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MODÈLES DE DÉPENDANCE DANS LA THÉORIE DU RISQUE

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Résumé

Initialement, la théorie du risque supposait l'indépendance entre les différentes variables aléatoires et autres paramètres intervenant dans la modélisation actuarielle. De nos jours, cette hypothèse d'indépendance est souvent relâchée afin de tenir compte de possibles interactions entre les différents éléments des modèles. Dans cette thèse, nous proposons d'introduire des modèles de dépendance pour différents aspects de la théorie du risque. Dans un premier temps, nous suggérons l'emploi des copules comme structure de dépendance. Nous abordons tout d'abord un problème d'allocation de capital basée sur la *Tail-Value-at-Risk* pour lequel nous supposons un lien introduit par une copule entre les différents risques. Nous obtenons des formules explicites pour le capital à allouer à l'ensemble du portefeuille ainsi que la contribution de chacun des risques lorsque nous utilisons la copule Farlie-Gumbel-Morgenstern. Pour les autres copules, nous fournissons une méthode d'approximation. Au deuxième chapitre, nous considérons le processus aléatoire de la somme des valeurs présentes des sinistres pour lequel les variables aléatoires du montant d'un sinistre et de temps écoulé depuis le sinistre précédent sont liées par une copule Farlie-Gumbel-Morgenstern. Nous montrons comment obtenir des formes explicites pour les deux premiers moments puis le moment d'ordre m de ce processus. Le troisième chapitre suppose un autre type de dépendance causée par un environnement extérieur. Dans le contexte de l'étude de la probabilité de ruine d'une compagnie de réassurance, nous utilisons un environnement markovien pour modéliser les cycles de souscription. Nous supposons en premier lieu des temps de changement de phases de cycle déterministes puis nous les considérons ensuite influencés en retour par les montants des sinistres. Nous obtenons, à l'aide de la méthode d'erlangisation, une approximation de la probabilité de ruine en temps fini.

Abstract

Initially, it was supposed in risk theory that the random variables and other parameters of actuarial models were independent. Nowadays, this hypothesis is often relaxed to take into account possible interactions. In this thesis, we propose to introduce some dependence models for different aspects of risk theory. In a first part, we use copulas as dependence structure. We first tackle a problem of capital allocation based on the Tail-Value-at-Risk where the risks are supposed to be dependent according to a copula. We obtain explicit formulas for the capital to be allocated to the overall portfolio but also for the contribution of each risk when we use a Farlie-Gumbel-Morenstern copula. For the other copulas, we give an approximation method. In the second chapter, we consider the stochastic process of the discounted aggregate claims where the random variables for the claim amount and the time since the last claim are linked by a Farlie-Gumbel-Morgenstern copula. We show how to obtain exact expressions for the first two moments and for the moment of order m of the process. The third chapter assumes another type of dependence that is caused by an external environment. In the context of the study of the ruin probability for a reinsurance company, we use a Markovian environment to model the underwriting cycles. We suppose first deterministic cycle phase changes and then that these changes can also be influenced by the claim amounts. We use the erlangization method to obtain an approximation for the finite time ruin probability.

