\omega}}$ with the gluon equation of motion in Eq. (3.37). The operator $\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\ (i\partial_n\cdot i\partial_n)\Psi_{n,\hat{\omega}_2} = \bar{\Psi}_{n,\hat{\omega}_1}\not{q}\, i\not{\partial}_n i\not{\partial}_n\Psi_{n,\hat{\omega}_2}$, and is removed by the quark equation of motion in Eq. (3.35). For the operators with two $\mathcal{G}'s$ only the structures in $\mathbf{Q}_{4-7}$ have expansions that start at twist-4. For example, $\mathcal{G}^{\mu\alpha}_{n,\hat{\omega}_3}\mathcal{G}_{n,\hat{\omega}_4\alpha\nu}$ at LO is proportional to $(g\mathcal{B}^\alpha_\perp)_{\omega_3}(g\mathcal{B}^\perp_\alpha)_{\omega_4}n^\mu n^\nu$ so closing the indexes with γ^μ or γ^ν generates a $\not{n}$ that next to χ_n gives zero.

It is less obvious that operators with one $\mathcal{G}_n$ and one $i\partial_n$ are redundant and can be eliminated from the RPI basis. Consider the operator

$$\mathbf{Q}_* = \bar{\Psi}_{n,\hat{\omega}_1}\not{q}\,q_\mu ig\mathcal{G}^{\mu\nu}_{n,\hat{\omega}_3}i\partial^n_\nu\Psi_{n,\hat{\omega}_2}\,. \qquad (3.84)$$

To remove it we use a manipulation discussed by Jaffe and Soldate in Ref. [63]. First we write

$$\mathcal{G}_n^{\mu\nu} i\partial_\nu^n = \mathcal{G}_n^{\mu\nu} i\hat{\mathcal{D}}_\nu^n - \mathcal{G}_n^{\mu\nu}[1/(q\cdot i\partial_n) q_\alpha ig\mathcal{G}_{n\nu}^\alpha] \,, \tag{3.85}$$

and note that the term with two $\mathcal{G}_n$'s can be ignored since it is already in our basis. Next using the definition (3.20) we can write

$$q_\mu ig\mathcal{G}_n^{\mu\nu} i\hat{\mathcal{D}}_\nu^n = q^\mu [i\hat{\mathcal{D}}_\mu^n, i\hat{\mathcal{D}}_n^\nu] i\hat{\mathcal{D}}_\nu^n = \frac{1}{2}\big\{-q^\mu[i\hat{\mathcal{D}}_\nu^n, [i\hat{\mathcal{D}}_\mu^n, i\hat{\mathcal{D}}_n^\nu]] - (i\hat{\mathcal{D}}_n)^2 iq\cdot\partial_n + iq\cdot\partial_n (i\hat{\mathcal{D}}_n)^2\big\} \,. \tag{3.86}$$

The double commutator term is turned into a four-quark operator by the gluon equations of motion in Eq. (3.37). For the remaining terms we write $(i\hat{\mathcal{D}}_n)^2 = i\,\hat{\not{\mathcal{D}}}_n \; / \; i\hat{\mathcal{D}}_n + \frac{i}{2}\sigma_{\mu\nu}\mathcal{G}_n^{\mu\nu}$, where the $\sigma_{\mu\nu}$ term gives $\mathbf{Q}_3$, and terms involving $(i\hat{\not{\mathcal{D}}}_n)^2$ are turned into the operators $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_4$, and $\mathbf{Q}_6$ by the quark equation of motion in Eq. (3.35). (They are not simply set to zero, since $[i\hat{\not{\mathcal{D}}}_n]$ does not commute with $\delta(\hat{\omega}-2q\cdot i\partial_n)$.) Finally, we can also rule out the only other non-trivial operator $\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\, i\not{\partial}_n \gamma_\mu q_\nu \mathcal{G}_{n,\hat{\omega}_3}^{\mu\nu}\Psi_{n,\hat{\omega}_2}$. Using the gluon equation of motion we write

$$\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\, i\not{\partial}\gamma_\mu q_\nu \mathcal{G}_{n,\hat{\omega}_3}^{\mu\nu}\Psi_{n,\hat{\omega}_2} + \bar{\Psi}_{n,\hat{\omega}_1}\not{q}\gamma_\mu i\not{\partial}\mathcal{G}_{n,\hat{\omega}_3}^{\mu\nu} q_\nu \Psi_{n,\hat{\omega}_2} = -2\mathbf{Q}_* + \dots \,, \tag{3.87}$$

where the ellipsis denotes operators with two $\mathcal{G}_n$'s or four-quark fields that are part of the basis. The Bianchi identity in Eq. (3.38) gives another relation for the two operators on the LHS of Eq. (3.87) and implies that they can be written in terms of $\mathbf{Q}_3$, $\mathbf{Q}_*$, $\mathbf{Q}_4$, and $\mathbf{Q}_6$. Thus both the operators $\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\, i\not{\partial}_n\gamma_\mu q_\nu \mathcal{G}_{n,\hat{\omega}_3}^{\mu\nu}\Psi_{n,\hat{\omega}_2}$ and $\bar{\Psi}_{n,\hat{\omega}_1}\not{q}\,\gamma_\mu i\not{\partial}_n \mathcal{G}_{n,\hat{\omega}_3}^{\mu\nu} q_\nu \Psi_{n,\hat{\omega}_2}$ are redundant. Finally we note that the order of the $\mathcal{G}_n$'s in an operator like $\mathbf{Q}_6$ is not important, since we can always symmetrize or antisymmetrize its Wilson coefficient in $\hat{\omega}_3$ and $\hat{\omega}_4$. Note that when considering the transformation of the operators under charge conjugation one must consider both the operator and its Wilson coefficient. We discuss an example below in Eq. (3.94).

The number of independent RPI operators in Eq. (3.81) is smaller than in the basis of homogeneous operators in Eq. (3.83), implying that there exist further constraints on the Wilson coefficients of the homogeneous basis at twist-4. To find the constraints we must expand the operators in Eq. (3.83) in terms of those in Eq. (3.81). We start with $\mathbf{Q}_4$ through $\mathbf{Q}_{11}$ which are in one-to-one correspondence with operators in the homogeneous basis,

$$\begin{aligned}
\mathbf{Q}_4 &= \frac{\omega_3\omega_4}{4(n\cdot q)}\,O_5\,, &\quad \mathbf{Q}_5 &= \frac{\omega_3\omega_4}{4(n\cdot q)}\,O_6\,, &\quad \mathbf{Q}_6 &= -\frac{\omega_3\omega_4}{4(n\cdot q)}\,O_7\,,\\
\mathbf{Q}_7 &= \frac{-\omega_3\omega_4}{4(n\cdot q)}\,O_8\,, & \mathbf{Q}_8 &= \frac{1}{(n\cdot q)^2}\,O_9\,, & \mathbf{Q}_9 &= \frac{1}{(n\cdot q)^2}\,O_{10}\,,\\
\mathbf{Q}_{10} &= \frac{1}{(n\cdot q)^2}\,O_{12}\,, & \mathbf{Q}_{11} &= \frac{1}{(n\cdot q)^2}\,O_{11}\,.
\end{aligned} \tag{3.88}$$

Here the order of the $\hat{\omega}_i$ subscripts in operators on the left exactly matches up with the ω_i subscripts on the right. For the remaining operators whose expansions start at twist-4 and for Q_1 that starts at twist-2, we have

$$\begin{aligned}
\mathbf{Q}_2(\hat{\omega}_1,\hat{\omega}_3,\hat{\omega}_2) &= \frac{1}{(n\cdot q)^2}\bigg[\frac{\omega_3}{2\omega_2}O_{4a}(\omega_1,\omega_3,\omega_2) + \frac{\omega_3}{2\omega_1}O_{4b}(\omega_1,\omega_3,\omega_2) \\
&\qquad - O_{3a}(\omega_1,\omega_3,\omega_2) + O_{3b}(\omega_1,\omega_3,\omega_2)\bigg] + \dots\,,\\
\mathbf{Q}_3(\hat{\omega}_1,\hat{\omega}_3,\hat{\omega}_2) &= \frac{2}{(n\cdot q)^2}\bigg[\frac{\omega_1}{\omega_2}O_{4a}(\omega_1,\omega_3,\omega_2) + O_{4b}(\omega_1,\omega_3,\omega_2) - 2O_{3b}(\omega_1,\omega_3,\omega_2)\bigg] + \dots\,,\\
\mathbf{Q}_1(\hat{\omega}_1,\hat{\omega}_2) &= \frac{1}{n\cdot q}O_1(\omega_1,\omega_2) + \frac{\bar{n}\cdot q}{(n\cdot q)^2}\bigg[\Big\{\frac{-1}{\omega_1\omega_2} + \frac{d}{d\omega_1}\frac{1}{\omega_1} + \frac{d}{d\omega_2}\frac{1}{\omega_2}\Big\}O_2(\omega_1,\omega_2)\\
&\quad + \Big\{\frac{2}{(\omega_a-\omega_2)^2} - \frac{d}{d\omega_2}\frac{2}{\omega_a-\omega_2}\Big\}\Big\{O_{3a}(\omega_1,\omega_a-\omega_2,\omega_a) - O_{3b}(\omega_1,\omega_a-\omega_2,\omega_a)\Big\}\\
&\quad + \Big\{\frac{-2}{(\omega_1-\omega_a)^2} - \frac{d}{d\omega_1}\frac{2}{\omega_1-\omega_a}\Big\}\Big\{O_{3a}(\omega_a,\omega_1-\omega_a,\omega_2) - O_{3b}(\omega_a,\omega_1-\omega_a,\omega_2)\Big\}\\
&\quad + \Big\{\frac{-1}{\omega_1\omega_2} + \frac{d}{d\omega_1}\frac{1}{\omega_1}\Big\}O_{4a}(\omega_a,\omega_1-\omega_a,\omega_2) + \frac{d}{d\omega_2}\frac{1}{\omega_a}\,O_{4a}(\omega_1,\omega_a-\omega_2,\omega_a)\\
&\quad + \Big\{\frac{-1}{\omega_1\omega_2} + \frac{d}{d\omega_2}\frac{1}{\omega_2}\Big\}O_{4b}(\omega_1,\omega_a-\omega_2,\omega_a) + \frac{d}{d\omega_1}\frac{1}{\omega_a}\,O_{4b}(\omega_a,\omega_1-\omega_a,\omega_2)\bigg]\\
&\quad + \cdots\,.
\end{aligned} \tag{3.89}$$

Here the ellipses indicate terms involving operators O_{5-12} that have already occurred in $\mathbf{Q}_{4-11}$ and hence they are no longer important for determining the linear independent combinations. It is interesting to note that expanding the operator $\mathbf{Q}_*$ gives the same combination of O_{3a} and O_{3b} that appears in $\mathbf{Q}_2 - 4\omega_1/\omega_3\mathbf{Q}_3$, so even if we had not eliminated $\mathbf{Q}_*$ from the RPI basis, the implications for the homogeneous basis would be the same.

The three RPI operators in Eq. (3.89) have expansions in terms of six homogeneous operators O_1, O_2, O_{3a}, O_{3b}, O_{4a}, and O_{4b}, so there are three RPI relations. The Wilson coefficients of these six homogeneous operators are determined by three coefficients, $\hat{C}_{1,2,3}$ in the RPI basis. It is convenient to trade $\hat{C}_{1,2,3}$ for the three coefficients C_1, C_{3a}, and C_{3b}. The remaining coefficients C_2, C_{4a}, and C_{4b} are then determined by RPI. We find

$$
\begin{aligned}
C_2(\omega_1,\omega_2) &= -\frac{\bar{n}\cdot q}{n\cdot q}\Big\{\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1}\frac{d}{d\omega_1}+\frac{1}{\omega_2}\frac{d}{d\omega_2}\Big\}C_1(\omega_1,\omega_2)\,,\\
C_{4a}(\omega_1,\omega_3,\omega_2) &= -\frac{1}{2}C_{3a}(\omega_1,\omega_3,\omega_2)-\frac{\omega_1}{2\omega_2}C_{3b}(\omega_1,\omega_3,\omega_2)+\frac{\bar{n}\cdot q}{n\cdot q\,\omega_2\omega_3}\,C_1(\omega_1,\omega_2-\omega_3)\\
&\quad -\frac{\bar{n}\cdot q\,(\omega_2+\omega_3)}{n\cdot q\,\omega_3(\omega_2)^2}\,C_1(\omega_1+\omega_3,\omega_2)\,,\\
C_{4b}(\omega_1,\omega_3,\omega_2) &= -\frac{\omega_2}{2\omega_1}C_{3a}(\omega_1,\omega_3,\omega_2)-\frac{1}{2}C_{3b}(\omega_1,\omega_3,\omega_2)+\frac{\bar{n}\cdot q\,(\omega_1-\omega_3)}{n\cdot q\,\omega_3(\omega_1)^2}\,C_1(\omega_1,\omega_2-\omega_3)\\
&\quad -\frac{\bar{n}\cdot q}{n\cdot q\,\omega_1\omega_3}\,C_1(\omega_1+\omega_3,\omega_2)\,. \qquad (3.90)
\end{aligned}
$$

We have cross-checked the relation for C_2 with a tree-level matching computation. Note that $C_2(\omega_1,\omega_2)$ multiplies a matrix element that gives $\delta(\omega_1-\omega_2)$, while $C_{4a,4b}(\omega_1,\omega_3,\omega_2)$ multiplies a $\delta(\omega_1+\omega_3-\omega_2)$, and that we have used these δ-functions at various intermediate steps. That is, the result in Eq. (3.90) applies for a basis of operators, whose matrix elements have vanishing total derivatives.

Our operator bases can be compared to the flavor singlet and parity even basis of Jaffe and Soldate in Ref. [63] which has one operator at twist-2, and 12 operators at twist-4.[3] There is a simple correspondence between the 11 operators in our RPI

[3]The notation in Eq. (3.81) suggests that all quark bilinears are flavor singlet contractions if χ_n has multiple flavor components. To incorporate other possibilities for the flavor indices is straightforward [63]. We consider χ_n as a doublet of SU(2) flavor, or a triplet of SU(3) flavor, with elements

basis in Eq. (3.82,3.83) and the QCD operators in their basis. The correspondence is one-to-one for $\mathbf{Q}_1$, the four-quark operators $\mathbf{Q}_{8-11}$, and the operators $\mathbf{Q}_{2,3}$ that have one $\mathcal{G}_n^{\mu\nu}$. For the operators with two $\mathcal{G}_n^{\mu\nu}$'s we have four operators compared to their six, but the difference is accounted for by the way in which the twist towers are enumerated. We used continuous $\hat{\omega}_i$'s where even and odd symmetry under the interchange $\hat{\omega}_3 \leftrightarrow \hat{\omega}_4$ encodes two possible color structures with f^{ABC} and d^{ABC}, while Ref. [63] uses a discrete basis with integer powers of $(i\bar{n} \cdot D_n)$, where the choice of which operators to eliminate by integration by parts implies that the two color structures yield different operators. Our homogeneous basis has 14 operators up to twist-4, and most closely corresponds to an enumeration of an operator basis in terms of the so-called "good" quark and gluon fields. The good quark and gluon fields have been discussed in Refs. [7, 64, 70]. In this basis the power counting is manifest. From the three RPI relations in Eq. (3.90) the number of independent short distance Wilson coefficients is 11, and so encodes the same amount of information as the OPE basis from Ref. [63]. Note that there is no room in the traditional OPE in DIS for a correspondence with higher order operators with soft fields. In our language, the validity of the OPE for DIS with generic x implies that soft degrees of freedom are not needed, and one can consider that fluctuations from that region are reabsorbed into the collinear fields.

When the basis of bilinear quark operators is considered in the forward proton matrix element it can be reduced even further as discussed in detail in Ref. [42]. In this process it is found that the matrix elements of operators like O_2, O_{4a}, and O_{4b} do not provide independent information. Hence at this level the RPI relations in Eq. (3.90) do not appear to have practical implications.

χ_n^f. For photon currents one has a charge matrix in flavor space in each QCD current, which is $\hat{Q} = \mathrm{diag}(2/3, -1/3, -1/3)$ for SU(3). Thus, at leading order in the electromagnetic interactions one must simply introduce a $\hat{Q}^2$ in all bilinear-quark operators, O^1 through O^8 in Eq. (3.81). When counting the four-quark operators O^9 to O^{12} induced by photons we double the number of operators because there are two possibilities, $\hat{Q}^2 \otimes 1$ and $\hat{Q} \otimes \hat{Q}$. In this notation the flavor singlet contraction for the four-quark operators is $1 \otimes 1$. For the RPI basis of operators the analysis of flavor structures is identical, and hence flavor does not modify the constraints in Eq. (3.90).

3.7.4 Deep Inelastic Scattering for Gluons at Twist-4

Next let us consider the minimal basis for pure gluon DIS operators up to twist-4. We proceed in a similar manner to our construction for quarks, first writing the homogeneous basis and then the RPI basis to check if reparametrization invariance provides constraints on the homogeneous operators. The homogeneous basis is

$$\begin{aligned}
O_1 &= \mathrm{Tr}\big[(g\mathcal{B}_{n\perp})_{\omega_1}\cdot(g\mathcal{B}_{n\perp})_{\omega_2}\big]\,, \qquad (3.91)\\
O_2 &= \mathrm{Tr}\big[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}\mathcal{P}^2_\perp(g\mathcal{B}^{\perp}_{n\,\mu})_{\omega_2}\big]\,,\\
O_{3,4} &= \mathrm{Tr}\big[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}(g\mathcal{B}^{\nu}_{n\perp})_{\omega_2}\mathcal{P}^\alpha_\perp(g\mathcal{B}^{\beta}_{n\perp})_{\omega_3}\big]\Gamma^{1,2}_{\mu\nu\alpha\beta}\,,\\
O_{5,6} &= \mathrm{Tr}\big[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}(g\mathcal{B}^{\nu}_{n\perp})_{\omega_2}(g\mathcal{B}^{\alpha}_{n\perp})_{\omega_3}(g\mathcal{B}^{\beta}_{n\perp})_{\omega_4}\big]\Gamma^{1,2}_{\mu\nu\alpha\beta}\,,\\
O_{7,8} &= \mathrm{Tr}\big[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}(g\mathcal{B}^{\nu}_{n\perp})_{\omega_2}\big]\mathrm{Tr}\big[(g\mathcal{B}^{\alpha}_{n\perp})_{\omega_3}(g\mathcal{B}^{\beta}_{n\perp})_{\omega_4}\big]\Gamma^{1,2}_{\mu\nu\alpha\beta}\,,\\
O_9 &= \mathrm{Tr}\big[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}\mathcal{P}^\perp_\mu\mathcal{P}^\nu_\perp(g\mathcal{B}^{\perp}_{n\,\nu})_{\omega_2}\big]\,,
\end{aligned}$$

where $\Gamma^{1,2}_{\mu\nu\alpha\beta} = \{g_{\mu\nu}g_{\alpha\beta}, g_{\mu\alpha}g_{\nu\beta}\}$ and the traces are over color. Recall that the equations of motion (3.32) were used to eliminate the operators $gn\cdot\mathcal{B}_n$ and $in\cdot\partial_n(g\mathcal{B}^\mu_{n\perp})$. Again since the basis is designed for taking forward matrix elements we are free to integrate by parts and hence we do not consider $\mathcal{P}^{\mu\dagger}_\perp$. There is a third tensor structure, $\Gamma^3_{\mu\nu\alpha\beta} = g_{\mu\beta}g_{\alpha\nu}$, that can also be considered for O_{3-8}, but which can always be eliminated. For $O_{3,4}$ this is done using integration by parts and the cyclic trace, giving

$$\mathrm{Tr}\big[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}(g\mathcal{B}^{\nu}_{n\perp})_{\omega_2}\mathcal{P}^\alpha_\perp(g\mathcal{B}^{\beta}_{n\perp})_{\omega_3}\big]\Gamma^{3}_{\mu\nu\alpha\beta} = -O_4(\omega_2,\omega_3,\omega_1) - O_3(\omega_3,\omega_1,\omega_2)\,. \quad (3.92)$$

For O_{5-8} the cyclic property of the trace suffices to eliminate $\Gamma^3_{\mu\nu\alpha\beta}$ in an analogous manner. The operator $\mathrm{Tr}[(g\mathcal{B}^{\mu}_{n\perp})_{\omega_1}\mathcal{P}^\alpha_\perp(g\mathcal{B}^{\nu}_{n\perp})_{\omega_2}(g\mathcal{B}^{\beta}_{n\perp})_{\omega_3}]$ is also not needed in the basis because it can be put into the form of the operators O_3 and O_4. This is done by acting with the $\mathcal{P}^\alpha_\perp$ on the two $\mathcal{B}_\perp$'s to the right, using the cyclic property of the trace, and again noting that $O_{3,4}$ encode all orderings for the ω_i subscripts.

For forward spin averaged matrix elements the RPI basis of gluon operators up to twist-4 is

$$
\begin{aligned}
\mathbf{Q}_1 &= q_\nu q_\beta \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2}\big] g_{\mu\alpha}\,, \qquad (3.93)\\
\mathbf{Q}_2 &= \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2}\big] g_{\mu\alpha} g_{\nu\beta}\,,\\
\mathbf{Q}_3 &= q_\nu q_\sigma \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2} ig\mathcal{G}^{\rho\sigma}_{n,\hat\omega_3}\big] g_{\mu\alpha} g_{\beta\rho}\,,\\
\mathbf{Q}_4 &= q_\nu q_\beta q_\sigma \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2} i\partial_\mu ig\mathcal{G}^{\rho\sigma}_{n,\hat\omega_3}\big] g_{\alpha\rho}\,,\\
\mathbf{Q}_{5,6} &= q_\nu q_\beta q_\sigma q_\lambda \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2} ig\mathcal{G}^{\rho\sigma}_{n,\hat\omega_3} ig\mathcal{G}^{\tau\lambda}_{n,\hat\omega_4}\big] \Gamma^{1,2}_{\mu\alpha\rho\tau}\,,\\
\mathbf{Q}_{7,8} &= q_\nu q_\beta q_\sigma q_\lambda \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2}\big] \mathrm{Tr}\big[ig\mathcal{G}^{\rho\sigma}_{n,\hat\omega_3} ig\mathcal{G}^{\tau\lambda}_{n,\hat\omega_4}\big] \Gamma^{1,2}_{\mu\alpha\rho\tau}\,.
\end{aligned}
$$

Here we remove a possible operator $q_\nu q_\beta \mathrm{Tr}\big[(ig\mathcal{G}_{n,\hat\omega_1})^{\nu}{}_{\alpha}(i\partial)^2 ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2}\big]$ by writing $(i\partial)^2\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2} = (i\partial_\mu)i\partial^\mu\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2}$, and then using the Bianchi identity in Eq. (3.38) to rewrite this operator in terms of operator with two $\mathcal{G}_n$'s, plus $(i\partial_\mu)i\partial^\alpha\mathcal{G}^{\beta\mu}_{n,\hat\omega_2}$ and $(i\partial_\mu)i\partial^\beta\mathcal{G}^{\mu\alpha}_{n,\hat\omega_2}$. The last two terms are removed by the gluon equation of motion. There is no need to include the analog of $\mathbf{Q}_4$ with the $i\partial_\mu$ acting on $ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2}$, because it is related to $\mathbf{Q}_4$ by integration by parts up to a term, $i\partial_\mu ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_1}$ that reduces to other operators through the gluon equation of motion. Again the cyclic nature of the trace allows one to remove $\Gamma^3_{\mu\alpha\rho\tau}$ for $\mathbf{Q}_{5-8}$.

In order to consider the effect of charge conjugation on these basis one must consider the transformation of

$$
\int [d\hat\omega_j]\hat C_i(\hat\omega_j)\,\mathbf{Q}_i(\hat\omega_j)\,, \qquad \text{or} \qquad \int [d\omega_j] C_i(\omega_j)\, O_i(\omega_j)\,, \qquad (3.94)
$$

where $\hat C_i$ is the Wilson coefficient associated with $\mathbf{Q}_i$, and C_i the Wilson coefficient associated with O_i. We can impose constraints on $\hat C_i(\hat\omega_j)$ and $C_i(\omega_j)$ such that (3.94) is C-invariant. For example, note that under charge conjugation $\mathbf{Q}_3$ transforms into

$$
-q_\nu q_\sigma \mathrm{Tr}\big[ig\mathcal{G}^{\mu\nu}_{n,\hat\omega_3} ig\mathcal{G}^{\alpha\beta}_{n,\hat\omega_2} ig\mathcal{G}^{\rho\sigma}_{n,\hat\omega_1}\big] g_{\mu\alpha} g_{\beta\rho}\,, \qquad (3.95)
$$

so to make it C-invariant we impose that $\hat C_3(\hat\omega_1,\hat\omega_2,\hat\omega_3) = -\hat C_3(\hat\omega_3,\hat\omega_2,\hat\omega_1)$. Similar considerations apply to the homogeneous basis. For example, the combinations $O_3(\omega_1,\omega_2,\omega_3) - O_3(\omega_2,\omega_1,\omega_3)$ and $O_4(\omega_1,\omega_2,\omega_3) + O_4(\omega_1,\omega_3,\omega_2) + O_3(\omega_3,\omega_2,\omega_1)$ are even under charge conjugation.

Next we must expand the RPI basis in Eq. (3.93) in terms of the homogeneous basis in Eq. (3.91) to find possible constraints. We first expand $\mathbf{Q}_{5-8}$, they have only operators with four $g\mathcal{B}_\perp^\mu$'s, that is O_{5-8},

$$\mathbf{Q}_{5,6} = \frac{\omega_1\omega_2\omega_3\omega_4}{16}\, O_{5,6}\,, \qquad \mathbf{Q}_{7,8} = \frac{\omega_1\omega_2\omega_3\omega_4}{16}\, O_{7,8}\,. \tag{3.96}$$

Next we expand $\mathbf{Q}_{3,4}$ to find

$$\begin{aligned}
\mathbf{Q}_3 &= \frac{\omega_1\omega_3}{4(n\cdot q)}\big[- O_4(\omega_1,\omega_2,\omega_3) - O_4(\omega_3,\omega_1,\omega_2) - O_3(\omega_2,\omega_3,\omega_1)\big] + \cdots\,, \\
\mathbf{Q}_4 &= \frac{\omega_1\omega_2\omega_3}{8}\Big[O_4(\omega_1,\omega_2,\omega_3) - \frac{\omega_3}{\omega_1} O_3(\omega_2,\omega_3,\omega_1)\Big] + \ldots\,,
\end{aligned} \tag{3.97}$$

where we integrate over the repeated ω_a variable. The ellipses in Eq. (3.97) indicate terms involving operators O_{5-8} that have already occurred in $\mathbf{Q}_{5-8}$ and hence are no longer important for determining the linear independent combinations. Eq. (3.97) implies that O_3 and O_4 have Wilson coefficients that are independent of other operators in the basis. When we expand the remaining RPI operators $\mathbf{Q}_{1,2}$, we may also have terms with $O_{1,2,9}$ which have two $g\mathcal{B}_\perp^\mu$'s. We find

$$\begin{aligned}
\mathbf{Q}_1(\hat{\omega}_1,\hat{\omega}_2) &= \frac{\omega_1\omega_2}{4} O^1(\omega_1,\omega_2) + \frac{(\bar{n}\cdot q)}{4(n\cdot q)}\Big(2 - \frac{d}{d\omega_1}\omega_1 - \frac{d}{d\omega_2}\omega_2\Big) O^2(\omega_1,\omega_2) + \ldots\,, \\
\mathbf{Q}_2(\hat{\omega}_1,\hat{\omega}_2) &= \ldots\,,
\end{aligned} \tag{3.98}$$

where the ellipsis indicates terms involving operators O_{3-8} that have already occurred in $\mathbf{Q}_{3-8}$. The fact that O^9 does not occur in the expansion of any of the RPI operators indicates that it is ruled out by RPI (explaining why we listed it last in the basis). Furthermore, the operators O^1 and O^2 only enter in the combination obtained from expanding $\mathbf{Q}^1$, and so their Wilson coefficients are related by

$$\begin{aligned} C_2(\omega_1,\omega_2) &= \frac{\bar{n}\cdot q}{n\cdot q}\Big(2+\omega_1\frac{d}{d\omega_1}+\omega_2\frac{d}{d\omega_2}\Big)\frac{C_1(\omega_1,\omega_2)}{\omega_1\omega_2} \\ &= \frac{\bar{n}\cdot q}{\omega_1\omega_2\, n\cdot q}\Big(\omega_1\frac{d}{d\omega_1}+\omega_2\frac{d}{d\omega_2}\Big)C_1(\omega_1,\omega_2)\,. \end{aligned} \tag{3.99}$$

For the gluon DIS operators the RPI relations are similar to that for the quark basis, namely it is the collinear operators with $\mathcal{P}_\perp$'s that are constrained. This was also observed in Ref. [4] for the heavy-to-light currents at second order in the power counting. Overall there are eight homogeneous operators for spin-averaged gluon DIS up to twist-4, and seven independent Wilson coefficients.

An analysis of twist-4 gluon matrix elements was done in Ref. [6] using leading-order Feynman diagram, based on the methods of Ref. [42]. To the best of our knowledge, the complete linear independent bases of twist-4 pure glue operators given in Eq. (3.91) and (3.93) have not been given earlier in the literature.

3.7.5 Two Jet production: n-n' operators

An important application for operators with two-collinear directions, n-n', is the study of two jet phenomena and event shapes. The effective theory SCET has been used to study jets at leading order in the power expansion and various orders in the α_s expansion in Refs. [12, 13, 22, 47, 48, 59, 65, 71, 72, 90, 96]. Another interesting application is to describing parton showers with SCET [16, 17], where both leading and subleading operators with two-collinear directions play some role. In this Section we study the leading and first power suppressed quark operators with two-collinear directions. For two jet processes it is convenient to use the center-of-momentum (CM) frame where the two jets are back to back. In this frame we can take $n'=\bar{n}$ so that $n'\cdot n=2$. Our main interest will be in the operators that do not vanish in this frame, however part of our discussion touches on the additional operators that do.

To be concrete we consider operators that appear in two jet production from a virtual photon of momentum q^μ in $e^+e^-\to J_n J_{n'}$. In QCD the fundamental hadronic operator is the current $J^\mu = \bar{\psi}\gamma^\mu\psi$, which is conserved $\partial_\mu J^\mu = 0$ or $q_\mu J^\mu = 0$, is odd under charge-conjugation, and transforms as a vector under parity and time-reversal. To describe high-energy jet production this current is matched onto a series of SCET currents $J_\ell^{(k)}(\omega_i) \sim \lambda^k$ with Wilson coefficients $C_\ell(\omega_i)$,

$$J^\mu = \sum_{n,n'} \sum_{k=0}^{\infty} \sum_{\ell} \int [\prod_i d\omega_i]\, C_\ell(\omega_i)\, \big[J_\ell^{(k)}(\omega_i)\big]^\mu_{\text{2-jet}} \,. \tag{3.100}$$

Here k denotes the power in λ, the subscript ℓ denotes members of the basis at a given order, and the ω_i are the set of gauge invariant momentum fractions upon which the operator depends. We also sum over all collinear directions n and n', and the appropriate ones for a given computation are picked out by the jet-momenta in the states. Because of this sum we are free to swap $n \leftrightarrow n'$ when considering symmetry implications. The C, P, and T symmetry properties of $C_\ell(\omega_i)J_\ell^{(k)}(\omega_i)$ are the same as J^μ, and they also satisfy current conservation, $q_\mu[J_\ell^{(k)}(\omega_i)]^\mu = 0$. Finally, since the matching takes place at a hard scale where perturbation theory is valid, the SCET operators should have the same $LL + RR$ chirality as J^μ.

We first construct a basis of SCET operators that is homogeneous in the power counting and with even chirality. For the construction of this basis it is convenient to define

$$\begin{aligned} g_T^{\mu\nu} &= g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\,, & \gamma_T^\mu &= \gamma^\mu - \frac{q^\mu \not{q}}{q^2}\,, \\ r_-^\mu &= \frac{n\cdot q}{2}\, n'^\mu - \frac{n'\cdot q}{2}\, n^\mu\,, & r_+^\mu &= \frac{n\cdot q}{2}\, n'^\mu + \frac{n'\cdot q}{2}\, n^\mu\,, \end{aligned} \tag{3.101}$$

where r_-^μ is odd under $n \leftrightarrow n'$ and r_+ is even. We also define $s_\pm^\mu$ as $r_\pm^\mu$ with $n \to \bar{n}$ and $n' \to \bar{n}'$. Four of these objects are transverse to q^μ, $q_\mu g_T^{\mu\nu} = 0$, $q_\mu \gamma_T^\mu = 0$, and $q\cdot r_- = q\cdot s_- = 0$, which is helpful for satisfying current conservation. For constructing the homogeneous basis it suffices to consider the vectors $\{r_-, q, s_-, s_+\}$ in place of $\{n, \bar{n}, n', \bar{n}'\}$. When we specialize to the CM frame, $q_{n\perp}^\mu = q_{n'\perp}^\mu = 0$, $s_\pm^\mu = \mp r_\pm^\mu$, and the vector $r_+^\mu = q^\mu$, and hence r_+^μ and $s_\pm^\mu$ do not need to be considered.

In a general frame the LO operator is $\bar{\chi}_{\bar{n},\omega_1}\Gamma^\mu \chi_{n,\omega_2}$ with $\Gamma^\mu = \{\gamma_T^\mu, r_-^\mu \not{q}, r_-^\mu\, /$

$r_-, g_T^{\mu\nu} r_\nu^+ \not{q}, r_-^\mu \not{r}_+, g_T^{\mu\nu} r_\nu^+ \not{r}_-, g_T^{\mu\nu} r_\nu^+ \not{r}_+\}$ plus terms where r_+ or r_- are replaced by $s_\pm$. No terms with q^μ are allowed by current conservation. Things become much simpler if we focus on operators that are non-zero in the CM frame. In the CM frame $\bar{\chi}_{n',\omega_1}\not{q}\,\bar{\chi}_{n,\omega_2} = 0$, $\bar{\chi}_{n',\omega_1}\not{r}_-\bar{\chi}_{n,\omega_2} = 0$, and the vectors r_+ and $s_\pm$ become redundant, so there is only one operator at lowest order

$$J_0^{(0)} = \bar{\chi}_{n',\omega_1}\gamma_T^\mu \chi_{n,\omega_2}\,. \tag{3.102}$$

Here $\omega_i = \{\omega_1, \omega_2\}$ and for brevity we suppress the index μ on the LHS.

To construct a homogeneous basis at NLO we again consider only operators which are non-vanishing in the CM frame. In the CM frame we can take the total transverse momentum of the jet equal to zero, so we have the relations $\bar{\chi}_{n',\omega_1}\Gamma_\mu \mathcal{P}_\perp^\mu \chi_{n,\omega_2} = \bar{\chi}_{n',\omega_1}\Gamma_\mu \mathcal{P}_\perp^{\dagger\,\mu}\chi_{n,\omega_2} = 0$, with Γ_μ any gamma structure, and hence do not need to consider operators with a single $\mathcal{P}_\perp$. Again all operators with a $\not{q}$ or $\not{r}_-$ vanish, as do those with $q\cdot(g\mathcal{B}_{n\perp})$ and $r_-\cdot(g\mathcal{B}_{n\perp})$, and the analogs with $n \to n'$. Operators with three γ's can all reduce to operators with a single γ plus terms that are zero in the CM frame. This implies that at NLO there are only two operators

$$\begin{aligned} J_1^{(1)} &= r_-^\mu\, \bar{\chi}_{n',\omega_1}\gamma_\nu (g\mathcal{B}_{n\perp}^\nu)_{\omega_3}\chi_{n,\omega_2}\,, \\ J_2^{(1)} &= r_-^\mu\, \bar{\chi}_{n',\omega_1}\gamma_\nu (g\mathcal{B}_{n'\perp}^\nu)_{\omega_3}\chi_{n,\omega_2}\,. \end{aligned} \tag{3.103}$$

Linear combinations of these two SCET currents can both be made odd under charge conjugation by imposing appropriate conditions on their coefficients under $\omega_1 \leftrightarrow -\omega_2$.

To see if there are constraints on the Wilson coefficients we write down a basis of RPI operators up to NLO. The objects γ_T^μ and $g_T^{\mu\nu}$ are invariant under RPI and can be used for this construction, but the object $r_\pm^\mu$ cannot. We find the basis

$$\mathbf{J}_0^{(0)} = \bar{\Psi}_{n',\hat{\omega}_1}\gamma_T^\mu \Psi_{n,\hat{\omega}_2}\,, \tag{3.104}$$

$$\mathbf{J}_1^{(1)} = \bar{\Psi}_{n',\hat{\omega}_1} g_{\mu\alpha}^T \gamma_\beta ig\mathcal{G}_{n,\hat{\omega}_3}^{\alpha\beta}\Psi_{n,\hat{\omega}_2}\,, \qquad \mathbf{J}_2^{(1)} = \bar{\Psi}_{n',\hat{\omega}_1} g_{\mu\alpha}^T \gamma_\beta ig\mathcal{G}_{n',\hat{\omega}_3}^{\alpha\beta}\Psi_{n,\hat{\omega}_2}\,,$$

$$\mathbf{J}_3^{(1)} = g_{\mu\lambda}^T \bar{\Psi}_{n',\hat{\omega}_1}\gamma_\alpha q_\beta \big[i\partial_n^\lambda ig\mathcal{G}_{n,\hat{\omega}_3}^{\alpha\beta}\big]\Psi_{n,\hat{\omega}_2}\,, \quad \mathbf{J}_4^{(1)} = g_{\mu\lambda}^T \bar{\Psi}_{n',\hat{\omega}_1}\gamma_\alpha q_\beta \big[i\partial_{n'}^\lambda ig\mathcal{G}_{n',\hat{\omega}_3}^{\alpha\beta}\big]\Psi_{n,\hat{\omega}_2}\,,$$

$$\mathbf{J}_5^{(1)} = g^T_{\mu\lambda}\bar{\Psi}_{n',\hat{\omega}_1}\gamma_\alpha q_\beta\, ig\mathcal{G}^{\alpha\beta}_{n,\hat{\omega}_3} i\partial^\lambda_n \Psi_{n,\hat{\omega}_2}\,, \qquad \mathbf{J}_6^{(1)} = g^T_{\mu\lambda}\bar{\Psi}_{n',\hat{\omega}_1}(-i\overleftarrow{\partial}^\lambda_{n'})\gamma_\alpha q_\beta\, ig\mathcal{G}^{\alpha\beta}_{n',\hat{\omega}_3}\,\Psi_{n,\hat{\omega}_2}\,.$$

Here we do not write down RPI operators which vanish in the CM frame when expanded, such as $\bar{\Psi}_{n'} g^T_{\mu\nu} i\partial^\nu_n \Psi_n$ or operators with only the Dirac structure $\not{q}$. This set also includes three γ operators since in $\mathbf{J}_1^{(1)}$ replacing $\Gamma_{\mu\alpha\beta} = g^T_{\mu\alpha}\gamma_\beta$ by $\Gamma_{\mu\alpha\beta} = \gamma^\mu_T\gamma_\alpha\gamma_\beta$ gives an operator that vanishes in the CM frame, and any other order for the γ's is then redundant. The same is true for $\Gamma_{\mu\alpha\beta} = \gamma^T_\mu \not{q}\gamma_\alpha q_\beta - q^2 g^T_{\mu\alpha}\gamma_\beta$. Analogous arguments rule out three γ terms replacing the tensor in $\mathbf{J}_2^{(1)}$. There are no others operators with $i\partial^\lambda_n$ or $i\partial^\lambda_{n'}$ besides $\mathbf{J}^{(1)}_{3-6}$ at LO. To see why, notice that for the operators with $i\partial^\lambda_n$ and $i\partial^\lambda_{n'}$, only the contraction with $g^T_{\mu\lambda}$ has the potential to give a LO term. Momentum conservation requires $q^\lambda = i\partial^\lambda_n + i\partial^\lambda_{n'}$ and because $q^\lambda g^T_{\mu\lambda} = 0$, we can exchange $i\partial^\lambda_{n'}$ and $i\partial^\lambda_n$. The operators $\mathbf{J}^{(1)}_{3-6}$ correspond to keeping $i\partial^\lambda_n$ when we have a $\mathcal{G}^{\alpha\beta}_{n,\hat{\omega}}$, and $i\partial^\lambda_{n'}$ when we have a $\mathcal{G}^{\alpha\beta}_{n',\hat{\omega}}$.

The number of operators in Eq. (3.104) is greater than that in the homogeneous basis, and when expanded $\mathbf{J}_{0,1,2} \to J_{0,1,2}$. Thus the operators in Eqs. (3.102) and (3.103) are not connected by RPI. For two jet production the constraints imposed by considering the CM frame are strong enough that RPI provides no further information. (RPI could still constrain the homogeneous basis of operators in a general frame, but does not have practical implications for determining the basis of operators for an analysis to be carried out with homogeneous operators in the CM frame.)

3.7.6 Three Jet Production: n_1-n_2-n_3 operators

Here we analyze operators for three jet production. As in the two jet case, we consider production of jets from e^+e^- scattering through a virtual photon. To construct the minimal SCET basis needed for a matching we could proceed like in the previous cases, by writing down both the most general homogeneous basis and RPI basis consistent with the symmetry of the process, and expanding the RPI basis to find possible connections. In the two jet processes the interesting terms in the homogeneous basis are made of only two operators up to NLO. However, for three jets the homogeneous basis has many operators at LO since we have three distinct directions n_1, n_2 and

n_3. With only two directions we could greatly reduced the number of operators by focusing on the ones that do not vanish in the center of momentum frame, meaning those that do not vanish when $n_1 \to n$, $n_2 \to \bar{n}$, and $q_\perp \to 0$. This choice rules out many operators because of the relations $\bar{\chi}_{\bar{n}}\Gamma_\perp \not{n}\chi_n = \bar{\chi}_{\bar{n}}\Gamma_\perp \not{\bar{n}}\chi_n = 0$, where $\Gamma_\perp$ is a Dirac structure without $\not{\bar{n}}$ or $\not{n}$ factors. With three directions there is more freedom, for example the perpendicular direction of n_1-$\bar{n}_1$ is not the same as the one of n_2-$\bar{n}_2$, or of n_3-$\bar{n}_3$. Now to construct the homogeneous basis, we can use n_1, $\bar{n}_1$, n_2, $\bar{n}_2$, n_3, $\bar{n}_3$, γ^μ, so in the three jet case we have a bigger set of objects available.

On the other hand the RPI basis still has a reasonable number of objects, namely Ψ_{n_i}, $\mathcal{G}^{\mu\nu}_{n_i}$, $\partial^\mu_{n_i}$, γ^μ, and q^μ. The $\partial^\mu_{n_i}$ operators and q^μ are connected by momentum conservation, $i\partial^\mu_{n_1} + i\partial^\mu_{n_2} + i\partial^\mu_{n_3} = q^\mu$, so one of them can be eliminated. Hence we expect that reparametrization invariance will give a large number of connections on the homogeneous basis, so many in fact that it is not even convenient to write down the homogeneous basis. It is much quicker to just write only the RPI basis and expand it to determine a basis of allowed homogeneous operators.

The RPI basis for three jets at LO is made of two quarks fields and a gluon field (we do not consider here the case with pure gluon jets). n_1 and n_2 will be the directions of the quark and antiquark jets, and n_3 will be the direction of the gluon jet. As for the two jet case, because of current conservation, the only objects that can carry the vector index and are RPI invariant are $g_T^{\mu\nu}$ and γ_T^μ. The RPI basis is

$$
\begin{aligned}
\mathbf{J}_1 &= \bar{\Psi}_{n_1,\hat{\omega}_1}\gamma_\nu \not{q}\,\gamma^T_\mu q_\sigma ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3}\Psi_{n_2,\hat{\omega}_2}\,, &
\mathbf{J}_2 &= \bar{\Psi}_{n_1,\hat{\omega}_1}\gamma^T_\mu \not{q}\,\gamma_\nu q_\sigma ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3}\Psi_{n_2,\hat{\omega}_2}\,, \\
\mathbf{J}_3 &= \bar{\Psi}_{n_1,\hat{\omega}_1}\gamma_\sigma\gamma_\nu\gamma^T_\mu ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3}\Psi_{n_2,\hat{\omega}_2} &
\mathbf{J}_4 &= \bar{\Psi}_{n_1,\hat{\omega}_1}\gamma^T_\mu\gamma_\nu\gamma_\sigma ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3}\Psi_{n_2,\hat{\omega}_2}\,, \\
\mathbf{J}_5 &= g^T_{\mu\alpha}\bar{\Psi}_{n_1,\hat{\omega}_1}\gamma_\nu q_\sigma\, i\overleftarrow{\partial}^\alpha_{n_1} ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3}\Psi_{n_2,\hat{\omega}_2}\,, &
\mathbf{J}_6 &= g^T_{\mu\alpha}\bar{\Psi}_{n_1,\hat{\omega}_1}\gamma_\nu q_\sigma ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3} i\partial^\alpha_{n_2}\Psi_{n_2,\hat{\omega}_2}\,, \\
\mathbf{J}_7 &= g^T_{\mu\alpha}\bar{\Psi}_{n_1,\hat{\omega}_1}\not{q}\,\gamma_\nu\gamma_\sigma i\overleftarrow{\partial}^\alpha_{n_1} ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3}\Psi_{n_2,\hat{\omega}_2}\,, &
\mathbf{J}_8 &= g^T_{\mu\alpha}\bar{\Psi}_{n_1,\hat{\omega}_1}\not{q}\,\gamma_\nu\gamma_\sigma ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3} i\partial^\alpha_{n_2}\Psi_{n_2,\hat{\omega}_2}\,, \\
\mathbf{J}_9 &= g^T_{\mu\nu}\bar{\Psi}_{n_1,\hat{\omega}_1}\not{q}\, ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3} i\overleftrightarrow{\partial}_{12\sigma}\Psi_{n_2,\hat{\omega}_2}\,, &
\mathbf{J}_{10} &= \bar{\Psi}_{n_1,\hat{\omega}_1}\gamma^T_\mu\not{q}\,\gamma_\nu ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3} i\overleftrightarrow{\partial}_{12\sigma}\Psi_{n_2,\hat{\omega}_2}\,, \\
\mathbf{J}_{11} &= \bar{\Psi}_{n_1,\hat{\omega}_1}\gamma^T_\mu q_\nu ig\mathcal{G}^{\nu\sigma}_{n3,\hat{\omega}_3} i\overleftrightarrow{\partial}_{12\sigma}\Psi_{n_2,\hat{\omega}_2}\,. & &
\end{aligned}
\tag{3.105}
$$

For the first four operators we chose the Dirac structures $\{\gamma_\nu \not{q}\,\gamma^T_\mu q_\sigma,\ \gamma^T_\mu \not{q}\,\gamma_\nu q_\sigma,\ \gamma_\sigma\gamma_\nu\gamma^T_\mu,$

$\gamma_\mu^T\gamma_\nu\gamma_\sigma\}$ in order to simplify the transformation of the basis under charge conjugation. Since $\{\gamma_\mu^T, \not{q}\} = 0$ the sum of the first two structures gives $-g_{\mu\nu}^T\not{q}q_\sigma$, and using the antisymmetry of $\mathcal{G}_{n3}^{\nu\sigma}$ the sum of the last two gives $4g_{\mu\nu}^T\gamma_\sigma$, so structures with a $g_{\mu\nu}^T$ are redundant. Other three γ operators are also redundant. We have used the equations of motion and Bianchi identity in Eqs. (3.35,3.38) to eliminate $i\not{\partial}_{n_i}$, and momentum conservation to eliminate $i\partial_{n3}^\mu = q^\mu - i\partial_{n_1}^\mu - i\partial_{n_2}^\mu$. For the operators $\mathbf{J}_{9-11}$ we have a derivative contracted with $ig\mathcal{G}_{n3}^{\nu\sigma}$, and we can use the gluon equation of motion, $i\partial_{n3\sigma}\mathcal{G}_{n3}^{\sigma\nu} = (q_\sigma - i\partial_{n_1\sigma} - i\partial_{n_2\sigma})\mathcal{G}_{n3}^{\sigma\nu} = \ldots$, where the ellipsis denotes higher twist terms, to eliminate $(i\partial_{n_1}^\sigma + i\partial_{n_2}^\sigma)$ and leave only $i\overleftrightarrow{\partial}_{12}^\sigma \equiv (i\overleftarrow{\partial}_{n_1}^\sigma - i\partial_{n_2}^\sigma)$. Note that we cannot use the trick used in DIS for $\mathbf{Q}_*$, to eliminate $\mathbf{J}_{9-11}$, because here $i\partial_{n_{1,2}}^\alpha$ and $\mathcal{G}_{n3}^{\nu\sigma}$ have different collinear directions. Operators with two or more derivatives are redundant for the construction of the LO basis of RPI currents with one-vector index μ, and hence do not need to be considered.

We can match the three jet RPI basis of currents with the basis of homogeneous SCET currents by writing

$$\sum_{n_1,n_2,n_3}\sum_\ell \int [\prod_i d\hat{\omega}_i]\, \hat{C}_\ell(\hat{\omega}_i)\, [\mathbf{J}_\ell(\hat{\omega}_i)]^\mu = \sum_{n_1,n_2,n_3}\sum_\ell \int [\prod_i d\omega_i]\, C_\ell(\omega_i)\, [\mathcal{J}_\ell(\omega_i)]^\mu_{\text{3-jet}} + \ldots . \tag{3.106}$$

On the RHS the integration variable was changed using $\hat{\omega}_i = n{\cdot}q\,\omega_i$ and any additional $n\cdot q$ factors were absorbed into the Wilson coefficients $C_\ell(\omega_i)$. We can determine the currents $[\mathcal{J}_l(\omega_i)]^\mu_{\text{3-jet}}$ of the homogeneous basis, whose form is as in Eq. (3.70), by just expanding the currents (3.105) using Eqs. (3.65) and (3.67). This yields the homogeneous operator basis

$$\mathcal{J}_1 = \bar{\chi}_{n_1,\omega_1}(g\mathcal{B}_{n3}^\perp)_{\omega_3}\not{q}\,\gamma_T^\mu\,\chi_{n_2,\omega_2}\,, \tag{3.107}$$

$$\mathcal{J}_2 = \bar{\chi}_{n_1,\omega_1}\gamma_T^\mu\not{q}(g\mathcal{B}_{n3}^\perp)_{\omega_3}\chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_3 = \omega_3\,\bar{\chi}_{n_1,\omega_1}(g\mathcal{B}_{n3}^\perp)_{\omega_3}\not{n}_3\gamma_T^\mu\,\chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_4 = \omega_3\,\bar{\chi}_{n_1,\omega_1}\gamma_T^\mu\,\not{n}_3(g\mathcal{B}_{n3}^\perp)_{\omega_3}\chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_5 = \omega_1\, \bar\chi_{n_1,\omega_1} n_{1T}^{\mu} (g\mathcal{B}\!\!\!/_{n_3}^{\perp})_{\omega_3} \chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_6 = \omega_2\, \bar\chi_{n_1,\omega_1} n_{2T}^{\mu} (g\mathcal{B}\!\!\!/_{n_3}^{\perp})_{\omega_3} \chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_7 = \omega_1\, \bar\chi_{n_1,\omega_1} n_{1T}^{\mu}\, \not{a}\, \not{n}_3 (g\mathcal{B}\!\!\!/_{n_3}^{\perp})_{\omega_3} \chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_8 = \omega_2\, \bar\chi_{n_1,\omega_1} n_{2T}^{\mu}\, \not{a}\, \not{n}_3 (g\mathcal{B}\!\!\!/_{n_3}^{\perp})_{\omega_3} \chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_9 = \omega_3\, \bar\chi_{n_1,\omega_1}\, \not{a} \Big[n_3^{\nu} (g\mathcal{B}_{n_3\perp}^{\mu})_{\omega_3} - n_{3T}^{\mu} (g\mathcal{B}_{n_3\perp}^{\nu})_{\omega_3} \Big] (n_{2\nu}\omega_2 + n_{1\nu}\omega_1) \chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_{10} = \omega_3\, \bar\chi_{n_1,\omega_1} \gamma_T^{\mu}\, \not{a} \Big[n_3^{\nu} (g\mathcal{B}\!\!\!/_{n_3}^{\perp})_{\omega_3} - \not{n}_3 (g\mathcal{B}_{n_3\perp}^{\nu})_{\omega_3} \Big] (n_{2\nu}\omega_2 + n_{1\nu}\omega_1) \chi_{n_2,\omega_2}\,,$$

$$\mathcal{J}_{11} = \bar\chi_{n_1,\omega_1} \gamma_T^{\mu} (g\mathcal{B}_{n_3\perp}^{\nu})_{\omega_3} (n_{2\nu}\omega_2 + n_{1\nu}\omega_1) \chi_{n_2,\omega_2}\,,$$

where $n_{1T}^{\mu} = n_1^{\mu} - q^{\mu}(n_1{\cdot}q)/q^2$, $n_{2T}^{\mu} = n_2^{\mu} - q^{\mu}(n_2{\cdot}q)/q^2$ and $\not{a} = (\bar n_3\cdot q)\not{n}_3^{\mu}/2 + (n_3\cdot q)\bar{\not{n}}_3^{\mu}/2$. To simplify the results we did not bother to write out the terms with $q_\mu (g\mathcal{B}_{n_3\perp}^{\mu})$ in Eq. (3.107), which are terms that vanish in a frame where $q_{\perp n_3} = 0$. In some cases we have absorbed RPI factors in the Wilson coefficients $C_\ell(\omega_i)$ when carrying out the expansion.

The tree level matching from QCD to SCET for three jets comes from matching two Feynman diagrams in QCD onto the operator basis in Eq. (3.107), and is done at the hard scale $\mu = Q$. This gives

$$C_1 = C_6 = \frac{-2}{n_1{\cdot}n_3\,\omega_1\omega_3}\,, \quad C_2 = -C_5 = \frac{-2}{n_2{\cdot}n_3\,\omega_2\omega_3}\,, \quad C_{3,4} = C_{7-11} = 0\,. \qquad (3.108)$$

The results for these Wilson coefficients are invariant under type-III RPI as expected.

The above matching computation can be compared with the tree level SCET computations for parton showers in Ref. [17], where three final state jets are considered. To compare the calculations we take the two stages of matching of Ref. [17] both at $\mu = Q$, and we split the operators in Eqs. (27,28) of Ref. [17] into two parts, $\mathcal{O}_3 = \mathcal{O}_{3a} + \mathcal{O}_{3b}$ and $\mathcal{O}_3^{(2)} = \mathcal{O}_{3a}^{(2)} + \mathcal{O}_{3b}^{(2)}$. The matching computation of Ref. [17] used a frame $q_{n_2\perp} = 0$ for $\mathcal{O}_{3a}$ and $\mathcal{O}_{3a}^{(2)}$ and a frame $q_{n_1\perp} = 0$ for $\mathcal{O}_{3b}$ and $\mathcal{O}_{3b}^{(2)}$. With these frame choices, we confirm that $C_1\mathcal{J}_1 + C_6\mathcal{J}_6 = \mathcal{O}_{3a} + \mathcal{O}_{3a}^{(2)}$ and $C_2\mathcal{J}_2 + C_5\mathcal{J}_5 = \mathcal{O}_{3b} + \mathcal{O}_{3b}^{(2)}$, providing a cross-check on the results in Eq. (3.108).

3.7.7 Two Jets from Gluon Fusion: $gg \to q\bar{q}$ operators

Next we consider the example of the production of two quark jets from gluon fusion, which is relevant for the LHC. In this application we will see that RPI substantially constrains the number and structure of operators. This basis of operators have not yet been constructed. The factorization theorem for $pp \to 2\,\mathrm{jets}$ has been discussed in Ref. [68], and were also considered recently in Ref. [11] using SCET. SCET has also been used to resum electroweak Sudakov logarithms by solving RGE equations for four quark collinear operators in Refs. [35, 98, 99], and to consider Higgs production from pp collisions [1].

