UNIVERSITÉ LAVAL

Domain Theory 101 : An ideal exploration of this domain

Mémoire

Loïc Ricaud

Maîtrise en informatique - avec mémoire
Maître ès sciences (M. Sc.)

Québec, Canada

Résumé

Les problèmes logiciels sont frustrants et diminuent l'expérience utilisateur. Par exemple, la fuite de données bancaires, la publication de vidéos ou de photos compromettantes peuvent affecter gravement une vie. Comment éviter de telles situations ? Utiliser des tests est une bonne stratégie, mais certains bogues persistent. Une autre solution est d'utiliser des méthodes plus mathématiques, aussi appelées méthodes formelles. Parmi celles-ci se trouve la sémantique dénotationnelle. Elle met la sémantique extraite de vos logiciels préférés en correspondance avec des objets mathématiques. Sur ceux-ci, des propriétés peuvent être vérifiées. Par exemple, il est possible de déterminer, sous certaines conditions, si votre logiciel donnera une réponse. Pour répondre à ce besoin, il est nécessaire de s'intéresser à des théories mathématiques suffisamment riches. Parmi les candidates se trouvent le sujet de ce mémoire : la théorie des domaines. Elle offre des objets permettant de modéliser formellement les données et les instructions à l'aide de relations d'ordre. Cet écrit présente les concepts fondamentaux tout en se voulant simple à lire et didactique. Il offre aussi une base solide pour des lectures plus poussées et contient tout le matériel nécessaire à sa lecture, notamment les preuves des énoncés présentés.

Abstract

Bugs in programs are definitively annoying and have a negative impact on the user experience. For example, leaks of bank data or leaks of compromising videos or photos have a serious effect on someone's life. How can we prevent these situations from happening? We can do tests, but many bugs may persist. Another way is to use mathematics, namely formal methods. Among them, there is denotational semantics. It links the semantics of your favorite program to mathematical objects. On the latter, we can verify properties, e.g., absence of bugs. Hence, we need a rich theory in which we can express the denotational semantics of programs. Domain Theory is a good candidate and is the main subject of this master thesis. It provides mathematical objects for data and instructions based on order relations. This thesis presents fundamental concepts in a simple and pedagogical way. It is a solid basis for advanced readings as well as containing all the needed knowledge for its reading, notably proofs for all presented statements.

Contents

List of Figures

Acknowledgments

Graduate studies require directors. In my case, Nadia Tawbi is my director and Josée Desharnais is my co-director. I thank them so much for the experience and for the time they invested in me. This master thesis is a journey that has started long ago when I finished my first undergraduate year. From this point, Nadia has been looking after me, advising me, pushing my capacities and offering me opportunities. Thank you Nadia for always believing in me! Josée has also been there a lot: supporting my internships, my scholarships and offering her expertise whenever I needed it. Allowing me to be an teaching assistant in her course MAT-1919 has been a great opportunity as well as developing my comprehension of concepts. Thank you Josée for these years. Overall, I thank my directors so much for everything!

During my master studies, I have been working at LSFM. I met coworkers that I would like to thank. I thank Andrew Bedford for his kindness and his help, Souad El Hatib for sharing her experiences and Alireza Yeganehparast for bringing a positive presence to the lab and practicing my English. Outside of LSV, I thank Mathieu Alain for being realistic and being a great classmate. I address a very special thanks to Gabriel Dion-Bouchard who has been my classmate for more than five years, my coworker for more than three years, my roommate, my confident and my supporter. Thank you!

During the summer of 2019, I did an internship at LSV in Cachan, France. Jean Goubault-Larrecq was my supervisor. I thank him for the extraordinary experience, the advices and the time he invested in me. At the same time, I met his post-doctorate student Zhenchao Lyu. I thank him for being an awesome coworker, an adviser and a great table tennis mate. Or, as they say in Chinese, “Xiè xie”! Finally, this internship was possible thanks to Josée and to Nadia. So, I thank them so much again!

My graduated studies and my internship in France have been funded by NSERC. I thank them for it!

As the Latin proverb goes, “Mens sana in corpore sano”. It means a healthy mind in a healthy body. For this, I thank the Rouge et Or alpine ski team and the supporting staff. Being a part of it made me push my limits and improve my physical skills. It brought balance to my graduate studies. Special thanks go to Sven Pouliot, the head coach. In the same way, I thank Julien Massicotte. He has always been a great source of inspiration and motivation as well as a friend. Thank you!

Finally, but not the least, I thank my family. Its precious members supported me during this journey, even when times were hard. My last thank is to Sophie Major, my awesome little girlfriend. She pushed me, helped me, read my drafts and supported me, which is hard sometime. Thank you for being a part of my life!

Introduction

Imagine yourself using a program. Somehow, someway, on this particular day and moment, it does not work. For example, you are a student and you cannot access your homework which is due in five minutes. Another example would be waiting for a bus which is already gone, while some application tells you that it is arriving. We can go into darker examples. You have to do radiation therapy. The program responsible for the amount of radiation has a fault and gives you way too much radiation. It leads into more problems than before. Another example is a burning car whose doors are locked because their managing program is in a deadlock state. Such program faults are called bugs and can ruin your day, to say the least... Are they avoidable? Some are, fortunately! How? Well, the trend is to do automatic testing such as unit testing or integration testing. However, tests only prevent some cases. For example, if some function takes a polymorphic parameter, then there should be tests for every form of the parameter. Doing so is hard and often not done. It is the same problem when some parameter can have infinitely many values. Many cases are left untested. Hence, we need to find a solution which checks any possible case.

Mathematics provides a framework in which we can do proofs about objects. How can we use this domain for our problem? We could extract some mathematical representation of a program, i.e., we have an object. We already have properties, i.e., propositions on objects. An example of a desired property is that, at any state, the program is not in a deadlock. Finally, we would verify the properties, i.e., make proofs. Such proofs validate the absence of bugs. Hence, we obtain hundred percent guarantee on the absence of undesired behaviors.

In this master thesis, we focus more on the formal representation of programs rather than on the verification of properties. A formal representation calls for a mathematical theory. In our case, it is Domain Theory. We choose it because it has three interesting characteristics. The first is type representation. It is necessary to represent the types used in programs. Note that among all the types, we find the functions. Hence, Domain Theory already provides a way of mathematically representing programs. The second is a simple way to describe computability. Indeed, mathematical functions are not all computable. Thus, we would like to know which ones in a formal representation are exportable to computers. The third is a finite description of infinite objects. Since the latter arise in mathematics and will, probably, in a formal program model, we want represent them finitely for computers. Hence, Domain Theory is the framework that we are interested in and that is explained in this master thesis, notably in Chapter 2. In order to fully appreciate the reading, we provide a mathematical basis from Order Theory in Chapter 1. Note that Domain Theory is also inspired by Topology Theory. We explore this in Chapter 3. In the same time, we introduce any notion needed from Topology Theory as is done for Order Theory in Chapter 1.

This master thesis is written to help beginners learn Domain Theory. To do so, we need a lot of mathematical statements, namely propositions and theorems. We provide their proofs in two ways. The first is a scheme that can be viewed as the concatenation of hints through sentences. The second is fully detailed proofs presented in Appendices A, B and C. We try to make it easy to read with a smooth progression through difficulties. Finally, we aim to provide a self-contained document.

Chapter 1

Order Theory

Order Theory is a fundamental area usually learned at the beginning of a computer science degree. This mathematical branch is about ordering elements of some set. The classical example is ordering numbers: 1 is lower than or equal to 2. Orders also lead to a proof technique called induction. More information can be found in the book [1]. In this chapter, we focus on orders and their ramifications. We first introduce orders in Section 1.1. This concept leads to bounds, or limits, of a set, structures which contain their bounds and closures to obtain such structures. These are the topics of Sections 1.2, 1.4 and 1.5. In Section 1.3, we explain how to reason on bounds. Finally, we look at the same concepts but this time with functions in Section 1.6 and 1.7. All these topics are needed in Order Theory for a smooth progression to Domain Theory in Chapter 2.

1.1 Orders

This section presents orders. For example, $\leq$ is a partial order on the naturals. We will also see other examples.

Intuitively, we can think of an order as a directed graph where the nodes represent the elements considered and the edges represent the order between elements. Smaller elements are put below greater elements in a graphical representation in Figure 1.1. Hence, the first element of a path is lower than or equal to any other element in the path, in particular the last element. Such graphs are in the style of Hasse diagrams (see [2] for more on Hasse diagrams). Note that Hasse diagrams are finite, but we have a lot of infinite graphs. Hence, our use of Hasse diagrams is more an extension of them. For simplicity, we will refer to finite and infinite diagrams as Hasse diagrams. A concrete example with natural numbers and an abstract example are shown in Figure 1.1. The second should be understood as $\circ$ is lower than or equal to $\triangle$, which is lower than or equal to $\star$, which is lower than or equal to something, etc.

We start the formalization of this intuition by definition 1.1.1. It presents three well-known properties that serve to formalize orders (e.g., [3], [4] and [5]).

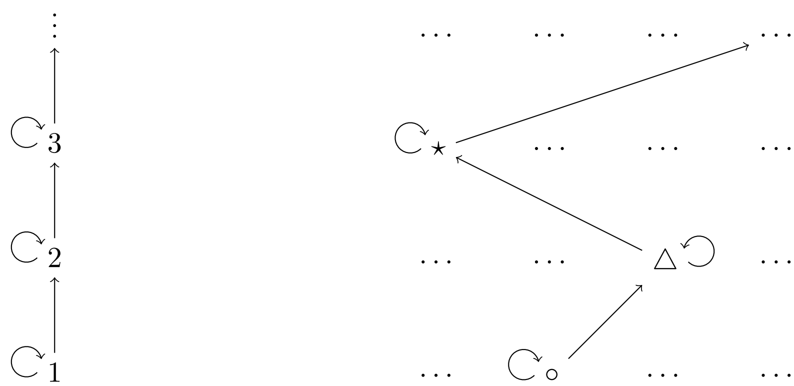

Figure 1.1: Examples of orders: natural numbers on the left and abstract on the right

Definition 1.1.1. *Let X be a set and $\mathcal{R} \subseteq X \times X$.*

1. *$\mathcal{R}$ is said to be* reflexive *if $(\forall x \in X \mid x\mathcal{R}x)$;*
2. *$\mathcal{R}$ is said to be* antisymmetric *if $(\forall x, y \in X \mid x\mathcal{R}y \Rightarrow (y\mathcal{R}x \Rightarrow x = y))$;*
3. *$\mathcal{R}$ is said to be* transitive *if $(\forall x, y, z \in X \mid x\mathcal{R}y \Rightarrow (y\mathcal{R}z \Rightarrow x\mathcal{R}z))$.*

What is the link between Hasse diagrams and the properties presented in Definition 1.1.1? Take a look at the loop over any element. It represents the fact that elements are lower than or equal to themselves, and greater than or equal to themselves. Hence, a loop over a single vertex represents the case of reflexivity. However, note that loops are usually absent in Hasse diagrams, because they are considered trivial or implicit. Antisymmetry is needed in order to choose at what height we should put an element. If an element x were to be greater than or equal to another element y and, at the same time, lower than or equal to y, then how would we arrange them in the graph? Would x be above y? Would y be above x? We would not know, but with antisymmetry we know they are the same element. Hence, such indecision about the arrangement of elements is impossible using antisymmetry. Finally, we want to say something like: if this number is lower than or equal to this one which is lower than or equal to this other one, then the first number should be lower than or equal to the third one. In other words, no matter how we choose a finite path in the diagram, a node in the path should be lower than or equal to the subsequent nodes. This is the reason why we need transitivity.

We have figured out intuitively what is an order. We now formalize the two classical types of ordered sets.

Definition 1.1.2 (Preordered set)**.** *A set X together with a relation $\mathcal{R} \subseteq X \times X$ is a* preordered set *if $\mathcal{R}$ is reflexive and transitive. In this case, $\mathcal{R}$ is called the preorder of X.*

Definition 1.1.3 (Poset)**.** *A set X together with a relation $\mathcal{R} \subseteq X \times X$ is a* partially ordered set*, poset for short, if $\mathcal{R}$ is reflexive, antisymmetric and transitive. In this case, $\mathcal{R}$ is called the partial order of X.*

Partial orders are often called "orders". However, we will stick with the "partial order" appellation to avoid any ambiguity. The reader may note that every partial order is a preorder.

Beside the partial order between naturals, there is another partial order used a lot between sets. It is the inclusion. Is it really a partial order? Let us look at the conditions. Any set is included in itself. Hence, inclusion is a reflexive relation. If we have two sets A and B such that $A \subseteq B$ and $B \subseteq A$, then A contains all elements of B and vice versa. By the axiom of extensionality, we know that this means $A = B$. Hence, inclusion is an antisymmetric relation. Finally, suppose we have three sets A, B and C such that $A \subseteq B$ and $B \subseteq C$. In this case, C contains all the elements of B. In particular, it contains the ones which are in A, because A is contained in B. Therefore, we have $A \subseteq C$, which means that inclusion is a transitive relation. Since inclusion is reflexive, antisymmetric and transitive, it is a partial order.

Writing P for the preorder and $\langle x, y \rangle \in P$ for the relation membership seems to be a burden. We need the simple notation 1.1.4 to capture this concept and Remark 1.1.5.

Notation 1.1.4. *Let (X, P) be a preordered set. We write $\sqsubseteq$ to note the preorder P on X. Moreover, we use it as an infix notation. For example, if $x, y \in X$, $x \sqsubseteq y$ means $\langle x, y \rangle \in P$. If there are multiple preorders in the context, we use subscripts to refer to their corresponding sets. Moreover, if we talk about any preorder $(X, \sqsubseteq)$, we omit $\sqsubseteq$. We just say that X is a preordered set and we use $\sqsubseteq$ as its preorder. Finally, if $x, y \in X$ and $x \neq y$, then we write $x \sqsubset y$ to mean $x \sqsubseteq y$ and $x \neq y$.*

The literature also uses "$\leq$" as notation for preorders, but we prefer "$\sqsubseteq$" to avoid confusion with the classical ordering on numbers.

Going back to an arbitrary preorder, how should we interpret and read it? Remark 1.1.5 answers this question.

Remark 1.1.5. *Let X be a preordered set and $x, y \in X$. If $x \sqsubseteq y$, then we say that x is lower than or equal to y with respect to $\sqsubseteq$. If $x \sqsubset y$, then we say that x is lower than y with respect to $\sqsubseteq$.*

We have defined what "lower than or equal to" is, but what does it mean to be "greater than or equal to"? This might seems trivial, but any concept has to be defined to avoid any confusion. Indeed, one can just unfold the definition of a concept whenever it is used in a proposition and verify its truth rather than deducing a definition that might not be the intended one. Let us return to the definition of "greater than or equal to". It is the same concept up to the reading way. Hence, in a preordered set X, for elements $x, y \in X$, saying "y is greater than or equal to x", noted $y \sqsupseteq x$, is the same as saying "x is lower than or equal to y". We can extend this reasoning to the opposite of a preordered set. Proposition 1.1.6 formalizes this thought which in return will produce dual notions in the next sections.

Proposition 1.1.6. *Let X be a preordered set (resp. poset). $(X, \sqsupseteq)$ is a preordered set (resp. poset).*

Proof. The three conditions of preorders are proved by rewriting and deducing the conditions from the properties of $\sqsubseteq$. $\square$

A full proof of Proposition 1.1.6 is written in Appendix A.1.

So far, we have seen basic preorders. We are ready to give more properties of them. For example, it is interesting to be able to compare any pair of elements. This is the totality property introduced in Definition 1.1.7.

Definition 1.1.7 (Total relation). *A binary relation $\mathcal{R}$ over a set X is* total *if*

$$(\forall r, s \in X \mid r\mathcal{R}s \vee s\mathcal{R}r \vee r = s)\,.$$

Since preorders are relations, they can be total. The order "$\leq$" on numbers is the classical example, because we can always say which number is lower than or equal to another. However, the reader must be careful to avoid falling into the trap of saying: if an element is not lower than or equal to another, then it is greater. This is only true for a total preorder, which is usually not the case. Hence, it is possible that two elements are incomparable. For example, consider the inclusion over the set $\mathcal{P}(\mathbb{N})$. As we saw, it is a partial order. Now, note that the sets $\{1, 2\}$ and $\{3, 4\}$ are incomparable, because neither $\{1, 2\} \subseteq \{3, 4\}$, nor $\{3, 4\} \subseteq \{1, 2\}$.

So far, we have already seen classical partial orders, i.e., the natural order $\leq$ over numbers and the inclusion relation $\subseteq$. Their reverses are still partial orders, so we have four examples. We give a fifth, a sixth and a seventh in Definitions 1.1.8, 1.1.9 and 1.1.10. For the latter, we exhibit its Hasse diagram on the alphabet $\{a, b\}$ in Figure 1.2. More examples are coming in the next sections.

Definition 1.1.8 (Discrete order). *The relation $\{\langle x, x\rangle \mid x \in X\}$ on a set X is the* discrete order.

The discrete order is reflexive, because it contains all the identical pairs. The discrete order is antisymmetric and transitive, because the preconditions are only fulfilled when elements are equals, hence the postconditions. Therefore, the discrete order is a partial order.

Definition 1.1.9 (Subgraph order). *The* subgraph order *on a collection $\mathcal{G}$ of graphs is the relation $\{\langle G, H\rangle \in \mathcal{G} \times \mathcal{G} \mid V(G) \subseteq V(H) \wedge E(G) \subseteq E(H)\}$, where $V(G)$ is the set of vertexes of G and $E(G)$ is the set of edges of G.*

A graph is its own subgraph, because inclusion is reflexive. Hence, the subgraph order is reflexive. The same goes for antisymmetry and transitivity. Both are consequences of inclusion being a partial order. Thus, the subgraph order is a partial order.

Definition 1.1.10 (Prefix order). *The relation $\{\langle u, v\rangle \in \Sigma^* \times \Sigma^* \mid (\exists w \in \Sigma^* \mid uw = v)\}$ on an alphabet Σ is the* prefix order.

A word is its own prefix. Indeed its suffix, the word "w" in Definition 1.1.10, is the empty word. Hence, the prefix order is reflexive. If a word u is the prefix of a word v and vice versa, then it means that the concatenated word in each case has to be empty. Otherwise, we can reach a contradiction. Hence, the prefix order is antisymmetric. Finally, it is transitive by concatenating both suffixes. Hence, the prefix order is a partial order.

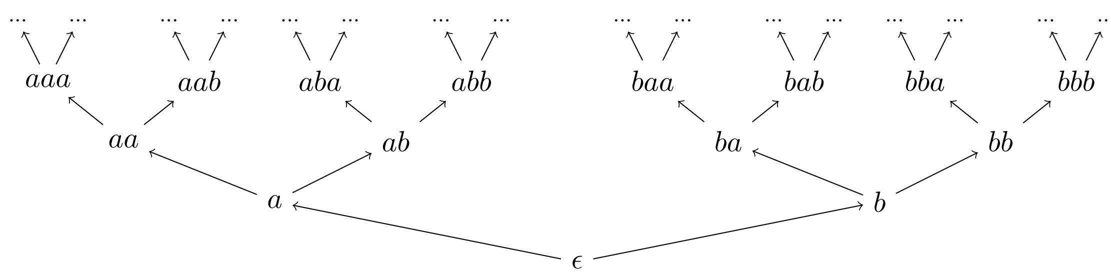

Figure 1.2: Hasse diagram of the prefix order on the alphabet $\{a, b\}$

1.2 Bounds

Now that we have defined what preordered sets are, it would be nice to talk about special elements. For example, which elements are lower than or equal to all others and the same thing for greater elements. In other words, we want to name some sets and some particular elements that bound a preordered set.

Definition 1.2.1 covers single elements with special characteristics and sets of limits. One must be careful when using these notions because, depending on the context, the desired elements might not exist.

Definition 1.2.1. *Let X be a preordered set and $A \subseteq X$.*

1. *An element $x \in X$ is an* upper bound *of A if $(\forall a \in A \mid a \sqsubseteq x)$.*
2. *An element $x \in X$ is a* lower bound *of A if $(\forall a \in A \mid x \sqsubseteq a)$.*
3. *The set of all upper bounds of A is*

$$\mathbf{ub}_X(A) ::= \{y \in X \mid (\forall a \in A \mid a \sqsubseteq y)\} \ .$$

4. *The set of all lower bounds of A is*

$$\mathbf{lb}_X(A) ::= \{y \in X \mid (\forall a \in A \mid y \sqsubseteq a)\} \ .$$

5. *The set of all least upper bounds of A is*

$$\mathbf{lub}_X(A) ::= \{y \in X \mid (\forall a \in A \mid a \sqsubseteq y) \wedge (\forall u \in \mathbf{ub}_X(A) \mid y \sqsubseteq u)\} \ .$$

6. The set of all greatest lower bounds of A is

$$\mathbf{glb}_X(A) ::= \{y \in X \mid (\forall a \in A \mid y \sqsubseteq a) \wedge (\forall l \in \mathbf{lb}_X(A) \mid l \sqsubseteq y)\} \ .$$

In the cases where X is deducible from the context, we write $\mathbf{ub}(A)$, $\mathbf{lb}(A)$, $\mathbf{lub}(A)$ or $\mathbf{glb}(A)$ without their ambient set X as subscript.

We illustrate Definition 1.2.1 on a set X equipped with the discrete order. Let $A \subseteq X$. If A contains more than one element, it has no upper bounds and no lower bounds. Indeed, an upper bound is "greater than or equal to" any element of A. But, by definition of the discrete order, this means equal to any element of A. Hence, if A contains more than one element, an upper bound would be equal to different elements which is impossible. If A contains one element, then the only upper bound is the element itself. The same reasoning apply to lower bounds.

Another example is the set $X = \{(ab)^n \mid n \in \mathbb{N}\}$ equipped with the prefix order. Consider the subset $A = \{(ab)^n \mid n \in \mathbb{N} - \{0\}\}$. The set A consists of words that are a repetition of ab one or more times, i.e., words of the form $abababab\ldots$. Among its lower bounds, there is the empty word. It is a prefix of any word. There is also the word ab, because any word is a repetition of it, that means a word begins with ab. To summarize, we have $\mathbf{lb}(A) = \{\epsilon, ab\}$. Note that the word "$a$" is not a lower bound of A, because it is not a word of X. We also have $\mathbf{ub}(A) = \emptyset$. Indeed, an upper bound needs to have all words of A as prefixes. This is impossible since A is infinite and X contains only finite words. More precisely, the sets of upper bounds of a subset of X is empty if the subset is infinite.

For the naturals with their usual order, lower bounds of a subset are all the elements "lower than or equal to" the least element of the subset. For example, the set of lower bounds of $\{n \in \mathbb{N} \mid 4 \leq n\}$ is $\{0, 1, 2, 3, 4\}$. For upper bounds, we have the same problem as for the prefix order. There is no upper bound for any infinite subset. For any finite subset, an upper bound is an element "greater than or equal to" the greatest element of the finite subset. For example, the set of upper bounds of $\{n \in \mathbb{N} \mid n \leq 100\}$ is $\{n \in \mathbb{N} \mid 100 \leq n\}$.

Least and greatest elements have special names, as pointed out in Notation 1.2.2. Examples of posets with such elements are the naturals equipped with their order "$\leq$", which are a poset with a least element, namely "0". The reals plus infinity equipped with their order "$\leq$" are a poset with a greastest element, namely "∞".

Notation 1.2.2. *The least element of a poset is usually called* bottom *and is noted $\bot$. The greatest element of a poset is usually called* top *and is noted $\top$.*

Be aware that Notation 1.2.2 uses a poset. The reader may also remark the use of the determinant "the" rather than "a". Indeed, in a poset, two least (resp. greatest) elements of the same subset are equal by antisymmetry. We enlighten this fact in Proposition 1.2.3. In a preorder, there could be more than one least element. They are referred as "minimal elements".

Proposition 1.2.3. *Let X be a poset and $A \subseteq X$. Then*

$$|\mathbf{lub}(A)| = 0 \vee |\mathbf{lub}(A)| = 1 \ .$$

Proof. If $\mathbf{lub}(A) = \emptyset$, then we are done. Thus, suppose $\mathbf{lub}(A) \neq \emptyset$. Then there exist $u, u' \in \mathbf{lub}(A)$. By definition, we have $u, u' \in \mathbf{lb}(A)$. Hence, by Definition 1.2.1 Item *5*, we have $u \sqsubseteq u'$ and $u' \sqsubseteq u$. By antisymmetry, we have $u = u'$. Therefore, there is only one element in $\mathbf{lub}(A)$. $\square$

Posets with a least element have a special name as Definition 1.2.4 points out. We will see that the pointed appellation is often used for structures having a least element.

Definition 1.2.4 (Pointed poset)**.** *A poset is a* pointed *poset if it has a least element.*

A bottom element can be used to create the flattening of a set. The procedure, defined in Definition 1.2.5, transforms a set into a partially ordered set by adding a bottom element. The term flat comes from the fact that elements are only comparable with bottom and themselves. Hence, as we show in the Hasse diagram of Figure 1.3, the set is flattened on a line with only a bottom element below.

Definition 1.2.5 (Flat partial order)**.** *For a set X, the* flat partially ordered set *of X is the pair $(X \cup \{\bot\}, \{\langle x, y\rangle \mid x = \bot \vee x = y\})$. Its partial order is the* flat partial order.

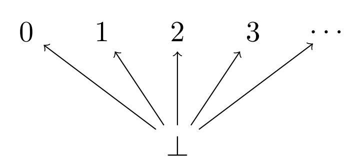

Figure 1.3: The flat naturals

Now that we have defined all the elements bounding a subset of a poset, we want to pick a least or a greatest one of them. They are named in Definition 1.2.6. Note that, by Proposition 1.2.3, we know that either they are unique or they do not exist at all.

Definition 1.2.6 (supremum and infimum)**.** *Let X be a poset and $A \subseteq X$ be such that the following bounds exist.*

1. *The* supremum *of A is its least upper bound, noted $\bigsqcup A$.*
2. *The* infimum *of A is its greatest lower bound, noted $\bigsqcap A$.*

Once again, be careful about existence. Proposition 1.2.3 is very explicit. Using supremum and infimum requires checking for existence before anything and everything. For example, $(\mathbb{R}, \leq)$ is a poset, but $\mathbb{R}$ itself has no supremum and no infimum.

The supremum is often call the *join* and can be noted $\bigvee$. The infimum is often called the *meet*, and can be noted $\bigwedge$. According to the website WolFramAlpha ([6]), the join and the meet appellations come from the algebraic side of Order Theory.

In this paragraph, we consider a poset X. In the powerset of X (i.e., $\mathcal{P}(X)$), there are two particular elements: $\emptyset$ and X. What happens when we use supremum and infimum functions on them? Answering this question is the goal of Proposition 1.2.7.

Proposition 1.2.7. *Let X be a poset.*

1. *If $\bigsqcup X$ exists, then X has a top element $\top$ and $\top = \bigsqcup X$.*
2. *If $\bigsqcap X$ exists, then X has a bottom element $\perp$ and $\perp = \bigsqcap X$.*
3. *If $\bigsqcup \emptyset$ exists, then X has a bottom element $\perp$ and $\perp = \bigsqcup \emptyset$.*
4. *If $\bigsqcap \emptyset$ exists, then X has a top element $\top$ and $\top = \bigsqcap \emptyset$.*

Proof. Each item is proved by unfolding the definitions. From there, we remark that upper bounds or lower bounds are either the whole set or the supposed existing bound. The rest of the proof is assigning the $\top$ or $\perp$ element to the assumed existing bound and verifying conditions of the definitions.

In each case, the bound exists by assumption. $\square$

A full proof of Proposition 1.2.7 is written in Appendix A.2.

1.3 Proof reasoning

A lot of proofs about suprema and infima are made via the antisymmetry property. The proof of Proposition 1.3.1 is a classical example using this argument.

Proposition 1.3.1. *Let X be a poset, $A, B \subseteq X$ and $\mathcal{U} \subseteq \mathcal{P}(X)$ be such that $\bigsqcup A$, $\bigsqcup B$, $\bigsqcap A$ and $\bigsqcap B$ exist.*

1. $A \subseteq B \Rightarrow \bigsqcup A \sqsubseteq \bigsqcup B$
2. $A \subseteq B \Rightarrow \bigsqcap B \sqsubseteq \bigsqcap A$
3. $\bigsqcup\left(\bigcup_{U \in \mathcal{U}} U\right) = \bigsqcup_{U \in \mathcal{U}}(\bigsqcup U)$

Before proving Proposition 1.3.1, we take a step back to reason about what to prove and how to do it.

When working with suprema and infima, we have to prove that they exist. In Proposition 1.3.1, we have an assumption telling us that infima and suprema exist. Suppose that we do not have the existence hypothesis. What can we do? We need to find some element that exists and prove that it is the supremum or the infimum. For the former, we need to prove that the element is an upper bound and that it is lower than or equal to any other upper bound. For the latter, we need to prove that the element is a lower bound and that it is greater than or equal to any other lower bound. In other word, we apply the definition of those functions.

In general, to prove that a supremum $\bigsqcup A$ is lower than or equal to an element e (i.e., $\bigsqcup A \sqsubseteq e$), the first step is to prove that e is an upper bound of A. To do so, it must be greater than or equal to every element of A. The second is to use the least property of suprema.

For the infimum, the recipe is nearly the same. We want to prove that an infimum $\bigsqcap A$ is greater than or equal to an element e (i.e., $\bigsqcap A \sqsubseteq e$). The first step is to prove that e is a lower bound of A, i.e., lower than or equal to every element of A. The second is to use the greatest property of infima.

For equality, antisymmetry is usually the tactic to use. Unfortunately, one direction is often a lot more difficult than the other.

We are ready to look at the proof of Proposition 1.3.1 in which we apply these recipes.

Proof. Let X be a poset, $A, B \subseteq X$ and $\mathcal{U} \subseteq \mathcal{P}(X)$ be such that $\bigsqcup A$, $\bigsqcup B$, $\bigsqcap A$ and $\bigsqcap B$ exist.

1. We have $\bigsqcup B \in \mathbf{ub}(A)$ because $A \subseteq B$ and $(\forall b \in B \mid b \sqsubseteq \bigsqcup B)$ by definition. Since $\bigsqcup A$ is the least upper bound of A, $\bigsqcup A \sqsubseteq \bigsqcup B$.

2. We have $\bigsqcap B \in \mathbf{lb}(A)$ using $A \subseteq B$. Since $\bigsqcap A$ is the greatest lower bound of A, $\bigsqcap B \sqsubseteq \bigsqcap A$.

3. Equality is proved via the antisymmetry property (def. 1.1.1).

   $\boxed{\sqsubseteq}$ We have $\big(\forall U \in \mathcal{U} \mid \bigsqcup U \sqsubseteq \bigsqcup_{U \in \mathcal{U}}(\bigsqcup U)\big)$ because a supremum is an upper bound. Hence, we deduce that $\big(\forall U \in \mathcal{U}, u \in U \mid u \sqsubseteq \bigsqcup_{U \in \mathcal{U}}(\bigsqcup U)\big)$ by transitivity. From this deduction, we have $\bigsqcup_{U \in \mathcal{U}}(\bigsqcup U) \in \mathbf{ub}(\bigcup_{U \in \mathcal{U}} U)$. Since $\bigsqcup\big(\bigcup_{U \in \mathcal{U}} U\big)$ is the least upper bound of $\bigcup_{U \in \mathcal{U}} U$ by definition, we have $\bigsqcup\big(\bigcup_{U \in \mathcal{U}} U\big) \sqsubseteq \bigsqcup_{U \in \mathcal{U}}(\bigsqcup U)$ as wanted.

   $\boxed{\sqsupseteq}$ Consider $U \in \mathcal{U}$. We have $\bigsqcup\big(\bigcup_{U \in \mathcal{U}} U\big) \in \mathbf{ub}(U)$ since $U \subseteq \bigcup_{U \in \mathcal{U}} U$. Since $\bigsqcup U$ is the least upper bound of U, we have $\bigsqcup U \sqsubseteq \bigsqcup\big(\bigcup_{U \in \mathcal{U}} U\big)$ by Item *1*.

   Since we considered an arbitrary $U \in \mathcal{U}$, the last property deduced is true for any $U \in \mathcal{U}$. Hence, we have $\bigsqcup\big(\bigcup_{U \in \mathcal{U}} U\big) \in \mathbf{ub}(\{\bigsqcup U \mid U \in \mathcal{U}\})$. Since $\bigsqcup_{U \in \mathcal{U}}(\bigsqcup U)$ is the least upper bound, we have $\bigsqcup_{U \in \mathcal{U}}(\bigsqcup U) \sqsubseteq \bigsqcup\big(\bigcup_{U \in \mathcal{U}} U\big)$ as wanted.

□

The purpose of the proofs of Proposition 1.3.1 is to develop proof reasoning. The reader may have figured out that the "$\sqsupseteq$" direction of Item *3* of Proposition 1.3.1 can be proved by using Item *1* of the same statement. Note that there is also a small use of suprema being least upper bounds.

Now, that we have these recipes, we can look at the two useful Propositions 1.3.2 and 1.3.4, which requires Definition 1.3.3. The first should be an easy exercise. The second is trickier and is a lot like a commutative property for the supremum. However, the key is to go step by step and to remember the proof reasoning we put in place above. When two sets or more are involved, the reader should be careful about the orders and the types. Most of the mistakes come from using the wrong one. Hence, using subscripts is a good idea.

Proposition 1.3.2. *Let X be a poset and $A, B \subseteq X$ be such that $\bigsqcup A$ and $\bigsqcup B$ exist, and $(\forall a \in A \mid (\exists b \in B \mid a \sqsubseteq b))$. Then*

$$\bigsqcup A \sqsubseteq \bigsqcup B\ .$$

Proof. It is a consequence of transitivity and suprema being least upper bounds. □

A full proof of Proposition 1.3.2 is written in Appendix A.3.

Definition 1.3.3 (Indexed sets)**.** *A set A is* indexed by *a set B if there is a surjective function f from B to A.*
In other words, when indexed sets are used, elements of the set A are described by an application of f to an element of the set B. The function f can be left implicit and a notation of the form a_b or $a(b)$ is for the element $a \in A$ such that $a = f(b)$, for some $b \in B$.
Note that a set can be indexed by itself. In this case, the surjective function can be the identity function (Definition 1.6.8).

Proposition 1.3.4. *Let A, B be sets and α be a poset indexed by A and B. If the following suprema exist and if $\alpha(a, b)$ is the element of α at the index $a \in A$ and $b \in B$, then*

$$\bigsqcup_{a \in A}\left(\bigsqcup_{b \in B}\alpha(a, b)\right) = \bigsqcup_{b \in B}\left(\bigsqcup_{a \in A}\alpha(a, b)\right)\ .$$

Proof. The first step is to use antisymmetry. The second step is to use the fact that a supremum is a least upper bound. Hence, we have to prove that one direction is an upper bound of the other direction without its first supremum. The third step is to use Proposition 1.3.2. We can because the remaining suprema are the same. The fourth step, easily forgotten, is to prove that the conditions of Proposition 1.3.2 are met. Note that both directions are symmetric modulo the sets used. Finally, the different suprema exist by assumption. □

A full proof of Proposition 1.3.4 is written in Appendix A.4.

1.4 Lattices

We have defined suprema and infima in Definition 1.2.6. We now define classical structures where these elements exist. We begin with two of those structures in Definition 1.4.1 and their merger in Definition 1.4.2.

Definition 1.4.1 (Semi-lattices)**.** *A partially ordered set is a*

- $\sqcup$-semi-lattice *if it is closed under suprema of finite subsets.*
- $\sqcap$-semi-lattice *if it is closed under infima of finite subsets.*

Definition 1.4.2 (Lattice)**.** *A partially ordered set is a* lattice *if it is closed under the supremum and the infimum functions for finite subsets.*

In other words, a lattice is a partially ordered set that is both a $\sqcup$-semi-lattice and a $\sqcap$-semi-lattice.

Lattices are often presented as being structures closed under suprema and infima of pairs of elements. Both definitions are equivalent and it can be proved by induction on the finite subsets' size. On the other hand, if we want to talk about any subset, including infinite ones, we need Definition 1.4.3.

Definition 1.4.3 (Complete lattice)**.** *A partially ordered set is a* complete lattice *if the supremum and the infimum of any of its subsets exist.*

Lattices can be drawn as extended Hasse diagrams as talked in Section 1.1. When a proof involves lattices, it is usually helpful to have a drawn scheme of the proof. This helps developing intuition for the proof writer and the proof readers.

We end this section by Proposition 1.4.4, about complete lattices. We give a graphical intuitive proof in Figure 1.4, as stated in the above paragraph.

Proposition 1.4.4. *Let L be a poset. L is a complete lattice if and only if the supremum of any subset of L exists.*

Proof. The "only if" part is a direct consequence of L being a complete lattice. For the "if" part, the infimum of a subset is the supremum of the lower bounds of the subset. A visual interpretation is shown in Figure 1.4. □

A full proof of Proposition 1.4.4 is written in Appendix A.5.

Some readers may wonder what happens when the set of lower bounds is empty in the previous proof. Proposition 1.4.4 still holds, because the supremum of the empty set exists by assumption. Hence, we have a bottom element as we stated in Proposition 1.2.7. Therefore, this case is impossible, because the bottom element is a lower bound for any element.

Examples of complete lattices are powersets with inclusion, the set of open sets in topology with inclusion and relations between any sets, seen as graphs, with the subgraph order. Complete lattices are also lattices. The set of naturals with their usual order is a lattice that is not complete. Indeed, any finite subset of $\mathbb{N}$ has a supremum and an infimum, because $\leq$ is total. However, it lacks a top element for infinite subsets. For example, the subset containing even numbers, or odd numbers, has no supremum.

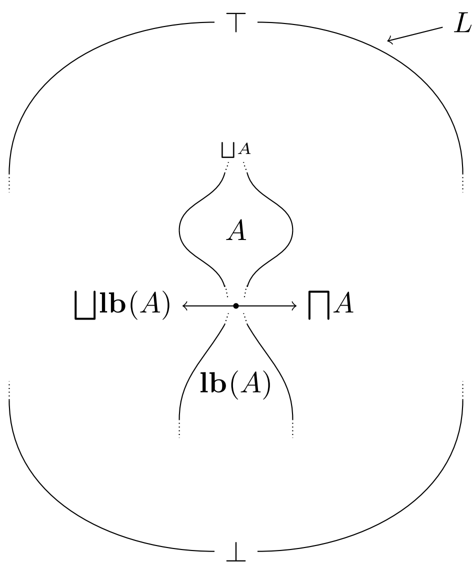

Figure 1.4: The infimum of a set A in a poset L is the supremum of its lower bounds

1.5 Upward and downward closures

So far, we have worked with upper bounds and lower bounds. We now work with elements that are greater than or equal (resp. lower than or equal) to at least one element of the related set. Definition 1.5.1 formalizes these concepts.

Definition 1.5.1. *Let X be a preordered set and $A \subseteq X$.*

1. *The* downward closure *of A is the set $\{x \in X \mid (\exists a \in A \mid x \sqsubseteq a)\}$, noted $\downarrow_X A$.*

2. *The* upward closure *of A is the set $\{x \in X \mid (\exists a \in A \mid a \sqsubseteq x)\}$, noted $\uparrow_X A$.*

3. *A is* downward closed *if $A = \downarrow_X A$.*

4. *A is* upward closed *if $A = \uparrow_X A$.*

5. *An element $x \in X$ is* maximal *if $((\uparrow_X \{x\})) \cap X = \{x\}$.*

6. *An element $x \in X$ is* minimal *if $((\downarrow_X \{x\}) \cap X) = \{x\}$.*

If the ambient set X is deducible from the context, then we omit it as a subscript.

We note $\downarrow x$ and $\uparrow x$ instead of $\downarrow \{x\}$ and $\uparrow \{x\}$ for any element x. Moreover, in the literature (e.g., [7]), upward closed sets are also called upper sets. Downward closed sets are called lower sets. We use the explicit name for easier reading.

Looking at Definition 1.5.1, we can observe that downward closure is the dual notion of upward closure. Hence, notions defined on a concept are usually dually defined on the other. Indeed, in the light of Proposition 1.1.6, we remark that an upward closed set (resp. downward closed set) is a downward closed set (resp. upward closed set) in the opposite preordered set. The reader should keep this remark in head as we advance through the subject.

Proposition 1.5.2 and 1.5.3 are classical facts about these two closures. They are also expected properties for closures, i.e., the closure contains the element from which it is obtained and the closure is idempotent.

Proposition 1.5.2. *Let X be a preordered set and $A \subseteq X$. Then $A \subseteq {\downarrow}A$ and $A \subseteq {\uparrow}A$.*

Proof. In each cases, all elements of A are lower than or equal to (resp. greater than or equal to) themselves by reflexivity. Hence they are in both closures. $\square$

Proposition 1.5.3. *Let X be a preordered set and $A \subseteq X$.*

1. ${\downarrow}A = {\downarrow}({\downarrow}A)$
2. ${\uparrow}A = {\uparrow}({\uparrow}A)$

Proof. Both items are consequences of reflexivity for the "$\subseteq$" part and transitivity for the "$\supseteq$" part. $\square$

A full proof of Proposition 1.5.3 is written in Appendix A.6.