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Avant-propos

Cette thèse traite de l'utilisation de modèles de dépendance dans la théorie du risque. Elle est constituée de 4 chapitres dont l'introduction générale. Le travail présenté s'articule autour de trois articles abordant l'emploi de modèles de dépendance dans la théorie du risque. Le Chapitre 2 est constitué d'un article co-écrit avec mon directeur de thèse à l'Université Laval Etienne Marceau et ma co-directrice de thèse de l'Université Laval Hélène Cossette , s'intitulant *TVaR based allocation with copula* et publié dans la revue *Insurance : Mathematics & Economics*, Bargès *et al.* (2009). Il expose un problème d'allocation de capital basée sur la mesure de risque TVaR lorsque les risques sont dépendants au travers d'une copule. Des formules exactes pour le capital à allouer à l'ensemble du portefeuille ainsi qu'à chacun des risques sont obtenues pour certaines paramétrisations du modèle. Dans le cas général, des méthodes d'approximation sont proposées. Le Chapitre 3 repose sur un article, Bargès *et al.* (2010a), actuellement soumis pour publication. Co-écrit avec Hélène Cossette, Etienne Marceau et Stéphane Loisel, mon co-directeur de thèse à l'Université Lyon 1, il aboutit à l'expression des moments de la valeur présente du montant total des sinistres lorsque les variables aléatoires représentant les montants de sinistre et les temps inter-sinistre sont liées par une copule de Farlie-Gumbel-Morgenstern. Ces deux chapitres de la thèse traitent de l'utilisation des copules. Le troisième chapitre considère l'emploi d'un environnement markovien pour refléter une influence externe au modèle étudié. Ce chapitre aborde le sujet de la théorie de la ruine en temps continu et fournit une approximation de la probabilité de ruine en temps fini dans un contexte de réassurance. Certains paramètres du modèle de risque peuvent être assujettis à des variations cycliques. Ce dernier travail est basé sur un article également soumis pour publication et réalisé en collaboration avec Stéphane Loisel et Xavier Venel de la Toulouse School of Economics, Bargès *et al.* (2010b).

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Chapitre 1

Introduction

Cette introduction générale se compose de deux sections. Après une description du contexte actuel de l'assurance et des enjeux liés à la réforme réglementaire européenne Solvabilité II, nous présentons les différents aspects de la théorie du risque abordés dans la thèse. Les modèles de dépendance employés dans les chapitres suivants sont exposés dans un second temps.

1.1 La théorie du risque

La théorie du risque a pour objectif de fournir différents outils mathématiques d'évaluation et de quantification des risques souscrits par les compagnies d'assurance et de réassurance. L'assurance consiste en une opération de transfert de risque d'un assuré vers un assureur qui, en contrepartie d'une prime, s'engage à supporter les coûts financiers de sinistres éventuels. Afin de garantir le respect de leurs engagements auprès des assurés, les assureurs et réassureurs ont recours à une modélisation actuarielle des risques. Des modèles mathématiques sont sans cesse développés pour répondre à une meilleure compréhension des risques et de leur évolution. Des ouvrages de référence dans le domaine de la théorie du risque sont Marceau (2009), Klugman *et al.* (2008), Kaas *et al.* (2001), McNeil *et al.* (2005), Mikosch (2009), Denuit et Charpentier (2004) et Denuit et Charpentier (2005).

1.1.1 Le contexte actuel lié à Solvabilité II

Le monde de l'assurance est actuellement en pleine mutation en raison du projet de réforme européenne Solvabilité II sur les réglementations des compagnies d'assurance et de réassurance qui entrera en vigueur à partir d'octobre 2012. Ce projet viendra modifier les réglementations précédentes de Solvabilité I mises en place en 2002 et basées sur des directives datant des années 70, 1973 pour l'assurance non-vie et 1979 pour l'assurance vie. Ces directives avaient pour but de prévoir une dotation additionnelle et prudentielle en capital par une approche forfaitaire. Une marge de solvabilité fonction proportionnelle des primes et des sinistres de l'assureur ou réassureur avait ainsi été établie afin de répondre à l'arrivée d'événements exceptionnels. Ces réglementations n'étant plus adaptées à l'évolution des risques actuels (tempêtes européennes, ouragan aux Etats-Unis, risque terroriste, crises financières), l'Union Européenne a initié au début des années 2000 le projet de réforme réglementaire Solvabilité II.

Ce projet de réforme Solvabilité II s'appuie principalement sur la prise en compte de l'exposition aux risques. Elle incitera ainsi les assureurs et réassureurs à une meilleure connaissance et gestion des risques et à accroître la transparence des informations liées à leurs activités. Solvabilité II intégrera dans ses directives la gestion des structures de groupe et établira une harmonisation de ses réglementations au niveau européen. Un processus de développement de la réforme dit Lamfalussy a été mis en place et se décline en 3 niveaux. Le premier niveau consiste en la définition des principes de base adoptés par le Conseil et le Parlement Européen. La détermination de mesures de mise en œuvre et leur validation par la Commission Européenne constitue le niveau 2. Le CEIOPS (*Committee of European Insurance and Occupational Pensions Supervisors* ou Comité Européen des Contrôleurs des Assurances et des Pensions Professionnelles) rendra au niveau 3 des avis et recommandations sur les mesures adoptées au niveau 2. Le CEIOPS participe également aux points techniques des niveaux 1 et 2. A ce jour, nous sommes à la phase 2 de ce processus.