We consider the incoming gluons to be collinear in different directions, which is appropriate for the high energy collision of energetic protons at the LHC, and we assume that the final state jets have a large perpendicular momentum relative to the beam axis. Hence the final jets are described by two additional collinear directions, making four in total. Unlike our previous examples, here there is not an external q^μ vector, the hard interaction takes place entirely between strongly interacting particles. Hence this is an example of the case ii) discussed above Eq. (3.7).

Similarly to the three jets case, it is convenient to directly write the RPI basis without first writing the homogeneous basis, because the presence of four collinear directions imply that there are a large number of homogeneous operators, many of which are restricted by RPI. Due to the absence of an external hard vector q^μ in this process, in the definition of the currents we make use the RPI delta function factors of Eq. (3.15), $\hat{\Delta}_{km}$. The general formula for matching the RPI operators onto homogeneous operators is

$$i \sum_{n_1,n_2,n_3,n_4} \sum_{\ell} \int [\prod_{i,j} d\hat{\omega}_{ij}]\, \hat{C}_\ell(\hat{\omega}_i)\, [\prod_{km} \hat{\Delta}_{km}] \mathbf{Q}_\ell = i \sum_{n_1,n_2,n_3,n_4} \sum_{\ell} \int [\prod_{i} d\omega_i]\, C_\ell(\omega_i)\, [O_\ell(\omega_i)]_{gg\to q\bar{q}} + \ldots\,, \tag{3.109}$$

where we use the same manipulations needed to get Eq. (3.11). Note that here we have divided the RPI operators into the δ-functions in $\hat{\Delta}_{km}$ which depend on $\hat{\omega}_{km}$, and the remainder of the operator $\mathbf{Q}_\ell$ that does not. The starting point for building a basis for $\mathbf{Q}_\ell$ is the object $\bar{\Psi}_{n_1}\mathcal{G}^{\mu\nu}_{n_3}\mathcal{G}^{\alpha\beta}_{n_4}\Psi_{n_2}$. We assume a $LL+RR$ chirality for the quarks which is suitable when strong interactions produce massless quarks, and hence include either γ^λ or $\gamma^\lambda\gamma^\sigma\gamma^\tau$. Since the overall operator is a scalar, all the vector indices on the field strengths and on the Dirac structure must be contracted with $g_{\mu\nu}$'s or $i\partial^\mu_{n_i}$'s. We can use the equations of motion and Bianchi identity in Eqs. (3.35,3.37,3.38) to eliminate terms with $i\partial\!\!\!/_{n_i}$ in any operator, and terms with $\partial_{n_3\mu}\mathcal{G}^{\mu\nu}_{n_3}$ or $\partial_{n_4\mu}\mathcal{G}^{\mu\nu}_{n_4}$. In addition, momentum conservation implies $i\partial^\mu_{n_1}+i\partial^\mu_{n_2}+i\partial^\mu_{n_3}+i\partial^\mu_{n_4}=0$, and we will use this to eliminate all operators with an $i\partial_{n_1}$. This leaves twenty operators for the RPI basis

$$
\begin{aligned}
\mathbf{Q}_1 &= \bar{\Psi}_{n_1}\gamma_\beta g_{\nu\alpha}\, ig\mathcal{G}^{\mu\nu}_{n_3} i\partial_{n_4\mu} ig\mathcal{G}^{\alpha\beta}_{n_4}\Psi_{n_2}\,, &\qquad \mathbf{Q}_2 &= \bar{\Psi}_{n_1}\gamma_\mu g_{\nu\alpha} i\partial_{n_3\beta}\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4}\Psi_{n_2}\,,\\
\mathbf{Q}_3 &= \bar{\Psi}_{n_1}\gamma_\mu g_{\nu\alpha}\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\beta}\Psi_{n_2}\,, & \mathbf{Q}_4 &= \bar{\Psi}_{n_1}\gamma_\beta g_{\nu\alpha}\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\mu}\Psi_{n_2}\,,\\
\mathbf{Q}_5 &= \bar{\Psi}_{n_1}\gamma_\nu\gamma_\alpha\gamma_\beta\, ig\mathcal{G}^{\mu\nu}_{n_3} i\partial_{n_4\mu} ig\mathcal{G}^{\alpha\beta}_{n_4}\Psi_{n_2}\,, & \mathbf{Q}_6 &= \bar{\Psi}_{n_1}\gamma_\mu\gamma_\nu\gamma_\alpha i\partial_{n_3\beta}\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4}\Psi_{n_2}\,,\\
\mathbf{Q}_7 &= \bar{\Psi}_{n_1}\gamma_\mu\gamma_\nu\gamma_\alpha\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\beta}\Psi_{n_2}\,, & \mathbf{Q}_8 &= \bar{\Psi}_{n_1}\gamma_\nu\gamma_\alpha\gamma_\beta\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\mu}\Psi_{n_2}\,,\\
\mathbf{Q}_9 &= \bar{\Psi}_{n_1}\gamma_\alpha\, ig\mathcal{G}^{\mu\nu}_{n_3} i\partial_{n_4\mu} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\nu} i\partial_{n_2\beta}\Psi_{n_2}\,, \\
\mathbf{Q}_{10} &= \bar{\Psi}_{n_1}\gamma_\mu i\partial_{n_3\alpha}\, ig\mathcal{G}^{\mu\nu}_{n_3} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\nu} i\partial_{n_2\beta}\Psi_{n_2}\,.
\end{aligned}
\tag{3.110}
$$

The other ten operators $\mathbf{Q}_{11-20}$ have the same structure as Eq. (3.110) but with a trace over color for the gluon operators, for example $\mathbf{Q}_{11}=\bar{\Psi}_{n_1}\gamma_\beta g_{\nu\alpha}\mathrm{Tr}[\, ig\mathcal{G}^{\mu\nu}_{n_3} i\partial_{n_4\mu} ig\mathcal{G}^{\alpha\beta}_{n_4}]\Psi_{n_2}$. Note that $\mathbf{Q}_{1-10}$ have $\mathcal{G}_{n_3}$ to the left of $\mathcal{G}_{n_4}$, so one might think that there are ten more operators with the $\mathcal{G}$'s in the other order. However, in Eq. (3.109) we sum over $n_{3,4}$ and integrate over $d\hat{\omega}_3 d\hat{\omega}_4$, and hence include operators obtained from the interchange $n_3\leftrightarrow n_4$, $\hat{\omega}_3\leftrightarrow\hat{\omega}_4$. Recall that the directions n_i are only determined by the matrix elements. So if we consider a matrix element with gluons in the n and n' direction then there is a contribution from $n_3=n$, $n_4=n'$, and from $n_3=n'$, $n_4=n$. Other possible operators might be $\bar{\Psi}_{n_1}\gamma_\alpha\, i\partial_{n_3\beta} ig\mathcal{G}^{\mu\nu}_{n_3} i\partial_{n_4\mu} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\nu}\Psi_{n_2}$, $\bar{\Psi}_{n_1}\gamma_\mu i\partial_{n_3\alpha}\, ig\mathcal{G}^{\mu\nu}_{n_3} i\partial_{n_4\nu} ig\mathcal{G}^{\alpha\beta}_{n_4} i\partial_{n_2\beta}\Psi_{n_2}$ and similarly with the trace. We can use the Bianchi identity (3.38) to rule them out. For example, in the first operator we have implicitly already used the Bianchi identity for the $i\partial_{n_4\mu} ig\mathcal{G}_{n_4}^{\alpha\beta}$ term because we did not write operators with $i\not\partial_{n_4} ig\mathcal{G}_{n_4}^{\alpha\beta}$. But we can apply the Bianchi identity to $i\partial_{n_3\beta} ig\mathcal{G}_{n_3}^{\mu\nu}$, that is not connected with γ's. In this way we can write this operator in terms of $\mathbf{Q}_1$, $\mathbf{Q}_4$ and operators with three gluon fields. Note that we do not need to consider operators with $i\partial_n \cdot i\partial_{n'}$ since all these contracted derivatives are contained in the $\hat{\Delta}_{km}$'s.

A natural frame for analyzing $gg \to q\bar{q}$ is the CM frame with the choices $\bar{n}_1 = n_2$, $\bar{n}_2 = n_1$, $\bar{n}_3 = n_4$, $\bar{n}_4 = n_3$. We expand the currents (3.110) with an eye towards using them in this frame. Actually, only the condition $\bar{n}_3 = n_4$, $\bar{n}_4 = n_3$ is necessary to find the following operators

$$
\begin{aligned}
O_1 &= \omega_4\, \bar{\chi}_{n_1,\omega_1} \not{n}_4\, (g\mathcal{B}^{\mu}_{n_3\perp})_{\omega_3} (g\mathcal{B}^{\perp}_{n_4\mu})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_2 &= \omega_3\, \bar{\chi}_{n_1,\omega_1} \not{n}_3\, (g\mathcal{B}^{\mu}_{n_3\perp})_{\omega_3} (g\mathcal{B}^{\perp}_{n_4\mu})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_3 &= \omega_2\, \bar{\chi}_{n_1,\omega_1} (g\not{\mathcal{B}}_{n_3\perp})_{\omega_3} (g n_2\cdot \mathcal{B}^{\perp}_{n_4})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_4 &= \omega_2\, \bar{\chi}_{n_1,\omega_1} (g n_2\cdot \mathcal{B}^{\perp}_{n_3})_{\omega_3} (g\not{\mathcal{B}}_{n_4\perp})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_5 &= \omega_4\, \bar{\chi}_{n_1,\omega_1} \not{n}_4 (g\not{\mathcal{B}}_{n_3\perp})_{\omega_3} (g\not{\mathcal{B}}_{n_4\perp})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_6 &= \omega_3\, \bar{\chi}_{n_1,\omega_1} \not{n}_3 (g\not{\mathcal{B}}_{n_3\perp})_{\omega_3} (g\not{\mathcal{B}}_{n_4\perp})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_7 &= \omega_2\omega_3\omega_4\, \bar{\chi}_{n_1,\omega_1} \not{n}_3\, \not{n}_4\, (g\not{\mathcal{B}}_{n_3\perp})_{\omega_3} (g n_2\cdot \mathcal{B}^{\perp}_{n_4})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_8 &= \omega_2\omega_3\omega_4\, \bar{\chi}_{n_1,\omega_1} \not{n}_3\, \not{n}_4 (g n_2\cdot \mathcal{B}^{\perp}_{n_3})_{\omega_3} (g\not{\mathcal{B}}_{n_4\perp})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_9 &= (\omega_2)^2\omega_4\, \bar{\chi}_{n_1,\omega_1} \not{n}_4\, (g n_2\cdot \mathcal{B}^{\perp}_{n_3})_{\omega_3} (g n_2\cdot \mathcal{B}^{\perp}_{n_4})_{\omega_4} \chi_{n_2,\omega_2}\,, \\
O_{10} &= (\omega_2)^2\omega_3\, \bar{\chi}_{n_1,\omega_1} \not{n}_3\, (g n_2\cdot \mathcal{B}^{\perp}_{n_3})_{\omega_3} (g n_2\cdot \mathcal{B}^{\perp}_{n_4})_{\omega_4} \chi_{n_2,\omega_2}\,.
\end{aligned}
\tag{3.111}
$$

O_{11-20} have the same structure of (3.111) but with a trace over color of the two gluon operators. O_i is given by the expansion of $\mathbf{Q}_i$ for $i = 1, 2, 5, 6$, by the expansion of a suitable linear combination of $\mathbf{Q}_i$ and $\mathbf{Q}_{i-1}$ for $i = 3, 6$, and of $\mathbf{Q}_i$ and $\mathbf{Q}_{i-3}$ for $i = 4, 8$. $O_{9/10}$ are given by the expansion of a suitable linear combination of $\mathbf{Q}_{9/10}$, $\mathbf{Q}_{1/2}$ and $\mathbf{Q}_{4/3}$. In some cases we have absorbed reparametrization invariant prefactors that appear in the expansion into the Wilson coefficients $C_l(\omega_i)$. By using momentum conservation it is possible to reduce these ten operators to just four independent operators at leading order in SCET.[4]

It is straightforward to carry out the matching from QCD onto the SCET operators in Eq. (3.111). At tree level there are three Feynman diagrams. The amplitude squared is also known analytically at one-loop [43], and a full matching computation at this order involves regulating infrared singularities in the same way for the loops in QCD and SCET before subtracting. The only point to be careful about is the sum over the n_i's in Eq. (3.109), since definite values for these n_i's should be determined by the states. For example, if we consider the tree level $gg \to q\bar{q}$ matrix element of O_1 with perpendicular polarization for the gluons then

$$\begin{aligned}
&\Big\langle q_{n_1'}(p_1')\bar{q}_{n_2'}(p_2)\Big| \sum_{n_i} \int [d\omega_j] i C_1(\omega_1,\omega_3,\omega_4,\omega_2) O_1(\omega_j) \Big| g^\perp_{n_3'}(p_3') g^\perp_{n_4'}(p_4')\Big\rangle \\
&= i g^2 C_1(\omega_1',\omega_3',\omega_4',\omega_2')\omega_4' \left[\epsilon^{\mu A}_{n_3'\perp}\epsilon^{B}_{n_4'\perp\mu}\right]\left[\bar{u}_{n_1'} \not{n}_4' T^A T^B u_{n_2'}\right] \\
&\quad + i g^2 C_1(\omega_1',\omega_4',\omega_3',\omega_2')\omega_3' \left[\epsilon^{\mu A}_{n_3'\perp}\epsilon^{B}_{n_4'\perp\mu}\right]\left[\bar{u}_{n_1'} \not{n}_3' T^B T^A u_{n_2'}\right] . \qquad (3.112)
\end{aligned}$$

The two terms come from the cases $n_{3,4} = n'_{3,4}$ and $n_{3,4} = n'_{4,3}$ respectively. Therefore to determine the C_ℓ's it suffices to compute terms contributing to the color structure $T^A T^B$ in QCD, which at tree level gives

$$\begin{aligned}
&C_1 = \frac{-1}{(n_3\cdot n_4)\omega_3\omega_4}\,, \quad C_2 = \frac{1}{(n_3\cdot n_4)\omega_3\omega_4}\,, \quad C_3 = \frac{2}{(n_2\cdot n_4)\,\omega_2\omega_4}\,, \quad C_5 = \frac{1}{(n_2\cdot n_4)\,\omega_2\omega_4}\,, \\
&C_4 = C_{6-20} = 0\,. \qquad (3.113)
\end{aligned}$$

Note that the results for the C_ℓ's are invariant under type-III RPI transformations as expected, and that in the frame used for our computation $n_3 \cdot n_4 = 2$. We have confirmed that a consistent result is obtained by considering the $T^B T^A$ terms. Eq. (3.112) expresses the interesting fact that with distinct collinear directions for all final state particles, only the color ordered QCD amplitudes are needed for the matching which

[4]We thank W. Waalewijn for his explicit derivation of this point.

determines the SCET Wilson coefficients.

3.8 Conclusion

In SCET the momenta of collinear particles are decomposed with light-like vectors n^μ and $\bar{n}^\mu$, where $\vec{n}$ is close to the direction of motion. The vectors n^μ and $\bar{n}^\mu$ are required to define collinear operators that have a definite order in the power counting. However, there is a freedom in defining n and $\bar{n}$, which leads to reparametrization constraints. The decomposition of operators in the theory must satisfy these constraints in order to be consistent. This reparametrization invariance gives nontrivial relations among the Wilson coefficients of collinear operators occurring at different orders in the power counting, and for situations with multiple collinear directions gives constraints on the form of operators making up a complete basis.

In this chapter we have constructed objects that are invariant under both collinear gauge transformations and reparametrization transformations, a superfields Ψ_{n_i} for fermions and a superfield $\mathcal{G}_{n_i}^{\mu\nu}$ for gluons. Here the subscript n_i denotes an equivalence class of light-like vectors under RPI. The superfields are invariant under collinear gauge transformations through a reparametrization invariant Wilson line $\mathcal{W}_{n_i}$ that is the generalization of the usual W_{n_i}. We constructed RPI operators out of these superfields by introducing reparametrization invariant δ-functions. The δ-functions act on the RPI operators to pick out large momenta, and are convoluted with hard Wilson coefficients that must be computed by matching computations. The power of the RPI operators is that they encode information about the minimal basis of Wilson coefficients. However, they do not have a definite power counting order. By expanding them in λ one obtains a minimal basis of operators with a good power counting, where all constraints on the Wilson coefficients are made explicit. The final basis of operators with a good power counting involves a two-component field χ_{n_i} for quarks, a field $\mathcal{B}^\mu_{n_i\perp}$ for the two physical gluon polarizations, derivatives $\mathcal{P}^\mu_{n_i\perp}$, and delta functions $\delta(\omega - \overline{\mathcal{P}}_{n_i})$ that pick out the large momenta of these collinear fields. That is it. Other field components such as $n_i \cdot \mathcal{B}_{n_i}$, and other derivatives such as

$in_i \cdot \partial_{n_i}$, are eliminated from the purely collinear operator basis using the equations of motion.

This procedure was applied to several processes. We studied spin-averaged DIS for quarks at twist-4, as a means of testing our setup in a framework where the power suppressed basis of operators is well understood. We then constructed a minimum basis of pure glue operators for DIS at twist-4. These applications involve a single collinear direction. For processes with multiple collinear directions we considered operator bases for jet production. Useful constraints from RPI were not found for the first power suppressed operators in $e^+e^- \to 2\,\text{jets}$. On the other hand, already at leading order in the power counting, RPI provided important constraints on the complete basis of operators for $e^+e^- \to 3\,\text{jets}$ with three distinct collinear directions. RPI was also very useful in constructing a complete basis of operators for gluon fusion producing two quark initiated jets, where there are four collinear directions. In this case the process of interest is $pp \to 2\,\text{jets}$, which will be studied at the LHC. We expect the complete bases of operators constructed here will be a useful ingredient in the study of factorization theorems for this process. The steps we used to construct complete basis will also be useful when considering factorization for processes with more jets in the final state. In general we found that RPI becomes more powerful for processes involving more jets, essentially because the number of vectors n_i and $\bar{n}_i$ proliferates faster than the number of objects that must be considered to build the RPI basis.

An interesting observation discussed in Section 3.7.7 is that when matching from QCD onto SCET operators describing multiple collinear directions n_i, the Wilson coefficient is determined by the color ordered QCD amplitude. Since results for multi-leg QCD amplitudes are often expressed in a color ordered form, this should simplify the matching of QCD amplitudes onto SCET.

Chapter 4

Parton Showers to NLO

4.1 Introduction

A final state shower MC is based on the "strongly-ordered limit" which describes the leading log contribution (accounting for soft emission by angular ordering or other approximations). In this kinematic configuration, each radiated particle comes off much more collinear to its parent than the previous one, a situation that can be formulated in terms of perpendicular momenta or invariant masses, *i.e.*

$$q_{0\perp} \gg q_{1\perp} \gg q_{2\perp} \gg \dots, \qquad \text{or} \qquad q_0^2 \gg q_1^2 \gg q_2^2 \gg \dots . \tag{4.1}$$

Furthermore, and important for practical computation, each emission is independent of the previous one to leading log order. Thus, if we have calculated the differential cross section for $(i-1)$-parton emission, $d\sigma_{i-1}$, then we can obtain the i-parton case as

$$d\sigma_i = \frac{P^{(0)}_{i\to jk}}{q^2_{i-1}} d\sigma_{i-1}, \tag{4.2}$$

where $P^{(0)}$ is the leading order (LO) "splitting function" that captures the probability of the $(i-1)^{th}$ emitted parton to split into two others, jk, and q^2_{i-1} is its virtuality. Thus, we can formulate the process in terms of a probabilistic Markov chain of $(i-1)$ $1 \to 2$ particle splittings. The probabilities are determined by the functions $P^{(0)}_{i\to jk}$, which are related to Altarelli-Parisi kernels. As an example, for $q \to qg$ in QCD, after averaging and summing over spins,

$$P^{(0)}_{q\to qg} = \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z}, \tag{4.3}$$

where z is the momentum fraction of the daughter with respect to the parent. This classical, probabilistic process gives rise to the parton shower algorithms used by event generators to model radiation through Monte Carlo, such as Pythia [91, 92], Herwig [5, 38], and Sherpa [54] (although by now more sophisticated generalizations such as the dipole shower [32, 33] are becoming more popular). Given some initial virtuality, q_0^2, and an initial momentum fraction, x_0, MCs generate the virtuality and the momentum fraction of the daughter particle after the spitting. The virtuality is determined by a Sudakov factor, $\Delta(q^2, q_0^2)$, which gives the probability of a parton to evolve from q_0^2 to q^2 without branching,

$$\Delta(q^2, q_0^2) = \exp\left[- \int_{q_0^2}^{q^2} \frac{dq'^2}{q'^2} \int dx \, \frac{\alpha_s}{2\pi} P^{(0)}_{jk}(x) \right]. \tag{4.4}$$

The traditional LL parton shower makes a difficult problem tractable, but has some shortcomings related to its dependence on the leading log approximation. Even though the splitting functions only dominate in soft and collinear limits, the shower is used everywhere in order to generate events that cover the full phase space. In addition, since each emission is independent from the previous one in the shower, the LL approximation does not include any spin or color correlations. Furthermore, the procedure is classical and the interference between different amplitudes is only probed beyond leading log.

The hierarchy of scales in the parton shower makes it amenable to an effective field theory treatment. Since the shower regime occurs for fields in the soft and collinear regions, we can describe it with Soft-Collinear Effective Theory (SCET) [8, 10, 14, 18]. SCET is the appropriate effective theory for studying parton showers because it is designed to reproduce exactly the limit of soft and collinear particles. Moreover, SCET is organized in an expansion of power counting parameter that makes it convenient for classifying all corrections. The first work on parton showers using SCET was Refs. [16, 17], where the authors showed how the splitting function and the Sudakov factor emerge naturally in SCET. They reproduced the LL parton showers using SCET, and showed how higher order virtual corrections can be encoded by matching onto Wilson coefficients in the effective theory. Unfortunately they introduced choices and approximations at several points along the way, which makes our task of identifying the full set of corrections beyond LL difficult, so some modification of their setup will be required.

Before discussing our approach, we give here a brief discussion of literature on improvements to the basic shower Monte Carlo picture. To begin with, consider a simple setup where one declares that an artificial scale μ_0 divides collinear from hard radiation. We describe emissions above μ_0 through fixed-order tree-level calculations, and those beneath by running shower Monte Carlo. Each regime would get an accurate treatment, but interfacing the two results in leading-log introduces sensitivity to the scale μ_0. This is because the LO (in α_s) result contains no LL-resummation. The CKKW algorithm [31] defines a method to include this information. One distributes the particles in an event according to the probabilities given by the exact tree-level matrix element, with μ_0^2 as a lower cutoff on the virutality between any two particles. One then clusters the event using the k_T algorithm [29] to determine the splitting virtualities, q_i^2. With these scales in hand, one reweights the event by multiplication by appropriate Sudakov factors, as well as factors of $\alpha_s(q_i)/\alpha_s(Q)$, where Q is some hard scale. We can then run a parton shower algorithm on these amplitudes, vetoing any splitting virtuality harder than μ_0^2 to avoid double counting. The resulting distribution depends on μ_0 only at the subleading order $\alpha_s(\alpha_s \ln^2(Q/\mu_0))^m$.

Another important effect concerns soft gluons which are also kinematically enhanced. Collinear emissions reinforce the picture of partonic radiation as an isolated jet as they get distributed within some narrow cone about the original hard parton. Apriori soft emissions have no preferred direction and can communicate between elements of the shower. Fortunately, azimuthal averaging confines these emissions to lie within the cone containing the nearest collinear gluon as well as the showering quark or gluon. This effect is called angular-ordering and can be accomodated by methods such as evolving the shower by decreasing angle monotonically, as is done in Herwig [81], or by enforcing it with a veto in a mass-ordered shower (rightmost expression in Eq. (4.1)), which is an option in Pythia [91]. Additional considerations treated in shower programs include putting α_s at the scale of each splitting, and encoding momentum conservation at each vertex, which give the parton shower information beyond an analytic LO/LL calculation. All of these are treated in different fashions by different MC codes.

There are of course further corrections to include to go to NLO in α_s and NLL in kinematic logs. The most effort to date has gone to working out the NLO/LL contribution to incorporate one-loop-corrected amplitudes. Adding α_s corrections involves the numerical challenge of combining real and virtual results which separately diverge. The basic resolution is to extract the pole-portion of the real emission of i-partons and include it along with the virtual contributions to the $i-1$ case. Unfortunately this does not sum leading logs. One cannot blindly extend the CKKW procedure to NLO/LL, as it leads to double-counting problems; the Sudakov factors in the reweighting contain a portion of the one-loop contributions. Thus seperately adding on the full one-loop result would clearly double count.

There are two main solutions to the NLO/LL merging problem. The older of the two, MC@NLO [52], works by means of subtraction, finding the places where the Sudakovs will contribute at NLO, and removing the splitting function contribution. While conceptually clear, this has numerical and theoretical complications. The full amplitude and splitting function portions are calculated separately before subtraction, which is time-consuming. Furthermore, since the subtractions occur for the amplitude squared, one cannot guarantee positivity of the result. Hence one deal with negatively weighted events (and in some pathological cases negative cross-sections, see eg. [20]). To avoid the computational difficulties of process-by-process subtraction and the problem of negative weights, an alternative is the POWHEG algorithm [82]. It keeps the IR-safe NLO cross-section manifest, and defines a Sudakov factor based on a modified splitting function to handle LL resummation. In this way, it makes use of quantities already obtained in the fixed order NLO calculation, requiring fewer additional steps for its implementation for each known process. The conservation of probability obeyed by the splittings and related Sudakov factor avoid double countings and give back $\sigma_{\rm NLO}$ upon integration.

Another approach to go beyond LO/LL is to incorporate subleading logs by including in the Sudakov the contribution of the $\mathcal{O}(\alpha_s^2)$ corrections to the Altarelli-Parisi splitting kernels, $P_{qq}^{(1)}$. Unfortunately, doing this alone would fail to conserve probability if one did not also correct the probability for real emission in Eq. (4.3). Some of the subleading contributions take the form of $1 \to 3$ splittings, requiring a modification of the usual $1 \to 2$ algorithm. The KRKMC group aims to incorporate these subleading contributions in shower MC [60, 61, 93]. Similar to CKKW, their corrections take the form of a multiplicative reweighting. For a particular configuration of partons in phase space, they reweight by a factor that includes the insertion of $1 \to 3$ "defects" as loop-corrected $1 \to 2$ splittings that account for the affects of $P_{qq}^{(1)}$. If ρ is the fully differential cross-section, they define a corrected weight for n partons, w_n as:

$$w_n = \frac{\rho_{\rm LO}(k_1, \dots, k_n) + \sum_{r=1}^{n/2} \rho_{{\rm N}^r{\rm LO}}(k_1, \dots, k_n)}{\rho_{\rm LO}(k_1, \dots, k_n)}, \tag{4.5}$$

where r determines the number of defect insertions in any configuration.

In this chapter we set up an EFT framework to classify and study perturbative and power corrections to parton showers. The results of this work will be presented in Ref. [21]. While the ultimate goal is to facilitate the implementation of a complete NLL/NLO parton shower algorithm, our goals here are much more modest. The main objective of our work is to explain a computational framework where we can enumerate and classify all the needed theoretical ingredients in a NLO/NLL shower at a theoretical rather than algorithmic level. When carrying out explicit computations within this framewrok we focus primarily on shower power corrections in the fully differential cross section for an arbitrary number of emissions, that is

$$\frac{d\sigma}{d\vec{p}_1^3 \cdots d\vec{p}_n^3}\,, \tag{4.6}$$

for n-partons. Similarly to [17], we use an operator approach based on SCET. A main issue to resolve is taking into account different possibilities for the kinematic configuration of subsequent emissions, to go beyond the strong ordering described in Eq. (4.1). We overcome this issue by setting up a tower of related soft collinear effective fields theory called SCET$_i$, which also helps us deal with several technical obstacles. We formulate the shower description as a standard matching procedure between operators in different SCET$_i$. Power corrections are encoded by performing matching computations at subleading order in the kinematic power expansion between different regions. The hierarchy between regions is expressed by a power counting parameter $\lambda \ll 1$. These power corrections modify the structures that initiate the shower, branching probabilities, as well as opening up the $1 \to 3$ splitting channel. Virtual perturbative α_s corrections are encoded by performing matching calculations beyond tree level between SCET$_i$ theories. Finally, corrections to the Sudakov no-branching probabilities are encoded through anomalous dimensions of leading and subleading operators at the appropriate order within the different SCET$_i$s. We will carry out the necessary computations for the power corrections, and a subset of the required calculations for anomalous dimensions occuring for operators beyond the leading shower. This analysis includes the leading corrections to the shower from interference and from spin correlations. As much as possible we attempt to give pointers for additional computations that are needed in places where our analysis is incomplete. For example, to simplify things we have not treated color correlations since doing so increases the basis of operators and the number of computations, but does not change the conceptual setup.

The outline of this chapter is as follows. We review the Bauer-Schwartz SCET shower method in subsection 4.2.1 and discuss the technical obstructions to extending it to the level needed to determine the desired power corrections. We present our

SCET_i framework in subsection 4.2.2 to resolve these issues. In subsection 4.3 we analyze the LL shower in the SCET_i framework, and show that the transition between SCETs, $\text{SCET}_i \to \text{SCET}_{i+1}$, can be encoded by operator replacement rules on single parton collinear fields. Soft emissions in SCET_i are discussed, and we summarize the correspondence between SCET_i objects and LL shower ingredients to show how the mapping works. In subsection 4.4, we use the SCET_i formulation to classify and compute various corrections to the shower to $\mathcal{O}(\lambda^2)$ in the cross-section. Two main categories of branching corrections emerge, which we refer to as hard scattering and jet-substructure corrections. We also discuss ingredients needed for renormalization group evolution, and derive all the LL anomalous dimensions for our subleading operators. A summary of the NLL/NLO corrections is presented as a table mapping ingredients needed for the subleading shower to those in SCET. Conclusions are given in subsection 4.6.

Many details are relegated to the appendices B and C. We describe finite reparametrization transformations in appendix B.1, we use RPI in our matching computations to disentangle kinematic choices from kinematic power corrections. Details on the matching of QCD $\to$ SCET_1, $\text{SCET}_1 \to \text{SCET}_2$, and $\text{SCET}_2 \to \text{SCET}_3$ with generalization to SCET_i to SCET_{i+1} can be found in appendices B.2, B.3, and B.4, respectively. Appendix C contains a cross-check on our results, where we integrate a subset of our power suppressed terms to rederive the abelian terms in $P^{(1)}_{q\to qg}$, namely the $\mathcal{O}(\alpha_s)$ correction to the $q \to qg$ splitting function [39].

Before we start analyzing parton showers in SCET, we introduce the operator notation we will use for this chapter. Most of our discusion will involve interactions with collinear fields, and we use the notation χ_n for quarks and $\mathcal{B}^{\mu}_{n\perp}$ for gluons defined in subsection 2.1.1. We proved in subsection 3.4 that we need only three key building blocks to construct operators in SCET, these are: χ_n, $\mathcal{B}^{\mu}_{n\perp}$, and $\mathcal{P}^{\mu}_{n\perp}$, each of which carry a power counting scaling of $\mathcal{O}(\lambda)$. A general notation for the i-parton operators we will consider in this chapter is

$$\mathcal{O}^{(j,k,\ell)}\big(n_1^{[\ell_1]},\dots,n_{j+k}^{[\ell_{j+k}]}\big) = \Bigg[\prod_{a=1}^{j/2}(\mathcal{P}_{n_a\perp})^{\ell_a}\chi_{n_a}\Bigg]\Bigg[\prod_{b=j/2+1}^{j}(\mathcal{P}_{n_b\perp})^{\ell_b}\bar{\chi}_{n_b}\Bigg]\Bigg[\prod_{c=1}^{k}(\mathcal{P}_{n_c\perp})^{\ell_c}g\mathcal{B}_{n_c\perp}\Bigg], \tag{4.7}$$

where the number of partons is the sum of quarks and gluons, $j+k=i$, and the total number of $\perp$ derivatives is $\ell=\sum_{m=1}^{j+k}\ell_m$. The arguments in brackets denote the collinear direction for each parton field, with the superscript denoting the number of $\perp$ derivatives acting upon it. The power counting scaling of these operators is $\mathcal{O}^{(j,k,\ell)}\sim\lambda^{j+k+\ell}$. These operators are tensors in the space of spinors and Lorentz vectors, and the indices get contracted with structures contained in the Wilson coefficient C for the operator. If $C\mathcal{O}$ is a Lorentz scalar then j is even. Since the collinear fields carry a label referring to a specific light-cone vector, these operators describe particles in a specific region of phase space. We have seen in chapter 3 that the Wilson coefficients and the operators are in general convoluted. In this chapter, because we only treat tree-level quantities there are now convolutions between Wilson coefficients and operators and we do not need to use the notation in Eq.(3.5).

4.2 Obtaining the Parton Shower with SCET

4.2.1 Bauer-Schwartz Method

The original application of SCET to study and improve the parton shower was carried out in Refs. [16, 17] by Bauer & Schwartz. The main reasons why SCET is useful here are:

- The SCET fields, soft and collinear quarks and gluons, encode the infrared contributions which are exactly where the parton shower amplitudes have their dominant contributions in phase space.

- Since SCET is improvable order-by-order in the kinematic expansion parameter, λ, one has the potential to systematically correct the shower.

We will give a short overview of the Bauer-Schwartz approach, and then discuss the complications that arise when trying to extend the analysis to NLO in the λ expansion. In this subsection we will use notation that is not found elsewhere in this chapter to retain consistency with Refs. [16, 17].

The procedure of [16, 17] starts by constructing i-parton operators, $\mathcal{O}_i$, through matching SCET to QCD at a hard scale. For example, their $\mathcal{O}_2$ will equal $\mathcal{O}^{(2,0,0)}(n_1,\, n_2)$ in the notation of Eq. (4.7), and $\mathcal{O}_3$ will be $\mathcal{O}^{(2,1,0)}(n_1,\, n_2,\, n_3)$. As we run $\mathcal{O}_i(\mu)$ down, the leading log renormalization group evolution (LL RGE) does not mix operators and the exponential evolution operator encodes the no-branching probability. The evolution continues until another parton becomes apparent at a scale $\mu = p_T$.

If we have an i-parton operator, $\mathcal{O}_i = \mathcal{O}^{(j,i-j,0)}(n_1, \ldots, n_i)$ with all n's distinct, then it has the RG solution $\mathcal{O}_i(Q) = U^{(j,i-j,0)}(Q, \mu)\mathcal{O}_i(\mu)$ with

$$U^{(j,i-j,0)}(Q,\, \mu) \;=\; \exp\left[-\int_{\mu}^{Q} \frac{d\mu'}{\mu'}\gamma^{(j,i-j,0)}(\mu')\right], \tag{4.8}$$

where $\gamma^{(j,i-j,0)}$ is the operator's anomalous dimension. The leading-log resummation effects of the Sudakov factor in the PS enter through one-loop operator running in SCET, as dictated by the cusp anomalous dimension. The one-loop cusp portion is especially easy to calculate in SCET as it depends solely on the number of collinear fields, even though the loop calculations do generically involve soft loops as well [16, 17],

$$\gamma_{\mathrm{LL}}^{(n_q,n_g,0)}(\mu) \;=\; -\frac{\alpha_s}{\pi}\left[\frac{n_q}{2}C_F + \frac{n_g}{2}C_A\right]\log\frac{\mu^2}{Q^2}\,. \tag{4.9}$$

This form of the kernel yields a product of Sudakov factors which are the no-branching probabilities for each parton in the operator:

$$U_{\mathrm{LL}}^{(j,i-j,0)}(Q,\, \mu) \;=\; \Delta_q^{\frac{j}{2}}(Q,\, \mu)\Delta_g^{\frac{i-j}{2}}(Q,\, \mu). \tag{4.10}$$

Thus, in agreement with Ref. [31], we can account for leading-log effects for any particle multiplcity by simply multiplying matrix elements by appropriate Sudakov factors.

As we run $\mathcal{O}_i(\mu)$ down, another parton becomes apparent at a scale $\mu = p_T$. To account for this, Bauer & Schwartz devised a "threshold matching" of $\mathcal{O}_i$ to a new, higher multiplicty operator, $\mathcal{O}_{i+1}^{(i)}$, where the subscript still denotes the number of partons in the operator and the superscript tracks the parent operator. The general threshold matching equation is [16, 17]

$$\left[C_n^{(j)}\langle\mathcal{O}_n^{(j)}\rangle\right]_{\mu=p_T+\epsilon} = \left[C_{n+1}^{(j)}\langle\mathcal{O}_{n+1}^{(j)}\rangle\right]_{\mu=p_T-\epsilon}. \tag{4.11}$$

After more running and threshold matching, we eventually wind up with $\mathcal{O}_n^{(i)}$ for various $n > i$. The $n-i$ particles emitted at increasingly lower scales by this process correspond to the parton showering of the original fields created at the hard scale by $\mathcal{O}_i$. Refs. [16, 17] also showed that an appropriate list of SCET operators ($\mathcal{O}_i$'s and $\mathcal{O}_i^{(n)}$'s) can interpolate between fixed-order QCD and parton shower (PS) calculations of IR-safe observables. Furthermore, Ref. [16, 17] carried out the important task of including $\mathcal{O}(\alpha_s)$ effects from matching QCD to SCET at one-loop.

That these subsequent emissions reproduce the usual parton shower splitting function emerges easily from SCET. Consider an operator $\mathcal{O}_i = \bar{\chi}_{n_0}\Omega$, where Ω is arbitrary and we have made explicit a single collinear quark field, $\bar{\chi}_{n_0}$. If we emit a collinear gluon from the quark, $q(q_0^\mu) \to q(q_1^\mu)g(k_1^\mu)$, the amplitude for the process is:

$$A_{LO}^{X+qg} = \bar{u}_n(q_1)\rho^\alpha \frac{\bar{q}_0}{q_0^2}\Omega, \tag{4.12}$$

where u_{n_0} is the collinear quark spinor, and ρ^α is the combination of the SCET gluon emission Feynman rule plus the $\bar{\chi}_{n_0}$ Wilson line emission (the quark $\mathcal{L}_n^{(0)}$ can be found in Eq. (2.16)),

$$\rho^\alpha = n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp}\gamma_{n_0\perp}^\alpha}{\bar{q}_1} + \frac{\gamma_{n_0\perp}^\alpha(\not{q}_0)_{n_0\perp}}{\bar{q}_1} - \frac{\bar{n}^\alpha}{\bar{q}_0}\left[\frac{q_0^2}{\bar{k}_1} + \frac{(\not{q}_1)_{n_0\perp}(\not{q}_0)_{n_0\perp}}{\bar{q}_1}\right]. \tag{4.13}$$

Note that ρ^α in SCET comes entirely from $\bar{\chi}_{n_0}$ without reference to anything residing in Ω. The subscript $(n_0\perp)$ refers to components perpendicular to n_0^μ and $\bar{n}^\mu$, which we denote by $\perp$ for the remainder of this computation. The amplitude in Eq. (4.12) is gauge invariant and $k_1^\alpha \rho_\alpha = 0$. Squaring A_{LO}^{X+qg} and summing over spins we have $\sum_{spin} \bar{u}_n(q_1) u_n(q_1) = \bar{q}_1 \not{n}_0/2$, and the gluon polarization sum denoted $\sum_{spin} \epsilon_\alpha \epsilon^*_\beta = d_{\alpha\beta}$. Since ρ^α commutes with $\not{n}_0$, we get an answer proportional to $\rho^\alpha \rho^{\dagger\beta} d_{\alpha\beta}$, where without loss of generality we can use a light-cone gauge, $d_{\alpha\beta} = -g_{\alpha\beta} + (\bar{n}_\alpha k_{1\beta} + k_{1\alpha}\bar{n}_\beta)/\bar{k}_1$. Crucially, this is a Dirac scalar:

$$\rho^\alpha \rho^{\dagger\beta} d_{\alpha\beta} \equiv |\rho|^2 = 2\Big(\frac{2\, q_0^2}{\bar{k}_1 \bar{q}_0} - \frac{q_{1\perp}^2}{\bar{q}_1^2} + \frac{2 q_{0\perp}\cdot q_{1\perp}}{\bar{q}_0 \bar{q}_1} - \frac{q_{0\perp}^2}{\bar{q}_0^2}\Big) \times \mathbb{I}_4, \tag{4.14}$$

where we have used the on-shell conditions $q_1^2 = 0$ and $k_1^2 = 0$.

In a frame where $q_0^\perp = 0$ we have $q_{1\perp} = -k_{1\perp}$ and $\bar{q}_0/q_0^2 = 1/(n_0 \cdot q_0)$. Here $n_0 \cdot q_0 = n_0 \cdot k_1 + n_0 \cdot q_1 = -k_{1\perp}^2/[\bar{q}_0\, z(1-z)]$, where $z \equiv \bar{q}_1/\bar{q}_0$. Thus we have the simpler expression

$$\rho^\alpha = n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp}\gamma^\alpha_{n_0\perp}}{\bar{q}_1} - \frac{\bar{n}^\alpha n_0 \cdot q_0}{\bar{k}_1}, \tag{4.15}$$

and

$$\rho^\alpha \rho^{\dagger\beta} d_{\alpha\beta} = 2\Big(\frac{2\, n\cdot k_1}{\bar{k}_1} - \frac{2 q_{1\perp}^2}{\bar{q}_1 \bar{k}_1} - \frac{q_{1\perp}^2}{\bar{q}_1^2}\Big) \times \mathbb{I}_4 = -\frac{2\, k_{1\perp}^2}{\bar{q}_0^2}\frac{(1+z^2)}{z^2(1-z)^2}. \tag{4.16}$$

Putting these properties together in the full amplitude squared we get

$$\begin{aligned} |A_{LO}^{X+qg}|^2 &= \frac{g^2 C_F}{(n_0\cdot q_0)^2}\frac{\bar{q}_1}{2}\,\mathrm{Tr}\left[\not{n}_0 \rho^\alpha \Omega\Omega^\dagger \rho^{\dagger\beta}\right] d_{\alpha\beta} = \frac{g^2 C_F\, \bar{q}_1}{(n_0\cdot q_0)^2}|\rho|^2\,\mathrm{Tr}\left[\frac{\not{n}_0}{2}\Omega\Omega^\dagger\right] \\ &= g^2 C_F\, 2z\,\frac{(1+z^2)}{|k_{1\perp}|^2}\,\mathrm{Tr}\left[\bar{q}_0 \frac{\not{n}_0}{2}\Omega\Omega^\dagger\right]. \end{aligned} \tag{4.17}$$

Thus, all information about the emission factors out to the front of the amplitude squared and is independent of the rest of the process encoded by Ω. Unlike the analogous computation in full QCD there was no need to expand the amplitude to obtain this result. In order to recover Eq. (4.3), we still need to include the z-dependence from phase space, since $P_{jk}^{(0)}(z)$ operates at the level of the cross section. Using $d^3k/(2E_k) = dk^- d^2k_\perp/(2k^-)$, for q_1 and k_1 we have

$$\frac{dq_1^- d^2 q_{1\perp}}{2\bar{q}_1}\frac{dk_1^- d^2 k_{1\perp}}{2\bar{k}_1} = \frac{dq_0^- d^2 q_{0\perp}}{2\bar{q}_0}\frac{dz\, d^2 k_{1\perp}}{2z(1-z)}. \tag{4.18}$$

Thus we recover the expected $1/(1-z)$ dependence from the measure. Combining pieces and performing the trivial azimuthal integral $d\phi_{k_1}$, we get the expected expression:

$$d\sigma_{X+qg} = dz\,\frac{dk_{1\perp}^2}{k_{1\perp}^2} P^{(0)}_{q\to qg}(z)\, d\sigma_{X+q}, \tag{4.19}$$

where $P^{(0)}_{q\to qg}(z)$ is the quark splitting function in Eq. (4.3). Here $d\sigma_{X+q}$ is the cross-section for the rest of the process with emission of a momentum q_0 quark, and the corresponding amplitude squared is $\mathrm{Tr}\left[\frac{\bar{q}_0}{2}\not{n}_0 \Omega\Omega^\dagger\right]$. Whether Ω represents a simple hard current or an entire chain of collinear splittings, we see that the $q \to qg$ emission factors out with the expected logarithmic singularities, as in Eq. (4.2).

In order to obtain their results, Bauer and Schwartz introduced choices and approximations at several points which obscure the path toward systematicaly computing NLO corrections in the SCET power-counting parameter, λ. Indeed, they concluded that obtaining these corrections "... may be prohibitively difficult" [16]. Some of the issues one encounters trying to work at higher orders are:

1. At NLO, it becomes crucial to distinguish which simplifications correspond to approximations with power corrections, and which involve a choice of coordinates where the final answer is coordinate independent. For example, in general a collinear state has nonzero momentum components perpendicular to the index n of the field that annihilates it. In Refs. [16, 17] a choice was made to have collinear SCET fields in the operators only create particles whose momenta perfectly aligns with their index direction, n:

$$\chi_n|q\rangle = \delta_{n,n_q}, \quad \text{where } n_q^\mu = q^\mu/E_q. \tag{4.20}$$

   Eq. (4.20) enforces certain kinematical restrictions on final state particles, and required that fermions fields be rotated to an appropriate n_q^μ via $\xi_n \to (\not{n}_q\not{\bar{n}}/4)\xi_{n_q}$.

2. At LO it was possible to avoid a potential double counting between collinear and soft fields by dropping soft emission and Wilson line emission, and taking only collinear emissions with transverse polarization. The threshold matching

procedure is designed to avoid double counting of collinear operators, such as a Lagrangian emission from $\mathcal{O}_2$ and direct emission from $\mathcal{O}_3^{(2)}$, since only one of these is allowed to operate at a time. However, the threshold matching in Eq. (4.11) makes the technical procedure for incorporating power corrections unclear.

3. Threshold matching contains another impediment to systematic improvement. The basic idea is that the initial operator $\mathcal{O}_2$ has nonzero projection onto Fock states of any multiplicity, but the number of particles created by an operator is a scale-dependent question. The matching scales are determined by the strong ordering kinematics, $p_{1\perp} \gg \ldots \gg p_{m\perp}$. At the scale of an emission, say $p_{1\perp}$, one matches to the operator $\mathcal{O}_2^{(3)}$. However, going to higher orders in the shower necessitates encoding departures from the strong-ordering condition needed for this procedure.

There is also an interplay between some of these. For example, if we wish to construct a $q\bar{q}g$ operator that is obtained from a time-ordered product of the SCET lagrangian with $\mathcal{O}_2$, then there is a kinematic conflict since the mother quark has nonzero invariant mass, but the quark in $\mathcal{O}_2$ is suppose to have null momentum to enforce Eq. (4.20). Encoding the emission in $\mathcal{O}_3^{(2)}$ via threshold matching avoids the conflict with the choice in 1. In carrying out their method, Bauer & Schwartz carefully enumerated the above approximations. They affect the ability to include corrections in λ, but do not impact the LO shower.

Building on the work of Refs. [16, 17], the main goal of the framework we develop in the next subsection is to overcome this list of issues so that we can determine the NLO corrections to the shower using SCET.

4.2.2 Using SCET$_i$

The main feature of the parton shower is the ability to capture the dominant physics of particles emitted in kinematically hierarchical regions of phase space. Our goal is to reformulate the SCET interface with the shower using a standard sequence of matching and running steps in different versions of SCET,

$$\mathrm{QCD} \to \mathrm{SCET}_1 \to \mathrm{SCET}_2 \to \cdots \to \mathrm{SCET}_N\,. \tag{4.21}$$

We refer to this as the SCET_i procedure. The key distinction between a SCET at one stage and the next is the definition of the corresponding resolution parameters $1 \gg \lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_N$, where λ^i is the power counting parameter of SCET_i. As we move down the chain, the corresponding SCET resolves smaller invariant masses $\sim (Q\lambda_j)^2$, and has a different meaning for its collinear sectors $\{[n_i]\}_{\mathrm{SCET}_j}$. To keep track of this we will attach a subscript to the operators to denote the SCET_i in which its fields live,

$$\mathcal{O}_i^{(j,k,\ell)}(n_1,\ldots,n_{j+k})\,. \tag{4.22}$$

Effectively with Eq. (4.21), we partiton the momenta of particles in an event into classes,

$$\Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_N\,, \tag{4.23}$$

where Ω_j is defined to contain the momenta of all propogators or particles with invariant mass $p^2 \sim (Q\lambda_j)^2$ or smaller, or an equivalent condition on relative perpendicular momenta.[1] The allowed momenta in Ω_i correspond to the IR particles of SCET_i. The sequence of SCET_j's is truncated when we resolve a scale of order the parton shower cutoff, $Q\lambda^N = p_T^{\mathrm{cut}} \simeq 1\,\mathrm{GeV}$, that is in SCET_N.

The strongly ordered configuration of partons in Eq. (4.1) corresponds with removing a single q_j^2 in Ω_j as we pass from $\Omega_j \to \Omega_{j+1}$. However with Eq. (4.23) nothing stops us from having multiple emissions at a single scale. If two mother particles, with q_j^2 and q_{j+1}^2, are associated to the same Ω_k, then when we integrate out the scale

[1]Note that we do not associate a strong hiearchy to the hard scales $\bar{p}_i$ in each SCET_i, and we consider $\bar{p}_i \sim Q$ despite the fact that the partons loose energy as they split and evolve. This does not imply that the hard scale for each SCET_j is fixed, but mearly that parametrically it does not tend to decrease as rapidly as the invariant masses encoded in the power counting parameter λ_j.

in SCET_k this configuration just contributes to an operator with a different parton multiplicity from the strongly ordered case. Thus, with the Eq. (4.21) setup there is no obstacle to considering corrections from an arbitrary assignment of q_j^2's to Ω_k's. This resolves issue 3. of Subsection 4.2.1.

To carry out calculations in the SCET_i framework it is convenient and sufficient to take a specific definition of the power counting parameters, $\lambda_i = (\lambda)^i$. The motivation for this is that we want the hierarchy between neighboring splittings to stay the same throughout the shower so as not to privilege any portion of it. We will see in Subsection 4.4.4 that this democratic setup allows us to interpret our $\mathcal{O}(\lambda)$ corrections to i-parton amplitudes as universal corrections to the splitting probability, given at LO by Eq. (4.3). Our defintion of collinearity will change as we go to lower scales, and from Eq. (2.2) fields collinear to n within Ω_i have:

$$(n \cdot q_i,\, \bar{q}_i,\, q_{i\perp}) \sim (\lambda^{2i},\, 1,\, \lambda^i)\, \bar{q}_i, \tag{4.24}$$

and virtuality $\sim (\bar{q}_i)^2\lambda^{2i}$. For convenience we use the same auxillary vector $\bar{n}^\mu$ defined for n-collinear fields in SCET_1 for all subsequent collinear fields in SCET_i's that descend from these n-collinear mothers. In SCET_i $\mathcal{L}_n^{(0)}$ again only couples collinear fields in the same direction n. Since different SCET_i's have different definitions of collinearity, our description of identical physical processes changes when we switch to a theory with a lower scale.

We depict this in Fig. 4-1, where the left panel is in SCET_i and the right panel is in SCET_{i+1}. In SCET_i, the quark ($\vec{q}_1$) and gluon ($\vec{k}_1$) are n_0-collinear. This means that at LO they are emitted from a qqg vertex in the LO SCET_i Lagrangian (or a Wilson line). Schematically, the amplitude for a $\perp$-polarized gluon looks like[2]

$$A^{q\bar{q}g} = C_{q\bar{q}} \int dx \langle 0 | T\{\mathcal{L}_{\text{SCET}_i}(x)\mathcal{O}^{q\bar{q}}\} | q\bar{q}g\rangle, \tag{4.25}$$

namely like the first term in Eq. (2.22). The right-hand panel of Fig. 4-1 denotes the same configuration as seen by SCET_{i+1}. The scale of this theory is lower and

[2] From here on we will drop the superscript (0) and the subscript n from the collinear Lagrangian.

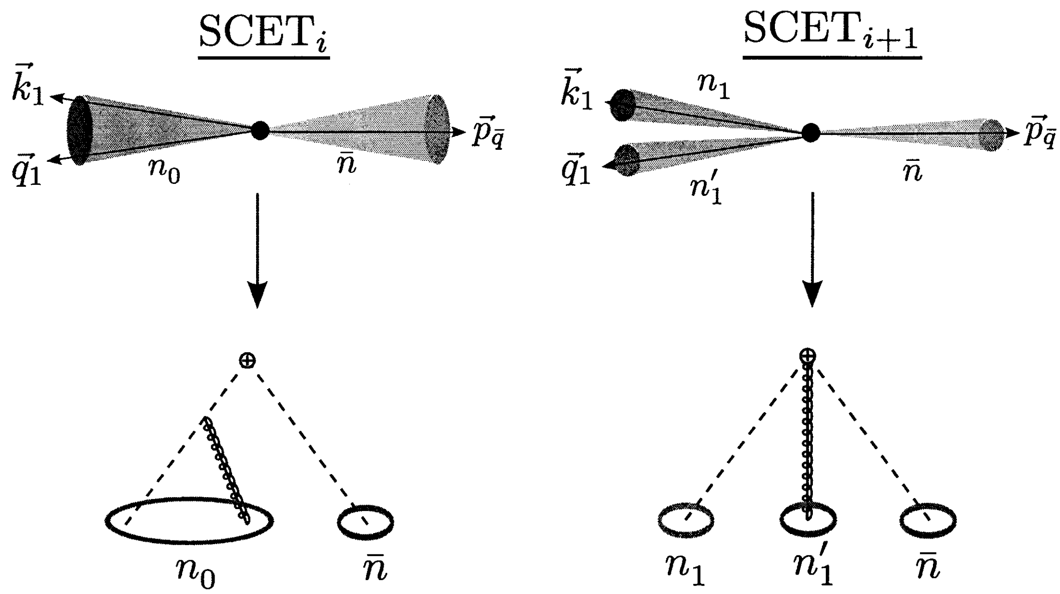

Figure 4-1: The same three-parton process as seen in two different SCET's, SCET$_i$ and SCET$_{i+1}$. Above: Kinematic configuration of the quarks and gluon. The solid cones represent the regions considered collinear to the vectors drawn. Below: Feynman diagrams for the corresponding amplitude. Note that in SCET$_{i+1}$ we have removed a propagating degree of freedom. The amplitude thus comes from a higher dimension operator $\mathcal{O}^{q\bar{q}g}$, rather than time-ordered product of $\mathcal{L}_{\mathrm{SCET}_i}$ with the current $J^{q\bar{q}}$, as seen for SCET$_i$.

the definition of collinearity stricter, so the quark and gluon are not collinear here. Therefore, the amplitude now comes from an operator with three partons,

$$A^{q\bar{q}g} = C_{q\bar{q}g}\langle 0|\mathcal{O}^{q\bar{q}g}|q\bar{q}g\rangle, \tag{4.26}$$

as in Eq. (2.23). As usual in EFT, continuity of the S-matrix is maintained by matching SCET$_i$ to SCET$_{i+1}$ to calculate $C_{q\bar{q}g}$.

Before moving on, we list some technical advantages to working in the SCET$_i$ framework:

1. Collinear fields in SCET with different n-labels, as well as soft fields, do not overlap in Hilbert space. SCET provides built in mechanisms to avoid double counting that will be inherited by terms in the SCET$_i$ chain, such as zero-bin subtractions [78]. This allows us to separate an i-jet process with i dis-

tinguished partons, from an $(i-1)$-jet process with i partons, where two are collinear and unresolved. Lower-scale SCET_i's distinguish processes more finely based on their stricter definition of collinearity. This resolves issue 2. from Subsection 4.2.1. This SCET property also illuminates simplified structures in the power corrections, such as the form of the amplitude interference (cf. subsection 4.4.4).