At this point, we can picture an order and a lattice with an extended Hasse diagrams. Can we do the same for upward and downward subsets? Informally, we can as done in Figure 1.5. An upward closed set is something going up while a downward closed set is something going down. Those representations are mostly informal, because some cases do not correspond to them. For example, if an upward closed set do not have a bottom element or a downward closed set do not have a top element, then the representation presented in Figure 1.5 is not valid.

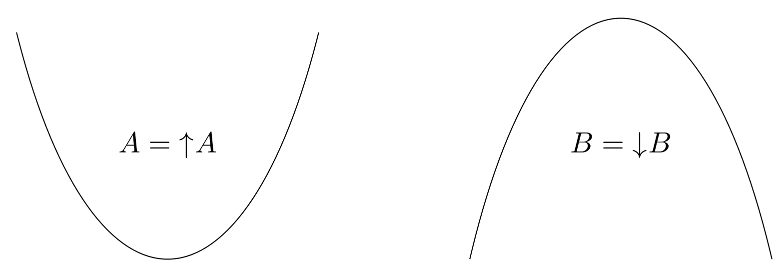

Figure 1.5: Informal representation of an upward closed set A on the left and a downward closed set B on the right

We head back to downward closure and upward closure being dual notions. We looked at this fact in terms of the opposite preordered set. We change perspective in Proposition 1.5.4 to look at it in the same preordered set.

Proposition 1.5.4. *In a preordered set X, a subset $A \subseteq X$ is upward closed if and only if $X - A$ is downward closed.*

Proof. The two directions are symmetric. We only prove the implication. Suppose that A is upward closed. Let $y \in X - A$. If an element lower than or equal to y is in A then y is in A by assumption. This would be a contradiction. Hence, all elements lower than or equal to y are in $X - A$. Thus, the latter is downward closed. $\square$

A full proof of Proposition 1.5.4 is written in Appendix A.7.

Finally, upward and downward closed sets have the wonderful property to be decomposable into their element closures. It is stated in Proposition 1.5.5. The proof is a good exercise and should be short and simple.

Proposition 1.5.5. *Let X be a preordered set and $A \subseteq X$.*

1. $A = \downarrow A \Rightarrow A = \bigcup_{a \in A} \downarrow a$
2. $A = \uparrow A \Rightarrow A = \bigcup_{a \in A} \uparrow a$

Proof. The "inclusion" direction is proved using reflexivity. The "reverse inclusion" direction is proved using the assumption about the closure of the subset. $\square$

A full proof of Proposition 1.5.5 is written in Appendix A.8.

Going back to suprema and infima from Section 1.2, we can exhibit simple facts in Proposition 1.5.6.

Proposition 1.5.6. *Let X be a poset and $A \subseteq X$ be such that the following bounds exist.*

1. $\bigsqcup A = \bigsqcup (\downarrow A)$
2. $\bigsqcap A = \bigsqcap (\uparrow A)$

Proof. Both are proved via antisymmetry. Proposition 1.3.1 and the hints given below it may be handy. Finally, the infima and the suprema exist by assumption. $\square$

A full proof of Proposition 1.5.6 is written in Appendix A.9.

Remark that the ambient preordered set X is upward and downward closed, because by definition we only consider elements inside X. Hence, preordered sets are good examples of upward and downward closed sets. We will meet more later.

1.6 Functions

Until this point, we were only interested by sets and ordering their elements. In this section, we concentrate on functions. We introduce some useful facts and notations. We first introduce Notation 1.6.1 which helps to note sets of functions.

Notation 1.6.1. *The set of functions from a set X to a set Y is noted $[X \to Y]$.*

We continue by Notation 1.6.2 which is used to abbreviate the application of a function to a subset of its domain.

Notation 1.6.2. *Let X, Y be sets and $f \in [X \to Y]$.*

1. *For any $A \subseteq X$, $f(A) = \{f(a) \mid a \in A\}$.*
2. *For any $y \in Y$, $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.*
3. *For any $A \subseteq Y$, $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$.*

We present with a new order presented in Definition 1.6.3 which is proved to be a preorder in Proposition 1.6.4.

Definition 1.6.3 (Pointwise order)**.** *Let P and Q be preordered sets. The pointwise order, noted $\sqsubseteq_{\mathrm{pt}}$, is the set $\{(f, g) \in [P \to Q] \times [P \to Q] \mid (\forall p \in P \mid f(p) \sqsubseteq_Q g(p))\}$.*

Remark that we impose P to be a preordered set in the previous definition. However, this hypothesis can be removed since it is unused. So why is it in Definition 1.6.3? It is just to stay in the spirit that pointwise relations are defined on the same kind of structures.

Proposition 1.6.4. *The pointwise order between preordered sets (resp. posets) is a preorder (resp. partial order).*

Proof. The two properties for preorder are proved using the preorder of the codomain. The proof of antisymmetry follows the same pattern. $\square$

A full proof of Proposition 1.6.4 is written in Appendix A.10.

The notation "$\sqsubseteq_{\mathrm{pt}}$" is specific to this master thesis. The subscript may add complexity, but we prefer to be unambiguous.

Looking at Definition 1.6.3 of the pointwise order, we have remarked that it only depends on the preorder of the codomain. Hence, we exhibit a link between bounds of the pointwise order and bounds of the codomain in Proposition 1.6.5. More explicitly, we exhibit a bottom element using the bottom element of the codomain or a top element using the top element of the codomain.

Proposition 1.6.5. *Let P be a set and Q be a poset.*

1. *If Q has a bottom element, then $([P \to Q], \sqsubseteq_{\mathrm{pt}})$ has a bottom element.*
2. *If Q has a top element, then $([P \to Q], \sqsubseteq_{\mathrm{pt}})$ has a top element.*

Proof. In the bottom case, the bottom element is the function mapping all elements of P on $\bot_Q$. In the top case, the top element is the function mapping all elements of P on $\top_Q$. $\square$

A full proof of Proposition 1.6.5 is written in Appendix A.11.

Functions have no special behavior on top or bottom elements. Hence, the image of a function can have a totally different structure than the domain. To force some kind of structure preservation, we need strict functions defined in Definition 1.6.6 and Notation 1.6.7.

Definition 1.6.6 (Strict function)**.** *A function between pointed posets is* strict *if it preserves least elements.*

Notation 1.6.7. *The set of strict functions between pointed posets P and Q is noted $[P \xrightarrow{\perp!} Q]$.*

One should note that Definition 1.6.6 is only about bottom elements. Have we forgotten top elements? No, because top elements are bottom elements in the reverse preorder. Hence, preservation of top elements can be expressed in terms of strictness on the reverse preorder.

We finish this section by giving classical examples of functions with interesting properties: the identity function and the constant functions.

Definition 1.6.8 (Identity function)**.** *The* identity *function on a set X, noted* **id***, is*

$$\begin{array}{rccc} \mathbf{id}: & X & \to & X \\ & x & \mapsto & x \end{array} \quad .$$

Definition 1.6.9 (Constant function)**.** *The* constant *function on $y \in Y$ between a set X and a set Y, noted c_y, is*

$$\begin{array}{rccc} \mathsf{c}_y: & X & \to & Y \\ & x & \mapsto & y \end{array} \quad .$$

Note that in Proposition 1.6.5, bottom and top elements are respectively the two constant functions $\mathsf{c}_\perp$ and $\mathsf{c}_\top$. Of course, it depends on the existence of $\perp$ and $\top$ in this case.

1.7 Monotonic functions

Order Theory requires a very nice property on functions: monotonicity. It is presented in Definition 1.7.1.

Definition 1.7.1 (Monotonic function)**.** *A function f between preordered sets P and Q is* monotonic *if $(\forall x, y \in P \mid x \sqsubseteq_P y \Rightarrow f(x) \sqsubseteq_Q f(y))$.*

Examples of monotonic functions are increasing functions. On the set of words of an alphabet equipped with the prefix order defined in Definition 1.1.10, the tail function is monotonic. The latter maps a sequence to the same sequence without its first element. To simplify things, we only consider sequences on the finite alphabet $\{a, b\}$ with the bottom element ϵ. If we consider that tailing ϵ returns ϵ, the tail function is monotonic. Take two elements related in Figure 1.2. Tailing them makes the elements go down one level. They are still related in the same direction, hence the monotonicity.

Note that monotonic functions preserve orders. A dual notion for reversing orders exists, but we will not use it. Just know that it exists.

We exhibit Corollary 1.7.2 about monotonic functions.

Corollary 1.7.2. *The set* $\left[P \xrightarrow{m} Q\right]$ *of monotonic functions between preordered sets (resp. posets) ordered pointwise is a preordered set (resp. poset).*

Proof. Monotonic functions are still functions. Hence it is a direct consequence of Proposition 1.6.4. $\square$

In the Definition 1.7.1 of monotonic functions, the assumption "$x \sqsubseteq_P y$" and the consequence is "$f(x) \sqsubseteq_Q f(y)$". They are respectively equivalent to $\bigsqcup \{x, y\} = y$ and to $\bigsqcup \{f(x), f(y)\} = f(y)$. By putting those two equations together, we obtain $\bigsqcup \{f(x), f(y)\} = f(y) = f(\bigsqcup \{x, y\})$. Hence there is a way to describe motononicity only in terms of distribution of a function over suprema. This is formalize in Proposition 1.7.3. Note that to use suprema, we need to work in posets.

Proposition 1.7.3. *A total function* f *between posets* P *and* Q *such that, for any* $A \subseteq P$ *such that* $\bigsqcup A$ *exists,* $f(\bigsqcup A) = \bigsqcup_{a \in A} f(a)$ *is monotonic.*

Proof. The proof is a consequence of the assumption as we justified above. $\square$

A full proof of Proposition 1.7.3 is written in Appendix A.12.

Chapter 2

Domain Theory

Domain Theory was invented by Dana Scott in his report [8] written in the early 70s. It was then published with more contents [9]. The goal was to provide semantics for the lambda calculus. Lambda calculus is as powerful as any programming language. Hence, we can use Domain Theory as a rigorous semantics for formal verification. This is our purpose as explained in the introduction.

Chapter 1 of Handbook of Logic in Computer Science [7] is the basis of this chapter. The latter aims to give an overview of classic Domain Theory. We present the basics of Domain Theory from Section 2.1 to Section 2.7, notably the way-below relation which is the base on which we build almost everything. Then, we conclude with a way to construct a domain from one of its adequate subsets in Section 2.8 and Section 2.9. We will see what are such *adequate* subsets in Section 2.5. Be patient until then!

After reading this chapter, the reader will be prepared to fully appreciate the reading of Chapter 1 of Handbook of Logic in Computer Science [7].

2.1 Directed and filtered sets

Now that we learned the basic notions, we start our exploration of Domain Theory. We start by defining sets with special properties in Definitions 2.1.1 and 2.1.2. Those sets are the foundation of Domain Theory.

Definition 2.1.1 (Directed set). *In a preordered set X, a subset $A \subseteq X$ is* directed *if $A \neq \emptyset$ and $(\forall a, b \in A \mid (\exists c \in A \mid a \sqsubseteq c \wedge b \sqsubseteq c))$.*

Definition 2.1.2 (Filtered set). *In a preordered set X, a subset $A \subseteq X$ is* filtered *if $A \neq \emptyset$ and $(\forall a, b \in A \mid (\exists c \in A \mid c \sqsubseteq a \wedge c \sqsubseteq b))$.*

Once again, we have a situation where dual notions appear. Hence, definitions based on one concept can be viewed on the other one. The one that makes statements easier to understand and to work with should be used.

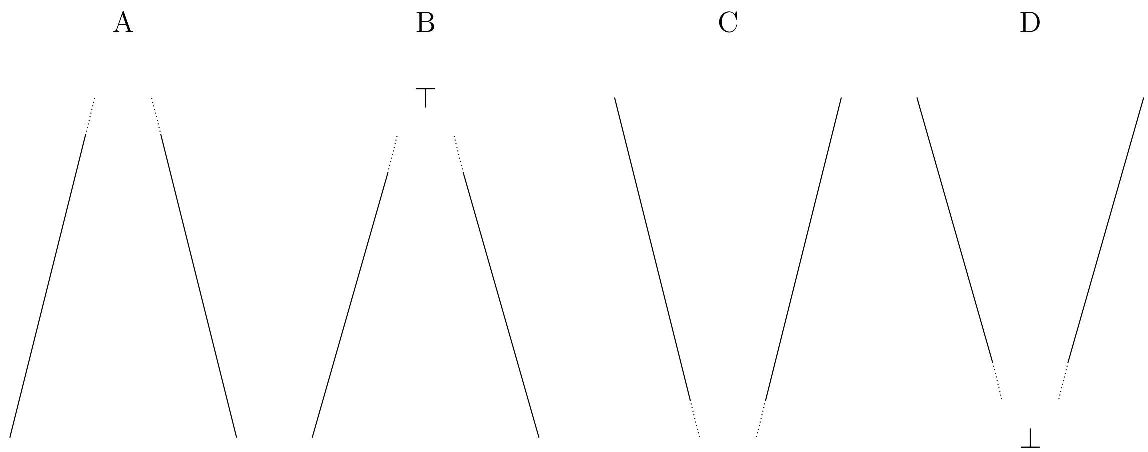

Figure 2.1: A: a directed set without a supremum, B: a directed set with a supremum, C: a filtered set without an infimum, D: a filtered set with an infimum

Definitions 2.1.1 and 2.1.2 are simple. However, the non-emptiness condition is easily forgotten during a proof.

Figure 2.1 contains a graphical representation of directed sets and filtered sets. The pictures are built using the definitions. For directed sets, think of it as having two elements and finding a greater one. Translate it in a Hasse diagram and a triangle is obtained. For filtered sets, the same process brings us to a reversed triangle. The set is an extension, i.e., a bigger triangle. However, we work with arbitrary preordered sets. Hence, we do not know if they have a supremum, or an infimum, or none of them. The reals equipped with their usual order are a good example. In this case, we are forced to let the top or the bottom of the triangle open. It goes on and on infinitely.

Directed sets are the main structure of study together with suprema. Hence, we use the commonly used Definition 2.1.3 to summarize those two aspects.

Definition 2.1.3 (Directed supremum)**.** *In a poset X, if a subset $A \subseteq X$ is directed and has a supremum, then the latter is a* directed supremum*, noted $\bigsqcup^{\uparrow} A$.*

Directed suprema are very useful to calculate arbitrary suprema as pointed out in Proposition 2.1.4.

Proposition 2.1.4. *Let X be a $\sqcup$-semi-lattice and $A \subseteq X$ such that $\bigsqcup A$ exists and $A \neq \emptyset$. Then*

$$\bigsqcup A = \bigsqcup^{\uparrow} \left\{ \bigsqcup F \mid F \subseteq A \ \wedge \ F \text{ is finite} \right\} .$$

Proof. First of all, we have to prove that $\{\bigsqcup F \mid F \subseteq A \ \wedge \ F \text{ is finite}\}$ is directed. Since $A \neq \emptyset$, then this set is also non-empty. For the second condition, we use the fact that the union of two finite sets is finite. So, it is in the set. The "$\sqsubseteq$" part is proved using Proposition 1.3.1 Item *1*. For the equality, we use antisymmetry. Every element of A is in its respective singleton, which is finite. Thus, it is lower than or equal to the directed supremum, which in return is an upper bound of A. Hence the "$\sqsubseteq$" direction is proved because the supremum is the least upper bound. The "$\sqsupseteq$" direction is directly proved, because the supremum is the least upper bound. Finally, all suprema exist by assumption. $\square$

A full proof of Proposition 2.1.4 is written in Appendix B.1.

Note that Proposition 2.1.4 implies that we can restrict our attention to finite subsets rather than the whole set to determine its supremum.

To add funkiness in this section, we present the nice Proposition 2.1.5. Intuitively, this proposition tells us that, whenever a finite set is included in a directed union of some sets, then it is already included in one of those sets.

Proposition 2.1.5. *Let X be a set, $M \subseteq X$ be a finite subset and $\mathcal{D} \subseteq \mathcal{P}(X)$ be directed with respect to $\subseteq$. Then*

$$M \subseteq \bigcup \mathcal{D} \Rightarrow (\exists D \in \mathcal{D} \mid M \subseteq D) \ .$$

Proof. By induction on the cardinality of M, we use the induction hypothesis to find some set containing all elements except one. The latter has to be in some other set. By directedness of $\mathcal{D}$, we find the subset D we are looking for. $\square$

A full proof of Proposition 2.1.5 is written in Appendix B.2.

In the same way, we introduce Proposition 2.1.6. It is an example where we extend the property of directedness on two elements to a finite set of elements. More precisely, a finite subset of a directed set has an upper bound in the latter.

Proposition 2.1.6. *Let X be a poset, $A \subseteq X$ be directed and $M \subseteq A$ be a finite subset. Then*

$$(\exists u \in A \mid u \in \mathbf{ub}(M)) \ .$$

Proof. We use the same pattern as Proposition 2.1.5. By induction on the cardinality of M, we use the induction hypothesis to find an upper bound for M except one element. By directedness of A, we find an upper bound for the whole set. $\square$

A full proof of Proposition 2.1.6 is written in Appendix B.3.

We continue this section by Proposition 2.1.7 using the notion of a monotonic function defined in Section 1.7. Interestingly, the image of a directed set by a monotonic function should be directed, this is why a monotonic function is also called *order preserving*.

Proposition 2.1.7. *Let P and Q be preordered sets, and $f \in \left[P \xrightarrow{m} Q\right]$ be monotonic. If a subset $A \subseteq \mathtt{dom}(f)$ is directed, then $\{f(a) \mid a \in A\}$ is directed.*

Proof. By directedness of A, $f(A)$ is non-empty.

For two elements of $f(A)$, we can find their corresponding elements in A. By directedness, we can find a "greater than or equal to" element. By monotonicity, the image of the latter is greater than or equal to both starting elements of the image. $\square$

A full proof of Proposition 2.1.7 is written in Appendix B.4.

Finally, we merge the new notions with those of Section 1.5. This leads to Definition 2.1.8.

Definition 2.1.8. *Let X be a preordered set, $x \in X$ and $A \subseteq X$.*

1. *A is an* ideal *if A is directed and downward closed.*
2. *An ideal is a* principal ideal *if it is the downward closure of a singleton.*
3. *A is a* filter *if A is filtered and upward closed.*
4. *A filter is a* principal filter *if it is the upward closure of a singleton.*

Any pointed poset is itself a filter. It is filtered using the bottom element and upward closure is satisfied by definition. Applying the same reasoning to the top element, we obtain ideals. Any non-empty totally preordered set is a filter and an ideal at the same time by totality. In this case, we do not need a top or bottom elements. The naturals with their usual order are such an example. In a collection of sets equipped with the inclusion ordering, if a non-empty subset is closed under finite unions (resp. finite intersections), then it is directed (resp. filtered).

Proposition 2.1.9 shows that Items *2* and *4* of Definition 2.1.8 are well defined.

Proposition 2.1.9. *Let X be a preordered set and $x \in X$.*

1. *$\downarrow x$ is a principal ideal*
2. *$\uparrow x$ is a principal filter.*

Proof. We only prove the case of a principal ideal, because arguments for principal filter are the same arguments applied dually.

By reflexivity, x is in its downward closure. It proves directedness. The downward closure is satisfied by Proposition 1.5.3 Item *1*. The singleton in the last condition is $\{x\}$. □

A full proof of Proposition 2.1.9 is written in Appendix B.5.

Ideals and filters are also defined in Order Theory for lattices. We concentrate our attention on the former. The latter, being a dual notion, is left as a good exercise for the reader. Order theoretic ideals are defined in Definition 2.1.10. One may observe that it is very close to Definition 2.1.8 Item *1*. Proposition 2.1.11 enlightens this point.

Definition 2.1.10 (Order theoretic ideal)**.** *A subset I of a lattice L is an* order theoretic ideal *if it is downward closed and $(\forall a, b \in I \mid \bigsqcup \{a, b\} \in I)$.*

Proposition 2.1.11. *Let L be a lattice and $D \subseteq L$. Then D is an ideal if and only if D is an order theoretic ideal.*

Proof. The "if" part is done by choosing the supremum as the "greater than or equal to" element. The "only if" part is done using the fact that the supremum is the least upper bound. $\square$

A full proof of Proposition 2.1.11 is written in Appendix B.6.

The notion of ideal is going to become very important in the last sections of this chapter. It will serve to construct a directed-complete partial order from one of its suitable subsets. Everything will be defined and put together in time.

2.2 Directed-complete partial order

When something is said to be complete, it often means that limits exists. We study completeness in the case of directed sets, presented in Section 2.1. In Definition 2.2.1, we define a poset where directed sets have a supremum.

Definition 2.2.1 (Directed-complete partial order)**.** *A poset is a* directed-complete partial order*, dcpo for short, if all its directed subsets have a supremum.*

Some may think that dcpos are rare but we meet them often. Complete lattices are dcpos by definition (see Definition 1.4.3). Topologies, which will be defined in Definition 3.1.1), or powersets ordered by inclusion are complete lattices. Hence, they are dcpos. Any finite poset is a dcpo. However, we must be careful to avoid falling in traps. The reals or the naturals ordered by their usual order (i.e., $\leq$) are not dcpos. Indeed, those sets are themselves directed. Hence, if those posets were dcpos, $\mathbb{N}$ and $\mathbb{R}$ should have a supremum, which is not the case. On the other hand, we can add the missing limits, i.e., $\mathbb{R} \cup \{\infty\}$ and $\mathbb{N} \cup \{\omega\}$ are dcpos. Note that "∞" and "ω" are two new elements greater than any other element. We also have the classic notations $\mathbb{R}^{\infty} = \mathbb{R} \cup \{\infty\}$ and $\mathbb{N}_{\omega} = \mathbb{N} \cup \{\omega\}$. In practice, adding limits, like we just did, is far from easy (e.g., Section 3 of the article [10]).

In the same way, we introduce the notion of ideal completion in Definition 2.2.2.

Definition 2.2.2 (Ideal completion)**.** *The* ideal completion *of a preordered set P is the set of all its ideals, noted* $\mathbf{Idl}(P)$*.*

Ideal completion can be viewed as adding limits. For example, the ideal completion of the naturals is isomorphic to $\mathbb{N}_{\omega}$. Indeed, any ideal is a principal ideal except one, $\mathbb{N}$ itself. Hence, the ideal completion of the naturals "contains" each natural in its principal ideal form and ω as the naturals themselves. The usual order between naturals is verified if and only if the inclusion between ideals is verified. Note that ω is greater than or equal to any natural, as wanted. Hence, we have an order isomorphism. Finally, it turns out that the ideal completion of a poset is a dcpo as stated in Proposition 2.2.3.

Proposition 2.2.3. *The ideal completion of a poset is a dcpo.*

Proof. The supremum of a directed set of ideals is their directed union, which is an ideal. □

A full proof of Proposition 2.2.3 is written in Appendix B.7.

We shall see other examples of dcpos as we continue our journey.

2.3 Scott-continuous functions

We saw monotonic functions in Section 1.7. They are order theoretic objects. We define Scott-continuous functions in Definition 2.3.1. They are the domain theoretic side.

Definition 2.3.1 (Scott-continuous function)**.** *A function f between dcpos is* Scott-continuous *if f is monotonic and, for any directed subset A of its domain, $f\left(\bigsqcup^{\uparrow} A\right) = \bigsqcup^{\uparrow}_{a \in A} f(a)$.*

Examples of Scott-continuous functions are the topological continuous functions in the Scott topology. We will learn this in Chapter 3. Other examples are any function with a mapping of the form "$n \mapsto n + c$" with $n, c \in \mathbb{N}_{\omega}$.

In the literature, Scott-continuous functions are also simply called continuous functions. However, we prefer the appellation Scott-continuous to avoid ambiguity with topological continuous functions or with classical continuous functions. Later, we will see that they can coincide under some assumptions.

Although Definition 2.3.1 requires monotonicity, the condition on directed suprema is enough. Proposition 2.3.2 enlightens this point. We deduce Corollary 2.3.3 from it.

Proposition 2.3.2. *Let f be a function between dcpos satisfying $f\left(\bigsqcup^{\uparrow} A\right) = \bigsqcup^{\uparrow}_{a \in A} f(a)$ for any directed subset A of its domain. Then f is monotonic.*

Proof. Since we start from comparable elements for monotonicity, they form a directed set. From this point, we use the assumption to conclude. □

A full proof of Proposition 2.3.2 is written in Appendix B.8.

Corollary 2.3.3. *Let f be a function between dcpos satisfying $f\left(\bigsqcup^{\uparrow} A\right) = \bigsqcup^{\uparrow}_{a \in A} f(a)$ for any directed subset A of its domain. Then f is Scott-continuous.*

Proof. Monotonicity is implied by Proposition 2.3.2. □

Actually, there is a third way to describe Scott-continuous functions. We state a very interesting fact about monotonicity in Proposition 2.3.4 and prove its link to Scott-continuity in Corollary 2.3.5. Everything relies on monotonicity. A monotonic function maps a supremum on a "greater than or equal to" element because the supremum is an upper bound and monotonicity preserves order.

Proposition 2.3.4. *Let P and Q be posets, $f \in \left[P \xrightarrow{m} Q\right]]$ be monotonic and $A \subseteq P$. If the following suprema exist, then*

$$\bigsqcup_{a \in A} f(a) \sqsubseteq_Q f\left(\bigsqcup A\right) .$$

Proof. We have $f(\bigsqcup A) \in \mathbf{ub}(\{f(a) \mid a \in A\})$ by monotonicity. Hence, we have $\bigsqcup_{a\in A} f(a) \sqsubseteq_Q f(\bigsqcup A)$, because the supremum is the least upper bound. $\square$

Corollary 2.3.5. *A monotonic function between dcpos is Scott-continuous if, for any directed subset A of its domain, $f\left(\bigsqcup^{\uparrow} A\right) \sqsubseteq \bigsqcup^{\uparrow}_{a\in A} f(a)$.*

Proof. Assume $f\left(\bigsqcup^{\uparrow} A\right) \sqsubseteq \bigsqcup^{\uparrow}_{a\in A} f(a)$, for any directed subset A. We have $\bigsqcup^{\uparrow}_{a\in A} f(a) \sqsubseteq f\left(\bigsqcup^{\uparrow} A\right)$ by Proposition 2.3.4. Hence, we have $f\left(\bigsqcup^{\uparrow} A\right) = \bigsqcup^{\uparrow}_{a\in A} f(a)$ by antisymmetry and we have f monotonic by assumption. We conclude that f is Scott-continuous. $\square$

To summarize, we have three ways to describe Scott-continuity: Definition 2.3.1, Corollary 2.3.3 and Corollary 2.3.5. It is not enough! Indeed, we will add a fourth way in Section 3.9. Concretely, we will only use Corollary 2.3.3, i.e., we will only show the condition on directed suprema to prove that a function is Scott-continuous.

Proposition 2.3.6 is an easy exercise to practice proving Scott-continuity. It will find its use later.

Proposition 2.3.6. *Any constant function is Scott-continuous.*

Proof. Let f be a constant function, $\mathtt{im}(f)$ be the element returned by applying f and A be a directed subset of its domain. Then $f\left(\bigsqcup^{\uparrow} A\right) = \mathtt{im}(f) = \bigsqcup^{\uparrow}_{a\in A} \mathtt{im}(f) = \bigsqcup^{\uparrow}_{a\in A} f(a)$ which proves Scott-continuity. $\square$

As we said in Section 2.2, we meet a new example of dcpo in Proposition 2.3.7.

Proposition 2.3.7. *The set of Scott-continuous functions between dcpos ordered pointwise is a dcpo.*

Proof. Let F be a directed set of Scott-continuous functions ordered pointwise. The directed supremum of F is the function mapping an element x to $\bigsqcup^{\uparrow}_{f\in F} f(x)$. The rest of the proof is to check the Scott-continuity condition, which is done by using Proposition 1.3.4. $\square$

A full proof of Proposition 2.3.7 is written in Appendix B.9.

2.4 Way-below relation

We have assembled the first pieces of Domain Theory with the notions of dcpo and Scott-continuous function. Both of them are really useful, but what can we say about the elements of a dcpo? Is there a way to talk about them without enumerating them? This would be quite convenient for a computer since its memory is finite and sets can be infinite. For this reason we need the fundamental way-below relation from Definition 2.4.1. We also draw the definition in Figure 2.2.

Definition 2.4.1 (Way-below relation)**.** *For a dcpo D and elements $x, y \in D$, x is* way below *y, noted $x \ll y$, if, for any $A \subseteq D$ directed, $y \sqsubseteq \bigsqcup^{\uparrow} A \Rightarrow (\exists a \in A \mid x \sqsubseteq a)$. If $x \ll y$, we also say that x approximates y. In the same way, for a subset $A \subseteq D$, we use $A \ll y$ to mean $(\forall a \in A \mid a \ll y)$. If needed, we add a subscript to note the ambient space of approximation, e.g., $x \ll_D y$ means that x approximates y in D.*

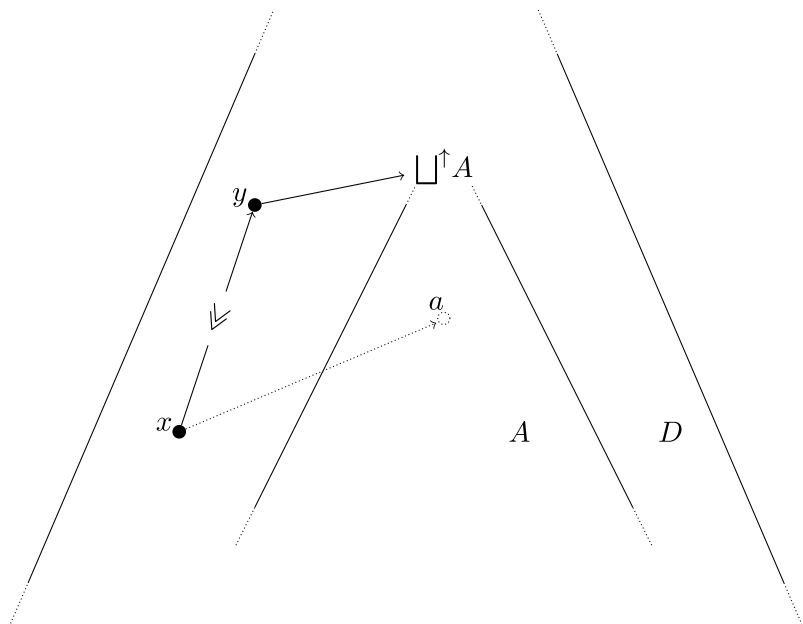

Figure 2.2: An example of an element $x \in D$ approximating an element $y \in D$ in a dcpo D with a directed subset A

Is the way-below relation reflexive? Unfortunately, it is not. To illustrate, consider the dcpo $\mathbb{R}^{\infty}$ with the natural ordering over the reals. Suppose "$\ll$" is reflexive. Then, we have $1 \ll 1$. However, consider the interval $[0, 1[$: it is a directed subset whose supremum is 1. Hence, there should be a real in $[0, 1[$ greater than or equal to 1. This is false. We can conclude that "$\ll$" is not reflexive. In this case, it is even irreflexive. Indeed, for any element $r \in \mathbb{R}^{\infty}$, suppose we have $r \ll r$. Then, in the directed interval $\{s \in \mathbb{R}^{\infty} \mid s < r\}$ that has r for supremum, there is an element greater than or equal to r. Since it is not the case, an element such as r does not exist. Then, is it irreflexive? No, it is not. To illustrate, we consider another dcpo: $\mathbb{N}_{\omega}$ with the natural ordering over naturals. Suppose "$\ll$" is irreflexive. Then we do not have $0 \ll 0$. However, since 0 is the bottom element of $\mathbb{N}_{\omega}$, it is lower than or equal to any element. Hence, the way-below condition is satisfied. We can conclude that "$\ll$" is neither reflexive, nor irreflexive.

In the dcpo $\mathbb{N}_{\omega}$ with the natural ordering over naturals, we said that $0 \ll 0$ in the previous paragraph. The justification was that 0 is a bottom element. Then, is any bottom element approximating itself? We prove it later in Proposition 2.4.7, but, to do so, we need more knowledge.

The way-below relation is not a reflexive order. However, what can we say about antisymmetry and transitivity? We can say that the way-below relation is antisymmetric and transitive. To see why, we need Proposition 2.4.2.

Proposition 2.4.2. *Let D be a dcpo and $x, x', y, y' \in D$.*

1. $x \ll y \Rightarrow x \sqsubseteq y$
2. $x' \sqsubseteq x \ll y \sqsubseteq y' \Rightarrow x' \ll y'$

Proof. Let D be a dcpo and $x, x', y, y' \in D$.

1. The singleton $\{y\}$ is directed and its supremum is y. If $x \ll y$, there is an element in $\{y\}$ greater than or equal to x. It can only be y. Hence, we have $x \sqsubseteq y$ as wanted.
2. Consider any directed set with supremum greater than or equal to y'. By transitivity, it is greater than or equal to y. Since $x \ll y$, there is an element of the directed set greater than or equal to x. By transitivity, this element is again greater than or equal to x'. We have proved the way-below condition. Hence $x' \ll y'$.

□

Although Proposition 2.4.2 is simple, it will be very useful in proofs.

We can now conclude about the antisymmetry and the transitivity of the way-below relation. They both hold as stated in Proposition 2.4.3.

Proposition 2.4.3. *The way-below relation over a dcpo is antisymmetric and transitive.*

Proof. Antisymmetry is a consequence of Proposition 2.4.2 Item *1* and antisymmetry of the partial order of the dcpo. Transitivity is a consequence of applying Proposition 2.4.2 Item *1*, then Item *2* with the clever substitutions. □

A full proof of Proposition 2.4.3 is written in Appendix B.10.

We add useful and classic extensions in Definitions 2.4.4 and 2.4.5 as well as new notations.

Definition 2.4.4. *Let D be a dcpo and $x \in D$.*

1. *An element $y \in D$ is an* approximant *of x if $y \ll x$.*
2. *An element $y \in D$ is* approximated *by x if $x \ll y$.*
3. *The* set of approximants *of x is the set $\{y \in D \mid y \ll x\}$, noted $\twoheaddownarrow x$.*
4. *The* set of approximated elements *by x is the set $\{y \in D \mid x \ll y\}$, noted $\twoheaduparrow x$.*
5. *The* order of approximation *on D is $\ll_D$.*

Definition 2.4.5 (Compact element)**.** *An element of a dcpo is* compact *if it is way-below itself. The set of compact elements of a dcpo D is noted $K(D)$.*

Some reader may have encountered compact elements under the name of "finite elements". The choice of naming will become clearer in Section 3.8.

To conclude this introduction on the way-below relation, we propose to exercise by proving Propositions 2.4.6, 2.4.7 and 2.4.9, and the little Corollary 2.4.8.

Proposition 2.4.6. *Let D be a dcpo and $x \in D$. Then*

1. $\twoheaduparrow x \subseteq \uparrow x$;
2. $\twoheaddownarrow x \subseteq \downarrow x$;
3. $x \in K(D) \Rightarrow \uparrow x = \twoheaduparrow x$;
4. $x \in K(D) \Rightarrow \downarrow x = \twoheaddownarrow x$.

Proof. It is a consequence of Proposition 2.4.2. □

A full proof of Proposition 2.4.6 is written in Appendix B.11. Moreover, it is a classical example of the utility of Proposition 2.4.2.

Proposition 2.4.7. *The bottom element of a pointed dcpo approximates any element.*

Proof. The bottom element is lower than or equal to every element. Hence, the way-below condition is trivially satisfied. □

Corollary 2.4.8. *The order of approximation of a pointed dcpo is non-empty.*

Proof. The way-below relation contains all the pairs of the form $(\perp, x)$ for any element x, since $\perp \ll x$ by Proposition 2.4.7. □

Proposition 2.4.9. *In a complete lattice, the approximants of an element form a $\sqcup$-semilattice.*

Proof. The supremum of a finite set A of approximants is an approximant too. Indeed, in a directed set whose supremum is greater than or equal to the approximated element, we use Proposition 2.1.6 to find an upper bound of A and the supremum being the least upper bound to conclude the proof. □

A full proof of Proposition 2.4.9 is written in Appendix B.12.

2.5 Basis

This section continues on the goal of expressing elements without enumerating them. We start with Definition 2.5.1.

Definition 2.5.1 (Basis)**.** *A subset B of a dcpo D is a* basis *of D if, for any element $x \in D$, there exists a directed subset of $B_x = B \cap \twoheaddownarrow x$ with supremum x.*

What does a basis look like? We go back to the previous examples of dcpos and try to find a basis for each.

- We argue that $\mathbb{N}$ is a basis for $\mathbb{N}_\omega$ with its natural order has a basis.

  1. Note that any non-empty finite subset of $\mathbb{N}_\omega$ contains its supremum. On the other hand, any infinite subset of $\mathbb{N}_\omega$ has "ω" for supremum.
  2. We argue that $B_n = \downarrow n$, for any $n \in \mathbb{N}$. The approximants of any $n \in \mathbb{N}$ are all the elements lower than or equal to it, then $\twoheaddownarrow n = \downarrow n$. Hence, the set of approximants is included in the basis, i.e., $\downarrow n \subseteq \mathbb{N}$. Thus, we have $B_n = \downarrow n$.
  3. We argue that B_n is directed, for any $n \in \mathbb{N}$. Since $\downarrow n$ is directed by Proposition 2.1.9 Item *1*, B_n is directed.
  4. We argue that $\bigsqcup^{\uparrow} B_n = n$, for any $n \in \mathbb{N}$. We have $n = \bigsqcup^{\uparrow} \downarrow n$. Hence $\bigsqcup^{\uparrow} B_n = n$.
  5. We argue $\bigsqcup^{\uparrow} B_\omega = \omega$. The approximants of "$\omega$" are all the naturals, then $\twoheaddownarrow \omega = \mathbb{N}$. Thus $B_\omega = \mathbb{N}$. The latter is directed and has "ω" for supremum. Hence $\bigsqcup^{\uparrow} B_\omega = \omega$

- We argue that $\mathbb{Q}$ is a basis for $\mathbb{R}^\infty$ with the natural order "$\leq$".

  1. Note that we argued that the way-below relation is irreflexive on $\mathbb{R}^\infty$ with the natural order "$\leq$" in Section 2.4 below Definition 2.4.1.
  2. We argue that $q \ll r \iff q < r$. The "$\Rightarrow$" direction is already proved by Proposition 2.4.2 Item *1*. Hence we focus on the "$\Leftarrow$" direction. For a real r and a rational q such that $q < r$, suppose we have $A \subseteq \mathbb{R}^\infty$ such that $\bigsqcup^{\uparrow} A = r$. We have two choices for A: either A contains r, or A contains an interval of the form $[s, r[$ with $s \in \mathbb{R}^\infty$. In both cases, there exists a real greater than or equal to q, as wanted.
  3. We argue that B_r is directed, for any $r \in \mathbb{R}$. Note that $B_r = \{q \in \mathbb{Q} \mid q < r\}$ by Item 2 of this argument. The proof follows from the fact that there is always a rational lower than or equal to a real for the emptiness condition and from totality of $\leq$ for the second condition.
  4. We argue that $\bigsqcup^{\uparrow} B_r = r$, for any $r \in \mathbb{R}$. We have that r is an upper bound of $\{q \in \mathbb{Q} \mid q < r\}$; hence of B_r. We also know that $\{q \in \mathbb{Q} \mid q < r\}$ is directed. Therefore, its supremum exists, call it u. Suppose u is lower than r. Then there exists a rational s between u and r, because rationals are dense in the reals. But $s \in \{q \in \mathbb{Q} \mid q < r\}$; hence B_r. We have reached a contradiction, because u is an upper bound. Therefore, we conclude that $\bigsqcup^{\uparrow} B_r = r$.
  5. We argue $\bigsqcup^{\uparrow} B_\infty = \infty$. The approximants of "$\infty$" are all the rationals, then $\twoheaddownarrow \infty = \mathbb{Q}$. Thus $B_\infty = \mathbb{Q}$. The latter is directed and has "∞" for supremum. Hence $\bigsqcup^{\uparrow} B_\infty = \infty$

- For any finite poset, the whole set is the basis, because any element approximates itself. Indeed, a directed finite set contains its supremum. This can be shown by induction on the cardinality of the set or as a corollary of Proposition 2.1.6. Hence, all elements are compact since, in this case, verifying the precondition of the way-below relation implies verifying its postcondition.

Proposition 2.5.2 provides useful facts on bases. It even adds more details on the structure of a basis and of its intersection with approximants. We add Corollary 2.5.3 about the greatest basis in a dcpo.

Proposition 2.5.2. *Let D be a dcpo and $B \subseteq D$ be a basis.*

1. *For any element $x \in D$, B_x is directed and $\bigsqcup^{\uparrow} B_x = x$.*

2. *$K(D) \subseteq B$.*

3. *For any subset $B' \subseteq D$, if $B \subseteq B'$ then B' is also a basis of D.*

Proof. Let D be a dcpo and $B \subseteq D$ be a basis.