A l'instar de la directive bancaire de Bâle II, Solvabilité II est basée sur 3 piliers qui constituent les objectifs principaux de la réforme (voir Figure 1.1). Le premier pilier concerne les exigences quantitatives requises pour les provisions techniques et les fonds propres des compagnies, notamment par l'établissement de deux seuils d'intervention prudentielle que sont le Capital de Solvabilité Requis ou *Solvability Capital Requirement* (SCR) et le Minimum de Capital Requis ou *Minimum Capital Requirement* (MCR). Le deuxième pilier fixe les exigences qualitatives dans la gestion des risques en interne par la mise en place d'un système de gouvernance adéquat et dans la supervision faite par les autorités de contrôle. Enfin, des conditions de communications publiques sur les activités financières, de gestion des risques et de solvabilité entraînant une discipline de

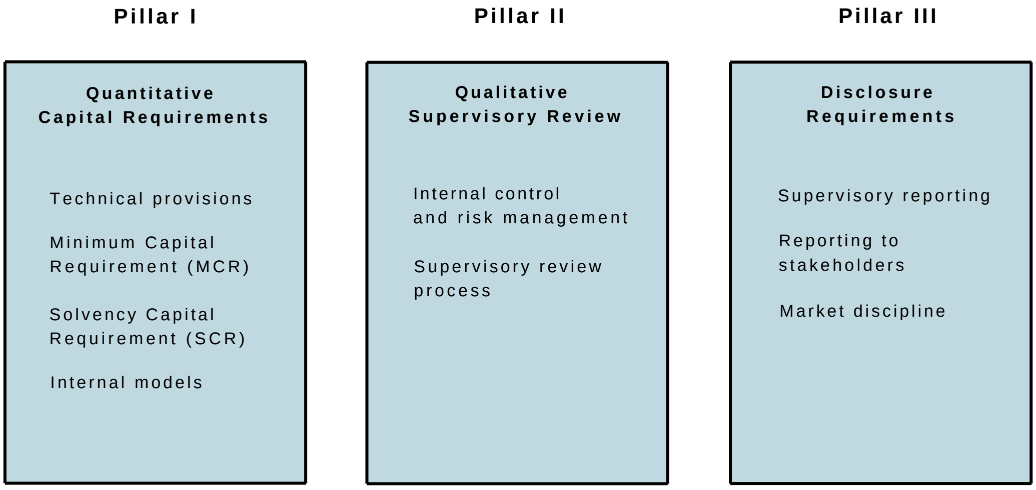

FIGURE 1.1 – Les trois piliers

marché établissent le troisième pilier.