2. Soft modes communicate between collinear sectors and threaten the factorization of different jets. Fortunately, SCET constrains the interactions they have with collinear fields. In fact, one can decouple them with soft Wilson lines in the LO SCET Lagrangian. At LO we maintain factorization and derive the angular ordering and coherent branching of soft emissions (cf. subsection 4.3.2). Soft interactions which are power suppressed can also be systematically studied in SCET with Lagrangians available in the literature [15, 24, 25].

3. In SCET_i we have a symmetry group RPI_i which corresponds to coordinate choices. In SCET_{i+1} only a subset of this $\text{RPI}_{i+1} \subset \text{RPI}_i$ remains a symmetry of the new theory. The kinematics in the coset portion $\text{RPI}_i/\text{RPI}_{i+1}$ within SCET_i become a set of higher-dimension operators in SCET_{i+1}, and describe configurations which would not otherwise be contained in the SCET_{i+1} Lagrangian (cf. subsection 4.3 and appendix B.1). This resolves issue 1. from Subsection 4.2.1.

4. In matching between SCET_i and SCET_{i+1}, higher order operators in the lower-scale theory are needed to reproduce the physics of the higher one. We proved that all higher order purely collinear operators can be built from quark fields (χ_n), perpendicular gluon fields $(\mathcal{B}_{\perp n})$, and the perpendicular momentum operators $(P_{\perp n})$ (*cf.* subsection 2.1.1). Thus the symmetries and equations of motion of SCET greatly simplify the operator basis one needs to consider at each order in λ (cf. subsection 4.4 and appendices B.2 and B.3).

The final SCET_N corresponds to the scale where the shower stops, *i.e.* where $Q\lambda^N \sim p_T^{\text{cut}}$. In SCET_N, we only require the coefficients of the operators where all collinear partons belong to distinct sectors and which have no $\mathcal{P}_{n\perp}$'s, $C_N^{(j,k,0)}\mathcal{O}_N^{(j,k,0)}$. Here we have $n_j^\mu = p_j^\mu/\bar{p}_j$ for each parton that is each perticle is collinear to a different direction, this is as in Eq. (4.20), but we only do this for operators which have just one parton in an equivalence class. The more general operators with more than one parton in a single collinear direction and $\perp$-momenta are required for the steps through intermediate SCETs, to calculate NLO corrections. The coefficients $C_N^{(j,k,0)}$ encode the history of the shower, including branching and evolution, and can be written entirely in terms of: dot products $n_i \cdot n_{i'}$ (that is equivalent to products of physical parton momenta), hard momenta $\bar{p}_i$, and the renormalization scale of dimensional regularization, μ. The dot products of ns carry the power counting in λ. It is also worth mentioning that SCET maintains gauge invariance for power corrections, so the Wilson coefficients encoding the shower information are gauge invariant.

Our goal is to carry out the transition $\text{SCET}_i \to \text{SCET}_{i+1}$ with a standard matching procedure, and to find a correspondence between the SCET_i setup and the information needed for carrying out the probabilistic parton shower. A traditional LO parton shower needs four basic ingredients: an evolution variable, no-branching Sudakov factors, the splitting function, and a notion of factorization yielding an iterative algorithm. Given a desired evolution variable, a correspondence must be set up with the $n_i \cdot n_{i'}$'s in the SCET_i procedure. At LO this is straightforward, but at NLO this will effect the manner in which the power corrections enter a probabilistic shower code. The no-branching Sudakov's are obtained by renormalization of operators in each SCET_i, with calculations that are along the same lines as those covered in detail in Refs.[16, 17]. At LO, only operators with distinct collinear sectors are required, while at NLO the renormalization becomes more complicated since operators may now contain two or more partons that are collinear in the same direction. The splitting functions appear in the matching between SCET_i and SCET_{i+1}. At NLO, this includes $1 \to 3$ splitting functions as well as corrections where more partons appear at the start of the shower. The majority of the splittings in a tree still factorize at NLO, and the properties of SCET control complications in the interference.

In the above discussion we did not identify the parameter λ with an outcome in the probabilistic shower. As far as the shower is concerned λ is merely a bookkeeping device which helps determine what pieces are needed beyond LO, and those pieces depend only on physical momentum and not on λ. One could try defining $\lambda_1 = k_{1\perp}/Q$, $\lambda_2 = k_{2\perp}/Q$, etc., but this is not ideal since probabilistically there is a chance for events where $k_{1\perp} \sim Q$ or $k_{1\perp} \sim p_T^{\rm cut}$. The organization in Eq. (4.23) instead exploits the fact that *on average* showers are strongly-ordered. Our expansion in λ will then *on average* give a description of the most likely first deviations from strong-ordering. The correspondence between shower ingredients and ingredients in SCET$_i$ implies a probabilistic interpretation, rather than fixed unique values for the scales $Q\lambda_i$. In the end, we pass to the shower the information obtained along the way to determining the operators of SCET$_N$ to NNLO in λ at the amplitude-squared level. Again, what is important is to obtain the ingredients in the shower, and in this sense SCET$_i$ is a crutch to derive the NLO shower ingredients and not a physical theory that outputs precisely the desired NLO shower.

Now that we have defined the basics of the SCET$_i$ setup and its advantages, we are in a position to give an operational overview of our results. Building on item 4 above, the bulk of this work is devoted to studying how one derives SCET$_{i+1}$ from SCET$_i$. We will find that many simplifications occur. We can write down the operator which reproduces the strongly-ordered parton shower in a closed form expression for an arbitrary number of partons (subsection 4.3.1). Improving the parton shower then simply corresponds to matching at higher orders. We will find one set of corrections that only involve fields near the hard-scattering process. One can determine them from matching to QCD amplitudes with a small number of partons (subsection 4.4.1). They represent a set of non-resummable contributions, which we can interpret as improving the initial condition from the hard-scale matrix element. We will also find a generalizable set of corrections for an arbitrary number of partons that admits a closed form expression (subsection 4.4.2). This leads to a correction of the substructure of the jet, and we can use it to cross-check it by known $\mathcal{O}(\alpha_s)$ corrections to the usual parton splitting functions. Lastly, when we construct the squared amplitude the interference between all of these operators in the SCET$_i$ picture is straightforward (subsection 4.4.4).

It is also worth commenting explicitly on shower ingredients that we do not compute in this work. We only treat the case of a showering quark $q \to qg$ and for the abelian portion ($\propto C_F^2$). We have left gluon splittings, $g \to q\bar{q}$ and $g \to gg$ out of this analysis, though the extension to these cases should be straightforward. We have not computed the evolution factors at NLL required for a full NLL shower with a proper resummation of double logs. We have also not determined the full effect of NLO power corrections from subleading soft interactions, although we briefly examine the factorized structure of such corrections in subsection (4.3.2). Finally, and most importantly, we do not attempt here to develop a realistic numerical algorithm for implementing an NLL shower. These items are all left to future investigations.

4.3 Parton Shower in SCET via Operator Replacement

In the previous subsection, we presented our approach of using a series of EFTs, the SCET_i, to handle processes with a hierarchy of many scales. We will now use this technique to calculate the leading contribution to a series of collinear emissions, as occurs in the parton shower. Our ultimate goal is to incorporate corrections, but as a starting point we want to easily reproduce the strongly-ordered configuration Eq. (4.1). We can do this if we declare that in a shower, the i^{th} particle decomposes as:

$$(n \cdot q_i,\, \bar{q}_i,\, q_{i\perp}) \sim (\lambda^{2i},\, 1,\, \lambda^i)\, Q, \tag{4.27}$$

and therefore has virtuality $q_i^2 \sim Q^2 \lambda^{2i}$. strongly-ordered i-gluon radiation in Fig. 4-6. This is exactly the same condition as Eq. (4.24), which we used to define the EFT, SCET_i. Thus, it is natural to treat the $i^{\rm th}$ emission in the theory SCET_i.

To calculate the operators that describe i emissions in the strongly-ordered limit, we will perform a series of matchings $\mathrm{SCET}_i \to \mathrm{SCET}_{i+1}$. We will find that the most efficient way to describe the process at LO is to be in SCET_i for i-parton radiation.

$p^2 \lesssim Q^2$	QCD
$p^2 \lesssim Q^2\lambda^2$	SCET_1
$p^2 \lesssim Q^2\lambda^4$	SCET_2
$p^2 \lesssim Q^2\lambda^6$	SCET_3
$\vdots$	$\vdots$
$p^2 \lesssim Q^2\lambda^{2i}$	SCET_i

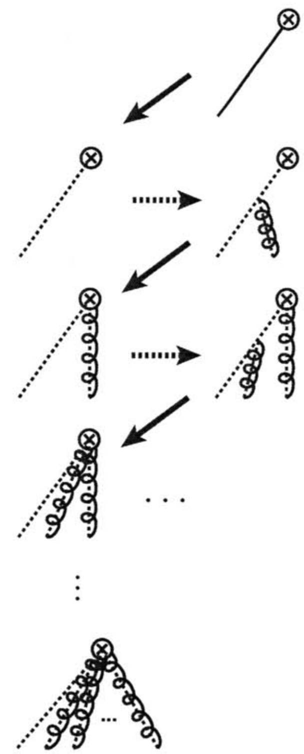

Figure 4-2: We construct the operators that reproduce strongly-ordered gluons through a series of matchings and emissions. *Horizontal* dashed arrows refer to the radiation of a gluon from a time-ordered product of the SCET_i lagrangian with the operator creating fields at the point marked by $\otimes$. *Diagonal* solid arrows denote the matching the process to a higher multiplicity operator in SCET_{i+1}.

Thus, we emit and match i-times in series, as shown by Fig. (4-2). At LO, we will show that one can implement this using an operator replacement rule. In the case of $q \to qg$ emission, it takes the form:

$$\chi_{n_1} \to c\, g\mathcal{B}^{\alpha}_{n_3\perp}\, \chi_{n_2}\,, \tag{4.28}$$

where χ_n and $\mathcal{B}^{\alpha}_{n\perp}$ are the SCET fields associated with collinear quarks and gluons respectively, and C is the Wilson coefficient whose indices are suppressed. Though we do not compute them, there should be similar $g\mathcal{B}^{\alpha}_{n_1\perp} \to c'\, \bar{\chi}_{n_2}\chi_{n_3} + c''\, g\mathcal{B}^{\beta}_{n_2\perp} g\mathcal{B}^{\gamma}_{n_3\perp}$ rules as well. In SCET, each collinear field carries the label n, which gives its direction of collinearity. Note that the quark field on the LHS of (4.28) has a different one from those on the RHS. This relates to the stricter definition of collinearity in

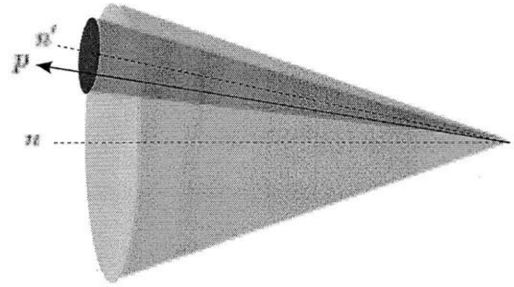

Figure 4-3: The opening angle of the light grey (blue) cone is λ^{2i}, and the aperture of the dark grey (red) one is $\lambda^{2(i+1)}$. The particle with momentum p is collinear to both n and n' in SCET_i, but only to n' in SCET_{i+1}. RPI in SCET_i allows us to move the index direction, n, anywhere inside the appropriate collinear cone while keeping the theory invariant.

SCET_{i+1} shown in Fig. 4-3. In order to perform the matching we will make use of the reparametrization invariance (RPI) to change fields' index labels, n.

4.3.1 LO Shower Revisited

We first want to reproduce the strongly-ordered contribution to i-gluon radiation from the quark in $q\bar{q}$ pair production. Our iterative matching procedure for multiple EFTs takes a particularly simple form at LO (LO, NLO, *etc.* refer to the expansion in λ). For our standard example, we take the process $e^+e^- \to$ jets. Starting in QCD, we couple the quarks to another sector via the operator, $J^\mu_{\text{QCD}} = \bar{q}\,\Gamma^\mu q$. This allows us to avoid complications that come from the initial state such a backward evolution. In SCET_1 (which is equivalent to the usual SCET), matching to QCD at tree-level converts the quark coupling to the following operator at LO: $\bar{\chi}_{n_0}\Gamma^\mu\chi_{\bar{n}}$, which produces q and $\bar{q}$ in different collinear directions, for details on the matching QCD to SCET_1 see appendix B.2. Using the notation in Eq. (4.7) we write the SCET_1 operator in the following way:

$$\bar{\chi}_{n_0}\Gamma^\mu\chi_{\bar{n}} = \left(C^{(2,0,0)}_{1,\text{LO}}\right)_{ij}\left(\mathcal{O}^{(2,0,0)}_1(n_0,\bar{n})\right)_{ij} \tag{4.29}$$

where

$$\left(\mathcal{O}_1^{(2,0,0)}(n_0,\bar n)\right)_{ij} = (\bar\chi_{n_0})_i(\chi_{\bar n})_j\,, \qquad (4.30)$$
$$\left(C_{1,\mathrm{LO}}^{(2,0,0)}\right)_{ij} = (\Gamma^\mu)_{ij}\,.$$

and i and j are spinor indices. The subscripts 1 in Eq. (4.30) means that the fields are defined in SCET_1. Our focus is on gluon emissions from the quark, and we always take the antiquark in the same direction, $\bar n$. We can therefore use the following shorthand notation:

$$\mathcal{O}_i^{(2,k,0)}(n_1,n_1',\ldots,n_k',\bar n) \equiv \mathcal{O}_i^{(k)}(n_1,n_1',\ldots,n_k')\,, \qquad (4.31)$$

where the subscript indicates that the operators are defined in SCET_i. In the rest of this chapter we will drop the spinor indices to make the notation more readable. Using the above convention we write the operator in Eq. (4.29) as

$$\bar\chi_{n_0}\Gamma^\mu\chi_{\bar n} = C_{1,\mathrm{LO}}^{(0)}\mathcal{O}_1^{(0)}(n_0)\,. \qquad (4.32)$$

The LO derivations are independent of the exact structure of Γ^μ. In fact, even the antiquark is a spectator, and we could just as easily use $\mathcal{O}^{(q)} = \bar\chi_{n_0}\Omega$, where Ω is arbitrary. However, as we will discuss in sec. 4.4, matching QCD to SCET_1 at higher orders requires us to specify Ω.

We start with single gluon radiation. In this case, shown in Fig. 4-4, the emission amplitude is[3]:

$$\begin{aligned} A_{\mathrm{LO}}^{q\bar q g} &= C_{1,\mathrm{LO}}^{(0)}\langle 0|\int dx\, T\{\mathcal{L}^{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(0)}(n_0)\}|q_{n_0}g_{n_0}\bar q_{\bar n}\rangle & (4.33)\\ &= \bar u_{n_0}(q_1)g\left(n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp}\gamma_{n_0\perp}^\alpha}{\bar q_1}\right)\frac{\bar q_0}{q_0^2}\Gamma^\mu v_{\bar n}(p_{\bar q}), & (4.34)\end{aligned}$$

where we have now labelled the collinear directions of the particles in the state

[3]All the amplitudes we write in this work refer only to the hadronic part of $e^+e^- \to$ jets, thus $A_{\mathrm{LO}}^{q\bar q g}$ is the amplitude of $\gamma^* \to q\bar q g$.

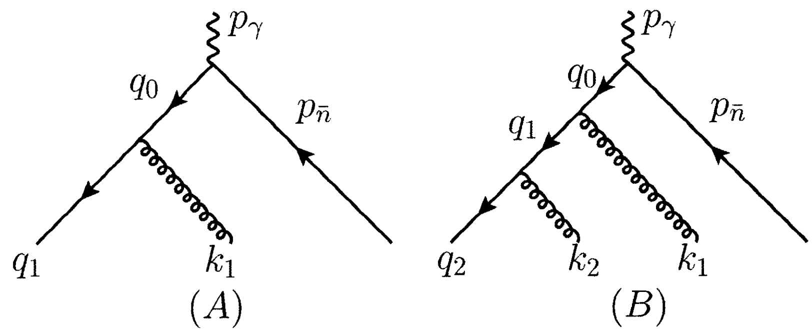

Figure 4-4: Vector labels for single (A) and double (B) gluon emission.

$|q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle$ for later convenience. The SCET_1 Lagrangian is given in Eq. (2.16). Here we study the process in the center of mass frame with $p_\gamma = (Q, 0, 0, 0)$ and the quark (q_0) and and antiquark ($p_{\bar{q}}$) along the directions $n_0 = (1, 0, 0, 1)$ and $\bar{n} = (1, 0, 0, -1)$, respectively:

$$\begin{aligned} p_\gamma^\mu &= \frac{Q}{2} n_0^\mu, + \frac{Q}{2} \bar{n}^\mu \\ p_{\bar{q}}^\mu &= \frac{n_0 \cdot p_{\bar{q}}}{2} \bar{n}^\mu, \\ q_0^\mu &= \frac{\bar{q}_0}{2} n_0^\mu + \frac{n_0 \cdot q_0}{2} \bar{n}^\mu . \end{aligned} \tag{4.35}$$

We decompose the emitted quark (q_1) and gluon (k_1) along the directions $(n_0, \bar{n})$

$$\begin{aligned} q_1^\mu &= \frac{\bar{q}_1}{2} n_0^\mu + (q_1)_{n_0\perp}^\mu + \frac{n_0 \cdot q_1}{2} \bar{n}^\mu , \\ k_1^\mu &= \frac{\bar{k}_1}{2} n_0^\mu + (k_1)_{n_0\perp}^\mu + \frac{n_0 \cdot k_1}{2} \bar{n}^\mu . \end{aligned} \tag{4.36}$$

The variables are defined in Fig. 4-4. By momentum conservation we have $(k_1)_{n_0\perp} = -(q_1)_{n_0\perp}$, $Q = \bar{q}_0 = \bar{k}_1 + \bar{q}_1$ and $n_0 \cdot p_{\bar{q}} = Q - n_0 \cdot q_1 - n_0 \cdot k_1$. We take all the external particle on -shell, thus $n_0 \cdot q_1 = -(q_1)_{n_0\perp}^2/\bar{q}_1$ and similarly for $n_0 \cdot k_1$. As we discussed in subsection 4.2.1, [16, 17] showed that single gluon emission in SCET reproduces the splitting function (Eq. (4.3)) and factorization behavior (Eq. (4.17)) of the standard parton shower. Take this simple behavior of a single radiation will reproduce the shower for an arbitrary number of gluons.

We proceed to two-gluon emission as this demonstrates the benefit of going to a

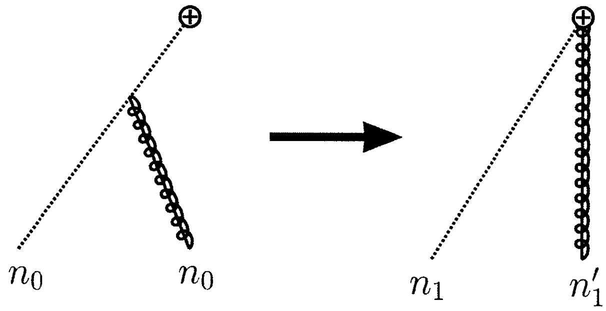

Figure 4-5: Left panel: Single gluon emission in SCET_1 comes from the time-ordered product of the lagrangian with a quark-creating operator, $A = \langle 0|T\{\mathcal{L}_{\text{SCET}_1}\mathcal{O}_1^{(0)}\}|q\bar{q}\rangle$. Right panel: For parent quarks which are too virtual for SCET_2, the gluon comes from the central vertex via a higher-dimensional operator, $A = \langle 0|\mathcal{O}_2^{(1)}|qg\bar{q}\rangle$.

lower-scale EFT as more gluons are radiated. If $k_{1\perp}$ and $k_{2\perp}$ are the perpendicular momenta with respect to their parents of the gluons emitted first and second, then strong ordering dictates that $k_{2\perp} \ll k_{1\perp}$. SCET_1 makes no parametric difference between the two, as all perpendicular momenta are assigned a factor λ. Since SCET_2 has a stricter definition of collinearity (a cone size of $\mathcal{O}(\lambda^2)$), it can distinguish that of the two fields. This allows us to integrate out the parent of k_1, as its offshellness is too hard $\mathcal{O}(Q^2\lambda^2)$ for SCET_2 and match to a higher-dimension operator. In this way, we remove information unneeded to reproduce strong-ordering. By systematically putting it back (subsection 4.4.2), we can improve upon the standard strongly-ordered parton shower.

We want to describe the less collinear emission (k_1) as coming from a higher-dimension SCET_2 operator (Fig. 4-5), in this case $C_{2,\text{LO}}^{(1)}\mathcal{O}_2^{(1)}(n_1, n_1')$. To find it, we just need to find the SCET_2 operator that reproduces the amplitude for single gluon emission in Eq. (4.34). As one can see, Eq. (4.34) makes explicit reference to the label direction n_0. In SCET_1, this labels the virtual quark (q_0), the real quark (q_1), and the gluon (k_1) directions. However, as we have discussed (*cf.* Fig. 4-1), we have a tighter definition of collinearity in SCET_2, such that q_1 and k_1 cannot share the same label direction with each other or q_0, which is not even a propagating degree of freedom in the theory. Spinors in SCET have a label dependence, and so we can have different spinors for q_1 according to whether we are in SCET_1 or

SCET$_2$. This makes it difficult to match at the level of an unintegrated quantity such as an amplitude. The momentum p decomposed along n' and $\bar{n}$ is This leads to a technical complication in the computation, but one that is easy to surmount, making use of finite reparametrization invariance transformations that allow us to rotate the label vector while keeping the theory invariant, Fig 4-3. We define reparametrization invariant in subsection 2.2 and we construct finite RPI transformation in appendix B.1. We define new labels n_1 and n_1' as the basis where the final quark, and gluon respectively have zero perpendicular momentum,

$$\begin{aligned} q_1 &= \bar{q}_1 \frac{n_1}{2}\,, \\ k_1 &= \bar{k}_1 \frac{n_1'}{2}\,. \end{aligned} \tag{4.37}$$

The momenta are especially simple in this basis, q_1 and k_1 are on-shell, which means $n_1 \cdot q_1 = n_1' \cdot k_1 = 0$. In SCET$_1$, we are free to use n_0 or n_1 to describe the q_1 quark and k_1 gluon because we can use the RPI symmetry to transform between this choice and other equivalent choices. Whatever choice we make the results for the Wilson coefficients in the matching will be the same. Since n_1 is also a valid index for the field in SCET$_2$, we can do the matching computation using the same spinor, $u_{n_1}(q_1)$, in both theories. The following results show how to express a spinor in the n_1 direction in terms of one in the n_0 direction and how to rotate the vector n_0 to $n_1(n_1')$ such that the quark(gluon) field with perpendicular momentum $(q_1)_{n_0\perp}(k_1)_{n_0\perp}$ in the n_0-basis has 0 perpendicular momentum in the $n_1(n_1')$-basis:

$$\begin{aligned} u_{n_0} &= \frac{\not{n}_0 \bar{\not{n}}}{4} u_{n_1}\,, \\ n_1^\alpha &= n_0^\alpha + \frac{2(q_1)_{n_0\perp}}{\bar{q}_1} - \frac{(q_1)^2_{n_0\perp}}{\bar{q}_1^2}\bar{n}^\alpha\,, \\ n_1'^\alpha &= n_0^\alpha + \frac{2(k_1)_{n_0\perp}}{\bar{k}_1} - \frac{(k_1)^2_{n_0\perp}}{\bar{k}_1^2}\bar{n}^\alpha\,. \end{aligned} \tag{4.38}$$

The two different n_i-vectors' directions lie within cones of size λ, and so we apply the transformation in SCET$_1$. It is simple to check that in the new basis, $(q_1)_{n_1\perp} =$

$q_1 - (n_1 \cdot p)\bar{n}/2 - \bar{q}_1\, n_1/2 \;=\; 0$ and similarly for $(k_1)_{n_1'\perp}$. In appendix B.1 we derive Eq. (4.38) and other rotation formulas. Acting on Eq. (4.34), we get:

$$A_{\rm LO}^{q\bar{q}g} = g\frac{\bar{q}_0}{q_0^2}\bar{u}_{n_1}\left(n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp}\gamma^\alpha_{n_1'\perp}}{\bar{q}_1}\right)\frac{\not{\bar{n}}\not{n}_0}{4}\Gamma^\mu v_{\bar{n}}\,, \tag{4.39}$$

where $q_0 \;=\; q_1 + k_1$. Terms with subscript n_0, n_1, n_1' are in the frame defined by q_0, q_1, k_1, respectively. Having changed bases, we can easily write the SCET_2 operator that reproduce the amplitude (4.39) $C_{2,\rm LO}^{(1)}(\mu)\mathcal{O}_2^{(1)}(n_1,n_1')$, where:[4]

$$\mathcal{O}_2^{(1)}(n_1,n_1') = (\bar{\chi}_{n_1})_j\, g\mathcal{B}^\alpha_{n_1'\perp}\,(\chi_{\bar{n}})_k\;, \tag{4.40}$$

$$C_{2,\rm LO}^{(1)} = U_{\rm LL}^{(0)}(n_0)(Q,\mu_1)\left[\frac{\bar{q}_0}{q_0^2}\left(n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp}\gamma^\alpha_{n_1'\perp}}{\bar{q}_1}\right)\frac{\not{\bar{n}}\not{n}_0}{4}\Gamma^\mu\right]_{jk}\Theta_{\lambda^2}[n_1\cdot n_1']\,. \tag{4.41}$$

We have included in the Wilson coefficient the leading log RG-kernel $U_{\rm LL}^{(0)}$ that comes from running down the SCET_1 operator $\mathcal{O}_1^{(0)}$ from the scale Q to the scale $\mu_1 \sim \lambda Q$ where we have the emission. From Eq. (4.10) we have

$$U_{\rm LL}^{(0)}(n_0)(Q,\mu_1) = \Delta_q(Q,\mu_1)\,, \tag{4.42}$$

where we have simplified the superscript as in Eq. (4.31), that is

$$U_{\rm LL}^{(2,k,0)}(n_1,n_1',\ldots,n_k',\bar{n}) \equiv U_{\rm LL}^{(k)}(n_1,n_1',\ldots,n_k')\,. \tag{4.43}$$

When $U_{\rm LL}$ refers to an operator where all the collinear directions are distinct, we will drop ns from the notation. In the interpretation of this factor as a Sudakov, we see that it accounts for the non-branching from the scale Q to the μ_1 of the emission. We can write the coefficient $C_{2,\rm LO}^{(1)}$ only in terms of scalar product of ns, this is shown in Eq. (B.53) in appendix B.3, for the running coefficient we can take $\mu_1 = \sqrt{|n_1\cdot n_1'|}Q$.

The function $\Theta_{\lambda^2}[n_1\cdot n_1']$ in Eq. (4.41) is a function that encodes the information

[4] See appendix B.3 for more detail on this matching.

that $(n_1 \cdot n_1') \lesssim \lambda^2$. The SCET$_2$ operator $\mathcal{O}_2^{(1)}(n_1, n_1')$ knows that the gluon and the quark are not collinear in SCET$_2$, that is $q_1 \cdot k_1 > \lambda^4 Q^2$ (or $n_1 \cdot n_1' > \lambda^4$), but it does not know that they were collinear in SCET$_1$. We calculated the coefficient $C_{2,\,\mathrm{LO}}^{(1)}$ in a phase space region where $q_1 \cdot k_1 \lesssim \lambda^2 Q^2$ (or $n_1 \cdot n_1' \lesssim \lambda^2$) in the matching to the SCET$_1$ $\mathcal{O}_1^{(0)}(n_0)$ and this information must be encoded in the Wilson coefficient. We encode that $n_1 \cdot n_1' \lesssim \lambda^2$ using the Θ function in $C_{2,\,\mathrm{LO}}^{(1)}$. For a LL shower we can take $\Theta = 1$, because there is not problem of overlap between different regions. However, once we start considering power corrections, we will need Θ to turn on and off particular regions of phase space. For later convenience we define the complement to the Θ function as

$$\tilde{\Theta}_{\lambda^k}[n_i \cdot n_j] = 1 \;-\; \Theta_{\lambda^k}[n_i \cdot n_j]\,. \tag{4.44}$$

Schematically, we have

$$\Theta_{\lambda^k}[n_i \cdot n_j] = \begin{cases} 1 & n_i \cdot n_j \lesssim \lambda^k \\ 0 & n_i \cdot n_j \gtrsim \lambda^k \end{cases}\,. \tag{4.45}$$

In principle, we could choose Θ to be the usual step-function, θ, but for practical integration, it is better to define a smoothed step. We give an example of such a function in Eq. (B.61), and plot it in Fig. 4-9.

The coefficient $C_{2,\mathrm{LO}}^{(1)}$ has a power counting in λ because of the presence of multiple SCETs. This is different from what happens in a usual SCET matching where all the coefficients are of order λ^0. $C_{2,\mathrm{LO}}^{(1)}$ has an overall weight λ^{-1}. We get λ^{-2} from the SCET$_1$ propagator, $1/q_0^2$. The numerator is proportional to λ and comes from the vertex: $\left(n_{q_0}^{\alpha} + (\not{q}_1)_{n_0\perp}\gamma^{\alpha}_{n_1'\perp}/\bar{q}_1\right)$. The second term is $\mathcal{O}(\lambda)$ because of the power counting of $(\not{q}_1)_{n_0\perp} \sim \lambda$. Since n_0^{α} also gets contracted with $\mathcal{B}^{\alpha}_{n_1'\perp}$, it only contributes its perpindicular component in the n_1' frame. From Eq. (4.38), we see that $(n_0)_{n_1'\perp} \sim n_0 - n_1' \sim (k_1)_{n_0\perp}/\bar{k}_1 \sim \lambda$. The vertex thus contributes the power of λ corresponding to the perpendicular momentum of the daughter with respect to its mother.

We can obtain $C_{2,\mathrm{LO}}^{(1)}\mathcal{O}_2^{(1)}(n_1, n_1')$ from the original two-parton operator, $C_{1,\mathrm{LO}}^{(0)}(n_0)\mathcal{O}_1^{(0)}(n_0)$, in two steps: first we multiply it by the running factor $U^{(0)}_{\rm LL}(n_0)(Q,\mu_1) = \Delta_q(Q,\mu_1)$, where the Δ_i are given in relation to the RG kernel in Eqs. (4.9-4.10), and second we apply the replacement rule

$$(\bar\chi_{n_0})_i \to (c^\alpha_{\rm LO}(n_0))_{ji}\,(\bar\chi_{n_1})_j g\mathcal{B}^{n_1'\perp}_\alpha, \tag{4.46}$$

where $c^\alpha_{\rm LO}$ is:

$$c^\alpha_{\rm LO}(n_0) = \frac{\bar q_0}{q_0^2}\left(n_0^\alpha + \frac{(q_1)^\mu_{n_0\perp}\gamma^\alpha_{n_1'\perp}}{\bar q_1}\right)\frac{\bar{n}\!\!\!/ n\!\!\!/_0}{4}\,\Theta_{\lambda^2}[n_1\cdot n_1']\,. \tag{4.47}$$

The relation (4.46) is the operator statement of splitting in the parton shower. The scale μ_1 defines the endpoint of running in the UV theory. As we evolve down, more partons become apparent. We can see this here by the presence of two fields where there had been one. It makes the basic aspects of the shower manifest. The replacement rule affects the quark alone, and so we see that the amplitude for splitting factorizes off from the rest of the process. The RG kernel gets modified to reflect the changing no-branching probabilities. We can interpret the vertex portion of $c^\alpha_{\rm LO}$ as the "square root" of the splitting function. The spinor projector ($\bar{n}\!\!\!/ n\!\!\!/_0/4$) in Eq. (4.47) rotates the spin-sum from $n\!\!\!/_1$ to $n\!\!\!/_0$ in accordance with Eq. (4.2). The remaining part of $c^\alpha_{\rm LO}$ after stripping off the Sudakovs is:

$$P_\alpha \equiv \frac{\bar q_0}{q_0^2}\left(n_0^\alpha + \frac{(q\!\!\!/_1)_{n_0\perp}\gamma^\alpha_{n_1'\perp}}{\bar q_1}\right), \tag{4.48}$$

which squares to a trivial Dirac structure. Furthermore, even though $\rho_\alpha(\bar q/q_0^2) \neq P_\alpha$ because of the RPI rotations we performed (where ρ is defined in Eq. (4.15)), where we square $|\rho|^2(\bar q_0/q_0^2)^2 = |P|^2$ with respect the gauge polarization sum, $d_{\alpha\beta}$, so

$$|P|^2 = \frac{1+z^2}{q_{1\perp}^2}. \tag{4.49}$$

Just as before, including the z-dependence from the measure and spin-sum, we recover the the standard splitting function $\propto (1+z^2)/(1-z)$. Thus, $c^\alpha_{\rm LO}$ weights the probability assigned to the expectation value of $C^{(1)}_{2,\mathrm{LO}}(\mu)\mathcal{O}^{(1)}_2(n_1,n'_1)$ appropriately.

Having computed the LO result for a single gluon, it is straightforward to proceed to an arbitrary number of emissions. In SCET_2, we know that two-gluon emission comes from the T-product of the lagrangian with $C^{(1)}_{2,\mathrm{LO}}(n_1,n'_1)\mathcal{O}^{(1)}_2(n_1,n'_1)$. We can proceed by emitting a third with another SCET_2 lagrangian insertion. The situation is now exactly parallel to when we were considering two-gluon emission in SCET_1. Once again, we are carrying around more information than we need to reproduce the strongly-ordered contribution, so we should integrate out the scale $\sim Q\lambda^2$ to get a new theory, SCET_3, that has a two-gluon operator, $C^{(2)}_{3,\mathrm{LO}}(n_2,n'_1,n'_2)\mathcal{O}^{(2)}_3(n_2,n'_1,n'_2)$ at LO. Similarly to before, the amplitude has the contribution,

$$A^{q\bar{q}gg}_{\mathrm{LO}} = C^{(1)}_{2,\mathrm{LO}}(\mu)\langle 0|\int dx\, T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}^{(1)}_2(n_2,n'_1)\}|q_{n_2}g_{n_2}g_{n'_1}\bar{q}_{\bar{n}}\rangle\,. \tag{4.50}$$

The vertex for gluon emission in the SCET_2 lagrangian is identical to that in SCET_1. Thus, integrating out the parent of the Lagrangian-emitted, second-most collinear gluon, we obtain a two-gluon SCET_3 operator exactly as before. Similar to the matching SCET_1 to SCET_2, we can obtain the SCET_3 operator $C^{(2)}_{3,\mathrm{LO}}\mathcal{O}^{(2)}_3(n_2,n'_1,n'_2)$ from the SCET_2 $C^{(1)}_{2,\mathrm{LO}}(\mu)\mathcal{O}^{(1)}_2(n_1,n'_1)$, just multiplying it by the running factor for $\mathcal{O}^{(1)}_1$,

$$U^{(1)}_{\mathrm{LL}} = \Delta_q(\mu_0,\mu_1)\,\Delta_g^{1/2}(\mu_0,\mu_1)\,, \tag{4.51}$$

and applying the replacement rule

$$\begin{aligned}
(\bar{\chi}_{n_2})_i &\rightarrow (c^{\alpha}_{\mathrm{LO}}(n_1))_{ji}\,(\bar{\chi}_{n_3})_j g\mathcal{B}^{n'_2\perp}_{\alpha}\,, \\
c^{\alpha}_{\mathrm{LO}}(n_1) &= \frac{\overline{q}_1}{q_1^2}\left(n_1^{\alpha} + \frac{(\not{q}_2)_{n_1\perp}\gamma^{\alpha}_{n'_2\perp}}{\overline{q}_2}\right)\frac{\not{\bar{n}}\not{n}_1}{4}\,\Theta_{\lambda^4}[n_2\cdot n'_2]\,,
\end{aligned} \tag{4.52}$$

where n_2 and n'_2 are the direction where the quark and the second emitted gluon are collinear, they are defined in appendix B.1. One can iterate this procedure to obtain the LO result for N-gluon emission. If we use the replacement rule $N-1$ times we go

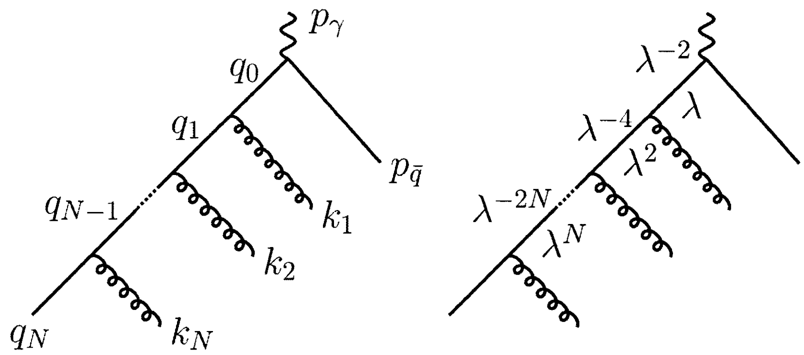

Figure 4-6: Left panel: our kinematical convention for a strongly ordered process. Quark momenta are denoted by q_i and gluon momenta by k_i. Right panel: power counting of the LO coefficient in SCET$_N$. The powers of λ with negative exponents refer to the propagator contribution to the amplitude. Those with positive exponents refer to the perpendicular momentum of the gluon with respect to its parent, since SCET vertices are proportional to it.

down to SCET$_N$ operator $C_{N,\mathrm{LO}}^{(N-1)}\mathcal{O}_N^{(N+1)}(n_N, n'_1, \ldots, n'_{N+1})$, after which Lagrangian emissions are no longer distinguished as separate particles. We have[5]

$$\begin{aligned}
\mathcal{O}_N^{(N-1)}(n_N, n'_1, \ldots, n'_{N-1}) =&\bar{\chi}_{n_N}\left(\prod_{k=1}^{N-1} g\mathcal{B}_{\alpha_k}^{n'_k\perp}\right)\chi_{\bar{n}}\,, \qquad (4.53)\\
C_{N,\mathrm{LO}}^{(N-1)} =& \left(\prod_{k=1}^{N-1} U_{\mathrm{LL}}^{(k-1)}(\mu_{k-1},\mu_k)\, c_{\mathrm{LO}}^{\alpha_k}(n_{k-1})\right)\Gamma^\mu,\\
c_{\mathrm{LO}}^{\alpha_k}(n_{k-1}) =& \frac{\bar{q}_{k-1}}{q_{k-1}^2}\left(n_{k-1}^{\alpha_k} + \frac{(\not{q}_{k\perp})_{n_{k-1}}\gamma_{n'_k\perp}^{\alpha_k}}{\bar{q}_k}\right)\frac{\not{\bar{n}}\not{n}_{k-1}}{4}\,\Theta_{\lambda^{2k}}[(n_k\cdot n'_k)]\,,\\
U_{\mathrm{LL}}^{(k-1)}(\mu_{k-1},\mu_k) =& \Delta_q(\mu_{k-1},\mu_k)(\Delta_g(\mu_{k-1},\mu_k))^{(k-1)/2}\,,
\end{aligned}$$

Where the variables for i emission are defined in Fig. 4-6, $\mu_0 = Q$, $\mu_k = \sqrt{|n_k\cdot n'_k|}Q$ and $q_k = \left(q_i + \sum_{j=k-i}^{n} k_j\right)^2$. n_k and n'_k are the directions where the quark q_k and the external quark k_k have not perpendicular component, they can be related to n_{k-1} through reparametrization invariant (*cf.* appendix B.1). We can extend the argument we used to calculated the power counting of $C_{2,\mathrm{LO}}^{(1)}$ to the SCET$_N$ coefficient in Eq. (4.53), just counting the contributions form vertices and the propagators,

[5] Because we only analyze gluon emissions from a quark, we have at SCET$_N$ exactly $N-1$ gluon emissions, so the SCET$_N$ operator is $\mathcal{O}_{N-1}^{(N)}$.

$C^{(N-1)}_{N,\text{LO}}$ has a power counting $\sum_i^{N-1} 1/\lambda^{-i} = \lambda^{-N(N-1)/2}$, see Fig. (4-6).

Similarly to the discussion above Eq. (4.49), we can extract the vertex part of $c^{\alpha_k}_{\text{LO}}$ to define P^{α_k}. We get that

$$|P^{\alpha_k}|^2 = \frac{1+z_k^2}{q^2_{k\perp\, n_{k-1}}}, \tag{4.54}$$

where $z_k \equiv \bar{q}_k/\bar{q}_{k-1}$. Thus, we get that the amplitude squared goes like the factorized product of the appropriate $1 \to 2$ splitting functions. Since $\mathcal{O}^{(i-1)}_i(n_{i-1}, n'_1, \ldots, n'_{i-1})$ is just built up from the repeated use of Eq. (4.46), we see that it requires no added information after we compute the first $q \to qg$ splitting. Thus, what we need to pass to a shower algorithm comes just from single real and single virtual gluon computations, as we list below in Subsection 4.3.3 in Table 4.1. Collinear splitting is entirely handled by the splitting rule in Eq. (4.46) at LL order.[6]

We have seen that to describe LO parton shower we only need operators with χ_n and $\mathcal{B}_n$ fields, but when we go to NLO, we will see that we can also have operator with $\mathcal{P}_{n\perp}$. In the final SCET$_N$ where all χ_n and $\mathcal{B}_n$ fields have their own n_i directions we do not need operators with factors of $\mathcal{P}_{\perp n_i}$ in the operator. Such operators encode redundant information that is specified by RPI. That is, in SCET$_N$ we only have the operator $\mathcal{O}^{(N-1)}_N(n_N, n'_1, \ldots, n'_{N-1})$ and all the corrections are encoded in the Wison coefficients.

4.3.2 Soft emissions

SCET describes soft degrees of freedom using soft quark and gluon field: $q_s(x)$ and $A_s(x)$. In this work we focus on fully differential cross section where we can always distinguish collinear and soft modes, but in an integrated cross section we have to

[6]It is straightforward to see that we do not have additional contributions at LO. Firstly, consider the possibility of operators that do not take the form of a single-field replacement rule. These would depend on the details of the hard process that produced the quark in the first place and could threaten the factorization of the shower. In fact, we will get such terms when we match QCD $\to$ SCET$_1$, but they are always suppressed, as we discuss in sec. 4.4. Returning to single-field replacement, let us consider matching SCET$_1$ $\to$ SCET$_2$, as results in this case will generalize to all SCET$_i$. Rule (4.46) sends $\chi_{n_1} \to C\mathcal{B}^{\mu}_{n_3\perp}\,\chi_{n_2}$. At LO, we cannot get such a replacement involving mutliple gluon fields, $\mathcal{B}_{n_j\perp}$, as this implies that we have integrated out multiple, hard ($\sim Q\lambda^2$) propagators. Such a contribution would not be strongly ordered, and is suppressed. We will see in sec. 4.4 that we do have such contributions at higher orders.

implement soft emissions with some form of zero-bin subtractions [78] to avoid interference between soft and collinear radiation. As we have seen in subsection 2.1, the collinear sector and the soft sector couple through the covariant derivative λ^2, $iD_s^\mu = i\partial^\mu + gA_s^\mu$, (Eq. (2.13)) acting on the collinear fields. At LO the collinear particles only couple to the $n \cdot A_s$ component of the soft gluons and the soft-collinear factorization guarantees that we can absorb all this couplings into Wilson line $Y(x)$ along the direction of the collinear particle, Eq. (2.15). In SCET this is accomplished by making the field redefinitions in Eqs. (2.14), so the new collinear fields no longer couple to soft gluons through their kinetic term. The outcome for the composite fields considered here is that

$$\chi_n \to Y_n \chi_n \,, \qquad \mathcal{B}_n^\mu \to Y_n \mathcal{B}_n^\mu Y_n^\dagger \,. \tag{4.55}$$

Note that here we consider nonabelian soft interactions which is why the soft Wilson lines do not cancel for the $\mathcal{B}_n^\mu$ field.

In matching SCET$_i$ to SCET$_{i+1}$ we will only consider external soft modes in SCET$_{i+1}$ with momenta $k^\mu \sim Q\lambda^{2(i+1)}$. These modes are contained within the soft modes in SCET$_i$. We do not consider particles with soft momenta $k^\mu \sim Q\lambda^{2i}$ that could not be encoded by onshell modes in SCET$_{i+1}$. Since such modes are forced to have larger momenta than the soft fields in SCET$_{i+1}$ they are not responsible for IR divergences, and any contributions from momenta of this type can be encoded in the Wilson coefficients of our SCET$_{i+1}$ operators with its field content.

In a given SCET$_i$ after making the field redefinition the effect of soft gluons is encoded by Wilson lines Y_n in the operators, with the form

$$\bar{\chi}_{n_N}^{(0)} Y_{n_N}^\dagger \prod_{k=1}^{N} Y_{n'_k} \mathcal{B}_{n'_k \perp}^{(0)\alpha_k} Y_{n'_k}^\dagger \, \Gamma_\mu Y_{\bar{n}} \chi_{\bar{n}} \,. \tag{4.56}$$

In the context of SCET$_i$, the angular ordering property of the soft emission and the coherent parton branching formalism for soft emission with multiple hard partons emerge naturally. If we take the Fourier transform of $Y_n(x)$ we get

$$Y = 1 + \sum_{m=1}^{\infty} \sum_{\text{perms}} \frac{(-g)^m}{m!} \frac{n \cdot A_s^{a_1} \cdots n \cdot A_s^{a_m}}{n \cdot k_1 n \cdot (k_1 + k_2) \cdots n \cdot (\sum_{i=1}^{m} k_i)} T^{a_m} \cdots T^{a_1} \tag{4.57}$$

where k_1, k_2, ... k_n are the momenta of the gluon fields. The eikonal structure of (4.57) leads to angular ordering. If a collinear particle with momentum q_i in the n_i direction emits a soft gluon of momentum k_s, the amplitude acquires a term proportional to

$$F_{\text{soft}} = \frac{n_i \cdot \varepsilon_s}{n \cdot k_s} = \frac{q_i \cdot \varepsilon_s}{q_i \cdot k_s} + O(\lambda)\,, \tag{4.58}$$

where ε_s is the polarization vector of the soft radiation and $q_i^\mu = \bar{q}\, n_i^\mu/2$ up to power corrections. If $A_n(q_1, q_2, \cdots, q_n)$ is the amplitude to emit n collinear particles with momenta $q_1, q_2, \cdots, q_n$ and A_{n+1} the amplitude with one more emission k_s in the soft region, we get $A_{n+1}(q_1, q_2, \cdots, q_n, k_s) \sim A_n(q_1, q_2, \cdots, q_n) \sum_{i=1}^{n} C_i\, q_i \cdot \varepsilon_s / q_i \cdot k$, where C_i is a color factor. For the cross section this implies

$$d\sigma_{n+1} = d\sigma_n \frac{dE_s}{E_s} \frac{d\Omega_s}{2\pi} \frac{\alpha_s}{2\pi} \sum_{i,j} C_{i,j} W_{i,j}\,, \tag{4.59}$$

where $d\Omega_s$ and E_s are the element of solid angle and the energy of the emitted soft gluon, $C_{i,i}$ is a color factor. Here

$$W_{i,j} = \frac{E_s^2\, q_i \cdot q_j}{q_i \cdot k_s\, q_j \cdot k_s} \tag{4.60}$$

is called the radiation function. After integration of $W_{i,j}$ over azimuthal angular variables the soft gluons only contribute when the gluon is confined to the cones centered in the direction of the particle i or j, and are hence angular ordered.

To see how coherent branching emerges we consider the effect of soft gluons in our leading order matching between SCET$_i$ and SCET$_{i+1}$. If we consider effects encoded by operators with exactly the same collinear field content in SCET$_i$ and SCET$_{i+1}$ then graphs involving soft gluons agree and there is no contribution to the matching. If we consider instead the calculations that lead to the LO replacement rule $\bar{\chi}_{n_0} \to c_{\rm LO}\bar{\chi}_{n_1}\mathcal{B}^{\perp}_{n_1'}$, then the soft gluons are encoded by

$$\text{SCET}_i: \quad \bar{\chi}_{n_0} Y^{\dagger}_{n_0}\,, \qquad\qquad \text{SCET}_{i+1}: \; c_{\rm LO}\bar{\chi}_{n_1} Y^{\dagger}_{n_1} Y_{n_1'} \mathcal{B}^{\perp}_{n_1'} Y^{\dagger}_{n_1'}\,. \tag{4.61}$$

For soft gluons at wide angles relative to n_0, n_1, and n_1' the effect of attachments to $Y^{\dagger}_{n_1} Y_{n_1'}$ are power suppressed because soft attachments to these two lines cancel up to terms that are power suppressed by $n_1 \cdot n_1' \sim \lambda^{2i}$. The remaining attachment to $Y^{\dagger}_{n_1'}$ looks the same as attachments to $Y^{\dagger}_{n_0}$ at leading power since $n_0 \cdot n_1' \sim \lambda^{2i}$. Thus wide angle soft gluons do not resolve the substructure revealed by matching to SCET_{i+1} and effectively only couple to the overall color charge of the parent quark $\bar{\chi}_{n_0}$. Soft radiation that is close in angle to n_1 and n_1' resolves the split quark $\bar{\chi}_{n_1}$ and gluon $\mathcal{B}^{\perp}_{n_1'}$, compensating for the $n_1 \cdot n_1'$ suppression by additional collinear singularities in its propagator factors. Thus the coherent branching formalism for soft gluons emerges naturally for amplitudes in our SCET_i picture.

From the SCET point of view it would be natural to distinguish soft and collinear radiation in the shower and treat them independently, being careful not to double count. For simplicity all available shower codes treat them in a coherent fashion. The effect of accounting for soft coherent branching in the shower typically leads to modifications of the Sudakov probability factors (see for example Ref. [44]), and effects the choice of evolution variable or by adding additional vetoes. In the context of SCET the implications of this were discussed recently in Ref. [19].

4.3.3 Summary LO Parton Showers

In table 4.1 we summarize results for the mapping between the LO/LL parton shower at and our SCET_i picture. In the first column we put the elements needed for showering, and in the central column the translation to elements in the SCET_i setup. The usual splitting function is related to our replacement rule $\bar{\chi}_{n_0} \to c^{\rm LO}\bar{\chi}_{n_1}\mathcal{B}^{\perp}_{n_1'}$, that in turn is related to the SCET_2 coefficient of the operator $\mathcal{O}_2^{(1)}$. The LL Sudakov comes from LL running factors related to the one-loop cusp anomoulous dimension as in Ref. [17]. At leading order soft emission in SCET_i is taken into account adding

Shower Ingredients	Quantity in SCET$_i$	Found In:
Splitting function	Replacement rule	Eq.(4.46)
LL Sudakov factor	One-loop cusp anomalous dimension	Eq. (4.10)
Soft emission	Soft amplitude	Eq. (4.59)
Evolution variables	$\bar{q},\ n_i \cdot n_j$	

Table 4.1: Mapping between parton shower and SCET$_i$ at LO/LL.

soft Wilson lines Y_n into our operators. This leads to angular ordering and coherent branching, which must be accounted for with modifications to the shower to account for the soft singular regions. Finally, showers are constructed with different choices of evolution variables and the choice effects the structure of power corrections. In SCET$_i$ we have seen that we can write all coefficients in terms of the large momenta ($\bar{q}$) and dot product of ns vectors ($n_i \cdot n_j$), which are natural variables in the SCET$_i$ picture. At LO/LL the translation to any appropriate set of evolution variables is straightforward. Beyond this order one must systematically account for additional power corrections induced by changing variable from $\bar{q}_i$ and $n_i \cdot n_j$ to the desired evolution variabbles for the shower.

4.4 SCET Power Corrections to the Shower

As we have seen in the previous section, we reproduce the usual parton shower by matching collinear gluon emissions to increasingly lower-scale EFTs, the SCET$_i$. Our goal is to catalog the leading corrections (in λ) to the cross section for the emission of an arbitrary number of collinear gluons to a quark. By this we mean all amplitude terms to LO and NLO, as well as those at NNLO that can interfere with LO. As we will argue in subsection. 4.4.4, in many cases of interest, there is no LO/NLO interference, and so we focus on NLO/NLO and LO/NNLO. Just as in the strongly-ordered case, it is convenient to integrate down to SCET$_{i+1}$ when describing the emission of i-gluons. We obtain these corrections by doing our matching computations at higher order. We will show that there are two distinct types of subleading matching, and they have a different physical interpretation:

- One set originates in matching QCD $\to$ SCET$_1$ at higher orders. This generates a set of subleading terms that remain suppressed as we move down to lower-scale SCET$_j$'s. We call them *hard-scattering* corrections as they involve the details of the hard-scale process that created our initial hadrons. Also, they are most important for partons radiated closest to the original vertex.

- The other set comes from the subleading matching SCET$_i$ $\to$ SCET$_{i+1}$. They involve processes described by the SCET$_i$ lagrangian, but ones that get integrated out into higher dimension operators at lower scales. These corrections are ubiquitous. They do not depend on the hard-scattering details, and we can determine them for arbitrary SCET$_i$ $\to$ SCET$_{i+1}$ once we have found them in SCET$_1$ $\to$ SCET$_2$. Furthermore, they relate to known $\mathcal{O}(\alpha_s)$ corrections to the $q \to qg$ splitting function. For this reason we call them *jet-structure* corrections.

Determining the above to NLO in the cross-section will only involve single and double gluon emission. Thus, we will never need to compute in a lower-scale theory than SCET$_3$. We perform all the QCD$\to$SCET$_1$$\to$SCET$_2$$\to$SCET$_3$ matchings for these amplitudes necessary to calculate the corrections in appendices B.2-B.4. Below, we discuss the final results for the corrections, with Subsection 4.4.1 focusing on the hard-scattering corrections and Subsection 4.4.2 on the jet-corrections ones. For these portions of the work, the matching is only done at tree level, though formulas in the appendices include one-loop RG kernels. We discuss the effects of LL running on correction terms in Subsection 4.4.3. Lastly, in Subsection 4.4.4, we will study the amplitude squared and we will see there is a great simplification of the interference structure in SCET$_N$. We will also describe steps going from our subleading operators toward the systematic improvement of the parton shower to NLL.

4.4.1 Hard-Scattering Corrections

Let us begin by examining the matching QCD$\to$SCET$_1$ for single gluon emission collinear to the quark. For this case, all corrections are of the hard-scattering type. Unlike LO, corrections can have dependence on the process that creates the $\bar{q}q$ pair.