1. Let $x \in D$. For directedness of B_x, we use the approximant property on the directed subset of B_x which has x as supremum. It returns two elements of a directed set. Hence, there is a greater than or equal to element which turns out to be the element we are looking for. The element x being the supremum is a consequence of Proposition 1.3.1 and 2.4.2 Item *1*.

2. Let $c \in K(D)$. We use the compact property to find an element $c' \in B_c$ greater than or equal to c. However, it is only possible if c' is c because c is the supremum of B_c.

3. Because a supset of a basis contains it, it also contains the directed set for all elements. Hence, the condition for a basis is respected.

□

A full proof of Proposition 2.5.2 is written in Appendix B.13.

Corollary 2.5.3. *In a dcpo D with basis B, D itself is the greatest basis.*

Proof. This is a direct consequence of Proposition 2.5.2 Item *3*. □

Considering Proposition 2.5.2, it is possible that a basis only contains compact elements. Hence, we introduce Notation 2.5.4 to put the emphasis on the compact elements of the basis that are approximating some element.

Notation 2.5.4. *In a dcpo D where $K(D)$ is a basis, for any element $x \in D$, the set $K(D) \cap {\downarrow} x$ is noted $K(D)_x$.*

In other words, Notation 2.5.4 stresses the fact that the compact elements are the basis.

Another feature of the basis concerns Scott-continuity. Thanks to Proposition 2.5.2 Item *1*, we can decompose elements into their directed basis elements. Hence, if we know the behavior of a Scott-continuous function on basis elements, then we can infer its behavior on all elements by looking at a subset of the basis. We formalize this intuition in Proposition 2.5.5.

Proposition 2.5.5. *The behavior of a Scott-continuous function on a dcpo with a basis is determined by its behavior on basis elements.*

Proof. It is a consequence of Scott-continuity and Proposition 2.5.2 Item *1*. □

A full proof of Proposition 2.5.5 is written in Appendix B.14.

2.6 Domains

We have everything we need to introduce domains in Definition 2.6.1.

Definition 2.6.1 (Domain)**.** *A structure that has notions of convergence and approximation is a* domain*.*

Dcpos are domains. They fit the definition perfectly with the existence of suprema of directed sets and the way-below relation.

The notion of domain is more abstract than the notions of dcpo and way-below relation. However, when reading domain theoretic articles as this one, the reader must be careful, because sometimes "domain" is just a different way of saying "dcpo". Note that domains could be more than dcpos, for example lattices that are dcpos. To avoid any ambiguity, we are only using "dcpos" rather than "domains".

The notion of basis, presented in Section 2.5, is very important for dcpos. There are two principal kinds of dcpos with basis presented in Definitions 2.6.2 and 2.6.3.

Definition 2.6.2 (Continuous dcpo)**.** *A dcpo is* continuous *if it has a basis. If the basis is countable, then the dcpo is* ω-continuous*.*

Definition 2.6.3 (Algebraic dcpo)**.** *A dcpo is* algebraic *if its compact elements form a basis. If the basis of compact elements is countable, then the dcpo is* ω-algebraic*.*

As often, there is another equivalent way of defining a continuous dcpo (resp. algebraic dcpo). It is to say that the approximants (resp. compact approximants) of some element are directed and have the latter as supremum. It is formalized in Proposition 2.6.4.

Proposition 2.6.4. *Let D be a dcpo.*

1. *D is continuous if and only if* $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} \twoheaddownarrow x\right)$.
2. *D is algebraic if and only if* $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} K(D)_x\right)$.

Proof. The algebraic item is easier. Indeed, the "only if" direction is a direct consequence of Proposition 2.5.2 Item *1* while the "if" direction is exactly like the continuous item. Hence, we focus on the latter. The ideas for algebraic remain the same and so does the proof except the use of $K(D)$ rather than a basis B.

The proof of the "only if" part is similar to the one of Proposition 2.5.2 Item *1*. For directedness, $\twoheaddownarrow x$ is not empty since it contains B_x, which is not empty by directedness. For the second condition, we apply the definition of approximation to deduce the existence of two elements in B_x greater than or equal to the two approximants and apply directedness of B_x to prove the existence of the element we are interested in. For the supremum, we use antisymmetry. The "lower than or equal to" direction is done by using Proposition 1.3.1 Item *1* and the fact that B_x is included in $\twoheaddownarrow x$ by definition. The "greater than or equal to" direction is a consequence of Proposition 2.4.2 Item *1*.

The "if" direction is done by choosing D itself as the basis. The condition of basis is proved by choosing the set D_x itself as the directed subset with x as supremum. Note that in this case we have $D_x = \twoheaddownarrow x$. □

A full proof of Proposition 2.6.4 is written in Appendix B.15.

Proposition 2.6.4 comes in handy for many proofs. Depending on the statement to prove, using it for continuity rather than its definition can make the proof simpler. If the reader is stuck with one, trying the other is a good idea.

As always, it is interesting to have examples. For continuous dcpos, we have all those of Section 2.5. We can add continuous lattices ([11]) as well as algebraic dcpos. The latter are continuous by definition. We also give a visual representation of a dcpo without a basis in Figure 2.3. Indeed, the order of approximation of this dcpo is empty. Without loss of generality, we focus on the left branch composed of circles of this dcpo. A circle does not approximate any other element of the dcpo. Indeed, any element is lower than or equal to the top element. Therefore, if a circle were to approximate an element, then we could consider the directed subset consisting of the triangles, excluding $\top$. Its directed supremum is the top element. Thus, there should be a triangle greater than or equal to the approximant circle. The last affirmation is impossible to satisfy since the two branches are disjoint. Hence, the order of approximation is empty and there is no basis.

As we stated when we were looking for a basis in a finite poset, all elements of a finite poset are compact. Hence, a finite poset is an algebraic dcpo. We get three more examples from Chapter 1 of Handbook of Logic in Computer Science [7]:

1. Any set equipped with the discrete order (see Definition 1.1.8) is an algebraic dcpo. Indeed any element approximates itself and only itself, because it is incomparable with the other elements. Hence, the basis is the whole set and contains only compact elements.

2. Functions can be represented as bipartite graphs. One side is the domain and the other side is the image. The set $[X \rightharpoonup Y]$ of partial functions between the sets X and Y

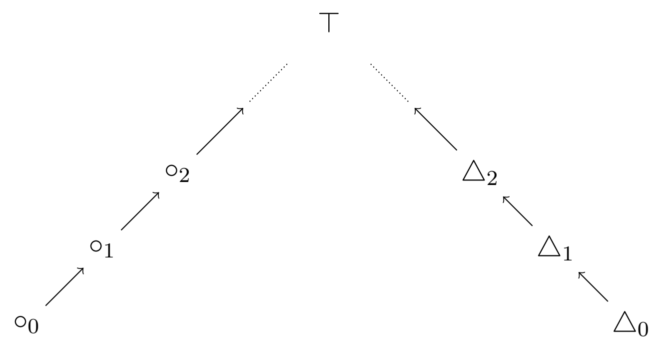

Figure 2.3: A non-continuous dcpo

ordered by subgraphs (see Definition 1.1.9) is an algebraic dcpo. Compact elements are functions with a finite carrier, i.e., with a finite domain. This appellation is confusing in the context, hence the need for another name. The supremum of a directed subset of partial functions ordered by subgraphs is their graph union. This union is directed. Hence, we can apply Proposition 2.1.5 to partial functions with a finite carrier. It proves that they are compact. Now we argue that functions with a finite carrier are a basis, say B. To be so, they have to be the directed supremum of the sets B_f, for all partial functions f. This is true, thanks to Proposition 2.1.4. The directedness of the sets B_f is proved by the fact that a union of two finite graphs is a finite graph and there is a bottom element. The latter is the function with an empty carrier.

3. The ideal completion of a poset P is an algebraic dcpo. We proved that it is a dcpo in Proposition 2.2.3. Its compact elements are the downward closure of singletons, namely principal ideals. Indeed, we know that the supremum of a directed subset of $\mathbf{Idl}(P)$ is its directed union. The latter contains particularly the unique element from which the principal is obtained, call it x. If a union contains x, it means that x is also in some ideal of the directed subset. Because an ideal is downward closed by definition, it contains $\downarrow x$. Another way of seeing it is to apply Proposition 2.1.5 with the singleton $\{x\}$ as the finite set. Then, by downward closure, the elements lower than or equal to x are also in the subset found. We have compact elements, but do they form a basis? A compact element C approximates an ideal I if C is included in I. We deduce that the sets B_I for all ideals I are of the form $\{\downarrow x \mid x \in I\}$. They are directed, because the ideals, from which they are obtained, are directed. The supremum is the union, which means that I is the union, as we saw in Proposition 1.5.5. Hence, we just proved textually that, for any poset P, $\mathbf{Idl}(P)$ is a dcpo.

2.7 Properties of dcpos

In this section, we give some properties on the concepts presented so far. It is inspired by Section 2.2.5 of Chapter 1 of Handbook of Logic in Computer Science [7].

Now that we know what dcpos are, we go back to bases. Elements of the basis of a continuous dcpo can serve as witnesses for the order. We illustrate this fact in Proposition 2.7.1. Note that we give an intuitive representation of the contraposition (i.e., $x \not\sqsubseteq y \iff (\exists b \in B_x \mid b \notin B_y)$) in Figure 2.4.

Proposition 2.7.1. *Let D be a dcpo, $B \subseteq D$ be a basis and $x, y \in D$. Then*

$$x \sqsubseteq y \iff B_x \subseteq B_y \ .$$

Proof. The "only if" direction is a consequence of Proposition 2.4.2 Item *2* by assigning x' to an element of B_x. The latter is then approximating y. Hence, we have the inclusion. The "if" direction is a consequence of Proposition 1.3.1 Item *1* and Proposition 2.5.2 Item *1*. □

A full proof of Proposition 2.7.1 is written in Appendix B.16.

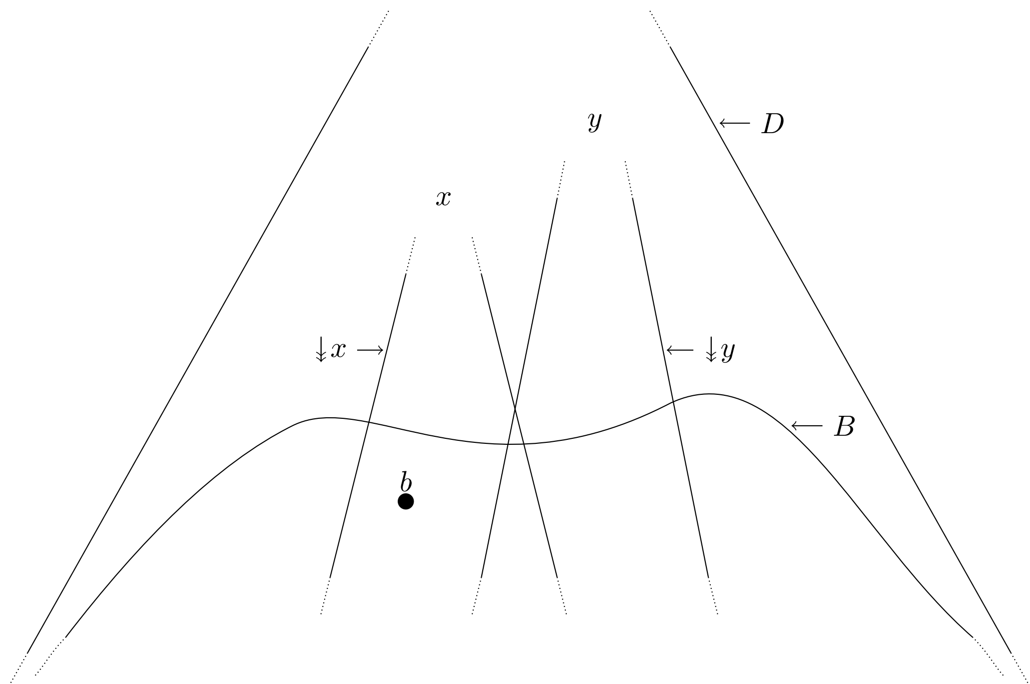

Figure 2.4: Basis element b witnessing $x \not\sqsubseteq y$ in a continuous dcpo D with basis B

We now take a look at the most important property of the way-below relation. It is called the *interpolation property*. Intuitively, it says that one can always find an approximant of an element y between y and any finite set approximating y. In other words, one can always approach y closer and closer. We state it formally in Proposition 2.7.2.

Proposition 2.7.2 (Interpolation property)**.** *Let D be a continuous dcpo, $y \in D$ and $M \subseteq D$ be a finite set such that $M \ll y$. Then*

$$(\exists x \in D \mid M \ll x \land x \ll y) \ .$$

Proof. The trick is to consider the set $A = \{a \in D \mid (\exists a' \in D \mid a \ll a' \land a' \ll y)\}$. It is directed and its supremum is y. From this, we can use the approximation property to find an element for each $m \in M$ greater than or equal to it in A. Since there is a finite number of them, they have an upper bound in A. From there, we use the definition of A to justify the existence of x. The fact that $M \ll x$ follows from Proposition 2.4.2 Item *2* while $x \ll y$ follows from the choice of x.
Note that, if $M = \emptyset$, then the set produced by the "for each $m \in M$" is empty. Then any element of A is an upper bound. □

A full proof of Proposition 2.7.2 is written in Appendix B.17.

We can specialize Proposition 2.7.2 to a basis element as shown by Proposition 2.7.3. The former is the most used form of the interpolation property. The latter is interesting to deduce smaller basis as we do later in Proposition 2.7.5.

Proposition 2.7.3 (Interpolation property)**.** *Let D be a dcpo, $B \subseteq D$ be a basis, $y \in D$ and $M \subseteq D$ be a finite set such that $M \ll y$. Then*

$$(\exists x \in B \mid M \ll x \land x \ll y) \ .$$

Proof. The proof is very close to the one of Proposition 2.7.2. The major difference is that the set considered is $\{a \in D \mid (\exists a' \in B \mid a \ll a' \land a' \ll y)\}$. □

A full proof of Proposition 2.7.3 is written in Appendix B.18. It is interesting to compare this proof to the one of Proposition 2.7.2 to fully see the little differences of using one direction or the other of Proposition 2.6.4 Item *1*.

We have even more in the case of algebraic dcpos! Indeed, we can characterize the way-below relation by finding a compact element between the elements related. We state it formally in Corollary 2.7.4.

Corollary 2.7.4. *Let D be an algebraic dcpo and $x, y \in D$. Then*

$$x \ll y \iff (\exists c \in K(D) \mid x \ll c \land c \ll y) \ .$$

Proof. The "only if" direction is a direct application of Proposition 2.7.3 with $K(D)$ as the basis. The "if" direction is a direct consequence of Proposition 2.4.2. □

An interesting fact about continuous dcpos is that any non-compact element is useless in the basis. In other words, we can erase those non-compact elements from the basis without losing the property of being a basis. We state this in Proposition 2.7.5.

Proposition 2.7.5. *In a dcpo D with a basis B, if an element $d \in B$ is not compact, then $B - \{d\}$ is still a basis of D.*

Proof. For any element $x \in D$, there must be a directed subset of $(B - \{d\})_x$ with supremum x. By using the interpolation property (see Proposition 2.7.3), we find an element d' different from d in the basis B approximating x. We note C_x the directed subset of B_x which has x as supremum. Consider the set $(C_x - \{d\}) \cup \{d'\}$. It is directed, has x as supremum and is a subset of $(B - \{d\})_x$. Hence, we are done. □

A full proof of Proposition 2.7.5 is written in Appendix B.19.

Proposition 2.7.5 can be used to deduce that the compact elements form the smallest basis possible with respect to inclusion. We state it in Proposition 2.7.6.

Proposition 2.7.6. *In a dcpo D with basis B, B is the smallest basis with respect to inclusion if and only if $B = K(D)$.*

Proof. The "only if" direction is proved by contradiction using Proposition 2.7.5 to enlighten the contradiction. The "if" direction is a consequence of Proposition 2.5.2 Item *2*. □

A full proof of Proposition 2.7.6 is written in Appendix B.20.

From Proposition 2.7.2, we can deduce that any element approximating the supremum of a directed set must approximate one of the elements of the set. This is formalized in Corollary 2.7.7.

Corollary 2.7.7. *Let D be a continuous dcpo and $A \subseteq D$ be directed. Then*

$$\twoheaddownarrow \bigsqcup^{\uparrow} A = \bigcup_{a \in A} \twoheaddownarrow a \ .$$

Proof. Let D be a continuous dcpo and $A \subseteq D$ be directed.

$\boxed{\subseteq}$ Let $b \in \twoheaddownarrow \bigsqcup^{\uparrow} A$. By the interpolation property (Proposition 2.7.2), there is $b' \in D$ such that $b \ll b'$ and $b' \ll \bigsqcup^{\uparrow} A$. Since $b' \ll \bigsqcup^{\uparrow} A$, there is $c \in A$ such that $b' \sqsubseteq c$. We have $b \ll c$ by Proposition 2.4.2 Item *2*. Hence, we have $b \in \bigcup_{a \in A} \twoheaddownarrow a$ as wanted.

$\boxed{\supseteq}$ Let $b \in \bigcup_{a \in A} \twoheaddownarrow a$. Then there exists $c \in A$ such that $b \ll c$. We have $b \ll \bigsqcup^{\uparrow} A$ by Proposition 2.4.2 Item *2*.

□

In the definition of the way-below relation, elements are approximated if they are lower than or equal to some directed supremum. However, in the case of a continuous dcpo, we can be more restrictive. Indeed, we can just look at those directed suprema that are equal to the approximated element. This intuition is formalized in Proposition 2.7.8.

Proposition 2.7.8. *Let D be a continuous dcpo and $x, y \in D$. Then $x \ll y$ if and only if, for any directed subset $A \subseteq D$, $y = \bigsqcup^{\uparrow} A \Rightarrow (\exists a \in A \mid x \sqsubseteq a)$.*

Proof. The "only if" direction is an application of the definition of approximation. The "if" direction is done by applying the assumption to $\downarrow y$ and applying Proposition 2.4.2 Item *2*. □

A full proof of Proposition 2.7.8 is written in Appendix B.21.

We end this section with some facts about compact elements. They are presented in Proposition 2.7.9, 2.7.10 and 2.7.11.

Proposition 2.7.9. *In a continuous dcpo, minimal upper bounds of finitely many compact elements are again compact.*

Proof. From Proposition 2.6.4 Item *1*, we know that the approximants of the minimal upper bounds are directed. We use Proposition 2.1.6 to find an upper bound of the compact elements. The latter is greater than or equal to the minimal upper bound considered. □

A full proof of Proposition 2.7.9 is written in Appendix B.22.

Proposition 2.7.10. *In a complete lattice, the supremum of finitely many compact elements is again compact.*

Proof. This is a consequence of Proposition 2.1.6, the compactness of the finite set and the supremum being the least upper bound. □

A full proof of Proposition 2.7.10 is written in Appendix B.23.

Proposition 2.7.11. *A complete lattice is algebraic if and only if every element is the supremum of compact elements.*

Proof. The "only if" direction is proved by taking $K(D)_x$, for any element x, as the compact elements. The "if" direction is proved using Proposition 2.6.4 Item *2* and Proposition 2.7.10. □

A full proof of Proposition 2.7.11 is written in Appendix B.24.

2.8 Properties of ideals

In this section, we briefly state some properties of ideals. Why are we interested in ideals? In fact, we are more interested in the ideal completion, which is defined in Definition 2.2.2. It will help us to build dcpos. The question will be answered in depth in Section 2.9. For now, the only thing we care about is to get some experience with the concept.

We begin slowly by constructing the order of approximation of a continuous dcpo based on the orders of approximation of its principal ideals. This implies that we can get principal ideals separately and construct a new dcpo from them. We formalize the idea in Proposition 2.8.1.

Proposition 2.8.1. *Let D be a continuous dcpo. Then*

$$\{\langle x,y\rangle \in D\times D \mid x \ll y\} = \bigcup_{d\in D} \left\{\langle x,y\rangle \in {\downarrow}d \times {\downarrow}d \mid x \ll_{{\downarrow}d} y\right\} \ .$$

Proof. The "inclusion" direction is the easiest one since any directed set of an ideal is a directed set in the whole dcpo. Hence, we can apply the way-below property to conclude the proof.

The "reverse inclusion" direction is proved by using Proposition 2.7.8. Note that a directed set is included in the downward closure of its supremum. □

A full proof of Proposition 2.8.1 is written in Appendix B.25.

With the same reasoning, we can extract continuity information from the principal ideals. More precisely, a dcpo is continuous if and only if its principal ideals are continuous. This is Proposition 2.8.2. Once again, we could construct a new continuous dcpo by merging many continuous principal ideals. Of course, it also works for algebraic dcpos, i.e., we can deduce "algebraic" from algebraic principal ideals. This is Proposition 2.8.3.

Proposition 2.8.2. *A dcpo is continuous if and only if its principal ideals are continuous.*

Proof. Both directions are proved using Proposition 2.6.4 Item *1*. In each direction, the goal is to deduce that approximants are the same. The conclusion comes from the assumption. □

A full proof of Proposition 2.8.2 is written in Appendix B.26.

Proposition 2.8.3. *A dcpo is algebraic if and only if its principal ideals are algebraic.*

Proof. The proof is exactly the same as that of Proposition 2.8.2. The difference resides in using Proposition 2.6.4 Item *2* instead of Item *1*. □

We now know that continuity of a dcpo depends on its principal ideals. Can we say more? The compact elements of a continuous dcpo can be inferred from the compact elements of its principal ideals. This means that assembling continuous principal ideals preserves continuity and compact elements! We formalize the idea in Proposition 2.8.4.

Proposition 2.8.4. *In a continuous dcpo, an element is compact if and only if it is compact in a principal ideal.*

Proof. The "only if" direction is proved by using the principal ideal of the element. The "if" direction is proved using Proposition 2.7.8. □

A full proof of Proposition 2.8.4 is written in Appendix B.27.

2.9 Constructing a dcpo from a basis

We worked with principal ideals in Section 2.8. Now, we enter the endgame of this section: building an entire dcpo from one of its basis and its order of approximation restricted to the basis. To do so, we need ideals and Scott-continuous function. But first, we present abstract bases in Definition 2.9.1 that are the main structures in this section.

Definition 2.9.1 (Abstract basis)**.** *A set B equipped with a transitive relation $\prec$ is an* abstract basis *if, for any finite subset $M \subseteq B$ and any element $y \in B$ such that $M \prec y$, there is $x \in B$ such that $M \prec x$ and $x \prec y$.*
For any subset $A \subseteq B$ and any element $y \in B$, $A \prec y$ stands for $(\forall a \in A \mid a \prec y)$.
In the case of an arbitrary abstract basis $(B, \prec)$, we omit $\prec$.

At this point, there are relevant questions that need answers. Why do we drop reflexivity? As we said, the goal is to build a continuous dcpo from its basis and the order of approximation restricted to the basis. As we saw in Section 2.4, the latter is not always reflexive. Hence, we cannot have reflexivity as an assumption. Then why do we work with abstract bases? The goal is to present general results. If we were to work with dcpos and their orders of approximation, we would restrict the generality of the results. Indeed, any preordered set is an abstract basis. The reflexivity makes the condition for abstract bases trivial by taking "y" for the element "x". Hence, anything presented on abstract bases can be apply not only to the basis of a continuous dcpo but also to preordered sets.

Even though we are working with abstract bases, we still need previously defined concepts. For simplicity, we are not going to make new definitions for this section only. Hence, if we use concepts that were defined on preordered sets, then what we mean is the same concept but with abstract bases instead of the preordered sets and their transitive relations instead of the preorders. For example, the definition of monotony on abstract bases is "A function f from an abstract basis B to an abstract basis C is *monotone* if, for any $x, y \in B$, $x \prec_B y$ implies $f(x) \prec_C f(y)$.". Note that a concept can use more than one structure. In this case, we mean the same concept but with the old structures substituted for the new ones. For example, the definition of monotony between abstract bases and preordered sets is "A function f from an abstract basis B to a preordered set P is *monotone* if, for any $x, y \in B$, $x \prec_B y$ implies $f(x) \sqsubseteq_P f(y)$.".

We said that we need ideals and Scott-continuous functions. Since we are now working with abstract bases, we have to describe ideals and a dcpo for Scott-continuity. These two descriptions are done in Propositions 2.9.2 and 2.9.3.

Proposition 2.9.2. *For any abstract basis B, for any $x \in B$, the set $\downarrow x$ is an ideal.*

Proof. The directedness condition of an ideal is a consequence of the property of the transitive relation equipped to B. The downward closure condition is also a consequence of this property and of transitivity. □

A full proof of Proposition 2.9.2 is written in Appendix B.28.

Proposition 2.9.3. *For any abstract basis B, the set* $\mathbf{Idl}(B)$ *equipped with inclusion is a dcpo.*

Proof. The proof is exactly the same as that of Proposition 2.2.3. The proof of the latter never uses reflexivity, nor antisymmetry. Therefore, it is valid for this proposition as well. $\square$

Mathematical concepts, such as preorders, can often be restricted to other concepts, such as partial orders. We want to do the same thing with monotonic functions, i.e., restrict them to Scott-continuous functions. The former are defined on elements and the latter on directed sets. Elements can be extended to directed sets using their principal ideals. Note that there are ideals that are not principal. Hence, the extension consists of mapping an ideal to the supremum of the image of its mapped elements. We formalize this intuition in Definition 2.9.5. We also need a new very simple function that sends an element on its ideal. It is defined in Definition 2.9.4.

Definition 2.9.4 (Function `i`)**.** *Let B be an abstract basis. The function* `i` *is the function mapping an element to its ideal. More formally, we have:*

$$\begin{array}{rccc} \mathtt{i}: & B & \to & \mathbf{Idl}(B) \\ & b & \mapsto & \downarrow b \ . \end{array}$$

Note that Definition 2.9.4 is valid thanks to Proposition 2.9.2.

Definition 2.9.5 (Hat function)**.** *Let D be a dcpo, B be an abstract basis and $f \in \left[B \xrightarrow{m} D\right]$. The function $\hat{f}$ is the function mapping an ideal to the directed supremum of its image by f. More formally, we have:*

$$\begin{array}{rccc} \hat{f}: & \mathbf{Idl}(B) & \to & D \\ & I & \mapsto & \bigsqcup^{\uparrow}_{i \in I} f(i) \ . \end{array}$$

Some readers may be surprised by the directed supremum used in Definition 2.9.5. However, it is a consequence of Proposition 2.1.7. Since an ideal is directed by definition, then so is its image by a monotonic function. Moreover, the directed supremum exists because the image is embedded in a dcpo.

We have defined a specialization in Definition 2.9.5. Of course, it has interesting properties! We start with its Scott-continuity (see Corollary 2.3.3) in Proposition 2.9.6.

Proposition 2.9.6. *Let D be a dcpo, B be an abstract basis and $f \in \left[B \xrightarrow{m} D\right]$. The function $\hat{f}$ is Scott-continuous.*

Proof. The proof is done using Corollary 2.3.3. The equality is proved by expanding the definition of $\hat{f}$. Then, the supremum of a directed set of ideals is its directed union. Then applying Proposition 1.3.1 Item *3* allows the insertion of a directed supremum instead of the union. This directed supremum is the function $\hat{f}$ applied to an ideal. This concludes the proof. $\square$

A full proof of Proposition 2.9.6 is written in Appendix B.29.

Scott-continuity is very important, because a lot of notions require it. Moreover, we will see other links between notions of continuity.

Proposition 2.9.7 enlightens a link between a monotonic function and its hat version.

Proposition 2.9.7. *Let D be a dcpo, B be an abstract basis and $f \in \left[B \xrightarrow{m} D\right]$. Then*

$$\hat{f} \circ \mathtt{i} \sqsubseteq_{\mathrm{pt}} f \ .$$

Proof. The proof is done by using monotonicity to deduce that f applied to an element is an upper bound and concluding using the definition of a supremum. $\square$

A full proof of Proposition 2.9.7 is written in Appendix B.30.

In Proposition 2.9.7, we might be tempted to replace "$\sqsubseteq_{\mathrm{pt}}$" by $=$. Unfortunately, equality does not hold, because we do not have $x \in \mathtt{i}(x)$ for any $x \in B$. For equality to hold, we need the transitive relation over B to be reflexive. In this case, we do have $x \in \mathtt{i}(x)$ for any $x \in B$. Hence, the directed supremum of $f(\mathtt{i}(x))$ is just $f(x)$. Indeed, the latter is an upper bound of $f(\mathtt{i}(x))$ by monotonicity. But $f(x)$ is in $f(\mathtt{i}(x))$, hence $f(x)$ is the supremum. Therefore, we have equality as wanted.

The next property is about comparing the hat function to other Scott-continuous functions that satisfy the inequality stated in Proposition 2.9.7, i.e., the composition with the function $\mathtt{i}$ is lower than or equal to f. In fact, the function of Definition 2.9.5 is the greatest with respect to the pointwise order. This is formalized in Proposition 2.9.9. Note that we need an adapted version of Proposition 1.5.5 Item *1* in the proof. We present it as Proposition 2.9.8.

Proposition 2.9.8. *For any abstract basis B and any ideal $I \in \mathbf{Idl}(B)$, $I = \bigcup_{i \in I} \mathtt{i}(i)$.*

Proof. The "$\subseteq$" direction is a consequence of the directedness of I. The "$\supseteq$" is a consequence of the downward closure of I. $\square$

A full proof of Proposition 2.9.8 is written in Appendix B.31.

Proposition 2.9.9. *Let D be a dcpo, B be an abstract basis, $f \in \left[B \xrightarrow{m} D\right]$ and $g \in [\mathbf{Idl}(B) \to D]$ be a Scott-continuous function such that $g \circ \mathtt{i} \sqsubseteq_{\mathrm{pt}} f$. Then*

$$g \sqsubseteq_{\mathrm{pt}} \hat{f} \ .$$

Proof. The trick is to use Proposition 2.9.8. It allows the use of Scott-continuity. Then the end of the proof is done using the assumption on g and the definition of $\hat{f}$. $\square$

A full proof of Proposition 2.9.9 is written in Appendix B.32.

We set up everything in order to construct a dcpo from its basis. However, what does constructing mean? Unfortunately, it does not mean equality. It means equivalent up to an *isomorphism*. We define it in Definition 2.9.10.

Definition 2.9.10 (Isomorphic sets)**.** *Two sets A and B are* isomorphic *if there are two functions $f \in [A \to B]$ and $g \in [B \to A]$ such that $f \circ g = \mathbf{id}_B$ and $g \circ f = \mathbf{id}_A$.*

Definition 2.9.10 comes from category theory which is way more abstract than needed in our context. Hence, in our case, the reader can read Definition 2.9.10 as being a bijection.

Finally, we introduce Theorem 2.9.11. As expected, it states that a dcpo is isomorphic to the ideal completion of one of its bases with respect to the approximation relation.

Theorem 2.9.11. *A continuous dcpo D with a basis B is isomorphic to $\mathbf{Idl}(\langle B, \ll \rangle)$.*

Proof. The function from D to $\mathbf{Idl}(\langle B, \ll \rangle)$ maps x to B_x. The function from $\mathbf{Idl}(\langle B, \ll \rangle)$ to D is $\hat{f}$ where f is a function that maps an element of B to the same element in D. □

A full proof of Theorem 2.9.11 is written in Appendix B.33.

Theorem 2.9.11 implies that we only need to stock the basis and the way-below relation in memory to construct any element. This is important for computers, which have a finite capacity. On the other hand, note that there are many infinite bases, as we saw in many examples.

2.10 Conclusion

We have presented the basic notions of Domain Theory, i.e., directed sets, dcpos, Scott-continuity, way-below relation, approximation and ideal completion. All these concepts are related and are easier to use and to understand with practice. Proving all presented propositions without the hints is a good exercise.

From now on, the reader should be able to start reading any book about Domain Theory. Since we based this thesis on Chapter 1 of Handbook of Logic in Computer Science [7], it could be a good start. Moreover, in the latter, abstract concepts, such as abstract basis, are presented in more depth. Even though the reader is ready for such readings, we strongly suggest to read the following Chapter 3 on topology. Indeed, there are many notions that are quite important.

Finally, there are nice properties for fixpoints in Domain Theory. Interested readers should give it a try.

Chapter 3

Topology Theory

In this chapter, we first aim at presenting basics of topology. Then, we apply them to the concepts of Domain Theory, presented in Chapter 2. This allows us to see the behavior of dcpos without speaking of their elements, but rather in terms of Scott-open sets. In other words, we try to do the same thing that we did in the basis section, i.e., Section 2.9. However, this time, we use topology and its rich development in mathematical literature.

3.1 Topologies and opens

Preorders are relations on sets. Topologies are subsets of a set and they are equipped with two very important properties. Definition 3.1.1 formalizes the concept of topology.

Definition 3.1.1 (Topology)**.** *A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is a* topology *on a set X if $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The sets of $\mathcal{T}$ are called opens or open sets. They are noted $\mathcal{O}(X)$.*

In the literature (e.g., [12]), topologies often have to include the empty set and the whole set. It is still the case with Definition 3.1.1 as stated in Proposition 3.1.2. The proof uses only the fact that we allow the empty union and the empty intersection in Definition 3.1.1.

Proposition 3.1.2. *Let X be a set and $\mathcal{T}$ be a topology on X.*

1. $\emptyset \in \mathcal{T}$
2. $X \in \mathcal{T}$

Proof. Since a topology is closed under arbitrary unions, the empty union is in $\mathcal{T}$. Hence, we have Item *1*, i.e., $\emptyset \in \mathcal{T}$. Since a topology is closed under finite intersections, the empty intersection is in $\mathcal{T}$. Hence, we have Item *2*, i.e., $X \in \mathcal{T}$. □

When we worked with a set equipped with a preorder, we abbreviated it by a preordered set. This is convenient, because it is shorter and allows us to not specify arbitrary preorders.

We do the same thing with a set equipped with a topology in Definition 3.1.3: we abbreviate it by a topological space.

Definition 3.1.3 (Topological space)**.** *A set is a* topological space *if it is equipped with a topology.*

We are ready for five classical examples, namely Definitions 3.1.4, 3.1.5, 3.1.6, 3.1.7 and 3.1.8. Note that the first two topologies are closed under arbitrary intersections and that the subspace topology is very important. We let as exercises to the reader to prove that they are indeed topologies.

Definition 3.1.4 (Discrete topology)**.** *For any set X, its powerset $\mathcal{P}(X)$ is the* discrete topology

Definition 3.1.5 (Trivial topology)**.** *For any set X, the set $\{\emptyset, X\}$ is the* trivial topology*.*

Definition 3.1.6 (Product topology)**.** *For two topological spaces X and Y equipped with their topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, the set containing all the unions of the sets in $\mathcal{T}_X \times \mathcal{T}_Y$, is the* product topology*.*

Definition 3.1.7 (Order topology)**.** *For any preordered set X, the set containing unions of open intervals, i.e., sets of the form $]x, y[::= \{z \in X \mid x \sqsubset z \sqsubset y\}$, and, if $\bot$ or $\top$ exists in X, of the forms $[\bot, y[::= \{z \in X \mid z \sqsubset y \vee z = \bot\}$ and $]x, \top] ::= \{z \in Z \mid x \sqsubset z \vee z = \top\}$, is the* order topology*.*

Definition 3.1.8 (Subspace topology)**.** *For any set X with a topology $\mathcal{T}_X$ and any subset $Y \subseteq X$, the set $\{U \cap Y \mid U \in \mathcal{T}_X\}$ is the* subspace topology *of Y.*

3.2 Closed sets

In topology, open sets are important objects of study (Definition 3.1.1). Open sets have a dual notion called closed sets (see Definition 3.2.1). In fact, a topology can be defined in two ways: either by stating what closed sets are or by stating what open sets are. Note that the former have different properties from the latter, as stated in Proposition 3.2.2.

Definition 3.2.1 (Closed set)**.** *A subset C of a topological space X is* closed *if its complement is open. The set of closed sets of a topological space X is noted $\Gamma(X)$.*

Proposition 3.2.2. *The set of closed sets of a topological space is closed under finite unions and arbitrary intersections.*

Proof. The trick is to replace closed sets by their complementary open sets. Then, by set theory, finite unions become finite intersections and arbitrary intersections become arbitrary unions. They are open by definition of a topology and their complements are closed. Hence, we are done. □

A full proof of Proposition 3.2.2 is written in Appendix C.1.

We now present two notable sets in topology. The first is the largest open set contained in some set. The second is the smallest closed set containing some set. Definitions 3.2.3 and 3.2.4 formally define them. Note that we can always add a subscript to clarify in which topological space the set is taken.

Definition 3.2.3 (Interior)**.** *For a subset $A \subseteq X$ of a topological space X, its* interior*, noted* $\mathbf{int}(A)$*, is the set* $\bigcup\{U \in \mathcal{O}(X) \mid U \subseteq A\}$.

Definition 3.2.4 (Closure)**.** *For a subset $A \subseteq X$ of a topological space X, its* closure*, noted* $\mathbf{cl}(A)$*, is the set* $\bigcap\{C \in \Gamma(X) \mid A \subseteq C\}$.

The closure of a subset A of a topological space is also written as $\bar{A}$ (e.g., [13]).

The interior of a set is open because it is a union of open sets. Furthermore, as we said, it is the largest open set contained in some set because all the other open sets are contained in the union. Finally, the interior of an open set is the open set itself by reflexivity of inclusion. We apply the same ideas on the closure: it is a closed set because it is the intersection of closed sets. It is also the smallest closed set containing some set because all the other containing closed sets contain the intersection. The closure of a closed set is the closed set itself. We formalize some of this discussion in Proposition 3.2.5 and Proposition 3.2.6.

Proposition 3.2.5. *Let X be a topological space and $A \subseteq X$.*

1. $\mathbf{int}(A) \subseteq A$
2. $A \subseteq \mathbf{cl}(A)$.

Proof. Both items are conclusions of set theory. Item *1* is true because any set containing a collection of sets contains their union. Item *2* is true because the intersection of a collection of sets containing a set contains the latter. □

Proposition 3.2.6. *In a topological space, a set is closed if and only if it contains its closure.*

Proof. The "only if" part is done using the definition of the closure and set theory. The "if" part is a corollary of Proposition 3.2.5 Item *2*. □

A full proof of Proposition 3.2.6 is written in Appendix C.2.

Can we represent open sets? Well, it depends on the topology. Knowing it helps presenting the open sets in an appropriate shape. However, there is a very used representation presented in Figure 3.1. In this image, open sets are circles without their outside border. In other words, it is the interior of the circle. On the other hand, closed sets are circles with their outside border. Note that circles are a shape we chose but, if the open sets where downward closed, we would have used Figure 1.5. The important thing to remember is the border, i.e., is it absent (open) or present (closed).

In Section 2.2, we stated that topologies are dcpos because they are complete lattices. We can now prove it via Proposition 3.2.7.

Figure 3.1: On the left an open circle for an open set U and, on the right, a closed circle for a closed set C

Proposition 3.2.7. *The set $\mathcal{O}(X)$ of a topological space X is a complete lattice.*

Proof. The supremum of a set of open sets is their union. By definition, it is open. Its infimum is the interior of its intersection. □

A full proof of Proposition 3.2.7 is written in Appendix C.3. Note that we could have simplified the proof a lot by using Proposition 1.4.4, but we wanted to show a use of the interior of a set.

We end this section with a last definition about opens and a proposition. In topology, we speak in terms of opens and not in terms of elements. Hence, the goal of Definition 3.2.8 is to replace elements by the opens containing them. We also present Definition 3.2.9, which is commonly used with Definition 3.2.8.

Definition 3.2.8 (Neighborhood)**.** *In a topological space X, the set $\{U \in \mathcal{O}(X) \mid x \in U\}$ is the* neighborhood *of the element $x \in X$, noted $\mathcal{N}_x$.*

Definition 3.2.9 (Neighbor)**.** *In a topological space X, an open subset $U \in \mathcal{O}(X)$ is a* neighbor *of an element x if $U \in \mathcal{N}_x$.*

Proposition 3.2.10 is complicated and uses every notion we have seen so far in Chapter 3. However, it enlightens the talk we had about borders before Figure 3.1. An element on the border is in the closure because we add elements not present in the interior, i.e., elements that are in all the opens intersecting a subset.

Proposition 3.2.10. *In a topological space X, an element $x \in X$ is in the closure of a subset $A \subseteq X$ if and only if $(\forall U \in \mathcal{N}_x \mid U \cap A \neq \emptyset)$.*

Proof. The trick is to prove the contraposition, i.e., $x \notin \mathbf{cl}(A) \iff (\exists U \in \mathcal{N}_x \mid U \cap A = \emptyset)$.

The "only if" direction is done by taking $X - \mathbf{cl}(A)$ as the open set. The rest is set theory and Proposition 3.2.5 Item *2*.