Nous nous intéressons ici plus particulièrement au Pilier I qui traite des obligations techniques que les assureurs et réassureurs seront tenus de suivre. Ces exigences sont axées autour d'une vision *market consistent* du bilan économique équilibré où les actifs couvrent adéquatement les passifs d'une compagnie telle que l'illustre la Figure 1.2. Les provisions techniques sont évaluées différemment selon la nature couvrable (*hedgeable*) du risque, c'est-à-dire un engagement pouvant faire l'objet d'un portefeuille répliquant sur le marché financier, ou non-couvrable (*non-hedgeable*). Les risques couvrables font ainsi l'objet d'une valorisation *market consistent* et ne requièrent aucune marge de prudence supplémentaire à l'évaluation de leurs provisions techniques. En revanche, une telle marge de prudence est demandée pour les risques non-couvrables qui représentent la majorité des risques des assurances (mortalité, risques non-vie,...). Les provisions techniques pour ce type de risques sont composées du *Best Estimate* (BE), déterminé comme la valeur actuelle probable des flux futurs, et de la Marge de Risque ou *Risk Margin* (RM). Celle-ci est évaluée soit par une approche Percentile (*Quantile Approach*), soit par une approche Coût du Capital (*Cost of Capital approach*) (voir le QIS4, CEIOPS (2008)). Le seuil MCR, dont l'évaluation standard n'est pas encore définitive, représente le niveau minimum de fonds propres en dessous duquel les autorités de contrôle pourront demander à la compagnie de se retirer. Enfin, le SCR, qui sera le principal indicateur de solvabilité, correspond au montant nécessaire pour absorber les pertes infligées par des sinistres exceptionnels. Il pourra être déterminé soit par une formule standard soit par un modèle interne qui demandera la validation des autorités de contrôle. Même si les compagnies devront fournir le SCR standard, elles seront encouragées à développer un modèle interne favorisant une meilleure connaissance et gestion de leurs risques.

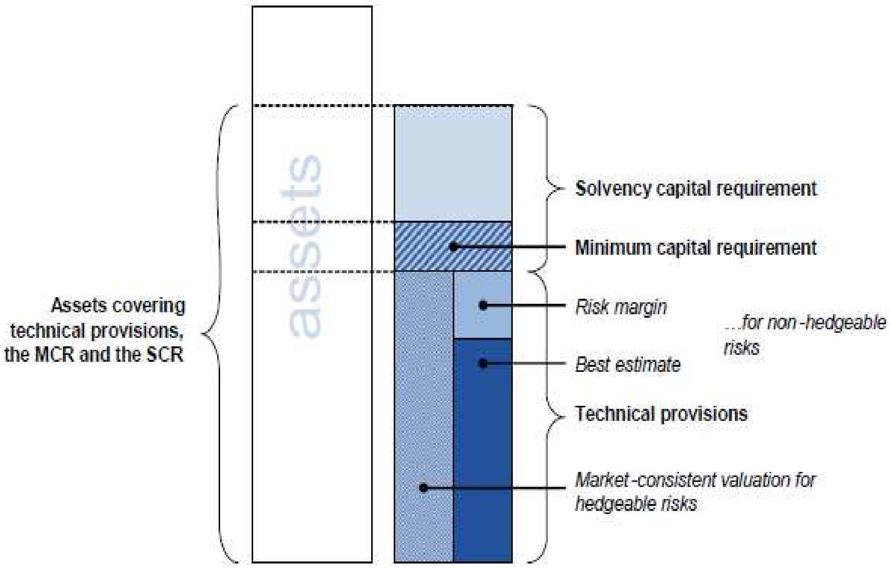

FIGURE 1.2 – Bilan économique (Source : CEIOPS' Advice to the European Commission in the Framework of the Solvency II project on Pillar I issues - further advice, graphic page 9 [© CEIOPS 2007])

La formule standard d'évaluation du SCR se construit autour d'une approche modulaire où chaque module représente un type de risque encouru par les assureurs et réassureurs comme le montre la Figure 1.3. Le SCR global se compose ainsi du BSCR (*Basic Solvability Capital Requirement*), de la charge de capital allouée au risque opérationnel, notée SCR_{op}, et d'un capital d'ajustement, noté Adj, correspondant aux taxes différées. En général, le BSCR constituera la plus grande part du SCR global. Le BSCR comprend, selon les risques souscrits par une compagnie, les SCR pour les modules de risque non-vie, de marché, santé, de défaut et vie qui sont eux-mêmes décomposés en sous-modules. L'évaluation du SCR par module se fait à l'aide de la mesure de risque *Value-at-Risk* (VaR) pour un niveau de confiance de 99.5 % sur la distribution des pertes du risque sur un horizon de temps de 1 an (voir le QIS4, CEIOPS (2008)). Le BSCR s'obtient ensuite par l'agrégation des différents capitaux liés à chacun des risques tel que

$$BSCR = \sqrt{\sum_i \sum_j \rho_{ij} SCR_i SCR_j}, \tag{1.1}$$

où les termes ρ_{ij} sont des paramètres définis dans le QIS4 reflètant une certaine corrélation entre les différents types de risque. De cette manière, le BSCR sera toujours plus petit que la somme des SCR par risque et traduira ainsi un effet de diversification. Le