For concreteness, we will consider the coupling of QCD quarks to the vector current $J^{\mu}_{\rm QCD} = \bar{q}\gamma^{\mu}q$. The matching is performed in the centre of mass frame with the initial virtual photon with momentum $p_\gamma = (Q,0,0,0)$. The full details of matching calculation for QCD to SCET$_1$ is performed in appendix B.2. To reproduce the full QCD current $J^{\mu}_{\rm QCD}$, we need an infinite tower of SCET$_1$ operators increasingly higher order in λ. However, to get the required amplitude to NNLO, we only need four:

$$
\begin{aligned}
A^{q\bar{q}g}_{\text{to NNLO}} = C^{(0)}_{1,\text{LO}} \int & dx \langle 0| T\{\mathcal{L}_{\text{SCET}_1}(x)\mathcal{O}^{(0)}_1(n_0)\}|\, q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle \\
& + C^{(1)}_1(n_0,n_0)\langle 0|\mathcal{O}^{(1)}_1(n_0,n_0)|\, q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle + C^{(1)}_{1,\mathcal{T}}(n_0,n_0)\langle 0|\mathcal{T}^{(1)}_1(n_0,n_0)|\, q_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle \\
& + C^{(1)}_1(n_1,n_1')\langle 0|\mathcal{O}^{(1)}_1(n_1,n_1')|\, q_{n_1} g_{n_1'} \bar{q}_{\bar{n}}\rangle ,
\end{aligned}
\tag{4.62}
$$

where

$$
\begin{aligned}
\mathcal{O}^{(0)}_1(n_0) &= \bar{\chi}_{n_0}\chi_{\bar{n}} \, , \\
\mathcal{O}^{(1)}_1(n_0,n_0) &= \bar{\chi}_{n_0}\, g\mathcal{B}^{\alpha}_{n_0\perp}\chi_{\bar{n}} \, , \\
\mathcal{T}^{(1)}_1(n_0,n_0) &= \bar{\chi}_{n_0}\left[\mathcal{P}^{\beta}_{n_0\perp}\, g\mathcal{B}^{\alpha}_{n_0\perp}\right]\chi_{\bar{n}} \, , \\
\mathcal{O}^{(1)}_1(n_1,n_1') &= \bar{\chi}_{n_1}\, g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}} \, .
\end{aligned}
\tag{4.63}
$$

We gave the expression for $C^{(0)}_{1,\text{LO}}$ in Eq. (4.41). The amplitude from the operator $\mathcal{O}^{(0)}_1(n_0)$ is shown in the first diagram in the SCET$_1$ column of Fig. 4-7, those from $\mathcal{O}^{(1)}_1(n_0,n_0)$ and $\mathcal{T}^{(1)}_1(n_0,n_0)$ in the second, and that for $\mathcal{O}^{(1)}_1(n_1,n_1')$ in the third. We call $\mathcal{O}^{(1)}_1(n_0,n_0)$ and $\mathcal{T}^{(1)}_1(n_0,n_0)$ "two-jet" operators as they are labeled only with two distinct collinear directions (n_0 and $\bar{n}$) (for the direction of the antiquark we do not put a symbol following the convention in Eq. (4.31)). They describe a gluon collinear to the quark. We obtain the coefficients $C^{(1)}_1(n_0,n_0)$ and $C^{(1)}_{1,\mathcal{T}}(n_0,n_0)$ by expanding the QCD amplitude in the limit of small gluon momentum transverse to the quark's direction with the usual SCET parametrics: $(n_0, \bar{k}_1, k_{1n_0\perp}) \sim (\lambda^2, 1, \lambda)Q$. There are an additional set of two-jet configurations corresponding to the gluon being collinear to the antiquark. These are trivial to obtain by charge conjugation and we do not discuss them separately here. The operator $\mathcal{O}^{(1)}_1(n_1,n_1')$ is a three-jet configuration,

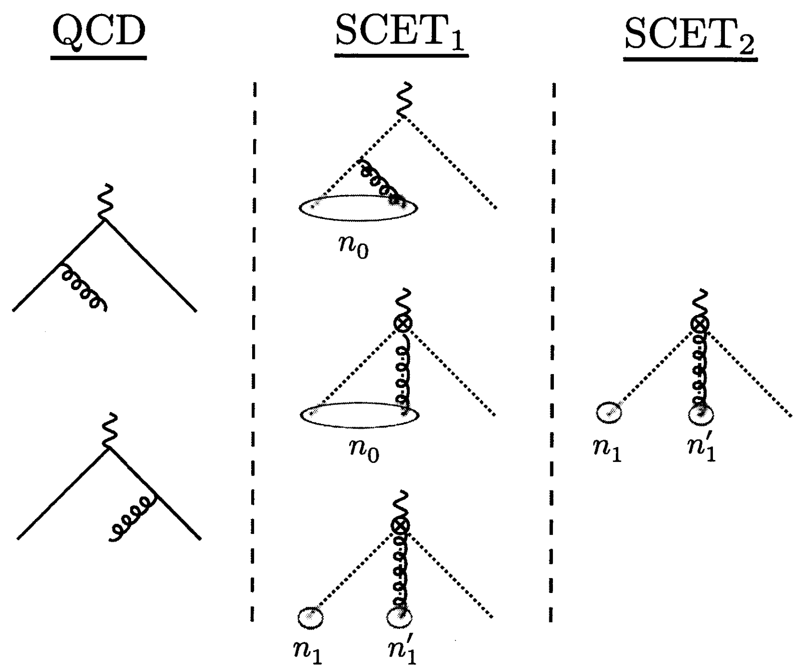

Figure 4-7: Matching QCD to SCET$_1$ to SCET$_2$ for one emission collinear to the quark direction. We depict the operator structures that lead to this process in each of the the three theories. The QCD contribution is standard. In SCET$_1$, we either emit a collinear gluon through the time-ordered product of the Lagrangian with an two-parton operator, or from three-parton operators. In SCET$_2$, this configuration only arises from a higher-dimension three-parton operators.

as it describes three distinct directions where the quark, the gluon and the antiquark are far apart according to the definition of collinearity in SCET$_1$. Whenever we have an operator where each field has its own index label, we can chose for the purpose of matching the n_i such that they are exactly aligned with the external particle momenta. The coefficients $C_1^{(1)}(n_0, n_0)$ and $C_1^{(1)}(n_1, n_1')$, are discussed in Eqs. (B.36) and give:

$$C_1^{(1)}(n_0, n_0) = \frac{1}{Q}(n_0^\mu - \bar{n}^\mu)\gamma^\alpha_{n_0\perp} ,$$
$$C_{1,\mathcal{T}}^{(1)}(n_0, n_0) = \frac{1}{\bar{q}_1 \bar{k}_1}\gamma^\mu_{n_0\perp}\gamma^\beta_{n_0\perp}\gamma^\alpha_{n_0\perp} - \frac{2}{\bar{q}_1 Q}\, g^{\beta\mu}\gamma^\alpha_{n_0\perp} , \tag{4.64}$$

We use the same kinematic variables defined in Fig. 4-4. For $C_1^{(1)}(n_0, n_0)$ and $C_{1,\mathcal{T}}^{(1)}(n_0, n_0)$ the initial current is not a spectator, so neither term is simply proportional to the γ^μ that we started with. This dependence on the details of the rest of process is a characteristic feature of these hard-scattering corrections. We give the coefficient

$C_1^{(1)}(n_1, n_1')$ of the operator with quark, gluon, and antiquark in distinct collinear directions in Eq. (B.39).

Going to SCET_2 for single gluon emission is straightforward. The basis of operators needed to reproduce the amplitude (4.62) is equal to (4.63), but where they are defined in SCET_2 instead of SCET_1: $\mathcal{O}_2^{(0)}(n_0)$, $\mathcal{O}_2^{(1)}(n_0, n_0)$, $\mathcal{T}_2^{(1)}(n_0, n_0)$, and $\mathcal{O}_2^{(1)}(n_1, n_1')$. As the computations get more complicated with subsequent emissions, we wish to minimize our effort by only including those terms necessary to give the corrections to a shower algorithm. This means we are only interested in the following:

1. Since we construct observables by squaring and integrating amplitudes, we will need to keep those NNLO contributions that can interfere with LO. These give terms at the same order as an NLO operator squared with itself. We do not compute NNLO amplitude terms which have zero interference with the LO amplitude. A list of the necessary computations is found in Appendix B.3.

2. Our ultimate goal is not a complete SCET_i theory from which one can do computations, but an improved shower algorithm. In Table 4.1, we give a list of those ingredients needed to construct a map between SCET_i and a LL parton shower. We will augment the map with items needed for corrections (Table 4.2 in subsection 4.5), but will not calculate contributions which only contain redundant information for the shower.

The latter point has important implications for the sorts of operator structures we need to consider. If we wanted to do computations in SCET_2, then of course we would need all operators and Wilson coefficients to the order we are working. As discussed above, single gluon contributions in SCET_2 where the gluon and the quark are collinear at $\mathcal{O}(\lambda^2)$ (*i.e.* $\mathcal{O}_2^{(1)}(n_0, n_0)$ or $\mathcal{T}_2^{(1)}(n_1, n_1)$) correspond to a quark which does not split until after the scale of matching $\mathrm{SCET}_1 \to \mathrm{SCET}_2$. However the corresponding no-branching probability is already determined in SCET_1, so the coefficients of these operators in SCET_2 are not required. Thus, we only need to calculate those single gluon contributions where each field has its own index label in SCET_2, meaning determine the Wilson coefficient $C_2^{(1)}(n_1, n_1')$ of $\mathcal{O}_2^{(1)}(n_1, n_1')$.

The matching equation for $C_2^{(1)}(n_1, n_1')$ in SCET$_2$ is:

$$\begin{aligned}
&C^{(0)}_{1,\,\mathrm{LO}}(n_0)\int dx^4 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(0)}(n_0)\}|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle + C_1^{(1)}(n_0,n_0)\langle 0|\mathcal{O}_1^{(1)}(n_0,n_0)|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle \\
&+ C_1^{(1)}(n_1,n_1')\langle 0|\mathcal{O}_1^{(1)}(n_1,n_1')|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle + C^{(1)}_{1,\mathcal{T}}(n_0,n_0)\langle 0|\mathcal{T}_1^{(1)}(n_0,n_0)|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle \\
&= C_2^{(1)}(n_1,n_1')\langle 0|\mathcal{O}_2^{(1)}(n_1,n_1')|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle\,.
\end{aligned} \tag{4.65}$$

It is convenient to decompose $C_2^{(1)}(n_1, n_1')$ as

$$C_2^{(1)} = C^{(1)}_{2,\,\mathrm{LO}} + C^{(1)H,a}_{2,\,\mathrm{NLO}} + C^{(1)H,b}_{2,\,\mathrm{NLO}} + C^{(1)H}_{2,\,\mathrm{NNLO}}\,, \tag{4.66}$$

where $C^{(1)}_{2,\,\mathrm{LO}}$ is the coefficient multiplying $\langle 0|\mathcal{O}_2^{(1)}(n_1,n_1')|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle$ that reproduces the second line in Eq. (4.65), $C^{(1)H,a}_{2,\,\mathrm{NLO}}$ the third line, *etc.* $C^{(1)}_{2,\,\mathrm{LO}}$ was calculated in (4.41) using RPI to rotate objects in the SCET$_1$ amplitude such that they can come from SCET$_2$ operators. The third and fourth terms give NLO coeffcients, which we can calculate in a similar manner to $C^{(1)}_{2,\,\mathrm{LO}}$. Their values are derived in Eqs. (B.57) and (B.59):

$$\begin{aligned}
C^{(1)H,a}_{2,\,\mathrm{NLO}} &= \frac{1}{Q}\Big(\frac{\bar k_1 n_1'^{\mu} + \bar q_1 n_1^{\mu}}{\bar q_0} - \Big(1+\frac{\bar q_1 \bar k_1}{2\,\bar q_0^2}(n_1\!\cdot\! n_1')\Big)\bar n^{\mu}\Big)\gamma^{\alpha}_{n_1'\perp}\Theta_{\lambda^2}[n_1\cdot n_1']\,, \qquad (4.67)\\
C^{(1)H,b}_{2,\,\mathrm{NLO}} &= -\frac{2}{(n_1\!\cdot\! n_1')\bar q_1 \bar k_1}\gamma^{\alpha}\not{p}_{\gamma}\,\gamma^{\mu}_T \\
&\quad + \Big[\frac{1}{(n\!\cdot\! p_{\bar q})\bar k_1}\Big(\gamma^{\mu}_T\not{p}_{\gamma} - \bar q_1\, n^{\mu}_{1T}\Big) + \frac{2(n\!\cdot\! p_{\bar q})}{(n_1\!\cdot\! n_1')\bar q_1 \bar k_1}\bar n^{\mu}_T\Big]\gamma^{\alpha}\tilde\Theta_{\lambda^2}[n_1\cdot n_1']\,,\\
C^{(1)H}_{2,\,\mathrm{NNLO}} &= \Big(\frac{1}{2\,Q}\Big(\gamma^{\mu}_{n_1'\perp}\sqrt{|n_1\cdot n_1'|}\not{v}_1 + \bar n^{\mu}\frac{\bar q_1}{Q}(n_1\cdot n_1')\Big)\gamma^{\alpha}_{n_1'\perp}\\
&\quad + \frac{\bar k_1}{Q^2}\Big(\sqrt{|n_1\cdot n_1'|}v_1^{\mu} - \bar n^{\mu}\,(n_1\!\cdot\! n_1')\frac{(\bar k_1^2-\bar q_1^2)}{2\,Q^2}\Big)\gamma^{\alpha}_{n_1'\perp}\Big)\Theta_{\lambda^2}[n_1\cdot n_1']\,.
\end{aligned}$$

Here n_1 and n_1' are aligned with the direction of the quark and the gluon for the matching, as in Eq. (4.37), and v_1 is defined in Eq. (B.14). In Eq. (4.67) we have not displayed the running factors that come from evolution of the SCET$_1$ operators. The SCET$_1$ operators in Eqs. (4.63) can have different running factors, in particular the two-jet and three-jet operators have different LL evolution. Therefore it is important

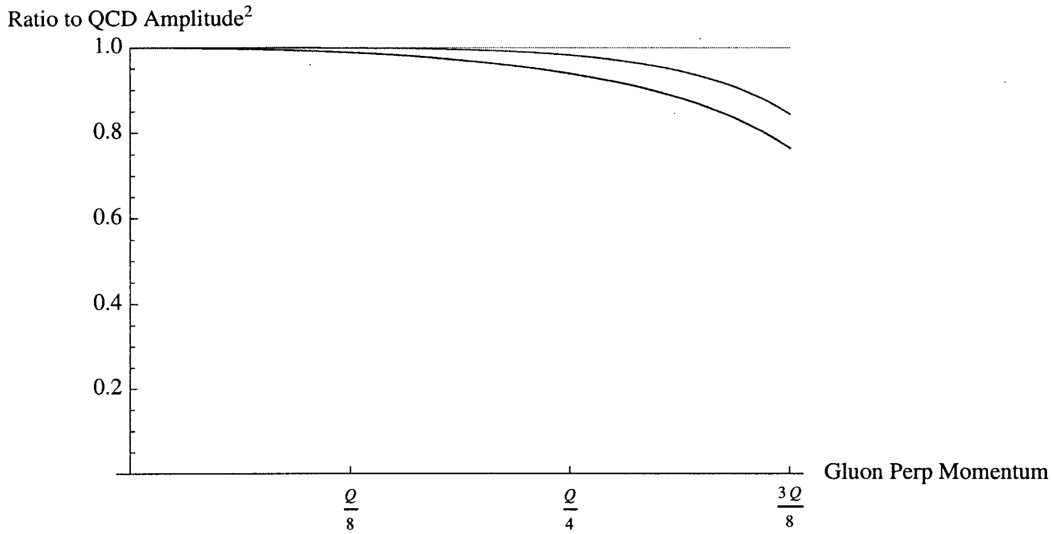

Figure 4-8: Plot of the ratios of the amplitudes squared for $\gamma^* \to q\bar{q}g$ for $R_{\rm LO} = |A^{q\bar{q}g}|^2_{\rm LO}/|A^{q\bar{q}g}|^2_{\rm QCD}$ (blue) and $R_{\rm NLO} = |A^{q\bar{q}g}|^2_{\rm NLO,2-jet}/|A^{q\bar{q}g}|^2_{\rm QCD}$ (red) versus $|k_1|_{n_0\perp}$, for $\bar{k}_1/\bar{q}_0 = 0.4$. The amplitudes are evaluated without running factors.

to decompose the coefficient $C_2^{(1)}$ as in Eq. (4.66) so that we can keep track of which SCET_1 evolution factor to include for each one. The running of these operators is discussed further in subsection 4.4.3.

To illustrate the effects of including hard-scattering corrections, in Fig. 4-8 we plot the ratios $R_{\rm LO} = |A^{q\bar{q}g}|^2_{\rm LO}/|A^{q\bar{q}g}|^2_{\rm QCD}$ and $R_{\rm NLO} = (|A^{q\bar{q}g}|^2_{\rm LO}+|A^{q\bar{q}g}|^2_{\rm to\,NNLO,2-jet})/|A^{q\bar{q}g}|^2_{\rm QCD}$ versus the gluon perp momentum. Here $|A^{q\bar{q}g}|^2_{\rm QCD}$ is the QCD amplitude squared for one-gluon emission, $|A^{q\bar{q}g}|^2_{\rm LO}$ is the SCET_2 amplitude squared for one-gluon emission from the LO coefficient $C^{(1)}_{2\rm LO}\mathcal{O}^{(1)}_2$ (from Eq. (B.62)), and $|A^{q\bar{q}g}|^2_{\rm to\,NLO,\,2-jet}$ is the NLO amplitude squared for one-gluon emission in the two-jet region that comes from the coefficients $C^{(1)H,a}_{2,\,\rm NLO}$ and $C^{(1)H}_{2,\,\rm NNLO}$ (given in Eq. (B.65)). Notice that including corrections up to NNLO in the amplitudes extends the region where tree-level SCET_1 and QCD agree. The advantage of using the one-gluon SCET amplitude over QCD is that it is incorporated into shower language of branching and no-branching, and also automatically avoids double counting with LL emissions.

In the above computation we have seen that the SCET_2 coefficient $C_2^{(1)}$ and corresponding operator $\mathcal{O}_2^{(1)}$ carry results that occured in several different operators in SCET_1. This operator carries the information that its fields are in distinct equivalence

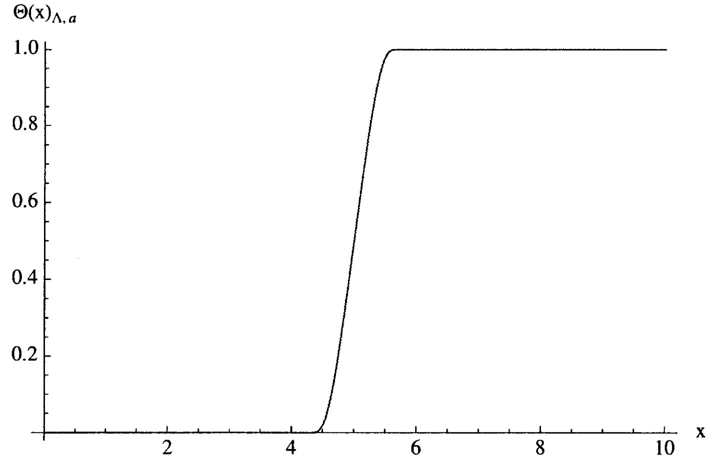

Figure 4-9: Plot of $\Theta(x)_{\Lambda,a}$ defined in Eq. (B.61) for $\Lambda = 5$ and $a = 0.8$. The parameter Λ tells where the function switches from 0 to 1 and the parameter a how fast it does it.

classes, $\{[n_j]\}$, with separations $\gg \lambda^4$. However it does not know just how inequivalent they are, whether they are separated by $\sim \lambda^2$ or by $\sim \lambda^0$, which is information that was stored in the operators in SCET$_1$. In SCET$_2$ this information must be stored in the Wilson coefficient in the matching procedure. For the case of $C_1^{(1)}(n_0, n_0)$ and $C_1^{(1)}(n_1, n_1')$, we need to keep their contributions separate as they correspond to parametrically distinct regions of phase space in SCET$_1$. To enforce this distinction, we use a Wilsonian cutoff on the dot product of collinear directions for different fields with the smoothed θ-function, $\Theta_{\lambda^j}[n_i, n_i']$ described in Subsection 4.3.1. The notation means that $\Theta \to 0$ once $n_i \cdot n_i' \gtrsim \lambda^j$, and $\tilde{\Theta} = 1 - \Theta$ behaves in the opposite fashion.

Therefore, $C_{2,\,\mathrm{NLO}}^{(1)H,a} \propto \Theta_{\lambda^2}[n_1, n_1']$ and $C_{2,\,\mathrm{NLO}}^{(1)H,b} \propto \tilde{\Theta}_{\lambda^2}[n_1, n_1']$, as these enforce where in the SCET$_1$ phase space these contributions originated. In appendix B.3, we define a smooth Θ function in Eq.(B.61), which is plotted in Fig. 4-9. It depends on two parameter, $\Theta(x)_{\Lambda,a}$, The parameter Λ tells where the function switches from 0 to 1 and the parameter a how fast it does it. By virtue of its appearance in the Wilson coefficients this Θ function merges the two-jet amplitude squared with a quark and gluon in one-jet, $(|A^{q\bar{q}g}|^2_{\mathrm{LO}} + |A^{q\bar{q}g}|^2_{\mathrm{to\,NNLO,\,2-jet}})$, with the three-jet amplitude squared $|A^{q\bar{q}g}|^2_{\mathrm{NNLO,\,3-jet}}$, that is the amplitude squared that come from using the coefficient $C_{2,\,\mathrm{NNLO}}^{(1)H,b}$. The full expression is given in Eq. (B.64). As shown in Fig. B-3, the theta

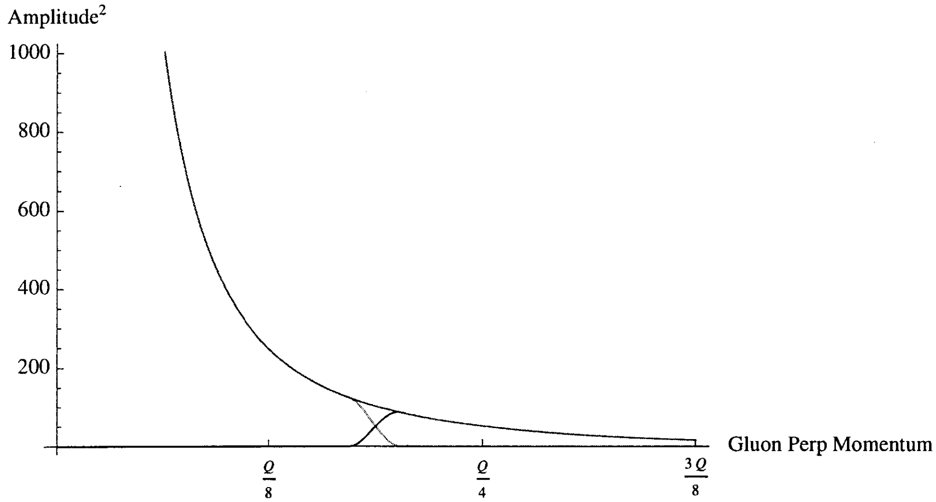

Figure 4-10: Merging of the two-jet and and three-jet amplitudes squared for $\gamma^* \to q\bar{q}g$ process using a smooth theta function. Plots of the two-jet amplitude square, $|A^{q\bar{q}g}|^2_{\rm LO} + |A^{q\bar{q}g}|^2_{\rm to\,NNLO,\,2-jet}$, (green), three-jet, $|A^{q\bar{q}g}|^2_{\rm NLO,\,3-jet}$, (blue) and sum (red) versus $|k_1|_{n_0\perp}$. The amplitudes are evaluated without running coefficients for $\bar{k}_1/\bar{q}_0 = 0.4$.

function smoothly merges the two squared amplitudes.

With two-gluon emission, the SCET$_1$ graphs will include jet-structure corrections in addition to hard-scattering ones. It is straightforward to distinguish the types as the former result from taking time-ordered products of the SCET$_1$ Lagrangian with operators generated by the LO replacement rule, Eq. (4.46), while the latter will come only from terms involving a power suppressed SCET$_1$ operator. To fully identify the subleading contributions to two-gluon emission, we must match down to SCET$_2$ where the LL contribution is first uniquely identified. We already know that the LO result comes from two applications of Eq. (4.46).

In Fig. (4-11), we show the contributions to two-gluon emission in QCD, SCET$_1$, SCET$_2$, and SCET$_3$. The first column in the SCET$_1$ category correspond to the jet-structure corrections to be considered in the next section. In the second column we have a set of hard-scattering corrections from taking the T-product of the SCET$_1$ Lagrangian with the suppressed single gluon operators we calculated above in Eqs. (4.64), $C_1^{(1)}\mathcal{O}_1^{(1)}$ and $C_{1,\mathcal{T}}^{(1)}\mathcal{T}_1^{(1)}$.

In considering the basis of operators in SCET$_2$ we do not need operators such as

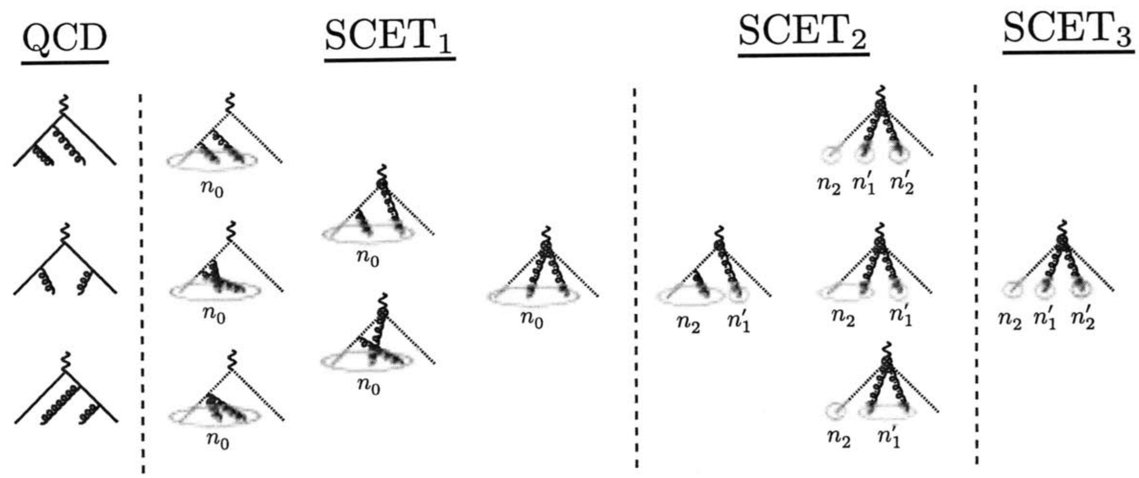

Figure 4-11: Matching QCD to SCET$_1$ to SCET$_2$ to SCET$_3$ for two-gluon emission collinear to the quark direction. Once again, we depict the operator structures that lead to this process in each of the theories. Gluons drawn away from the central vertex are emitted by the renormalizable lagrangian in that theory, while those coming from the vertex are due to higher-dimension operators.

$\mathcal{T}_2^{(1)}(n_1, n_1')$ since, $\mathcal{P}_{n_1'\perp}\mathcal{B}_{n_1'\perp} = 0$, with n_1' lying along the gluon momentum. The coefficient of such operators are determined by RPI, and we can use RPI in SCET$_2$ to make a coordinate choice where they are not necessary. As mentioned above in the single gluon matching section, our interest is only in calculating those terms needed to improve a shower algorithms, which precludes us from considering operators such as $\mathcal{T}_2^{(1)}(n_0, n_0)$ or $\mathcal{O}_2^{(1)}(n_0, n_0)$. Therefore, for double gluon emission we only need to calculate the coefficients of the following operators

$$\begin{aligned}
\mathcal{O}_2^{(1)}(n_1, n_1') &= \bar{\chi}_{n_1} g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}} \\
\mathcal{O}_2^{(2)}(n_2, n_2, n_1') &= \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_2\perp}\, g\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}\,, \\
\mathcal{O}_2^{(2)}(n_2, n_1', n_1') &= \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_1'\perp}\, g\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}\,, \\
\mathcal{O}_2^{(2)}(n_2, n_1', n_2') &= \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_1'\perp}\, g\mathcal{B}^{\beta}_{n_2'\perp}\chi_{\bar{n}}\,.
\end{aligned} \tag{4.68}$$

Thus in SCET$_2$, we are interested in *two* gluon operators where two fields can have the same label. When we pass to SCET$_3$ we can restrict our interest to only $\mathcal{O}_3^{(2)}(n_2, n_1', n_2')$, the operator whose collinear fields are all in distinct light-cone directions.

The coefficients of $\mathcal{O}_2^{(1)}(n_1, n_1')$ needed to compute the leading power corrections are given in Eqs. (B.59) and (B.57). We get an NLO contribution to the two gluon amplitude by computing the matrix element, $C^{(1)}_{2\,\rm NLO}\langle 0|T\{\mathcal{L}_{\rm SCET_2}\mathcal{O}^{(1)}_2\}|q\bar{q}gg\rangle$ (fifth column in Fig. 4-11). The power counting of the contribution does not change as the gluon from $\mathcal{L}_{\rm SCET_2}$ gives a vertex$\times$propagator factor of λ^{-3}, which is the same as LO. There are also coefficients we need from two-gluon matching calculations for the operator $\mathcal{O}^{(2)}_2$ (sixth column in Fig. 4-11). Putting in the index structures on the fields, these include $C^{(2)J}_{2,\rm NLO}(n_2,\, n_1',\, n_2')$ for $\mathcal{O}^{(2)}_2(n_2, n_2, n_1')$, $C^{(2)J}_{2,\rm NLO}(n_2,\, n_1',\, n_1')$ for $\mathcal{O}^{(2)}_2(n_2, n_1', n_1')$ and $C^{(2)J}_{2,\rm NLO}(n_2,\, n_2,\, n_2') + C^{(2)H}_{2,\rm NNLO}(n_2,\, n_2,\, n_2')$ for $\mathcal{O}^{(2)}_2(n_2, n_1', n_2')$. Since only the last one multiplies an operator that interferes with LO, it is the only one we will need to keep track of to NNLO for the hard scattering. (We will see in the next subsection that there are jet-structure corrections to all three terms in the sixth column at NLO.) All other hard-scattering contributions are beyond the order we need. Thus we only need to compute $C^{(2)H}_2(n_2,\, n_2,\, n_2')$ given by the matching equation

$$\begin{aligned}
&C^{(1)}_1 \int dx \langle 0|T\{\mathcal{L}_{\rm SCET_1}(x)\mathcal{O}^{(1)}_1(n_0, n_0)\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle \\
&- C^{(1)}_{2,\rm NLO}(n_2, n_1') \int dx \langle 0|T\{\mathcal{L}_{\rm SCET_2}(x)\mathcal{O}^{(1)}_2(n_2, n_1')\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle \\
&= C^{(2)H}_2(n_2, n_2, n_1')\langle 0|\mathcal{O}^{(2)}_2(n_2, n_2, n_1')|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle\,,
\end{aligned} \tag{4.69}$$

where we subtract the time-ordered product graph in $\rm SCET_2$ from the time-ordered product graph in $\rm SCET_1$. The result for $C^{(2)H}_2(n_2,\, n_2,\, n_2')$ is given in Eq. (B.104). It is easy to see why $\mathcal{O}^{(2)}_2$ only gets hard-scattering contributions up to this order. By defintion, hard-scattering has to involve a suppressed operator from the QCD $\rightarrow$ $\rm SCET_1$ matching, and so we begin at NLO at the lowest order. Including a second gluon, but demanding that we cannot write it as coming from a $\rm SCET_2$ lagrangian emission takes us to one order higher, namely NNLO.

All the contributions we have discussed so far have come from the hard-scattering, single-gluon, suppressed operators in $\rm SCET_1$. There are also those with two gluons. That is to say a process where neither gluon comes from the $\rm SCET_1$ Lagrangian, represented by the diagram in the third SCET_1 column in Fig. 4-11. One example is double emission from the antiquark, as shown in the third QCD graph of Fig. 4-11. We know from applying Eq. (4.46) twice, that LO for this process is at $\mathcal{O}(\lambda^{-3})$ counting only vertex×propagator factors, as these are all we need to compare different $q\bar{q}gg$ processes. We readily see that double antiquark emission is $\sim \lambda^0$. This follows from the same arguments we give above for single emission from the antiquark. Thus, they are N^{3}LO. This is higher order than we are analyzing. Besides antiquark vertices, we also have subleading emissions from the quark in QCD that arise from the suppressed SCET-spinor portion of the QCD quark propagator (*cf.* appendix B.2). In fact, we have already such contributions for single emission. However, if both emissions come from the suppressed propagator, once again, this is $\sim \lambda^0$ at lowest order, and so we can neglect it. Mixed antiquark/suppressed spinor contributions are also N^{3}LO. Thus, we do not need corrections to double emission collinear to the quark if they do not involve at least one SCET_1 Lagrangian insertion. We can extend this argument to further emissions without using the SCET_1 Lagrangian. Once again, these will go as $\mathcal{O}(\lambda^0)$, while LO goes like $\mathcal{O}(\lambda^{-\frac{i(i+1)}{2}})$. Thus, to the order we are working, we only need the single gluon hard-scattering correction (plus Lagrangian insertions). Furthermore, we only needed the NLO Wilson coefficient $C_1^{(1)}$, given in Eq. (4.64).

4.4.2 Jet-Structure Corrections

The *jet-structure* corrections only involve contributions from the SCET_1 Lagrangian. These arise from the graphs in the first SCET_1 column in Fig. 4-11. We specifically designed our leading order replacement rule in Eq. (4.46), so when it is used twice it only contains that part of double emission corresponding to the leading strongly-ordered limit. This occurs for the gluons having collinearities $\sim \lambda$, λ^2, respectively. However SCET_1 also describes other kinematic situations and in this section we compute the corrections from these regions.

The prescription for obtaining two-gluon jet-structure corrections is to compute the double gluon emission amplitude in SCET_1 coming from two lagrangian inserstions and take different limits on the relative collinearities of n_2, n_2', and n_1', where these labels refer to the null vectors exactly proportional the corresponding particle momenta. We can define:

$$A^{q\bar{q}gg}_{\rm NLO} = C^{(0)}_{1,\,\rm LO} \int dx_1 dx_2 \langle 0|T\{\mathcal{L}_{\rm SCET_1}(x_1)\mathcal{L}_{\rm SCET_1}(x_2)\mathcal{O}^{(0)}_1\}|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle, \qquad (4.70)$$

and then calculate,

$$\lim_{n_2\cdot n'_2\sim\lambda^2} A^{q\bar{q}gg}_{\rm NLO} = C^{(2)J}_{2,\,\rm NLO}(n_2,\, n'_1,\, n'_2)\langle 0|\mathcal{O}^{(2)}_2|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle \qquad (4.71)$$

$$\lim_{n'_1\cdot n'_2\sim\lambda^4} A^{q\bar{q}gg}_{\rm NLO} = C^{(2)J}_{2,\,\rm NLO}(n_2,\, n'_1,\, n'_1)\langle 0|\mathcal{O}^{(2)}_2|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle \qquad (4.72)$$

$$\begin{aligned}\lim_{n_2\cdot n'_2\sim\lambda^4} A^{q\bar{q}gg}_{\rm NLO} &= C^{(1)}_{2,\,\rm LO}\langle 0|T\{\mathcal{L}_{\rm SCET_2}\mathcal{O}^{(1)}_2\}|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle \\ &\quad + C^{(2)J}_{2,\,\rm NLO}(n_2,\, n_2,\, n'_1)\langle 0|\mathcal{O}^{(2)}_2|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle \qquad (4.73)\end{aligned}$$

We make a few things about the above equations. Firstly, there is a four-parton correction operator that has the same index structure as LO, $(n_2,\, n_2,\, n'_1)$. We cannot obtain it as a pure limit of $A^{q\bar{q}gg}_{\rm NLO}$, and we will need to subtract off the LO contributions. Secondly, the limit in Eq. (4.71) does not lead to an expansion of any part of $A^{q\bar{q}gg}_{\rm NLO}$, as the scaling of the n-indices' dot products is exactly that from $\rm SCET_1$. Even though it just gives back the same expression as the $\rm SCET_1$ amplitude $A^{q\bar{q}gg}_{\rm NLO}$, the $\rm SCET_2$ result for $C^{(2)J}_{2,\,\rm NLO}(n_2,\, n'_1,\, n'_2)\mathcal{O}^{(2)}_2$ tells us something more. This Wilson coefficient is proportional to $\tilde{\Theta}_{\lambda^2}[n'_1\cdot n'_2]\tilde{\Theta}_{\lambda^2}[n_2\cdot n'_2]$, where the $\tilde{\Theta}$'s only have support outside the phase space region of Eq. (4.72), as well as the strongly-ordered limit of Eq. (4.73), (see Eqs. (4.45) and (B.61) for the defintion of Θ, $\tilde{\Theta}$). The full results for the Wilson coefficients shown in Eq. (4.73) can be found in Eqs. (B.80), (B.94), and (B.102). At the amplitude level, given a particular phase space configuration for an external state, we will only ever need one of these terms for double gluon emission in $\rm SCET_2$. Squaring the result is straightforward as there will be no interference between them.

We will now examine how to improve the matching of SCET_i to SCET_{i+1}, and show that the jet-structure corrections computed here generalize to a matching result for this case. We first notice that the first two operators above do not interfere with the one giving LO, as they have different index structures. The subleading term in Eq. (4.73) does inhabit the same region of phase space as $C^{(1)}_{2,\,\mathrm{LO}}\mathcal{O}^{(1)}_2$, but as we will argue in Subsection 4.4.4 LO/NLO interference cancels out of most observables of interest. Before proceeding, we note that our description of corrections to two-gluon emission gets even simpler when we match to SCET_3. In SCET_3 the only operator we need has distinct collinear directions for all fields. Thus, we can write all hard and jet corrections to two-gluon emission we have found in the coefficient, $C^{(2)}_3$ for the operator $\mathcal{O}^{(2)}_3(n_2, n'_1, n'_2) = \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n'_1\perp} g\mathcal{B}^{\beta}_{n'_2\perp}\chi_{\bar{n}}$, as we do in Eqs. (B.110). The same will hold for i-gluon emission in SCET_{i+1}. Our NLO jet-structure operators therefore have the following form:

$$C^{(2)J,I}_{3,\,\mathrm{NLO}}(n_2,\, n_1,\, n'_1)\mathcal{O}^{(2)}_3 \;=\; h^{\alpha\beta}_I\,\bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n'_1\perp} g\mathcal{B}^{\beta}_{n'_2\perp}\Gamma^{\mu}\chi_{\bar{n}}\,, \tag{4.74}$$

where $h^{\alpha\beta}_I$ is given by Eq. (B.118). Here $I = \{1, 2, 3\}$ and we distinguish the coefficients $C^{(2)J,I}_{3,\,\mathrm{NLO}}$ depending from which SCET_2 operators they come from in order to properly account for their renormalization group evolution factors in SCET_2. For $I = 1$, it comes form the SCET_2 operator $\mathcal{O}^{(2)}_2(n_2, n'_2, n'_1)$, for $I = 2$ from $\mathcal{O}^{(2)}_2(n_2, n'_1, n'_1)$ and for $I = 3$ from $\mathcal{O}^{(2)}_2(n_2, n_2, n'_1)$.

When doing the LO matching for SCET_i to SCET_{i+1}, we found that the replacement rule to go from SCET_1 to SCET_2 generalized to the case of i-gluon strongly-ordered emission. Similarly, we can take the above operator, Eq. (4.74), and recast it as a replacement rule for our original current insertion, $C^{(0)}_{1,\,\mathrm{LO}}\mathcal{O}^{(0)}_1$. It takes the form of a $1 \to 3$ replacement rule:

$$\bar{\chi}_{n_0} \to h^{\alpha\beta}_I\,\bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n'_1\perp} g\mathcal{B}^{\beta}_{n'_2\perp}, \tag{4.75}$$

with contributions from $I = 1, 2, 3$.

If we want to consider the NLO radiation of $i{+}1$ gluons, we can perform a very similar matching between SCET_i and SCET_{i+2} to the one above for $\mathrm{SCET}_1 \to \mathrm{SCET}_3$ to obtain an operator $C^{(i+1)J}_{i+2,\,\mathrm{NLO}}\mathcal{O}^{(i+1)}_{i+2}$. Since the first $(i-1)$ emissions are strongly ordered, they completely factor out. Thus, the amplitude for the emission of the final two gluons will be identical to that for simple two-gluon emission. We can therefore take the $(i-1)$ gluon LO operator, $C^{(i-1)}_{i,\mathrm{LO}}\mathcal{O}^{(i-1)}_{i}$, and use the replacement rule in Eq. 4.75), to obtain $C^{(i+1)J}_{i+2,\mathrm{NLO}\,I}\mathcal{O}^{(i+1)}_{i+2}$. Our NLO replacement rule corresponds to violating strong ordering at any location in the shower, either by taking the j^{th} and $(j+1)^{\mathrm{th}}$ gluons to have the same parametric collinearity with respect to their parents, $k_{j+1\perp} \sim k_{j\perp}$, or by including the region of phase space where the propagator between them is hard, and so we get no collinear divergence as the quark and second gluon become collinear.[7]

It is not difficult to see that this gives an NLO contribution for any j. If we have i-gluon strongly-ordered emission, the propagators and vertices will go as $\lambda^{-i(i+1)/2}$, where the j^{th} gluon contributes λ^{-j}. If we violate strong ordering as we mention above for any two gluons, the product of their vertices times propagators goes like λ^{-2j} instead of $\lambda^{-(2j+1)}$. Thus, we can insert $\bar{\chi}_{n_0} \to h_I\,\bar{\chi}_{n_2} g\mathcal{B}_{n'_1\perp} g\mathcal{B}_{n'_2\perp}$ instead of two successive $\bar{\chi}_{n_0} \to c_{\mathrm{LO}}\bar{\chi}_{n_1}\mathcal{B}_{n'_1\perp}$'s in operator matching as a "defect" in strong ordering at any stage and obtain an NLO jet-structure correction. The Θ-functions contained in the Wilson coefficients, $C^{(i+1)J}_{i,\mathrm{NLO}}$, allow us to read off at which step in the shower we violated strong-ordering. In appendix C, we show how an integrated version of $h^{\alpha\beta}_{I}$ is related to the subleading splitting function which serves as a cross-check on our computations.

4.4.3 Operator Running

Up until now, our discussion of matching has taken place mostly at tree-level. Connecting to the leading-log parton shower, however, also requires that we include the effects of full one-loop and two-loop cusp needed for NLL running. For this reason, our final expressions for Wilson coefficients in Apps. B.2-B.4 all include the neces-

[7] At this point, one may ask why we do not go farther and consider the case $k_{j+1\perp} \gg k_{j\perp}$. In fact, we do not have to. Since the amplitude for i-gluon emission has an underlying Bose symmetry, we are free to partition phase space into $i!$ regions, each of which gives an identical contribution to the cross section. Thus, to get the final answer, we only need to integrate over one of them. While we can choose this region such that $k_{j+1\perp} \gg k_{j\perp}$ never occurs, we are forced to include $k_{j+1\perp} \sim k_{j\perp}$. If we do not wish to partition phase space in this manner then the Bose symmetry implies that the result for $k_{j+1\perp} \gg k_{j\perp}$ can be obtained from the configurations already discussed.

sary notation for evolution kernels. Identifying the power suppressed amplitudes as corresponding to perturbative corrections to more inclusive observables it is natural to take only LL evolution for power suppressed or α_s suppressed corrections, and include NLL evolution only for the leading shower terms. In this section we give arguments that determine the LL evolution for the subleading operators, and discuss what precisely is missing to obtain the full NLL evolution kernel.

To set the stage we consider as an example the running in SCET$_1$. We can consider our matching to QCD to have taken place at a scale Q, but we can run down to a lower scale μ, as is necessary before we further match to SCET$_2$. The zero and single gluon operators in SCET$_1$ acquire the following factors (*cf.* the tree-level version in Eq. (4.64)):

$$
\begin{aligned}
C_0^{(0)}(n_0) &= U^{(2,0,0)}(n_0)(Q,\mu)\,\gamma^\mu_{n_0\perp} \\
C_1^{(1)}(n_0,n_0) &= U^{(2,1,0)}(n_0,n_0)(Q,\mu)\otimes\frac{n_0^\mu-\bar{n}^\mu}{Q}\gamma^\alpha_{n_0\perp}\,, \\
C_{1,\mathcal{T}}^{(1)}(n_0,n_0) &= U_{\mathcal{T}}^{(2,1,1)}(n_0,n_0)(Q,\mu)\otimes\frac{1}{\bar{q}_1\bar{k}_1}\left(\gamma^\mu_{n_0\perp}\gamma^\beta_{n_0\perp}\gamma^\alpha_{n_0\perp}-\frac{2}{\bar{q}_1 Q}\,g^{\mu\beta}\gamma^\alpha_{n_0\perp}\right), \\
C_1^{(1)}(n_1,n_1') &= -U^{(2,1,0)}(n_1,n_1')(Q,\mu)\left(\frac{2}{(n_1\!\cdot\! n_1')\bar{q}_1\bar{k}_1}\gamma^\alpha_{n_0\perp}\,\not{\epsilon}_\gamma\,\gamma^\mu_T\right. \\
&\quad\left.+\left[\frac{1}{(n\!\cdot\! p_{\bar{q}})\bar{k}_1}\left(\gamma^\mu_T\not{\epsilon}_\gamma-\bar{q}_1\,n^\mu_{1T}\right)+\frac{2(n\!\cdot\! p_{\bar{q}})}{(n_1\!\cdot\! n_1')\bar{q}_1\bar{k}_1}\bar{n}^\mu_T\right]\gamma^\alpha_{n_0\perp}\right). \qquad (4.76)
\end{aligned}
$$

$U^{(j,k,l)}(n_1,n_2,\ldots,n_{j+k})$ is the RG-kernel of the operator $\mathcal{O}^{(j,k,l)}(n_1,n_2,\ldots,n_{j+k})$ defined in Eq. (4.7) and $U_{\mathcal{T}}^{(2,1,1)}$ is the RG-kernel for the operator $\mathcal{T}_1^{(1)}$. Since the antiquark is always collinear to $\bar{n}$, we avoid writing $\bar{n}$ in the RG-kernel's argument. We inserted the symbol $\otimes$ in the second and third line of Eq. (B.40), because when we have an operator where two or more fields share the same collinear direction, there can be a convolution in the fraction momentum $\bar{p}$ between the running factor and the coefficient. If the field are collinear in the same direction, they can talk each other in SCET and they can exchange momentum in running down from scale Q to μ. The anomalous dimension of an operator is independent of which SCET$_i$ it is defined, but does depend on the field content and in particular how many different collinear directions are in the operator. Thus, the RG-kernel of the operator $\bar{\chi}_{n_0} g\mathcal{B}^\alpha_{n_0\perp}\chi_{\bar{n}}$ is different from that of $\bar{\chi}_{n_1} g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}}$.

In Ref. [17] the LL part of $U^{(j,i-j,0)}(Q,\,\mu)$ was related to the Sudakov form factor Eq. (4.10) (up to accounting for the soft effects of angular ordering). The cusp term in the anomalous dimension gives the LL part, and comes from soft and collinear one-loop diagrams. The result from the soft diagrams is constrained by that of the collinear diagrams in order to cancel out infrared sensitivity that cannot be absorbed in local counterterms at the hard scale. Here we will use this same argument, but in reverse, in order to determine the LL anomalous dimension of various subleading operators.

Due to the soft-collinear factorization, the soft structure only depends on the number of collinear directions. After making the field redefinition operators like $\bar{\chi}_{n_0}\chi_{\bar{n}}$ and $\bar{\chi}_{n_0} g\mathcal{B}^{\alpha}_{n_0\perp}\chi_{\bar{n}}$ both have $Y^{\dagger}_{n_0}Y_{\bar{n}}$, and so both have the same soft divergences, and hence have the same one-loop cusp term. Thus the leading-log resummation only depends on the number of collinear index directions in the operator. We therefore have

$$U^{(2,0,0)}_{\rm LL}(n_0) = U^{(2,1,1)}_{\rm LL}(n_0,n_0) = U^{(2,1,1)}_{{\rm LL},\mathcal{T}}(n_0,n_0)\,. \tag{4.77}$$

Thus at LL order we have the full set of evolution kernels for subleading collinear operators and we account for these factors in the appendix expressions. Since this is a LL effect we expect the effects of soft radiation and angular ordering to be incorporated in a manner identical to the evolution factor in the LL shower.

An important consequence of this result for the LL evolution is that it justifies treating our hard-scattering corrections as improvements to the fixed-order, matrix-element calculation that goes into a shower algorithm. Correcting the two-jet amplitude with either $C^{(1)}_{1,\rm NLO}$ or $C^{(1)}_{1,\mathcal{T}}$, we see that the LL resummation is the same as that in the standard shower except that there is an extra parton already inside the leading jet. We thus get a shower correction just by using a matrix element improved by including our hard-scattering terms. This is different from simply running a LL shower on higher order matrix elements, as different anomalous dimensions control each operator's evolution. Some, operators, like those just mentioned with only n_0 and $\bar{n}$ collinear directions, run like two-jet configurations, that is with a quark-antiquark Sudakov. Others, (*e.g.* $C_1^{(1)}(n_1', n_1)\mathcal{O}_1^{(1)}$) have three-parton running since they have three distinct collinear directions. This latter set correpsonds to the usual implementation of fixed order corrections in parton showers, but the former is a novel type of shower improvement whose implementation will require further study.

On the other hand, the effect of jet-structure corrections is not to modify the initial scattering process, but to go hand in hand with the NLL change to the leading operators' running. The complication we must face for this calculation is that this correction to the evolution kernel must in principal be carried out in the same scheme used to distinguish the phase space regions for the jet-structure corrections, and hence may depend on the choice for the Θ functions. Furthermore it is likely that power suppressed soft effects will also have implications for the subleading evolution kernel. Our lack of an appropriate NLL evolution factor for the shower is due to these two issues. In addition at this level of accuracy one must take into account the appropriate scale choice for $\alpha_s(\mu)$ in the LL expressions.

To setup the distinction between kinematic regions we used Wilsonian type Θ functions, but from the point of view of evolution $\overline{\mathrm{MS}}$ would be simpler. Although it is not directly relevant to our setup it is nevertheless still interesting to consider how the NLL evolution kernel would arise in $\overline{\mathrm{MS}}$. As we will discuss in Appendix C, when integrated over phase space in dimensional regularization the jet-structure corrections give the real emission portion of $P_{qq}^{(1)}$, which is the $\mathcal{O}(\alpha_s)$ correction to the Altarelli-Parisi splitting kernel. Combined with known SCET results for single-emission at one-loop, we can recover all of $P_{qq}^{(1)}$. Obtaining this expression is important both conceptually and practically. It validates our formal expansion in λ, showing that corrections to $\mathcal{O}(\lambda^2)$, along with a set of previously calculated SCET one-loop diagrams, capture the contributions needed for NLL resummation. On the practical side it provides a cross check on the computations.

With $P_{qq}^{(1)}$ in hand we can extend the argument of [16, 17] that the Sudakov factor gives the LL part of the the RG kernel $U^{(i)}(Q, \mu)$ (Eq. (4.10)) to the NLL level, looking at $U^{(0)}(Q,\mu)$ for running of the operator $C^{(0)}_{1,\mathrm{LO}}\mathcal{O}^{(0)}_1$. Using the Sudakov factor of [31] for quarks, we have:

$$\Delta_q(Q,\mu) = \exp\left\{ -\frac{C_F}{2\pi}\int_\mu^Q \frac{d\mu'}{\mu'}\alpha_s(\mu') \int_{\frac{\sqrt{\mu'}}{Q}}^{1-\frac{\sqrt{\mu'}}{Q}} dz \frac{1+z^2}{1-z} \right\}, \tag{4.78}$$

where we recognize $P^{(0)}_{qq}$, Eq. (4.3). Performing the z integral and expanding in the limit of large Q gives:

$$\Delta_q(Q,\mu) \approx \exp\left\{ \frac{C_F}{\pi}\int_\mu^Q \frac{d\mu'}{\mu'}\alpha_s(\mu') \left[\log\left(\frac{\mu'^2}{Q^2}\right) + \frac{3}{2}\right]\right\}, \tag{4.79}$$

which is identical to $U^{(0)}(Q,\mu)$ at one-loop. The term in the exponent proportional to $\log(\mu'^2/Q^2)$ sums the leading logs in the parton shower. Interpreting Eq. (4.79) as an RG kernel, this log piece is coming from the one-loop cusp anomalous dimension, C_F. The factor of 3/2 is the remaining part of the one-loop anomalous dimension, and it sums part of the collinear NLL.[8] In order to get the full NLL summation, one also needs corrections corresponding to the two-loop cusp anomalous dimension. This is a known result in SCET for the operator $\bar{\chi}_n\chi_{\bar{n}}$, which we can relate to $P^{(1)}_{qq}$, by adding the subleading splitting function to the exponent of $\Delta_q(Q,\mu)$. Unlike Eq. (4.79), this new expression actually sums next-to-leading logs in the $\overline{\mathrm{MS}}$ scheme. We wish to stress, however, that the ultimate goal to improve parton showers through resummation is to include all next-to-leading-logs. In this work, we have not considered the effects of soft NLL, nor those related to the two-loop running of α_s. Furthermore, one needs the NLL running for all of our operators. While our formulas in Apps. B.2-B.4 do include all LL running, we only have NLL for $C^{(0)}_i\mathcal{O}^{(0)}_i$. The collinear-NLL-improved Sudakov corresponding to it is:

$$\Delta_q^{\mathrm{NLL}}(Q,\mu) = \exp\left\{ -\int_\mu^Q \frac{d\mu'}{\mu'} \int_{\frac{\sqrt{\mu'}}{Q}}^{1-\frac{\sqrt{\mu'}}{Q}} dz\, \left[P^{(0)}_{qq}(z,\mu') + P^{(1)}_{qq}(z,\mu')\right] \right\}, \tag{4.80}$$

[8] Since Eq. (4.79) resums some NLL contributions [31] calls it the NLL Sudakov factor.

where $P_{qq}^{(0)}$ given in Eq. (4.3) and $P_{qq}^{(1)}$ in [39]. Once again, we integrate in z, expanding in large Q to get:

$$\Delta_q^{\rm NLL}(Q,\mu) = \exp\left\{\int_\mu^Q \frac{d\mu'}{\mu'}\left[\frac{\alpha_s(\mu')}{\pi}C_F\left(\log\left(\frac{\mu'^2}{Q^2}\right)+\frac{3}{2}\right)\right.\right.$$
$$\left.\left. + \frac{\alpha_s^2(\mu')}{4\pi^2}C_F\left(C_g\left(\frac{67}{9}-\frac{\pi^2}{3}\right)-\frac{20}{9}C_F T_F\, n_F\right)\log\left(\frac{\mu'^2}{Q^2}\right)\right]\right\}, \quad (4.81)$$

where the term $\propto \alpha_s^2$ reproduces the known result for the two-loop cusp anomalous dimension.

4.4.4 From Operators Toward a Corrected Shower

As discussed previously, our end goal is to match down to an EFT, SCET$_N$, where each field has its own index direction. Further Lagrangian emission from these operators is physically meaningless, as the resolution scale is set $\sim \mathcal{O}(\text{GeV})$, below which we stop computing in perturbation theory and pass to a hadronization routine. Thus in SCET$_N$ we match everything to the single operator $\mathcal{O}_N^{(N-1)}(n_N, n'_1, \ldots, n'_{N-1})$ and all the information Lo and NLO is encoded in the Wilson coefficients. We have calculated the coefficients From this SCET$_N$, we define a map to different shower algorithm ingredients. We gave those needed for a LL shower Monte Carlo in Table 4.1, which only required single gluon considerations for collinear effects. To include the leading SCET power corrections and consistently resum collinear NLL, we will need operators with an arbitrary of partons in the mapping. However, as we saw in Secs. 4.4.1 and 4.4.2, we only needed to do two-gluon emission computations to obtain the amplitudes to NNLO that contribute the leading corrections in λ to observables. For the hard-scattering corrections, this was because modifying the emissions of particles further away from the initial hard-scale production led to increasingly suppressed terms. For jet-structure, while corrections can occur anywhere, the leading ones only involve the most minimal deviation from strong-ordering, that affecting nearest-neighbor The LO coefficient for SCET$_N$ is given in Eq. (4.53), and for NLO are given in appendix B.4. As we demonstrate below, upon squaring these amplitudes, the simplicity of the picture remains, and we only need to consider the non-trivial phase space of two particles at a time.