The "if" direction is done by using set theory to obtain $\mathbf{cl}(A) \subseteq X - U$ and $x \notin X - U$. □

A full proof of Proposition 3.2.10 is written in Appendix C.4.

3.3 Scott topology

We have everything we need to start looking at the Scott topology. We start by Definitions 3.3.1 and 3.3.2. We prove that the Scott topology is indeed a topology in Proposition 3.3.3.

Definition 3.3.1 (Scott-open). *A subset U of a dcpo D is* Scott-open *if*

1. *U is upward closed;*

2. *for any directed subset $A \subseteq D$ such that $\bigsqcup^{\uparrow} A \in U$, there exists an element $d \in U \cap A$.*

Definition 3.3.2 (Scott topology). *The topology where open sets are Scott-open is the* Scott topology.

Proposition 3.3.3. *The Scott topology is a topology.*

Proof. For finite intersections, upward closure is a consequence of the upward closures of Scott-opens. The intersection with directed sets is proved using the same property on all Scott-opens. From this point, we can use Proposition 2.1.6. The upper bound is in all Scott-opens. Hence, it is in their intersection.

For arbitrary unions, upward closure is obtained by applying upward closure of a Scott-open. Intersection with directed sets is proved using intersection with directed sets of a Scott-open. □

A full proof of Proposition 3.3.3 is written in Appendix C.5.

We still have a lot of concepts to see, but the Scott topology will be the main object of study and we will focus on it in all the following sections.

We have defined Scott-open sets, the next thing to do is to describe their complements, i.e., Scott-closed sets. They are defined in Definition 3.3.4.

Definition 3.3.4 (Scott-closed). *A subset C of a dcpo D is* Scott-closed *if its complement $X - C$ is Scott-open.*

Working with the complement of a Scott-open set to speak of a Scott-closed set can be annoying and misleading. Hence, we present another description in Proposition 3.3.5.

Proposition 3.3.5. *In a dcpo D equipped with its Scott topology, a subset $C \subseteq D$ is Scott-closed if and only if it is downward closed and it is closed under directed suprema.*

Proof. The downward closure of the "only if" part is an application of Proposition 1.5.4 and set theory. The closure under directed suprema is done by contradiction. If a supremum of a directed set is not in the Scott-closed set, then it is in its Scott-open complement. Hence, there is an element of the directed set in the Scott-open complement by definition. It is a contradiction. We conclude that the supremum is in the Scott-closed set.

The "if" part is done by showing that the complement of the set is Scott-open. The latter is done with the same arguments as for the "only if" part. □

A full proof of Proposition 3.3.5 is written in Appendix C.6.

As always, it is interesting to have examples of the presented concepts. In order to illustrate the Scott topology, we give an example of a Scott-open set in Proposition 3.3.6.

Proposition 3.3.6. *In a dcpo D, for any compact element $x \in K(D)$, $\twoheaduparrow x$ is Scott-open.*

Proof. The upward closure part is a consequence of Proposition 2.4.6 Items *1* and *3*, and antisymmetry. The intersection with a directed set is a consequence of compactness and upward closure. $\square$

A full proof of Proposition 3.3.6 is written in Appendix C.7.

To generalize Proposition 3.3.6 to any element, we need continuity in order to have the interpolation property as stated in Proposition 3.3.7.

Proposition 3.3.7. *In a continuous dcpo D, the sets $\twoheaduparrow x$, for any $x \in D$, are Scott-open.*

Proof. The upward closure part is a consequence of Proposition 2.4.2 Item *2*. The intersection with a directed set is a consequence of Corollary 2.7.7. $\square$

A full proof of Proposition 3.3.7 is written in Appendix C.8.

Note that continuity is needed in Proposition 3.3.7 to show Condition *2* of Scott-openness. Condition *1* is true in any dcpo by Proposition 2.4.2 Item *2*.

The set of elements approximated by a single element has another property described in Proposition 3.3.8. It is exactly the largest open set included in the principal filter related to the approximating element.

Proposition 3.3.8. *Let D be a continuous dcpo equipped with its Scott topology and $x \in D$. Then*

$$\twoheaduparrow x = \mathbf{int}(\uparrow x) \ .$$

Proof. For the "$\subseteq$" part, we use Proposition 3.3.7 to obtain $\twoheaduparrow x \in \mathcal{O}(D)$. But we also have $\twoheaduparrow x \subseteq \uparrow x$ by Proposition 2.4.6 Item *1*. This proof is done since the interior is the largest open set included in $\uparrow x$.

The "$\supseteq$" part is done by using property *2* of a Scott-open set thanks to Proposition 2.6.4 Item *1*. The element found allows us to apply Proposition 2.4.2 Item *2*. The element from the interior is approximated. Hence, we are done. $\square$

A full proof of Proposition 3.3.8 is written in Appendix C.9.

3.4 Bases

Again? We meet once again bases, and once again they will generate the related structure. In Section 2.5, bases are meant to generate dcpos. This time, they are meant to generate topologies. Hence, they generate a set of subsets of the initial set rather than elements of the latter. Knowing that, bases should contain subsets rather than elements. Intuitively, for any element, there should be a basis set which serves as a representative. In continuous dcpos, those sets are the directed approximants. We would also like to have the smallest pieces possible. Taking this into account, we exhibit Definition 3.4.1.

Definition 3.4.1 (Basis)**.** *A* basis $\mathcal{B}$ *for a topology of a set* X *is a subset of* $\mathcal{P}(X)$ *such that:*

1. $(\forall x \in X \mid (\exists B \in \mathcal{B} \mid x \in B))$;
2. $(\forall x \in X, B_1, B_2 \in \mathcal{B} \mid x \in B_1 \cap B_2 \Rightarrow (\exists B_3 \in \mathcal{B} \mid x \in B_3 \wedge B_3 \subseteq B_1 \cap B_2))$.

When we presented Figure 3.1, we said that open circles would find their uses. Indeed, they form a basis on $\mathbb{R} \times \mathbb{R}$. Any element is in a circle, e.g., the many ones centered at the element. Hence, Condition *1* of a basis is satisfied. If an element is in the intersection of two circles, there is a circle contained in it and containing the element. Hence, Condition *2* of a basis is satisfied. We represent this situation in Figure 3.2.

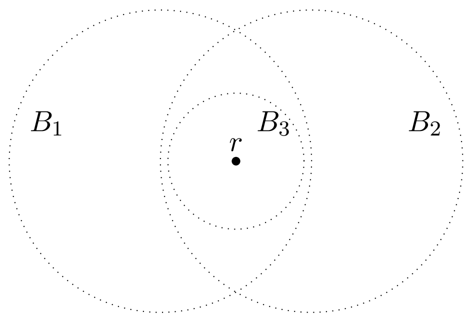

Figure 3.2: An element $r \in \mathbb{R} \times \mathbb{R}$ contained in an open circle B_3 contained in the intersection of two open circles B_1 and B_2

Some readers may be intrigued about the word "topological" in topological basis. Indeed, there is no link to any topology in Definition 3.4.1. To resolve this problem, we just have to define the topology generated by the basis as we did in Section 2.9. To do so, we introduce Definition 3.4.2.

Definition 3.4.2 (Topology generated by a basis)**.** *On a set* X *with a basis* $\mathcal{B}$*, the* topology generated by $\mathcal{B}$ *is the set* $\{U \in \mathcal{P}(X) \mid (\forall u \in U \mid (\exists B \in \mathcal{B} \mid u \in B \wedge B \subseteq U))\}$.

There we have a topology generated by a basis. Wait... Is it even a topology? Of course it is! We prove it in Proposition 3.4.3.

Proposition 3.4.3. *The topology generated by a basis of a set is a topology.*

Proof. For arbitrary unions, the proof is straightforward because the basis element containing an element is contained in the union by set theory. For finite intersections, the proof is done by induction using Property *2* of a basis. □

A full proof of Proposition 3.4.3 is written in Appendix C.10.

The topology generated by a basis is defined in terms of elements of the power set. However, we would like to describe it in terms of the basis elements. That is what we did in Section 2.9 for dcpos. It is simpler for topology because opens are just unions of basis elements. We formalize this in Proposition 3.4.4. It will also be useful when we look at the Scott topology.

Proposition 3.4.4. *The topology generated on a set X by a basis is exactly the set of all unions of elements of the basis.*

Proof. The "$\supseteq$" part is easier. Indeed, basis elements are open by definition. Hence, unions of them are also open.

For the "$\subseteq$" part, we exhibit all basis elements contained in an open and containing at least one of its elements. The union of these basis elements is the open set itself. $\square$

A full proof of Proposition 3.4.4 is written in Appendix C.11.

In the light of Proposition 3.4.4, did we meet topological bases in the earlier sections? Indeed, we did in Section 3.1 when we presented the product and the order topologies in Definitions 3.1.6 and 3.1.7. It is interesting to read them one more time and to focus on the way that we wrote those definitions, i.e., "[...] set containing all the unions of [...]".

We use Scott topology to illustrate another basis on a continuous dcpo. An important thing in continuous dcpos is the way-below relation. Can we think of using it to find a basis for the Scott topology? To answer, we take a look back at Proposition 3.3.7. We know that sets of the form $\twoheaduparrow x$ for any element x are Scott-open. They look like the small pieces we are looking for and, indeed, they form a basis. We formalize those ideas in Proposition 3.4.5.

Proposition 3.4.5. *In a continuous dcpo D, the set $\{\twoheaduparrow d \mid d \in D\}$ is a basis.*

Proof. Condition *1* of a basis is respected because the set of approximants of an element is not empty by Proposition 2.6.4 Item *1*. Condition *2* is respected because of the interpolation property. $\square$

A full proof of Proposition 3.4.5 is written in Appendix C.12.

So far, we have proved that the sets of the form $\twoheaduparrow x$ of any element x of a continuous dcpo form a basis for it. However, does this basis really generate the Scott topology on the continuous dcpo? We first prove Proposition 3.4.6. Then, we have the desired result for free, that is Corollary 3.4.7.

Proposition 3.4.6. *Let U be a Scott-open set of a continuous dcpo. Then*

$$U = \bigcup_{u \in U} \twoheaduparrow u \ .$$

Proof. The "$\subseteq$" direction is done using Proposition 2.6.4 Item *1* and Property *2* of a Scott-open set.

The "$\supseteq$" direction is done using Proposition 2.4.2 Item *1* and Property *1* of a Scott-open set. $\square$

A full proof of Proposition 3.4.6 is written in Appendix C.13.

Corollary 3.4.7. *The set $\{\twoheaduparrow d \mid d \in D\}$ form a basis for the Scott topology on a continuous dcpo D.*

Proof. The basis part is Proposition 3.4.5. The Scott topology part is Proposition 3.4.4 applied in Proposition 3.4.6. □

3.5 Finer and coarser topologies

We started this thesis by comparing elements. What if we consider topologies as elements? The simplest way to compare topologies is by looking at their open sets. The latter are compared through inclusion. Since we can talk about which topology is included in which or which topology includes which, there are two names for the comparison. They are presented in Definitions 3.5.1 and 3.5.2.

Definition 3.5.1 (Finer topology)**.** *A topology $\mathcal{T}$ is* finer *than a topology $\mathcal{T}'$ if $\mathcal{T}' \subseteq \mathcal{T}$.*

Definition 3.5.2 (Coarser topology)**.** *A topology $\mathcal{T}$ is* coarser *than a topology $\mathcal{T}'$ if $\mathcal{T} \subseteq \mathcal{T}'$.*

How to remember those strange name or understand them? We use the analogy from the book [14]. Imagine a bag of stones representing the open sets. Take a stone and split it into pieces. Finer stones are obtained from this operation. The bag contains more stones than the previous one. This is where the *finer* appellation comes from. In the same way, take two stones from the bag and merge them. It becomes a coarser stone. This is where the *coarser* appellation comes from. Note that merging can refer to union. However, if a topology $\mathcal{T}$ is finer than another topology $\mathcal{T}'$, then $\mathcal{T}$ contains opens that can intersect or be disjoint from the opens of $\mathcal{T}'$.

Another way of remembering is to interpret finer as giving more information. Note that it is simply the dictionaries' definition of "finer". Coarser can be interpreted as being more abstract. Note once again that it is very close to the dictionaries' definition. For example, take the trivial topology and the discrete topology. The former is coarser than the latter, because we are lacking refinement, i.e., the topology does not give much information on its related space. Or, the latter is finer than the former, because it provides more information.

As we said in Section 3.3, we now exemplify on Scott topology. But, first, we need to introduce the Alexandroff topology in Definition 3.5.3.

Definition 3.5.3 (Alexandroff topology)**.** *The Alexandroff topology is the one whose opens are upward closed.*

Another way of defining the Alexandroff topology is saying that the closed sets are downward closed. Anyway, we can see that the Alexandroff topology is finer than the Scott topology as stated in Proposition 3.5.4.

Proposition 3.5.4. *The Alexandroff topology is finer than the Scott topology.*

Proof. Any Scott-open set is upward closed by definition. Hence, it is Alexandroff-open. □

3.6 Specialization preordering

We started out this thesis by presenting preorders in Section 1.1. We are now addressing topological spaces. They are two very different concepts. But any topological space can be turned into a preordered set. To do so, we need a preorder that works for any topological space. We present one in Definition 3.6.1.

Definition 3.6.1 (Specialization preordering)**.** *In a topological space X, the* specialization preordering *is the relation $\{\langle x, y\rangle \in X \times X \mid (\forall U \in \mathcal{O}(X) \mid x \in U \Rightarrow y \in U)\}$.*

Notation 3.6.2. *In a topological space X, a pair $\langle x, y\rangle \in X \times X$ in the specialization preordering is noted $x \sqsubseteq_{\text{topo}} y$. It can be read as x is a specialization of y.*

In Definition 3.6.1, to order elements, we are interested in all open sets containing an element. It is similar to the neighborhood (see Definition 3.2.8). So, for opens, we look at all the opens containing an element, say "x". If they all contain another element, say "y", then y is greater than or equal to x.

Note that Notation 3.6.2 is invented in the same spirit of simplicity and non-ambiguity of the notation of the pointwise order. On the other hand, note that the symbol $\leq$ is usually used for specialization preordering (e.g., [15] and [16]).

In Proposition 3.6.3 we provide a proof that the specialization preordering is a preorder.

Proposition 3.6.3. *The specialization preordering is a preorder.*

Proof. Reflexivity and transitivity are direct consequences of the definition of the specialization preordering. □

A full proof of Proposition 3.6.3 is written in Appendix C.14.

Having multiple definitions of a concept might be helpful depending on the context. Hence, we give alternate definitions of the specialization preordering in Propositions 3.6.4, 3.6.5, and Corollary 3.6.6. They are not difficult, so the reader can prove all of them.

Proposition 3.6.4. *Let X be a topological space and $x, y \in X$. Then*

$$x \sqsubseteq_{\text{topo}} y \iff \mathcal{N}_x \subseteq \mathcal{N}_y \ .$$

Proof. Both parts of the proposition are consequences of first order logic and set theory. □

A full proof of Proposition 3.6.4 is written in Appendix C.15.

Proposition 3.6.5. *In a topological space X, an element $x \in X$ is a specialization of an element $y \in X$ if and only if x is contained in all closed sets containing y.*

Proof. The trick is to use the fact that the complement of a closed set is open and to apply contraposition on the right hand side, i.e., $(\forall C \in \Gamma(X) \mid y \notin C \Rightarrow x \notin C)$. Then the results are their respective assumption. □

A full proof of Proposition 3.6.5 is written in Appendix C.16.

Corollary 3.6.6. *Let X be a topological space and $x, y \in X$. Then*

$$x \sqsubseteq_{\text{topo}} y \iff x \in \mathbf{cl}(\{y\}) \ .$$

Proof. The "only if" direction is a consequence of Proposition 3.6.5 applied from left to right and the fact that $\mathbf{cl}(\{y\})$ is a closed set containing y by definition. The "if" direction is again a consequence of Proposition 3.6.5 applied from right to left and by definition of the closure, i.e., $\mathbf{cl}(\{y\})$ is contained in all closed sets containing y. □

Note that Propositions 3.6.4 and 3.6.5 can be proved by a sequence of equivalences as we do in the appendix. On the other hand, note that Corollary 3.6.6 could be turned into a proposition independent from Proposition 3.6.5. In this case, it would have been proved by using Proposition 3.2.10. It is an exercise left to the reader. In any case, this corollary has a particular form, reminding of Definition 1.5.1. It is a form of downward closure with respect to the specialization preordering. We highlight this fact in Corollary 3.6.7. However, do not get confused if the topological space is already a preordered set. It is the downward closure using the specialization preordering and not the preorder on the underlying set.

Corollary 3.6.7. *Let X be a topological space and $y \in X$. Then*

$$\downarrow_{\sqsubseteq_{\text{topo}}} y = \mathbf{cl}(\{y\}) \ .$$

Proof. It is a direct consequence of Corollary 3.6.6. □

There are topologies where the preorder of the underlying set and the specialization preordering coincide. A good example is the Alexandroff Topology presented in Definition 3.5.3.

We know that the specialization preordering is a preorder and we know multiple definitions. What is next? We can try to add properties, for instance to specialize the specialization preordering to a specialization partial ordering. Fortunately for us, it happens magically that the specialization preordering on the Scott topology is a partial ordering! We state it in Proposition 3.6.9. But, in order to prove the latter, we first need to realize that principal ideals are Scott-closed as we do in Proposition 3.6.8.

Proposition 3.6.8. *A principal ideal is Scott-closed.*

Proof. The proof relies on Proposition 3.3.5. The downward closure is the result of the definition of a principal ideal while the closure under directed suprema relies on the supremum being the least upper bound. □

A full proof of Proposition 3.6.8 is written in Appendix C.17.

Proposition 3.6.9. *The specialization preordering on a dcpo equipped with its Scott topology is a partial ordering.*

Proof. Reflexivity and transitivity are done in Proposition 3.6.3. Hence, this leaves only antisymmetry to be proved. Suppose we have x and y two elements of the dcpo such that $x \sqsubseteq_{\text{topo}} y$ and $y \sqsubseteq_{\text{topo}} x$. We use Proposition 3.6.5, Proposition 3.6.8 and antisymmetry of the dcpo to conclude. □

A full proof of Proposition 3.6.9 is written in Appendix C.18.

3.7 Separation axioms

In Section 3.6, we have presented a preorder that works on any topological space. Hence, we can order elements in topological spaces. However, this is not enough to distinguish elements. The specialization preordering is very permissive on that point. For example, consider two elements such that one is in an open set if and only if the other is in the open set. They are topologically indistinguishable. Therefore, can we add stronger restrictions on a topological space to differentiate elements between them and on a topological view?

This section answers this question by adding hypotheses on topological spaces. These assumptions are called *separation axioms*. There are three main separation axioms, namely T_0, T_1 and T_2. They are respectively defined in Definitions 3.7.1, 3.7.2 and 3.7.3. T_2 spaces are widely called Hausdorff spaces as written in Definition 3.7.4.

Definition 3.7.1 (T_0 axiom)**.** *A topological space X is T_0 if*

$$(\forall x, y \in X \mid \mathcal{N}_x = \mathcal{N}_y \Rightarrow x = y) \ .$$

Definition 3.7.2 (T_1 axiom)**.** *A topological space X is T_1 if*

$$(\forall x, y \in X \mid x \neq y \Rightarrow \mathcal{N}_x - \mathcal{N}_y \neq \emptyset \wedge \mathcal{N}_y - \mathcal{N}_x \neq \emptyset) \ .$$

Definition 3.7.3 (T_2 axiom)**.** *A topological space X is T_2 if*

$$(\forall x, y \in X \mid x \neq y \Rightarrow (\exists U \in \mathcal{N}_x, V \in \mathcal{N}_y \mid U \cap V = \emptyset)) \ .$$

Definition 3.7.4 (Hausdorff space)**.** *A topological space is* Hausdorff *if it is T_2.*

Note that, for the T_0 axiom, we could have chosen to replace the implication by an if and only if. However, the "if" direction is not very interesting, because it is always true by rewriting and reflexivity. For T_2 spaces, we are requiring the existence of disjoint neighbors for different elements. For T_1 spaces, for different elements, we are requiring the existence of neighbors not containing the different elements. Intuitively, T_2 seems stronger than T_1. Following the same logic, we would like to say that T_1 seems stronger than T_0. And, in fact, they are! We formalize these ideas in Propositions 3.7.5 and 3.7.6 and Corollary 3.7.7.

Proposition 3.7.5. *A T_2 topological space is T_1.*

Proof. Because the space is T_2, there are disjoint neighbors for distinct elements. Those neighbors prove that the difference of the neighborhoods is non-empty. $\square$

A full proof of Proposition 3.7.5 is written in Appendix C.19.

Proposition 3.7.6. *A T_1 topological space is T_0.*

Proof. Using contraposition, we can apply the T_1 assumption to prove the inequality between neighborhoods. $\square$

A full proof of Proposition 3.7.6 is written in Appendix C.20.

Corollary 3.7.7. *A T_2 topological space is T_0.*

Proof. The result follows from the assumption after applying Propositions 3.7.6 and 3.7.5. $\square$

In the introduction of this section, we justified the T_0, T_1 and T_2 axioms based on a weak restrictions to the specialization preordering. What is its link with separation axioms? To answer this question, we take a look at Proposition 3.6.4 about the inclusion of neighborhoods and Definition 3.7.1 of a T_0 space. There is a difference between the two. In one, we have $\sqsubseteq_{\text{topo}}$ and $\subseteq$, and, in the other, we have $=$ and $=$. Equalities can be obtained with antisymmetry. Unfortunately, the specialization preordering is only a preorder. Hence, making it a partial order resolves the problem, as stated in Proposition 3.7.8.

Proposition 3.7.8. *A topological space is T_0 if and only if $\sqsubseteq_{\text{topo}}$ is a partial ordering.*

Proof. The "only if" direction is done by using Proposition 3.6.4, the antisymmetry of $\subseteq$ and the T_0 assumption. For the "if" direction, for two elements x and y of the topological space, we suppose $\mathcal{N}_x = \mathcal{N}_y$. Hence, we can apply Proposition 3.6.4 to obtain $x \sqsubseteq_{\text{topo}} y$ and $y \sqsubseteq_{\text{topo}} x$. Since the specialization preordering is a partial ordering by assumption, we conclude $x = y$. $\square$

Now, how can we link the separation axioms to the Scott topology? We have Corollary 3.7.9. It follows directly from Proposition 3.6.9, i.e., a dcpo equipped with its Scott topology is always a T_0 space.

Corollary 3.7.9. *A dcpo equipped with its Scott topology is T_0.*

Proof. We apply Proposition 3.7.8 followed by Proposition 3.6.9. $\square$

We also have a result for the T_2 axiom. Indeed, the only way for a dcpo equipped with its Scott topology to form a T_2 space is when the partial order of the dcpo is equality. We state this in Proposition 3.7.10.

Proposition 3.7.10. *A dcpo equipped with its Scott topology is T_2 if and only if its order is equality.*

Proof. The "only if" direction is done by contraposition. If the order is not equality, there is a pair of different elements in it. All neighbors of the first element are neighbors of the second. Hence, the space is not T_2.

The "if" direction is done by taking different elements. Each singleton is Scott-open and they are disjoint. Hence, we are done. $\square$

A full proof of Proposition 3.7.10 is written in Appendix C.21. Note that we used contraposition for the "if" direction, but the proof is also feasible by contradiction or directly by taking the contraposition on the T_2 assumption.

The sad part of the Scott topology not being T_2 leads to a happy part: the study of Non-Hausdorff topologies. We will not go in any more details. We refer the reader to [17] to acquire further knowledge.

3.8 Compactness

In Section 3.7, we studied opens containing some element and not some other. In this section, we study sets contained in opens.

We start right away with the most general concept: the compactness of a set. Before that, we need Definitions 3.8.1 and 3.8.2. The second one may seem obvious, but we prefer to be very explicit about any concept.

Definition 3.8.1 (Cover)**.** *Let X be a set. A family of subsets $\mathcal{C} \subseteq \mathcal{P}(X)$ is a* cover *of a subset $A \subseteq X$ if $A \subseteq \bigcup \mathcal{C}$.*

Definition 3.8.2 (Open cover)**.** *On a topological space X, a cover $\mathcal{C}$ of a subset is an* open cover *if $\mathcal{C} \subseteq \mathcal{O}(X)$.*

Now, we are ready to introduce the concept of compact subset of topological spaces in Definition 3.8.3 and of compact spaces in Definition 3.8.4.

Definition 3.8.3 (Compact set)**.** *In a topological space X, a subset $A \subseteq X$ is* compact *if, for all open covers $\mathcal{C}$ of A, there is a finite subcover $\mathcal{F} \subseteq \mathcal{C}$ of A.*

Definition 3.8.4 (Compact space)**.** *A topological space X is* compact *if X is compact.*

One could think that any topological space is compact, because the whole set is an open set and forms a finite subcover. However, the quantification is on all open covers which means also the ones not containing the whole set. Hence, not all topological spaces are compact. For example, take the reals with their order topology. This topology has been defined in Definition 3.1.7 of Section 3.1 as the one whose open sets are unions of open intervals $]x, y[$ with $x, y \in \mathbb{R}$. Knowing this, we can say that the family of open intervals of the form $]r, r + k[$, with $r, k \in \mathbb{R}$ and $k > 0$, forms an open cover of $\mathbb{R}$. However, it is impossible to extract a finite subcover. On the other hand, take the naturals with their usual order equipped with the

Alexandroff topology. The union of any open cover of $\mathbb{N}$ must contain zero. Thus, there is an open in the open cover that contains zero. Then it contains the naturals by upward closure. Hence, the naturals are compact. To generalize, any pointed preordered set equipped with the Alexandroff topology is compact. The reasoning applied is the same, except that zero is replaced by a least element.

Knowing a new concept does not always imply mastering it. Hence, we add some propositions on compactness to practice. They also help to gain knowledge on this notion and, maybe, to build links between concepts. We start by the easiest one, i.e., Proposition 3.8.5. We continue with a harder one, i.e., Proposition 3.8.6. Unfortunately, it is not an if and only if. We need the additional T_2 axiom as stated in Proposition 3.8.7.

Proposition 3.8.5. *A finite space is compact.*

Proof. An open cover is finite by set theory. Hence, it is itself a finite subcover. □

Proposition 3.8.6. *Any closed subset of a compact space is compact.*

Proof. The trick is to use the definition of a closed set, i.e., its complement is open. Then, from the cover to which we add the open set, a cover for the space is obtained. By compactness, there is a finite subcover for the space. This finite subcover is also covering the closed set. □

A full proof of Proposition 3.8.6 is written in Appendix C.22.

Proposition 3.8.7. *Any compact subset of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and $C \subseteq X$ be compact. The trick is to prove that $X - C$ is open. To do so, for each element of $X - C$, we find open disjoint neighbors for every element of C. The union forms an open cover of C. There is a finite subcover. The intersection of the disjoint neighbors of the corresponding neighbors of the elements of $X - C$ contains the element and is disjoint from the finite subcover. Hence, it is disjoint from C, i.e., it is in $X - C$. This finite intersection is also open by definition. By repeating this procedure for all the elements of $X - C$, we obtain many open sets. Their union is open by definition and is $X - C$ by set theory. □

A full proof of Proposition 3.8.7 is written in Appendix C.23.

We are already familiar with compactness. Indeed, we introduced a different concept with the same name in Section 2.4. In the latter, an element is compact if it approximates itself. Are those compact elements related to some compact sets? In fact, they are as we state in Corollary 3.8.9. However, we start by Proposition 3.8.8, which holds for any elements.

Proposition 3.8.8. *In a preordered set P equipped with its Alexandroff topology, the set $\uparrow x$ is compact for any element $x \in P$.*

Proof. The proof is straightforward because an open cover contains an open which contains x which, by definition, is equivalent to containing $\uparrow x$. □

A full proof of Proposition 3.8.8 is written in Appendix C.24.

Corollary 3.8.9. *In a dcpo D equipped with its Alexandroff topology, the set $\uparrow x$ is compact for any compact element $x \in K(D)$.*

Proof. A dcpo is a preordered set. Hence, we apply Proposition 3.8.8 to obtain that $\uparrow x$ is compact. This means that $\uparrow x$ is compact by Proposition 2.4.6 Items *1* and *3*, and antisymmetry. $\square$

Since the Scott topology is coarser than the Alexandroff topology, any open cover in the former is an open cover in the latter. Therefore, we exhibit Proposition 3.8.10 and Corollary 3.8.11.

Proposition 3.8.10. *In a dcpo D equipped with its Scott topology, the set $\uparrow x$ is compact for any element $x \in D$.*

Proof. The proof is exactly the same as the one of Proposition 3.8.8. $\square$

Corollary 3.8.11. *In a dcpo D equipped with its Scott topology, the set $\uparrow x$ is compact for any compact element $x \in K(D)$.*

Proof. The proof is exactly the same as the one of Proposition 3.8.9. $\square$

We take a quick look back at Proposition 3.8.8 and Corollary 3.8.9. What is the idea in the proof? It is that the cover contains the least element of the set. By upward closure, this means that an open in the cover contains all the principal filters, w.r.t. to the order. This open is the finite subcover. We generalize this idea in Proposition 3.8.12.

Proposition 3.8.12. *A pointed subset of a preordered set equipped with its Alexandroff topology is compact.*

Proof. Any open cover of a pointed subset contains an open containing the least element. This open set is the finite subcover. $\square$

Compactness depends on sets containing sets. There is a similar concept for sets containing elements. For this purpose, we introduce the concept of local compactness in Definition 3.8.13 and we apply it to a topological space in Definition 3.8.14. Furthermore, we illustrate Definition 3.8.14 in Figure 3.3.

Definition 3.8.13 (Locally compact set)**.** *In a topological space X, a subset $A \subseteq X$ is* locally compact *if, for any element $x \in A$, there is a compact subset $K \subseteq A$ such that $(\exists U \in \mathcal{N}_x \mid U \subseteq K)$.*

Definition 3.8.14 (Locally compact space)**.** *A topological space X is* locally compact *if X is locally compact.*

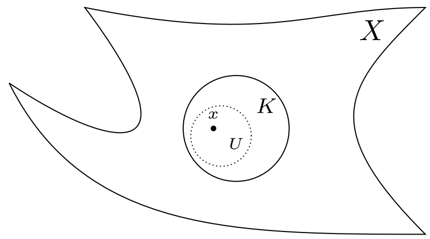

Figure 3.3: An example of local compactness of a space X on a point x where K is compact and U is open

Now that we have two notions of compactness in topology, a good question is how do they relate to each other? In other words, is there one which is stronger than the other? The intuition is to see that one property is defined on sets of sets while the other is defined on sets. From this perspective, we prove that compactness is stronger than locally compactness in Proposition 3.8.15.

Proposition 3.8.15. *A compact space is locally compact.*

Proof. For any element X, the compact set containing a neighbor of x is the whole space. □

A full proof of Proposition 3.8.15 is written in Appendix C.25.

Compact spaces are locally compact by Proposition 3.8.15. Are locally compact spaces compact? Not necessarily. What are non-compact locally compact spaces? Any infinite set equipped with the discrete topology (see Definition 3.1.4) is locally compact. Indeed, singletons are compact, because any open cover contains at least one open set containing the element. It is also open in the discrete topology. This proves the locally compact condition. However, an infinite set with the discrete topology is not compact! Indeed, one open cover is the collection of singletons. Since there are infinitely many of them, there is no finite subcover. This argument fails to be true if the set is finite as we proved in Proposition 3.8.5.

We are far, far away from Scott topology. Wait, are we? We are not, because we have a lot of tools at our disposition to go back to Scott topology. First, we need continuity to find open sets easily. Second, we have to find compact sets in the Scott topology. We have them as we stated previously. Hence, we are ready to prove Proposition 3.8.16.

Proposition 3.8.16. *A continuous dcpo equipped with its Scott topology is locally compact.*

Proof. For any element, there is an approximant by continuity. The upward closure of this approximant is compact and contains the Scott-open consisting of all the approximated elements of the approximant. Particularly, it contains the starting element. Hence, local compactness is proved. □

A full proof of Proposition 3.8.16 is written in Appendix C.26.

3.9 Continuous functions

So far, we have studied properties of open sets. In this section, we take a look at functions. Scott-continuous functions, presented in Section 2.3, are domain theoretic functions. For topology, they are simply called continuous functions. We define them in Definition 3.9.1 and we illustrate one in Figure 3.4.

Definition 3.9.1 (Continuous function)**.** *A function* $f \in [X \to Y]$ *between topological spaces* X *and* Y *is* continuous *if* $\big(\forall V \in \mathcal{O}(Y) \mid f^{-1}(V) \in \mathcal{O}(X)\big)$.

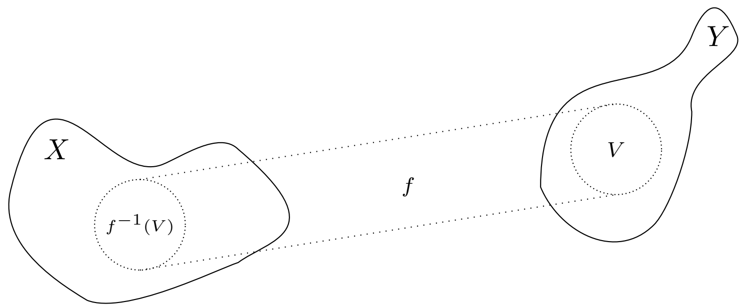

Figure 3.4: An example of a continuous function f, between two topological spaces X and Y, on an open set V

What if f is not surjective? Can it still be continuous? Indeed, it can. If an element is not in the image, then it will not be considered in the preimage set. What happens when the open set in the codomain is not "reachable"? In this case, the preimage is empty and the empty set is open by definition. Hence, a function that is not surjective can still be continuous. For example, consider the codomain as being at least a two-element set equipped with the discrete topology and the function as being the constant function. The latter is not surjective because it only reaches one element. The preimage of a set by the constant function is either the empty set or the whole set. Both are open. Hence, the constant function is continuous.

What if f is not injective? Then f^{-1} might not be a function. However, it does not affect continuity. Indeed, as pointed out in Notation 1.6.2, we are interested by a subset of the domain that is mapped by f to the desired open set of Y. Consider the example of the previous paragraph. The constant function is not injective, since it only reaches one of the element of the codomain. But it is continuous.

Examples of continuous functions are the classical continuous functions. We will see another example latter. Continuous functions can be built under some assumptions. For example, the composition of continuous functions is continuous. If the reader is interested, we recommend reading Section 2.18 of the book "Topology: A First Course" [14].

We now focus on properties of continuous functions. We begin by two alternative definitions given in Propositions 3.9.2 and 3.9.3.

Proposition 3.9.2. *A function $f \in [X \to Y]$ between topological spaces X and Y is continuous if and only if* $\big(\forall C \in \Gamma(Y) \mid f^{-1}(C) \in \Gamma(X)\big)$.

Proof. The proof is done by using the definition of a closed set, i.e., it is the complement of an open set. $\square$

A full proof of Proposition 3.9.2 is written in Appendix C.27.

Proposition 3.9.3. *A function $f \in [X \to Y]$ between topological spaces X and Y is continuous if and only if* $(\forall A \subseteq X \mid f(\mathbf{cl}(A)) \subseteq \mathbf{cl}(f(A)))$.

Proof. The "only if" direction is done by taking an element $a \in f(\mathbf{cl}(A))$. Then, we apply Proposition 3.2.10. The preimage of a neighbor of a is open by continuity of f and contains the preimage of a. Hence, it contains an element of A by Proposition 3.2.10. This element is both in the neighbor of a and in $f(A)$.

The "if" direction is done using the fact that a set is open if and only if its complement is closed. Proposition 3.2.6 is used to prove the closed condition. Finally, the proof of the inclusion is completed by using Proposition 3.2.10. $\square$

A full proof of Proposition 3.9.3 is written in Appendix C.28.

We have seen compactness properties in Section 3.8. It would be nice if, in some way, a function could preserve those properties. Fortunately, a continuous function does preserve the full compactness property as we stated in Proposition 3.9.4.

Proposition 3.9.4. *The image of a compact topological space by a continuous function is compact.*

Proof. Since an open cover of the image is open, its preimage is also open by continuity of the function. But the preimage is a compact topological space. More precisely, it is an open cover of the compact space. Compactness returns a finite subcover whose image is the finite subcover that we are looking for. $\square$

A full proof of Proposition 3.9.4 is written in Appendix C.29.

For bijective functions, we can apply continuity to their inverses. If a bijective function is continuous as well as its inverse, then it is called a homeomorphism, as defined in Definition 3.9.5. To end our little detour on homeomorphisms, we give an alternate definition in Proposition 3.9.6.

Definition 3.9.5 (Homeomorphism)**.** *A bijective function f is a* homeomorphism *if f and f^{-1} are continuous.*

Proposition 3.9.6. *A bijective function $f \in [X \to Y]$ between two topological spaces X and Y is a homeomorphism if and only if* $(\forall U \subseteq X \mid U \in \mathcal{O}(X) \iff f(U) \in \mathcal{O}(Y))$.

Proof. The "only if" part is done by using the continuity of f^{-1} and the fact $f = \left(f^{-1}\right)^{-1}$.

The "if" part is proved using set theory, notably $f = \left(f^{-1}\right)^{-1}$. Openness follows from the assumption. $\square$

A full proof of Proposition 3.9.6 is written in Appendix C.30.

Proposition 3.9.3 seems familiar. Having the image of the closure included in the closure of the image is close to a pattern that we saw in Section 2.3. The closure of a set A is a limit by adding all the elements on the edge of A, i.e., the least closed set containing A. Thus, we obtain a form very close to Corollary 2.3.5. We are still missing monotonicity but continuous functions have some link with Scott-continuous functions. Indeed, Scott-continuity of a function is the same thing as topological continuity. We investigate this in Proposition 3.9.7.

Proposition 3.9.7. *A function $f \in [D \to E]$ between two dcpos D and E equipped with their respective Scott topology is continuous if and only if it is Scott-continuous.*

Proof. The "only if" part is done using Corollary 2.3.5. Monotonicity and the "lower than or equal to" condition are proved by Proposition 3.6.8, which is used to find Scott-closed sets, and Proposition 3.9.2, which is used to deduce that their preimage is also Scott-closed. For monotonicity, we also use the downward closure property of Scott-closed sets to conclude. For the condition on directed sets, we also use the fact that Scott-closed sets contain the directed suprema of their directed subsets.

The "if" part is done by showing that the preimage of a Scott-open set is Scott-open. For upward closure, the upward closure property of the image is used to conclude. For the condition on intersecting directed sets, the intersecting directed set property of the image is used to conclude. To summarize, both properties are proved by mapping an element to its image and applying Scott-openness properties of the image. $\square$

A full proof of Proposition 3.9.7 is written in Appendix C.31.

3.10 Sobriety

In this section, we study the notion of sobriety. This property is very useful to construct a topological space from its open sets. We also study the notion of sobrification. It can be seen as some kind of completion in the sense that it produces a set containing its maximal points.

In topology, there are closed sets and open sets. These are very general notions without many properties. For sobriety, we ask for more specific closed sets. They are the irreducible closed sets defined in Definition 3.10.1.

Definition 3.10.1 (Irreducible closed set)**.** *In a topological space X, a closed set C is* irreducible *if $C \neq \emptyset$ and* $(\forall C_1, C_2 \in \Gamma(X) \mid C \subseteq C_1 \cup C_2 \Rightarrow C \subseteq C_1 \vee C \subseteq C_2)$.

Note that an irreducible closed set is not empty. This condition is often forgotten as for directedness[1].

Before giving classic examples, we introduce an alternative definition of irreducible closed set in Proposition 3.10.2. This time, we describe irreducible closed sets in terms of open sets. Therefore, depending on the context, we can adapt easily.

Proposition 3.10.2. *In a topological space X, a closed set C is irreducible if and only if $C \neq \emptyset$ and $(\forall U_1, U_2 \in \mathcal{O}(X) \mid C \cap U_1 \neq \emptyset \wedge C \cap U_2 \neq \emptyset \Rightarrow C \cap (U_1 \cap U_2) \neq \emptyset)$.*

Proof. The proof is done via pushing symbols through a succession of equivalences. The major hint is that a direction of the "if and only if" is the contraposition of the other and vice versa. □

A full proof of Proposition 3.10.2 is written in Appendix C.32.

The alternate definition of an irreducible closed set given in Proposition 3.10.2 might seem to be always true, but it is not. An easy counterexample in set theory is to consider the three sets $\{1, 2\}$, $\{1\}$ and $\{2\}$. We have $\{1, 2\} \cap \{1\} = \{1\} \neq \emptyset$ and $\{1, 2\} \cap \{2\} = \{2\} \neq \emptyset$, but $\{1, 2\} \cap (\{1\} \cap \{2\}) = \{1, 2\} \cap \emptyset = \emptyset$. Hence, this property of irreducible closed sets is not a tautology.

The closure of a set A is the smallest closed set containing A. But when A is a singleton, in any union of closed sets that contains the element of A, there is a closed set containing the single element of A. Hence, it should contain the closure of A. This reflection leads to Proposition 3.10.3.