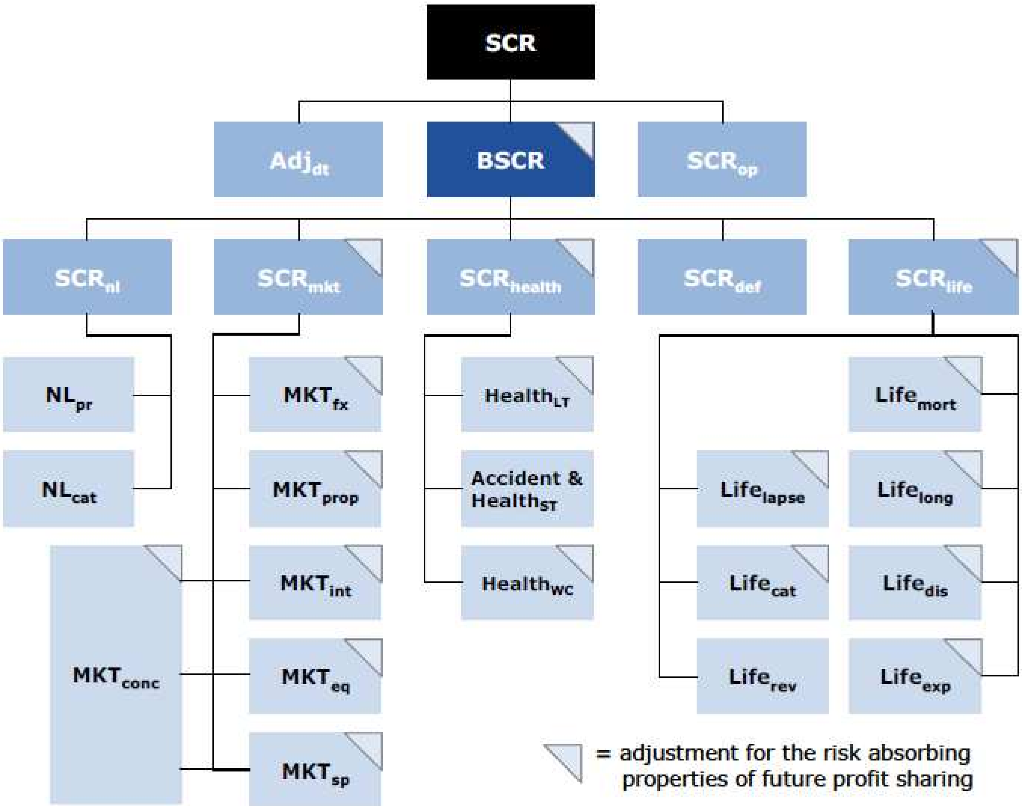

FIGURE 1.3 – Architecture modulaire du SCR (Source : CEIOPS' Report on its fourth Quantitative Impact Study (QIS4) for Solvency II, graphic page 31 [© CEIOPS 2008])

SCR global est ensuite calculé par

$$SCR = BSCR + Adj + SCR_{op}. \tag{1.2}$$

Ainsi, l'évaluation de cette marge de prudence du capital par la formule standard de Solvabilité II permet déjà une certaine prise en compte de la dépendance qui peut intervenir entre les différents éléments en jeu dans la modélisation actuarielle. Cette nouvelle réforme incitera les compagnies d'assurance et de réassurance, par le biais notamment du recours possible à un modèle interne pour le SCR, à mieux connaître et évaluer leurs risques. Cela passe par l'utilisation de modèles évolués de dépendance telle qu'il est proposé dans cette thèse.