Interference for LO^2 and Jet-Structure Corrections

It is a general statement about SCET fields with different n index labels that they have no overlap in Hilbert space. As an example, we can take two different operators, $\mathcal{O}_{n_1}$ and $\mathcal{O}_{n_2}$. All the fields in $\mathcal{O}_{n_1}$, except that labeled by n_1 (*e.g.* χ_{n_1}), have exact analogs in $\mathcal{O}_{n_2}$, creating the same type of particle and having the same index label. Instead, $\mathcal{O}_{n_2}$ contains a field labeled by n_2, which may or may not be the same as n_1. We thus have:[9]

$$\begin{aligned} &\langle q_1, q_2, \ldots, q_m|\mathcal{O}^\dagger_{n_1}|0\rangle\langle 0|\mathcal{O}_{n_2}|q_1, q_2, \ldots, q_m\rangle = \\ &\delta_{n_1,n_2}\langle q_1, q_2, \ldots, q_m|\mathcal{O}^\dagger_{n_1}|0\rangle\langle 0|\mathcal{O}_{n_2}|q_1, q_2, \ldots, q_m\rangle. \end{aligned} \tag{4.82}$$

It is to guarantee a relation like Eq. (4.82) that our Wilson coefficients contain Θ-functions (*cf.* Eqs. (4.45) and (B.61)), which will cutoff the overlap regions in phase space once we begin integrating. The amplitude squared is particularly easy in SCET$_N$, where we have only the operator $\mathcal{O}^{(N-1)}_N(n_N, n'_1, \ldots, n'_{N-1})$, and where each particle is defined in a different collinear direction.

SCET$_i$ simplifies the amplitude squared by distinguishing at the operator level which configurations are strongly-ordered and which are not. This means that we have no interference between $C^{(N-1)}_{N,\mathrm{LO}}\mathcal{O}^{(N-1)}_N$ and $C^{(N-1)J}_{N,\mathrm{NLO}}\mathcal{O}^{(N-1)}_N$ where $C^{(N-1)}_{N,\mathrm{LO}}$ is the LO SCET$_N$ coefficient defined in Eq. (4.53), and $C^{(N-1)J}_{N,\mathrm{NLO}}$ is the NLO SCET$_N$ coefficient defined in Eq.(B.119). Even though the $\mathcal{O}$'s are the same, the Θ-functions in the C's enforce different conditions, where the former is strongly ordered, while the latter is not. Thus, in the analog of Eq. (4.82), the Kronecker delta will give zero.

We get a further simplification when we square the NLO contributions. Looking

[9] By RPI, n_1 and n_2 do not have to be exactly equal, but must concur up to an angle of $\mathcal{O}(\lambda^i)$ in SCET$_i$.

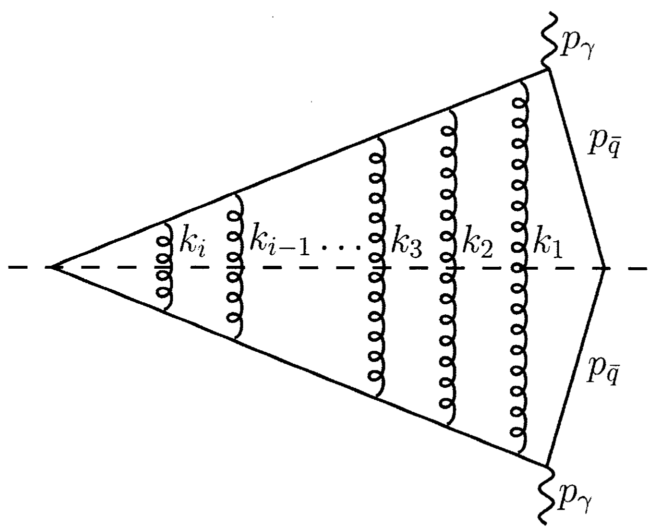

Figure 4-12: Amplitude squared for the LO operator $C^{(i)}_{N\,\mathrm{LO}}\mathcal{O}^{(i)}_N$. Though we do the amplitude computation in SCET$_N$, we illustrate the process here with a cut SCET$_1$ Feynman diagram in order to emphasize the simple ladder structure.

at $C^{(N-1)J}_{N,\,\mathrm{NLO}}$ in detail, we have:

$$C^{(N-1)J}_{N,\,\mathrm{NLO}} = \sum_{l=1}^{N-2} C^{(N-1)J}_{N,\,\mathrm{NLO}}(l)\,, \tag{4.83}$$

where

$$\begin{aligned} C^{(N-1)J}_{N,\,\mathrm{NLO}}(l) = \sum_{I=1}^{3} \Bigg[\Bigg(\prod_{k=1}^{l-1} U^{(k-1)}_{\mathrm{LL}}(\mu_{k-1},\,\mu_k) c^{\alpha_k}_{\mathrm{LO}}(n_{k-1}) \Bigg) U^{(l-1)}_{\mathrm{LL}}(\mu_{k-1},\,\mu_k) \otimes h^{\alpha,\beta}_I(n_{l+1}, n'_l, n'_{l+1}) \\ \times \Bigg(\prod_{k=l+1}^{N-1} U^{(k-1)}_{\mathrm{LL}}(\mu_{k-1},\,\mu_k) c^{\alpha_k}_{\mathrm{LO}}(n_{k-1}) \Bigg) \Bigg] \Gamma^\mu\,. \end{aligned} \tag{4.84}$$

In $C^{(N-1)J}_{N,\,\mathrm{NLO}}(l)$ we have made explicit that the $l, (l+1)^{\mathrm{th}}$ gluons violate strong-ordering and come with the factor $h^{\alpha\beta}$ of the subleading splitting rule, Eq. (4.75). The sum in the last term over I counts the different types of NLO jet-structure terms given in Eq. (4.74). The $c^{\alpha_k}_{\mathrm{LO}}$ are defined in Eqs. (B.115), and the U's are running factors. The complete explanation of the symbols in Eq. (4.84) can be found in the discussion around Eq. (B.120). Since different l correspond to a violation of strong-ordering at different points in the shower, each of the $C^{(i+1)J}_{i,\,\mathrm{NLO}}(l, l+1)$ codes a different index structure. Therefore, there is no interference for different values of l, and we have that the amplitude squared to NNLO for jet-structure corrections is just the sum of squares of the individual operators:

$$|A^{q(N-1)g\bar{q}J}|^2_{\text{to NNLO}} = |A^{q(N-1)g\bar{q}}|^2_{\text{LO}} + |A^{q(N-1)g\bar{q}J}|^2_{\text{NNLO}}\,, \tag{4.85}$$

where

$$\begin{aligned} |A^{q(N-1)g\bar{q}}|^2_{\text{LO}} &= |C^{(N-1)}_{N\,\text{LO}}|^2 |\langle 0|\mathcal{O}^{(N-1)}_N|q(N-1)g\bar{q}\rangle|^2\,, \\ |A^{q(N-1)g\bar{q}J}|^2_{\text{NNLO}} &= \sum_{l=1}^{N-2} |C^{(N-1)J}_{N,\,\text{NLO}}(l)|^2 |\langle 0|\mathcal{O}^{(N-1)}_N|q(N-1)g\bar{q}\rangle|^2, \end{aligned} \tag{4.86}$$

where with$|q(N-1)g\bar{q}\rangle$ we indicate a state with $N-1$ gluon emission. The simplification even extends inside each of the terms, since the j^{th} gluon only gets contracted with itself. Diagramatically, this means there are zero nearest-neighbor crossings in the $|\text{LO}|^2$ diagram, as we see in Fig. 4-12 and a maximum of *one* in the $|\text{NLO}|^2$ case shown in Fig. 4-13. We thus only slightly modify the factorized emission formula, Eq. (4.2).

We have proved that at NLO for the jet-structure corrections the only non trivial part on the amplitude squared involves at most two near gluons. We can see why terms that have interference with more than two gluons are suppressed, just looking at the internal propagators in the amplitude. The amplitude for $i+1$ emissions has a factor $1/q_1^2 \times 1/q_2^2 \times \cdots \times 1/q_i^2$ due to the propagators. The LO term comes form the strong-ordered region where each new emission is more collinear then the previous one, that means where $q_1^2 \gg q_2^2 \gg \cdots \gg q_i^2$, Eq. (4.1). The jet-structure NLO is given when $q_j^2 \sim q_{j+1}^2$ and this allows the two gluons k_{j+1} and k_{j+2} to share the same region of the phase space and so to interference. Let us suppose that we instead want to interference the gluon k_{j+1} with k_{j+3}. When these two gluons have the same momentum we have $q_j^2 \sim q_{j+1}^2 \sim q_{j+2}^2$, and therefore the associated amplitude is clearly suppressed with respect to the LO and jet-structure NLO amplitudes.

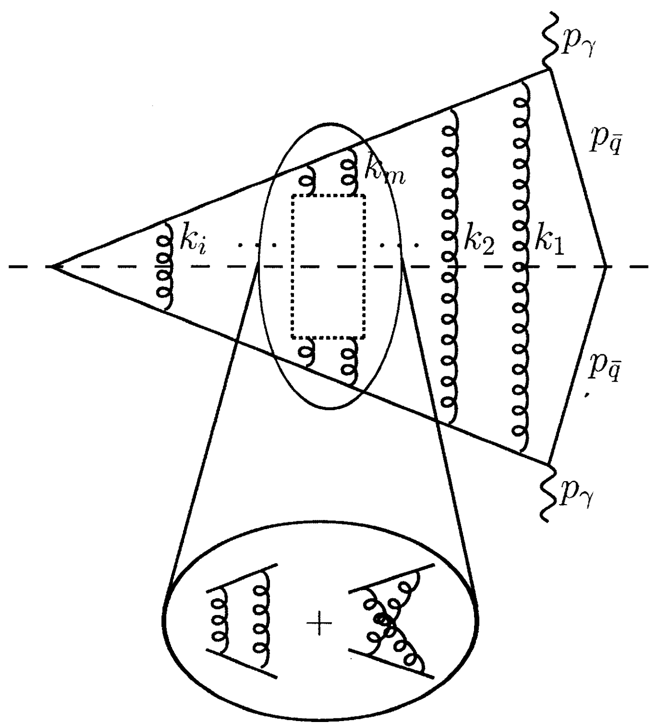

Figure 4-13: Contribution to the amplitude squared of the jet-structure piece at NLO. We are using a cut SCET_1 Feynman diagram to show that the SCET_N operator, $C^{(N)J}_{N,\mathrm{NLO}}\mathcal{O}^{(N)}_N$, contains only a single deviation from the simple ladder structure apparent at LO (Fig. 4-12).

Interference for Hard-Scattering Corrections

The corrections to the cross-section to $\mathcal{O}(\lambda^2)$ involve hard-scattering ones as well. Unlike the jet-structure case, these involve NNLO amplitude terms, as well. As we argued above, they only involve the gluons closest to the hard interaction. Thus, we will not need to sum over many terms as we do in Eq. (4.86). In fact, for hard-scattering corrections, we only need to worry about interfering SCET_i operators that arise from acting with the LO replacement rule Eq. (4.46) on $C^{(0)}_{1,\mathrm{LO}}$, $C^{(1)}_{1,\mathrm{NLO}}$, and $C^{(1)}_{1,\mathcal{T}}$, which are given in Eqs. (4.30) and (4.64). Since the 2^{nd} through i^{th} gluons arise from the LO rule for all three coefficients, they proceed as in the LO/LO case. The interference to look at in detail is that of the first two gluons. We only need to compute single emission graphs, as we can add the other emissions to the amplitude squared by multiplying the appropriate splitting functions and propagators. In SCET_N, we have:

$$|A^{q(N-1)g\bar{q}H}|^2_{\text{to NNLO}} = \big(|C^{(N-1)\dagger}_{N,\text{LO}} C^{(N+1)H}_{N,\text{NLO}} + C^{(N-1)H\dagger}_{N,\text{NLO}} C^{(N-1)}_{N,\text{LO}}| + |C^{(N-1)H}_{N,\text{NLO}}|^2 \\ + |C^{(N-1)\dagger}_{N,\text{LO}} C^{(N-1)H}_{N,\text{NNLO}} + C^{(N-1)H\dagger}_{N,\text{NNLO}} C^{(N-1)}_{N,\text{LO}}|\big)|\langle 0|\mathcal{O}^{(N-1)}_N|q(N-1)g\bar{q}\rangle|^2 \tag{4.87}$$

The Wilson coefficients are found in Eqs. (B.114), (B.116), and (B.122), respectively. The only nontrivial interference in Eq. (4.87) occurs between the first two-gluon emissions.

The interference between LO and NLO simplifies in many cases of interest. For example for one-gluon emission

$$|A^{qg\bar{q}}_{\text{LO/NLO}}|^2_{\mu\nu} = \frac{4\bar{q}_1\bar{p}_{\bar{q}}}{q_0^2} k_{1\perp\,\nu}(n_\mu - \bar{n}_\nu). \tag{4.88}$$

If we can cleanly separate the initial and final states (*e.g.* $e^+e^- \to$ jets), then by a classic proof (*i.e.* that in [84]) involving the Ward identity, once we have integrated over final state vector quantities (we are allowed to keep scalars such as z_i unintegrated), the resulting differential observable depends on $g^{\mu\nu}|A_{\text{LO/NLO}}|^2_{\mu\nu}$, which for Eq. (4.88) is zero. While this is quite straightforward for leptonic initial states, one may be able to extend it to certain hadronic ones as well. Once again, as we said in sec. 4.4.1, one can account for these corrections by modifying the hard-scale matrix element and then running a parton shower modified to include the different no-branching probabilities for different phase space configurations of the same particle content. Also unlike standard shower corrections, we can avoid double counting issues, in principle, because all contributions, whether LO, hard-scattering, or jet-structure corrections are kept separately in distinct operators.

4.5 Correction Map at NLO/NLL

We summarize our results for parton shower in Table 4.2 including ingredients necessary for NLO/NLL accuracy in the cross-section. Since it is easier, in the table we use the language of SCET_1 to discuss the corrections, rather than referring to terms in the

Category	Shower Ingredients	Quantity in SCET$_i$	Found In:
Hard Scattering	Hard matrix elements with more partons	Wilson coeff. of $\bar{\chi}_{n_1}\mathcal{B}_{n_1'}\chi_{\bar{n}}$ in SCET$_1$	Eq. (4.67)
	Power correction to initial branching	Wilson coeff. of $\bar{\chi}_{n_0}\mathcal{B}_{n_0}\chi_{\bar{n}}$ and $\bar{\chi}_{n_0}[\mathcal{P}_\perp\mathcal{B}_{n_0}]\chi_{\bar{n}}$	Eq. (4.67)
	within the leading jet $\mathcal{O}(\alpha_s)$ hard virtual correction	One-loop matching for $\bar{\chi}_{n_1}\chi_{\bar{n}}$	See [16, 17]
Jet Structure	$1 \to 3$ Splitting functions	Double gluon real emission in SCET$_1$	Eq. (4.74)
	$\mathcal{O}(\alpha_s)$ virtual correction for LO $1 \to 2$ splittings	One-loop correction to $1 \to 2$ replacement rule	Left for future work
	Probability of $1 \to 2$ vs. $1 \to 3$	Compare SCET amplitudes	Left for future work
No Branching Probabilities	NLL Sudakov factor for leading branching	NLL anomalous dimension for leading operators	Left for future work
	LL Sudakovs for subleading branching	LL anomalous dimensions for subleading operators	Eqs.(4.10) and (4.77)
Soft Emission	Subleading corrections from soft gluons	Include effects of soft emission from subleading SCET soft Lagrangians	Left for future work

Table 4.2: Mapping between ingredients for a NLO parton shower algorithm and computations in SCET$_i$. For exclusive cross-sections these ingredients would together yield results accurate to NLO in α_s, NLO in the power expansion (λ), and with corresponding NLL resummation.

final SCET$_N$. In SCET$_N$ the features of SCET operators that avoid double counting and allow the various contributions to be distinguished are encoded by Θ functions in the Wilson coefficients. In considering the totally differential cross-section we found at NLO two kinds of power corrections. This includes a set of matrix-element corrections that we called hard-scattering corrections (Section 4.4.4), and a set of contributions that improve double real emissions that we called jet-structure corrections (Section 4.4.4).

In the the hard category we have overall three different kinds of corrections. The first is due the the SCET$_1$ operator $\bar{\chi}_{n_1}\mathcal{B}_{n_1'}\chi_{\bar{n}}$ that gives the SCET$_2$ coefficient $C_{2,\mathrm{NLO}}^{(1)H,b}$ in Eq. (4.67). It is an improvement of the hard matrix element that takes into account an extra parton. The second is due to the SCET$_1$ operators $\bar{\chi}_{n_0}\mathcal{B}_{n_0}\chi_{\bar{n}}$ and $\bar{\chi}_{n_0}[\mathcal{P}_\perp\mathcal{B}_{n_0}]\chi_{\bar{n}}$ that give the SCET$_2$ coefficients $C_{2,\mathrm{NLO}}^{(1)H,a}$ and $C_{2,\mathrm{NNLO}}^{(1)H}$ in Eq. (4.67).

This correction also accounts for more partons, but it describes a situation where they are initially emitted close to the collinear quark rather than widely separated, a correction to the initial branching within the leading jet. It is important to note that the two kinds of corrections have different renormalization group evolution and thus different Sudakov no-branching factors. For a full NLL resummation we also need a third type of hard scattering correction, the one-loop virtual corrections to the leading shower operator. For the required operator $\bar{\chi}_{n_0}\chi_n$ these were discussed in Refs. [16, 17].

For the category of jet-structure corrections there are again several ingredients to consider. We derived a replacement rule for two emissions $1 \to 3$, which should be added on top of two leading order $1 \to 2$ splittings. This correction takes into account emissions in a region of the phase space that is not strongly-ordered. In addition at NLO/NLL we require the $\mathcal{O}(\alpha_s)$ virtual correction to the LO splitting rule. This would be derived from a one loop matching computation that should be straightforward, but was not considered here. In addition the shower requires a new type of probability for when to do a $1 \to 3$ splitting versus a standard $1 \to 2$ splitting. The shower we are discussing has positive weights, since the square of amplitudes is positive, and it is this additional probability function that accounts for situations where the original LL $1 \to 2$ shower yields a result that is larger than the physical observable of interest. A full investigation of this is left to the future.

We also saw that the Sudakov factors, that give the no branching probability, are associated with the running factors of the operator and in turn with their anomalous dimension (Section 4.4.3). To NLL we need the NLL Sudakovs factor for leading branching and the LL Sudakovs for subleading branching that are associated repsectively to the NLL anomalous dimensions for the leading operators and LL anomalous dimensions for the subleading operators. At LL we have the Sudakovs for subleading branching (Eqs. (4.10) and (4.77)), but we have not yet calculated the NLL Sudakov for leading branching in the scheme appropriate for our setup, as described in Subsection 4.4.3. The last item in the table is the treatment of soft radiation at NLO. This can be achieved by considering time-ordered products for the matching of QCD to SCET_1 and SCET_i to SCET_{i+1} that involve soft gluons and subleading soft Lagrangians that are known in SCET [15, 24, 25]. One must then work out the effect that these NLO soft amplitudes have on interference. The associated soft calculations and investigations have been left for future work.

Not included in table 4.2, but worth recalling, is the fact that our NLO results have been expressed in terms of a particular set of variables, namely dot products of light-like vectors $n_i \cdot n_j$ and large light-cone momenta $\bar{q}_i$. As mentioned in Subsection 4.3.3 there will be corrections at NLO/NLL that arise because of the choice of shower evolution variables and the translation of the SCET_i results to this choice, and these must be considered on a case by case basis. They are not accounted for elsewhere in our NLO corrections.

4.6 Conclusion

In this work we developed a rigorous framework based on a tower of independent but related EFTs, the SCET_i, to study corrections to the parton shower. The work of [16, 17] showed how to formulate the LL parton shower in terms of SCET, and how virtual corrections are straightforward to incorporate by one-loop matching. The SCET_i framework extends these ideas in a manner that makes it easy to deal with double counting issues, the issue of disentangling coordinate choices from kinematic power corrections, and the construction of a complete set of operators for corrections at a desired order. The interference structure and hence the leading corrections to spin correlations and color correlations are also straightforward to work out in the SCET_i setup.

The SCET_i are iteratively used to integrate out the characteristic scale, $Q\lambda^i$ for increasing i. This approach allows one to perform a systematic expansion which can correct both the hard-scale process that produce partons and the parton shower itself. We described the parton shower through the operators $\mathcal{O}_j^{(i)}$ in SCET_j and used standard matching procedures to make the transition from SCET_j to SCET_{j+1} where more partons become apparent. Performing the matching relied crucially on the RPI symmetry of SCET, and we extended the usual infinitesimal version to carry out the finite rotations in each SCET_i that we needed. A summary of ingredients required for the NLO/NLL shower are given in Table 4.2, including both calculations carried out here as well as those left for future work. The main result of our work are:

1. An easy replacement rule for the LL shower, $\chi_{n_0} \to c^{\alpha}_{\mathrm{LO}}\chi_{n_1} g\mathcal{B}^{\alpha}_{n'_1\perp}$, where c_{LO} is related to the standard LO splitting-function. Also a demonstration that angular ordering and coherent branching for soft emissions emerge naturally in the SCET_i framework.

2. At NLO we found two kinds of corrections: hard-scattering corrections and jet-structure corrections. The hard-scattering corrections depend on the hard process and appear near the top of the shower tree. They came from matching QCD to SCET_1 at higher order. Since they only occur at the top of the shower one can treat these as a modified form of matrix-element correction. A subset of these corrections correspond to the usual implementation of fixed-order matrix elements, while the remaining ones give power corrections to the initial branching in the LL shower.

3. The jet-structure corrections are independent from what happens at the hard scale, hence they are universal for any process we want to study. They come from matching SCET_i to SCET_{i+1} at higher order for any i. They can appear anywhere in the shower tree and they take into account emissions in regions of the phase space that are not strongly-ordered. For these corrections we found that the NLO operator are related to the LO operator via a replacement rule for two emissions: $\chi_{n_0} \to h^{\alpha\beta}_I \chi_{n_2} g\mathcal{B}^{\alpha}_{n'_1\perp} g\mathcal{B}^{\beta}_{n'_2\perp}$.

4. The SCET_i picture allows us to easily take into account interference. Once we reach the final SCET_N theory all the fields are labeled in a different collinear directions. Because in SCET we can only can contract collinear fields that share the same collinear direction, in SCET_N calculating the amplitude squared becomes very easy. Kinematic information that is encoded by the shower history

from passing through earlier SCET$_i$s is encoded by Θ functions in the final SCET$_N$ Wilson coefficients. We proved that when emitting an arbitrary number of partons, the non-trivial part of the amplitude squared involves at most two fields.

The framework developed here allows for systematic improvement to arbitrary orders in the kinematic expansion. There are still several important steps to take, though, before this picture can lead to a practical implementation, including additional computations that we have outlined. We list here three topics whose development would be particularly useful, and which we believe should be straightforward to approach:

1. This work has only considered $q \to qg$ splittings and an abelian theory. One should include the full nonabelian results and compute the coefficients required for gluon splitting as well. This is required to properly treat color correlation corrections in a manner determined by the NLO interference pattern. For collinear particles we expect that one can include the dominant part of these effects by considering nearest-neighbor interference, leaving the rest of the shower as before.

2. Only a subset of the terms required for a full NLL resummation were considered. We computed LL evolution for subleading operators, but did not carry out the computation of the NLL evolution of the leading operator in a scheme that is consistent with our power corrections (we only considered it in $\overline{\mathrm{MS}}$). In order for a consistent treatment as a probabilistic process, the real emission probabilities and Sudakov no-branching corrections must go hand in hand. We also did not consider how to include the effects of our NLO jet-structure $1 \to 3$ replacements in an algorithm. The non-trivial task is how to simultaneously implement $1 \to 2$ and $1 \to 3$ splittings in a consistent manner.

3. Since soft modes in SCET can communicate between different collinear jets, they carry the ability to spoil their factorization. Fortunately, this did not

happen when we included their LO couplings. In fact, we were able to derive the condition of angular ordering and coherent branching in SCET_i. It is open question as to what extent NLO soft couplings can be factorized in the shower tree and the necessary SCET computations were not carried out here. The treatment of soft NLO interactions in SCET in the past has always led to factorized structures, so we remain optimistic that such effects will be tractable for the shower.

We also briefly comment on how the corrections in Table 4.2 relate to corrections already implemented in parton shower codes in the literature. In most cases the goal of these codes differed from the strict NLO/NLL shower corrections considered here. This makes a strict association impossible, but there is still a general correspondence that can be made. CKKW [31] is a $\text{LO}(\alpha_s)$/LL procedure whose goal is to merge matrix elements involving multiple partons with a parton shower in a manner that avoids double counting. In our language this corresponds to the real emission hard-scattering corrections in the first row of Table 4.2. The $\bar{\chi}_{n_0}\chi_{\bar{n}}$ and $\bar{\chi}_{n_1}\mathcal{B}_{n_1'}^{\perp}\chi_{\bar{n}}$ operators describe processes with different numbers of initial well-separated jets. In CKKW a parameter y_{cut} is used to separate the extra emission in the matrix element from emissions in the shower. In our analysis the contributions from showering $\bar{\chi}_{n_0}\chi_{\bar{n}}$ does not interfere with the direct contribution from $\bar{\chi}_{n_1}\mathcal{B}_{n_1'}^{\perp}\chi_{\bar{n}}$, and this is encoded by Θ functions in the Wilson coefficient of SCET_N.

In MC@NLO [52] and POWHEG [82], virtual and real matrix element corrections at NLO in α_s are incorporated into the shower, with the goal of ensuring that it reproduces an associated cross-section completely at NLO in α_s. The implementation includes careful handling of the cancellation of real and virtual IR divergences. In Table 4.2 our goal was to implement corrections at NLO in powers and NLL in logs/α_s, but for all jets from the shower rather than just the first jet needed for the NLO cross-section. At NLL we have only terms up to $\mathcal{O}(\alpha_s \log)$ in the total cross-section, and hence does not encode the entire NLO result (in α_s). In our language the corrections that contribute to the NLO cross section correspond to the hard scattering corrections in the first plus second row, and third rows of Table 4.2. In order to compute the

NLO cross section it is not necessary to distinguish between the terms in the first and second rows of the table, and these terms are indeed considered simultaneously in MC@NLO and POWHEG.

The hard scattering corrections in our second row correspond to how to treat real radiation at higher precision that has the same strongly ordered kinematic configuration as the LO shower. The work of KRKMC group [60, 61, 93], on the other side, aims to improve the shower algorithm taking into account the Altarelli-Parisi splitting function at NLO: $P_{qq}^{(1)}$. In our language this corresponds to our jet-structure corrections and we have seen in Appendix C how the replacement rule in Eq. (4.75) is related to $P_{qq}^{(1)}$. In our method we can consistently take into account both correction at order α_s in the $1 \to 2$ splitting function, as well as the subleading contribution that comes from the $1 \to 3$ splitting function.

Chapter 5

Conclusions

At hadron colliders we compare data to theoretical predictions for exclusive quantities in order to take into account experimental cuts and detector effects that are necessary to isolate the signal from the the background. Monte Carlo event generators have been proved indispensable for making exclusive theoretical prediction and they are heavily used by experimentalists to simulate events. They are based on a leading log (LL) parton shower algorithm that is defined in the soft-collinear limit and uses a probabilistic Markov chain of $1 \to 2$ particle splittings to recursively generate partons. There have been several improvements to LL parton showers, but a systematic way to resum next-to-leading log (NLL) is missing in the literature and there is not even a clear method to catalog all the necessary corrections. An improvement in the traditional event generators up to NLL, would allow us to better distribute particles in phase space, and to have a better normalization of the shower. This will in turn improve the shape of prediction for cross-section at the LHC and thus yield more precise theoretical predictions.

In this thesis we developed a rigorous framework to study corrections to partons showers using soft collinear effective field theory (SCET) to pave the way for an implementation of a NLL parton shower algorithm. Our work is based on a tower of independent but related EFTs, the SCET_i. We described the parton shower through the operators $\mathcal{O}_i^{(j)}$ in SCET_i, where j is the number of parton in the shower, and we used standard matching procedures to make the transition from SCET_i to SCET_{i+1}

where more partons become apparent. Even if we have many EFTs, we will use only a single power counting parameter and we calculated corrections to next-to-leading order (NLO) in the power counting for the parton shower.

At LO in the power counting we recover the usual LL shower. We also proved that angular ordering and coherent branching for soft emissions emerge naturally in the SCET_i framework. At NLO we found two kinds of corrections: hard-scattering corrections and jet sub-structure corrections. The hard-scattering corrections depend on the initial hard process and one can treat these as a modified form of matrix-element correction. A subset of these corrections correspond to the usual implementation of fixed-order matrix elements, while the remaining ones give power corrections to the initial branching in the LL shower. The jet-structure corrections are the same for any process we want to study. They can appear anywhere in the shower tree and they take into account emissions in regions of the phase space that are not strongly-ordered. For these corrections we found that determing the NLO operator can be encoding by a $1 \to 3$ particle splitting replacement rule. The SCET_i picture allows us to easily take into account interference. We proved that when emitting an arbitrary number of partons, the non-trivial part of the amplitude squared involves at most two fields.

Thus, our work represents an improvement of parton shower both at the interface with the matrix element and in the parton splitting. In Table 4.2 we enlisted the corrections we calculated as well as all the missing ingredients for a full NLL shower. The next things to calculate are: virtual collection for the LO $1 \to 2$ splittings, the NLL Sudakov factor for the leading branching and the subleading correction from soft gluons. The last step towards a NLL shower is to implement these corrections in a NLL shower algorithm that is able to disentangle the probability of $1 \to 3$ splittings versus a standard $1 \to 2$ splittings.

Performing the matching between different SCET_i relied on the reparametrization invariant (RPI) symmetry of SCET. We extensively studied RPI in Chapter 3. We constructed operators that are invariant under reparametrization transformations and we use them to reduce the number of operators in SCET. We constructed a minimum basis of pure glue operators for DIS at twist-4, for production of two and three jets from e^+e^-, and for production of two jets form gluon fusion.

Appendix A

Invariance to the choice of hard-vector q^μ

From the construction in section 3.3, a natural question arises about the special role of q^μ in Eq. (3.15). What happens if there is more than one possible choice for q^μ in a given process? Say we have a q^μ and a q'^μ with Wilson coefficients that can depend on q^2, q'^2, and $q \cdot q'$, where $q_\perp \sim q'_\perp \sim \lambda$. It turns out that in this case any linear combination of q^μ and q'^μ in Eq. (3.15) is equally good, and is equivalent to any other choice. Hence, one choice suffices. To prove this we consider the expansion of the reparametrization invariant variable

$$\xi \equiv \frac{2q\cdot q'}{q^2} \pm \sqrt{\Big(\frac{2q\cdot q'}{q^2}\Big)^2 - \frac{4q'^2}{q^2}} = \frac{n\cdot q'}{n\cdot q} + \mathcal{O}(q_\perp^2)\,, \tag{A.1}$$

where we take the plus sign if the expansion is done with $n \cdot q'/n \cdot q > \bar{n} \cdot q'/\bar{n} \cdot q$ and the minus sign otherwise. Now use this variable to define

$$q' \cdot i\partial - \xi\,(q \cdot i\partial) \equiv \hat{Q}^{[2]}_{\rm INV}\,, \tag{A.2}$$

where the operator $\hat{Q}^{[2]}_{\rm INV}$ is RPI and its expansion starts at order λ^2. Thus

$$\int d\omega\, C(\omega)\, \delta(\omega - q' \cdot i\partial) = \int d\omega\, C(\omega)\, \delta\big(\omega - \xi\, q \cdot i\partial - \hat{Q}^{[2]}_{\rm INV}\big)$$

$$
\begin{aligned}
&= \int d\omega' \, C(\xi\omega') \, \delta(\omega' - q\cdot i\partial - \hat{Q}^{[2]}_{\rm INV}/\xi) \\
&= \int d\omega' \Big[\tilde{C}(\omega') \, \delta(\omega' - q\cdot i\partial) + \tilde{B}(\omega') \, \hat{Q}^{[2]}_{\rm INV} \, \delta'(\omega' - q\cdot i\partial) + \ldots \Big] \,,
\end{aligned}
\tag{A.3}
$$

where in the second line we changed the dummy variable to $\omega' = \xi\omega$, In the last line both terms are RPI, and the ellipsis denotes higher order terms which are also RPI order by order in λ. Eq. (A.3) demonstrates that we can swap the parameter $q' \to q$ in the δ-function, since the change is compensated by a change of notation in the leading order Wilson coefficient $C \to \tilde{C}$. Given that we imagine starting with a complete basis of RPI operators built with $\delta(\omega - q'{\cdot}i\partial)$ or with $\delta(\omega' - q{\cdot}i\partial)$, the higher terms in the series in Eq. (A.3), like $\tilde{B}$, also simply change a Wilson coefficient in our basis. Thus, the choice of q or q' in the δ-function just corresponds to a different choice of the basis for the invariant operators, and one choice suffices.

Appendix B

Matching SCET_i to SCET_{i+1}

In this appendix we perform the matching QCD to SCET_1 in subsection B.2, SCET_1 to SCET_2 in subsection B.3, and SCET_2 to SCET_3 to SCET_N in subsection B.4. A key element that we use in the SCET_i to SCET_{i+1} matching is the finite RPI transformations that are defined in subsection B.1.

B.1 Finite RPI

We have seen in chapter 2 that we can always decompose the momentum p of a particle in two light-cone directions n and $\bar{n}$

$$p^\mu = n\!\cdot\!p\frac{\bar{n}^\mu}{2} + \bar{p}\frac{n^\mu}{2} + p_\perp^\mu\,, \tag{B.1}$$

with $n^2 = 0$, $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. We define p collinear to the direction n if the components of (B.1) scale as $(n\!\cdot\!p,\, \bar{p},\, p_\perp) \sim (\lambda^2,\, 1,\, \lambda)\, Q$, where Q is the hard scale and $\lambda \ll 1$. The vector n has physical meaning, $\vec{n}$ is the direction where most of the momentum is allocated, that is the direction $\vec{p}$ is inside a cone of opening angle λ around $\vec{n}$, see Fig (2-1). $\bar{n}$ does not carry any physical meaning and it is only needed to decompose the momentum in (B.1). The power counting λ defines the level of the collinearity. The decomposition (B.1) is not unique, we can shift n by an amount λ and the particle we still be collinear to the direction n. Pictorially this means that if we move the vector n inside the cone in Fig. (2-1), p is always collinear to it. The transformations that move the n inside the collinear cone are the reparametrization invariant (RPI) transformation of type-I, Eq. (2.24). Thus if a particle is collinear to a direction n, it is also collinear to any directions n' that is related to n with an RPI transformation.

Two collinear sectors in SCET, n_1 and n_2, are distinct if [9]

$$n_1 \cdot n_2 \gg \lambda^2 \,, \tag{B.2}$$

We will label the external state with the direction n where the particles are collinear to and with a subscript the magnitude of collinearity, so for example $|q_{n_1} q_{n_2}\rangle_1$ is a state with two quarks, one collinear to n_1 and one to n_2 where $n_1 \cdot n_2 \gg (\lambda)^2$. A collinear field in SCET labeled with n_i can create a particle collinear to the n_i direction, but this means that state is also collinear to any $n_j \in [n_i]$, where $[n_i]$ is the equivalence class defined in Eq. (2.25), so we have

$$\xi_{n_i} |q_{n_j}\rangle = \begin{cases} u_{n_i} & \text{if } n_j \in [n_i] \\ 0 & \text{otherwise} \end{cases} \,, \tag{B.3}$$

The type-I RPI transformation connects vectors that belong to the same class $[n_j]$.

For each $\{n, \bar{n}\}$, the type-I RPI infinitesimal transformations are given in Eq. (2.24) These transformations preserve the relations $n^2 = 0$, $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. In the matching from SCET_i to SCET_{i+1} we need to rotate the direction n to a direction n' that is closer to the momentum of the particle, such that p is collinear to n' in SCET_{i+1}, The RPI transformations allow us to do it. Thus the finite RPI transformations in SCET_i is crucial to match the two theories and hence is not simply a convention. On the other hand the choice within SCET_{i+1} is purely a convention. We have some freedom in choosing n', if λ^{i+1} is the power counting of SCET_{i+1} any n' such that $\vec{p}$ is inside a cone of aperture λ^{i+1} around $\vec{n}'$ is fine, see Fig. (4-3). The momentum p decomposed along n' and $\bar{n}$ is

$$p^\mu = n'\!\cdot\! p\frac{\bar{n}^\mu}{2} + \bar{p}\frac{n'^\mu}{2} + p^\mu_{n'\perp}\,. \tag{B.4}$$

For this convention we will pick n' as that direction n_p where p as zero perpendicular momentum, that is

$$p = \bar{p}\frac{n_p^\mu}{2} + n_p\!\cdot\! p\frac{\bar{n}^\mu}{2}\,. \tag{B.5}$$

The relation between n and n_p is

$$n_p^\mu = n^\mu + 2\,\frac{p_\perp^\mu}{\bar{p}} - \bar{n}^\mu\frac{(p_\perp)^2}{\bar{p}^2}\,. \tag{B.6}$$

This is a RPI finite transformation (instead Eq. (2.24) is infinitesimal). It is easy to check that $n_p^2 = 0$, $n_p\!\cdot\!\bar{n} = 2$ and that $p^\mu_{n_p\perp} = p^\mu - n_p\!\cdot\! p\,\bar{n}^\mu/2 - \bar{p}\,n_p^\mu/2 = 0$.

We will derive similar relations for other quantities. To see how the quark field transform we use the RPI invariant fermion field [80]

$$\psi_n = \Big(1 + \frac{\not{D}_n^\perp}{\bar{n}\!\cdot\! D_n}\frac{\not{\bar{n}}}{2}\Big)\xi_n\,. \tag{B.7}$$

Because (B.7) is invariant under RPI, $\psi_n = \psi_{n_p}$ and we can write

$$\Big(1 + \frac{\not{D}_n^\perp}{\bar{n}\!\cdot\! D_n}\frac{\not{\bar{n}}}{2}\Big)\xi_n = \Big(1 + \frac{\not{D}_{n_p}^\perp}{\bar{n}\!\cdot\! D_{n_p}}\frac{\not{\bar{n}}}{2}\Big)\xi_{n_p}\,. \tag{B.8}$$

Multiplying (B.8) by the projector $\not{n}\not{\bar{n}}/4$ we get the finite RPI relation

$$\xi_n = \frac{\not{n}\not{\bar{n}}}{4}\xi_{n_p}\,. \tag{B.9}$$

The relation (B.9) is in agreement with the spinor equation (A7) in [17] once you choose $\bar{n}_1 = \bar{n}_2$. Object with a full Lorentz index like p^μ or γ^μ, are RPI invariant because there is no reference to the light-cone vectors n and $\bar{n}$, but on object in the perpendicular direction, like $p_\perp^\mu$ or $\gamma_\perp^\mu$ are not, because $\perp$ is defines relative to n and

$\bar{n}$. Using the relation $\gamma_\perp^\mu = \gamma^\mu - \bar{n}^\mu \not{n}/2 - n^\mu \not{\bar{n}}/2$, we derive the expression

$$\gamma_\perp^\mu = \gamma_{n_p\perp}^\mu + \bar{n}^\mu \frac{\not{p}_\perp}{\bar{p}} + p_\perp^\mu \frac{\not{\bar{n}}}{\bar{p}} - \bar{n}^\mu \frac{(p_\perp)^2}{\bar{p}^2} \not{\bar{n}} \,. \tag{B.10}$$

Now we focus on one gluon emission. We consider the case of a virtual quark with momentum q_0 emitting an external gluon and quark with momentum k_1 and q_1 respectively. In Fig. (B-1-A) we portrayed one gluon emission where the initial quark q_0 comes from a QCD current $\bar{q}\gamma^\mu q$. We call n_0, n_1' and n_1 the directions where q_0,

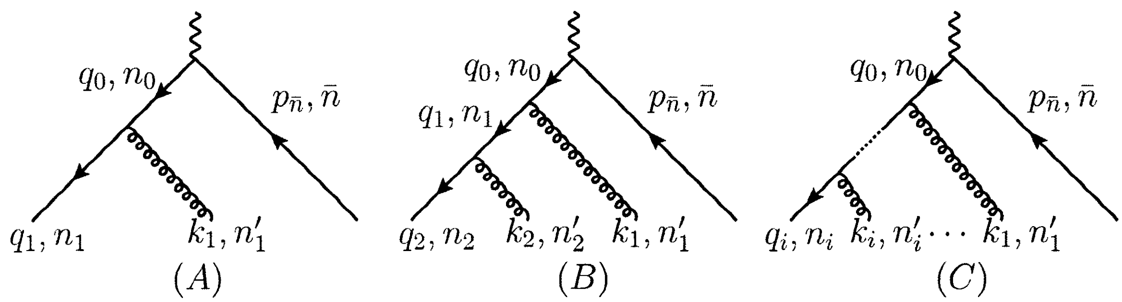

Figure B-1: Kinematic variable for one gluon emission (A), two-gluon emission (B) and i gluon emission (C). The ns vector are defined such that the momentum has not perpendicular component along that directions $n - \bar{n}$.

k_1 and q_1 zero have perpendicular direction, that is

$$\begin{aligned} q_0 &= \bar{q}_0 \frac{n_0^\mu}{2} + n_0 \cdot q_0 \frac{\bar{n}^\mu}{2} \,, \\ k_1 &= \bar{k}_1 \frac{n_1'^\mu}{2} \,, \\ q_1 &= \bar{q}_1 \frac{n_{q_1}^\mu}{2} \,, \end{aligned} \tag{B.11}$$

where $\bar{q}_0 = \bar{q}_1 + \bar{k}_1$ by momentum conservation in the $\bar{n}$ direction, and q_0 has a $\bar{n}$ component because it is off-shell. Using the Eq. (B.6) we can relate n_1' and n_1 to n_0

$$\begin{aligned} n_1'^\mu &= n_0^\mu - 2\frac{(q_1)_{n_0\perp}^\mu}{\bar{k}_1} - \bar{n}^\mu \frac{(q_1)_{n_0\perp}^2}{\bar{k}_1^2} \,, \\ n_1^\mu &= n_0^\mu + 2\frac{(q_1)_{n_0\perp}^\mu}{\bar{q}_1} - \bar{n}^\mu \frac{(q_1)_{n_0\perp}^2}{\bar{q}_1^2} \,, \end{aligned} \tag{B.12}$$

where we have used the equality $(k_1)^\mu_{n_0\perp} = -(q_1)^\mu_{n_0\perp}$. Some useful relations are

$$\begin{aligned}
n_1\cdot n_1' &= n_0\cdot n_1' \frac{\bar{q}_0^2}{\bar{k}_1^2} = n_0\cdot n_1 \frac{\bar{q}_0^2}{\bar{q}_1^2} = -2\frac{(q_1)^2_{n_0\perp}\bar{q}_0^2}{\bar{q}_1^2\,\bar{k}_1^2}\,, \\
n_0^\mu &= \frac{\bar{k}_1 n_1'^\mu + \bar{q}_1 n_1^\mu}{\bar{q}_0} - \bar{n}^\mu\,(n_1\cdot n_1')\frac{\bar{q}_1\bar{k}_1}{2\,\bar{q}_0^2}\,, \\
(q_1)^\mu_{n_0\perp} &= \frac{1}{2}\frac{\bar{q}_1\bar{k}_1}{\bar{q}_1+\bar{k}_1}\sqrt{|n_1\cdot n_1'|}v_1^\mu - \bar{n}^\mu\,(n_1\cdot n_1')\frac{\bar{q}_1\bar{k}_1(\bar{k}_1^2-\bar{q}_1^2)}{4\,\bar{q}_0^3}\,, \\
\gamma^\mu_{n_0\perp} &= \gamma^\mu_{n_1'\perp} - \bar{n}^\mu\frac{(\not{q}_1)_{n_0\perp}}{\bar{k}_1} - (q_1)^\mu_{n_0\perp}\frac{\not{\bar{n}}}{\bar{k}_1} - \bar{n}^\mu\,(n_1\cdot n_1')\frac{\bar{q}_1^2}{2\,\bar{q}_0^2}\not{\bar{n}}\,,
\end{aligned} \tag{B.13}$$

where

$$v_1^\mu = \frac{n_1^\mu - n_1'^\mu}{\sqrt{|n_1\cdot n_1'|}}\,, \tag{B.14}$$

is a vector with $|v_1^2| = 2$. Another useful relation is

$$q_0^2 = (q_1+k_1)^2 = n_1'\cdot n_1\frac{\bar{q}_1\bar{k}_1}{2}\,. \tag{B.15}$$

Notice that we can express all quantities in terms of the vectors n_1', n_1 and the momenta $\bar{q}_1$ and $\bar{k}_1$.

In two-gluon emissions the kinematic variables are defined in Fig. (B-1-B). We called n_0, n_1', n_2' and n_1 the directions where q_0, k_1, k_2 and q_1 have perpendicular direction zero, that is:

$$\begin{aligned}
q_0 &= \bar{q}_0\frac{n_0^\mu}{2} + n_0\cdot q_0\frac{\bar{n}^\mu}{2}\,, \\
k_1 &= \bar{k}_1\frac{n_1'^\mu}{2}\,, \\
q_1 &= \bar{q}_1\frac{n_1^\mu}{2} + n_1\cdot q_1\frac{\bar{n}^\mu}{2}\,, \\
k_2 &= \bar{k}_2\frac{n_2'^\mu}{2}\,, \\
q_2 &= \bar{q}_2\frac{n_2^\mu}{2}\,,
\end{aligned} \tag{B.16}$$

now q_1 is also off-shell and have a component along $\bar{n}$. The definition (B.12) is still valid, and we can define n_2' and n_2 as

$$n_2'^\mu = n_1^\mu - 2\frac{(q_2)^\mu_{n_1\perp}}{\bar{k}_2} - \bar{n}^\mu \frac{(q_2)^2_{n_1\perp}}{\bar{k}_2^2}\,,$$
$$n_2^\mu = n_1^\mu + 2\frac{(q_2)^\mu_{n_1\perp}}{\bar{q}_2} - \bar{n}^\mu \frac{(q_2)^2_{n_1\perp}}{\bar{q}_2^2}\,, \tag{B.17}$$

where $(k_2)_{n_1\perp} = -(q_2)_{n_1\perp}$. The relations (B.13) are still valid, and also the relations with $0 \to 1$ and $1 \to 2$, these are

$$n_2 \cdot n_2' = n_1 \cdot n_2' \frac{\bar{q}_0^2}{\bar{k}_2^2} = n_0 \cdot n_1 \frac{\bar{q}_0^2}{\bar{q}_2^2} = -2\frac{(q_2)^2_{n_1\perp}\bar{q}_1^2}{\bar{q}_1^2\,\bar{k}_2^2}\,,$$
$$n_1^\mu = \frac{\bar{k}_2 n_2'^\mu + \bar{q}_2 n_2^\mu}{\bar{q}_1} - \bar{n}^\mu\,(n_2 \cdot n_2')\frac{\bar{q}_2\bar{k}_2}{2\,\bar{q}_1^2}\,,$$
$$(q_2)^\mu_{n_1\perp} = \frac{1}{2}\frac{\bar{q}_2\bar{k}_2}{\bar{q}_2+\bar{k}_2}\sqrt{|n_2\cdot n_2'|}v_2^\mu - \bar{n}^\mu\,(n_2\cdot n_2')\frac{\bar{q}_2\bar{k}_2(\bar{k}_2^2-\bar{q}_2^2)}{4\,\bar{q}_1^3}\,,$$
$$\gamma^\mu_{n_1\perp} = \gamma^\mu_{n_2'\perp} - \bar{n}^\mu\frac{(\not{q}_2)_{n_1\perp}}{\bar{k}_2} - (q_2)^\mu_{n_1\perp}\frac{\not{\bar{n}}}{\bar{k}_2} - \bar{n}^\mu\,(n_2\cdot n_2')\frac{\bar{q}_2^2}{2\,\bar{q}_1^2}\not{\bar{n}}\,, \tag{B.18}$$

where

$$v_2^\mu = \frac{n_2^\mu - n_2'^\mu}{\sqrt{|n_2 \cdot n_2'|}}\,, \tag{B.19}$$

is a vector with $|v_2^2| = 2$. We can write $n_1 \cdot n_1'$ and v_1 in terms of n_2, n_1' and n_2':

$$n_1 \cdot n_1' = \frac{\bar{k}_2(n_2'\cdot n_1') + \bar{q}_2(n_2\cdot n_1')}{\bar{q}_1} - (n_2\cdot n_2')\frac{\bar{q}_1\bar{k}_1}{\bar{q}_0^2}\,, \tag{B.20}$$
$$v_1^\mu = \frac{2\,\bar{k}_2\bar{q}_1\,n_2'^\mu + 2\,\bar{q}_1\bar{q}_2 n_1^\mu - \bar{k}_2\bar{q}_2\,(n_2\cdot n_2')\,\bar{n}^\mu - 2\,\bar{q}_1 n_1'^\mu}{\sqrt{|n_1\cdot n_1'|}}\,. \tag{B.21}$$

(B.15) for two emissions is modified to

$$q_0^2 = (q_2+k_1+k_2)^2 = n_1'\cdot n_2\frac{\bar{q}_2\bar{k}_1}{2} + n_2'\cdot n_2\frac{\bar{q}_2\bar{k}_2}{2} + n_1\cdot n_2'\frac{\bar{k}_1\bar{k}_2}{2}\,. \tag{B.22}$$

Others useful relations are

$$\gamma^\mu_{n_0\perp} = \gamma^\mu_{\perp_2} + \bar{n}^\mu\Big(\frac{(\not{q}_1)_{n_0\perp}}{\bar{q}_1} - \frac{(\not{q}_2)_{n_1\perp}}{\bar{k}_2}\Big) + \Big(\frac{(q_1)^\mu_{n_0\perp}}{\bar{q}_1} - \frac{(q_2)^\mu_{n_1\perp}}{\bar{k}_2}\Big)\not{\bar{n}} - \bar{n}^\mu\not{\bar{n}}\Big(\frac{(q_1)^2_{n_0\perp}}{\bar{q}_1^2} + \frac{(q_2)^2_{n_1\perp}}{\bar{k}_2^2}\Big)\,,$$

$$
\begin{aligned}
q_1^2 = (k_1 + q_2)^2 &= (n_1'\cdot n_2)\frac{\bar{k}_1\bar{q}_2}{2}\,, \\
(k_2 + q_2)^2 &= (n_2'\cdot n_2)\frac{\bar{k}_2\bar{q}_2}{2}\,.
\end{aligned}
\tag{B.23}
$$

For i gluon emissions Fig. (B-1-C), we call n_k' the light-cone vector parallel to the k-gluon, n_i the light-cone vector parallel to the external quark, and n_k the light cone vector such that the k internal virtual quark has zero perpendicular momentum with respect to $(n_k,\, \bar{n})$. To calculate n_i, n_i' we can iterate the formulas above up to i emissions. That is we can calculate n_i, n_i' from n_{i-1} using Eq. (B.12) with $0 \to (i-1)$, $1 \to i$.

B.2 Matching QCD to SCET$_1$

To study the process of $q \to qg$ emission, we match the QCD current,

$$J^{\mu}_{\rm QCD} = \bar{q}\gamma^{\mu}q\,, \tag{B.24}$$

to SCET$_1$ operators for a final state with a quark, antiquark, and gluon. The particle momenta are q_1 for the quark, $p_{\bar{q}}$ for the antiquark, and k_1 for the gluon, (*cf.* Fig. B-2A). We do the matching in the center of mass frame with

$$p_{\gamma} = q_1 + p_{\bar{q}} + k_1 = (Q,0,0,0)\,. \tag{B.25}$$

SCET$_1$, being equivalent to the usual SCET, is formulated as an expansion in the parameter λ. The current in Eq. (B.24) matches onto an infinite series of SCET$_1$ operators. We will perform the matching up to NNLO for one gluons emission, and focus only on the cases when the gluon is either collinear to the quark or far form both the quark and the antiquark. Obtaining the limit of gluon-antiquark collinearity from our work is a simple exercise in charge conjugation. We can construct the SCET$_1$ operators out of a few building blocks: the quark field χ_n, the gluon field $\mathcal{B}^{\alpha}_{n\perp}$ and the perpendicular momentum operator $\mathcal{P}^{\alpha}_{n\perp}$, plus Dirac structures. χ_n, $\mathcal{B}^{\alpha}_{n\perp}$ and $\mathcal{P}^{\alpha}_{n\perp}$ all scale as λ. The basis of SCET$_1$ operators for one emission up to NNLO is [80]. [1].

$$\begin{aligned}
\mathcal{O}^{(0)}_1(n_0) &= \bar{\chi}_{n_0}\chi_{\bar{n}}\,,\\
\mathcal{O}^{(1)}_1(n_0,n_0) &= \bar{\chi}_{n_0}\, g\mathcal{B}^{\alpha}_{n_0\perp}\chi_{\bar{n}}\,,\\
\mathcal{T}^{(1)}_1(n_0,n_0) &= \bar{\chi}_{n_0}\left[\mathcal{P}^{\beta}_{n_0\perp}\, g\mathcal{B}^{\alpha}_{n_0\perp}\right]\chi_{\bar{n}}\,,\\
\mathcal{O}^{(1)}_1(n_1,n_1') &= \bar{\chi}_{n_1} g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}}\,,
\end{aligned} \tag{B.26}$$

$\mathcal{O}^{(0)}_1(n_0)$ standss for $\mathcal{O}^{(0)}_1(n_0,\bar{n})$ and similarly for the other operators, we do not write the antiquark direction as it is always $\bar{n}$. $\mathcal{O}^{(0)}_1$ is the LO operator and scales as λ^2,

[1] $\mathcal{T}^{(1)}_1(n_1,n_1')$ is zero because we can choose the directions n_1 and n_1' to align perfectly with particle momenta such that *e.g.*. $\mathcal{P}_{n_1\perp}\mathcal{B}_{n_1\perp} = 0$

$\mathcal{O}_1^{(1)}(n_0, n_0)$ and $\mathcal{O}_1^{(1)}(n_1, n_1')$ are the NLO operators, scaling like λ^3, and $\mathcal{T}_1^{(1)} \propto \lambda^4$.