Proposition 3.10.3. *In a topological space, the closure of a singleton is irreducible.*

Proof. Suppose that the closure of a singleton is contained in the union of two closed sets C_1 and C_2. Then C_1 or C_2 contains the singleton. Because the closure is the smallest closed set containing the singleton, it is included in C_1 or C_2. □

A full proof of Proposition 3.10.3 is written in Appendix C.33.

The topological spaces such that irreducible closed sets are closures of singletons are called sober as defined in Definition 3.10.4.

Definition 3.10.4 (Sober space)**.** *A topological space is* sober *if every irreducible closed set is the closure of a unique singleton.*

Sobriety is a nice property for a space. A lot comes from the uniqueness of the singleton. It means that whenever an irreducible closed set equals the closure of many elements they must be equal. This looks like a property we have already seen: antisymmetry. However, antisymmetry is about a partial order. Fortunately for us, we have one, i.e., the specialization partial ordering. There is only one requirement for the specialization preordering to be a

[1] The author made this mistake in his first proof of Proposition 3.10.18.

partial order: the space must be T_0. As Proposition 3.10.6 points out, sobriety allows us to have T_0 spaces. Note that we need Proposition 3.10.5 which encapsulates a part of the reasoning we just had.

Proposition 3.10.5. *Let X be a topological space. Then*

$$(\forall x, y \in X \mid \mathcal{N}_x = \mathcal{N}_y \Rightarrow \mathbf{cl}(\{x\}) = \mathbf{cl}(\{y\})) \ .$$

Proof. The proof is done by going from a closure to another using Proposition 3.2.10 and rewriting the intersection as the belonging to a neighborhood. □

A full proof of Proposition 3.10.5 is written in Appendix C.34.

Proposition 3.10.6. *A sober space is T_0.*

Proof. Two elements having the same neighborhood have the same closure by Proposition 3.10.5. Hence, they are equal by sobriety. □

A full proof of Proposition 3.10.6 is written in Appendix C.35.

Sober spaces are not just related to the T_0 axiom. Indeed, they are also related to the T_2 axiom. We remark that, in a T_2 space, if a set A contains more than one element, then there exist disjoint opens intersecting A. This remark and Proposition 3.10.2 lead to Proposition 3.10.7. Then Proposition 3.10.8 follows directly. In the end, we have the implication diagram presented in Figure 3.5.

Proposition 3.10.7. *In a T_2 space, an irreducible closed set is a singleton.*

Proof. By definition, irreducible closed sets are not empty. If they were to have more than two elements, we could apply the T_2 assumption to obtain disjoint neighbors. This leads to a contradiction using Proposition 3.10.2. □

A full proof of Proposition 3.10.7 is written in Appendix C.36.

Proposition 3.10.8. *A T_2 space is sober.*

Proof. The fact that irreducible closed sets are closures of singletons comes from Proposition 3.10.7. The uniqueness of the element comes from set theory and the definition of the closure. □

A full proof of Proposition 3.10.8 is written in Appendix C.37.

What are examples of sober spaces? We can start by a T_2 space such as the reals with the order topology. A sober space that is not T_2 is $\mathbb{N}_\omega$ equipped with the Alexandroff topology. It is not T_2 because, for two elements x and y such that $x \leq y$, any neighbor of x is a neighbor of y. Note that the case "$y \leq x$" is symmetric. $\mathbb{N}_\omega$ equipped with the Alexandroff topology is sober, because any closed set is the downward closure of an element. On the other hand, $\mathbb{N}$

$$
\begin{tikzcd}
 & \text{Sober} \arrow[dr, Rightarrow] & \\
T_2 \arrow[ur] \arrow[r, Rightarrow] & T_1 \arrow[r, Rightarrow] & T_0
\end{tikzcd}
$$

Figure 3.5: Implication diagrams between separation axioms and sobriety

equipped with the Alexandroff topology fails to be sober. Indeed, $\mathbb{N}$ is an irreducible closed subset of itself, but it is not the closure of a unique element.

Other examples of sober spaces are continuous dcpos equipped with their Scott topology, as stated in Proposition 3.10.10. If the reader is interested by the proof without hints, it is a good idea to read the proof of 2.7.2. It is done by considering some set. To simplify the proof of Proposition 3.10.10, we introduce Proposition 3.10.9.

Proposition 3.10.9. *In a dcpo equipped with its Scott topology, a principal ideal of an element is exactly its closure.*

Proof. The "inclusion" part is done using the downward closure property of a Scott-closed set. The "reverse inclusion" part is done using Proposition 3.6.8. $\square$

A full proof of Proposition 3.10.9 is written in Appendix C.38.

Proposition 3.10.10. *A continuous dcpo equipped with its Scott topology is sober.*

Proof. Let C be an irreducible Scott-closed set. Consider the set $C' = \bigcup_{c \in C} \twoheaddownarrow c$. C' is directed and has the same supremum as C. Hence, C is the closure of $\bigsqcup^{\uparrow} C'$. $\square$

A full proof of Proposition 3.10.10 is written in Appendix C.39.

Now, what about spaces that are not sober? There is a way to obtain a sober space from them using something call sobrification. We define it in Definition 3.10.11.

Definition 3.10.11 (Sobrification)**.** *In a topological space X, the set of irreducible closed subsets is the* sobrification *of X, noted $\mathcal{S}(X)$.*

As we said in the introduction to this section, the sobrification of a space can be seen as a completion where limits are added. We said that the naturals with their Alexandroff topology is not a sober space. However, $\mathbb{N}_\omega$ equipped with the Alexandroff topology is a sober space. But $\mathbb{N}_\omega$ is isomorphic to $\mathcal{S}(\mathbb{N})$. Each irreducible closed set having a maximal element is mapped to this element and $\mathbb{N}$ is mapped to ω. The inverse is to map a natural to its principal ideal and ω to $\mathbb{N}$. A more formal proof is a good exercise for the reader to confirm our sayings.

The sobrification of a space is sober, as stated in Proposition 3.10.18. However, we need a few more concepts to get there. We start with a new operator on open sets that is defined in Definition 3.10.12. We extend the latter to sets of open sets in Notation 3.10.13.

Definition 3.10.12 (Diamond open)**.** *In a topological space X, for any $U \in \mathcal{O}(X)$, the set $\{C \in \mathcal{S}(X) \mid U \cap C \neq \emptyset\}$ is the* diamond open set *of U, noted $\diamond U$.*

Notation 3.10.13. *In a topological space X, for any $\mathcal{U} \subseteq \mathcal{O}(X)$, the set $\{\diamond U \mid U \in \mathcal{U}\}$ is noted $\diamond \mathcal{U}$.*

There are interesting properties about the diamond operator. It looks like it can commute with unions and intersections and inclusion is preserved in both ways. We state those observations in Proposition 3.10.14.

Proposition 3.10.14. *Let X be a topological space.*

1. $(\forall \mathcal{U} \subseteq \mathcal{O}(X) \mid \diamond \bigcup \mathcal{U} = \bigcup \diamond \mathcal{U})$
2. $(\forall \mathcal{U} \subseteq \mathcal{O}(X) \mid \diamond \bigcap \mathcal{U} = \bigcap \diamond \mathcal{U})$
3. $(\forall U_1, U_2 \in \mathcal{O}(X) \mid U_1 \subseteq U_2 \iff \diamond U_1 \subseteq \diamond U_2)$

Proof. Items *1* and *2* are proved by set theory. The "only if" direction of Item *3* is proved by Item *1* and Proposition 1.7.3. The "if" direction of Item *3* is proved by contradiction. Suppose $u \in U_1$ and $u \in X - U_2$. The closure of u is in $X - U_2$ and in $\diamond U_1$. By assumption, it is also in $\diamond U_2$. This means $\mathbf{cl}(\{u\}) \cap U_2 \neq \emptyset$. But this is a contradiction with $\mathbf{cl}(u) \subseteq X - U_2$. Thus, we have $u \in U_2$. □

A full proof of Proposition 3.10.14 is written in Appendix C.40.

The topology on the sobrification of a space is the lower Vietoris topology. It is defined in Definition 3.10.15.

Definition 3.10.15 (Lower Vietoris topology)**.** *In a topological space X, the* lower Vietoris topology *is the set $\diamond \mathcal{O}(X) = \{\diamond U \mid U \in \mathcal{O}(X)\}$.*

Note that the lower Vietoris topology depends on the original topology of a space. It is also the case of the subspace topology.

We prove that the lower Vietoris topology is indeed a topology in Proposition 3.10.16.

Proposition 3.10.16. *The lower Vietoris topology is a topology.*

Proof. The lower Vietoris topology being closed under finite intersections and arbitrary unions is a consequence of Proposition 3.10.14. □

A full proof of Proposition 3.10.16 is written in Appendix C.41.

The lower Vietoris topology is a topology as we just stated in Proposition 3.10.16. This means that a set equipped with it has a specialization preordering (see Definition 3.6.1). In particular, we are interested in the sobrification which contains irreducible closed sets. Hence, the specialization preordering is between sets. It would be nice and simpler if it was the traditional inclusion, would it not? We are lucky, it is as stated in Proposition 3.10.17.

Proposition 3.10.17. *The specialization preordering on the sobrification of a space equipped with the lower Vietoris topology is the inclusion.*

Proof. In symbols, we want to show $(\forall C_1, C_2 \in \mathcal{S}(X) \mid C_1 \sqsubseteq_{\text{topo}} C_2 \iff C_1 \subseteq C_2)$. The "only if" part is done by contradiction. It means that $C_1 \in \diamond(X - C_2)$, which leads to a contradiction using the assumption. The "if" part is proved using the definition of the specialization preordering. If C_1 is in a diamond open set, then its intersection with the open sets of the space is not empty. It is the same for C_2 by assumption. Hence, it is also in the diamond open set as wanted. $\square$

A full proof of Proposition 3.10.17 is written in Appendix C.42.

We have all the requirements to prove that the sobrification is sober. We state it in Proposition 3.10.18.

Proposition 3.10.18. *The sobrification of a space equipped with the lower Vietoris topology is sober.*

Proof. An irreducible set $\mathcal{C}$ of the sobrification is the complement of a diamond open set $\diamond U$. The closure of the complement of U is $\mathcal{C}$. Uniqueness is proved using Proposition 3.10.17. The fact that the complement of U is irreducible is harder. Non-emptiness follows from the irreducibility of $\mathcal{C}$. For the rest of the definition, the trick is to work with the diamond version of the complement $C_1 \cup C_2$. It contains the complement of U. $\mathcal{C}$ is included in the complement of this open set. Without loss of generality, it makes $\mathcal{C}$ included in the closed set obtained from C_1 by irreducibility. The rest is set theory using Proposition 3.10.14 Item *3*. $\square$

A full proof of Proposition 3.10.18 is written in Appendix C.43.

3.11 Conclusion

We have seen the basics of topology in Sections 3.1 and 3.2. Then, we introduced the object of study, i.e., the Scott topology, in Section 3.3. Topologies are like dcpos, they are generated by bases as presented in Section 3.4. To continue on our mindset of ordering elements, we have presented terms to compare topologies on their open sets in Section 3.5. We have presented the specialization preordering, a preorder based on open sets, in Section 3.6. It can be turned into a partial order with additional axioms presented in Section 3.7. They also help to differentiate elements based on their neighborhood. We then studied the compactness of topological spaces in Section 3.8 and how functions can preserve topological properties in Section 3.9. Finally, we have presented a completion based on the irreducible closed sets in Section 3.10. Everything we have done gives us Theorem 3.11.1.

Theorem 3.11.1. *A continuous dcpo equipped with its Scott topology is locally compact and sober.*

Proof. The local compactness is proved in Proposition 3.8.16 and the sobriety in Proposition 3.10.10. □

This study of topology is a good basis for any book in topology such as [14] and [17]. The second one is more than recommended. Another good idea is to read the master thesis [18]. Their approach and their notations are different, but all the concepts are well explained, well exemplified and proved!

Conclusion

This master thesis gives all the needed concepts to explore the literature about Domain Theory. We are interested in this mathematical subject of computer science to use denotational semantics for security purposes. Malware programs can be coded in different languages with different syntax. To be free of syntax, we abstract malware programs and we consider its semantics. Domain Theory provides a good basis for semantics. In this branch of computer science, objects with nice properties such as being computable, which is a must for automation, are provided. From this point, we can verify properties by making proofs on the mathematical representation of programs. Hence, this master thesis provides definitions and detailed properties of Domain Theory. It is self-contained.

We started this master thesis by presenting Order Theory in Chapter 1. The most important concept is partial orders. Then we can add properties to obtain structures such as lattices. We also presented proof tactics to work with bounds. Finally, we presented the concept of monotonic functions.

From this starting point, we presented Domain Theory in Chapter 2. We started by presenting dcpos, Scott-continuous functions and the way-below relation. Domains are often at least dcpos. We presented additional properties to obtain different dcpos. The most important property is the continuity of a dcpo, because it implies the existence of a set, named the basis, from which we can construct any other element. From a computer science point of view, this means that we only need to stock this basis in memory rather than all the elements. The construction of a continuous dcpo from its basis is the final stone added to Chapter 2.

Domain Theory is strongly linked with topology. Hence, in Chapter 3, whenever we presented a topological concept, we enlightened its links to Domain Theory through the Scott topology on a dcpo. Moreover, this chapter shows how much continuity is important for a dcpo. It implies local compactness and sobriety which is rather non-trivial.

Where does this master thesis lead to? It leads to book chapters such as [7] or, more generally, to journals, such as [19]. The latter is state-of-the-art research on domain theory, which makes it a must read. A very interesting article in this conference is [16]. On its fourth page, the reader can find a summary figure of the different inclusions between spaces. There is a lot to cover! On a learning side, we highly recommend the book [17]. On the other hand, we have presented other topics which are interesting. Order Theory leads to books such as [3] and [11]. Proofs and computation lead to proof assistants such as Coq ([20]) or Isabelle/HOL

([21]). Formal verification leads to formal models such as WSTS presented in the trilogy [10], [22] and [23]. Topology Theory leas to books such as [14] and [24]. To summarize, this thesis leads to any domain presented and, in particular, the one that the reader had the most fun reading!

Bibliography

[1] D. Makinson, *Sets, Logic and Maths for Computing, Second Edition.* Undergraduate Topics in Computer Science, Springer, 2012.

[2] J. L. Balcázar and C. Tîrnăucă, "Border algorithms for computing Hasse diagrams of arbitrary lattices," in *Formal Concept Analysis* (P. Valtchev and R. Jäschke, eds.), (Berlin, Heidelberg), pp. 49–64, Springer Berlin Heidelberg, 2011.

[3] V. K. Garg, *Introduction to Lattice Theory with Computer Science Applications.* Hoboken, New Jersey: Wiley, 1st ed., June 2015.

[4] R. Kossak, *Mathematical Logic on Numbers, Sets, Structures, and Symmetry*, vol. 3 of *Springer Graduate Texts in Philosophy.* Springer International Publishing, 1st ed., 2018.

[5] G. O'Regan, *Mathematics in Computing an Accessible Guide to Historical, Foundational and Application Contexts.* London: Springer, 2013.

[6] M. Insall and E. W. Weisstein, "Lattice." http://mathworld.wolfram.com/Lattice.html. From MathWorld – A Wolfram Web Resource.

[7] A. Jung and S. Abramsky, *Handbook of Logic in Computer Science*, vol. 3, Chapter Domain Theory. Clarendon Press, 1995.

[8] D. Scott, "A type-theoretical alternative to ISWIM, CUCH, OWHY," Manuscript, University of Oxford, 1969.

[9] D. Scott, "A type-theoretical alternative to ISWIM, CUCH, OWHY," *Theor. Comput. Sci.*, vol. 121, pp. 411–440, Dec. 1993.

[10] A. Finkel and J. Goubault-Larrecq, "Forward analysis for WSTS, part I: Completions," *Leibniz International Proceedings in Informatics, LIPIcs*, vol. 3, Mar. 2009.

[11] G. Gierz, *Continuous Lattices and Domains*, vol. 93 of *Encyclopedia of mathematics and its applications.* Cambridge, U.K: Cambridge University Press, 2003.

[12] D. Farenick, *Fundamentals of Functional Analysis.* Universitext, Cham, Switzerland: Springer International Publishing, 2016.

[13] K. P. Hart, J. v. Mill, and P. Simon, *Recent Progress in General Topology III.* Atlantis Press, 1st ed., 2014.

[14] J. Munkres, *Topology: A First Course.* Prentice-Hall, 1974.

[15] J. Goubault-Larrecq, "On Noetherian spaces," in *Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science (LICS'07)*, (Wrocław, Poland), pp. 453–462, IEEE Computer Society Press, July 2007.

[16] M. de Brecht, J. Goubault-Larrecq, X. Jia, and Z. Lyu, "Domain-complete and LCS-complete spaces," in *Proceedings of the International Symposium on Domain Theory (ISDT'19)*, vol. 345 of *Electronic Notes in Theoretical Computer Science*, (Yangzhou, China), pp. 3–35, Elsevier Science Publishers, June 2019.

[17] J. Goubault-Larrecq, *Non-Hausdorff Topology and Domain Theory - Selected Topics in Point-Set Topology*, vol. 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.

[18] W. M. Al-Hanafi, "Scott topology and its relation to the Alexandroff topology," Master thesis, Islamic University of Gaza, Aug. 2008.

[19] A. Jung, Q. Li, L. Xu, and G. Zhang, eds., *The proceedings of ISDT 2019, the 8th International Symposium on Domain Theory and Its Applications, ISDT 2019*, vol. 345 of *Electronic Notes in Theoretical Computer Science*, (Yangzhou, China), Elsevier B.V., June 2019.

[20] G. Huet, G. Kahn, and C. Paulin-Mohring, "The Coq Proof Assistant: A Tutorial," *Research report*, vol. 178, Apr. 2013.

[21] T. Nipkow, L. C. Paulson, and M. Wenzel, *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*. Springer-Verlag, June 2019.

[22] A. Finkel and J. Goubault-Larrecq, "Forward analysis for WSTS, part II: Complete WSTS," *Logical Methods in Computer Science*, vol. 8, Sept. 2012.

[23] M. Blondin, A. Finkel, and J. Goubault-Larrecq, "Forward analysis for WSTS, part III: Karp-Miller trees," in *Proceedings of the 37th Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'17)* (S. Lokam and R. Ramanujam, eds.), vol. 93 of *Leibniz International Proceedings in Informatics*, (Kanpur, India), Leibniz-Zentrum für Informatik, Dec. 2017.

[24] W. Ballmann and W. Stern, *Introduction to Geometry and Topology*. Compact textbooks in mathematics, Cham, Switzerland: Birkhäuser, 2018.

Appendix A

Proofs of Chapter 1

This appendix contains detailed proofs of some propositions stated in Chapter 1. The statements to prove are all written in the first lines of their corresponding proof.

A.1 Proof of Proposition 1.1.6

Proof. Let $(P, \sqsubseteq)$ be a preordered set (resp. poset). We show that $(P, \sqsupseteq)$ is a preordered set (resp. poset).

1. Let $x \in P$. We show $x \sqsupseteq x$, i.e., $x \sqsubseteq x$ by definition. This is true since $\sqsubseteq$ is reflexive.

2. Suppose that $(P, \sqsubseteq)$ is a poset. Let $x, y \in P$ be such that $x \sqsupseteq y$ and $y \sqsupseteq x$. We show $x = y$. We have $x \sqsubseteq y$ and $y \sqsubseteq x$ by definition of $\sqsupseteq$. We have $x = y$, because $\sqsubseteq$ is antisymmetric.

3. Let $x, y, z \in P$ be such that $x \sqsupseteq y$ and $y \sqsupseteq z$. We show $x \sqsupseteq z$, i.e., $z \sqsubseteq x$ by definition. We have $y \sqsubseteq x$ and $z \sqsubseteq y$ by definition of $\sqsupseteq$. We have $z \sqsubseteq x$, because $\sqsubseteq$ is transitive.

$\square$

A.2 Proof of Proposition 1.2.7

Proof. Let X be a poset.

1. Suppose that $\bigsqcup X$ exists. We show that $\bigsqcup X$ is the top element of X. It is greater than or equal to any element of X because it is in $\mathbf{ub}(X)$ by definition of the supremum. It is unique by antisymmetry.

2. Suppose that $\bigsqcap X$ exists. We show that $\bigsqcap X$ is the bottom element of X. It is lower than or equal to any element of X because it is in $\mathbf{lb}(X)$ by definition of the infimum. It is unique by antisymmetry.

3. Suppose that $\bigsqcup\emptyset$ exists. We show that $\bigsqcup\emptyset$ is the bottom element of X. The upper bounds of the empty set are all the elements of X. Hence, $\bigsqcup\emptyset$ is lower than or equal to any element by definition of the supremum. It is unique by antisymmetry.

4. Suppose that $\bigsqcap\emptyset$ exists. We show that $\bigsqcap\emptyset$ is the top element of X. The lower bounds of the empty set are all the elements of X. Hence, $\bigsqcap\emptyset$ is greater than or equal to any element by definition of the infimum. It is unique by antisymmetry.

□

A.3 Proof of Proposition 1.3.2

Proof. Let X be a poset and $A, B \subseteq X$ be such that $(\forall a \in A \mid (\exists b \in B \mid a \sqsubseteq b))$. Suppose that $\bigsqcup A$ and $\bigsqcup B$ exist. We show $\bigsqcup A \sqsubseteq \bigsqcup B$. Let $a \in A$. Then there exists $b \in B$ such that $a \sqsubseteq b$. Thus, we have $a \sqsubseteq \bigsqcup B$ by transitivity and the fact that a supremum is an upper bound. Because we took an arbitrary element of A, we have $\bigsqcup B \in \mathbf{ub}(A)$. Hence, we have $\bigsqcup A \sqsubseteq \bigsqcup B$, because the supremum is the least upper bound. □

A.4 Proof of Proposition 1.3.4

Proof. Let A, B be sets and α be a poset indexed by A and B. We show that if the following suprema exist, then $\bigsqcup_A \bigsqcup_B \alpha = \bigsqcup_B \bigsqcup_A \alpha$. We note by $\alpha(a, b)$ the element of α at the index $a \in A$ and $b \in B$. Using antisymmetry, we prove both directions.

$\boxed{\sqsubseteq}$ We show $\bigsqcup_A \bigsqcup_B \alpha \sqsubseteq \bigsqcup_B \bigsqcup_A \alpha$. It is equivalent to show that the right hand side is an upper bound of $\left\{\bigsqcup_{b \in B} \alpha(a, b) \mid a \in A\right\}$, because the supremum is the least upper bound. Let $x \in A$. Consider $\bigsqcup_{b \in B} \alpha(x, b) \in \left\{\bigsqcup_{b \in B} \alpha(a, b) \mid a \in A\right\}$. We have to show that $\bigsqcup_{b \in B} \alpha(x, b) \sqsubseteq \bigsqcup_{b \in B} \bigsqcup_{a \in A} \alpha(a, b)$. By applying Proposition 1.3.2, we only have to prove $\big(\forall c \in \{\alpha(x, b) \mid b \in B\} \mid \big(\exists d \in \left\{\bigsqcup_{a \in A} \alpha(a, b) \mid b \in B\right\} \mid c \sqsubseteq d\big)\big)$. Let $c \in \{\alpha(x, b) \mid b \in B\}$. Then, $c = \alpha(x, y)$ for some $y \in B$. Choose $d = \bigsqcup_{a \in A} \alpha(a, y)$. It exists by assumption. We have $c \sqsubseteq d$ because a supremum is an upper bound.

$\boxed{\sqsupseteq}$ We show $\bigsqcup_A \bigsqcup_B \alpha \sqsupseteq \bigsqcup_B \bigsqcup_A \alpha$. The argument is symmetric to the one of the "$\sqsubseteq$" direction with A and B interchanged.

□

A.5 Proof of Proposition 1.4.4

Proof. Let L be a poset.

$\boxed{\Rightarrow}$ Suppose that L is a complete lattice. We show that, for any subset $A \subseteq L$, $\bigsqcup A$ exists. This is true by Definition 1.4.3 of a complete lattice.

$\boxed{\Leftarrow}$ Suppose that, for any subset $A \subseteq L$, $\bigsqcup A$ exists. We show that L is a complete lattice.

1. We show that, for any subset $A \subseteq L$, $\bigsqcup A$ exists. This is true by assumption.

2. We show that, for any subset $A \subseteq L$, $\bigsqcap A$ exists. Let $A \subseteq L$. By definition of the lower bounds, we have $(\forall a \in A \mid a \in \mathbf{ub}(\mathbf{lb}(A)))$. We have $\bigsqcup \mathbf{lb}(A) \in \mathbf{lb}(A)$, because the supremum is the least upper bound. Note that $\bigsqcup(\mathbf{lb}(A))$ exists because all suprema exist and $\mathbf{lb}(A) \neq \emptyset$ by Proposition 1.2.7 Item *3*. Since a supremum is an upper bound, we have $(\forall l \in \mathbf{lb}(A) \mid l \sqsubseteq \bigsqcup \mathbf{lb}(A))$. We have completed the proof because $\bigsqcup(\mathbf{lb}(A))$ exists and is $\bigsqcap A$ by definition.

$\square$

A.6 Proof of Proposition 1.5.3

Proof. Let X be a preordered set and $A \subseteq X$.

1. We show $\downarrow A = \downarrow(\downarrow A)$.

   $\boxed{\subseteq}$ Let $a \in \downarrow A$. We want $a \in \downarrow(\downarrow A)$. By reflexivity, we have $a \sqsubseteq a$. Hence, we have $a \in \downarrow(\downarrow A)$ by definition.

   $\boxed{\supseteq}$ Let $a \in \downarrow(\downarrow A)$. We want $a \in \downarrow A$. There exists $c \in \downarrow A$ such that $a \sqsubseteq c$, because $a \in \downarrow(\downarrow A)$. Thus, there exists $d \in A$ such that $c \sqsubseteq d$. We have $a \sqsubseteq d$ by transitivity. Hence, we have $a \in \downarrow A$ by definition.

2. We show $\uparrow A = \uparrow(\uparrow A)$. The proof is symmetric to that of Item 1 of this proof.

$\square$

A.7 Proof of Proposition 1.5.4

Proof. Let X be a preordered set and $A \subseteq X$. We show that A is upward closed if and only if $X - A$ is downward closed.

$\boxed{\Rightarrow}$ Suppose that A is upward closed. We want $(\forall x \in X, y \in X - A \mid x \sqsubseteq y \Rightarrow x \in X - A)$. Consider $x \in X$ and $y \in X - A$ such that $x \sqsubseteq y$. We prove by contradiction that $x \in X - A$. Hence, we suppose that $x \notin X - A$, i.e., $x \in A$. Since A is upward closed, we have $y \in A$, because $x \sqsubseteq y$. But this is a contradiction with $y \in X - A$. Hence $x \in X - A$ as wanted.

$\boxed{\Leftarrow}$ Suppose that $X - A$ is downward closed. We want $(\forall x \in X, a \in A \mid a \sqsubseteq x \Rightarrow x \in A)$. Consider $x \in X$ and $a \in A$ such that $a \sqsubseteq x$. We prove by contradiction that $x \in A$. Hence, we suppose that $x \notin A$, i.e., $x \in X - A$ by set theory. Since $X - A$ is downward closed, we have $a \in X - A$, because $a \sqsubseteq x$. But this is a contradiction with $a \in A$. Hence $x \in A$ as wanted.

$\square$

A.8 Proof of Proposition 1.5.5

Proof. Let X be a preordered set and $A \subseteq X$.

1. Suppose $A = \downarrow A$. We show $A = \bigcup_{a \in A} \downarrow a$.

   $\boxed{\subseteq}$ Let $b \in A$. We have $b \in \downarrow b$ by reflexivity. Hence, we have $b \in \bigcup_{a \in A} \downarrow a$.

   $\boxed{\supseteq}$ Let $b \in \bigcup_{a \in A} \downarrow a$. Then there exists $c \in A$ such that $b \sqsubseteq c$. Since A is downward closed, we have $b \in A$.

2. Suppose $A = \uparrow A$. We show $A = \bigcup_{a \in A} \uparrow a$. The proof is symmetric to that of Item 1 of this proof.

$\square$

A.9 Proof of Proposition 1.5.6

Proof. Let X be a poset and $A \subseteq X$ in which the follow bounds exist.

1. We show $\bigsqcup A = \bigsqcup(\downarrow A)$.

   $\boxed{\sqsubseteq}$ Since $A \subseteq \downarrow A$ by Proposition 1.5.2, we apply Proposition 1.3.1 Item *1* to conclude this direction.

   $\boxed{\sqsupseteq}$ By construction of $\downarrow A$, all its elements are lower than or equal to at least one element of A, which is lower than or equal to $\bigsqcup A$ by definition. By transitivity, all the elements of $\downarrow A$ are lower than or equal to $\bigsqcup A$, i.e., $\bigsqcup A \in \mathbf{ub}(\downarrow A)$. Since $\bigsqcup(\downarrow A)$ is the least upper bound of $\downarrow A$, $\bigsqcup A \sqsubseteq \bigsqcup(\downarrow A)$.

2. We show $\bigsqcap A = \bigsqcap(\uparrow A)$.

   $\boxed{\sqsubseteq}$ By construction of $\uparrow A$, all its elements are greater than or equal to at least one element of A, which is greater than or equal to $\bigsqcap A$ by definition. By transitivity, all the elements of $\uparrow A$ are greater than or equal to $\bigsqcap A$, i.e., $\bigsqcap A \in \mathbf{lb}(\uparrow A)$. Since $\bigsqcap(\uparrow A)$ is the greatest lowerbound of $\uparrow A$, $\bigsqcap A \sqsubseteq \bigsqcap(\uparrow A)$.

   $\boxed{\sqsupseteq}$ Since $A \subseteq \uparrow A$ by Proposition 1.5.2, we apply Proposition 1.3.1 Item *1* to conclude this direction.

$\square$

A.10 Proof of Proposition 1.6.4

Proof. Let P and Q be preordered sets. We show that the pointwise order over $[P \to Q]$ is a preorder. If P and Q are posets, then the pointwise order is a partial order.

1. We show reflexivity of $\sqsubseteq_{\mathrm{pt}}$. Let $f \in [P \to Q]$ and $x \in P$. Then, we have $f(x) \sqsubseteq_Q f(x)$ since $\sqsubseteq_Q$ is a preorder. Hence, $f \sqsubseteq_{\mathrm{pt}} f$ and reflexivity is proved.

2. We show transitivity of $\sqsubseteq_{\mathrm{pt}}$. Let $f, g, h \in [P \to Q]$ be such that $f \sqsubseteq_{\mathrm{pt}} g$ and $g \sqsubseteq_{\mathrm{pt}} h$, and $x \in P$. We have $f(x) \sqsubseteq_Q g(x)$ and $g(x) \sqsubseteq_Q h(x)$. Hence, we have $f(x) \sqsubseteq_Q h(x)$ since $\sqsubseteq_Q$ is a preorder. We conclude $f \sqsubseteq_{\mathrm{pt}} h$ and transitivity is proved.

3. Suppose P and Q are posets. We show antisymmetry of $\sqsubseteq_{\mathrm{pt}}$. Let $f, g \in [P \to Q]$ be such that $f \sqsubseteq_{\mathrm{pt}} g$ and $g \sqsubseteq_{\mathrm{pt}} f$. We show that $f = g$, i.e., that $(\forall x \in P \mid f(x) = g(x))$. Let $x \in P$. We have $f(x) \sqsubseteq_Q g(x)$ and $g(x) \sqsubseteq_Q f(x)$ by definition of $\sqsubseteq_{\mathrm{pt}}$. Then, we have $f(x) = g(x)$ since $\sqsubseteq_Q$ is a partial order. Hence, we have $f = g$ and antisymmetry is proved.

$\square$

A.11 Proof of Proposition 1.6.5

Proof. Let P be a set and Q be a poset.

1. Suppose that Q has a bottom element. We show that $([P \to Q], \sqsubseteq_{\mathrm{pt}})$ has a bottom element. Consider the function $f = \{\langle p, \bot_Q \rangle \mid p \in P\}$. Let $g \in [P \to Q]$ and $p \in P$ be an element. We have $f(p) = \bot_Q \sqsubseteq_Q g(p)$ by definition of f and $\bot_Q$. Hence, we have $f \sqsubseteq_{\mathrm{pt}} g$ which means that f is a bottom element as wanted.

2. Suppose that Q has a top element. We show that $([P \to Q], \sqsubseteq_{\mathrm{pt}})$ has a top element. The proof is symmetric to that of Item 1 of this proof.

$\square$

A.12 Proof of proposition 1.7.3

Proof. Let P and Q be posets and $f \in [P \to Q]$ be total and such that, for any $A \subseteq P$ such that $\bigsqcup A$ exists, $f(\bigsqcup A) = \bigsqcup_{a \in A} f(a)$. We show that f is monotonic. Let $x, y \in P$. The following equivalences and implication show the result:

$$
\begin{array}{cll}
 & x \sqsubseteq_P y & \\
\iff & & \langle \text{Order theory} \rangle \\
 & \bigsqcup_P \{x, y\} = y & \\
\Rightarrow & & \langle f \text{ is a function} \rangle \\
 & f(\bigsqcup_P \{x, y\}) = f(y) & \\
\iff & & \langle \text{Assumption} \rangle \\
 & \bigsqcup_Q \{f(x), f(y)\} = f(y) & \\
\iff & & \langle \text{Order theory} \rangle \\
 & f(x) \sqsubseteq_Q f(y) \ . &
\end{array}
$$

Note that by "Order theory" we mean "Definition 1.2.6 and the proof reasoning of Section 1.3 are used to prove the result.". □

Appendix B

Proofs of Chapter 2

This appendix contains detailed proofs of some propositions stated in Chapter 2. The statements to prove are all written in the first lines of their corresponding proof.

B.1 Proof of Proposition 2.1.4

Proof. Let X be a $\sqcup$-semi-lattice and $A \subseteq X$ be such that $\bigsqcup A$ exists and $A \neq \emptyset$. Consider $\text{FinSup} = \{\bigsqcup F \mid F \subseteq A \wedge F \text{ is finite}\}$. We show $\bigsqcup A = \bigsqcup^{\uparrow} \text{FinSup}$.

1. We show that FinSup is directed.

   (a) We show $\text{FinSup} \neq \emptyset$. Since $A \neq \emptyset$, there is $a \in A$. Then $a \in \text{FinSup}$ because $a = \bigsqcup \{a\}$.

   (b) We show $(\forall a, b \in \text{FinSup} \mid (\exists c \in \text{FinSup} \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in \text{FinSup}$. Then there exist finite subsets $F_a, F_b \subseteq A$ such that $a = \bigsqcup F_a$ and $b = \bigsqcup F_b$. Choose $c = \bigsqcup(F_a \cup F_b)$. It exists since X is a $\sqcup$-semi-lattice. We have $F_a \cup F_b \subseteq A$ since both are subsets of A. We have $F_a \cup F_b$ finite because the union of two finite sets is finite. Finally, we have $a = \bigsqcup F_a \sqsubseteq \bigsqcup(F_a \cup F_b) = c$ and $b = \bigsqcup F_b \sqsubseteq \bigsqcup(F_a \cup F_b) = c$ by applying Proposition 1.3.1 Item *1*.

2. We show $\bigsqcup A = \bigsqcup^{\uparrow} \text{FinSup}$.

   $\boxed{\sqsubseteq}$ We have $(\forall a \in A \mid a = \bigsqcup \{a\} \wedge \bigsqcup \{a\} \in \text{FinSup})$ and thus $(\forall a \in A \mid a \in \text{FinSup})$. Therefore, we have $\bigsqcup^{\uparrow} \text{FinSup} \in \mathbf{ub}(A)$. Hence, we conclude $\bigsqcup A \sqsubseteq \bigsqcup^{\uparrow} \text{FinSup}$, because the supremum is the least upper bound.

   $\boxed{\sqsupseteq}$ We have $(\forall f \in \text{FinSup} \mid f \sqsubseteq \bigsqcup A)$ by Proposition 1.3.1 Item *1*. Hence, $\bigsqcup A \in \mathbf{ub}(\text{FinSup})$. We conclude $\bigsqcup A \sqsupseteq \bigsqcup^{\uparrow} \text{FinSup}$, because the supremum is the least upper bound.

$\square$

B.2 Proof of Proposition 2.1.5

Proof. Let X be a set, $M \subseteq X$ be a finite subset and $\mathcal{D} \subseteq \mathcal{P}(X)$ be directed with respect to inclusion such that $M \subseteq \bigcup \mathcal{D}$. We show $(\exists D \in \mathcal{D} \mid M \subseteq D)$. We use induction on the size of M.

- **Base case:** Suppose that $|M| = 0$. Then $M = \emptyset$. Take D to be any element of $\mathcal{D}$. D exists since $\mathcal{D}$ is non-empty by directedness. We have $M \subseteq D$ as wanted.
- **Induction hypothesis:** For all sets $M' \subseteq \bigcup \mathcal{D}$ of cardinality n, there exists $D \in \mathcal{D}$ such that $M' \subseteq D$.
- **Induction step:** Suppose that $|M| = n + 1$. Then there exists $m \in M$. By the induction hypothesis, there is $D_1 \in \mathcal{D}$ such that $M - \{m\} \subseteq D_1$. Take $D_2 \in \mathcal{D}$ such that $m \in D_2$. D_2 exists since $M \subseteq \bigcup \mathcal{D}$. Take $D \in \mathcal{D}$ such that $D_1 \subseteq D$ and $D_2 \subseteq D$. D exists by directedness of $\mathcal{D}$. Finally, by set theory, we have $M = (M - \{m\}) \cup \{m\} \subseteq D_1 \cup D_2$ and, thus, $M \subseteq D$.

□

B.3 Proof of Proposition 2.1.6

Proof. Let X be a poset, $A \subseteq X$ be directed and $M \subseteq A$ be a finite subset. We show that there exists $u \in A$ such that $u \in \mathbf{ub}(M)$. We use induction on the size of M.

- **Base case:** Suppose that $|M| = 0$. Then $M = \emptyset$. Since $A \neq \emptyset$ by directedness, there is $a \in A$. We have $a \in \mathbf{ub}(M)$ as wanted.
- **Induction hypothesis:** For all sets $M' \subseteq A$ of cardinality n, there exists $u \in A$ such that $u \in \mathbf{ub}(M')$.
- **Induction step:** Suppose that $|M| = n + 1$. Then there exists $m \in M$. And, we have $m \in A$, because $M \subseteq A$. By the induction hypothesis, there is $u_1 \in A \cap \mathbf{ub}(M - \{m\})$. By directedness of A, there is $u \in A$ such that $m \sqsubseteq u$ and $u_1 \sqsubseteq u$. From $u_1 \in A \cap \mathbf{ub}(M - \{m\})$ and $u_1 \sqsubseteq u$, we obtain $u \in \mathbf{ub}(M - \{m\})$. But since $m \sqsubseteq u$, we have $u \in \mathbf{ub}(M)$.

□

B.4 Proof of Proposition 2.1.7

Proof. Let P and Q be preordered sets, $f \in \left[P \xrightarrow{m} Q\right]$ and $A \subseteq \texttt{dom}(f)$ be a directed subset. Consider $F_A = \{f(a) \mid a \in A\}$. We show that F_A is directed.

1. We show $F_A \neq \emptyset$. Since A is directed, we have $A \neq \emptyset$. Then there exists $a \in A$. Hence, $f(a)$ exists, because $A \subseteq \mathtt{dom}(f)$, and $F_A \neq \emptyset$.

2. We show the second condition for directedness. Let $f(a), f(b) \in F_A$. By directedness of A, there is $c \in A$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. Note that $f(c)$ is defined since $A \subseteq \mathtt{dom}(f)$. By monotonicity, we have $f(a) \sqsubseteq f(c)$ and $f(b) \sqsubseteq f(c)$ as wanted.

□

B.5 Proof of Proposition 2.1.9

Proof. Let X be a preordered set and $x \in X$.

1. We show that $\downarrow x$ is a principal ideal, i.e., directed, downward closed and the downward closure of a singleton.
   (a) We show directedness
      i. We show $\downarrow x \neq \emptyset$. This is true since $x \in \downarrow x$ by reflexivity.
      ii. We show $(\forall a, b \in \downarrow x \mid (\exists c \in \downarrow x \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in \downarrow x$. Choose $c = x$. It exists since x exists. By definition of the downward closure, we have $a \sqsubseteq c \wedge b \sqsubseteq c$.
   (b) We show downward closure, i.e., $\downarrow x = \downarrow\downarrow x$. This is true by Proposition 1.5.3 Item *1*.
   (c) We show that $\downarrow x$ is the downward closure of a singleton. The singleton is $\{x\}$. Then $\downarrow x = \downarrow \{x\}$ by reflexivity.