In SCET_1, two particles are collinear if they are inside a cone of aperture λ, or equivalently if $p_1 \cdot p_2 \lesssim (Q\lambda)^2$. Usually, we formulate this condition with dimensionless quantities, $n_{p_1} \cdot n_{p_2} \lesssim \lambda^2$, where n_{p_i} is exactly proportional to the particle momentum. To distinguish a two-jet from a three-jet state, we label the external states with the direction to which the particles are collinear. A state $|q_{n_0}\rangle_1$ indicates a state where a quark with momentum q_1 is collinear to the direction n_0, that is $(\bar{q}_1, n_0 \cdot q_1, (q_1)_{n_0\perp}) \sim (1, \lambda^2, \lambda)Q$, and this state can be annihilated by any operator, χ_n, where n and n_0 are in the same SCET_1 equivalence class, $\{[n]\}$. The subscript, 1, tells us that we are using the SCET_1 classes here. As we will see when we match to lower-scale SCET_i, we will change this number appropriately. A two-jet state with a collinear quark and gluon, and an antiquark along $\bar{n}$ is given by $|q_{n_0}\, g_{n_0}\, \bar{q}_{\bar{n}}\rangle_1$. The fact that the quark and gluon share an index label implies that $q_1 \cdot k_1 \lesssim (\lambda)^2$. A three-jet state is indicated by $|q_{n_1}\, g_{n_1'}\, \bar{q}_{\bar{n}}\rangle_1$, where each particle is collinear to a different direction. The operators $\mathcal{O}_1^{(0)}(n_0)$, $\mathcal{O}_1^{(1)}(n_0, n_0)$ and $\mathcal{T}_1^{(1)}(n_0, n_0)$ are two-jet operators, that is they can only create a two-jet state, whereas $\mathcal{O}_1^{(1)}(n_1, n_1')$ is for three-jets. Multiplying the operator in (B.26) by the Wilson coefficients, we have

$$\begin{aligned} J^\mu_{\mathrm{QCD}} = &\, C^{(0)}_{1,\mathrm{LO}}(n_0)\mathcal{O}_1^{(0)}(n_0) + C_1^{(1)}(n_0, n_0)\mathcal{O}_1^{(1)}(n_0, n_0) + C^{(1)}_{1,\mathcal{T}}(n_0, n_0)\mathcal{T}_1^{(1)}(n_0, n_0) \\ & + C_1^{(1)}(n_1, n_1')\mathcal{O}_1^{(1)}(n_1, n_1') + \dots\,, \end{aligned} \tag{B.27}$$

where the ellipses indicate higher order terms in λ. We begin by looking at two-jet operators in detail. For this region, because we are in the center of mass frame, the two jets are back to back. We define the kinematics as follows, the antiquark is exactly parallel to $\bar{n} = (1, 0, 0, -1)$, while the quark and the gluon are collinear to $n_0 = (1, 0, 0, 1)$, such that $q_0 = q_1 + k_1$ has no component perpendicular to n_0 and $\bar{n}$:

$$\begin{aligned} p^\mu_{\bar{q}} &= n_0 \cdot p_{\bar{q}} \frac{\bar{n}^\mu}{2}\,, \\ q_1^\mu &= \bar{q}_1 \frac{n_0^\mu}{2} + n_0 \cdot q_1 \frac{\bar{n}^\mu}{2} + (q_1)^\mu_{n_0\perp}\,, \end{aligned}$$

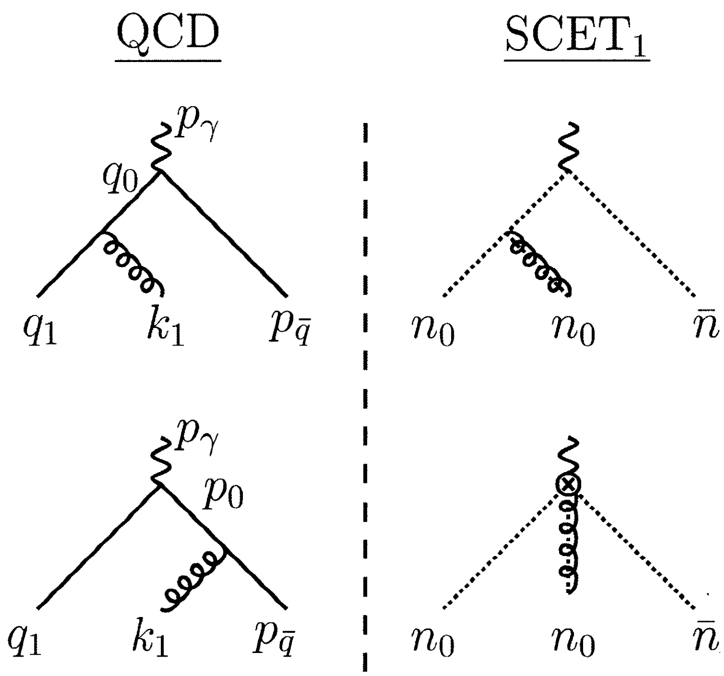

Figure B-2: Matching QCD to SCET_1 for the two-jet configuration: In the first column there are the two Feynman graphs for one-gluon emission in QCD, labelled by the 4-momenta. In the second column there are the two Feynman graphs in SCET_1 that reproduce the same amplitude in the case the quark and gluon are collinear along the direction n_0. The first graph come from the operator $\mathcal{O}_1^{(0)}$ with the insertion of the SCET_1 Lagrangian, the second graph comes from the operators $\mathcal{O}_1^{(1)}$ and $\mathcal{T}_1^{(1)}$.

$$k_1^\mu = \bar{q}_1 \frac{n_0^\mu}{2} + n_0 \cdot k_1 \frac{\bar{n}^\mu}{2} + (k_1)^\mu_{n_0\perp} \,, \tag{B.28}$$

where $(n_0 \cdot q_1, \bar{q}_1, q_{1\perp})$ and $(n_0 \cdot k_1, \bar{k}_1, k_{1\perp})$ scale as $(\lambda^2, 1, \lambda)$, and $(q_1)^\mu_{n_0\perp} = -(k_1)^\mu_{n_0\perp}$ by momentum conservation. The Wilson coefficients are defined through the equation

$$\begin{aligned}\langle 0|J^\mu_{\mathrm{QCD}}|q_{n_0}\, g_{n_0}\, \bar{q}_{\bar n}\rangle_1 =& C^{(0)}_{1,\mathrm{LO}}(n_0, n_0) \int dx^4 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(0)}\}|q_{n_0}\, g_{n_0}\, \bar{q}_{\bar n}\rangle_1 \\ &+ C_1^{(1)}(n_0, n_0)\langle 0|\mathcal{O}_1^{(1)}|q_{n_0}\, g_{n_0}\, \bar{q}_{\bar n}\rangle_1 + C^{(1)}_{1,\mathcal{T}}(n_0, n_0)\langle 0|\mathcal{T}_1^{(1)}|q_{n_0}\, g_{n_0}\, \bar{q}_{\bar n}\rangle_1 \,.\end{aligned} \tag{B.29}$$

Calculating the C's for this two-jet process goes as follows. We decompose the QCD amplitude along n_0 and $\bar n$ using Eq. (B.28) and we write the QCD spinor in terms of the SCET_1 spinor Eq. (B.31). Expanding in λ up to NNLO, on the RHS, we compute the amplitudes for the three SCET_1 terms. The coefficient $C^{(0)}_{1,\mathrm{LO}}$ was already determined from matching QCD to $\mathrm{SCET_1}$ for zero gluon emission, it is

$$C^{(0)}_{1,\mathrm{LO}} = \gamma^\mu \,. \tag{B.30}$$

The coefficients $C_1^{(1)}$ and $C_{1,\mathcal{T}}^{(1)}$ come from solving Eq. (B.29) at NLO and NNLO, respectively. Since $\mathcal{O}_1^{(1)}$ and $\mathcal{T}_1^{(1)}$ are at different orders in λ, there are no ambiguities in solving Eq. (B.29) for both $C_1^{(1)}$ and $C_{1,\mathcal{T}}^{(1)}$.

In order to do the matching, we need the relation between the QCD and SCET spinors. Using Eq. (B.7), we can write:

$$u(p) = \Big(1 + \frac{\not{p}_\perp \bar{\not{n}}}{2\,\bar{p}}\Big) u_n(p) \,, \tag{B.31}$$

where $u(p)$ is the QCD spinor and $u_n(p)$ is the $\mathrm{SCET_1}$ one. It easy to see that the SCET spinor satisfies

$$\begin{aligned}
\frac{\bar{\not{n}}\not{n}}{4} u_n &= 0 \,, \\
\frac{\not{n}\bar{\not{n}}}{4} u_n &= 0 \,, \\
\sum_s \bar{u}_n^s u_n^s &= \bar{p}\frac{\bar{\not{n}}}{2} \,.
\end{aligned} \tag{B.32}$$

Note that the normalization in the spin sum does not introduce NLO terms.

The QCD amplitude for $\gamma^* \to q\bar{q}g$ is shown in Fig. B-2:

$$A^{q\bar{q}g}_{\mathrm{QCD}} = \bar{u}(q_1) ig\gamma^\alpha \frac{i\not{q}_0}{q_0^2}\gamma^\mu v(p_{\bar{q}}) - \bar{u}(q_1) ig\gamma^\mu \frac{i\not{p}_0}{p_0^2}\gamma^\alpha v(p_{\bar{q}}) \,. \tag{B.33}$$

The amplitude (B.33) is intended to be multiplied by the polarization vector for the gluon, that in general we do not explicitly write. Using Eqs. (B.28) and (B.31) in (B.33) and expanding to NNLO in λ we get

$$A^{q\bar{q}g}_{\mathrm{QCD}} = A^{q\bar{q}g}_{\mathrm{LO}} + A^{q\bar{q}g}_{\mathrm{NLO}} + A^{q\bar{q}g}_{\mathrm{NNLO}} \,, \tag{B.34}$$

where

$$
\begin{aligned}
A^{q\bar{q}g}_{\mathrm{LO}} &= -g\bar{u}_{n_0}\Big[\Big(n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp}}{\bar{q}_1}\gamma^\alpha_{n_0\perp}\Big)\frac{\bar{q}_0}{q_0^2} + \frac{\bar{n}^\alpha}{\bar{k}_1}\Big]\gamma^\mu_{n_0\perp}v_{\bar{n}}\,,\\
A^{q\bar{q}g}_{\mathrm{NLO}} &= g\frac{n_0^\mu - \bar{n}^\mu}{Q}\bar{u}_{n_0}\Big(\gamma^\alpha_{n_0\perp} - \frac{(\not{k}_1)_{n_0\perp}}{\bar{k}_1}\bar{n}^\alpha\Big)v_{\bar{n}}\,,\\
A^{q\bar{q}g}_{\mathrm{NNLO}} &= g\Big(\frac{1}{\bar{q}_1} + \frac{1}{\bar{k}_1}\Big)\frac{1}{Q}\bar{u}_{n_0}\gamma^\mu_{n_0\perp}(\not{k}_1)_{n_0\perp}\Big(\gamma^\alpha_{n_0\perp} - \frac{(\not{k}_1)_{n_0\perp}}{\bar{k}_1}\bar{n}^\alpha\Big)v_{\bar{n}}\\
&\quad - g\frac{2}{\bar{q}_1 Q}\bar{u}_{n_0}(k_1)^\mu_{n_0\perp}\Big(\gamma^\alpha_{n_0\perp} - \frac{(\not{k}_1)_{n_0\perp}}{\bar{k}_1}\bar{n}^\alpha\Big)v_{\bar{n}}\,.
\end{aligned}
\tag{B.35}
$$

From (B.35), and knowing $C^{(0)}_{1,\mathrm{LO}}$, it is easy to determine the other two Wilson coefficients to reproduce $A^{q\bar{q}g}_{\mathrm{QCD}}$, they are

$$
\begin{aligned}
C^{(1)}_1(n_0, n_0) &= \frac{1}{Q}(n_0^\mu - \bar{n}^\mu)\gamma^\alpha_{n_0\perp}\,,\\
C^{(1)}_{1,\mathcal{T}}(n_0, n_0) &= \frac{1}{\bar{q}_1\bar{k}_1}\gamma^\mu_{n_0\perp}\gamma^\beta_{n_0\perp}\gamma^\alpha_{n_0\perp} - \frac{2}{\bar{q}_1 Q}g^{\beta\mu}\gamma^\alpha_{n_0\perp}\,,
\end{aligned}
\tag{B.36}
$$

where we have used the relation $\bar{q}_1 + \bar{k}_1 = Q$.

For the three-jet operator $O^{(1)}_1(n_1, n_1')$, the matching was done in subsection 3.7.6, but we will translate to the notation used here. In this case we need three distinct directions in SCET to describe the three external particles, and there is no small parameter to expand in. This means that the amplitude for this operator is exactly equal to the tree-level QCD amplitude for a $q\bar{q}g$ process. One may wonder then, why we simply do not apply this everywhere instead of just the three-jet region. The answer has to do with running effects. The RG kernels of our two-jet operators, $\mathcal{O}^0_1$, $\mathcal{O}^1_1$, and $\mathcal{T}^1_1$, will resum the large collinear logarithms of those configurations (*cf.* Subsection 4.4.3). It is for this reason that we gain by keeping track of these as separate contributions.

Since the three external particles are collinear to three different directions, in principle we have to decompose each particle using its own pair of light-cone vectors, see Eq. (2.1). However, we can show that we need only four independent ones. In the center of mass frame, $q_0 = q_1 + k_1$ is back to back with the antiquark, $p_{\bar{q}} \propto \bar{n}$. We decompose q_0 along (n_0, $\bar{n}$) such that it has no component perpendicular to them:

$q_0 = n\,\bar{n}\cdot q_0/2 + \bar{n}\,n_0\cdot q_0/2$. Using Eqs. (B.12), we define the light-cone vectors n_1 and n_1' such that they are parallel to q_1 and k_1, respectively. We can actually decompose the quark along $(n_1,\,\bar{n})$ and the gluon into $(n_1',\,\bar{n})$. Unlike the two-jet case, where $(q_1)_{n_0\perp} \lesssim \lambda$, since the quark was collinear to n_0, here $(q_1)_{n_0\perp} > \lambda$. Summarizing, we have:

$$
\begin{aligned}
q_1^\mu &= \bar{n}\cdot q_1 \frac{n_1^\mu}{2}\,, \\
p_{\bar{q}}^\mu &= n\cdot p_{\bar{q}} \frac{\bar{n}^\mu}{2}\,, \\
k_1^\mu &= \bar{n}\cdot k_1 \frac{n_1'^\mu}{2}\,,
\end{aligned}
\tag{B.37}
$$

where $\bar{n}\cdot q_1$, $n\cdot p_{\bar{q}}$ and $\bar{n}\cdot k_1$ are $\mathcal{O}(1)$. Since a three-jet configuration has $q_1\cdot k_1 > \lambda^2$, we have that $n_1\cdot n_1' > \lambda^2$. Choosing the directions n_1, n_1' and $\bar{n}$ exactly parallel to the external momenta means that we do not need $\mathcal{T}_1^{(1)}(n_1,n_1') = \bar{\chi}_{n_1}\mathcal{P}_{n_1'}\mathcal{B}^\alpha_{n_1'\perp}\chi_{\bar{n}}$, as $\mathcal{P}_{n_1'}$ acting on the gluon field gives zero.

The matching is therefore given by

$$
\langle 0|J^\mu_{\rm QCD}|q_{n_1} q_{\bar{n}}\, g_{n_1'}\rangle_1 = C_1^{(1)}(n_1,n_1')\langle 0|O_1^{(1)}(n_1,n_1')|q_{n_1} q_{\bar{n}}\, g_{n_1'}\rangle_1\,, \tag{B.38}
$$

and the Wilson coefficient is

$$
\begin{aligned}
C_1^{(1)}(n_1,n_1') = &-\frac{2}{(n_1\cdot n_1')\bar{q}_1\bar{k}_1}\gamma^\alpha \not{p}_\gamma\,\gamma_T^\mu \\
&+\Big[\frac{1}{(n\cdot p_{\bar{q}})\bar{k}_1}\Big(\gamma_T^\mu \not{p}_\gamma - \bar{q}_1\, n_{1T}^\mu\Big) + \frac{2(n\cdot p_{\bar{q}})}{(n_1\cdot n_1')\bar{q}_1\bar{k}_1}\bar{n}_T^\mu\Big]\gamma^\alpha_{n_1'\perp}\,,
\end{aligned}
\tag{B.39}
$$

where the subscript T applied to a generic four vector f^μ means: $f_T^\mu \equiv f^\mu - p_\gamma^\mu(f\cdot p_\gamma)/p_\gamma^2$ and p_γ is defined in Eq. (B.25). We notice that all the Wilson coefficients in SCET$_1$ are of order λ^0. We described the effects of adding running to them in Subsection 4.4.3. As we discuss further in Appendix B.3, we do not need to compute any suppressed two-gluon operators in SCET$_1$ to the order at which we are working. In the next Appendix we will mach SCET$_1$ to SCET$_2$. Before doing it we have to run down the SCET$_1$ operators form the scale Q to the scale μ_1 where we have the first emission:

$$
\begin{aligned}
C_0^{(0)}(n_0) &= U^{(2,0,0)}(n_0)(Q,\mu)\,\gamma^\mu_{n_0\perp} \\
C_1^{(1)}(n_0,n_0) &= U^{(2,1,0)}(n_0,n_0)(Q,\mu)\otimes\frac{n_0^\mu-\bar n^\mu}{Q}\gamma^\alpha_{n_0\perp}\,, \\
C_{1,\mathcal{T}}^{(1)}(n_0,n_0) &= U_{\mathcal{T}}^{(2,1,1)}(n_0,n_0)(Q,\mu)\otimes\frac{1}{\bar q_1\bar k_1}\Big(\gamma^\mu_{n_0\perp}\gamma^\beta_{n_0\perp}\gamma^\alpha_{n_0\perp}-\frac{2}{\bar q_1 Q}\,g^{\mu\beta}\gamma^\alpha_{n_0\perp}\Big)\,, \\
C_1^{(1)}(n_1,n_1') &= -U^{(1)}(n_1,n_1')(Q,\mu)\Big(\frac{2}{(n_1\!\cdot\! n_1')\bar q_1\bar k_1}\gamma^\alpha_{n_0\perp}\not{p}_\gamma\,\gamma^\mu_T \\
&\quad+\Big[\frac{1}{(n\!\cdot\! p_{\bar q})\bar k_1}\Big(\gamma^\mu_T\not{p}_\gamma-\bar q_1\,n^\mu_{1T}\Big)+\frac{2(n\!\cdot\! p_{\bar q})}{(n_1\!\cdot\! n_1')\bar q_1\bar k_1}\bar n^\mu_T\Big]\gamma^\alpha_{n_0\perp}\Big)\,.
\end{aligned}
\tag{B.40}
$$

For the definition of the running factors $U^{(i,j,k)}(Q,\mu)$ see Section 4.4.3.

B.3 Matching SCET$_1$ to SCET$_2$

B.3.1 One-Gluon Emission

We now match SCET$_1$ to SCET$_2$ for one-gluon emission and two-gluon emissions. In this section we only deal with one emission. The basis of SCET$_2$ operators necessary for the matching up to NNLO is equal to Eq. (B.26) but with the operators defined in SCET$_2$ instead of SCET$_1$: $\mathcal{O}_2^{(0)}(n_0)$, $\mathcal{O}_2^{(1)}(n_0,n_0)$, $\mathcal{T}_2^{(1)}(n_0,n_0)$,$\mathcal{O}_2^{(1)}(n_1,n_1')$.[2] In the previous section we have matched QCD to SCET$_1$ for one emission and we saw that in SCET$_1$ we can have either a two-jet ($|q_{n_0}\,g_{n_0}\,\bar q_{\bar n}\rangle_1$) or three-jet configuration ($|q_{n_1}\,g_{n_1'}\,\bar q_{\bar n}\rangle_1$), depending on the collinearity of the external particles. When we go to SCET$_2$ we reduce the magnitude of collinearity, now particles with momenta p_1 and p_2 can be define collinear only if $p_1\cdot p_2\lesssim(\lambda^2)^2$, and the SCET$_2$ theory distinguishes between the states $|q_{n_0}\,g_{n_0}\,\bar q_{\bar n}\rangle_2$ and $|q_{n_1}\,g_{n_1'}\,\bar q_{\bar n}\rangle_2$. Because of the difference of collinearity between SCET$_1$ and SCET$_2$, a two-jet configuration in SCET$_1$ can be matched both onto $|q_{n_0}\,g_{n_0}\,\bar q_{\bar n}\rangle_2$ and $|q_{n_1}\,g_{n_1'}\,\bar q_{\bar n}\rangle_2$ in SCET$_2$, instead the three-jet configuration in SCET$_1$ can be only matched to a three state $|q_{n_1}\,g_{n_1'}\,\bar q_{\bar n}\rangle_2$ in SCET$_2$. The matching

[2] As before we do not consider operators like $\mathcal{O}_2^{(1)}(n_0,\bar n)$ that describe a gluon collinear to the antiquark.

is given by

$$
\begin{aligned}
J^{\mu}_{\rm QCD} =& C^{(0)}_{1,\rm LO}(n_0)\mathcal{O}^{(0)}_1(n_0) + C^{(1)}_1(n_0,n_0)\mathcal{O}^{(1)}_1(n_0,n_0) + C^{(1)}_1(n_1,n_1')\mathcal{O}^{(1)}_1(n_1,n_1') \\
&+ C^{(1)}_{1,\mathcal{T}}(n_0,n_0)\mathcal{T}^{(1)}_1(n_0,n_0) + \cdots \qquad \text{(B.41)}\\
=& C^{(0)}_2(n_0)\mathcal{O}^{(0)}_2(n_0) + C^{(1)}_2(n_0,n_0)\mathcal{O}^{(1)}_2(n_0,n_0) + C^{(1)}_2(n_1,n_1')\mathcal{O}^{(1)}_2(n_1,n_1') \\
&+ C^{(1)}_{2,\mathcal{T}}(n_0,n_0)\mathcal{T}^{(1)}_2(n_0,n_0) + \cdots\,, \qquad \text{(B.42)}
\end{aligned}
$$

where in the first equation we have written the original QCD current in terms of SCET_1 operators and in the second equation in terms of SCET_2 operators. The ellipses indicate higher order operators. If we close Eq. (B.42) with the state $|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2$ we get

$$
\begin{aligned}
& C^{(0)}_{1,\rm LO}(n_0)\int dx^4 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}^{(0)}_1(n_0)\}|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2 \\
&+ C^{(1)}_1(n_0,n_0)\langle 0|\mathcal{O}^{(1)}_1(n_0,n_0)|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2 + C^{(1)}_{1,\mathcal{T}}(n_0,n_0)\langle 0|\mathcal{T}^{(1)}_1(n_0,n_0)|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2 \\
&= C^{(0)}_2(n_0)\int dx^4 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}^{(0)}_2(n_0)\}|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2 \\
&+ C^{(1)}_2(n_0,n_0)\langle 0|\mathcal{O}^{(1)}_2(n_0,n_0)|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2 + C^{(1)}_{2,\mathcal{T}}(n_0,n_0)\langle 0|\mathcal{T}^{(1)}_2(n_0,n_0)|q_{n_0}\, g_{n_0}\, \bar q_{\bar n}\rangle_2\,,
\end{aligned}
\qquad \text{(B.43)}
$$

Because the structure of the operators in Eq. (B.43) is the same in the LHS and RHS, we simply get

$$
\begin{aligned}
C^{(0)}_2(n_0) &= C^{(0)}_{1,\rm LO}(n_0)\,,\\
C^{(1)}_2(n_0,n_0) &= C^{(1)}_1(n_0,n_0)\,,\\
C^{(1)}_{2,\mathcal{T}}(n_0,n_0) &= C^{(1)}_{1,\mathcal{T}}(n_0,n_0)\,.
\end{aligned}
\qquad \text{(B.44)}
$$

If we close Eq. (B.42) with the state $|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle_2$ we have

$$
\begin{aligned}
& C^{(0)}_{1,\rm LO}(n_0)\int dx^4 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}^{(0)}_1(n_0)\}|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle_2 + C^{(1)}_1(n_0,n_0)\langle 0|\mathcal{O}^{(1)}_1(n_0,n_0)|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle_2 \\
&+ C^{(1)}_1(n_1,n_1')\langle 0|\mathcal{O}^{(1)}_1(n_1,n_1')|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle_2 + C^{(1)}_{1,\mathcal{T}}(n_0,n_0)\langle 0|\mathcal{T}^{(1)}_1(n_0,n_0)|q_{n_1}\, g_{n_1'}\, \bar q_{\bar n}\rangle_2
\end{aligned}
$$

$$= C_2^{(1)}(n_1, n_1')\langle 0|\mathcal{O}_2^{(1)}(n_1, n_1')|q_{n_1}\, g_{n_1'}\, \bar{q}_{\bar{n}}\rangle_2 \,. \tag{B.45}$$

We decompose $C_2^{(1)}(n_1, n_1')$ as

$$C_2^{(1)} = C_{2,\,\mathrm{LO}}^{(1)} + C_{2,\,\mathrm{NLO}}^{(1)H,a} + C_{2,\,\mathrm{NLO}}^{(1)H,b} + C_{2,\,\mathrm{NNLO}}^{(1)H} \,, \tag{B.46}$$

where $C_{2,\,\mathrm{LO}}^{(1)}$ is the coefficient that multiplied by $\langle 0|\mathcal{O}_2^{(1)}(n_1, n_1')|q_{n_1}\, g_{n_1'}\, \bar{q}_{\bar{n}}\rangle_2$ reproduces the second line in Eq. (B.45), $C_{2,\,\mathrm{NLO}}^{(1)H,a}$ the third line and so on. All the SCET_2 coefficients in Eq. (B.44) scale as λ^0 like in SCET_1, but we will see that the SCET_2 coefficients in Eq. (B.46) scale with different powers of λ, thus what is meaningful is not the power counting of the operator by itself but the power counting of the Wilson coefficient$\times$operator. We will prove that

$$\begin{aligned}
C_2^{(0)}(n_0)\mathcal{O}_2^{(0)}(n_0) &\sim \lambda^4 \,, & C_{2,\,\mathrm{LO}}^{(1)}(n_1, n_1')\mathcal{O}_2^{(1)}(n_1, n_1') &\sim \lambda^5 \,, \\
C_2^{(1)}(n_0, n_0)\mathcal{O}_2^{(1)}(n_0, n_0) &\sim \lambda^6 \,, & C_{2,\,\mathrm{NLO}}^{(1)H,a}(n_1, n_1')\mathcal{O}_2^{(1)}(n_1, n_1') &\sim \lambda^6 \,, \\
C_{2,\mathcal{T}}^{(1)}(n_0, n_0)\mathcal{T}_2^{(1)}(n_0, n_0) &\sim \lambda^8 \,, & C_{2,\,\mathrm{NLO}}^{(1)H,b}(n_1, n_1')\mathcal{O}_2^{(1)}(n_1, n_1') &\sim \lambda^6 \,, \\
& & C_{2,\,\mathrm{NNLO}}^{(1)H}(n_1, n_1')\mathcal{O}_2^{(1)}(n_1, n_1') &\sim \lambda^7 \,.
\end{aligned} \tag{B.47}$$

In the second column we have only one operator $\mathcal{O}_2^{(1)}(n_1, n_1')$ and we have decomposed its coefficient according to Eq. (B.46). It is important to notice that the matching constrains but does not conserve the power counting, this is because in SCET_1 the fields scale as λ, but in SCET_2 they scale as λ^2, for example we have that the LO operator in SCET_1 is $C_1^{(0)}\mathcal{O}_1^{(0)} \sim \lambda^2$, but for the LO operator in SCET_2 we have $C_2^{(0)}\,\mathcal{O}_2^{(0)} \sim \lambda^5$.[3]

If we have to calculate cross section with a fixed number of external particles we have to take into account all the SCET_2 operators in Eq .(B.42). What we are interested in this work is to reproduce parton shower. To reproduce the LL parton shower we only need $C_{2,\,\mathrm{LO}}^{(1)}(n_1, n_1')\mathcal{O}_2^{(1)}(n_1, n_1')$. To see why we can emit a second gluon, k_2 from it. Before doing it we have to run it down to the scale of the second

[3] A consequence of it is that for two emissions we will have NLO operator in SCET_1 that contributes only to NNLO operators in SCET_2

emission using the factor $U^{(1)}(k_{1\perp}, k_{2\perp})$. Because the first gluon comes form the operator $\mathcal{O}_2^{(1)}(n_1, n_1')$ whose coefficient was defined in the matching form SCET_1, and the second gluons comes form inserting the SCET_2 Lagrangian, we know that $k_1^2 \gg k_2^2$, as the parton showering condition requires. The operators $\mathcal{O}_2^{(0)}(n_0)$ and $\mathcal{O}_2^{(1)}(n_0, n_0)$ carry information that we do not need to describe parton shower. For example $\mathcal{O}_2^{(0)}(n_0)$ describes a quark which had not emitted until after the scale of matching $k_{1\perp}$. However, the RG kernels already give us the no-branching probability so, even if they have lower order in λ, we do not use them. For this reason we call $C_{2,\,\mathrm{LO}}^{(1)}(n_1, n_1')\mathcal{O}_2^{(1)}(n_1, n_1')$ our LO operator in λ and we say that LO is at order λ^5. The coefficients $C_{2,\,\mathrm{NLO}}^{(1),H}(n_1, n_1')$ and $C_{2,\,\mathrm{NNLO}}^{(1),H}(n_1, n_1')$ give corrections to the parton shower for one emission, and they describe a three-parton process. This means that if we want to find correction to a parton shower process that start from a matrix element with two partons, we have to take into account also parton shower starting from a matrix element of three partons. Therefore in the rest of the section we will focus only on the operator we need: $\mathcal{O}_2^{(1)}(n_1, n_1')$.

We now turn to calculate the Wilson coefficient (B.46). We do it in three steps: first we calculate the amplitudes in SCET_1 on the RHS of (B.45), second we rotate it using the finite RPI transformation define in Appendix B.1, so that they overlap with the SCET_2 states, and third we calculate the SCET_2 amplitudes in the LHS of (B.45) and calculate the Wilson coefficients necessary to make the two side of Eq. (B.45) equal. We do it order by order and we start calculating the coefficient $C_{2,\,\mathrm{LO}}^{(1)}$. The SCET_1 amplitude in the first line of the LHS (B.45) is

$$A_{\mathrm{LO}}^{q\bar{q}g} = U^{(0)}(n_0)(Q, \mu_1)\, g\, \bar{\xi}_{n_0} \left(n_0^\alpha + \frac{(\not{q}_1)_{n_0\perp} \gamma_{n_0\perp}^\alpha}{\bar{q}_1} \right) \gamma_{n_0\perp}^\mu \xi_{\bar{n}}\,. \tag{B.48}$$

$U^{(0)}(n_0)$ is the running factor (see Section 4.4.3), and $\mu_1 \sim \lambda Q$ is at the scale of the emission. In (B.48) we have omitted the term proportional to $\bar{n}^\alpha$ because we work in the light-cone gauge where the Wilson lines are equal to the identity. The amplitude (B.48) is written in terms of objects projected in the n and $\bar{n}$ directions. As discussed in Appendix B.1, these directions are not suitable for a SCET_2 states but we can use the formulas (B.9) and write (B.48) in terms of the directions n_1 and n_1' that are the directions where the quark and gluon have zero perpendicular component, this gives

$$A_{\rm LO}^{q\bar{q}g} = U^{(0)}(n_0)(Q,\mu_1)\, g\, \bar{\xi}_{n_1} \left(n_1'^{\alpha} + 2\frac{(q_1)^{\alpha}_{n_0\perp}}{\bar{k}_1} + \frac{(\not{q}_1)_{n_0\perp}\gamma^{\alpha}_{n_1'\perp}}{\bar{q}_1} \right) \frac{\bar{q}_0}{q_0^2}\gamma^{\mu}_{n_0\perp}\xi_{\bar{n}}\,. \tag{B.49}$$

In (B.49) we have rotated the spinor in the n_1 direction, $\gamma_{n_0\perp}$ in the n_1' direction and we have written n_0 in terms of n_1, n_1' and $q_{n_0\perp}$. We have dropped all the terms proportional to $\bar{n}^{\alpha}$ and we made use of relations $\bar{\not{n}}\bar{\not{n}} = 0$ and $\bar{\xi}_{n_1}\not{n}_1 = 0$. Because the momentum of the gluon is parallel to $n_1'^{\mu}$, only the polarizations in the perpendicular direction with respect to $n_1'^{\mu}$ are physical, thus we can neglect the term proportional to $n_1'^{\mu}$ in Eq. (B.49). The SCET$_2$ amplitude $\langle 0|\bar{\chi}_{n_1} g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}}|q_{n_1}g_{n_1'}\bar{q}_{\bar{n}}\rangle$ is

$$\langle 0|\bar{\chi}_{n_1} g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}}|q_{n_1}g^{\alpha}_{n_1'}\bar{q}_{\bar{n}}\rangle = g\bar{\xi}_{n_1}\epsilon^{\alpha}_{n_1'\perp}\xi_{\bar{n}}\,, \tag{B.50}$$

where in Eq. (B.50) we have explicitly written the polarization vector for the gluon. From Eq. (B.49) and Eq. (B.50), we can see that the LO Wilson coefficient is

$$C^{(1)}_{2,\,\rm LO} = U^{(0)}(n_0)(Q,(k_1)_{n_0\perp})c^{\alpha}_{\rm LO}(n_0)\frac{\bar{q}_0}{q_0^2}\gamma^{\mu}_{n_0\perp}\,, \tag{B.51}$$

where

$$c^{\alpha}_{\rm LO}(n_0) = \left(2\frac{(q_1)^{\alpha}_{n_0\perp}}{\bar{k}_1} + \frac{(\not{q}_1)_{n_0\perp}\gamma^{\alpha}_{n_1'\perp}}{\bar{q}_1}\right)\frac{\bar{\not{n}}\not{n}_0}{4}\,\Theta_{\lambda^2}[n_1\cdot n_1']\,. \tag{B.52}$$

$\Theta_{\lambda^2}[n_1\cdot n_1']$ is a function that encodes the information that $(n_1\cdot n_1') \lesssim \lambda^2$, we will say more about it below. Because the matching come from a SCET$_1$ operator, $(q_1)_{n_0\perp}$ can go up to order λ and q_0^2 up to λ^2, thus $C^{(1)}_{2,\,\rm LO}$ has power counting λ^{-1}. Using formulas (B.13), we can write (B.51) only in terms of n_1 and n_1', this gives

$$C^{(1)}_{2,\,\rm LO} = U^{(0)}(n_0)(Q,\,\mu_1)\left(\frac{\bar{q}_1}{Q}\sqrt{|n_1\cdot n_1'|}v_1^{\alpha} + \frac{\bar{k}_1}{2Q}\sqrt{|n_1\cdot n_1'|}\not{v}_1\,\gamma^{\alpha}_{n_1'\perp}\right)\frac{2\bar{q}_0}{(n_1\!\cdot\! n_1')\,\bar{q}_1\bar{k}_1}$$
$$\times\left(\gamma^{\mu}_{n_1'} - \bar{n}^{\mu}\frac{1}{2}\frac{\bar{q}_1}{\bar{q}_0}\sqrt{|n_1\cdot n_1'|}\not{v}_1\right). \tag{B.53}$$

where v_1^μ is defined in Eq. (B.14), $\bar{q}_1 + \bar{k}_1 = Q$. For μ_1 we have to take a quantity that scale as λQ, we choose $\mu_1 = \sqrt{|n_1 \cdot n_1'|}Q$. Because $|v_1^2| = 2$, the power counting of (B.53) is only from the scalar product $n_1 \cdot n_1'$ that here is of order λ^2. In a similar way we can calculate $C_{2,\,\mathrm{NLO}}^{(1)H,a}$ and $C_{2,\,\mathrm{NNLO}}^{(1)H}$.

We have done the matching starting from the QCD current $J^\mu_{\mathrm{QCD}} = \bar{q}\,\gamma^\mu q$. If we had started from a general current, $\bar{q}\,\Gamma^\mu q$, the results (B.51) for $C_{2,\,\mathrm{LO}}^{(1)}$ would have the same upon the substitution

$$\gamma^\mu_{n_0\perp} \to \frac{\bar{\not{n}}\not{n}_0}{4}\Gamma^\mu . \tag{B.54}$$

We can obtain $C_{2,\,\mathrm{LO}}^{(1)}\mathcal{O}_2^{(2)}(n_1, n_1')$ from the SCET$_1$ operator $\bar{\chi}_{n_0}\Gamma^\mu\chi_{\bar{n}}$ running down from scale Q to scale μ_1 using the factor $U^{(0)}(Q,\,\mu_1)(n_0)$, and subsequently using the replacement rule

$$(\bar{\chi}_{n_0})_i \to (c^\alpha_{\mathrm{LO}})_{ji}(n_0)(\bar{\chi}_{n_1})_j\, g\mathcal{B}_\alpha^{n_1'\perp} . \tag{B.55}$$

On the other side we will see that the coefficients $C_{2,\,\mathrm{NLO}}^{(1)H,\,a}$, $C_{2,\,\mathrm{NLO}}^{(1)H,\,b}$, $C_{2,\,\mathrm{NNLO}}^{(1)H}$ are much different in the presence of a different QCD current. We labeled the coefficients whose structures depend on the hard scattering with the upper-script H, that means hard-correction. Starting from two-gluon emissions, there are corrections whose structures are independent from the initial currents and we will call them jet correction.

We now calculate $C_{2,\,\mathrm{NLO}}^{(1)H,a}$ and $C_{2,\,\mathrm{NNLO}}^{(1)H}$. For the NLO and NNLO amplitudes in the second and third line of the LHS of Eq. (B.45) we have

$$\begin{aligned}
A^{q\bar{q}g}_{\mathrm{NLO}} &= U^{(2,1,0)}(n_0, n_0)(Q,\,\mu_1)\frac{n_0^\mu - \bar{n}^\mu}{Q} g\bar{u}_{n_0}\gamma^\alpha_{n_0\perp}v_{\bar{n}} = \frac{n_0^\mu - \bar{n}^\mu}{Q} g\bar{u}_{n_1}\gamma^\alpha_{n_1'\perp}v_{\bar{n}}\,, \\
A^{q\bar{q}g}_{\mathrm{NNLO}} &= U_T^{(2,1,1)}(n_0, n_0)(Q,\,\mu_1) \\
&\quad\times \left(\frac{1}{\bar{q}_1} + \frac{1}{\bar{k}_1}\right)\frac{1}{Q} g\bar{u}_{n_0}\gamma^\mu_{n_0\perp}(\not{k}_1)_{n_0\perp}\gamma^\alpha_{n_0\perp}v_{\bar{n}} - \frac{2}{\bar{q}_1 Q} g\bar{u}_{n_0}(k_1)^\mu_{n_0\perp}\gamma^\alpha_{n_0\perp}v_{\bar{n}} \\
&= U_T^{(2,1,1)}(n_0, n_0)(Q,\,\mu_1) \\
&\quad\times \left(\frac{1}{\bar{q}_1} + \frac{1}{\bar{k}_1}\right)\frac{1}{Q} g\bar{u}_{n_1}\gamma^\mu_{n_0\perp}(\not{k}_1)_{n_0\perp}\gamma^\alpha_{n_1'\perp}v_{\bar{n}} - \frac{2}{\bar{q}_1 Q} g\bar{u}_{n_1}(k_1)^\mu_{n_0\perp}\gamma^\alpha_{n_1'\perp}v_{\bar{n}}\,,
\end{aligned} \tag{B.56}$$

where in the last equation we have performed the rotation to the directions n_1 and n_1'. The SCET_2 coefficients needed to reproduce the amplitudes in Eq. (B.56) are

$$\begin{aligned} C^{(1)H,a}_{2,\,\mathrm{NLO}} &= U^{(2,1,0)}(n_0,n_0)(Q,\,\mu_1)c^{H,a}_{2,\,\mathrm{NLO}} \\ C^{(1)H}_{2,\,\mathrm{NNLO}} &= U^{(2,1,1)}_{\mathcal{T}}(n_0,n_0)(Q,\,\mu_1)c^{H}_{2,\,\mathrm{NNLO}} \end{aligned} \tag{B.57}$$

where

$$\begin{aligned} c^{H,a}_{2,\,\mathrm{NLO}} &= U^{(2,1,0)}(n_0,n_0)(Q,\,\mu_1)\frac{n_0^\mu-\bar n^\mu}{Q}\gamma^\alpha_{n_0\perp}\Theta_{\lambda^2}[n_1\cdot n_1'] \\ &= U^{(2,1,0)}(n_0,n_0)(Q,\,\mu_1)\frac{1}{Q}\Big(\frac{\bar k_1 n_1'^\mu+\bar q_1 n_1^\mu}{\bar q_0}-\Big(1+\frac{\bar q_1\bar k_1}{2\,\bar q_0^2}(n_1\!\cdot\! n_1')\Big)\bar n^\mu\Big)\gamma^\alpha_{n_1'\perp}\Theta_{\lambda^2}[n_1\cdot n_1']\,, \\ c^{H}_{2,\,\mathrm{NNLO}} &= U^{(2,1,1)}_{\mathcal{T}}(n_0,n_0)(Q,\,\mu_1)\Big(\Big(\frac{1}{\bar q_1}+\frac{1}{\bar k_1}\Big)\frac{1}{Q}\gamma^\mu_{n_0\perp}(\not k_1)_{n_0\perp}\gamma^\alpha_{n_1'\perp}-\frac{2}{\bar q_1 Q}(k_1)^\mu_{n_0\perp}\gamma^\alpha_{n_1'\perp}\Big)\Theta_{\lambda^2}[n_1\cdot n_1'] \\ &= -U^{(2,1,1)}_{\mathcal{T}}(n_0,n_0)(Q,\,\mu_1)\Big(\frac{1}{2\,Q}\Big(\gamma^\mu_{n_1'\perp}\sqrt{|n_1\cdot n_1'|}\not v_1+\bar n^\mu\frac{\bar q_1}{Q}(n_1\cdot n_1')\Big)\gamma^\alpha_{n_1'\perp} \\ &\quad+\frac{\bar k_1}{Q^2}\Big(\sqrt{|n_1\cdot n_1'|}v_1^\mu-\bar n^\mu\,(n_1\!\cdot\! n_1')\frac{(\bar k_1^2-\bar q_1^2)}{2\,Q^2}\Big)\gamma^\alpha_{n_1'\perp}\Big)\Theta_{\lambda^2}[n_1\cdot n_1']\,. \end{aligned} \tag{B.58}$$

$C^{(1)H,a}_{2,\,\mathrm{NLO}}$ has power counting λ^0 and $C^{(1)H}_{2,\,\mathrm{NNLO}}\sim\lambda^1$.

For the coefficient $C^{(1)H,b}_{2,\,\mathrm{NLO}}$, the matching comes from the SCET_1 three-jet operators where $n_1\cdot n_1'$ is now of order λ^0. Because $C^{(1)}_1(n_1,n_1')$ defined in Eq. (B.39) is already labeled in terms of the directions n_1, n_1' and $\bar n$ exactly parallel to the external particles, we can simply write

$$C^{(1)H,b}_{2,\,\mathrm{NLO}}(n_1,n_1') = C^{(1)}_1(n_1,n_1')\tilde\Theta_{\lambda^2}[n_1\cdot n_1']\,, \tag{B.59}$$

where $\tilde\Theta_{\lambda^2}[n_1\cdot n_1']$ carries the information that $(n_1\cdot n_1')>\lambda$. Knowing that now $n_1\cdot n_1'\sim\lambda^0$, $C^{(1)H,b}_{2,\,\mathrm{NLO}}(n_1,n_1')$ scale as λ^0 and

$$C^{(1)H,b}_{2,\,\mathrm{NLO}}(n_1,n_1')\mathcal{O}^{(1)}_2(n_1,n_1')\sim\lambda^6\,. \tag{B.60}$$

The SCET_2 coefficients (B.53) and (B.57) comes from matching SCET_1 to SCET_2 for the two-jet SCET_1 phase space region where $q_1\cdot k_1\lesssim\lambda^2Q^2$ that implies $n_1\cdot n_1'\lesssim\lambda^2$, instead the coefficient $C^{(1)H,b}_{2,\,\mathrm{NLO}}$ comes for the three-jet SCET$_1$ region where $n_1 \cdot n_1' > \lambda^2$. Because the operator $\mathcal{O}_2^{(1)}(n_1, n_1')$ only knows that $n_1 \cdot n_1' > \lambda^4$, it is not able to distinguish the two regions, and this information has to be encoded in the Wilson coefficients, and we have done it inserting the function Θ and $\tilde{\Theta}$. We can think of $\tilde{\Theta}_{\lambda^2}[x]$ as usual theta function: $\Theta_{\lambda^2}[x] = \theta[\lambda^2 - x]$ and $\tilde{\Theta}_{\lambda^2}[x] = 1 - \Theta_{\lambda^2}[x]$, but if we have to calculate inclusive quantities we need to integrate the phase space and having hard theta functions in the integrand could be problematic. We can define a smoother theta function, an example is in Fig. 4-9

$$\Theta_{\Lambda,a}(x) = \begin{cases} 0 & \text{if } x < \Lambda - a \\ -\mathrm{Sign}(x-\Lambda)e^{\frac{2a\mathrm{Sign}(x-\Lambda)}{(x-\Lambda)-a\,\mathrm{Sign}(x-\Lambda)}} + (\mathrm{Sign}(x-\Lambda)+1)e^{-2} & \text{if } \Lambda - a < x < \Lambda + a \\ 1 & \text{if } x > \Lambda + a \end{cases}, \tag{B.61}$$

and $\tilde{\Theta}_{\Lambda,a}(x) = 1 - \Theta_{\Lambda,a}(x)$. The parameter Λ tells where the function switches from 0 to 1 and the parameter a how fast it does it. To see how this theta function works, we integrate the amplitude squared up to NLO. The LO amplitude squared is

$$|A^{q\bar{q}g}|^2_{\mathrm{LO}} = |\hat{C}^{(1)}_{2,\mathrm{LO}}(n_1, n_1')|^2 G(q_1, k_1, k_2, p_{\bar{q}}) \,, \tag{B.62}$$

where

$$G(q_1, k_1, k_2, p_{\bar{q}}) = {}_2\langle q_{n_1}\, g_{n_1'}\, \bar{q}_{\bar{n}}|\mathcal{O}_1^{(1)\dagger}(n_1, n_1')|0\rangle\langle 0|\mathcal{O}_1^{(1)}(n_1, n_1')|q_{n_1}\, g_{n_1'}\, \bar{q}_{\bar{n}}\rangle_2 \,. \tag{B.63}$$

The NLO and NNLO amplitude squared is

$$|A^{q\bar{q}g}|^2_{\mathrm{NLO}} + |A^{q\bar{q}g}|^2_{\mathrm{NNLO}} = |A^{q\bar{q}g}|^2_{\mathrm{to\,NNLO,\,2-jet}} + |A^{q\bar{q}g}|^2_{\mathrm{NNLO,\,3-jet}} \,, \tag{B.64}$$

where

$$\begin{aligned} |A^{q\bar{q}g}|^2_{\mathrm{to\,NNLO,\,2-jet}} = {} & (C^{(1)\dagger}_{2,\mathrm{LO}}(n_1, n_1')(C^{(1),H}_{2,\mathrm{NNLO}}(n_1, n_1') + C^{(1)H,a\dagger}_{2,\mathrm{NLO}}(n_1, n_1')) \\ & + (C^{(1)H,a\dagger}_{2,\mathrm{NLO}}(n_1, n_1') + C^{(1),H\dagger}_{2,\mathrm{NNLO}}(n_1, n_1'))C^{(1)}_{2,\mathrm{NLO}}(n_1, n_1') \end{aligned}$$

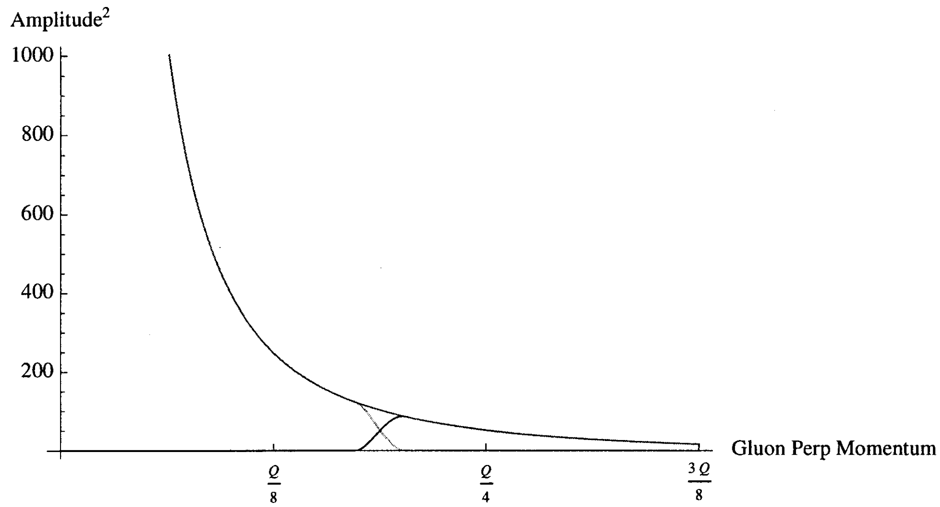

Figure B-3: Merging of the two-jet and and three-jet amplitudes squared for $q\bar{q}g$ process using a smooth theta function. Plots of the two-jet amplitude square, $|A^{q\bar{q}g}|^2_{\rm LO} + |A^{q\bar{q}g}|^2_{\rm NLO,\,2-jet}$, (green), three-jet, $|A^{q\bar{q}g}|^2_{\rm NLO,\,3-jet}$, (blue) and sum (red) versus $k_{1\perp_{n_0}}$. The amplitudes are evaluated two-jet and and three-jeted without running coefficients for $\bar{k}_1/\bar{q}_0 = 0.4$.

$$
\begin{aligned}
&+ |\hat{C}^{(1),Ha}_{2,\rm NLO}(n_1, n_1')|^2)G(q_1, k_1, k_2, p_{\bar{q}})\,, \\
|A^{q\bar{q}g}|^2_{\rm NNLO,\,3-jet} &= |C^{(1),Hb}_{2,\rm NLO}(n_1, n_1')|^2 G(q_1, k_1, k_2, p_{\bar{q}})\,.
\end{aligned}
\tag{B.65}
$$

$|A^{q\bar{q}g}|^2_{\rm NLO,\,2-jet}$ is the contribution to the amplitude squared from the two-jet region and $|A^{q\bar{q}g}|^2_{\rm NLO,\,3-jet}$ is the contribution from the three-jet region. In Fig. 4-8 we plot the ratios $|A^{q\bar{q}g}|^2_{\rm LO}/|A^{q\bar{q}g}|^2_{\rm QCD}$ and $(|A^{q\bar{q}g}|^2_{\rm LO} + |A^{q\bar{q}g}|^2_{\rm NLO,\,2-jet})/|A^{q\bar{q}g}|^2_{\rm QCD}$ versus $k_{1\perp_{n_0}}$, We notice that including NLO corrections extends the region where tree-level SCET and QCD agree. In Fig. B-3 we plot the the merging of the two-jet and and three-jet amplitude squared using the theta function. The theta function defined above smoothly merges the two amplitude squares. In Fig. B-4 we plot $|A^{q\bar{q}g}|^2_{\rm LO} + |A^{q\bar{q}g}|^2_{\rm NLO}$ with and without running factors.

B.3.2 Two-Gluon Emissions

We now match $\rm SCET_1$ to $\rm SCET_2$ for two-gluon emissions up to NNLO. In order to do it we first should calculate the matching QCD to $\rm SCET_1$ for two emissions up to NNLO. Besides the operators in (B.26), for two emissions in $\rm SCET_1$ there is the

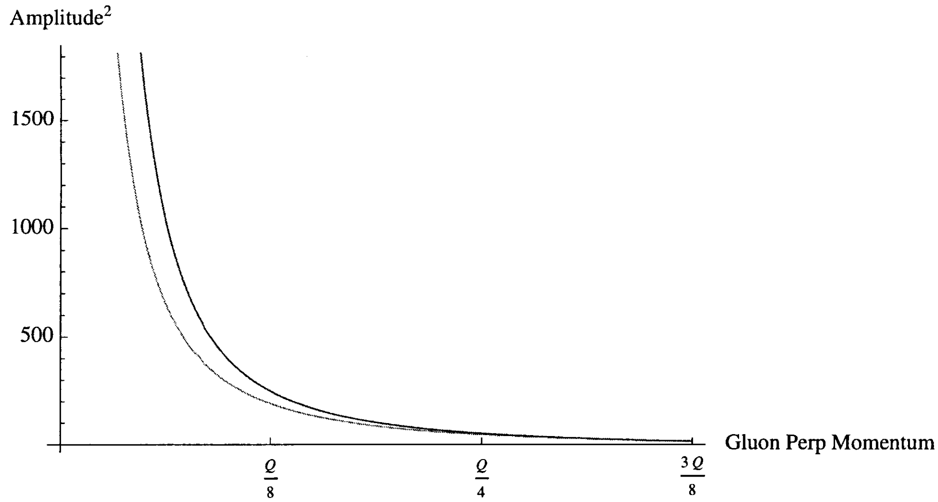

Figure B-4: Plot of the SCET$_2$ amplitudes square up to NLO, $|A^{q\bar{q}g}|^2_{\rm LO} + |A^{q\bar{q}g}|^2_{\rm NLO}$, with (green) and without (red) running factors versus $k_{1\perp_{n_0}}$ for $\bar{k}_1/\bar{q}_0 = 0.4$.

additional NNLO operator

$$\mathcal{O}_1^{(2)}(n_0, n_0, n_0) = \bar{\chi}_{n_0} g\mathcal{B}^{\alpha}_{n_0\perp} g\mathcal{B}^{\beta}_{n_0\perp}\chi_{\bar{n}}\,, \tag{B.66}$$

but this operators in not relevant for the matching SCET$_1$ to SCET$_2$ at NNLO. (B.66) matches to SCET$_2$ operators with two gluon field, like for example $\mathcal{O}_2^{(2)}(n_1, n_1', n_1') = \bar{\chi}_{n_1}\mathcal{B}^{\alpha}_{n_1\perp}\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}$, with Wilson coefficient $C_2^{(2)}(n_1, n_1, n_1')$ of order λ^0, that is (B.66) matches to a Wilson coefficient×operators in SCET$_2$ of order λ^8 that is N^{3}LO according to the power counting in (B.47). Thus the SCET$_1$ operators (B.26) are enough for the matching to SCET$_2$ up to two emissions.

Besides the operators necessary for the matching for one emission, the SCET$_2$ basis for two emissions has the additional two gluons operators

$$\begin{aligned}
\mathcal{O}_2^{(2)}(n_2, n_2, n_1') &= \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_2\perp} g\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}\,, \\
\mathcal{O}_2^{(2)}(n_2, n_1', n_1') &= \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_1'\perp} g\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}\,, \\
\mathcal{O}_2^{(2)}(n_2, n_1', n_2') &= \bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_1'\perp} g\mathcal{B}^{\beta}_{n_2'\perp}\chi_{\bar{n}}\,. \\
\mathcal{O}_2^{(2)}(n_0, n_0, n_0) &= \bar{\chi}_{n_0} g\mathcal{B}^{\alpha}_{n_0\perp} g\mathcal{B}^{\beta}_{n_0\perp}\chi_{\bar{n}}\,.
\end{aligned} \tag{B.67}$$

In the first operator in (B.67) the quark and the gluon are collinear and it does

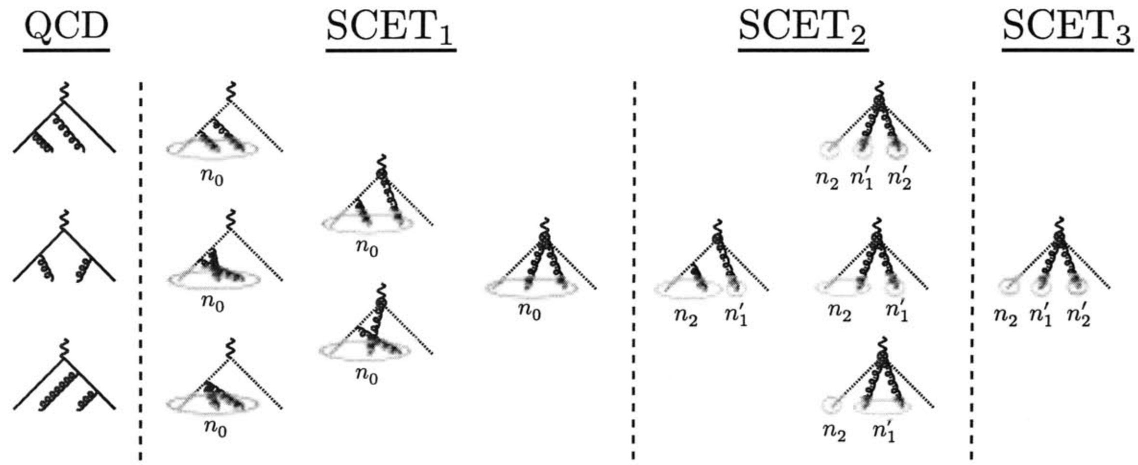

Figure B-5: Matching SCET$_1$ to SCET$_2$ to SCET$_3$ for two emissions for the two-jet configuration. In the first column there are the QCD Feyman diagrams; in the second column the SCET$_1$ diagrams from the operator $\mathcal{O}_1^{(0)}(n_0)$; in the third column the SCET$_1$ diagrams from the operator $\mathcal{O}_1^{(1)}(n_0, n_0)$ and $\mathcal{T}_1^{(1)}(n_0, n_0)$; in the fourth column the SCET$_1$ diagram from the operator $\mathcal{O}_1^{(2)}(n_0, n_0, n_0)$, this operator contributes only at N^{3}LO to the SCET$_2$ matching; in the fifth column the SCET$_2$ diagram from the operator $\mathcal{O}_1^{(1)}(n_2, n_1')$; In the sixth column the SCET$_2$ diagrams from the operators $\mathcal{O}_2^{(2)}(n_2, n_1', n_2')$, $\mathcal{O}_2^{(2)}(n_2, n_2, n_1')$ and $\mathcal{O}_2^{(2)}(n_2, n_1', n_1')$; in the seventh column the SCET$_3$ diagram from the operator $\mathcal{O}_3^{(2)}(n_1, n_1', n_2')$.

annihilate the state $|q_{n_2} g_{n_2} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2$, in the second the two gluons are collinear and it does annihilate the state $|q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2$, and in the third all the particles are far apart and it does annihilate the state $|q_{n_2} g_{n_1'} g_{n_2'} \bar{q}_{\bar{n}}\rangle_2$. The last SCET$_2$ operator in (B.67) is not necessary for the matching at NNLO. It can only be closed with the state $|q_{n_0} g_{n_0} g_{n_0} \bar{q}_{\bar{n}}\rangle_2$ that describes two emissions both collinear in SCET$_2$, his coefficient can only come form the SCET$_1$ operator $\mathcal{O}_1^{(2)}(n_0, n_0, n_0)$ and so it only contribites at N^{3}LO. The Wilson coefficients of the operators (B.67) are defined such that

$$\begin{aligned} J^\mu_{\rm QCD} &= C^{(0)}_{1,\rm LO}(n_0)\mathcal{O}^{(0)}_1(n_0) + C^{(1)}_1(n_0,n_0)\mathcal{O}^{(1)}_1(n_0,n_0) + C^{(1)}_{1,\mathcal{T}}\mathcal{T}^{(1)}_1(n_0,n_0) \qquad \text{(B.68)} \\ &\quad + C^{(1)}_1(n_1,n_1')\mathcal{O}^{(1)}_1(n_1,n_1') + \cdots \\ &= C^{(0)}_2(n_0)\mathcal{O}^{(0)}_2(n_0) + C^{(1)}_1(n_0,n_0)\mathcal{O}^{(2)}_2(n_0,n_0) + C^{(2)}_{2,\mathcal{T}}\mathcal{T}^{(2)}_2(n_0,n_0) \qquad \text{(B.69)} \\ &\quad + C^{(1)}_2(n_1,n_1')\mathcal{O}^{(1)}_2(n_1,n_1') + C^{(2)}_2(n_2,n_2,n_1')\mathcal{O}^{(2)}_2(n_2,n_2,n_1') \\ &\quad + C^{(2)}_2(n_2,n_1',n_1')\mathcal{O}^{(2)}_2(n_2,n_1',n_1') + C^{(2)}_2(n_2,n_1',n_2')\mathcal{O}^{(2)}_2(n_2,n_1',n_2') + \cdots \end{aligned}$$

where in the first equation we have written the original QCD current in terms of SCET_1 operators and in the second equation in terms of SCET_2 operators. The ellipses indicate higher order operators.