2. We show that $\uparrow x$ is a principal filter, i.e., filtered, upward closed and the upward closure of a singleton.
   (a) We show filteredness.
      i. We show $\uparrow x \neq \emptyset$. This is true since $x \in \uparrow x$ by reflexivity.
      ii. We show $(\forall a, b \in \uparrow x \mid (\exists c \in \uparrow x \mid a \sqsupseteq c \wedge b \sqsupseteq c))$. Let $a, b \in \uparrow x$. Choose $c = x$. It exists since x exists. By definition of the upward closure, we have $a \sqsupseteq c \wedge b \sqsupseteq c$.
   (b) We show upward closure, i.e., $\uparrow x = \uparrow\uparrow x$. This is true by Proposition 1.5.3 Item *2*.
   (c) We show that $\uparrow x$ is the upward closure of a singleton. The singleton is $\{x\}$. Then $\uparrow x = \uparrow \{x\}$ by reflexivity.

□

B.6 Proof of Proposition 2.1.11

Proof. Let L be a lattice and $D \subseteq L$. We show that D is an ideal if and only if D is an order theoretic ideal.

$\boxed{\Rightarrow}$ Suppose that D is an ideal. We show that D is an order theoretic ideal.

1. We show $D \neq \emptyset$. This is true since D is non-empty by idealness.
2. We show $D = \downarrow D$. This is true by idealness of D.
3. We show $(\forall a, b \in D \mid \bigsqcup \{a, b\} \in D)$. Since D is directed, there is $c \in D$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. Then, we have $c \in \mathbf{ub}(\{a, b\})$. Thus, we have $\bigsqcup \{a, b\} \sqsubseteq c$, because the supremum is the least upper bound. Hence, we have $\bigsqcup \{a, b\} \in D$ by downward closure of D.

$\boxed{\Leftarrow}$ Suppose that D is an order theoretic ideal. We show that D is an ideal.

1. We show that D is directed.
   (a) We show $D \neq \emptyset$. This is true since D is an order theoretic ideal.
   (b) We show $(\forall a, b \in D \mid (\exists c \in D \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in D$. Choose $c = \bigsqcup \{a, b\}$. It exists since D is an order theoretic ideal. We have $a \sqsubseteq c \wedge b \sqsubseteq c$ by definition of the supremum.
2. We show $D = \downarrow D$. This is true by order theoretic idealness of D.

□

B.7 Proof of Proposition 2.2.3

Proof. Let P be a poset. We show that $\mathbf{Idl}(P)$ is a dcpo. Let $\mathcal{A} \subseteq \mathbf{Idl}(P)$ be directed with respect to inclusion. We show $\bigcup \mathcal{A} \in \mathbf{Idl}(P)$ and $\bigcup \mathcal{A}$ is $\bigsqcup^{\uparrow} \mathcal{A}$.

1. We show $\bigcup \mathcal{A} \in \mathbf{Idl}(P)$, i.e., it is directed and downward closed.
   (a) We show $\bigcup \mathcal{A}$ is directed, i.e., $(\forall a, b \in \bigcup \mathcal{A} \mid (\exists c \in \bigcup \mathcal{A} \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in \bigcup \mathcal{A}$. Then there exist $I_a, I_b \in \mathcal{A}$ such that $a \in I_a$ and $b \in I_b$. By directedness of $\mathcal{A}$, there is $I_c \in \mathcal{A}$ such that $I_a \subseteq I_c$ and $I_b \subseteq I_c$. Take $c \in I_c$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. It exists and respects all conditions by directedness of I_c.
   (b) We show $\bigcup \mathcal{A} = \downarrow (\bigcup \mathcal{A})$.
   $\boxed{\subseteq}$ Let $a \in \bigcup \mathcal{A}$. We show $a \in \downarrow (\bigcup \mathcal{A})$. There is $I \in \mathcal{A}$ such that $a \in I$ by set theory. Then there is $c \in I$ such that $a \sqsubseteq c$ by directedness of I. Hence, $a \in \downarrow (\bigcup \mathcal{A})$.
   $\boxed{\supseteq}$ Let $a \in \downarrow (\bigcup \mathcal{A})$. We show $a \in \bigcup \mathcal{A}$. There is $b \in \bigcup \mathcal{A}$ such that $a \sqsubseteq b$ by definition of the downward closure. There is $I \in \mathcal{A}$ such that $b \in I$ by set theory. Since I is an ideal, it is downward closed. Hence, $a \in I$ and $a \in \bigcup \mathcal{A}$.

2. We show that $\bigcup \mathcal{A}$ is $\bigsqcup^{\uparrow} \mathcal{A}$.

   (a) We show $\bigcup \mathcal{A} \in \mathbf{ub}(\mathcal{A})$. This is true since any element of $\mathcal{A}$ is in the union, i.e., $\bigcup \mathcal{A}$.

   (b) We show $(\forall U \in \mathbf{ub}(\mathcal{A}) \mid \bigcup \mathcal{A} \subseteq U)$. Let $U \in \mathbf{ub}(\mathcal{A})$. We have $(\forall A \in \mathcal{A} \mid A \subseteq U)$ by definition of an upper bound. By set theory, we have $\bigcup \mathcal{A} \subseteq U$ as wanted.

□

B.8 Proof of Proposition 2.3.2

Proof. Let D, E be dcpos and $f \in [D \to E]$ be such that, for any $A \subseteq D$ directed, $f\left(\bigsqcup^{\uparrow} A\right) = \bigsqcup^{\uparrow}_{a \in A} f(a)$. We show that f is monotonic. Let $x, y \in D$ be elements such that $x \sqsubseteq_D y$. Consider the set $\{x, y\}$. It is non-empty and, for any pair, there is an upper bound in the set. Hence, the set $\{x, y\}$ is directed and $\bigsqcup^{\uparrow} \{x, y\} = y$. From this, we use the assumption on f and the fact that a supremum is an upper bound to obtain $f(x) \sqsubseteq_E \bigsqcup^{\uparrow}_{z \in \{x,y\}} f(z) = f\left(\bigsqcup^{\uparrow} \{x, y\}\right) = f(y)$. By transitivity, we conclude $f(x) \sqsubseteq_E f(y)$. □

B.9 Proof of Proposition 2.3.7

Proof. Let D, E be dcpos and let $\left[D \xrightarrow{cont} E\right]$ be the set of Scott-continuous functions between D and E. We show that $\left[D \xrightarrow{cont} E\right]$ ordered pointwise is a dcpo. Let $F \subseteq \left[D \xrightarrow{cont} E\right]$ be directed. We show that $\bigsqcup^{\uparrow} F$ exists and $\bigsqcup^{\uparrow} F \in \left[D \xrightarrow{cont} E\right]$.

1. We show that g is $\bigsqcup^{\uparrow} F$ where g maps an element $d \in D$ to $\bigsqcup^{\uparrow}_{f \in F} f(d)$. Note that the set $\{f(d) \mid f \in F\}$ is directed because F is directed.

   (a) We show $g \in \mathbf{ub}(F)$, i.e., $(\forall f \in F \mid f \sqsubseteq_{\text{pt}} g)$. Let $f \in F$ and $d \in D$. We have $f(d) \sqsubseteq_E \bigsqcup^{\uparrow}_{f \in F} f(d) = g(d)$ because the supremum is an upper bound. Hence, since we took an arbitrary d, we have $f \sqsubseteq_{\text{pt}} g$ as wanted.

   (b) We show $(\forall u \in \mathbf{ub}(F) \mid g \sqsubseteq_{\text{pt}} u)$. Let $u \in \mathbf{ub}(F)$ and let $d \in D$. Because u is an upper bound of F, we have $(\forall f \in F \mid f(d) \sqsubseteq_E u(d))$. We have $g(d) = \bigsqcup^{\uparrow}_{f \in F} f(d) \sqsubseteq_E u(d)$, because the supremum is the least upper bound. Hence, since we took an arbitrary d, we have $g \sqsubseteq_{\text{pt}} u$ as wanted.

   Note that, because g exists, so does $\bigsqcup^{\uparrow} F$.

2. We show that $\bigsqcup^{\uparrow} F \in [D \xrightarrow{cont} E]$. Let $A \subseteq D$ be directed. The following equalities

shows the Scott-continuity condition.

$$
\begin{array}{ll}
 & \bigsqcup^{\uparrow} F\left(\bigsqcup^{\uparrow} A\right) \\
= & \langle \text{Definition of } \bigsqcup^{\uparrow} F \rangle \\
 & \bigsqcup^{\uparrow}_{f \in F} f\left(\bigsqcup^{\uparrow} A\right) \\
= & \langle \text{Scott-continuity of } f \rangle \\
 & \bigsqcup^{\uparrow}_{f \in F} \bigsqcup^{\uparrow}_{a \in A} f(a) \\
= & \langle \text{Proposition 1.3.4} \rangle \\
 & \bigsqcup^{\uparrow}_{a \in A} \bigsqcup^{\uparrow}_{f \in F} f(a) \\
= & \langle \text{Definition of } \bigsqcup^{\uparrow} F \rangle \\
 & \bigsqcup^{\uparrow}_{a \in A} \left(\bigsqcup^{\uparrow} F(a)\right)
\end{array}
$$

□

B.10 Proof of Proposition 2.4.3

Proof. Let D be a dcpo. We show that the way-below relation over D is antisymmetric and transitive.

1. We show $(\forall x, y \in D \mid x \ll y \wedge y \ll x \Rightarrow x = y)$. Let $x, y \in D$ be such that $x \ll y$ and $y \ll x$. By Proposition 2.4.2 Item *1*, we have $x \sqsubseteq y$ and $y \sqsubseteq x$. Hence, since $\sqsubseteq$ is a partial order, we have $x = y$.

2. We show $(\forall x, y, z \in D \mid x \ll y \wedge y \ll z \Rightarrow x \ll z)$. Let $x, y, z \in D$ be such that $x \ll y$ and $y \ll z$. By Proposition 2.4.2 Item *1* and reflexivity, we have $x \sqsubseteq y$ and $z \sqsubseteq z$. We conclude that $x \ll z$ by applying Proposition 2.4.2 Item *2* with $x' := x$, $x := y$, $y := z$ and $y' := z$.

□

B.11 Proof of Proposition 2.4.6

Proof. Let D be a dcpo and $x \in D$.

1. We show $\twoheaduparrow x \subseteq \uparrow x$. Let $y \in \twoheaduparrow x$. By Proposition 2.4.2 Item *1*, we have $x \sqsubseteq y$. Hence, we have $y \in \uparrow x$.

2. We show $\twoheaddownarrow x \subseteq \downarrow x$. Let $y \in \twoheaddownarrow x$. By Proposition 2.4.2 Item *1*, we have $y \sqsubseteq x$. Hence, we have $y \in \downarrow x$.

3. Suppose $x \in K(D)$. We only show $\uparrow x \subseteq \twoheaduparrow x$ since the "$\supseteq$" direction is proved above. Let $y \in \uparrow x$ and $A \subseteq D$ be directed and such that $y \sqsubseteq \bigsqcup^{\uparrow} A$. By transitivity, we have

$x \sqsubseteq \bigsqcup^{\uparrow} A$. Since x is compact, we have $(\exists a \in A \mid x \sqsubseteq a)$. Hence, we conclude $x \ll y$, i.e., $y \in \twoheaduparrow x$.

4. Suppose $x \in K(D)$. We only show $\downarrow x \subseteq \twoheaddownarrow x$ since the "$\supseteq$" direction is proved above. Let $y \in \downarrow x$ and $A \subseteq D$ be directed and such that $x \sqsubseteq \bigsqcup^{\uparrow} A$. Since x is compact, we have $(\exists a \in A \mid x \sqsubseteq a)$. By transitivity, we have $y \sqsubseteq a$. Hence, we conclude $y \ll x$, i.e., $y \in \twoheaddownarrow x$.

□

B.12 Proof of Proposition 2.4.9

Proof. Let L be a complete lattice and $x \in L$. We show that $\twoheaddownarrow x$ is a $\sqcup$-semi-lattice. Let $M \subseteq \twoheaddownarrow x$ be a finite subset. We have to show $\bigsqcup M \in \twoheaddownarrow x$. Note that $\bigsqcup M$ exists since L is a complete lattice. Let $A \subseteq L$ be directed such that $x \sqsubseteq \bigsqcup^{\uparrow} A$. We have to show $(\exists a \in A \mid \bigsqcup M \sqsubseteq a)$. Since $M \ll x$, there is a "greater than or equal to" element in A for each element in M. Hence, consider $M_A = \{a_m \in A \mid m \in M \wedge m \sqsubseteq a_m\}$. M_A is finite, since M is finite, and $M_A \subseteq A$. Then there exists $a \in A \cap \mathbf{ub}(M_A)$ by Proposition 2.1.6. We have $\mathbf{ub}(M_A) \subseteq \mathbf{ub}(M)$ by definition of an upper bound and transitivity. Hence, we have $a \in \mathbf{ub}(M)$ and we conclude $\bigsqcup M \sqsubseteq a$, because the supremum is the least upper bound. □

B.13 Proof of Proposition 2.5.2

Proof. Let D be a dcpo and $B \subseteq D$ be a basis.

1. Let $x \in D$.

   (a) We show that B_x is directed, i.e., $(\forall a, b \in B_x \mid (\exists c \in B_x \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in B_x$. Since B is a basis, there is $B'_x \subseteq B_x$ directed and such that $\bigsqcup^{\uparrow} B'_x = x$. Take $a', b' \in B'_x$ such that $a \sqsubseteq a'$ and $b \sqsubseteq b'$. They exist because $a \ll x$, $b \ll x$ and $\bigsqcup^{\uparrow} B'_x = x$. By directedness of B'_x, there is $c \in B'_x$ such that $a' \sqsubseteq c$ and $b' \sqsubseteq c$. We have $c \in B_x$, since $B'_x \subseteq B_x$. Finally, we have $a \sqsubseteq c$ and $b \sqsubseteq c$ by transitivity.

   (b) We show that $\bigsqcup^{\uparrow} B_x = x$ by antisymmetry. Note that $\bigsqcup^{\uparrow} B_x$ exists since D is a dcpo.

   $\boxed{\sqsubseteq}$ By Proposition 2.4.2 Item *1*, we have $x \in \mathbf{ub}(B_x)$. We have $\bigsqcup^{\uparrow} B_x \sqsubseteq x$, because the supremum is the least upper bound.

   $\boxed{\sqsupseteq}$ Since there is a directed subset with supremum x in B_x, we have $x \sqsubseteq \bigsqcup^{\uparrow} B_x$ by Proposition 1.3.1 Item *1*.

2. Let $c \in K(D)$. We show $c \in B$. Since B is a basis, there is $B'_c \subseteq B_c$ directed and such that $\bigsqcup^{\uparrow} B'_c = c$. Take $c' \in B'_c$ such that $c \sqsubseteq c'$. It exists since $c \ll c$ and $\bigsqcup^{\uparrow} B'_c = c$. We

also have $c' \sqsubseteq \bigsqcup^{\uparrow} B'_c = c$ because the supremum is an upper bound by definition. Hence, we have $c' = c$ by antisymmetry. We conclude $c \in B$, because $c \in B'_c$ and $B'_c \subseteq B$.

3. Let $B' \subseteq D$ be such that $B \subseteq B'$. We show that B' is also a basis of D. Let $x \in D$. We have $B_x \subseteq B'_x$ since the approximants stay the same. Hence, we have that B'_x contains a directed subset with supremum x. Therefore, B' is a basis.

$\square$

B.14 Proof of Proposition 2.5.5

Proof. Let D be a dcpo with a basis B, E be a dcpo, $f \in [D \to E]$ be Scott-continuous and $d \in D$. We show that $f(d)$ is computable with only elements of B. Indeed, we have $f(d) = f\left(\bigsqcup^{\uparrow} B_d\right) = \bigsqcup^{\uparrow}_{b \in B_d} f(b)$ using Proposition 2.5.2 Item *1* and Scott-continuity. $\square$

B.15 Proof of Proposition 2.6.4

Proof. Let D be a dcpo.

1. We show that D is continuous if and only if $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x\right)$.

   $\boxed{\Rightarrow}$ Suppose that D is continuous, i.e., it has a basis B. We show $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x\right)$. Let $x \in D$.

   (a) We show that $\mathord{\downarrow\!\!\downarrow} x$ is directed.

      i. We show $\mathord{\downarrow\!\!\downarrow} x \neq \emptyset$. By Proposition 2.5.2 Item *1*, we have B_x directed. Hence, we have $B_x \neq \emptyset$. We also have $B_x \subseteq \mathord{\downarrow\!\!\downarrow} x$ by definition. Therefore, we conclude $\mathord{\downarrow\!\!\downarrow} x \neq \emptyset$.

      ii. We show $(\forall a, b \in \mathord{\downarrow\!\!\downarrow} x \mid (\exists c \in \mathord{\downarrow\!\!\downarrow} x \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in \mathord{\downarrow\!\!\downarrow} x$. Take $a', b' \in B_x$ such that $a \sqsubseteq a'$ and $b \sqsubseteq b'$. They exist because $a \ll x$, $b \ll x$ and $\bigsqcup^{\uparrow} B_x = x$ by Proposition 2.5.2 Item *1*. By the same proposition same item, there is $c \in B_x$ such that $a' \sqsubseteq c$ and $b' \sqsubseteq c$. We have $c \in \mathord{\downarrow\!\!\downarrow} x$ because $B_x \subseteq \mathord{\downarrow\!\!\downarrow} x$. We have $a \sqsubseteq c$ and $b \sqsubseteq c$ by transitivity.

   (b) We show $x = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x$. Note that the latter exists since D is a dcpo.

      $\boxed{\sqsubseteq}$ We have $\bigsqcup^{\uparrow} B_x \sqsubseteq \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x$ by Proposition 1.3.1 Item *1*. By Proposition 2.5.2 Item *1*, we conclude $x \sqsubseteq \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x$.

      $\boxed{\sqsupseteq}$ We have $(\forall y \in \mathord{\downarrow\!\!\downarrow} x \mid y \sqsubseteq x)$ by Proposition 2.4.2 Item *1*. Hence, we have $x \in \mathbf{ub}(\mathord{\downarrow\!\!\downarrow} x)$. We conclude $\bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x \sqsubseteq x$, because the supremum is the least upper bound.

   $\boxed{\Leftarrow}$ Suppose $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x\right)$. We show that D is continuous, i.e., D has a basis B. Choose $B = D$. B exists since D does. Let $y \in D$. We have to show that there is a directed subset of B_y with y as supremum. This subset is B_y itself. Indeed, we have $B_y = B \cap \downarrow\!\!\downarrow y = D \cap \downarrow\!\!\downarrow y = \downarrow\!\!\downarrow y$, and $\downarrow\!\!\downarrow y$ is directed and has y as supremum.

2. We show that D is algebraic if and only if $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} K(D)_x\right)$.

   $\boxed{\Rightarrow}$ Suppose that D is algebraic. We show $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} K(D)_x\right)$. It is true by Proposition 2.5.2 Item *1* with $K(D)$ as the basis.

   $\boxed{\Leftarrow}$ Suppose $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} K(D)_x\right)$. We show that D is algebraic, i.e., $K(D)$ is a basis. Let $y \in D$. We have to show that there is a directed subset of $K(D)_y$ with y as supremum. This subset is $K(D)_y$ itself since it is directed and has y as supremum.

□

B.16 Proof of Proposition 2.7.1

Proof. Let D be a continuous dcpo, $B \subseteq D$ be a basis and $x, y \in D$. We show $x \sqsubseteq y \iff B_x \subseteq B_y$.

$\boxed{\Rightarrow}$ Suppose $x \sqsubseteq y$. Let $b \in B_x$. We show that $b \in B_y$. We already have $b \in B$. We just miss $b \ll y$. The latter is proved by applying Proposition 2.4.2 Item *2* with $x' = b$, $x = b$, $y = x$ and $y' = y$.

$\boxed{\Leftarrow}$ Suppose $B_x \subseteq B_y$. We show $x \sqsubseteq y$. As wanted, we have $x = \bigsqcup^{\uparrow} B_x \sqsubseteq \bigsqcup^{\uparrow} B_y = y$ by Proposition 2.5.2 Item *1* and Proposition 1.3.1 Item *1*.

□

B.17 Proof of Proposition 2.7.2

Proof. Let D be a continuous dcpo, $y \in D$ and $M \subseteq D$ be a finite subset such that $M \ll y$. We show $(\exists x \in D \mid M \ll x \wedge x \ll y)$. Consider $A = \{a \in D \mid (\exists a' \in D \mid a \ll a' \wedge a' \ll y)\}$.

1. We show that A is directed.

   (a) We show $A \neq \emptyset$. We have $\downarrow\!\!\downarrow y \neq \emptyset$ by directedness. Applying the same reasoning, we have $(\forall z \in \downarrow\!\!\downarrow y \mid \downarrow\!\!\downarrow z \neq \emptyset)$. But $(\forall z \in \downarrow\!\!\downarrow y \mid \downarrow\!\!\downarrow z \subseteq A)$ and we conclude $A \neq \emptyset$.

   (b) We show $(\forall b, c \in A \mid (\exists d \in A \mid b \sqsubseteq d \wedge c \sqsubseteq d))$. Let $b, c \in A$. Since $b, c \in A$, there are $b_D, c_D \in D$ such that $b \ll b_D$, $b_D \ll y$, $c \ll c_D$ and $c_D \ll y$. By directedness, there is $d_D \in \downarrow\!\!\downarrow y$ such that $b_D \sqsubseteq d_D$ and $c_D \sqsubseteq d_D$. Note that $b \ll d_D$ and $c \ll d_D$ by Proposition 2.4.2 Item *2*. Thus, by directedness, there are $b', c' \in \downarrow\!\!\downarrow d_D$ such that $b \sqsubseteq b'$ and $c \sqsubseteq c'$. Again by directedness, there is $d \in \downarrow\!\!\downarrow d_D$ such that $b' \sqsubseteq d$ and $c' \sqsubseteq d$. We have $d \in A$, because $d \ll d_D$ and $d_D \ll y$. By transitivity, we also have $b \sqsubseteq d$ and $c \sqsubseteq d$ as wanted.

2. We show $\bigsqcup^{\uparrow} A = y$. Note that $\bigsqcup^{\uparrow} A$ exists since D is a dcpo and A is directed.

$\boxed{\sqsubseteq}$ Let $a \in A$. By Proposition 2.4.3, we have $a \ll y$. Hence, we have $a \sqsubseteq y$ by Proposition 2.4.2 Item *1*. Since we considered an arbitrary element a, we have $y \in \mathbf{ub}(A)$. We have $\bigsqcup^{\uparrow} A \sqsubseteq y$, because the supremum is the least upper bound.

$\boxed{\sqsupseteq}$ We showed $(\forall z \in \downarrow\!\!\!\downarrow y \mid \downarrow\!\!\!\downarrow z \subseteq A)$ in Item 1a of this proof. By set theory, we have $\bigcup_{z \in \downarrow\!\!\!\downarrow y} \downarrow\!\!\!\downarrow z \subseteq A$. Hence, we have $y = \bigsqcup^{\uparrow} \downarrow\!\!\!\downarrow y = \bigsqcup^{\uparrow}_{z \in \downarrow\!\!\!\downarrow y} \left(\bigsqcup^{\uparrow} \downarrow\!\!\!\downarrow z \right) = \bigsqcup \left(\bigcup_{z \in \downarrow\!\!\!\downarrow y} \downarrow\!\!\!\downarrow z \right) \sqsubseteq \bigsqcup^{\uparrow} A$ by Proposition 2.6.4 Item *1*, Proposition 1.3.1 Item *3* and Proposition 1.3.1 Item *1*.

3. We show $(\exists x \in B \mid M \ll x \wedge x \ll y)$. We have $(\forall m \in M \mid (\exists a_m \in A \mid m \sqsubseteq a_m))$, because $\bigsqcup^{\uparrow} A = y$ and $M \ll y$. By Proposition 2.1.6, there is $a \in A$ such that $a \in \mathbf{ub}(\{a_m \in A \mid m \in M \wedge m \sqsubseteq a_m\})$ (if $M = \emptyset$, then $\mathbf{ub}(\{a_m \in A \mid m \in M \wedge m \sqsubseteq a_m\}) = A$). Since $a \in A$, there is $x \in D$ such that $a \ll x$ and $x \ll y$. We have $M \ll x$ by Proposition 2.4.2 Item *2*.

□

B.18 Proof of Proposition 2.7.3

Proof. Let D be a dcpo, $B \subseteq D$ be a basis, $y \in D$ and $M \subseteq D$ be a finite subset such that $M \ll y$. We show $(\exists x \in B \mid M \ll x \wedge x \ll y)$.
Consider $A = \{a \in D \mid (\exists a' \in B \mid a \ll a' \wedge a' \ll y)\}$.

1. We show that A is directed.

   (a) We show $A \neq \emptyset$. Since B is a basis, we have $B_y \neq \emptyset$. Applying the same reasoning, we have $(\forall b \in B_y \mid B_b \neq \emptyset)$. Hence, we have $(\forall b \in B_y \mid B_b \subseteq A)$. Therefore, we conclude $A \neq \emptyset$.

   (b) We show $(\forall b, c \in A \mid (\exists d \in A \mid b \sqsubseteq d \wedge c \sqsubseteq d))$. Let $b, c \in A$. Since $b, c \in A$, there are $b_B, c_B \in B$ such that $b \ll b_B$, $b_B \ll y$, $c \ll c_B$ and $c_B \ll y$. Take $b_{B_y}, c_{B_y} \in B_y$ such that $b_B \sqsubseteq b_{B_y}$ and $c_B \sqsubseteq c_{B_y}$. They exist because $b_B \ll y$, $c_B \ll y$ and $\bigsqcup^{\uparrow} B_y = y$ by Proposition 2.5.2 Item *1*. Since B_y is directed by the same proposition, there is $d_{B_y} \in B_y$ such that $b_{B_y} \sqsubseteq d_{B_y}$ and $c_{B_y} \sqsubseteq d_{B_y}$. We have $b \ll d_{B_y}$ and $c \ll d_{B_y}$ by Proposition 2.4.2 Item *2* and by transitivity. Take the elements $b', c' \in B_{d_{B_y}}$ such that $b \sqsubseteq b'$ and $c \sqsubseteq c'$. They exist because of approximation and Proposition 2.5.2 Item *1*. Since $B_{d_{B_y}}$ is directed by the same proposition, there is $d \in B_{d_{B_y}}$ such that $b' \sqsubseteq d$ and $c' \sqsubseteq d$. We have $d \in A$ because $d \ll d_{B_y}$ and $d_{B_y} \ll y$. We have $b \sqsubseteq d$ and $c \sqsubseteq d$ by transitivity.

2. We show $\bigsqcup^{\uparrow} A = y$. Note that $\bigsqcup^{\uparrow} A$ exists since D is a dcpo and A is directed.

$\boxed{\sqsubseteq}$ Let $a \in A$. By Proposition 2.4.3, we have $a \ll y$. Hence, we have $a \sqsubseteq y$ by Proposition 2.4.2 Item *1*. Since we considered an arbitrary element a, we have $y \in \mathbf{ub}(A)$. We have $\bigsqcup^{\uparrow} A \sqsubseteq y$, because the supremum is the least upper bound.

$\boxed{\sqsupseteq}$ From Item 1a of this proof, we have $(\forall b \in B_y \mid B_b \subseteq A)$. By set theory, we have $\bigcup_{b\in B_y} B_b \subseteq A$. Hence, we have $y = \bigsqcup^{\uparrow} B_y = \bigsqcup^{\uparrow}_{b\in B_y}\left(\bigsqcup^{\uparrow} B_b\right) = \bigsqcup\left(\bigcup_{b\in B_y} B_b\right) \sqsubseteq \bigsqcup^{\uparrow} A$ by Proposition 2.5.2 Item *1*, Proposition 1.3.1 Item *3* and Proposition 1.3.1 Item *1*.

3. We show $(\exists x \in B \mid M \ll x \wedge x \ll y)$. We have $(\forall m \in M \mid (\exists a_m \in A \mid m \sqsubseteq a_m))$, because $\bigsqcup^{\uparrow} A = y$ and $M \ll y$. By Proposition 2.1.6, there is $a \in A$ such that $a \in \mathbf{ub}(\{a_m \in A \mid m \in M \wedge m \sqsubseteq a_m\})$ (if $M = \emptyset$, then $\mathbf{ub}(\{a_m \in A \mid m \in M \wedge m \sqsubseteq a_m\}) = A$). Since $a \in A$, there is $x \in B$ such that $a \ll x$ and $x \ll y$. We have $M \ll x$ by Proposition 2.4.2 Item *2*.

□

B.19 Proof of Proposition 2.7.5

Proof. Let D be a dcpo, $B \subseteq D$ be a basis and $d \in B - K(D)$. We show that $B - \{d\}$ is still a basis of D. Let $x \in D$. If $d \notin \mathord{\downarrow\!\!\downarrow} x$, then we are done, because $(B - \{d\}) \cap \mathord{\downarrow\!\!\downarrow} x = B_x$. Hence, suppose $d \in \mathord{\downarrow\!\!\downarrow} x$. Since B is a basis, there is $C_x \subseteq B_x$ directed and such that $\bigsqcup^{\uparrow} C_x = x$. By Proposition 2.7.3 with $[M = \{d\}]$, there is $d' \in B$ such that $d \ll d'$ and $d' \ll x$. Note that $d \neq d'$ because $d \notin K(D)$ and $d \ll d'$. Consider the set $E_x = (C_x - \{d\}) \cup \{d'\}$. We claim that E_x is directed, has x as supremum and is a subset of $(B - \{d\})_x$.

1. We prove that E_x is directed.

   (a) We show $E_x \neq \emptyset$. This is true since E_x contains at least d'.

   (b) We show $(\forall a, b \in E_x \mid (\exists c \in E_x \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in E_x$.

      i. Suppose $a = d'$. Take $e \in C_x$ such that $d' \sqsubseteq e$. It exists since $d' \ll x$ and $\bigsqcup^{\uparrow} C_x = x$. By directedness of C_x, there is $c \in C_x$ such that $e \sqsubseteq c$ and $b \sqsubseteq c$. Note that $c \neq d$ otherwise we would have $d = d'$ which is a contradiction. Hence, we have $c \in E_x$. We have $a \sqsubseteq c$ by transitivity.

      ii. Suppose $b = d'$. This case is symmetric to that of Item 1(b)i of this proof.

      iii. Suppose $a \neq d'$ and $b \neq d'$. By directedness of C_x, there is $e \in C_x$ such that $a \sqsubseteq e$ and $b \sqsubseteq e$.

         A. Suppose $e = d$. Choose $c = d'$. It exists since d' does. We have $c \in E_x$ because $d' \in E_x$. We also have $a \sqsubseteq c$ and $b \sqsubseteq c$ by transitivity and Proposition 2.4.2 Item *1*.

         B. Suppose $e \neq d$. Choose $c = e$. It exists since e does. We have $c \in E_x$, $a \sqsubseteq c$ and $b \sqsubseteq c$, because $a \sqsubseteq e$ and $b \sqsubseteq e$.

2. We prove $\bigsqcup^{\uparrow} E_x = x$. Note that $\bigsqcup^{\uparrow} E_x$ exists because D is a dcpo.

   $\boxed{\sqsubseteq}$ Since $E_x \subseteq \mathord{\downarrow\!\!\downarrow} x$ by definition, we have $\bigsqcup^{\uparrow} E_x \sqsubseteq \bigsqcup \mathord{\downarrow\!\!\downarrow} x$ by Proposition 1.3.1 Item *1*. We have $\bigsqcup^{\uparrow} E_x \sqsubseteq x$ as wanted by Proposition 2.6.4 Item *1*.

$\boxed{\sqsupseteq}$ We want $\bigsqcup^{\uparrow} E_x \sqsupseteq x = \bigsqcup^{\uparrow} C_x$. By applying Proposition 1.3.2, we only have to prove $(\forall a \in C_x \mid (\exists b \in E_x \mid a \sqsubseteq b))$. Let $a \in C_x$.

(a) Suppose $a = d$. Choose $b = d'$. It exists since d' does. We have $b \in E_x$ by definition. We have $a \sqsubseteq b$ by Proposition 2.4.2 Item *1*.

(b) Suppose $a \neq d$. Choose $b = a$. It exists since a does. We have $b \in E_x$ by definition and by assumption. We have $a \sqsubseteq b$ by reflexivity.

3. We prove $E_x \subseteq (B - \{d\})_x$. Let $e \in E_x$.

   (a) We show $e \in B - \{d\}$. If $e = d'$, then we are done, because $d \neq d'$ and $d' \in B$. If not, we have $e \in C_x - \{d\}$ by definition. Since $C_x \subseteq B$ by definition, we have $e \in B - \{d\}$ by set theory.

   (b) We show $e \in \twoheaddownarrow x$. If $e = d'$, then we are done, because $d' \ll x$. If not, we have $e \in C_x - \{d\}$ by definition. Since $C_x - \{d\} \subseteq \twoheaddownarrow x$ by definition, we have $e \in \twoheaddownarrow x$ by set theory.

We have proved that, for an arbitrary element $x \in D$, its approximants in $B - \{d\}$ contains a directed subset with x as supremum. Hence, we proved that $B - \{d\}$ is a basis. □

B.20 Proof of Proposition 2.7.6

Proof. Let D be a dcpo and $B \subseteq D$ be a basis. We show that B is the smallest basis with respect to inclusion if and only if $B = K(D)$.

$\boxed{\Rightarrow}$ We show this direction by contraposition. Suppose $B \neq K(D)$. We show that B is not the smallest basis. Then there exists $b \in B - K(D)$. Then, $B - \{b\}$ is still a basis by Proposition 2.7.5 and $B - \{b\} \subset B$. Therefore, B is not the smallest basis.

$\boxed{\Leftarrow}$ Suppose $B = K(D)$. We show that B is the smallest basis. Let B' be another basis of D. We need to show that $B \subseteq B'$. By Proposition 2.5.2 Item *2*, we have $B = K(D) \subseteq B'$.

□

B.21 Proof of Proposition 2.7.8

Proof. Let D be a continuous dcpo and $x, y \in D$. We show $x \ll y$ if and only if, for any directed subset $A \subseteq D$, $y = \bigsqcup^{\uparrow} A \Rightarrow (\exists a \in A \mid x \sqsubseteq a)$.

$\boxed{\Rightarrow}$ Suppose $x \ll y$. Let $A \subseteq D$ be directed and such that $y = \bigsqcup^{\uparrow} A$. We have $(\exists a \in A \mid x \sqsubseteq a)$ using $x \ll y$ and reflexivity.

$\boxed{\Leftarrow}$ Suppose that, for any directed subset $A \subseteq D$, $y = \bigsqcup^{\uparrow} A \Rightarrow (\exists a \in A \mid x \sqsubseteq a)$. We show $x \ll y$. By Proposition 2.6.4 Item *1*, we have $y = \bigsqcup^{\uparrow} \twoheaddownarrow y$. Then there exists $z \in \twoheaddownarrow y$ such that $x \sqsubseteq z$ by assumption. We conclude $x \ll y$ by applying Proposition 2.4.2 Item *2*.

□

B.22 Proof of Proposition 2.7.9

Proof. Let D be a continuous dcpo, $M \subseteq K(D)$ be a finite subset and $u \in \mathbf{ub}(M)$ be minimal. We show $u \in K(D)$. By Proposition 2.4.6 Item *3*, we have $(\forall m \in M \mid u \in \uparrow m)$. Hence, we have $M \ll u$, i.e., $M \subseteq \mathord{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} u$. We have that $\mathord{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} u$ is directed by Proposition 2.6.4 Item *1*. Hence, by Proposition 2.1.6, there exists $u' \in \mathord{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} u$ such that $u' \in \mathbf{ub}(M)$. We have $u \sqsubseteq u'$, because u is a minimal upper bound of M. Using Proposition 2.4.2 Item *1*, we have $u' \sqsubseteq u$. So, we have $u = u'$ by antisymmetry. Therefore, since $u' \in \mathord{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} u$, we conclude $u \ll u$ that is $u \in K(D)$ as wanted. $\square$

B.23 Proof of Proposition 2.7.10

Proof. Let L be a complete lattice and $M \subseteq K(L)$ be finite. We show $\bigsqcup M \in K(L)$, i.e., for any directed subset $A \subseteq L$ such that $\bigsqcup M \sqsubseteq \bigsqcup^{\uparrow} A$, there is $a \in A$ such that $\bigsqcup M \sqsubseteq a$. Note that $\bigsqcup M$ exists since L is a complete lattice. Let $A \subseteq L$ be directed and such that $\bigsqcup M \sqsubseteq \bigsqcup^{\uparrow} A$. Note that $\bigsqcup^{\uparrow} A$ exists since L is a complete lattice. We have $\left(\forall m \in M \mid m \sqsubseteq \bigsqcup^{\uparrow} A\right)$ by transitivity. Then, since M is composed of compact elements, there is a finite subset $A' = \{a_m \in A \mid m \in M \wedge m \sqsubseteq a_m\}$. By Proposition 2.1.6, there exists $a \in A$ such that $a \in \mathbf{ub}(A')$. We have $A \in \mathbf{ub}(M)$ by transitivity. We have $\bigsqcup M \sqsubseteq a$, because the supremum is the least upper bound. Hence, we conclude that $\bigsqcup M \in K(L)$. $\square$

B.24 Proof of Proposition 2.7.11

Proof. Let L be a complete lattice. We show that L is algebraic if and only if every element is the supremum of compact elements.

$\boxed{\Rightarrow}$ Suppose that L is algebraic. We show that every element is the supremum of compact elements. Let $x \in L$. We have $x = \bigsqcup^{\uparrow} K(L)_x$ by Proposition 2.6.4 Item *2*. Hence, x is the supremum of compact elements and we are done.

$\boxed{\Leftarrow}$ Suppose that every element is the supremum of compact elements. We show that L is algebraic. Let $x \in L$. We have to show $x = \bigsqcup^{\uparrow} K(L)_x$.

1. We show that $K(L)_x$ is directed.
   (a) We show $K(L)_x \neq \emptyset$. This is true since $\perp \in K(L)_x$ by Proposition 2.4.7.
   (b) We show $(\forall a, b \in K(L)_x \mid (\exists c \in K(L)_x \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in K(L)_x$. We can take $c = \bigsqcup \{a, b\}$, since L is a lattice. We have $c \in K(L)$ by Proposition 2.7.10. Since $x \in \mathbf{ub}(\{a, b\})$ by Proposition 2.4.2 Item *1*, we have $c \sqsubseteq x$, because the supremum is the least upper bound. Hence, we have $x \in \uparrow c$ by Proposition 2.4.6 Item *3*. Therefore, we conclude $c \in K(L)_x$.
2. We show $x = \bigsqcup^{\uparrow} K(L)_x$. Note that the latter exists since L is a complete lattice.

$\boxed{\sqsubseteq}$ By assumption, there is $C_x \subseteq K(L)$ such that $\bigsqcup C_x = x$. Then we have $(\forall c \in C_x \mid x \in \uparrow c)$ by Proposition 2.4.6 Item *3*. Hence, we have $C_x \subseteq K(L)_x$ and we conclude $x = \bigsqcup C_x \sqsubseteq \bigsqcup^{\uparrow} K(L)_x$ by Proposition 1.3.1 Item *1*.

$\boxed{\sqsupseteq}$ We have $x \in \mathbf{ub}(K(L)_x)$ by Proposition 2.4.2 Item *1*. Hence, we have $x \sqsupseteq \bigsqcup^{\uparrow} K(L)_x$, because the supremum is the least upper bound.

□

B.25 Proof of Proposition 2.8.1

Proof. Let D be a continuous dcpo. We show

$$\{\langle x, y\rangle \in D \times D \mid x \ll y\} = \bigcup_{d \in D} \left\{\langle x, y\rangle \in \downarrow d \times \downarrow d \mid x \ll_{\downarrow d} y\right\} .$$

$\boxed{\subseteq}$ Let $\langle x, y\rangle \in D \times D$ be such that $x \ll y$. We show $x \ll_{\downarrow y} y$. Let $A \subseteq \downarrow y$ be directed and such that $y \sqsubseteq \bigsqcup^{\uparrow} A$. Then there exists $a \in A$ such that $x \sqsubseteq a$, because $x \ll y$. Hence, we have $x \ll_{\downarrow y} y$ and $\langle x, y\rangle \in \bigcup_{d \in D} \left\{\langle x, y\rangle \in \downarrow d \times \downarrow d \mid x \ll_{\downarrow d} y\right\}$.

$\boxed{\supseteq}$ Let $d \in D$ and $\langle x, y\rangle \in \downarrow d \times \downarrow d$ be such that $x \ll_{\downarrow d} y$. We show $x \ll y$. Since D is a continuous dcpo, we use Proposition 2.7.8. Let $A \subseteq D$ be directed and such that $y = \bigsqcup^{\uparrow} A$. Since $y \in \downarrow d$, we have $A \subseteq \downarrow d$ by transitivity. Since $x \ll_{\downarrow d} y$, there is $a \in A$ such that $x \sqsubseteq a$. Hence, we have proved that $x \ll y$.

□

B.26 Proof of Proposition 2.8.2

Proof. Let D be a dcpo. We show that D is continuous if and only if its principal ideals are continuous.