We divide the Wilson coefficients in two category: jet and hard. We labeled the jet coefficients with the upper script J, and the hard with H. As already mention in the previous section, the jet coefficients are those whose structures do not depend on the QCD current. That is, if we had starting the matching from a general QCD current, $\bar{q}\,\Gamma^\mu q$ instead of $J^\mu_{\mathrm{QCD}} = \bar{q}\,\gamma^\mu q$, the jet coefficient would be the same upon the substitution in Eq. (B.54). The corrections that come from these coefficients are universal in the sense that they are independent of the information about the hard scattering that happens at the beginning of the shower. Instead the hard coefficients are those who depend on the QCD current, that it they give corrections that depend on the hard scattering. All the jet coefficients come from the matching with the SCET_1 operators $\mathcal{O}_1^{(0)}(n_0)$ because it can emits two gluons only through the Largangian insertion and the Lagrangian does not dependent on the initial QCD current.

We have seen in the previous section that the LO coefficient$\times$operator that we need for the parton shower is $C_2^{(1)}(n_1)\mathcal{O}_2^{(1)}(n_1, n_1')$, that is of order λ^5, Eq. (B.47). We are interested to calculate the amplitude squared at NNLO, that is we only need to calculate the NNLO amplitudes that interference with the LO amplitude. The LO amplitude for two emissions is

$$A^{qgg\bar{q}}_{\mathrm{LO}} = C^{(1)}_{2,\mathrm{LO}}(n_1, n_1') \int dx \langle 0| T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}_2^{(1)}(n_1, n_1')\}|q_{n_2} g_{n_2} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2 \,. \tag{B.70}$$

Because $A^{qgg\bar{q}}_{\mathrm{LO}}$ comes from closing the operator with the state $|q_{n_2} g_{n_2} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2$, it can only interferences with the amplitude that comes from $\mathcal{O}_2^{(2)}(n_2, n_2, n_1')$. Thus we will only calculate the component of the coefficients $C_2^{(2)}(n_2, n_1', n_2')$, and $C_2^{(2)}(n_2, n_1', n_1')$ that give NLO contribution, that is such that $C_2^{(2)}\mathcal{O}_2^{(2)}(n_2, n_1', n_2')$ and $C_2^{(2)}\mathcal{O}_2^{(2)}(n_2, n_1', n_2')$ is of order λ^6. Knowing that $\mathcal{O}_2^{(2)} \sim \lambda^8$, we have to calculate $C_2^{(2)}(n_2, n_1', n_2')$, and $C_2^{(2)}(n_2, n_1', n_1')$ only up to λ^{-2}. For $\mathcal{O}_2^{(2)}(n_2, n_2, n_1')$ instead we have also to calculate his coefficient up to λ^{-1}. We will use the lower script NLO/NNLO/N^{3}LO if the coefficient scales as $\lambda^{-2}/\lambda^{-1}/\lambda^{0}$.

We now turn to calculate the coefficient of (B.67), it can be defined by closing Eq. (B.69) with the state $|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2$. We decompose $C_2^{(2)}(n_2, n'_1, n'_2)$ as

$$C_2^{(2)}(n_2, n'_1, n'_2) = C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n'_1, n'_2) + C_2^{(2)H}(n_2, n'_1, n'_2)\,, \tag{B.71}$$

where

$$\begin{aligned} &C_{2,\,\mathrm{NLO}}^{(2)J}\langle 0|\mathcal{O}_2^{(2)}(n_2, n'_1, n'_2)|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2 = \\ &C_{1,\,\mathrm{LO}}^{(0)}(n_0) \int dx_1 dx_2 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x_1)\mathcal{L}_{\mathrm{SCET}_1}(x_2)\mathcal{O}_1^{(0)}(n_0, n_0)\}|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2 \end{aligned} \tag{B.72}$$

and

$$\begin{aligned} &C_2^{(2)H}(n_2, n'_1, n'_2)\langle 0|\mathcal{O}_2^{(2)}(n_2, n'_1, n'_2)|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2 = \\ &+ C_1^{(1)}(n_0, n_0) \int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(1)}(n_0, n_0)\}|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2 \\ &+ C_{1,\mathcal{T}}^{(1)}(n_0, n_0) \int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{T}_1^{(1)}(n_0, n_0)\}|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2 \\ &+ C_1^{(1)}(n_1, n'_1) \int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(1)}(n_1, n'_1)\}|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2\,. \end{aligned} \tag{B.73}$$

We decompose $C_2^{(2)H}(n_2, n'_1, n'_2)$ as

$$C_2^{(2)H}(n_2, n'_1, n'_2) = C_{2,\,\mathrm{NNLO}}^{(2)H,\,a}(n_2, n'_1, n'_2) + C_{2,\,\mathrm{N^3LO}}^{(2)H}(n_2, n'_1, n'_2) + C_{2,\,\mathrm{NNLO}}^{(2)H,\,b}(n_2, n'_1, n'_2)\,, \tag{B.74}$$

where $C_{2,\,\mathrm{NNLO}}^{(2)H,\,a}(n_2, n'_1, n'_2)$ is the coefficient that multiplied by $\langle 0|\mathcal{O}_2^{(2)}(n_2, n'_1, n'_2)|q_{n_2} g_{n'_1} g_{n'_2} \bar{q}_{\bar{n}}\rangle_2$ reproduces the the second line in the Eq. (B.73), $C_{2,\,\mathrm{N^3LO}}^{(2)H}(n_2, n'_1, n'_2)$ the third line and so on. In Eqs. (B.71-B.74) we put the labels in the coefficients to indicate their power counting. Because the operator $\mathcal{O}_2^{(2)}(n_2, n'_1, n'_2)$ does not interference with the LO operator we only need the coefficient $C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n'_1, n'_2)$ to calculate the amplitude squared at NLO. We calculate $C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n'_1, n'_2)$ and also $C_{2,\,\mathrm{NNLO}}^{(2)H,\,a}(n_2, n'_1, n'_2)$ because it will be useful later, and we prove that are of order λ^{-2} and λ^{-1} respectively. We

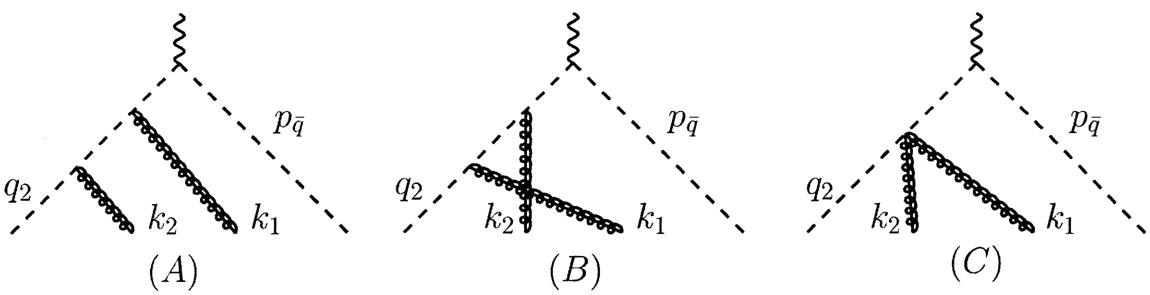

Figure B-6: Feynman diagrams for two emissions in SCET_1 from the operator $\mathcal{O}_1^{(0)}$.

will not calculate here the other coefficients, however it is easy to do it just repeating the method we will show below and one can prove that they have the power counting labelled in Eq. (B.74). $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2, n_1', n_2')$ and $C^{(2)H,\,a}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')$ come from two-jet operators in SCET_1, thus they are defined in a phase space region where $n_2 \cdot n_1'$, $n_2 \cdot n_2'$, $n_1' \cdot n_2' \lesssim \lambda^2$. As we have done for the coefficients calculated in the one gluon matching, we have to encode this information in the Wilson coefficients because the SCET_2 operator $\mathcal{O}_2^{(2)}(n_2, n_1', n_2')$ only knows that $n_2 \cdot n_1'$, $n_2 \cdot n_2'$, $n_1' \cdot n_2' > \lambda^4$. We will do it inserting Θ functions in the coefficients like for the one gluon matching.

To calculate the coefficients we proceed like we have done for one-gluon emission: on the LHS of Eqs. (B.72) and (B.73) we calculate the SCET_1 amplitude, we rewrite it along the direction n_2, n_1', n_2' where the quark and the two gluons are aligned using the finite RPI defined in Appendix (B.1); on the RHS we write the SCET_2 amplitude and calculate the Wilson coefficient necessary for the matching. We decompose the SCET_1 amplitude

$$A^{q\bar{q}gg}_{\mathrm{NLO}} = C_1^{(0)} \int dx_1 dx_2 \langle 0 | T\{\mathcal{L}_{\mathrm{SCET}_1}(x_1)\mathcal{L}_{\mathrm{SCET}_1}(x_2)\mathcal{O}_1^{(0)}\} | q_{n_2} g_{n_1'} g_{n_2'} \bar{q}_{\bar{n}} \rangle_2 \,, \qquad \text{(B.75)}$$

in

$$A^{q\bar{q}gg}_{\mathrm{NLO}} = A^{q\bar{q}gg}_{\mathrm{NLO},\,A} + A^{q\bar{q}gg}_{\mathrm{NLO},\,B} + A^{q\bar{q}gg}_{\mathrm{NLO},\,C} \,, \qquad \text{(B.76)}$$

where A, B, C correspond to the three graphs in Fig. B-6. Using the SCET Feynman rules we have

$$
\begin{aligned}
A^{q\bar q gg}_{\mathrm{NLO},A} &= U^{(0)}(n_0)(Q,(k_1)_{n_0\perp})\, g^2\, \bar u_{n_0}\Big[n_0^\beta + \gamma^\beta_{n_0\perp}\frac{(\not q_1)_{n_0\perp}}{\bar q_1} + \frac{(\not q_2)_{n_0\perp}}{\bar q_2}\gamma^\beta_{n_0\perp}\Big] \\
&\times \Big[n_0^\alpha + \frac{(\not q_1)_{n_0\perp}}{\bar q_1}\gamma^\alpha_{n_0\perp}\Big]\frac{\bar q_1}{q_1^2}\frac{\bar q_0}{q_0^2}\Gamma^\mu v_{\bar n}\,, \\
A^{q\bar q gg}_{\mathrm{NLO},B} &= U^{(0)}(n_0)(Q,(k_1)_{n_0\perp})\, g^2\, \bar u_{n_0}\Big[n_0^\alpha + \gamma^\alpha_{n_0\perp}\frac{(\not q_2+\not k_1)_{n_0\perp}}{\bar q_2+\bar k_1} + \frac{(\not q_2)_{n_0\perp}}{\bar q_2}\gamma^\alpha_{n_0\perp}\Big]\frac{\bar q_2+\bar k_1}{(q_2+k_1)^2} \\
&\times \Big[n_0^\beta + \frac{(\not q_2+\not k_1)_{n_0\perp}}{\bar q_2+\bar k_1}\gamma^\beta_{n_0\perp}\Big]\frac{\bar q_0}{q_0^2}\Gamma^\mu v_{\bar n}\,, \\
A^{q\bar q gg}_{\mathrm{NLO},C} &= U^{(0)}(n_0)(Q,(k_1)_{n_0\perp})\, g^2\, \bar u_{n_0}\Big[\frac{1}{\bar q_0-\bar k_2}\gamma^\alpha_{n_0\perp}\gamma^\beta_{n_0\perp} + \frac{1}{\bar k_2+\bar q_2}\gamma^\beta_{n_0\perp}\gamma^\alpha_{n_0\perp}\Big]\frac{\bar q_0}{q_0^2}\Gamma^\mu v_{\bar n}\,,
\end{aligned}
\tag{B.77}
$$

where $q_1 = q_2 + k_2$ and $q_0 = q_1 + q_2 + k_2$. As before we do not write terms with $\bar n^\alpha$ and $\bar n^\beta$, they are not necessary for the matching because the operator $\bar n\cdot A_n$ is constrained by gauge invariant to be only in the Wilson line. Now we rotate the amplitude (B.77) to the directions n_2 and n_1' and n_2' parallel to the quark and the two gluons, as described in Eq. (B.16)

$$
\begin{aligned}
A^{q\bar q gg}_{\mathrm{NLO},A} &= U^{(0)}(n_0)(Q,\mu_1)\, g^2\, \bar u_{n_2}\Big[\frac{\bar q_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\, v_2^\beta + \frac{\bar k_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\,\frac{\not v_2}{2}\gamma^\beta_{n_2'\perp}\Big] \\
&\times \Big[\frac{\bar q_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\, v_1^\alpha + \frac{\bar k_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,\frac{\not v_1}{2}\gamma^\alpha_{n_1'\perp}\Big]\frac{\bar q_1}{q_1^2}\frac{\bar q_0}{q_0^2}\,\gamma^\mu_{n_0\perp}v_{\bar n}\,, \\
A^{q\bar q gg}_{\mathrm{NLO},B} &= U^{(0)}(n_0)(Q,\mu_1)\, g^2\, \bar u_{n_2}\Big[\frac{\bar q_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\, v_1^\alpha + \frac{\bar k_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\,\frac{\not v_2}{2}\gamma^\alpha_{n_1'\perp} \\
&+ \gamma^\alpha_{n_1'\perp}\frac{\bar k_2\,\bar q_2}{(\bar q_2+\bar k_1)(\bar q_2+\bar k_2)}\sqrt{|n_2\cdot n_2'|}\,\frac{\not v_2}{2} - \gamma^\alpha_{n_1'\perp}\frac{\bar k_1\,\bar q_1}{(\bar q_2+\bar k_1)(\bar q_1+\bar k_1)}\sqrt{|n_1\cdot n_1'|}\,\frac{\not v_1}{2}\Big] \\
&\times \Big[\frac{\bar q_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\, v_2^\beta - \frac{\bar k_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\, v_1^\beta + \frac{\bar k_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,\frac{\not v_1}{2}\gamma^\beta_{n_2'\perp}\Big] \\
&\times \frac{\bar q_2+\bar k_1}{(q_2+k_1)^2}\frac{\bar q_0}{q_0^2}\gamma^\mu_{n_0\perp}v_{\bar n}\,, \qquad\qquad \text{(B.78)} \\
A^{q\bar q gg}_{\mathrm{NLO},C} &= U^{(0)}(n_0)(Q,\mu_1)\, g^2\, \bar u_{n_2}\Big[\frac{1}{\bar q_0-\bar k_2}\gamma^\alpha_{n_1'\perp}\gamma^\beta_{n_2'\perp} + \frac{1}{\bar k_2+\bar q_2}\gamma^\beta_{n_2'\perp}\gamma^\alpha_{n_1'\perp}\Big]\frac{\bar q_0}{q_0^2}\,\gamma^\mu_{n_0\perp}v_{\bar n}\,.
\end{aligned}
$$

$n_1\cdot n_1'$ is defined in terms of n_2, n_1' and n_2' in Eqs. (B.20),

v_1 and v_2 are defined in Eqs.(B.14) and (B.19), and the values of q_0^2, q_1^2 and

$(q_1+k_2)^2$ are given in Eqs.(B.22) and (B.23). As for the one-gluon emission, we can neglect the term with $n_1'^{\alpha}$ and $n_2'^{\beta}$ because not physical. The SCET$_2$ amplitude for $\langle 0|\mathcal{O}_2^{(2)}(n_2,n_1',n_2')|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_{\bar{n}}\rangle$ is

$$\langle 0|\bar{\chi}_{n_2} g\mathcal{B}^{\alpha}_{n_2'\perp} g\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}|q_{n_2}g_{n_1'}g_{n_2'}\bar{q}_{\bar{n}}\rangle = g^2\bar{u}_{n_2}\epsilon^{\alpha}_{n_1'\perp}\epsilon^{\beta}_{n_2'\perp}v_{\bar{n}}\,. \tag{B.79}$$

In Eq. (B.79) we have expicitly written the polarizations vectors of the external gluons in the state. Thus the Wilson coefficients $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2,n_1',n_2')$ is

$$C^{(2)J}_{2,\,\mathrm{NLO}}(n_2,n_1',n_2') = U^{(0)}(n_0)(Q,\mu_1)\, d_1^{J\,\alpha\beta}(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2\cdot n_1']\Theta_{\lambda^2}[n_2\cdot n_2']\Theta_{\lambda^2}[n_2'\cdot n_1']\,, \tag{B.80}$$

where

$$d_1^{J\,\alpha\beta}(n_2,n_1',n_2') = d_{1,A}^{J\,\alpha\beta}(n_2,n_1',n_2') + d_{1,B}^{J\,\alpha\beta}(n_2,n_1',n_2') + d_{1,C}^{J\,\alpha\beta}(n_2,n_1',n_2')\,, \tag{B.81}$$

with

$$\begin{aligned} d_{1,A}^{J\,\alpha\beta}(n_2,n_1',n_2') &= \Big[\frac{\bar{q}_2}{\bar{q}_2+\bar{k}_2}\sqrt{|n_2\cdot n_2'|}\,v_2^{\beta} + \frac{\bar{k}_2}{\bar{q}_2+\bar{k}_2}\sqrt{|n_2\cdot n_2'|}\,\frac{\not{v}_2}{2}\gamma^{\beta}_{n_2'\perp}\Big] \\ &\times\Big[\frac{\bar{q}_1}{\bar{q}_1+\bar{k}_1}\sqrt{|n_1\cdot n_1'|}\,v_1^{\alpha} + \frac{\bar{k}_1}{\bar{q}_1+\bar{k}_1}\sqrt{|n_1\cdot n_1'|}\,\frac{\not{v}_1}{2}\gamma^{\alpha}_{n_1'\perp}\Big]\frac{\bar{q}_1}{q_1^2}\frac{\bar{q}_0}{q_0^2}\,\gamma^{\mu}_{n_0\perp}\,, \end{aligned} \tag{B.82}$$

$$\begin{aligned} d_{1,B}^{J\,\alpha\beta}(n_2,n_1',n_2') &= \Big[\frac{\bar{q}_1}{\bar{q}_1+\bar{k}_1}\sqrt{|n_1\cdot n_1'|}\,v_1^{\alpha} + \frac{\bar{k}_2}{\bar{q}_2+\bar{k}_2}\sqrt{|n_2\cdot n_2'|}\,\frac{\not{v}_2}{2}\gamma^{\alpha}_{n_1'\perp} \\ &+\gamma^{\alpha}_{n_1'\perp}\frac{\bar{k}_2\,\bar{q}_2}{(\bar{q}_2+\bar{k}_1)(\bar{q}_2+\bar{k}_2)}\sqrt{|n_2\cdot n_2'|}\,\frac{\not{v}_2}{2} - \gamma^{\alpha}_{n_1'\perp}\frac{\bar{k}_1\,\bar{q}_1}{(\bar{q}_2+\bar{k}_1)(\bar{q}_1+\bar{k}_1)}\sqrt{|n_1\cdot n_1'|}\,\frac{\not{v}_1}{2}\Big] \\ &\times\Big[\frac{\bar{q}_2}{\bar{q}_2+\bar{k}_2}\sqrt{|n_2\cdot n_2'|}\,v_2^{\beta} - \frac{\bar{k}_1}{\bar{q}_1+\bar{k}_1}\sqrt{|n_1\cdot n_1'|}\,v_1^{\beta} + \frac{\bar{k}_1}{\bar{q}_1+\bar{k}_1}\sqrt{|n_1\cdot n_1'|}\,\frac{\not{v}_1}{2}\gamma^{\beta}_{n_2'\perp}\Big] \\ &\times\frac{\bar{q}_2+\bar{k}_1}{(q_2+k_1)^2}\frac{\bar{q}_0}{q_0^2}\gamma^{\mu}_{n_0\perp}, \\ d_{1,C}^{J\,\alpha\beta}(n_2,n_1',n_2') &= \Big[\frac{1}{\bar{q}_0-\bar{k}_2}\gamma^{\alpha}_{n_1'\perp}\gamma^{\beta}_{n_2'\perp} + \frac{1}{\bar{k}_2+\bar{q}_2}\gamma^{\beta}_{n_2'\perp}\gamma^{\alpha}_{n_1'\perp}\Big]\frac{\bar{q}_0}{q_0^2}\,\gamma^{\mu}_{n_0\perp}\,. \end{aligned}$$

The Θ functions in Eq. (B.80) are necessary to encode the information that $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2,n_1',n_2')$

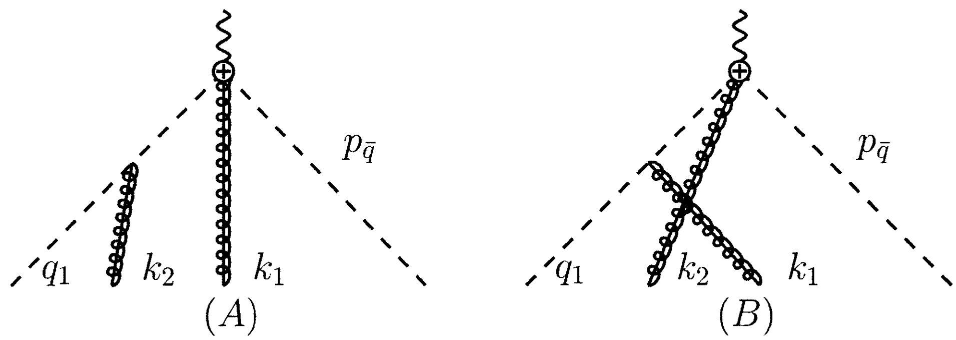

Figure B-7: Feynman diagrams for two emissions in SCET_1 from the operator $\mathcal{O}_1^{(1)}$.

comes from the two-jet SCET_1 operators. To calculate the power counting of $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2, n_1', n_2')$, we have to consider that this coefficient comes from matching SCET_1 to SCET_2 in the region where $n_2 \cdot n_1' \sim n_2 \cdot n_2' \sim n_1' \cdot n_2' \sim \lambda^2$, thus we have

$$C^{(2)J}_{2,\,\mathrm{NLO}}(n_2, n_1', n_2') \sim \lambda^{-2}\,. \tag{B.83}$$

We proceed similarly to calculate the coefficient $C^{(2),H,a}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')$ and prove that it has power counting λ^{-1}. We decompose the SCET_1 amplitude

$$A^{q\bar q gg}_{\mathrm{NNLO}} = C_1^{(1)}(n_0, n_0) \int dx \langle 0 | T\{\mathcal{L}_{\mathrm{SCET}_1}(x) \mathcal{O}_1^{(1)}(n_0, n_0)\} | q_{n_2} g_{n_1'} g_{n_2'} \bar q_{\bar n} \rangle_2\,,$$

in

$$A^{q\bar q gg}_{\mathrm{NNLO}} = A^{q\bar q gg}_{\mathrm{NLO},\,A} + A^{q\bar q gg}_{\mathrm{NLO},\,B}\,, \tag{B.84}$$

where A, B correspond to the two graphs in Fig. B-7. We have

$$\begin{aligned} A^{q\bar q gg}_{\mathrm{NNLO},\,A} &= U^{(2,1,0)}(n_0, n_0)(Q, \mu_1)\, g^2\, \bar u_{n_2} \Big[\frac{\bar q_2}{\bar q_2 + \bar k_2} \sqrt{|n_2 \cdot n_2'|}\, v_2^\beta \\ &\quad + \frac{\bar k_2}{\bar q_2 + \bar k_2} \sqrt{|n_2 \cdot n_2'|}\, \frac{\not{v}_2}{2} \gamma^\beta_{n_2'\perp} \Big] \gamma^\alpha_{n_1'\perp} \times \frac{\bar q_1\, \bar q_0}{q_1^2\, q_0^2} \frac{n^\mu - \bar n^\mu}{Q} \gamma^\mu_{n_0\perp} v_{\bar n}\,, \\ A^{q\bar q gg}_{\mathrm{NNLO},\,B} &= U^{(2,1,0)}(n_0, n_0)(Q, \mu_1)\, g^2\, \bar u_{n_2} \Big[\frac{\bar q_1}{\bar q_1 + \bar k_1} \sqrt{|n_1 \cdot n_1'|}\, v_1^\alpha \\ &\quad + \frac{\bar k_2}{\bar q_2 + \bar k_2} \sqrt{|n_2 \cdot n_2'|}\, \frac{\not{v}_2}{2} \gamma^\alpha_{n_1'\perp} + \gamma^\alpha_{n_1'\perp} \frac{\bar k_2\, \bar q_2}{(\bar q_2 + \bar k_1)(\bar q_2 + \bar k_2)} \sqrt{|n_2 \cdot n_2'|}\, \frac{\not{v}_2}{2} \end{aligned} \tag{B.85}$$

$$- \gamma^{\alpha}_{n_1'\perp} \frac{\bar{k}_1 \, \bar{q}_1}{(\bar{q}_2 + \bar{k}_1)(\bar{q}_1 + \bar{k}_1)} \sqrt{|n_1 \cdot n_1'|} \, \frac{\not{v}_1}{2} \Big] \gamma^{\beta}_{n_1'\perp} \frac{\bar{q}_1}{q_1^2} \frac{\bar{q}_0}{q_0^2} \frac{n^\mu - \bar{n}^\mu}{Q} \gamma^{\mu}_{n_0\perp} v_{\bar{n}} \,,$$

where in Eq. (B.85) we have already rotate the amplitude to the directions n_2, n_1' and n_2'. From Eqs. (B.79) and (B.85) we can see that the Wilson coefficient $C^{(2)H,\,a}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')$ is

$$\begin{aligned} C^{(2)H,\,a}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2') =& U^{(2,1,0)}(n_0, n_0)(Q, \mu_1) d_1^{H\,\alpha\beta}(n_2, n_1', n_2') \\ & \times \Theta_{\lambda^2}[n_2 \cdot n_1'] \Theta_{\lambda^2}[n_2 \cdot n_2'] \Theta_{\lambda^2}[n_2' \cdot n_1'] \,, \end{aligned} \tag{B.86}$$

where

$$d_1^{H\,\alpha\beta}(n_2, n_1', n_2') = d_{1,A}^{H\,\alpha\beta}(n_2, n_1', n_2') + d_{1,B}^{H\,\alpha\beta}(n_2, n_1', n_2') \tag{B.87}$$

with

$$\begin{aligned} d_{1,A}^{H\,\alpha\beta}(n_2, n_1', n_2') =& \Big[\frac{\bar{q}_2}{\bar{q}_2 + \bar{k}_2} \sqrt{|n_2 \cdot n_2'|} \, v_2^\beta + \frac{\bar{k}_2}{\bar{q}_2 + \bar{k}_2} \sqrt{|n_2 \cdot n_2'|} \, \frac{\not{v}_2}{2} \gamma^{\beta}_{n_2'\perp} \Big] \gamma^{\alpha}_{n_1'\perp} \frac{\bar{q}_1}{q_1^2} \frac{\bar{q}_0}{q_0^2} \frac{n^\mu - \bar{n}^\mu}{Q} \gamma^{\mu}_{n_0\perp} \,, \\ d_{1,B}^{H\,\alpha\beta}(n_2, n_1', n_2') =& \Big[\frac{\bar{q}_1}{\bar{q}_1 + \bar{k}_1} \sqrt{|n_1 \cdot n_1'|} \, v_1^\alpha + \frac{\bar{k}_2}{\bar{q}_2 + \bar{k}_2} \sqrt{|n_2 \cdot n_2'|} \, \frac{\not{v}_2}{2} \gamma^{\alpha}_{n_1'\perp} \\ & + \gamma^{\alpha}_{n_1'\perp} \frac{\bar{k}_2 \, \bar{q}_2}{(\bar{q}_2 + \bar{k}_1)(\bar{q}_2 + \bar{k}_2)} \sqrt{|n_2 \cdot n_2'|} \, \frac{\not{v}_2}{2} \\ & - \gamma^{\alpha}_{n_1'\perp} \frac{\bar{k}_1 \, \bar{q}_1}{(\bar{q}_2 + \bar{k}_1)(\bar{q}_1 + \bar{k}_1)} \sqrt{|n_1 \cdot n_1'|} \, \frac{\not{v}_1}{2} \Big] \gamma^{\beta}_{n_2'\perp} \frac{\bar{q}_2 + \bar{k}_1}{(q_2 + k_1)^2} \frac{\bar{q}_0}{q_0^2} \frac{n^\mu - \bar{n}^\mu}{Q} \gamma^{\mu}_{n_0\perp} \,. \end{aligned} \tag{B.88}$$

For the power counting of $C^{(2)H,\,a}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')$, as for the previous case, we have to consider that the matching is done in a region where $n_2 \cdot n_1' \sim n_2 \cdot n_2' \sim n_1' \cdot n_2' \sim \lambda^2$, this implies

$$C^{(2)H,\,a}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2') \sim \lambda^{-1} \,. \tag{B.89}$$

We now turn to calculate the coefficient $C_2^{(2)}(n_1, n_1', n_1')$. We will follow the same path done above. We decompose $C_2^{(2)}(n_1, n_1', n_1')$ as

$$C_2^{(2)}(n_2, n_1', n_1') = C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n_1', n_1') + C_2^{(2)H}(n_2, n_1', n_1')\,, \tag{B.90}$$

where

$$\begin{aligned} &C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n_1', n_1')\langle 0|\mathcal{O}_2^{(2)}(n_2, n_1', n_1')|q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2 \\ &= C_{1,\,\mathrm{LO}}^{(0)} \int dx_1 dx_2 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x_1)\mathcal{L}_{\mathrm{SCET}_1}(x_2)\mathcal{O}_1^{(0)}(n_0, n_0)\}|q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2\,, \end{aligned} \tag{B.91}$$

and

$$\begin{aligned} &C_2^{(2)H}(n_2, n_1', n_1')\langle 0|\mathcal{O}_2^{(2)}(n_2, n_1', n_1')|q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2 \\ &= C_1^{(1)} \int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(1)}(n_0, n_0)\}|q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2 \\ &+ C_{1,\mathcal{T}}^{(1)} \int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{T}_1^{(1)}(n_0, n_0)\}|q_{n_2} g_{n_1'} g_{n_1'} \bar{q}_{\bar{n}}\rangle_2\,. \end{aligned} \tag{B.92}$$

We decompose $C_2^{(2)}(n_2, n_1', n_1')$ as

$$C_2^{(2)H}(n_2, n_1', n_1') = C_{2,\,\mathrm{NNLO}}^{(2)H}(n_2, n_1', n_1') + C_{2,\,\mathrm{N^3LO}}^{(2)H}(n_2, n_1', n_1')\,, \tag{B.93}$$

where $C_{2,\,\mathrm{NLO}}^{(2)H}(n_2, n_1', n_1')$ is the coefficient that multiplied by $\langle 0|\mathcal{O}_2^{(2)}(n_2, n_1', n_2')|q_{n_2} g_{n_1'} g_{n_2'} \bar{q}_{\bar{n}}\rangle_2$ reproduces the the second line in the Eq. (B.92) and $C_{2,\,\mathrm{NLO}}^{(2)H}(n_2, n_1', n_1')$ the third line. We will only calculate $C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n_1', n_1')$ and prove that scales as λ^{-2}. This is the only operator that we need to calculate the amplitude squared at NLO. We will not calculate $C_{2,\,\mathrm{NNLO}}^{(2)H}(n_2, n_1', n_1')$ and $C_{2,\,\mathrm{NNLO}}^{(2)H}(n_2, n_1', n_1')$ here, but it is not difficult to do it and to prove that they scale as λ^{-1} and λ^0.

To calculate the amplitude in the second line in Eq. (B.91), we can use Eqs. (B.78) that is written in terms of the directions n_2, n_1' and n_2' that are the directions parallel to the external particles and take the limit $n_2' \cdot n_1' \to \lambda^4$. This is because in this case the two gluons are collinear in SCET$_2$, that is we have $k_1 \cdot k_2 \sim \lambda^4 Q$ or equivalently $n_1' \cdot n_2' \sim \lambda^4$ with $n_1 \cdot n_1'$ and $n_1 \cdot n_2'$ still of order λ^2. Thus we can define $C_{2,\,\mathrm{LO}}^{(2)J}(n_2, n_1', n_1')$

as

$$C_{2,\,\mathrm{NLO}}^{(2)J}(n_2,n_1',n_1') = U^{(0)}(n_0)(Q,\mu_1)d_2^{J\,\alpha\beta}(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2\cdot n_1']\Theta_{\lambda^2}[n_2\cdot n_2']\,, \tag{B.94}$$

where

$$\begin{aligned}
d_2^{J\,\alpha\beta}(n_2,n_1',n_2') &= \lim_{n_2'\cdot n_1'\to\lambda^4} d_1^{\alpha\beta}(n_2,n_1',n_2') &&\text{(B.95)}\\
&= \Big(\Big[\frac{\bar q_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\,v_2^\beta + \frac{\bar k_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\,\frac{\not{v}_2}{2}\gamma^\beta_{n_2'\perp}\Big] &&\text{(B.96)}\\
&\times\Big[\frac{\bar q_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,v_1^\alpha + \frac{\bar k_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,\frac{\not{v}_1}{2}\gamma^\alpha_{n_1'\perp}\Big]\\
&\frac{\bar q_1}{q_1^2}\frac{4\bar q_0}{\bar q_2\bar k_1(n_2\cdot n_1')+\bar q_2\bar k_2(n_2\cdot n_2')}\gamma^\mu_{n_0\perp}\,,\\
&+\Big[\frac{\bar q_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,v_1^\alpha + \frac{\bar k_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\,\frac{\not{v}_2}{2}\gamma^\alpha_{n_1'\perp}\\
&+\gamma^\alpha_{n_1'\perp}\frac{\bar k_2\,\bar q_2}{(\bar q_2+\bar k_1)(\bar q_2+\bar k_2)}\sqrt{|n_2\cdot n_2'|}\,\frac{\not{v}_2}{2} - \gamma^\alpha_{n_1'\perp}\frac{\bar k_1\,\bar q_1}{(\bar q_2+\bar k_1)(\bar q_1+\bar k_1)}\sqrt{|n_1\cdot n_1'|}\,\frac{\not{v}_1}{2}\Big]\\
&\times\Big[\frac{\bar q_2}{\bar q_2+\bar k_2}\sqrt{|n_2\cdot n_2'|}\,v_2^\beta - \frac{\bar k_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,v_1^\beta + \frac{\bar k_1}{\bar q_1+\bar k_1}\sqrt{|n_1\cdot n_1'|}\,\frac{\not{v}_1}{2}\gamma^\beta_{n_2'\perp}\Big]\\
&\times\frac{\bar q_2+\bar k_1}{(q_2+k_1)^2}\frac{4\bar q_0}{\bar q_2\bar k_1(n_2\cdot n_1')+\bar q_2\bar k_2(n_2\cdot n_2')}\gamma^\mu_{n_0\perp},\\
&+\Big[\frac{1}{\bar q_0-\bar k_2}\gamma^\alpha_{n_1'\perp}\gamma^\beta_{n_2'\perp} + \frac{1}{\bar k_2+\bar q_2}\gamma^\beta_{n_2'\perp}\gamma^\alpha_{n_1'\perp}\Big]\frac{4\bar q_0}{\bar q_2\bar k_1(n_2\cdot n_1')+\bar q_2\bar k_2(n_2\cdot n_2')}\gamma^\mu_{n_0\perp}\Big)\\
&\times\Theta_{\lambda^4}[n_1'\cdot n_2']\,.
\end{aligned}$$

In Eq. (B.94) there is a mismatch of notation between the LHS and RHS. In the LHS we have the quark labeled with n_2 and the two gluons with n_1' because the coefficient (B.80) is for the operator $\mathcal{O}_2^{(2)}(n_2,n_1',n_1')$, where the tow gluons are labeled with the same direction n_1' because they are collinear in SCET$_2$. In the RHS of (B.80) n_2, n_1' and n_2' are the directions parallel to the quarks and the two gluons as defined in Appendix B.1. We encode the information that the two gluons are collinear using the theta function in the RHS of equation Eq. (B.96) that carries the information that $n_1'\cdot n_2' \lesssim \lambda$. In the RHS of Eq. (B.96) we could decompose n_2 in terms of n_1 and $\bar n$, writing everything only in terms of this two directions, and in this way we can avoid to insert the theta function, however it is convenient to leave explicitly n_2, n_1' and n_2', because it will make easier for the matching to SCET$_3$. We notice that the RHS of Eq. (B.96) is just equal to the coefficient $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2, n_1', n_2')$ defined in Eq. (B.80) with the substitution $q_0^2 \to \bar{q}_2\bar{k}_1(n_2\cdot n_1')/4 + \bar{q}_2\bar{k}_2(n_2\cdot n_2')/4$. Knowing that $n_1'\cdot n_2' \sim \lambda^4$, $n_1\cdot n_2' \sim \lambda^2$ and $n_1\cdot n_1' \sim \lambda^2$, it is easy to check that Eq. (B.96) scales as λ^{-2}. Because $C^{(2)J}_{2,\,\mathrm{LO}}(n_2, n_1', n_1')$ comes from a two-jet SCET$_1$ operator, we have encoded the information the $n_2,\cdot n_1' \lesssim \lambda^2$ in Eq. (B.94) on the Θ functions.

For the coefficient $C_2^{(2)}(n_2, n_2, n_1')$, we decompose it as

$$C_2^{(2)}(n_2, n_2, n_1') = C^{(2)J}_{2,\,\mathrm{NLO}}(n_2, n_2, n_1') + C_2^{(2)H}(n_2, n_2, n_1')\,, \tag{B.97}$$

where

$$\begin{aligned}
&C^{(2)J}_{2,\,\mathrm{NLO}}(n_2, n_2, n_1')\langle 0|\mathcal{O}_2^{(2)}(n_2, n_2, n_1')|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2 \\
&= C^{(0)}_{1,\,\mathrm{LO}}\int dx_1 dx_2 \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x_1)\mathcal{L}_{\mathrm{SCET}_1}(x_2)\mathcal{O}_1^{(0)}(n_0, n_0)\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2 \\
&- C^{(1)}_{2,\mathrm{LO}}(n_2, n_1')\int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}_2^{(1)}(n_2, n_1')\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2\,,
\end{aligned} \tag{B.98}$$

and

$$\begin{aligned}
&C_2^{(2)H}(n_2, n_2, n_1')\langle 0|\mathcal{O}_2^{(2)}(n_2, n_2, n_1')|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2 \\
&= C_1^{(1)}\int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{O}_1^{(1)}(n_0, n_0)\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2 \\
&- C^{(1)}_{2,\mathrm{NLO}}(n_2, n_1')\int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}_2^{(1)}(n_2, n_1')\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2 \\
&+ C^{(1)}_{1,\mathcal{T}}\int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}(x)\mathcal{T}_1^{(1)}(n_0, n_0)\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2 \\
&- C^{(1)}_{2,\mathrm{NNLO}}(n_2, n_1')\int dx \langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}_2^{(1)}(n_2, n_1')\}|q_{n_2}g_{n_2}g_{n_1'}\bar{q}_{\bar{n}}\rangle_2\,,
\end{aligned} \tag{B.99}$$

We decompose $C_2^{(2)H}(n_2, n_2, n_1')$ as

$$C_2^{(2)}(n_2, n_2, n_1') = C^{(2)H}_{2,\,\mathrm{NNLO}}(n_2, n_2, n_1') + C^{(2)H}_{2,\,\mathrm{N^3LO}}(n_2, n_2, n_1')\,, \tag{B.100}$$

where $C^{(2)H}_{2,\,\mathrm{NNLO}}(n_2,n_2,n'_1)$ is the coefficient that multiplied by $\langle 0|\mathcal{O}^{(2)}_2(n_2,n'_1,n'_2)|q_{n_2}g_{n'_1}g_{n'_2}\bar{q}_{\bar{n}}\rangle_2$ reproduces the the second and third line in the Eq. (B.99), and $C^{(2)H}_{2,\,\mathrm{N^3LO}}$ the forth and fifth line. As for the previous cases, the coefficient $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2,n_2,n'_1)$ scale as λ^{-2}, $C^{(2)H}_{2,\,\mathrm{NNLO}}(n_2,n_2,n'_1)$ as λ^{-1} and $C^{(2)H}_{2,\,\mathrm{N^3LO}}(n_2,n_2,n'_1)$ as λ^0. Because the operator $\mathcal{O}^{(2)}_2(n_2,n_2,n'_1)$ interferences with the LO operator $\mathcal{O}^{(1)}_2(n_1,n'_1)$, to have the amplitude squared up to NLO we need both $C^{(2)J}_{2,\,\mathrm{NLO}}$ and $C^{(2)H}_{2,\,\mathrm{NNLO}}$. Let us star with $C^{(2)J}_{2,\,\mathrm{NLO}}$. To calculate the amplitude in the second line in Eq. (B.98), we use Eq. (B.78) and take the limit $n_2\cdot n'_2 \to \lambda^4$ with $n_2\cdot n'_1 \sim n'_1\cdot n'_2 \sim \lambda^2$ because now we have the quark and a gluon are collinear in SCET_2.[4] It is easy to check that

$$\lim_{n_2\cdot n'_2\to\lambda^4} C^{\alpha\beta}_{\mathrm{LO},\,A}\mathcal{O}^{(2)}_2(n_2,n_2,n'_1) = C^{(1)}_{2,\mathrm{LO}}\int dx\langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}^{(1)}_2\}|q_{n_2}g_{n_2}g_{n'_1}\bar{q}_{\bar{n}}\rangle_2\,. \tag{B.101}$$

Thanks to Eq. (B.101), we can write $C^{(2)J}_{2\,,\mathrm{NLO}}(n_2,n_2,n'_1)$ as

$$C^{(2)J}_{2,\,\mathrm{NLO}}(n_2,n_2,n'_1) = U^{(0)}(n_0)(Q,\mu_1)\, d^{J\,\alpha\beta}_3(n_2,n'_1,n'_2)\Theta_{\lambda^2}[n_2\cdot n'_2]\Theta_{\lambda^2}[n'_1\cdot n'_2] \tag{B.102}$$

where

$$\begin{aligned} d^{J\,\alpha\beta}_3(n_2,n'_1,n'_2) &= \lim_{n_2\cdot n'_2\to\lambda^4}(d^{J\,\alpha\beta}_{1B}(n_2,n'_1,n'_2) + d^{J\,\alpha\beta}_{1C}(n_2,n'_1,n'_2)) \\ &= \Big(\Big[\frac{\bar{q}_1}{\bar{q}_1+\bar{k}_1}v^{\alpha}_1 - \gamma^{\alpha}_{n'_1\perp}\frac{\bar{k}_1\,\bar{q}_1}{(\bar{q}_2+\bar{k}_1)(\bar{q}_1+\bar{k}_1)}\frac{\not{v}_1}{2}\Big] \\ &+ \Big[\frac{1}{\bar{q}_0-\bar{k}_2}\gamma^{\alpha}_{n'_1\perp}\gamma^{\beta}_{n'_2\perp} + \frac{1}{\bar{k}_2+\bar{q}_2}\gamma^{\beta}_{n'_2\perp}\gamma^{\alpha}_{n'_1\perp}\Big] \\ &\times \frac{4\bar{q}_0}{\bar{k}_1(\bar{k}_2(n'_2\cdot n'_1)+\bar{q}_2(n_2\cdot n'_1))}\,\gamma^{\mu}_{n_0\perp}\Big)\Theta_{\lambda^4}[n_2\cdot n'_1]\,. \end{aligned} \tag{B.103}$$

The limit (B.102) is define for $(n_2 n'_1) \sim (n'_2 n'_1) \sim \lambda^2$, thus the coefficient $C^{(2)J}_{2,\,\mathrm{NLO}}(n_2,n_2,n'_1)$ is of order λ^{-2}. As previously for $C^{(2)J}_{2,\,\mathrm{NLO}}(n_1,n'_1,n'_1)$, we prefer leaving (B.102) in terms of the direction n_2, n'_1 and n'_2. To calculate $C^{(2)H}_{2\,,\mathrm{NNLO}}(n_2,n_2,n'_1)$ we proceed in the same

[4]We could alternatively take the limit $n_2\cdot n'_1 \to \lambda^4$ with $n_2\cdot n'_2 \sim n'_2\cdot n'_1 \sim \lambda^2$

way. We have

$$C^{(2)H}_{2,\mathrm{NNLO}}(n_2,n_2,n_1') = U^{(2,1,0)}(n_0,n_0)(Q,\mu_1)d_3^{H\,\alpha\beta}(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2\cdot n_2']\Theta_{\lambda^2}[n_1'\cdot n_2']\,, \tag{B.104}$$

where

$$\begin{aligned} d_3^{H\,\alpha\beta}(n_2,n_1',n_2') &= \lim_{n_2\cdot n_2'\to\lambda^4} d_{1B}^{H\,\alpha\beta}(n_2,n_1',n_2') \\ &= \Big(\Big[\frac{\bar q_1}{\bar q_1+\bar k_1}\sqrt{|n_1\!\cdot\! n_1'|}\,v_1^\alpha - \gamma^\alpha_{n_1'\perp}\frac{\bar k_1\,\bar q_1}{(\bar q_2+\bar k_1)(\bar q_1+\bar k_1)}\sqrt{|n_1\!\cdot\! n_1'|}\,\frac{\not v_1}{2}\Big] \\ &\times \gamma^\beta_{n_2'\perp}\frac{\bar q_2+\bar k_1}{(q_2+k_1)^2}\frac{\bar q_0}{q_0^2}\,\frac{n^\mu-\bar n^\mu}{Q}\gamma^\mu_{n_0\perp}\Big)\Theta_{\lambda^4}[n_2\!\cdot\! n_1']\,. \end{aligned} \tag{B.105}$$

In Eq. (B.104) we have use the fact that

$$\lim_{n_2\cdot n_2'\to\lambda^4} C^{\alpha\beta}_{\mathrm{NNLO},\,A}\mathcal{O}_2^{(2)}(n_2,n_2,n_1') = C^{(1)}_{2,\mathrm{NLO}}\int dx\langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}(x)\mathcal{O}_2^{(1)}\}|q_{n_2}g_{n_2}g_{n_1'}\bar q_{\bar n}\rangle_2\,. \tag{B.106}$$

In Eqs. (B.102, B.104) there is a mismatch of notation between the LHS and RHS of the same kind of Eq. (B.94). In the LHS we have the quark and one gluon labeled with n_2 and the other gluon with n_1' because the coefficient (B.100) is for the operator $\mathcal{O}_2^{(2)}(n_2,n_2,n_1')$, where the quark and a gluon are labeled with the same direction n_2 because they are collinear in SCET_2. In the RHS of (B.80) n_2, n_1' and n_2' are the directions parallel to the quarks and the two gluons as defined in Appendix B.1. We encode the information that the quark and a gluon are collinear in the theta function in the RHS of equation Eqs. (B.103, B.104) that carries the information that $n_2\cdot n_1'\lesssim\lambda^4$. Because $C^{(2)J}_{2,\,\mathrm{LO}}(n_2,n_2,n_1')$ and $C^{(2)H}_{2,\mathrm{NNLO}}(n_2,n_2,n_1')$ comes from two-jet SCET_1 operators, we encoded the information that $n_2,\cdot n_1'\lesssim\lambda^2$ in Eqs. (B.102,B.104) in the Θ functions.

We have that all the NLO terms from the two gluon matching come from the SCET_1 operator $\mathcal{O}_1^{(0)}(n_0)$ and are jet corrections and at NNLO we have only hard corrections. Before matching SCET_2 to SCET_3, we have to insert in the coefficients the SCET$_2$ running factors. Below we list all the SCET$_2$ coefficients at NLO that we have calculated with the appropriated SCET$_2$ running factors. From the matching of one-gluon emission we have the coefficients:

$$\begin{aligned} C^{(1)}_{2,\mathrm{LO}}(n_1,n_1') &= U^{(2,1,0)}(n_1,n_1')(\mu_1,\mu)U^{(0)}(n_0)(Q,\mu_1)c_{\mathrm{LO}}(n_0)\frac{\bar{q}_0}{q_0^2}\gamma^\mu_{n_0\perp}\,, \\ C^{(1)H,a}_{2,\mathrm{NLO}}(n_1,n_1') &= U^{(2,1,0)}(n_1,n_1')(\mu_1,\mu)U^{(2,1,0)}(n_0,n_0)(Q,\,\mu_1)\otimes c^{H,a}_{2,\mathrm{NLO}}\,, \\ C^{(1)H}_{2,\mathrm{NNLO}}(n_1,n_1') &= U^{(2,1,0)}(n_1,n_1')(\mu_1,\mu)U^{(2,1,0)}(n_0,n_0)(Q,\,\mu_1)\otimes c^{H}_{2,\mathrm{NNLO}}\,, \\ C^{(1)H,b}_{2,\mathrm{NLO}}(n_1,n_1') &= U^{(2,1,0)}(n_1,n_1')(\mu_1,\mu)C^{(1)}_1(n_1,n_1')\tilde{\Theta}_{\lambda^2}[n_1\cdot n_1']\,, \end{aligned} \tag{B.107}$$

where the first coefficient in (B.107) is defined in Eq. (B.51), the second and third in Eqs. (B.57), and the last in (B.59). From the matching of two-gluon emission we have the coefficients:

$$\begin{aligned} C^{(2)J}_{2,\mathrm{NLO}}(n_2,n_2',n_1') &= U^{(2,1,0)}(n_2,n_2',n_1')(\mu_1,\mu)\,U^{(0)}(n_0)(Q,\mu_1) \\ &\quad\times d^J_1(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2{\cdot}n_1']\Theta_{\lambda^2}[n_2{\cdot}n_2']\Theta_{\lambda^2}[n_2'{\cdot}n_1']\,, \\ C^{(2)J}_{2,\mathrm{NLO}}(n_2,n_1',n_1') &= U^{(2,1,0)}(n_2,n_1',n_1')(\mu_1,\mu)\otimes U^{(0)}(n_0)(Q,\mu_1) \\ &\quad\times d^J_2(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2\cdot n_1']\Theta_{\lambda^2}[n_2\cdot n_2']\,, \\ C^{(2)J}_{2,\mathrm{NLO}}(n_2,n_2,n_1') &= U^{(2,1,0)}(n_2,n_2,n_1')(\mu_1,\mu)\otimes U^{(0)}(n_0)(Q,\mu_1) \\ &\quad\times d^J_3(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2\cdot n_2']\Theta_{\lambda^2}[n_1'\cdot n_2']\,, \\ &\quad\times d^H_1(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2{\cdot}n_1']\Theta_{\lambda^2}[n_2{\cdot}n_2']\Theta_{\lambda^2}[n_2'{\cdot}n_1']\,, \\ C^{(2)H}_{2,\mathrm{NNLO}}(n_2,n_2,n_1') &= U^{(2,1,0)}(n_2,n_2,n_1')(\mu_1,\mu)\otimes U^{(2,1,0)}(n_0,n_0)(Q,\mu_1) \\ &\quad\times d^H_3(n_2,n_1',n_2')\Theta_{\lambda^2}[n_2\cdot n_2']\Theta_{\lambda^2}[n_1'\cdot n_2']\,, \end{aligned} \tag{B.108}$$

where the coefficients are defined in Eqs. (B.80, B.94, B.102, B.104).