$\boxed{\Rightarrow}$ Suppose that D is continuous. Let $d \in D$. We show that $\downarrow d$ is continuous. By Proposition 2.6.4 Item *1*, it is enough to show $\left(\forall x \in \downarrow d \mid x = \bigsqcup^{\uparrow} \twoheaddownarrow_{\downarrow d} x\right)$. Let $x \in \downarrow d$. We have $\downarrow x \subseteq \downarrow d$, because $x \sqsubseteq d$. Hence, we have $\twoheaddownarrow x \subseteq \downarrow d$ by transitivity and Proposition 2.4.6 Item *2*. So, we have $\twoheaddownarrow x \subseteq \twoheaddownarrow_{\downarrow d} x$. We have $\twoheaddownarrow_{\downarrow d} x \subseteq \twoheaddownarrow x$ by Proposition 2.8.1. Hence, we have $\twoheaddownarrow_{\downarrow d} x = \twoheaddownarrow x$. Since $\bigsqcup^{\uparrow} \twoheaddownarrow x = x$ by Proposition 2.6.4 Item *1*, we conclude $x = \bigsqcup^{\uparrow} \twoheaddownarrow_{\downarrow d} x$ as wanted.

$\boxed{\Leftarrow}$ Suppose that any principal ideal of D is continuous. We show that D is continuous. By Proposition 2.6.4 Item *1*, it is enough to show $\left(\forall x \in D \mid x = \bigsqcup^{\uparrow} \twoheaddownarrow x\right)$. Let $x \in D$. We have $\twoheaddownarrow_{\downarrow x} x \subseteq \twoheaddownarrow x$ by Proposition 2.8.1. We also have $\twoheaddownarrow x \subseteq \twoheaddownarrow_{\downarrow x} x$ because any directed subset of $\downarrow x$ is also one of D by transitivity. Hence, we have $\twoheaddownarrow_{\downarrow d} x = \twoheaddownarrow x$. Since $\bigsqcup^{\uparrow} \twoheaddownarrow_{\downarrow x} x = x$ by Proposition 2.6.4 Item *1*, we conclude $x = \bigsqcup^{\uparrow} \twoheaddownarrow x$ as wanted.

□

B.27 Proof of Proposition 2.8.4

Proof. Let D be a continuous dcpo and $c \in D$. We show that $c \in K(D)$ if and only if $(\exists d \in D \mid c \in K(\downarrow d))$.

$\boxed{\Rightarrow}$ Suppose $c \in K(D)$. We show $c \ll_{\downarrow c} c$. Let $A \subseteq \downarrow c$ be directed and such that $c \sqsubseteq \bigsqcup^{\uparrow} A$. We have $(\exists a \in A \mid c \sqsubseteq a)$ as wanted because A is a directed subset of D by transitivity and $c \in K(D)$.

$\boxed{\Leftarrow}$ Let $d \in D$ be such that $c \in K(\downarrow d)$. We show $c \in K(D)$. We use Proposition 2.7.8 to show $c \ll c$. Let $A \subseteq D$ be directed and such that $c = \bigsqcup^{\uparrow} A$. We have $A \subseteq \downarrow d$ by transitivity. Hence, we have $(\exists a \in A \mid c \sqsubseteq a)$ because $c \in K(\downarrow d)$.

$\square$

B.28 Proof of proposition 2.9.2

Proof. Let B be an abstract basis and $x \in B$. We show that $\downarrow x$ is an ideal.

1. We show that $\downarrow x$ is directed.

   (a) We show $\downarrow x \neq \emptyset$. Since B is an abstract basis and $\emptyset$ is a finite subset of B such that $\emptyset \prec x$, there is $y \in B$ such that $\emptyset \prec y$ and $y \prec x$. Hence, $y \in \downarrow x$ and $\downarrow x \neq \emptyset$.

   (b) Let $a, b \in \downarrow x$. We show $(\exists c \in \downarrow x \mid a \prec c \land b \prec c)$. Since B is an abstract basis and $\{a, b\}$ is a finite subset of B such that $\{a, b\} \prec x$, there is $c \in B$ such that $\{a, b\} \prec c$ and $c \prec x$. Then, $c \in \downarrow x$.

2. We show that $\downarrow x$ is downward closed.

   $\boxed{\subseteq}$ Let $a \in \downarrow x$. We show $a \in \downarrow(\downarrow x)$, i.e., $(\exists b \in \downarrow x \mid a \prec b)$. Since B is an abstract basis and $\{a\}$ is a finite subset of B such that $\{a\} \prec x$, there is $b \in B$ such that $\{a\} \prec b$ and $b \prec x$. Then, $b \in \downarrow x$ as wanted.

   $\boxed{\supseteq}$ Let $a \in \downarrow(\downarrow x)$. We show $a \in \downarrow x$. Then, there is $b \in \downarrow x$ such that $a \prec b$. By transitivity of "$\prec$", $a \prec x$. Hence, $a \in \downarrow x$.

$\square$

B.29 Proof of Proposition 2.9.6

Proof. Let D be a dcpo, B be an abstract basis and $f \in \left[B \xrightarrow{m} D\right]$. We show that the function $\hat{f}$ is Scott-continuous. Let $\mathcal{I} \subseteq \mathbf{Idl}(B)$ be directed with respect to inclusion. We have to show that $\hat{f}\left(\bigsqcup^{\uparrow}\mathcal{I}\right) = \bigsqcup^{\uparrow}_{I \in \mathcal{I}} \hat{f}(I)$. The proof is concluded by the following equalities:

$$\hat{f}\left(\bigsqcup^{\uparrow}\mathcal{I}\right)$$

$$
\begin{aligned}
&= && \langle \text{Proof of Proposition 2.2.3} \rangle \\
&\quad \hat{f}\big(\textstyle\bigcup_{I\in\mathcal{I}} I\big) \\
&= && \langle \text{Definition of } \hat{f} \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow} \{f(i) \mid i \in \bigcup_{I\in\mathcal{I}} I\} \\
&= && \langle \text{Set theory} \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow} \bigcup_{I\in\mathcal{I}} \{f(i) \mid i \in I\} \\
&= && \langle \text{Proposition 1.3.1 Item } \mathit{3} \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow}_{I\in\mathcal{I}} \bigsqcup^{\uparrow} \{f(i) \mid i \in I\} \\
&= && \langle \text{Notation} \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow}_{I\in\mathcal{I}} \bigsqcup^{\uparrow}_{i\in I} f(i) \\
&= && \langle \text{Definition of } \hat{f} \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow}_{I\in\mathcal{I}} \hat{f}(I)\ .
\end{aligned}
$$

□

B.30 Proof of Proposition 2.9.7

Proof. Let D be a dcpo, B be an abstract basis, $f \in \left[B \xrightarrow{m} D\right]$. We show $\left(\hat{f} \circ \mathtt{i}\right) \sqsubseteq_{\mathrm{pt}} f$. Let $b \in B$. We show $\left(\hat{f} \circ \mathtt{i}\right)(b) \sqsubseteq_D f(b)$. We have $f(b) \in \mathbf{ub}(\{f(b') \mid b' \in {\downarrow} b\})$ by monotonicity. Hence, we have $\bigsqcup^{\uparrow}_{b' \in \downarrow b} f(b') \sqsubseteq_D f(b)$, because the supremum is the least upper bound. Note that the directed supremum exists, because D is a dcpo. We conclude $\left(\hat{f} \circ \mathtt{i}\right)(d) = \hat{f}({\downarrow} d) = \bigsqcup^{\uparrow}_{d' \in \downarrow d} f(d') \sqsubseteq_D f(d)$ by definition of functions $\mathtt{i}$ and $\hat{f}$. □

B.31 Proof of proposition 2.9.8

Proof. Let B be an abstract basis and $I \in \mathbf{Idl}(B)$. We show $I = \bigcup_{i \in I} \mathtt{i}(i)$.

$\boxed{\subseteq}$ Let $a \in I$. We show $a \in \bigcup_{i \in I} \mathtt{i}(i)$. There is $c \in I$ such that $a \prec c$ by directedness of I. Then $a \in {\downarrow} c$. Thus $a \in \mathtt{i}(i)$ by definition of the function $\mathtt{i}$. Hence $a \in \bigcup_{i \in I} \mathtt{i}(i)$.

$\boxed{\supseteq}$ Let $a \in \bigcup_{i \in I} \mathtt{i}(i)$. We show $a \in I$. There exists $b \in I$ such that $a \in \mathtt{i}(b)$ by set theory. Then $a \prec b$ by definition of the function $\mathtt{i}$. Therefore $a \in I$ by downward closure of I.

□

B.32 Proof of Proposition 2.9.9

Proof. Let D be a dcpo, B be an abstract basis, $f \in \left[B \xrightarrow{m} D\right]$, $g \in [\mathbf{Idl}(B) \to D]$ be Scott-continuous and such that $g \circ \mathtt{i} \sqsubseteq_{\mathrm{pt}} f$, and $I \in \mathbf{Idl}(B)$. We show $g(I) \sqsubseteq \hat{f}(I)$. The proof is concluded by the following equalities and inequality:

$$g(I)$$

$$
\begin{aligned}
&= && \langle \text{Proposition 2.9.8} \rangle \\
&\quad g\big(\textstyle\bigcup_{i \in I} \mathtt{i}(i)\big) \\
&= && \langle \text{Scott-continuity of } g \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow}_{i \in I} g(\mathtt{i}(i)) \\
&\sqsubseteq && \langle \text{Assumption on } g \rangle \\
&\quad \textstyle\bigsqcup^{\uparrow}_{i \in I} f(i) \\
&= && \langle \text{Definition of } \hat{f} \rangle \\
&\quad \hat{f}(I)\ .
\end{aligned}
$$

□

B.33 Proof of Theorem 2.9.11

Proof. Let D be a continuous dcpo with basis B. We show that D is isomorphic to $\mathbf{Idl}(\langle B, \ll \rangle)$. Consider the following functions:

$$
\begin{aligned}
f : \quad D &\to \mathbf{Idl}(\langle B, \ll \rangle) \\
d &\mapsto B_d
\end{aligned}
$$

and

$$
\begin{aligned}
g : \quad B &\to D \\
b &\mapsto b\ .
\end{aligned}
$$

To show the isomorphism, we show $f \circ \hat{g} = \mathbf{id}_{\mathbf{Idl}(\langle B, \ll \rangle)}$ and $\hat{g} \circ f = \mathbf{id}_D$.

1. We show $f \circ \hat{g} = \mathbf{id}_{\mathbf{Idl}(\langle B, \ll \rangle)}$. Let $I \in \mathbf{Idl}(\langle B, \ll \rangle)$. Hence, I is an ideal of B with respect to the way-below relation. Note that Proposition 2.4.2 Item *1* is an implication. Therefore, in D, I is only a directed set. But it is enough to say that $\bigsqcup^{\uparrow} I$ exists.

   (a) We show $B_{\bigsqcup^{\uparrow} I} = I$.

   $\boxed{\subseteq}$ Let $b \in B_{\bigsqcup^{\uparrow} I}$. By Corollary 2.7.7, there is $i \in I$ such that $b \ll i$. Since I is an ideal with respect to "$\ll$", we have $b \in I$ as wanted.

   $\boxed{\supseteq}$ Let $i \in I$. Since $I \subseteq B$, we have $i \in B$. Hence, we have to show $i \ll \bigsqcup^{\uparrow} I$. By Corollary 2.7.7, it is enough to show $(\exists j \in I \mid i \ll j)$. This is true by applying the second property of directedness to i.

   (b) We show $(f \circ \hat{g})(I) = \mathbf{id}_{\mathbf{Idl}(\langle B, \ll \rangle)}(I)$. This item is concluded by the following equalities:

$$
\begin{aligned}
&\quad (f \circ \hat{g})(I) \\
&= && \langle \text{Definition of } \hat{g} \rangle \\
&\quad f\Big(\textstyle\bigsqcup^{\uparrow}_{i \in I} g(i)\Big) \\
&= && \langle \text{Definition of } g \rangle
\end{aligned}
$$

$$
\begin{aligned}
& f\left(\bigsqcup\nolimits^{\uparrow}_{i \in I} i\right) \\
= \quad & \langle \text{Definition of } f \rangle \\
& B_{\bigsqcup^{\uparrow} I} \\
= \quad & \langle \text{Item 1a of this proof} \rangle \\
& I \\
= \quad & \langle \text{Definition 1.6.8} \rangle \\
& \mathbf{id}_{\mathbf{Idl}(\langle B, \ll \rangle)}(I) \ .
\end{aligned}
$$

2. We show $\hat{g} \circ f = \mathbf{id}_D$. Let $d \in D$. We have $(\hat{g} \circ f)(d) = \hat{g}(B_d)$ by definition of f. We have $\hat{g}(B_d) = \bigsqcup^{\uparrow}_{b \in B_d} g(b)$ by definition of $\hat{g}$. We have $\bigsqcup^{\uparrow}_{b \in B_d} g(b) = \bigsqcup^{\uparrow}_{b \in B_d} b = \bigsqcup^{\uparrow} B_d$ by definition of g and by notation. Since D is a continuous dcpo, we have $\bigsqcup^{\uparrow} B_d = d$ by Proposition 2.6.4 Item *1.* We have $d = \mathbf{id}_D(d)$ by definition. Hence, we have $\hat{g} \circ f = \mathbf{id}_D$.

$\square$

Appendix C

Proofs of Chapter 3

This appendix contains detailed proofs of some propositions stated in Chapter 3. The statements to prove are all written in the first lines of their corresponding proof.

C.1 Proof of Proposition 3.2.2

Proof. Let X be a topological space. We show that closed sets are closed under finite unions and arbitrary intersections. Let $\mathcal{C} \subseteq \Gamma(X)$. For every $C \in \mathcal{C}$, let $U_C = X - C$. Note that $(\forall C \in \mathcal{C} \mid U_C \in \mathcal{O}(X))$ by definition.

1. Suppose that $\mathcal{C}$ is finite. We show $\bigcup \mathcal{C} \in \Gamma(X)$. We have $\bigcup \mathcal{C} = \bigcup_{C \in \mathcal{C}} X - U_c = X - \bigcap_{C \in \mathcal{C}} U_c$ by set theory. $\bigcap_{C \in \mathcal{C}} U_c$ is a finite intersection of open sets since $\mathcal{C}$ is finite. Hence, it is an open set and its complement, i.e., $\bigcup \mathcal{C}$, is a closed set as wanted.

2. We show that $\bigcap \mathcal{C} \in \Gamma(X)$. We have $\bigcap \mathcal{C} = \bigcap_{C \in \mathcal{C}} X - U_c = X - \bigcup_{C \in \mathcal{C}} U_c$ by set theory. $\bigcup_{C \in \mathcal{C}} U_c$ is an arbitrary union of open sets. Hence, it is an open set and its complement, i.e., $\bigcap \mathcal{C}$, is a closed set as wanted.

□

C.2 Proof of Proposition 3.2.6

Proof. Let X be a topological space and $C \subseteq X$. We show that $C \in \Gamma(X)$ if and only if $\mathbf{cl}(C) \subseteq C$.

$\boxed{\Rightarrow}$ Suppose $C \in \Gamma(X)$. We show that $\mathbf{cl}(C) \subseteq C$. Let $c \in \mathbf{cl}(C)$. By definition of the closure, c must be in all the closed sets containing C. Since C is closed and C contains itself by reflexivity, we have $c \in C$ as wanted.

$\boxed{\Leftarrow}$ Suppose $\mathbf{cl}(C) \subseteq C$. We show $C \in \Gamma(X)$. By Proposition 3.2.5 Item *2* and antisymmetry, we have $C = \mathbf{cl}(C)$. The latter is closed, as explained after Definition 3.2.4. Hence C is closed, as wanted.

□

C.3 Proof of Proposition 3.2.7

Proof. Let X be a topological space. We show that $\mathcal{O}(X)$ is a complete lattice. Let $\mathcal{U} \subseteq \mathcal{O}(X)$.

1. We show $\bigcup \mathcal{U} \in \mathcal{O}(X)$ and is $\bigsqcup \mathcal{U}$.
   (a) We show $\bigcup \mathcal{U} \in \mathcal{O}(X)$. It is true because any union of open sets is open by definition.
   (b) We show $\bigcup \mathcal{U} \in \mathcal{O}(X)$ is $\bigsqcup \mathcal{U}$.
      i. We show $\bigcup \mathcal{U} \in \mathbf{ub}(\mathcal{U})$. It is true by set theory.
      ii. We show $(\forall B \in \mathbf{ub}(\mathcal{U}) \mid \bigcup \mathcal{U} \subseteq B)$. Let $B \in \mathbf{ub}(\mathcal{U})$. It contains any open $U \in \mathcal{U}$ because it is an upper bound. Hence, it contains its union by set theory.
2. We show $\mathbf{int}(\bigcap \mathcal{U}) \in \mathcal{O}(X)$ and is $\bigsqcap \mathcal{U}$.
   (a) We show $\mathbf{int}(\bigcap \mathcal{U}) \in \mathcal{O}(X)$. It is true because the interior is a union of open sets.
   (b) We show $\mathbf{int}(\bigcap \mathcal{U}) \in \mathcal{O}(X)$ is $\bigsqcap \mathcal{U}$.
      i. We show $\mathbf{int}(\bigcap \mathcal{U}) \in \mathbf{lb}(\mathcal{U})$. It is true because the interior is contained in the intersection which is contained in any $U \in \mathcal{U}$ by set theory.
      ii. We show $(\forall B \in \mathbf{lb}(\mathcal{U}) \mid B \subseteq \mathbf{int}(\bigcap \mathcal{U}))$. Let $B \in \mathbf{lb}(\mathcal{U})$. Note that B is open, because we are working in $\mathcal{O}(X)$. B is contained in any open $U \in \mathcal{U}$ because it is a lower bound. Hence, it is contained the intersection by set theory. By definition of the interior, we have $B \subseteq \mathbf{int}(\bigcap \mathcal{U})$ as wanted.

□

C.4 Proof of Proposition 3.2.10

Proof. Let X be a topological space, $A \subseteq X$ and $x \in X$. We show $x \in \mathbf{cl}(A)$ if and only if $(\forall U \in \mathcal{N}_x \mid U \cap A \neq \emptyset)$. We use contraposition, i.e., we show $x \notin \mathbf{cl}(A)$ if and only if $(\exists U \in \mathcal{N}_x \mid U \cap A = \emptyset)$.

$\boxed{\Rightarrow}$ Suppose $x \notin \mathbf{cl}(A)$. We show $(\exists U \in \mathcal{N}_x \mid U \cap A = \emptyset)$. Let $U = X - \mathbf{cl}(A)$. U is open because it is the complement of a closed set. We have $x \in U$ by set theory. Hence, we have $U \in \mathcal{N}_x$. We have $U \cap A = \emptyset$ using set theory and Proposition 3.2.5 Item *2*.

$\boxed{\Leftarrow}$ Suppose $(\exists U \in \mathcal{N}_x \mid U \cap A = \emptyset)$. We show $x \notin \mathbf{cl}(A)$. Since $U \cap A = \emptyset$, we have $A \subseteq X - U$ by set theory. But since $X - U \in \Gamma(X)$ by definition, we have $\mathbf{cl}(A) \subseteq X - U$. We have $x \notin X - U$ because $U \in \mathcal{N}_x$. So, we have $x \notin \mathbf{cl}(A)$ by set theory as wanted.

□

C.5 Proof of proposition 3.3.3

Proof. Let D be a dcpo equipped with its Scott topology. We show that the Scott topology of D is a topology. Let $\mathcal{U} \subseteq \mathcal{O}(D)$.

1. Suppose that $\mathcal{U}$ is finite. We show $\bigcap \mathcal{U} \in \mathcal{O}(D)$.

    (a) We show that $\bigcap \mathcal{U}$ is upward closed. Let $x \in \bigcap \mathcal{U}$ and $y \in D$ be such that $x \sqsubseteq y$. We have $(\forall U \in \bigcap \mathcal{U} \mid x \in U)$. Thus, we have $(\forall U \in \mathcal{U} \mid y \in U)$ by upward closure of Scott-opens. Hence, we have $y \in \bigcap \mathcal{U}$ as wanted.

    (b) Let $A \subseteq D$ be directed and such that $\bigsqcup^{\uparrow} A \in \bigcap \mathcal{U}$. We show $(\exists a \in A \mid a \in \bigcap \mathcal{U})$. We have $\left(\forall U \in \mathcal{U} \mid \bigsqcup^{\uparrow} A \in U\right)$. Hence, we have $(\forall U \in \mathcal{U} \mid (\exists a_U \in A \mid a_U \in U))$ by Property *2* of Scott-opens. Consider the set $A' = \{a_U \mid U \in \mathcal{U}\}$. We have that A' is finite, since $\mathcal{U}$ is finite, and $A' \subseteq A$. Then, there is $a \in A$ such that $a \in \mathbf{ub}(A')$ by Proposition 2.1.6. We have $(\forall U \in \mathcal{U} \mid a \in U)$ by upward closure of Scott-opens. Therefore, we conclude $a \in \bigcap \mathcal{U}$.

2. We show $\bigcup \mathcal{U} \in \mathcal{O}(D)$.

    (a) We show that $\bigcup \mathcal{U}$ is upward closed. Let $x \in \bigcup \mathcal{U}$ and $y \in D$ be such that $x \sqsubseteq y$. There exists $U \in \mathcal{U}$ such that $x \in U$. Then, we have $y \in U$ by upward closure of Scott-opens. Hence, we have $y \in \bigcup \mathcal{U}$ as wanted,

    (b) Let $A \subseteq D$ be directed and such that $\bigsqcup^{\uparrow} A \in \bigcup \mathcal{U}$. We show $(\exists a \in A \mid a \in \bigcup \mathcal{U})$. There exists $U \in \mathcal{U}$ such that $\bigsqcup^{\uparrow} A \in U$. Then, there exists $a \in A$ such that $a \in U$ by Property *2* of Scott-opens. Hence, we have $a \in \bigcup \mathcal{U}$.

□

C.6 Proof of proposition 3.3.5

Proof. Let D be a dcpo equipped with its Scott topology and $C \subseteq D$. We show that C is Scott-closed if and only if C is downward closed and C is closed under directed suprema.

$\boxed{\Rightarrow}$ Suppose that C is Scott-closed. Then $D - C$ is Scott-open by definition.

1. We show that C is downward closed. We have that $D - C$ is upward closed, because it is a Scott-open set. Hence, the result is true by Proposition 1.5.4.
2. We show that C is closed under directed suprema. Let $A \subseteq C$ be directed. Note that $\bigsqcup^{\uparrow} A$ exists because D is a dcpo. By way of contradiction, suppose $\bigsqcup^{\uparrow} A \notin C$. By set theory, we have $\bigsqcup^{\uparrow} A \in D - C$. The latter is Scott-open, hence $(\exists c \in A \mid c \in D - C)$ by definition of a Scott-open set. It is a contradiction, because $A \subseteq C$. We then conclude $\bigsqcup^{\uparrow} A \in C$.

$\boxed{\Leftarrow}$ Suppose that C is downward closed and closed under directed suprema. We show that C is Scott-closed, i.e., $D - C$ is Scott-open.

1. We show $D - C = \uparrow(D - C)$. Since C is downward closed, the result is true by Proposition 1.5.4.
2. Let $A \subseteq D$ be directed and such that $\bigsqcup^{\uparrow} A \in D - C$. We show $(\exists a \in A \mid a \in D - C)$. By way of contradiction, suppose $(\forall a \in A \mid a \in C)$. Then we have $\bigsqcup^{\uparrow} A \in C$, because C is closed under directed suprema. We have a contradiction since $\bigsqcup^{\uparrow} A \in D - C$. Hence, we have $(\exists a \in A \mid a \in D - C)$.

□

C.7 Proof of Proposition 3.3.6

Proof. Let D be a dcpo and $x \in K(D)$. We show that $\twoheaduparrow x$ is Scott-open.

1. We show that $\twoheaduparrow x$ is upward closed. By Proposition 2.4.6 Items *1* and *3*, and antisymmetry, we have $\twoheaduparrow x = \uparrow x$. Since the latter is upward closed by definition, then so is the former.

2. Let $A \subseteq D$ be directed and such that $\bigsqcup^{\uparrow} A \in \twoheaduparrow x$. We show $(\exists a \in A \mid x \ll a)$. Since x is compact, there is $a \in A$ such that $x \sqsubseteq a$. By Proposition 2.4.6 Item *3*, we have $a \in \twoheaduparrow x$, i.e., $x \ll a$.

□

C.8 Proof of Proposition 3.3.7

Proof. Let D be a continuous dcpo and $x \in D$. We show that $\twoheaduparrow x$ is Scott-open.

1. We show that $\twoheaduparrow x$ is upward closed. Let $y \in \twoheaduparrow x$ and $z \in D$ be such that $y \sqsubseteq z$. We have $x \ll z$ as wanted using Proposition 2.4.2 Item *2*.

2. Let $A \subseteq D$ be directed and such that $\bigsqcup^{\uparrow} A \in \twoheaduparrow x$. We show $(\exists a \in A \mid x \ll a)$. We have $x \in \twoheaddownarrow \bigsqcup^{\uparrow} A$. Hence by Corollary 2.7.7 there is $a \in A$ such that $x \ll a$, as wanted.

□

C.9 Proof of Proposition 3.3.8

Proof. Let D be a continuous dcpo equipped with its Scott topology and $x \in D$. We show $\twoheaduparrow x = \mathbf{int}(\uparrow x)$.

$\boxed{\subseteq}$ We show $\twoheaduparrow x \subseteq \mathbf{int}(\uparrow x)$. We have $\twoheaduparrow x \subseteq \uparrow x$ by Proposition 2.4.6 Item *1*. We have $\twoheaduparrow x \in \mathcal{O}(D)$ by Proposition 3.3.7. Hence, we have $\twoheaduparrow x \subseteq \mathbf{int}(\uparrow x)$ by definition of the interior.

$\boxed{\supseteq}$ Let $y \in \mathbf{int}(\uparrow x)$. We show $y \in \twoheaduparrow x$. We have $y = \bigsqcup^{\uparrow} \twoheaddownarrow y$ by Proposition 2.6.4 Item *1*. Take $y' \in \mathbf{int}(\uparrow x)$ such that $y' \ll y$. It exists by Property *2* of Scott-openness. We have $x \sqsubseteq y'$, because $y' \in \uparrow x$ by definition of the interior. Hence, we apply Proposition 2.4.2 Item *2* to obtain $x \ll y$.

$\square$

C.10 Proof of Proposition 3.4.3

Proof. Let X be a set with a basis $\mathcal{B}$. We show that the topology $\mathcal{T}$ generated by $\mathcal{B}$, i.e., $\{U \in \mathcal{P}(X) \mid (\forall u \in U \mid (\exists B \in \mathcal{B} \mid u \in B \wedge B \subseteq U))\}$ is a topology on X.

1. Let $\mathcal{U} \subseteq \mathcal{T}$. We show $\bigcup \mathcal{U} \in \mathcal{T}$. We have $\bigcup \mathcal{U} \in \mathcal{P}(X)$ by set theory. Let $u \in \bigcup \mathcal{U}$. Then there exists $U \in \mathcal{U}$ such that $u \in U$. Take $B \in \mathcal{B}$ such that $u \in B$ and $B \subseteq U$. B exists because $U \in \mathcal{T}$. We have $B \subseteq \bigcup \mathcal{U}$ by set theory. Hence, we have $\bigcup \mathcal{U} \in \mathcal{T}$ as wanted.

2. Let $\mathcal{U} \subseteq \mathcal{T}$ be a finite subset. We show $\bigcap \mathcal{U} \in \mathcal{T}$ using induction on the size $n \in \mathbb{N}$ of $\mathcal{U}$.
Base case: Suppose $n = 0$. We have $\mathcal{U} = \emptyset$ by set theory. Hence $\bigcap \mathcal{U} = X$. We have $X \in \mathcal{T}$ by using Property *1* of a basis and set theory for inclusion. Hence, we have $\bigcap \mathcal{U} \in \mathcal{T}$.
Induction hypothesis: For all $k \in \mathbb{N}$ such that $k \leq n$, if a set $\mathcal{M}$ has cardinality k, then $\bigcap \mathcal{M} \in \mathcal{T}$.
Induction step: Suppose $|\mathcal{U}| = n+1$. Then there exists $U \in \mathcal{U}$. We have $\bigcap (\mathcal{U} - \{U\}) \in \mathcal{T}$ by the induction hypothesis. The only thing yet to prove is $\bigcap (\mathcal{U} - \{U\}) \cap U \in \mathcal{T}$. Let $u \in \bigcap (\mathcal{U} - \{U\}) \cap U$. Then there exist $B_1, B_2 \in \mathcal{B}$ such that $u \in B_1$, $u \in B_2$, $B_1 \subseteq \bigcap (\mathcal{U} - \{U\})$ and $B_2 \subseteq U$. Take $B_3 \in \mathcal{B}$ such that $u \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. B_3 exists by Property *2* of a basis. We have $B_3 \subseteq \bigcap (\mathcal{U} - \{U\})$ and $B_3 \subseteq U$ by set theory. Hence, we have $B_3 \subseteq \bigcap (\mathcal{U} - \{U\}) \cap U$ and we are done.

$\square$

C.11 Proof of Proposition 3.4.4

Proof. Let X be a set with a basis $\mathcal{B}$. We show that the topology $\mathcal{T}$ generated by $\mathcal{B}$, i.e., $\{U \in \mathcal{P}(X) \mid (\forall u \in U \mid (\exists B \in \mathcal{B} \mid u \in B \wedge B \subseteq U))\}$ is exactly the set of all unions of elements of $\mathcal{B}$.

$\boxed{\subseteq}$ Let $U \in \mathcal{T}$ be generated by $\mathcal{B}$. We show that U is the union of basis elements. For any $u \in U$, there is a basis element $B_u \in \mathcal{B}$ such that $u \in B_u$ and $B_u \subseteq U$. All the B_u exist by definition of $\mathcal{T}$. U is the union of all those B_u by set theory.

$\boxed{\supseteq}$ Let $\mathcal{U} \subseteq \mathcal{B}$. We show $\bigcup \mathcal{U} \in \mathcal{T}$. Let $u \in \bigcup \mathcal{U}$. Then there exists $B_u \in \mathcal{U}$ such that $u \in B_u$. Since $\mathcal{U} \subseteq \mathcal{B}$, we have $B_u \in \mathcal{B}$. Hence, we have $B_u \subseteq \bigcup \mathcal{U}$.

$\square$

C.12 Proof of Proposition 3.4.5

Proof. Let D be a continuous dcpo. We show that the set $\{\twoheaduparrow d \mid d \in D\}$ is a basis

1. Let $x \in D$. We show $(\exists d \in D \mid x \in \twoheaduparrow d)$. Take $d \in \twoheaddownarrow x$. It exists because $\twoheaddownarrow x \neq \emptyset$ by Proposition 2.6.4 Item *1*. We have $d \in D$ and $x \in \twoheaduparrow d$.

2. Let $x, d_1, d_2 \in D$ be such that $x \in \twoheaduparrow d_1 \cap \twoheaduparrow d_2$. We show that there exists $d_3 \in D$ such that $x \in \twoheaduparrow d_3$ and $\twoheaduparrow d_3 \subseteq \twoheaduparrow d_1 \cap \twoheaduparrow d_2$. By Proposition 2.7.2, there is $d_3 \in D$ such that $d_1 \ll d_3$, $d_2 \ll d_3$ and $d_3 \ll x$. Then, we have $x \in \twoheaduparrow d_3$ and $\twoheaduparrow d_3 \subseteq \twoheaduparrow d_1 \cap \twoheaduparrow d_2$ by Proposition 2.4.3.

$\square$

C.13 Proof of Proposition 3.4.6

Proof. Let D be a continuous dcpo and $U \subseteq D$ be Scott-open. We show $U = \bigcup_{u \in U} \twoheaduparrow u$.

$\boxed{\subseteq}$ Let $x \in U$. We show $u \in \bigcup_{u \in U} \twoheaduparrow u$. We know $\bigsqcup^{\uparrow} \twoheaddownarrow x = x$ by Proposition 2.6.4 Item *1*. Then there exists $x' \in U \cap \twoheaddownarrow x$ by definition of a Scott-open set. Since $x \in \twoheaduparrow x'$ and $x' \in U$, we have $x \in \bigcup_{u \in U} \twoheaduparrow u$.

$\boxed{\supseteq}$ Let $x \in \bigcup_{u \in U} \twoheaduparrow u$. Then there exists $x' \in U$ such that $x \in \twoheaduparrow x'$. We show $x \in U$. By Proposition 2.4.2 Item *1*, we have $x' \sqsubseteq x$. Since U is upward closed by definition of Scott-openness, we have $x \in U$.

$\square$

C.14 Proof of Proposition 3.6.3

Proof. Let X be a topological space. We show that $\sqsubseteq_{\text{topo}}$ on X is a preorder.

1. We prove reflexivity. Let $x \in X$. Let $U \in \mathcal{O}(X)$ be such that $x \in U$. As wanted, we have $x \in U$.

2. We prove transitivity. Let $x, y, z \in X$ be such that $x \sqsubseteq_{\text{topo}} y$ and $y \sqsubseteq_{\text{topo}} z$. We show $x \sqsubseteq_{\text{topo}} z$. Let $U \in \mathcal{O}(X)$ be such that $x \in U$. Since $x \sqsubseteq_{\text{topo}} y$, we have $y \in U$. Since $y \sqsubseteq_{\text{topo}} z$, we have $z \in U$.

$\square$

C.15 Proof of Proposition 3.6.4

Proof. Let X be a topological space and $x, y \in X$. We show $x \sqsubseteq_{\text{topo}} y \iff \mathcal{N}_x \subseteq \mathcal{N}_y$. The following equivalences show this result:

$$
\begin{array}{lcl}
 & x \sqsubseteq_{\text{topo}} y & \\
\iff & & \langle \text{Definition of } \sqsubseteq_{\text{topo}} \rangle \\
 & (\forall U \in \mathcal{O}(X) \mid x \in U \Rightarrow y \in U) & \\
\iff & & \langle \text{Definition of neighborhoods} \rangle \\
 & (\forall U \in \mathcal{O}(X) \mid U \in \mathcal{N}_x \Rightarrow U \in \mathcal{N}_y) & \\
\iff & & \langle \text{Definition of } \subseteq \text{ and } (\forall x \in X, U \in \mathcal{N}_x \mid U \in \mathcal{O}(X)) \rangle \\
 & \mathcal{N}_x \subseteq \mathcal{N}_y\ . &
\end{array}
$$

□

C.16 Proof of Proposition 3.6.5

Proof. Let X be a topological space and $x, y \in X$. We show $x \sqsubseteq_{\text{topo}} y$ if and only if x is contained in all closed sets containing y. The latter can be translated in $x \sqsubseteq_{\text{topo}} y$ if and only if $(\forall C \in \Gamma(X) \mid y \in C \Rightarrow x \in C)$. The following equivalences show this result:

$$
\begin{array}{lcl}
 & x \sqsubseteq_{\text{topo}} y & \\
\iff & & \langle \text{Definition of } \sqsubseteq_{\text{topo}} \rangle \\
 & (\forall U \in \mathcal{O}(X) \mid x \in U \Rightarrow y \in U) & \\
\iff & & \langle \text{Contraposition} \rangle \\
 & (\forall U \in \mathcal{O}(X) \mid \neg(y \in U) \Rightarrow \neg(x \in U)) & \\
\iff & & \langle \text{Set theory} \rangle \\
 & (\forall U \in \mathcal{O}(X) \mid y \in X - U \Rightarrow x \in X - U) & \\
\iff & & \langle \text{Definition of closed sets} \rangle \\
 & (\forall C \in \Gamma(X) \mid y \in C \Rightarrow x \in C)\ . &
\end{array}
$$

□

C.17 Proof of Proposition 3.6.8

Proof. Let D be a dcpo and $x \in D$. We show that $\downarrow x \in \Gamma(D)$ using Proposition 3.3.5.

1. We show $\downarrow x = \downarrow(\downarrow x)$. This is true by Proposition 1.5.3 Item *1.*
2. Let $A \subseteq \downarrow x$ be directed. We show $\bigsqcup^{\uparrow} A \in \downarrow x$. We have $x \in \mathbf{ub}(A)$ by definition of the downward closure and the choice of A. Hence, we have $\bigsqcup^{\uparrow} A \sqsubseteq x$, because the supremum is the least upper bound, and $\bigsqcup^{\uparrow} A \in \downarrow x$ by definition of the downward closure.

□

C.18 Proof of Proposition 3.6.9

Proof. Let D be a dcpo equipped with its Scott topology. We show that $\sqsubseteq_{\text{topo}}$ on D is a partial order. It is a preorder by Proposition 3.6.3. Only antisymmetry is left to prove. Let $a, b \in D$ be such that $a \sqsubseteq_{\text{topo}} b$ and $b \sqsubseteq_{\text{topo}} a$. We show $a = b$. We have $a \in \downarrow b$ by Proposition 3.6.5, because $\downarrow b$ is Scott-closed by Proposition 3.6.8 and $b \in \downarrow b$ by reflexivity of $\sqsubseteq$. Hence, we have $a \sqsubseteq b$ by definition of the downward closure. By the same reasoning, we have $b \sqsubseteq a$. By antisymmetry of $\sqsubseteq$, we have $a = b$. □

C.19 Proof of Proposition 3.7.5

Proof. Let X be a T_2 topological space. We show that X is T_1. Let $x, y \in X$ be such that $x \neq y$. By T_2 assumption, there are $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $U \cap V = \emptyset$. Then, we have $x \notin V$ and $y \notin U$. Thus, we have $U \in \mathcal{N}_x - \mathcal{N}_y$ and $V \in \mathcal{N}_y - \mathcal{N}_x$. Hence, we have $\mathcal{N}_x - \mathcal{N}_y \neq \emptyset \wedge \mathcal{N}_y - \mathcal{N}_x \neq \emptyset$. □

C.20 Proof of Proposition 3.7.6

Proof. Let X be a T_1 topological space. We show that X is T_0 by contraposition. Let $x, y \in X$ be such that $x \neq y$. By T_1 assumption, there is $U \in \mathcal{N}_x - \mathcal{N}_y$. Hence we have $\mathcal{N}_x \neq \mathcal{N}_y$. □

C.21 Proof of Proposition 3.7.10

Proof. Let D be a dcpo equipped with its Scott topology. We show that D is a T_2 space if and only if its order is equality.

$\boxed{\Rightarrow}$ We use contraposition. Suppose that the order on D is not equality. We show that D is not T_2. Take $x, y \in D$ such that $x \sqsubset y$. They exist because the order on D is not equality. Let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. Because Scott-open sets are upward closed, we have $y \in U$. Hence, we have $y \in U \cap V$ and $U \cap V \neq \emptyset$ as wanted.

$\boxed{\Leftarrow}$ Suppose that the order on D is equality. We show that D is T_2. Let $x, y \in D$ be such that $x \neq y$. Choose $U = \{x\}$ and $V = \{y\}$. We have $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ because the singletons $\{x\}$ and $\{y\}$ are Scott-open. Indeed, they are upward closed, because the order is "=", and, for the same reason, the only directed sets intersecting them are themselves. Finally, we have $U \cap V = \emptyset$ because $x \neq y$.

□

C.22 Proof of Proposition 3.8.6

Proof. Let X be a compact space and $C \in \Gamma(X)$. We want to show that C is compact. Let $\mathcal{U} \subseteq \mathcal{O}(X)$ be an open cover of C, i.e., $C \subseteq \bigcup \mathcal{U}$. Since C is closed, there is $U \in \mathcal{O}(X)$ such that $X - U = C$. We have $X \subseteq (\bigcup \mathcal{U}) \cup U$ by set theory. By definition of a topology, $\bigcup \mathcal{U}$ is open and so is $(\bigcup \mathcal{U}) \cup U$. Therefore, the latter is an open cover of X. Since X is compact, there is $\mathcal{U}' \subseteq \mathcal{U}$ a finite subcover of X. But $\mathcal{U}'$ is also a finite subcover of C because $C \subseteq X$. $\square$

C.23 Proof of Proposition 3.8.7

Proof. Let X be a Hausdorff space and $K \subseteq X$ be a compact subset. We show that K is closed, i.e., that $X - K$ is open. Let $x \in X - K$ and $y \in K$. Note that x and y exist, otherwise $K = X$ or $K = \emptyset$, and we are done. We have $x \neq y$ by set theory. By the T_2 assumption, there are $U_y \in \mathcal{N}_x$ and $V_y \in \mathcal{N}_y$ such that $U_y \cap V_y = \emptyset$. By repeating this procedure for every $y \in K$, we obtain a cover of K, namely $\bigcup_{y \in K} V_y$. By compactness of K, there is a finite subcover of some size $n \in \mathbb{N}$. Consider the set $U_x = U_{y_1} \cap \cdots \cap U_{y_n}$ formed by taking the U_{y_i} such that $U_{y_i} \cap V_{y_i} = \emptyset$ and V_{y_i} is in the finite subcover, with $1 \leq i \leq n$. By definition of open sets, U_x is an open set. It contains x, because any U_{y_i} contains x. It is also disjoint from the finite subcover of K, because the intersection is included in any U_{y_i} which is disjoint of V_{y_i}, i.e., disjoint of their union, which is the finite subcover. We have $U_x \subseteq X - K$ by set theory, because U_x is disjoint from the finite subcover of K, which means disjoint of K by set theory. Now, we take the union and we obtain $\bigcup_{x \in X-K} U_x = X - K$ by set theory. But $\bigcup_{x \in X-K} U_x$ is open by definition, then so is $X - K$ as wanted. $\square$

C.24 Proof of Proposition 3.8.8

Proof. Let P be a preordered set equipped with its Alexandroff topology and $x \in P$. We show that $\uparrow x$ is compact. Let $\mathcal{C} \subseteq \mathcal{O}(P)$ be an open cover. We have $x \in \bigcup \mathcal{C}$ by definition of a cover and reflexivity. Then there exists $U \in \mathcal{C}$ such that $x \in U$. We also have $\uparrow x \subseteq U$ because the latter is upward closed by definition. Hence, we found a finite subcover, namely U. $\square$

C.25 Proof of Proposition 3.8.15

Proof. Let X be a compact space. We show that X is locally compact. Let $x \in X$. Choose X itself to be the compact neighbor of x. Finally, we have $X \subseteq X$ by reflexivity. $\square$

C.26 Proof of Proposition 3.8.16

Proof. Let D be a continuous dcpo equipped with its Scott topology. We show that D is locally compact. Let $x \in D$. Take $y \in \twoheaddownarrow x$. It exists because $\twoheaddownarrow x$ is directed by continuity of D, hence not empty. The principal filter $\uparrow y$ is compact by Proposition 3.8.10. The set $\twoheaduparrow y$ is Scott-open by Proposition 3.3.7, is contained in the compact set $\uparrow y$ by Proposition 2.4.6 Item *1* and contains x since $y \in \twoheaddownarrow x$. Hence, local compactness is proved. $\square$

C.27 Proof of Proposition 3.9.2

Proof. Let X and Y be topological spaces and $f \in [X \to Y]$. We show that f is continuous if and only if $\big(\forall C \in \Gamma(Y) \mid f^{-1}(C) \in \Gamma(Y)\big)$.

$\boxed{\Rightarrow}$ Suppose that f is continuous. Let $C \in \Gamma(Y)$. We show that $f^{-1}(C) \in \Gamma(X)$. We have $f^{-1}(Y - C) \in \mathcal{O}(X)$ by continuity of f and because $Y - C \in \mathcal{O}(Y)$ by definition. Therefore, we have that $X - f^{-1}(Y - C) \in \Gamma(X)$. In extension, this closed set is $\{x \in X \mid f(x) \notin Y - C\}$. This is exactly the set $\{x \in X \mid f(x) \in C\}$ which is $f^{-1}(C)$. Hence, the latter is closed.

$\boxed{\Leftarrow}$ Suppose $\big(\forall C \in \Gamma(Y) \mid f^{-1}(C) \in \Gamma(Y)\big)$. Let $V \in \mathcal{O}(Y)$. We show $f^{-1}(V) \in \mathcal{O}(X)$. We have that $f^{-1}(Y - V) \in \Gamma(X)$ by assumption. Therefore $X - f^{-1}(Y - V) \in \mathcal{O}(X)$. In extension, this open set is $\{x \in X \mid f(x) \notin Y - V\}$. This is the set $\{x \in X \mid f(x) \in V\}$ which is $f^{-1}(V)$. Hence, the latter is open.

$\square$

C.28 Proof of Proposition 3.9.3

Proof. Let X and Y be topological spaces and $f \in [X \to Y]$. We show that f is continuous if and only if $(\forall A \subseteq X \mid f(\mathbf{cl}(A)) \subseteq \mathbf{cl}(f(A)))$.

$\boxed{\Rightarrow}$ Suppose that f is continuous. We show $(\forall A \subseteq X \mid f(\mathbf{cl}(A)) \subseteq \mathbf{cl}(f(A)))$. Let $A \subseteq X$ and $y \in f(\mathbf{cl}(A))$. We want to show $y \in \mathbf{cl}(f(A))$. By Proposition 3.2.10, this is equivalent to show $(\forall V \in \mathcal{N}_y \mid V \cap f(A) \neq \emptyset)$. Let $V \in \mathcal{N}_y$. We can take $x \in \mathbf{cl}(A)$ such that $y = f(x)$. Moreover, $f^{-1}(V)$ is open by continuity of f and $x \in f^{-1}(V)$, because $x = f(y)$. Take $z \in f^{-1}(V) \cap A$. It exists by Proposition 3.2.10 where x is the element and $f^{-1}(V)$ is the neighbor of x. Hence, we have $f(z) \in V \cap f(A)$ which means $V \cap f(A) \neq \emptyset$ as wanted.

$\boxed{\Leftarrow}$ Suppose $(\forall A \subseteq X \mid f(\mathbf{cl}(A)) \subseteq \mathbf{cl}(f(A)))$. We show that f is continuous. Let $V \in \mathcal{O}(Y)$. We show $f^{-1}(V) \in \mathcal{O}(X)$. This is equivalent to show that $X - f^{-1}(V) \in \Gamma(X)$ which is equivalent to $\mathbf{cl}\big(X - f^{-1}(V)\big) \subseteq X - f^{-1}(V)$ by Proposition 3.2.6. Let $x \in \mathbf{cl}\big(X - f^{-1}(V)\big)$. We show $x \in X - f^{-1}(V)$. Then, we have $f(x) \in f\big(\mathbf{cl}\big(X - f^{-1}(V)\big)\big)$ and hence $f(x) \in \mathbf{cl}\big(f\big(X - f^{-1}(V)\big)\big)$ by assumption. We have $V \notin \mathcal{N}_{f(x)}$; otherwise, we would have $V \cap f\big(X - f^{-1}(V)\big) \neq \emptyset$ by Proposition 3.2.10 which is impossible by set theory. This means $f(x) \notin V$, which means $x \notin f^{-1}(V)$, which means $x \in X - f^{-1}(V)$, as wanted.

$\square$

C.29 Proof of Proposition 3.9.4

Proof. Let X be a compact topological space, Y be a topological space and $f \in [X \to Y]$ be continuous. We show that $f(X)$ is compact. Let $\mathcal{V} \subseteq \mathcal{O}(Y)$ be an open cover of $f(X)$. By continuity, the collection $\mathcal{U} = \{f^{-1}(V) \in \mathcal{O}(X) \mid V \in \mathcal{V}\}$ is an open cover of X. By compactness, there is a finite subcover $\mathcal{U}' \subseteq \mathcal{U}$ of X. Consider the collection $\mathcal{V}' = \{V \in \mathcal{V} \mid f^{-1}(V) \in \mathcal{U}'\}$. $\mathcal{V}'$ is included in $\mathcal{V}$ and is a finite subcover of $f(X)$. Indeed, any element $x \in X$ such that $f(x) \in f(X)$ is in a $f^{-1}(V')$ for some $V' \in \mathcal{V}'$ because $\mathcal{V}'$ is a cover. Hence, we have $f(x) \in V'$ which means that $f(X)$ is covered by $\mathcal{V}'$. $\square$

C.30 Proof of Proposition 3.9.6

Proof. Let X and Y be two topological spaces and $f \in [X \to Y]$ be bijective. We show that f is a homeomorphism if and only if $(\forall U \subseteq X \mid U \in \mathcal{O}(X) \iff f(U) \in \mathcal{O}(Y))$.

$\boxed{\Rightarrow}$ Suppose that f is a homeomorphism. Let $U \subseteq X$.

- $\boxed{\Rightarrow}$ Suppose $U \in \mathcal{O}(X)$. We show $f(U) \in \mathcal{O}(Y)$. We have $\left(f^{-1}\right)^{-1}(U) \in \mathcal{O}(X)$ since f^{-1} is continuous. But we have $\left(f^{-1}\right)^{-1} = f$. Therefore, we have $f(U) = \left(f^{-1}\right)^{-1}(U)$ and hence $f(U) \in \mathcal{O}(X)$.
- $\boxed{\Leftarrow}$ Suppose $f(U) \in \mathcal{O}(Y)$. We show $U \in \mathcal{O}(X)$. We have $f^{-1}(f(U)) \in \mathcal{O}(X)$ since f is continuous. But, since f is bijective, we have $f^{-1}(f(U)) = U$. Therefore, we have $U \in \mathcal{O}(X)$, as wanted.

$\boxed{\Leftarrow}$ Suppose $(\forall U \subseteq X \mid U \in \mathcal{O}(X) \iff f(U) \in \mathcal{O}(Y))$.

1. We show that f is continuous. Let $V \in \mathcal{O}(Y)$. We have $f\big(f^{-1}(V)\big) = V$ because f is bijective. Since f is continuous, we have $f^{-1}(V) \in \mathcal{O}(X)$.
2. We show that f^{-1} is continuous. Let $U \in \mathcal{O}(X)$. We have $f(U) \in \mathcal{O}(Y)$ by assumption. We also have $f(U) = \left(f^{-1}\right)^{-1}(U)$ because f is a bijection. Hence, we have $\left(f^{-1}\right)^{-1}(U) \in \mathcal{O}(Y)$ as wanted.

$\square$

C.31 Proof of Proposition 3.9.7

Proof. Let D and E be dcpos equipped with their respective Scott topology and $f \in [D \to E]$. We show that f is continuous if and only if f is Scott-continuous.

$\boxed{\Rightarrow}$ Suppose that f is continuous. We show that f is Scott-continuous. We use Corollary 2.3.5.

1. We show that f is monotonic, i.e., $(\forall x, y \in D \mid x \sqsubseteq_D y \Rightarrow f(x) \sqsubseteq_E f(y))$. Let $x, y \in D$ be such that $x \sqsubseteq_D y$. Consider the set $\downarrow_E f(y)$. It is Scott-closed by Proposition 3.6.8. Since f is continuous, we apply Proposition 3.9.2 to obtain that $f^{-1}(\downarrow_E f(y))$ is Scott-closed. We have $f^{-1}(\downarrow_E f(y)) = \{d \in D \mid f(d) \sqsubseteq_E f(y)\}$ by definition of the inverse. We have $y \in \{d \in D \mid f(d) \sqsubseteq_E f(y)\}$ by reflexivity. Hence, we have $x \in \{d \in D \mid f(d) \sqsubseteq_E f(y)\}$ by downward closure of a Scott-closed set. Therefore, we have $f(x) \sqsubseteq_E f(y)$ as wanted.
2. Let $A \subseteq D$ be directed. We show $f\left(\bigsqcup^{\uparrow} A\right) \sqsubseteq_E \bigsqcup^{\uparrow}_{a\in A} f(a)$. Consider the set $\downarrow_E \left(\bigsqcup^{\uparrow}_{a\in A} f(a)\right)$. It is Scott-closed by Proposition 3.6.8. Then, since f is continuous, we apply Proposition 3.9.2 to obtain that $f^{-1}\left(\downarrow_E \left(\bigsqcup^{\uparrow}_{a\in A} f(a)\right)\right)$ is Scott-closed. We have $f^{-1}\left(\downarrow_E \left(\bigsqcup^{\uparrow}_{a\in A} f(a)\right)\right) = \left\{d \in D \mid f(d) \sqsubseteq_E \bigsqcup^{\uparrow}_{a\in A} f(a)\right\}$ by definition of the inverse. For all $a \in A$, we have $a \in \left\{d \in D \mid f(d) \sqsubseteq_E \bigsqcup^{\uparrow}_{a\in A} f(a)\right\}$ because the supremum is an upper bound. Hence, we have $\bigsqcup^{\uparrow} A \in \left\{d \in D \mid f(d) \sqsubseteq_E \bigsqcup^{\uparrow}_{a\in A} f(a)\right\}$ by Proposition 3.3.5. Therefore, we have $f\left(\bigsqcup^{\uparrow} A\right) \sqsubseteq_E \bigsqcup^{\uparrow}_{a\in A} f(a)$ as wanted.

$\boxed{\Leftarrow}$ Suppose that f is Scott-continuous. We show that f is continuous. Let $V \in \mathcal{O}(E)$. We show $f^{-1}(V) \in \mathcal{O}(D)$. More precisely, we want to show that $f^{-1}(V)$ is Scott-open.

1. We show that $f^{-1}(V)$ is upward closed. Let $x \in f^{-1}(V)$ and $y \in D$ be such that $x \sqsubseteq_D y$. We have $f(x) \sqsubseteq_E f(y)$ because f is continuous and hence monotonic. Since V is upward closed by Scott-openness and $f(x) \in V$, we have $f(y) \in V$. Hence, we have $y \in f^{-1}(V)$ as wanted.
2. Let $A \subseteq X$ be directed and such that $\bigsqcup^{\uparrow} A \in f^{-1}(V)$. We show $A \cap f^{-1}(V) \neq \emptyset$. We have $f(\bigsqcup A) \in V$. Therefore, we have $\bigsqcup_{a\in A} f(a) \in V$ by Scott-continuity of f. Take $a \in A$ such that $f(a) \in f(A) \cap V$. It exists because V is Scott-open and contains the supremum of a directed set that, by definition, means it contains at least one element of the directed set. Hence, we have $a \in A \cap f^{-1}(V)$ by set theory. Therefore, we have $A \cap f^{-1}(V) \neq \emptyset$ as wanted.

□

C.32 Proof of Proposition 3.10.2

Proof. Let X be a topological space and $C \in \Gamma(X)$. We show that C is irreducible if and only if $C \neq \emptyset$ and $(\forall U_1, U_2 \in \mathcal{O}(X) \mid C \cap U_1 \neq \emptyset \wedge C \cap U_2 \neq \emptyset \Rightarrow C \cap (U_1 \cap U_2) \neq \emptyset)$. We use A^c to note $X - A$. The following equivalences show the desired result.

$$C \text{ is irreducible}$$

$$\iff \qquad \langle \text{Definition 3.10.1} \rangle$$

$$C \neq \emptyset \wedge (\forall C_1, C_2 \in \Gamma(X) \mid$$
$$C \subseteq C_1 \cup C_2 \Rightarrow C \subseteq C_1 \vee C \subseteq C_2)$$

$\iff$ $\langle$Definition of closed sets$\rangle$

$$C \neq \emptyset \wedge (\forall U_1, U_2 \in \mathcal{O}(X) \mid$$
$$C \subseteq (U_1)^c \cup (U_2)^c \Rightarrow C \subseteq (U_1)^c \vee C \subseteq (U_2)^c)$$

$\iff$ $\langle$Set theory$\rangle$

$$C \neq \emptyset \wedge (\forall U_1, U_2 \in \mathcal{O}(X) \mid$$
$$C \subseteq (U_1 \cap U_2)^c \Rightarrow C \subseteq (U_1)^c \vee C \subseteq (U_2)^c)$$

$\iff$ $\langle$Set theory$\rangle$

$$C \neq \emptyset \wedge (\forall U_1, U_2 \in \mathcal{O}(X) \mid$$
$$C \cap (U_1 \cap U_2) = \emptyset \Rightarrow C \cap U_1 = \emptyset \vee C \cap U_2 = \emptyset)$$

$\iff$ $\langle$Contraposition$\rangle$

$$C \neq \emptyset \wedge (\forall U_1, U_2 \in \mathcal{O}(X) \mid$$
$$C \cap U_1 \neq \emptyset \wedge C \cap U_2 \neq \emptyset \Rightarrow C \cap (U_1 \cap U_2) \neq \emptyset$$

□

C.33 Proof of Proposition 3.10.3

Proof. Let X be a topological space and $x \in X$. We show that $\mathbf{cl}(\{x\})$ is irreducible. Let $C_1, C_2 \in \Gamma(X)$ be such that $\mathbf{cl}(\{x\}) \subseteq C_1 \cup C_2$. We have $x \in C_1 \cup C_2$ by set theory and definition of the closure.

1. Suppose $x \in C_1$. So $\{x\} \subseteq C_1$. Hence, we have $\mathbf{cl}(\{x\}) \subseteq C_1$ because $\mathbf{cl}(\{x\})$ is the smallest closed set containing $\{x\}$ by definition and C_1 is closed.

2. Suppose $x \in C_2$. This case is symmetric as that of Item 1 of this proof.

□

C.34 Proof of Proposition 3.10.5

Proof. Let X be a topological space and $x, y \in X$ be such that $\mathcal{N}_x = \mathcal{N}_y$. We show $\mathbf{cl}(\{x\}) = \mathbf{cl}(\{y\})$. The following equalities show the desired result.

$$\mathbf{cl}(\{x\})$$

$=$ $\langle$Set theory$\rangle$

$$\{z \in X \mid z \in \mathbf{cl}(\{x\})\}$$

$=$ $\langle$Proposition 3.2.10$\rangle$

$$\{z \in X \mid (\forall U \in \mathcal{N}_z \mid U \cap \{x\} \neq \emptyset)\}$$

$=$ $\langle$Set theory$\rangle$

$$\{z \in X \mid (\forall U \in \mathcal{N}_z \mid x \in U)\}$$

$$
\begin{array}{ll}
= & \langle\text{Definition 3.2.8}\rangle \\
\quad \{z \in X \mid (\forall U \in \mathcal{N}_z \mid U \in \mathcal{N}_x)\} & \\
= & \langle\text{Assumption: } \mathcal{N}_x = \mathcal{N}_y\rangle \\
\quad \{z \in X \mid (\forall U \in \mathcal{N}_z \mid U \in \mathcal{N}_y)\} & \\
= & \langle\text{Similar reasoning}\rangle \\
\quad \mathbf{cl}(\{y\}) &
\end{array}
$$

□

C.35 Proof of Proposition 3.10.6

Proof. Let X be a sober space. We show that X is T_0. Let $x, y \in X$ be such that $\mathcal{N}_x = \mathcal{N}_y$. Consider $C = \mathbf{cl}(\{x\})$. C is irreducible by Proposition 3.10.3. We have $C = \mathbf{cl}(\{y\})$ by Proposition 3.10.5. Hence, C is a irreducible closed set which is the closure of x and y. Therefore, since X is sober, we have $x = y$. □

C.36 Proof of Proposition 3.10.7

Proof. Let X be a T_2 space. We show that, for any irreducible closed set C, $|C| = 1$. Let C be an irreducible closed set. Note that $C \neq \emptyset$ by definition of irreducibility. By way of contradiction, suppose $|C| > 1$. Then, there are $c_1, c_2 \in C$ such that $c_1 \neq c_2$. Since X is T_2, there are $U \in \mathcal{N}_{c_1}$ and $V \in \mathcal{N}_{c_2}$ such that $U \cap V = \emptyset$. By Proposition 3.10.2, we have $C \cap (U \cap V) \neq \emptyset$. This is a contradiction with the choice of U and V. Hence, we have $|C| = 1$. □

C.37 Proof of Proposition 3.10.8

Proof. Let X be a T_2 space. We show that X is sober. Let C be an irreducible closed subset of X.

1. We show $(\exists c \in X \mid C = \mathbf{cl}(\{c\}))$. By Proposition 3.10.7, there is $c \in X$ such that $C = \{c\}$. We have $C = \mathbf{cl}(\{c\})$ by Propositions 3.2.5 Item *2* and 3.2.6, and antisymmetry of $\subseteq$.

2. Suppose $(\exists c \in X \mid C = \mathbf{cl}(\{c\}))$. We show that c is unique. Let $c' \in X$ be such that $C = \mathbf{cl}(\{c'\})$. By definition of the closure, we have $c \in C$ and $c' \in C$. Then $C = \{c\}$ and $C = \{c'\}$ by Proposition 3.10.7. Hence, $c' = c$.

□

C.38 Proof of Proposition 3.10.9

Proof. Let D be a dcpo equipped with its Scott topology and $d \in D$. We show $\downarrow d = \mathbf{cl}(\{d\})$.

$\boxed{\subseteq}$ We show $\downarrow d \subseteq \mathbf{cl}(\{d\})$. We know that $\mathbf{cl}(\{d\})$ is downward closed by Scott-closedness and contains d by definition. Hence, $\mathbf{cl}(\{d\})$ also contains $\downarrow d$.

$\boxed{\supseteq}$ We show $\mathbf{cl}(\{d\}) \subseteq \downarrow d$. We know that $\downarrow d$ is Scott-closed by Proposition 3.6.8 and contains d by reflexivity. Hence, we have $\mathbf{cl}(\{d\}) \subseteq \downarrow d$ by definition of the closure.

□

C.39 Proof of Proposition 3.10.10

Proof. Let D be a continuous dcpo equipped with its Scott topology. We show that D is sober. Let $C \subseteq D$ be an irreducible Scott-closed set. Consider the set $C' = \bigcup_{c \in C} \twoheaddownarrow c$. We show that $\bigsqcup^{\uparrow} C'$ is the unique element of D whose closure is C.

1. We show that C' is directed.

   (a) We show $C' \neq \emptyset$. By irreducibility, there is $c \in C$. Take $c' \in \twoheaddownarrow c$. It exists because $\twoheaddownarrow c$ is directed by continuity of D. We have $c' \in C'$.

   (b) We show $(\forall a, b \in C' \mid (\exists c \in C' \mid a \sqsubseteq c \wedge b \sqsubseteq c))$. Let $a, b \in C'$. Then, we have $C \cap \twoheaduparrow a \neq \emptyset$ and $C \cap \twoheaduparrow b \neq \emptyset$. Take $c' \in C$ such that $c' \in \twoheaduparrow a \cap \twoheaduparrow b$. It exists by Proposition 3.10.2 and since $\twoheaduparrow a, \twoheaduparrow b \in \mathcal{O}(D)$ by Proposition 3.3.7. This means $a, b \in \twoheaddownarrow c'$. Choose $c \in \twoheaddownarrow c'$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. It exists by directedness of $\twoheaddownarrow c'$ and by continuity of D. We also have $c \in C'$ because $c' \in C$ and $c \in \twoheaddownarrow c'$

2. We show $C' \subseteq C$. Let $c' \in C'$. By definition of C', there is a $c \in C$ such that $c' \in \twoheaddownarrow c$. We have $c' \sqsubseteq c$ by Proposition 2.4.2 Item *1*. Hence, we have $c' \in C$ by downward closure of a Scott-closed set.

3. We show $\mathbf{ub}(C) = \mathbf{ub}(C')$.

   $\boxed{\subseteq}$ Let $u \in \mathbf{ub}(C)$. We show $u \in \mathbf{ub}(C')$. Let $c' \in C'$. We have $c' \in C$ by Item 2 of this proof. Hence, we have $c' \sqsubseteq u$ by definition of an upper bound.

   $\boxed{\supseteq}$ Let $u \in \mathbf{ub}(C')$. We show $u \in \mathbf{ub}(C)$. Let $c \in C$. We have $\twoheaddownarrow c \subseteq C'$. Thus, we have $u \in \mathbf{ub}(\twoheaddownarrow c)$. We have $c = \bigsqcup^{\uparrow} \twoheaddownarrow c$ by Proposition 2.6.4 Item *1*. Hence, we have $c \sqsubseteq u$, because the supremum is the least upper bound.

4. We show $C = \mathbf{cl}\left(\left\{\bigsqcup^{\uparrow} C'\right\}\right)$. Note that $\bigsqcup^{\uparrow} C'$ exists because C' is directed by Item 1 of this proof and D is a dcpo.

   $\boxed{\subseteq}$ Let $c \in C$. We show $c \in \mathbf{cl}\left(\left\{\bigsqcup^{\uparrow} C'\right\}\right)$. We have $c \sqsubseteq \bigsqcup^{\uparrow} C'$ by Item 3 of this proof. We have $\bigsqcup^{\uparrow} C' \in \mathbf{cl}\left(\left\{\bigsqcup^{\uparrow} C'\right\}\right)$ by definition. Hence, we have $c \in \mathbf{cl}\left(\left\{\bigsqcup^{\uparrow} C'\right\}\right)$ by downward closure of a Scott-closed set.

$\boxed{\supseteq}$ Since C' is directed by Item 1 of this proof and $C' \subseteq C$ by Item 2 of this proof, we have $\bigsqcup^{\uparrow} C' \in C$ by Scott-closedness. Hence, we have $\mathbf{cl}\left(\left\{\bigsqcup^{\uparrow} C'\right\}\right) \subseteq C$ because the closure of $\bigsqcup^{\uparrow} C'$ is contained in all closed sets containing $\bigsqcup^{\uparrow} C'$.

5. We show $\left(\forall d \in D \mid C = \mathbf{cl}(\{d\}) \Rightarrow d = \bigsqcup^{\uparrow} C'\right)$. Let $d \in D$ be such that $C = \mathbf{cl}(\{d\})$. We have $C = {\downarrow} d$ and $C = {\downarrow} \bigsqcup^{\uparrow} C'$ by Proposition 3.10.9 and Item 4 of this proof. Hence, we have $d \sqsubseteq \bigsqcup^{\uparrow} C'$ and $\bigsqcup^{\uparrow} C' \sqsubseteq d$. Therefore, we have $d = \bigsqcup^{\uparrow} C'$ by antisymmetry.

□

C.40 Proof of Proposition 3.10.14

Proof. Let X be a topological space.

1. We show $(\forall \mathcal{U} \subseteq \mathcal{O}(X) \mid \diamond \bigcup \mathcal{U} = \bigcup \diamond \mathcal{U})$. Let $\mathcal{U} \subseteq \mathcal{O}(X)$. The following equalities show the result.

$$
\begin{array}{cl}
 & \diamond \bigcup \mathcal{U} \\
= & \langle \text{Definition of } \diamond \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \cap \bigcup \mathcal{U} \neq \emptyset\} \\
= & \langle \text{Set theory} \rangle \\
 & \left\{C \in \mathcal{S}(X) \mid \bigcup_{U \in \mathcal{U}} (C \cap U) \neq \emptyset\right\} \\
= & \langle \text{Set theory} \rangle \\
 & \bigcup_{U \in \mathcal{U}} \{C \in \mathcal{S}(X) \mid C \cap U \neq \emptyset\} \\
= & \langle \text{Definition of } \diamond \rangle \\
 & \bigcup_{U \in \mathcal{U}} \diamond U \\
= & \langle \text{Notation} \rangle \\
 & \bigcup \diamond \mathcal{U}
\end{array}
$$

2. We show $(\forall \mathcal{U} \subseteq \mathcal{O}(X) \mid \diamond \bigcap \mathcal{U} = \bigcap \diamond \mathcal{U})$. Let $\mathcal{U} \subseteq \mathcal{O}(X)$.

   (a) Suppose $\mathcal{U} \neq \emptyset$. The following equalities show the result.

$$
\begin{array}{cl}
 & \diamond \bigcap \mathcal{U} \\
= & \langle \text{Definition of } \diamond \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \cap \bigcap \mathcal{U} \neq \emptyset\} \\
= & \langle \text{Set theory} \rangle \\
 & \left\{C \in \mathcal{S}(X) \mid \bigcap_{U \in \mathcal{U}} (C \cap U) \neq \emptyset\right\} \\
= & \langle \text{Set theory} \rangle \\
 & \bigcap_{U \in \mathcal{U}} \{C \in \mathcal{S}(X) \mid C \cap U \neq \emptyset\} \\
= & \langle \text{Definition of } \diamond \rangle \\
 & \bigcap_{U \in \mathcal{U}} \diamond U \\
= & \langle \text{Notation} \rangle \\
 & \bigcap \diamond \mathcal{U}
\end{array}
$$

(b) Suppose $\mathcal{U} = \emptyset$. The following equalities show the result.

$$
\begin{array}{ll}
 & \diamond \bigcap \mathcal{U} \\
= & \langle \text{Definition of } \diamond \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \cap \bigcap \mathcal{U} \neq \emptyset\} \\
= & \langle \text{Argument 2(b)i} \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \cap X \neq \emptyset\} \\
= & \langle \text{Set theory} \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \neq \emptyset\} \\
= & \langle C \neq \emptyset \text{ by irreducibility} \rangle \\
 & \{C \in \mathcal{S}(X) \mid \text{True}\} \\
= & \langle \text{Set theory} \rangle \\
 & \mathcal{S}(X) \\
= & \langle \text{Argument 2(b)ii} \rangle \\
 & \bigcap_{U \in \mathcal{U}} \diamond U \\
= & \langle \text{Notation} \rangle \\
 & \bigcap \diamond \mathcal{U}
\end{array}
$$

Note that an empty intersection of subsets is equal the whole set.

i. Since $\mathcal{U} = \emptyset$, it is an empty intersection of subsets of X. Hence it is equal to X itself.

ii. Since $\mathcal{U} = \emptyset$, it is an empty intersection of subsets of $\mathcal{S}(X)$. Hence it is equal to $\mathcal{S}(X)$ itself.

3. We show $(\forall U_1, U_2 \in \mathcal{O}(X) \mid U_1 \subseteq U_2 \iff \diamond U_1 \subseteq \diamond U_2)$. Let $U_1, U_2 \in \mathcal{O}(X)$.

$\boxed{\Rightarrow}$ Suppose $U_1 \subseteq U_2$. We show $\diamond U_1 \subseteq \diamond U_2$. Consider the "$\diamond$" operator as a function from $\mathcal{O}(X)$ to $\mathcal{P}(\mathcal{S}(X))$. In this case, it fulfills the requirements of Proposition 1.7.3. Hence the "$\diamond$" operator as a function is monotonic. Therefore $\diamond U_1 \subseteq \diamond U_2$.

$\boxed{\Leftarrow}$ Suppose $\diamond U_1 \subseteq \diamond U_2$. We show $U_1 \subseteq U_2$. Let $u \in U_1$. We have $\mathbf{cl}(\{u\}) \in \mathcal{S}(X)$ by Proposition 3.10.3 and $\emptyset \neq \{u\} \subseteq \mathbf{cl}(\{u\}) \cap U_1$ since $u \in U_1$. Hence, we have $\mathbf{cl}(\{u\}) \in \diamond U_1$. Since $\diamond U_1 \subseteq \diamond U_2$, we have $\mathbf{cl}(\{u\}) \in \diamond U_2$. By definition of the diamond operator, we have $\mathbf{cl}(\{u\}) \cap U_2 \neq \emptyset$. Suppose $u \notin U_2$. Hence, we have $u \in X - U_2$ by set theory and $\mathbf{cl}(\{u\}) \subseteq X - U_2$ because the closure is the smallest closed set containing u. This is a contradiction with $\mathbf{cl}(\{u\}) \cap U_2 \neq \emptyset$. Hence, we have $u \in U_2$ as wanted.

$\square$

C.41 Proof of Proposition 3.10.16

Proof. Let X be a topological space. We prove that the lower Vietoris topology from X is a topology. Let $\mathcal{U} \subseteq \mathcal{O}(X)$.

1. We show $\bigcup \diamond \mathcal{U}$ is open in the lower Vietoris topology from X. We have $\bigcup \diamond \mathcal{U} = \diamond \bigcup \mathcal{U}$ by Proposition 3.10.14 Item *1*. But $\bigcup \mathcal{U} \in \mathcal{O}(X)$ by definition of a topology. Hence $\bigcup \diamond \mathcal{U}$ is open in the lower Vietoris topology by definition of the latter.

2. Suppose that U is finite. We show $\bigcap \diamond \mathcal{U}$ is open in the lower Vietoris topology from X. We have $\bigcap \diamond \mathcal{U} = \diamond \bigcap \mathcal{U}$ by Proposition 3.10.14 Item *2*. But $\bigcap \mathcal{U} \in \mathcal{O}(X)$ by definition of a topology. Hence $\bigcap \diamond \mathcal{U}$ is open in the lower Vietoris topology by definition of the latter.

□

C.42 Proof of Proposition 3.10.17

Proof. Let X be a topological space. We show $(\forall C_1, C_2 \in \mathcal{S}(X) \mid C_1 \sqsubseteq_{\text{topo}} C_2 \iff C_1 \subseteq C_2)$ on $\mathcal{S}(X)$ equipped with the lower Vietoris topology. Let $C_1, C_2 \in \mathcal{S}(X)$.

$\boxed{\Rightarrow}$ Suppose $C_1 \sqsubseteq_{\text{topo}} C_2$. We show $C_1 \subseteq C_2$. By way of contradiction, suppose $C_1 \cap (X - C_2) \neq \emptyset$. This means $C_1 \in \diamond(X - C_2)$, since $X - C_2 \in \mathcal{O}(X)$ by definition of a closed set. Hence, we have $C_2 \in \diamond(X - C_2)$ because $C_1 \sqsubseteq_{\text{topo}} C_2$. By definition of the diamond operator, we have $C_2 \cap (X - C_2) \neq \emptyset$. This is a contradiction. Therefore, we have $C_1 \subseteq C_2$.

$\boxed{\Leftarrow}$ Suppose $C_1 \subseteq C_2$. We show $C_1 \sqsubseteq_{\text{topo}} C_2$. Let $U \in \mathcal{O}(X)$ be such that $C_1 \in \diamond U$. We have $C_1 \cap U \neq \emptyset$ by definition of the diamond operator. Thus, we have $C_2 \cap U \neq \emptyset$ by set theory. Hence, we have $C_2 \in \diamond U$ as wanted.

□

C.43 Proof of Proposition 3.10.18

Proof. Let X be a topological space. We show that $\mathcal{S}(X)$ equipped with the lower Vietoris topology is sober. Let $\mathcal{C} \in \mathcal{S}(\mathcal{S}(X))$. Since $\mathcal{C}$ is closed, there is a $U \in \mathcal{O}(X)$ such that $\mathcal{C} = \mathcal{S}(X) - \diamond U$. We show that $X - U$ is the unique element of $\mathcal{S}(X)$ whose closure in $\mathcal{S}(X)$ is $\mathcal{C}$.

1. We show $X - U \in \mathcal{S}(X)$.

   (a) We show $X - U \neq \emptyset$. By way of contradiction, suppose $X - U = \emptyset$. Thus, $U = X$ and $\diamond U = \diamond X = \mathcal{S}(X)$. This means that $\mathcal{C} = \mathcal{S}(X) - \diamond U = \mathcal{S}(X) - \mathcal{S}(X) = \emptyset$. It is a contradiction with the irreducibility of $\mathcal{C}$.

(b) Let $C_1, C_2 \in \Gamma(X)$ be such that $X - U \subseteq C_1 \cup C_2$. We show $X - U \subseteq C_1 \vee X - U \subseteq C_2$. Consider $\mathcal{C}' = \mathcal{S}(X) - \diamond(X - (C_1 \cup C_2))$. We have $\mathcal{C}' \in \Gamma(\mathcal{S}(X))$ because $\diamond(X - (C_1 \cup C_2))$ is open.

i. We show $\mathcal{C} = \{C \in \mathcal{S}(X) \mid C \subseteq X - U\}$. The following equalities show the result.

$$\begin{array}{lll} & \mathcal{C} & \\ = & & \langle\text{Definition of } \mathcal{C}\rangle \\ & \mathcal{S}(X) - \diamond U & \\ = & & \langle\text{Set theory}\rangle \\ & \{C \in \mathcal{S}(X) \mid C \notin \diamond U\} & \\ = & & \langle\text{Definition of } \diamond U \text{ and set theory}\rangle \\ & \{C \in \mathcal{S}(X) \mid C \cap U = \emptyset\} & \\ = & & \langle\text{Set theory}\rangle \\ & \{C \in \mathcal{S}(X) \mid C \subseteq X - U\} & \end{array}$$

ii. We show $\mathcal{C}' = \{C \in \mathcal{S}(X) \mid C \subseteq C_1\} \cup \{C \in \mathcal{S}(X) \mid C \subseteq C_2\}$. The following equalities show the result.

$$\begin{array}{lll} & \mathcal{C}' & \\ = & & \langle\text{Definition of } \mathcal{C}'\rangle \\ & \mathcal{S}(X) - \diamond(X - (C_1 \cup C_2)) & \\ = & & \langle\text{Set theory}\rangle \\ & \mathcal{S}(X) - \diamond((X - C_1) \cap (X - C_2)) & \\ = & & \langle\text{Proposition 3.10.14 Item } \mathit{2}\rangle \\ & \mathcal{S}(X) - (\diamond(X - C_1) \cap \diamond(X - C_2)) & \\ = & & \langle\text{Set theory}\rangle \\ & (\mathcal{S}(X) - \diamond(X - C_1)) \cup (\mathcal{S}(X) - \diamond(X - C_2)) & \\ = & & \langle\text{Definition of } \diamond \text{ and set theory}\rangle \\ & \{C \in \mathcal{S}(X) \mid C \cap (X - C_1) = \emptyset\} \cup \{C \in \mathcal{S}(X) \mid C \cap (X - C_2) = \emptyset\} & \\ = & & \langle\text{Set theory}\rangle \\ & \{C \in \mathcal{S}(X) \mid C \subseteq C_1\} \cup \{C \in \mathcal{S}(X) \mid C \subseteq C_2\} & \end{array}$$

iii. Let $i \in \{1, 2\}$. We show $\{C \in \mathcal{S}(X) \mid C \subseteq C_i\} \in \Gamma(\mathcal{S}(X))$. By the three last equalities of Item 1(b)ii of this proof, we have $\{C \in \mathcal{S}(X) \mid C \subseteq C_i\} = \mathcal{S}(X) - \diamond(X - C_i)$. Since $C_i \in \Gamma(X)$, we have $X - C_i \in \mathcal{O}(X)$. Then we have $\diamond(X - C_i) \in \mathcal{O}(\mathcal{S}(X))$. Thus $\mathcal{S}(X) - \diamond(X - C_i) \in \Gamma(\mathcal{S}(X))$. Hence $\{C \in \mathcal{S}(X) \mid C \subseteq C_i\} \in \Gamma(\mathcal{S}(X))$ as wanted.

iv. We show $X - U \subseteq C_1 \vee X - U \subseteq C_2$. Since $X - U \subseteq C_1 \cup C_2$, we obtain $\mathcal{C} \subseteq \mathcal{C}'$ from Items 1(b)i and 1(b)ii of this proof. This is equivalent to $\mathcal{C} \subseteq \{C \in \mathcal{S}(X) \mid C \subseteq C_1\} \cup \{C \in \mathcal{S}(X) \mid C \subseteq C_2\}$ by Item 1(b)ii of this proof. The last union is the union of two closed sets of $\mathcal{S}(X)$ by Item 1(b)iii of this proof. Hence, $\mathcal{C} \subseteq \{C \in \mathcal{S}(X) \mid C \subseteq C_1\} \vee \mathcal{C} \subseteq \{C \in \mathcal{S}(X) \mid C \subseteq C_1\}$ by irreducibility of $\mathcal{C}$.

A. Suppose $\mathcal{C} \subseteq \{C \in \mathcal{S}(X) \mid C \subseteq C_1\}$. This is equivalent to $\mathcal{S}(X) - \diamond U \subseteq \mathcal{S}(X) - \diamond(X - C_1)$ by definition of $\mathcal{C}$ for the left hand side of "=" and as proved in the three last equalities of Item 1(b)ii of this proof for the right hand side of "=". This is equivalent to $\diamond(X - C_1) \subseteq \diamond U$ by set theory [1]. By Proposition 3.10.14 Item *3*, this is equivalent to $X - C_1 \subseteq U$. Therefore, we conclude $X - U \subseteq C_1$ by set theory (see Footnote 1).

B. Suppose $\mathcal{C} \subseteq \{C \in \mathcal{S}(X) \mid C \subseteq C_2\}$. This case is symmetric to that of Item 1(b)ivA of this proof.

2. We show $\mathcal{C} = \mathbf{cl}_{\mathcal{S}(X)}(\{X - U\})$. The following equalities show the result.

$$
\begin{array}{lll}
 & \mathcal{C} & \\
= & & \langle \text{Item 1(b)i of this proof} \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \subseteq X - U\} & \\
= & & \langle \text{Item 1 of this proof} \\
 & & \ \text{and Proposition 3.10.17} \rangle \\
 & \{C \in \mathcal{S}(X) \mid C \sqsubseteq_{\text{topo}} X - U\} & \\
= & & \langle \text{Definition of the downward closure} \rangle \\
 & \downarrow_{\mathcal{S}(X), \sqsubseteq_{\text{topo}}} (X - U) & \\
= & & \langle \text{Corollary 3.6.7} \rangle \\
 & \mathbf{cl}_{\mathcal{S}(X)}(\{X - U\}) &
\end{array}
$$

3. Let $C \in \mathcal{S}(X)$ be such that $\mathbf{cl}_{\mathcal{S}(X)}(\{C\}) = \mathcal{C}$. We show $C = X - U$. By Corollary 3.6.6 and Items 1 and 2 of this proof, we have $C \sqsubseteq_{\text{topo}} X - U$ and $X - U \sqsubseteq_{\text{topo}} C$. By Proposition 3.10.17, we have $C \subseteq X - U$ and $X - U \subseteq C$. Hence, by antisymmetry, we have $C = X - U$.

□

[1] In set theory, for any set X and $A, B \subseteq X$, $A \subseteq B \iff X - B \subseteq X - A$.