B.4 Matching SCET$_2$ to SCET$_3$ to SCET$_N$

In this Appendix we match SCET$_2$ to SCET$_3$. All the SCET$_3$ operators necessary for the matching up to two-gluon emission are: $\mathcal{O}^{(0)}_3(n_0)$, $\mathcal{O}^{(1)}_3(n_0,n_0)$, $\mathcal{O}^{(1)}_3(n_1,n_1')$,

$\mathcal{O}_3^{(2)}(n_2, n_1', n_2')$, $\mathcal{O}_3^{(2)}(n_1, n_1, n_1')$, $\mathcal{O}_3^{(2)}(n_1, n_1', n_1')$. We have seen that to describe parton shower for one emission we only need the information of the coefficient of the SCET$_2$ operator $\mathcal{O}_2^{(1)}(n_1, n_1')$. We can apply the same argument here and infer that in SCET$_3$ the only information that we need, is in the the coefficient of the operator $\mathcal{O}_3^{(2)}(n_2, n_2', n_1')$, that is the only one that we calculate. We can follow the same steps made for matching SCET$_1$ to SCET$_2$ to calculate the Wilson coefficients $C_3^{(2)}(n_2, n_2', n_1')$. In this way it is no difficult to show that

$$\begin{aligned} C_3^{(2)}(n_2, n_2', n_1') = {} & C_{3,\,\mathrm{LO}}^{(2)} \\ & + C_{3,\,\mathrm{NLO}}^{(2)H,a} + C_{3,\,\mathrm{NLO}}^{(2)H,b} + C_{3,\,\mathrm{NLO}}^{(2)J\,a} + C_{3,\,\mathrm{NLO}}^{(2)J\,b} + C_{3,\,\mathrm{NLO}}^{(2)J\,c} \\ & + C_{3,\,\mathrm{NNLO}}^{(2)H,a} + C_{3,\,\mathrm{NNLO}}^{(2)H,b}\,, \end{aligned} \tag{B.109}$$

where

$$\begin{aligned} C_{3,\,\mathrm{LO}}^{(2)}(n_2, n_1', n_2') &= C_{3,\,\mathrm{LO}}^{(1)}(n_2, n_2') C_{2,\,\mathrm{LO}}^{(1)}(n_1, n_1')\,, \\ C_{3,\,\mathrm{NLO}}^{(2)H,\,a}(n_2, n_1', n_2') &= C_{3,\,\mathrm{LO}}^{(1)}(n_2, n_2') C_{2,\,\mathrm{NLO}}^{(1)H,a}(n_1, n_1')\,, \\ C_{3,\,\mathrm{NLO}}^{(2)H,\,b}(n_2, n_1', n_2') &= C_{3,\,\mathrm{LO}}^{(1)}(n_2, n_2') C_{2,\,\mathrm{NLO}}^{(1)H,b}(n_1, n_1')\,, \\ C_{3,\,\mathrm{NLO}}^{(2)J,\,1}(n_2, n_1', n_2') &= C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n_1', n_2')\tilde{\Theta}_{\lambda^4}[n_2\!\cdot\! n_2']\tilde{\Theta}_{\lambda^4}[n_2\!\cdot\! n_1']\tilde{\Theta}_{\lambda^4}[n_2'\!\cdot\! n_1']\,, \\ C_{3,\,\mathrm{NLO}}^{(2)J,\,2}(n_2, n_1, n_2') &= C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n_1', n_1')\tilde{\Theta}_{\lambda^4}[n_2\!\cdot\! n_2']\tilde{\Theta}_{\lambda^4}[n_2\!\cdot\! n_1']\Theta_{\lambda^4}[n_2'\!\cdot\! n_1']\,, \\ C_{3,\,\mathrm{NLO}}^{(2)J,\,3}(n_2, n_1', n_2') &= C_{2,\,\mathrm{NLO}}^{(2)J}(n_2, n_2, n_1')\Theta_{\lambda^4}[n_2\!\cdot\! n_2']\tilde{\Theta}_{\lambda^4}[n_2\!\cdot\! n_1']\tilde{\Theta}_{\lambda^4}[n_2'\!\cdot\! n_1']\,, \\ C_{3,\,\mathrm{NNLO}}^{(2)H,\,a}(n_2, n_1', n_2') &= C_{3,\,\mathrm{LO}}^{(1)}(n_2, n_2') C_{2,\,\mathrm{NNLO}}^{(1)}(n_1, n_1')\,, \\ C_{3,\,\mathrm{NNLO}}^{(2)H,\,b}(n_2, n_1', n_2') &= C_{2,\,\mathrm{NNLO}}^{(2)H}(n_2, n_2, n_1')\Theta_{\lambda^4}[n_2\!\cdot\! n_2']\tilde{\Theta}_{\lambda^4}[n_2\!\cdot\! n_1']\tilde{\Theta}_{\lambda^4}[n_2'\!\cdot\! n_1']\,, \end{aligned} \tag{B.110}$$

and

$$C_{3,\,\mathrm{LO}}^{(1)}(n_2, n_2') = \left(2\frac{(q_2)^{\beta}_{n_1\perp}}{\bar{k}_2} + \frac{(\not{q}_2)_{n_0\perp}\gamma^{\beta}_{n_2'\perp}}{\bar{q}_2} \right) \frac{\bar{q}_1}{q_1^2}\frac{\not{\bar{n}}\not{n}_1}{4}\,\Theta_{\lambda^4}[n_2\!\cdot\! n_2']\,. \tag{B.111}$$

In the LHS of the equations in the first, second and third line of (B.110) we can write n_1 in terms of n_2, n_2' and n_1' using the formulas in (B.18). The SCET$_2$ coefficients

$C^{(2)J}_{3,\,\mathrm{NLO}}$ and $C^{(2)H,b}_{3,\,\mathrm{NNLO}}$ are defined in Eqs. (B.108) and thy were already defined in terms of n_2, n_2' and n_1' (see Eqs. (B.80, B.94, B.102, B.104). As for the SCET$_1$ to SCET$_2$ matching, there is the problem of encoding additional information into the Wilson coefficient. The operator $\mathcal{O}_3^{(2)}(n_2, n_2', n_1')$ only knows that $n_2 \cdot n_1'$, $n_2 \cdot n_2'$, $n_1' \cdot n_2' > \lambda^6$, but we can see that the Wilson coefficients defined in Eqs. (B.110) come from matching SCET$_2$ to SCET$_3$ in different regions of the phase space. $C^{(2)}_{3,\,\mathrm{LO}}$, $C^{(2)H,a}_{3,\,\mathrm{NLO}}$, $C^{(2)H,a}_{3,\,\mathrm{NNLO}}$ are defined for $n_2 \cdot n_2' \lesssim \lambda^4$ and $\lambda^4 < n_2 \cdot n_1'$, $n_1' \cdot n_2' \lesssim \lambda^2$, $C^{(2)J,\,a}_{3,\,\mathrm{NLO}}$ for $\lambda^4 < n_2 \cdot n_1'$, $n_2 \cdot n_2'$, $n_1' \cdot n_2' \lesssim \lambda^2$; $C^{(2)J,\,b}_{3,\,\mathrm{NLO}}$ for $n_1' \cdot n_2' \lesssim \lambda^4$ and $\lambda^4 < n_2 \cdot n_1'$, $n_2 \cdot n_2' \lesssim \lambda^2$; $C^{(2)J,\,c}_{3,\,\mathrm{NLO}}$ for $n_2 \cdot n_2' \lesssim \lambda^4$ and $\lambda^4 < n_2 \cdot n_1'$, $n_1' \cdot n_2' \lesssim \lambda^2$. Some information about the scalar product of ns is already present in the theta functions of the SCET$_2$ coefficients, we add the missing information in the Θ functions present in Eqs. (B.110). All the coefficients SCET$_3$ defined above are for the same SCET$_3$ operator, thus now all the power counting is in the Wilson coefficients. The LO coefficient scales as λ^{-3}, the NLO coefficients as λ^{-2} and the NNLO coefficients as λ^{-1}.

At LO the coefficient$\times$operator in SCET$_3$ is given by the LO coefficient$\times$operator in SCET$_2$, $C^{(1)}_{2,\,\mathrm{LO}}\mathcal{O}_2^{(1)}(n_1, n_1')$ multiplying it by the running function $U^{(2,1,0)}(n_1, n_1')(\mu_1,\, \mu)$ and applying the replacement

$$(\bar\chi_{n_2})_i \to (c^{\alpha}_{\mathrm{LO}})_{ji}(n_1)(\bar\chi_{n_1})_j g\mathcal{B}^{n_1'\perp}_{\alpha}\,, \tag{B.112}$$

where $c^{\alpha}_{\mathrm{LO}}(n_1)$ is

$$c^{\alpha}_{\mathrm{LO}}(n_1) = \left(2\frac{(q_2)^{\alpha}_{n_0\perp}}{\bar k_2} + \frac{(\not q_2)_{n_1\perp}\gamma^{\alpha}_{n_2'\perp}}{\bar q_2}\right)\frac{\bar{\not n}\not n_{k+1}}{4}\,\Theta_{\lambda^4}[n_2 \cdot n_2']\,. \tag{B.113}$$

Eq. (B.112) has the same structure than Eq. (B.55). If we go on with the matching down to SCET$_N$ we find that the LO result would be given applying the above replacement $N-1$ times. At SCET$_N$ we could match everything to the operator $\mathcal{O}_N^{(N-1)}(n_{N-1}, n_1', \ldots, n_{N-1}')$, and the LO coefficient is

$$C^{(N-1)}_{N,\,\mathrm{LO}} = \prod_{k=1}^{N-1} U^{(k-1)}(\mu_{k-1},\, \mu_k) c^{\alpha_k}_{\mathrm{LO}}(n_{k-1})\Gamma^{\mu}\,, \tag{B.114}$$

with $\mu_k = \sqrt{|n_{k-1}\cdot n'_{k-1}|}Q$ and

$$c^{\alpha}_{\rm LO}(n_k) = \left(2\frac{(q_{k+1})^{\alpha}_{n_0\perp}}{\bar{k}_{k+1}} + \frac{(\not{q}_{k+1})_{n_k\perp}\gamma^{\alpha}_{n'_{k+1}\perp}}{\bar{q}_{k+1}}\right)\frac{\bar{\not{n}}\not{n}_k}{4}\,\Theta_{(\lambda^{(k+1)})^2}[n_{k+1}\cdot n'_{k+1}]\,. \qquad \text{(B.115)}$$

At NLO we have two kinds of corrections: hard and jet. The hard coefficients in Eqs. (B.110) depend on the QCD current, that is on the hard process at the top of the shower. We notice that the coefficient×operator in SCET$_3$ is just given by the coefficient×operator of the hard corrections in SCET$_2$ with the application of the replacement rule (B.112). If we go on with the matching down to SCET$_N$ we find that the NLO hard correction would be given applying the above replacement rules $N-2$ times to the SCET$_2$ hard correction operators. Thus we can consider this correction as a correction on the matrix elements from whom we start the shower:

$$C^{(N-1)H}_{N,\rm NLO} = \left(C^{(1)H,a}_{2,\rm NLO}(n_1,n'_1) + C^{(1)H,b}_{2,\rm NLO}(n_1,n'_1)\right)\left(\prod_{k=2}^{N-1} U^{(k-1)}(\mu_{k-1}\,,\mu_k)c^{\alpha_k}_{\rm LO}(n_{k-1})\right). \qquad \text{(B.116)}$$

The NLO jet corrections do not depend on the QCD current. In this case the SCET$_3$ coefficient×operators $\hat{C}^{(2)J,I}_{3,\rm NLO}\mathcal{O}^{(2)}_3(n_2,n'_1,n'_2)$, where $I=\{1,2,3\}$, are given by the LO SCET$_1$ operator $\bar{\chi}_{n_0}\gamma^{\mu}\chi_{\bar{n}}$ in three steps. First we multiply it for the running factor $U^{(2,1,0)}(n_0)(Q,\,\mu_1)$, second we apply the replacements

$$(\bar{\chi}_{n_2})_i \to (h^{\alpha\beta}_I)_{ji}(n_2,n'_1,n'_2)(\bar{\chi}_{n_1})_j\, g\mathcal{B}^{n'_1\perp}_{\alpha} g\mathcal{B}^{n'_2\perp}_{\beta}\,, \qquad \text{(B.117)}$$

where

$$\begin{aligned} h^{\alpha\beta}_1(n_2,n'_1,n'_2) &= d^{\alpha\beta}_1(n_2,n'_1,n'_2)\tilde{\Theta}_{\lambda^4}[n_2\cdot n'_2]\tilde{\Theta}_{\lambda^4}[n_2\cdot n'_1]\tilde{\Theta}_{\lambda^4}[n'_2\cdot n'_1]\,, \qquad \text{(B.118)}\\ h^{\alpha\beta}_1(n_2,n'_1,n'_2) &= d^{\alpha\beta}_2(n_2,n'_1,n'_2)\tilde{\Theta}_{\lambda^4}[n_2\cdot n'_2]\tilde{\Theta}_{\lambda^4}[n_2\cdot n'_1]\Theta_{\lambda^4}[n'_2\cdot n'_1]\,,\\ h^{\alpha\beta}_3(n_2,n'_1,n'_2) &= d^{\alpha\beta}_3(n_2,n'_1,n'_2)\Theta_{\lambda^4}[n_2\cdot n'_2]\tilde{\Theta}_{\lambda^4}[n_2\cdot n'_1]\tilde{\Theta}_{\lambda^4}[n'_2\cdot n'_1]\,. \end{aligned}$$

The $d^{\alpha\beta}_I$ coefficients are defined in Eqs.(B.81, B.96, B.103). Third we multiply the operartors that come from applying Eqs. (B.118) by the second running factor. This depends on the SCET$_2$ operator so each replacement rule (B.117) is followed by a different factor: $h_1^{\alpha\beta}$ by $U^{(2,1,0)}(n_2, n_1', n_2')(\mu_1, \mu_2)$, $h_2^{\alpha\beta}$ by $U^{(2,2,0)}(n_2, n_1', n_1')(\mu_1, \mu_2)$ and $h_3^{\alpha\beta}$by $U^{(2,2,0)}(n_2, n_2, n_1')(\mu_1,\, \mu_2)$. Because these corrections are independent of the hard scattering, we would encounter the same calculations we have done in the previous section for SCET$_1$ to SCET$_2$, at any matching SCET$_i$ to SCET$_{i+1}$. Thus the NLO jet coefficients for the SCET$_N$ final operator is

$$C_{N,\,\mathrm{NLO}}^{(N-1)J} = \sum_{l=1}^{N-2} C_{N,\,\mathrm{NLO}}^{(N)J}(l)\,, \tag{B.119}$$

where

$$\begin{aligned} C_{N,\,\mathrm{NLO}}^{(N-1)J}(l) = \sum_{I=1}^{3} \Big[\Big(\prod_{k=1}^{l-1} U^{(k-1)}(\mu_{k-1},\, \mu_k) c_{\mathrm{LO}}^{\alpha_k}(n_{k-1}) \Big) U^{(l-1)}(\mu_{k-1},\, \mu_k) \otimes h_I^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') \\ \times \Big(\prod_{k=l+1}^{N-1} U^{(k-1)}(\mu_{k-1},\, \mu_k) c_{\mathrm{LO}}^{\alpha_k}(n_{k-1}) \Big) \Big] \Gamma^\mu. \end{aligned} \tag{B.120}$$

and

$$\begin{aligned} h_1^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') &= d_1^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}\cdot n_{l+1}'] \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}\cdot n_l'] \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}'\cdot n_1']\,, \\ h_1^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') &= d_2^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}\cdot n_{l+1}'] \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}\cdot n_l'] \Theta_{(\lambda^{l+1})^2}[n_{l+1}'\cdot n_l']\,, \\ h_3^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') &= d_3^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}') \Theta_{(\lambda^{l+1})^2}[n_{l+1}\cdot n_{l+1}'] \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}\cdot n_l'] \tilde{\Theta}_{(\lambda^{l+1})^2}[n_{l+1}'\cdot n_l']\,. \end{aligned} \tag{B.121}$$

The coefficients $d_I^{\alpha\beta}(n_{l+1}, n_l', n_{l+1}')$ are equal to the coefficients $d_I^{\alpha\beta}(n_2, n_1', n_2')$ defined in Eqs.(B.81, B.96, B.103) upon the substitution $(n_2, n_1', n_2') \to (n_{l+1}, n_l', n_{l+1}')$ and $\lambda^2 \to \lambda^{2l}$. At NLO we calculated only the coefficients for the operators that interference with the LO operator, and we found only hard corrections. The considerations we have done above for the NLO hard corrections apply also to the NNLO hard correction, and we can write for the NNLO hard coefficient at SCET_N

$$
\begin{aligned}
C^{(N-1)H}_{N,\mathrm{NNLO}} = {}& C^{(1)H.a}_{2,\,\mathrm{NNLO}}(n_1, n_1')\Big(\prod_{k=2}^{N-2} U^{(k-1)}(\mu_{k-1},\,\mu_k)c^{\alpha_k}_{\mathrm{LO}}(n_{k-1})\Big) \\
&+ C^{(1)H.b}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')\Big(\prod_{k=3}^{N-3} U^{(k-1)}(\mu_{k-1},\,\mu_k)c^{\alpha_k}_{\mathrm{LO}}(n_{k-1})\Big)\,,
\end{aligned}
\tag{B.122}
$$

where the coefficients $C^{(1)H.a}_{2,\,\mathrm{NNLO}}(n_1, n_1')$ and $C^{(1)H.b}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')$ are defined in Eqs. (B.108).

Appendix C

$\mathcal{O}(\alpha_s^2)$ Correction to Splitting Function

In this appendix we rederive the (Abelian part of) the $\mathcal{O}(\alpha_s^2)$ correction to the $q \to qg$ splitting function, $P_{qq}^{(1)}$. This follows from obtaining the NLO (in λ, the SCET power counting parameter) correction to two-gluon emission. For comparison, we have chosen the classic result of Curci *et al.* [39]. The full expression for $P_{qq}^{(1)}$ involves many real and virtual contributions. We only calculated the $\sim C_F^2$ component of $P_{qq}^{(1)}$, and found that it agrees. Fortunately, [39] tabulated the different terms by graph, and so we were able to determine that we matched exactly for those calculations we did. Obtaining the full result would be a straightforward calculation of various one-loop renormalizations and non-Abelian emissions in SCET and combining them with our result. Ref. [39] splits the Abelian, two-gluon, real emission contributions to $P_{qq}^{(1)}$ into two topologically inequivalent diagrams, the box and crossed graphs, Fig. C-1. We calculated each of these individually.

The SCET_1 amplitude contains three graphs for two-gluon emission. These are shown in Fig. B-6, and we give the corresponding amplitudes in Eqs. (B.77). In order to obtain $P_{qq}^{(1)}$, we will need to square the amplitudes and integrate over phase space. Thus, we need to choose an explicit kinematics. We redraw, in figure (C-2), our vector labels for two-gluon emission. We choose a somewhat nonstandard assignment for our variables. This is to aid in the comparison with [39]. The final state parton

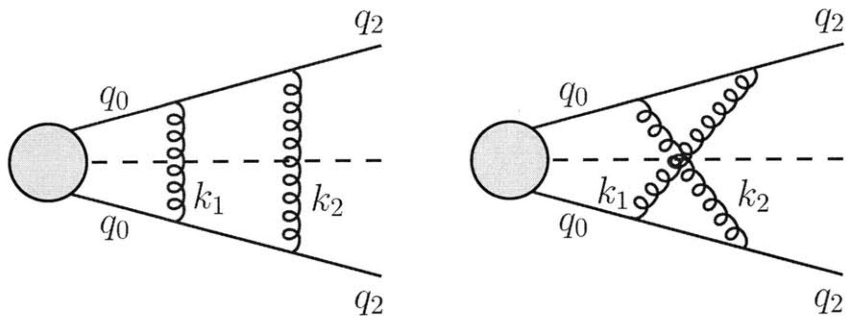

Figure C-1: Two distinct real emission contributions to $P_{qq}^{(1)}$ drawn as amplitudes squared. They are referred to as the box (**L**) and crossed (**R**) contributions.

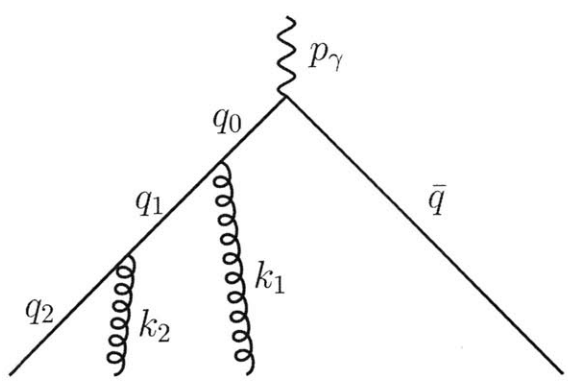

Figure C-2: Kinematics for double gluon emission. This particular diagram corresponds to the "A" graph of figure (B-6).

shower occurs for timelike virtual particles, and momentum fractions decrease the farther we are from the initial hard scattering. By contrast, [39] considered a DIS-type interaction where the shower is spacelike. Since the radiation in that case comes from initial states, the momentum fractions decrease toward the hard interaction. Only at LO in α_s are the spacelike and timelike splitting functions equal. This is the Gribov-Lipatov relation [56]. At higher orders, this gets violated, but there is a straightforward conversion procedure, detailed in [39, 94]. We, however, choose our kinematics such that our variable relations are equivalent to those for a spacelike process. For example, $P_{qq}^{(1)}$ is a function of $x \equiv \bar{q}_0/\bar{q}_2$. In a spacelike process, $x \in [0, 1]$. Rather than convert our answer, we will also define x as above, even though this means for us $x \in [1, \infty)$. Other integration variables will have their ranges shifted so that they have the same relation with x as in DIS, and thus they enter into our expression in the same way. Lastly, we do not do the phase space integration for q_2. While this *is* necessary for the timelike splitting function, the analogous particle for a spacelike process is a fixed initial state. Thus, for comparison purposes, we can leave it undone. Our vectors are as follows (note that this is a different frame from the one used previously for matching):

$$\begin{aligned}
q_2 &= \{p, 0, 0, p\} \\
k_1 &= \left\{-z_1 p - \frac{k_{1\perp}^2}{4pz_1},\, k_{1\perp}\cos(\phi_1),\, k_{1\perp}\sin(\phi_1),\, -z_1 p + \frac{k_{1\perp}^2}{4pz_1}\right\} \\
k_2 &= \left\{-z_2 p - \frac{k_{2\perp}^2}{4pz_2},\, k_{2\perp},\, 0, -z_2 p + \frac{k_{2\perp}^2}{4pz_2}\right\} \\
q_0 &= \left\{x\,p + \frac{q_0^2 + |\overrightarrow{\mathbf{k_1}}_\perp + \overrightarrow{\mathbf{k_2}}_\perp|^2}{4px}, \overrightarrow{\mathbf{k_1}}_\perp + \overrightarrow{\mathbf{k_2}}_\perp, x\,p - \frac{q_0^2 + |\overrightarrow{\mathbf{k_1}}_\perp + \overrightarrow{\mathbf{k_2}}_\perp|^2}{4px}\right\}. \quad \text{(C.1)}
\end{aligned}$$

Before proceeding, we wish to note some things about our assignment. First of all, while it is redundant to include $q_0 = k_1 + k_2 + q_2$, we will integrate over $d^4 q_0$ and wanted to present our parametrization. We see that $x = 1 - z_1 - z_2$. This is for consistency with the spacelike case, but here, $z_1, z_2 \in (-\infty, 0]$, hence the minus signs in k_1 and k_2. Additionally, only the relative azimuthal angle between k_1 and k_2 is physical. Thus, to simplify our formulas, we fix k_2 in the $x - z$ plane. A trivial factor of 2π will enter from the k_2 phase space once we integrate.

As a last step before squaring and integrating, we will introduce our measure and integral parametrization. While one could integrate the full final state phase space, we found such an approach prohibitively difficult. Instead, we can exploit the factorization of the the cross-Subsection into a hard interaction and a jet-function for an appropriate definition of each. We need only integrate the latter, and it will remain independent of the details of the former. We split up the cross-Subsection as follows ($d \equiv 4 + \epsilon$):

$$\sigma = \int \frac{dx}{x} \mathcal{H}(x) \left(\mathcal{J}_{\mathrm{LO}}(x, q^2) + \mathcal{J}_{J,\mathrm{NLO}}(x, q^2) + \ldots\right) q_{B,F}(x), \quad \text{(C.2)}$$

where

$$\mathcal{J}_{\mathrm{LO}} = \sum_i \frac{2}{\bar{q}_i} \int \prod_{j=1}^{i} \frac{đ^{d-1} k_j}{z_j} đ^d q_0 \, dq^2 \, \mathrm{PP}\left[C_{N,\mathrm{LO}}^{(N)} |\langle 0|\mathcal{O}_N^{(N)}|q(Ng)\bar{q}\rangle|^2\right] \tag{C.3}$$
$$\times \delta(x - \bar{q}_0/\bar{q}_i)\delta(q^2 - (q_i + \sum_{j=1}^{i} k_j)^2)\,(2\pi)^m \delta^{(m)}(q_0 - \sum_{j=1}^{i} k_j)),$$

$$\mathcal{J}_{J,\mathrm{NLO}} = \sum_i \frac{2}{\bar{q}_i} \int \prod_{j=1}^{i} \frac{đ^{d-1} k_j}{z_j} đ^d q_0 \, dq^2 \, \mathrm{PP}\left[C_{N,\mathrm{NLO}}^{(N)J} |\langle 0|\mathcal{O}_N^{(N)}|q(Ng)\bar{q}\rangle|^2\right] \tag{C.4}$$
$$\times \delta(x - \bar{q}_0/\bar{q}_i)\delta(q^2 - (q_i + \sum_{j=1}^{i} k_j)^2)\,(2\pi)^m \delta^{(m)}(q_0 - \sum_{j=1}^{i} k_j)),$$

and the q_i phase space and spin-sum gets moved into $\mathcal{H}$. We define z_j analogously to Eqs. (C.1) and (C.1). The function $q_{B,F}(x)$ is the bare fragmentation function which determines how the partons arrange themselves into hadrons. We call it 'bare' as it contains singularities necessary to cancel the collinear divergences from parton splitting to give a finite observable. The $\mathcal{J}$ terms defined above consist of only the pole portions of the corresponding operator expectation values. The reason we extract only the pole terms is that these are precisely what give the expression for $P_{qq}^{(1)}$.

By fixing the virtuality of $q_0^2 \equiv q^2$, we can obtain an expression without having to know its exact limits, which will depend on the details of the hard scattering. For $P_{qq}^{(1)}$, one only needs to calculate one-loop corrections to single emission and double emission, and we now specialize to the latter case. We perform the d-dimensional δ-function over $d^d q_0$ and rewrite the integral in terms of $k_{1\perp}$ and $k_{2\perp}$ dependent functions with $z_{1,2}$-dependent coefficients. Using the parametrization of [45], we can write:

$$\mathcal{J}_{J,\mathrm{NLO}}^{q\to qgg} = \frac{1}{(16\pi^2)^2} \int dq^2 \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{d^{d-2}\mathbf{k}_{1\perp}}{\pi} \frac{d^{d-2}\mathbf{k}_{2\perp}}{\pi} \delta(1 - x - z_1 - z_2)$$
$$\times \delta\left(q^2 - (a_1 \mathbf{k}_{1\perp}^2 + a_2 \mathbf{k}_{2\perp}^2 - \mathbf{k}_{1\perp}\cdot\mathbf{k}_{2\perp})\right)$$
$$\times \Bigg(A(z_1,z_2) + B(z_1,z_2)\frac{\mathbf{k}_{1\perp}\cdot\mathbf{k}_{2\perp}}{\mathbf{k}_{1\perp}^2} + C(z_1,z_2)\frac{\mathbf{k}_{1\perp}\cdot\mathbf{k}_{2\perp}}{\mathbf{k}_{2\perp}^2}$$
$$+ D(z_1,z_2)\frac{(\mathbf{k}_{1\perp}\cdot\mathbf{k}_{2\perp})^2}{\mathbf{k}_{1\perp}^2\mathbf{k}_{2\perp}^2} + E(z_1,z_2)\frac{\mathbf{k}_{1\perp}^2}{\mathbf{k}_{2\perp}^2} + F(z_1,z_2)\frac{\mathbf{k}_{2\perp}^2}{\mathbf{k}_{1\perp}^2}\Bigg)\frac{1}{q^4} \tag{C.5}$$

$$- [\mathrm{LO}]\,, \tag{C.6}$$

where

$$a1 = -\frac{1-z_2}{z_1}, \;\; a2 = -\frac{1-z_1}{z_2}. \tag{C.7}$$

The functions A, B, C, D are defined in [45], and their corresponding $\mathbf{k_{i\perp}}$ integrals are finite. We can check the intermediate step of their integration with [45]. The terms in our q^2 δ-function have a relative sign compared to theirs, as our $q^2 > 0$. We found it simplest to calculate in SCET$_3$ where only $C^{(2)J,1}_{3,\mathrm{NLO}}$ in Eq. (B.110) contributes. This corresponds to taking limits such that only its Θ-function gives support, while the jet-structure coeffecients are zero. By doing this, we wind up integrating over the strongly-ordered region of phase space, but since we know the LO contribution from $C^{(2)}_3$, we can formulate the subtraction in Eq. (C.6) at the operator level. There is just one subtlety, which we describe below, having to do with the appropriate treatment of dim reg in the standard scheme for calculating $P^{(1)}_{qq}$.

As a computational aside, we found it easiest to pass to a change of variables: $(u \equiv k_{1\perp} k_{2\perp},\; w \equiv k_{1\perp}/k_{2\perp})$. Then the δ-function just enforces:

$$u \;=\; u_0 \;\equiv\; \frac{q^2\, w}{a_1 w^2 + a_2 - 2\, w \cos(\phi_1)}. \tag{C.8}$$

Performing all but the dz_i integrals in $\mathcal{J}$, we get Table C.1, which corresponds to [45]'s Table 5. We thus reproduce the earlier result except for what we believe is a

Function of $\mathbf{k_{i\perp}}$ in integrand of equation (C.6)	Contribution to $\mathcal{J}$ multiplying $\frac{q^2}{(16\pi^2)^2 x} \int dz_1\, dz_2\, \delta(1 - z_1 - z_2 - x)$
1	$A(z_1, z_2)$
$\frac{\mathbf{k_{1\perp}} \cdot \mathbf{k_{2\perp}}}{\mathbf{k_1}^2_\perp}$	$-\frac{z_2}{1-z1} B(z_1, z_2)$
$\frac{\mathbf{k_{1\perp}} \cdot \mathbf{k_{2\perp}}}{\mathbf{k_2}^2_\perp}$	$-\frac{z_1}{1-z2} C(z_1, z_2)$
$\frac{(\mathbf{k_{1\perp}} \cdot \mathbf{k_{2\perp}})^2}{\mathbf{k_1}^2_\perp \mathbf{k_2}^2_\perp}$	$\left(1 + \frac{x}{2z_1 z_2} \ln \left[\frac{x}{(1-z_1)(1-z_2)}\right]\right) D(z_1, z_2)$

Table C.1: Purely finite contributions to $\mathcal{J}$

typo on their part swapping the B and C entries.

The E, F functions multiply integrals that lead to single ϵ poles after the $dk_{i\perp}$ integral (and double poles after integrating q^2), and so we must be more careful in treating them. These double poles correspond to the LO contribution, which we are explicitly subtracting as it does not contribute to $P_{qq}^{(1)}$. We discuss it in detail below Eq. (C.17). For now we concentrateon the divergent integrals multiplying E and F. When we did our computations for Table (C.1), we were helped by the finiteness of the expressions under the $dk_{i\perp}$ integration. We could thus take $\epsilon \to 0$ for these terms, which greatly simplifies their integrals. By contrast, we will need to keep the ϵ-dependence of the E, F terms, which results in an intractable computation. To get around this, one can introduce subtraction functions, which simply reproduce the ϵ poles (these are merely a computational aid and are not related to the subtraction of LO). We will need to take care that they do not remove any finite pieces. Secondly, since their full contribution to $\mathcal{J}$ is $\propto 1/\epsilon^2$, we will need to include for E and F any terms $\propto \epsilon$ that multiply $\frac{\mathbf{k_1}_\perp^2}{\mathbf{k_2}_\perp^2}$ or $\frac{\mathbf{k_1}_\perp^2}{\mathbf{k_2}_\perp^2}$. These arise from doing Dirac algebra in m-dimensions.

To do the integrals in $\mathcal{J}$ multiplying and E and F, we will change variables to u, w, and perform the u integration as well as the trivial ϕ_2 azimuthal one. We get for this contribution to $\mathcal{J}$:

$$\begin{aligned}\mathcal{J}|_{E,F} = & \frac{1}{(16\pi^2)^2}\frac{2}{\pi}\int dq^2\,\frac{dz_1}{z_1}\frac{dz_2}{z_2}\,d\phi_1\,dw\,\delta(1-x-z_1-z_2) \\ & \times\left(\frac{w\,u_0^{2+\epsilon}}{2\,q^2}E(z_1,z_2) + \frac{u_0^{2+\epsilon}}{2\,w\,q^2}F(z_1,z_2)\right)\frac{1}{q^4},\end{aligned} \tag{C.9}$$

where u_0 is defined by equation (C.8). Unfortunately, we could not manage to perform the w and ϕ_1 integrals for the functions multiplying E, F and obtain a result in terms of elementary functions. However, we only need the leading poles in ϵ, and so we will define subtraction functions to reproduce the poles of $\frac{w\,u_0^{2+\epsilon}}{2\,q^2}$, $\frac{u_0^{2+\epsilon}}{2\,w\,q^2}$, respectively:

$$\begin{aligned}\mathcal{S}_E &= \frac{q^2}{2\,a_1^2}\frac{w^{-\epsilon}}{(w+1)} \\ \mathcal{S}_F &= \frac{q^2}{2\,a_2^2}\frac{w^{\epsilon}}{(w+w^2)}.\end{aligned} \tag{C.10}$$

Integrating these in w gives us a pure $1/\epsilon$ term. Subtracting them from the functions in equation (C.9):

$$\begin{aligned}
\mathcal{A}_E &\equiv \tfrac{w\, u_0^{2+\epsilon}}{2\, q^2} &&= \frac{q^2\, w^3}{2(a_2 + a_1 w^2 - 2w\cos(\phi_1))^2}\left(\frac{w\, q^2}{a_2 + a_1 w^2 - 2w\cos(\phi_1)}\right)^{\epsilon} \quad \text{(C.11)}\\
\mathcal{A}_F &\equiv \tfrac{u_0^{2+\epsilon}}{2\, w\, q^2} &&= \frac{q^2}{2\, w(a_2 + a_1 w^2 - 2w\cos(\phi_1))^2}\left(\frac{w\, q^2}{a_2 + a_1 w^2 - 2w\cos(\phi_1)}\right)^{\epsilon},
\end{aligned}$$

leads to finite integrals, allowing us to pass to the $\epsilon \to 0$ limit prior to integration making the calculation tractable. Note that the subtraction terms are purely computational aides and are different from the factorization scheme mentioned above, which is designed to absorb a collinear divergence into a fragmentation function. After integrating w and ϕ_1, we want the $\epsilon^{-1,0}$ pieces as these turn into the single and double poles upon doing the q^2 integral and contribute to $\mathcal{J}$. The ϵ^0 piece, has one contribution besides that from $(\mathcal{A}_{E,F} - \mathcal{S}_{E,F})|_{\epsilon=0}$ ($\mathcal{S}_{E,F}$ contributes a pure $1/\epsilon$ pole). Our w integration goes from 0 to ∞, and we obtained $\mathcal{S}_{E,F}$ by expanding $\mathcal{A}_{E,F}$ in the appropriate $w \to 0, \infty$ limit to pick up the pole, while carefully regulating the other integration limit so as not to contribute its own spurious divergence or any subleading terms. However, we see that in equation (C.12), taking these limits actually results in factors $(a_1 w)^{-\epsilon}$ and $(w/a_2)^{\epsilon}$. Expanding the $a_i^{\pm\epsilon}$ to LO in ϵ does not affect $\mathcal{S}_{E,F}$. Nonetheless, since the subtraction functions have $1/\epsilon$ poles, including the NLO piece will yield an ϵ^0 contribution. We do not have it in $\mathcal{A}_{E,F}|_{\epsilon=0}$ since that sends $u_0^{\epsilon} \to 1$. Thus, we have the following addition to the contributions from the integration of $\mathcal{J}|_{E,F}$:

$$\mathcal{B}_E = -\epsilon\ln(a_1)\frac{q^2}{2\, a_1^2}\frac{w^{-\epsilon}}{(w+1)} \tag{C.12}$$

$$\mathcal{B}_F = -\epsilon\ln(a_2)\frac{q^2}{2\, a_2^2}\frac{w^{\epsilon}}{(w+w^2)}. \tag{C.13}$$

In the end, our $\epsilon^{-1,0}$ contributions after w and ϕ_1 integration come from: $\mathcal{S}_{E,F} + \mathcal{B}_{E,F} + (\mathcal{A}_{E,F} - \mathcal{S}_{E,F})|_{\epsilon=0}$. For integrating the first two terms, we leave the full ϵ dependence as this is tractable. Once we have accounted for where it is needed, we can set $\epsilon = 0$ in the last term to pick up the remaining finite result. Collecting all terms, we can obtain the counterpart to Table C.1 for E, F. The origin of the contributions in terms of $\mathcal{S}$, $\mathcal{B}$, and $\mathcal{S}-\mathcal{A}$ should be straightforward.

Function of $\mathbf{k_{i\perp}}$ in integrand of equation (C.6)	Contribution to $\mathcal{J}$ multiplying $\frac{q^2}{(16\pi^2)^2 x}\int dz_1\, dz_2\, \delta(1-z_1-z_2-x)$
$\frac{\mathbf{k_1}_\perp^2}{\mathbf{k_2}_\perp^2}$	$\left[\frac{2\,x\,z_1}{(1-z_2)^2 z_2\,\epsilon}\left(1-\epsilon\ln\left[-\frac{1-z_2}{z_1}\right]\right) + \frac{z_1}{z_2(1-z_2)^2}\left(z_1 z_2 + x\left(\ln\left[\frac{z_2(1-z_2)}{z_1 x}\right]-1\right)\right)\right] E(z_1,z_2)$
$\frac{\mathbf{k_2}_\perp^2}{\mathbf{k_1}_\perp^2}$	$\left[\frac{2\,x\,z_2}{(1-z_1)^2 z_1\,\epsilon}\left(1-\epsilon\ln\left[-\frac{1-z_1}{z_2}\right]\right) + \frac{z_2}{z_1(1-z_1)^2}\left(z_1 z_2 + x\left(\ln\left[\frac{z_1(1-z_1)}{z_2 x}\right]-1\right)\right)\right] F(z_1,z_2)$

Table C.2: Contributions to $\mathcal{J}|_{E,F}$

Having set up this much of the integration, we can take the amplitude squared from the process of interest and decompose it in terms of the $A(z_1,z_2)$, $B(z_1,z_2)$, *etc.* basis. We then simply have to read off the results from Tables C.1 and C.2, and perform the $z_{1,2}$ integrals. One of these is made trivial by the remaining x-dependent δ-function. As mentioned at the beginning of this Appendix, Ref. [39] recognizes two topologically distinct contributions, which we shall refer to as box and crossed, Fig. C-1, because of their appearance as cut two-loop diagrams. We can identify them in our calculation by their color structure (C_F^2 and $C_F^2-\frac{1}{2}C_F C_A$, respectively). In fact, we can already calculate the entire crossed contribution as it only involves terms from Table C.1, having no double pole contribution to $\mathcal{J}$. Determining the box graph, however, involves treating the LO subtraction properly.

As this subtraction is one of the more subtle points of the computation, we will present it in some detail. Its handling is tied up with what one means precisely by a "subleading splitting function." At LO in α_s, the definition is clear. The same splitting function that gives us the probability for a $1\to 2$ radiation also determines the running with scale of parton densities:

$$Q^2\frac{\partial}{\partial Q^2}f(x,Q^2) = \int_x^1 \frac{dz}{z}\,P_{qq}\left(\frac{x}{z},\alpha_s(Q^2)\right)\,f(z,Q^2), \tag{C.14}$$

where the $\mathcal{O}(\alpha_s)$ part of P_{qq}, $P_{qq}^{(0)}$ is given by Eq. (4.3). To determine $P_{qq}^{(1)}$, we have had to calculate a $1\to 3$ splitting, thus the probabilistic interpretation in terms of radiation is nontrivial as it involves a mix of $1 \to 2$ and $1 \to 3$ processes. At the level of Eq. (C.14) though, we see that we are just correcting PDF (or fragmentation function) evolution. In addition to the real-emission calculation that we are pursuing, one can alternatively determine P_{qq} from the anomalous dimension of certain twist-2 operators [50, 51]. Ref. [39] made a comparison to this approach and found agreement to $\mathcal{O}(\alpha_s^2)$. Since $P_{qq}^{(1)}$ is thus a two-loop object, it has the scheme dependence one would expect at this order, and so we need to make sure that we compute in the same one. In SCET, one could attempt the same cross-check from a straightforward two-loop calculation after fixing the renormalization scheme.

We will now show how to subtract the LO portion in the calculation of $\mathcal{J}_{J,\,\mathrm{NLO}}^{q\to qgg}$ We get a double collinear pole associated with the strongly-ordered emission of two gluons. We want to write this as removing the emission from our LO operator, $C_{3,\,\mathrm{LO}}^{(2)}\mathcal{O}_3^{(2)}$. As with any subtraction scheme, while the pole is unambiguous, we need to make sure to remove the appropriate finite pieces. We note that c_{LO}^{α} defined by Eq. (4.47) contains NLO pieces (in SCET$_3$ power counting) which come from the offshellness of the intermediate quark. It is true that the LO replacement rule, Eq. (4.46), gives only the splitting function times the logarithmic, collinear divergence. Nonetheless, the Wilson coefficients given by Eq. (4.53) for offshell quarks have additional terms. From the point of view of amplitdue matching, this poses no problem. However, if we want to copy [39]'s scheme, then we can only subtract poles associated with the pure LO result. As an operator subtraction in SCET$_3$, this means we need to change $\mathcal{O}_3^{(2)}$. In order to recover the correct splitting function with no NLO contribution, we will need to project the offshell quark momentum to an onshell one with the same $\bar{p}$-fraction. This alone, though, does not specify the spatial orientation of the vector and will not necessarily kill the subleading terms. To do that, we write the replacement rule, but in the limit that the offshell quark's daughters are exactly collinear with it. Equivalently, if we are in the frame determined by $\bar{n} = \{1, 0, 0, -1\}$, we can project the quark momentum along $n = \{1, 0, 0, 1\}$, *i.e.* $q_i \to \frac{\bar{q}_i}{2} n = q_i'$. Since the replacement rule also makes reference to the quark's parent's momentum, we also need to project it to what it would be if it had emitted an onshell quark with q_i'.

Thus, $q_{i-1} \to k_i + q'_i = q'_{i-1}$. In the end, this changes our replacement rule coefficient for the j^{th} quark to:

$$c'_{\text{LO}}{}^{\alpha_{j+1}} = \frac{\bar{q}_j}{q'^{\,2}_j}\left(n^{\alpha}_{q'_j} + \frac{\not{q}'_{j+1\perp_{n_{q'_j}}}\gamma^{\alpha}_{\perp j+1}}{\bar{q}_{j+1}}\right)\frac{\not{\bar{n}}\not{n}_{q_j}}{4}\,. \tag{C.15}$$

This changes the expression for $C^{(2)}_{3,\,\text{LO}}\mathcal{O}^{(2)}_3$ to involve c'_{LO} instead of c'_{LO} (*cf.* Eq. 4.53)

After the dq^2 integration, the $1/\epsilon$ term in $\mathcal{J}^{q\to qgg}_{J,\,\text{NLO}}$ will allow us to read off $P^{(1)}_{qq}$. The subtraction operator lives in SCET$_3$ as both gluons have their own direction and Wilson coefficient in that theory. We need it because our NLO operator, $C^{(2)J,\,1}_{3,\,\text{NLO}}\mathcal{O}^{(2)}_3$ is supported over all of phase space, and there contains LO portions. We therefore have

$$\mathcal{J}^{q\to qgg}_{J,\,\text{NLO}} = \int d\Pi_{k_1,\,k_2,\,q_0}\left[|C^{(2)J,\,1}_{3,\,\text{NLO}}\langle 0|\mathcal{O}^{(2)}_3|qgg\bar{q}\rangle|^2 - \left(|c'_{\text{LO}}{}^{\alpha_1}\,c'_{\text{LO}}{}^{\alpha_2}\langle 0|\mathcal{O}^{(2)}_3\Gamma^{\mu}|qgg\bar{q}\rangle|^2\right)_{\overline{\text{MS}}}\right] \tag{C.16}$$

The $\overline{\text{MS}}$ indicates that we are only subtracting pole parts of the LO contribution, with no finite pieces. However, there is still an amibiguity over *which* pole parts we subtract, as even though the LO contribution has a double pole from its two collinear divergences, we are at some liberty to decide which single pole parts we remove as well. As we expect, this subtraction operator squared takes the form of a convolution of two splitting functions:

$$\int d\Pi\left(|c'_{\text{LO}}{}^{\alpha_1}\,c'_{\text{LO}}{}^{\alpha_2}\langle 0|\mathcal{O}^{(2)}_3\Gamma^{\mu}|q\bar{q}gg\rangle|^2\right)_{\overline{\text{MS}}} = 2\int dq^2\,dy\,x\,p\,(1-y)^{\frac{\epsilon}{2}}\,(q^2)^{-1+\epsilon/2}\frac{\alpha^2}{2\pi^2} \times \frac{1}{y}P^{(0)}_{d,qq}(y)\frac{P^{(0)}_{4,qq}(x/y)}{\epsilon}. \tag{C.17}$$

What may seem surprising is that different splitting functions live in different dimensions. The reason for this particular scheme for regluating phase space has to do with the alternate, two-loop method for calculating $P^{(1)}_{qq}$, which was the original approach. For that result, in $\overline{\text{MS}}$ we would subtract a simple pole counterterm, regulate the loop integral in d-dimensions, and leave external particles in 4d. Since the phase space integrals are related to loops by cuts, we see above that our y-integral is, in fact, in d-dimensions, but the splitting involving two external particles is left simply in four.

Looking at the SCET$_1$ diagrams for the process (Fig. B-6), the amplitude ${c'_{\mathrm{LO}}}^{\alpha_1}\,{c'_{\mathrm{LO}}}^{\alpha_2}\langle 0|\mathcal{O}_3^{(2)}\Gamma^\mu|qgg\bar{q}\rangle$ comes from a subset of diagrams A^2 and B^2. The expression for subtraction is thus:

$$\int d\Pi\left(|{c'_{\mathrm{LO}}}^{\alpha_1}\,{c'_{\mathrm{LO}}}^{\alpha_2}\langle 0|\mathcal{O}_{\mathrm{S}_3}^{(2)}\Gamma^\mu|q\bar{q}gg\rangle|^2\right)_{\overline{\mathrm{MS}}} = \int dq^2\,dz_1 x\,p\left(\frac{z_1}{x+z_1}\right)^{\frac{\epsilon}{2}}$$
$$\times\frac{(q^2)^{-1+\epsilon/2}}{\epsilon}\frac{\alpha^2}{2\pi^2}\frac{1}{x+z_1}\left[\frac{1+\left(\frac{x}{x+z_1}\right)^2}{\frac{x}{x+z_1}-1}+\frac{\epsilon}{2}\left(1-\frac{x}{x+z_1}\right)\right]\left(\frac{1+(x+z_1)^2}{x+z_1-1}\right)$$
$$+\,z_1\leftrightarrow z_2. \tag{C.18}$$

We can note several things about this expression. For concreteness, we discuss the z_1-dependent term corresponding to graph A^2, Fig. B-6. The $\bar{p}$ fraction of q_0 relative to q_1 is $x/(x+z_1)$, and that of q_1 to q_2 is $x+z_1$, in terms of the variables in Eq. (C.17), $y' = x/(x+z_1)$. Performing the integrals leads to double and single poles. For later use, we write down the result of doing the dq^2, dz_i integrals, where one of latter is trivial since we have $\delta(1-x-z_1-z_2)$ sitting inside Π (*cf.* Eq. C.6).

$$\int d\Pi\left(|{c'_{\mathrm{LO}}}^{\alpha_1}\,{c'_{\mathrm{LO}}}^{\alpha_2}\langle 0|\mathcal{O}_3^{(2)}\Gamma^\mu|q\bar{q}gg\rangle|^2\right)_{\overline{\mathrm{MS}}} =$$
$$2\,x\,p\,\frac{\alpha^2}{2\pi^2}\left(\frac{1}{\epsilon^2}\frac{2}{x-1}\left[\left(-2\left(2\left(x^2+1\right)\log(\lambda)+(x-1)^2\right)\right.\right.\right.$$
$$\left.+\,4\left(x^2+1\right)\log(x-1)-\left(x^2-1\right)\log(x)\right)\right]$$
$$+\,\frac{2}{\epsilon}\frac{1}{(x-1)^2x}\left[2x(x-1)\left(2\left(x^2+1\right)\left(\mathrm{Li}_2(1-x)-\mathrm{Li}_2\left(\frac{1}{x}\right)\right)+\left(x^2-1\right)\mathrm{Li}_2\left(\frac{x-1}{x}\right)\right)\right.$$
$$+\,x\left(-2\left(x^2+1\right)(x-1)\log^2(\lambda)+4\left(\left(x^2+1\right)\left(\log\left(\frac{x-1}{x}\right)+x\log\left(\frac{x}{x-1}\right)\right)\right.\right.$$
$$\left.-\,(x-1)^3\right)\log(\lambda)-\left(3x^2+5\right)(x-1)\log^2(x)+2(x-1)^2\log(x)\Big)$$
$$+\,6x\left(x^2+1\right)(x-1)\log^2(x-1)-2x(x-1)^3$$
$$\left.\left.-\,2x(x+1)(x-1)^2\log(x-1)\log(x)((x-1)^2x)\right]\right) \tag{C.19}$$

where we have done the dz_i integrals between $1-x+\lambda$ and $-\lambda$ to regulate soft divergences. All λ-dependence cancels out of the final answer, which gives us a consistency check on the scheme.

Before comparing $P_{qq}^{(1)}$, we can check our setup with $P_{qq}^{(0)}$, by looking at the $\mathcal{O}(\alpha_s)$ contribution to $\mathcal{J}_{\mathrm{LO}}$ We see that [39] gets the following contribution:

$$P_{qq}^{(0)} = \left(\frac{\alpha_s}{2\pi}\right)\frac{2}{\epsilon}\frac{1+x^2}{1-x}. \tag{C.20}$$

Calculating in SCET_1, we get the following amplitude squared:

$$A_{q\to qg} = \frac{\bar{q}_0}{q_0^2}\left(\frac{2n\cdot k_1}{\bar{k}_1} + \frac{2k_{1\perp}\cdot q_{1\perp}}{\bar{q}_1\bar{k}_1} - \frac{q_{1\perp}^2}{\bar{q}_1^2}\right)\mathrm{Tr}[\not{\bar{n}}\Omega\Omega^\dagger] \tag{C.21}$$

With our definition of $\mathcal{J}_{\mathrm{LO}}$ in Eq. (C.6), we get:

$$P_{qq}^{(0)} = \left(\frac{\alpha_s}{2\pi}\right)\frac{2}{\epsilon}\frac{1+x^2}{x-1}. \tag{C.22}$$

The overall minus sign between eqns. (C.20) and (C.22) is due to the difference between the spacelike and timelike processes. It arises in the dz_i integral. Even though the z_i dependence is the same in the two calculations, and the integration limits are the same, 0 and $1-x$. For us, $1-x<0$, but in Ref. [39], it is positive.

We will compare the different contributions to double emission separately. In SCET_1, the C graph in Fig. B-6 will give box and crossed terms when interfered with itself and the A and B ones. We identify the crossed contribution by inserting the color structure and taking those terms proportional to $C_F^2 - \frac{1}{2}C_FC_A$. As mentioned above, it only contains the integrals in Table C.1. In terms of its notation, we have: The

Function defined in Eq. (C.6)	Value in crossed diagram
$A(z_1,z_2)$	$-\frac{16x\left(x^2+xz_1+(z_1-1)z_1+1\right)}{z_1(x+z_1-1)}$
$B(z_1,z_2)$	$\frac{8\left(x^2(z_1-2)-xz_1+z_1-1\right)}{x+z_1-1}$
$C(z_1,z_2)$	$\frac{8\left(x\left(x^2+(x-1)z_1+2\right)+z_1\right)}{z_1}$
$D(z_1,z_2)$	$16\left(x^2+1\right)$

Table C.3: Contributions to crossed amplitude squared diagram

box contribution additionally contains the functions in Table C.2, though we are only interested in the finite parts. Their z_i dependence is: For the crossed contribution,

Function defined in Eq. (C.6)	Value in box diagram
$A(z_1,z_2)$	$12x^2+8xz_1+8(z_1-1)z_1+12$
$B(z_1,z_2)$	$\frac{8(z_1-1)\left(x^2+(z_1-2)z_1+2\right)}{x+z_1-1}$
$C(z_1,z_2)$	$\frac{8(x+z_1)\left(2x^2+2xz_1+z_1^2+1\right)}{z_1}$
$D(z_1,z_2)$	0
$E(z_1,z_2)$	$4\left[\frac{\left(2x^4+6x^3z_1+x^2\left(7z_1^2+2\right)+2x\left(2z_1^3+z_1\right)+z_1^4+z_1^2\right)}{z_1^2}\right]$ $+4\epsilon\left[\frac{\left(x^2(x+z_1-1)^2+z_1^2\left((x+z_1-1)^2+x+z_1\right)+xz_1(x+z_1-1)^2\right)}{z_1^2}\right]$
$F(z_1,z_2)$	$4\left[\frac{\left(z_1^2-2z_1+2\right)\left(x^2+(z_1-1)^2\right)}{(x+z_1-1)^2}\right]$ $+4\epsilon\left[\frac{\left(x^2((z_1-1)z_1+1)+x(z_1-1)((z_1-2)z_1+2)+(z_1-1)^2((z_1-1)z_1+1)\right)}{(x+z_1-1)^2}\right]$

Table C.4: Contributions to box amplitude squared diagram

we perform the multiplication in Table C.1 with the functions defined in Table C.3 and integrate dz_1, having already done the trivial dz_2 integral. We again use a cutoff to avoid soft divergences, thus its range is between $1-x+\lambda$ and $-\lambda$. In the end, we obtain:

$$P^{(1)}_{qq\,\text{crossed}} = \left(\frac{\alpha_s}{2\pi}\right)^2\left[\left(\frac{1+x^2}{x-1}\right)\left(4\ln(x-1)-\ln^2(x)-\ln(\lambda)\right)-2(x+1)\ln(x)\right]. \tag{C.23}$$

The λ-dependent pieces will cancel against those from the box contribution. The other terms agree with [39] up to the previously discussed minus sign, and wherever $\ln(1-x)$ appears in the spacelike calculation, we get $\ln(x-1)$. Since our integrand and integration region are real, the imaginary pieces generated by $\ln(1-x)$ when making $x>1$ all must cancel.

The box calculation proceeds similarly except that we also include the terms proportional to $E(z_1,z_2)$ and $F(z_1,z_2)$ and perform the internal collinear subtraction, which changes the overall single pole term. Doing all this, we get:

$$P^{(1)}_{qq\,\text{box}} = \left(\frac{\alpha_s}{2\pi}\right)^2\left[\left(\frac{1+x^2}{x-1}\right)\left(\ln(\lambda)-\ln(x-1)\right)+2(2x-1)\ln(x)\right]. \tag{C.24}$$

The soft divergent pieces cancel against the crossed contribution, and once again we agree with [39] up to an overall sign, and the continuation $\ln(1-x) \to \ln(x-1)$.

In addition to these real emission contributions to the C_F^2 portion of $P_{qq}^{(1)}$, there are also single-emission, one-loop diagrams, Fig. C-3. We can account for their con-

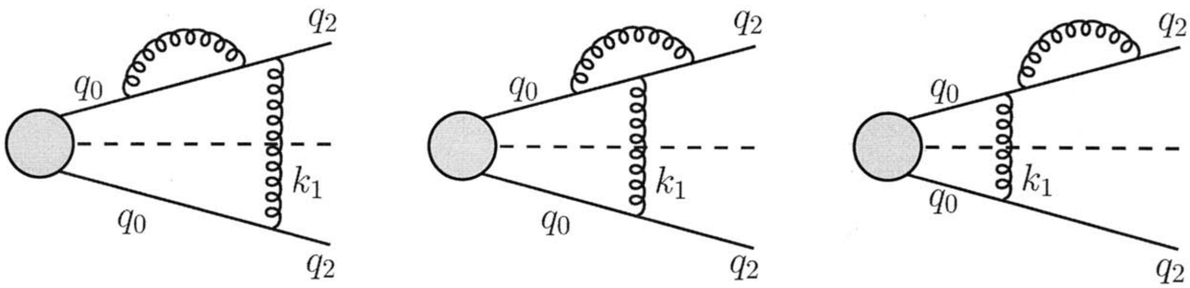

Figure C-3: Single emission, one-loop contributions to $P_{qq}^{(1)}$.

tributions in SCET easily. We have already derived the the same expression for single emission (Eqs. C.20 and C.22). Furthermore, both the quark wavefuncton renormalization and the vertex renormalization are the same in SCET as in QCD [10]. Thus, we recover the entire, gauge-invarant, $\propto \alpha_s^2 C_F^2$ contribution to the splitting function [39],

$$P_{qq\,\mathrm{abelian}}^{(1)} = C_F^2 \frac{\alpha_s^2}{2\pi} \left[(1-x)\ln(x) - \frac{3}{2}\frac{1+x^2}{1-x}\ln(x) - 2\frac{1+x^2}{1-x}\ln(x)\,\ln(1-x) \right.$$
$$\left. - \frac{1}{2}(1+x)\ln^2(x) - 5(1-x) - \frac{5}{2}(1+x)\ln(x) \right], \tag{C.25}$$

where we have written it with its usual sign conventions for spacelike evolution.

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