1.1.2 Les mesures de risque

Les mesures de risques sont des outils de quantification de risque. Elles permettent d'évaluer un niveau de dangerosité d'un risque mais également de comparer différents risques entre eux et de les classer selon le niveau de dangerosité. Quantification et comparaison des risques peuvent ensuite être utilisées à plusieurs fins telles que l'évaluation de prime, l'allocation de capital, la détermination de marges pour les transactions financières ou encore la sélection des risques d'un portefeuille d'assurance ou de réassurance. Une mesure de risque peut être définie comme suit.

Définition 1.1. *Une mesure de risque $\rho(X)$ est une fonctionnelle ρ qui attribue une valeur réelle à la variable aléatoire X des pertes associées à un risque telle que*

$$\rho : X \to \mathbb{R}. \tag{1.3}$$

Dans le contexte actuariel, on peut restreindre ρ en une fonctionnelle associant une valeur strictement positive à une variable aléatoire strictement positive. De nombreuses mesures de risque ont été proposées ces dernières années. Nous présentons ici celles qui sont principalement étudiées dans la littérature mais également utilisées en pratique.

L'une de plus populaires mesures de risque est la *Value-at-Risk* (VaR), appelée également "Valeur à Risque". Elle est associée à un niveau de confiance κ, $0 \leq \kappa < 1$. On définit la VaR par $VaR_\kappa(X) = F_X^{-1}(\kappa)$ où F_X^{-1} est la fonction de répartition inverse de X telle que

$$F_X^{-1}(\kappa) = \inf\{x \in \mathbb{R}, F_X(x) \geq \kappa\}. \tag{1.4}$$

Ainsi la VaR représente le $\kappa^{\text{ème}}$ quantile de la distribution de la variable aléatoire (v.a.) des coûts X. On peut également écrire

$$\kappa = Pr\left(X \leq VaR_\kappa(X)\right).$$

Toutefois, le contexte actuariel nécessite l'étude de la queue des distributions. En effet, par nature, un risque qui a une probabilité significative de causer de forts sinistres est dangereux et les compagnies d'assurance et de réassurance ont besoin d'évaluer ce niveau de dangerosité. L'épaisseur de la queue de la distribution des sinistres est donc un élément fondamental dans l'évaluation du niveau de danger d'un risque. Or, la VaR ne donne qu'une information ponctuelle au quantile κ de la distribution du sinistre et aucune information au delà de ce point. De plus, comme nous le verrons plus tard, la VaR ne satisfait pas en général toutes les propriétés requises à une mesure de risque dite cohérente selon Artzner *et al.* (1999). C'est pourquoi d'autres mesures de risque ont été proposées.

La *Conditional-Tail-Expectation* (CTE), appelée aussi la *Tail-Conditional-Expectation*, est une mesure de risque définie comme l'espérance de la v.a. X au delà de la VaR. Pour un niveau de confiance κ, $0 \leq \kappa < 1$, elle représente la moyenne des $(1-\kappa)100\%$ sinistres les plus élevés. Elle s'écrit de la manière suivante :

$$CTE_\kappa(X) = E\left[X | X > VaR_\kappa(X)\right]. \tag{1.5}$$

Cette mesure de risque, contrairement à la VaR, donne des informations sur la distribution de X au delà de la VaR et donc sur l'épaisseur de la queue de distribution.

Une autre mesure de risque qui est de plus en plus considérée dans la littérature est la *Tail-Value-at-Risk* (TVaR). Celle-ci est également souvent référencée comme l'*Expected-Shortfall*. La TVaR est définie pour un niveau de confiance κ, $0 \leq \kappa < 1$, comme

$$TVaR_\kappa(X) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(X)du. \tag{1.6}$$

Elle correspond à la moyenne arithmétique des valeurs de la VaR pour les niveaux de confiance au delà de κ. Elle peut également s'écrire comme suit :

$$TVaR_\kappa(X) = \frac{E\left[X\mathbf{1}_{\{X>VaR_\kappa(X)\}}\right] + VaR_\kappa(X)\Big(Pr\left(X \leq VaR_\kappa(X)\right) - \kappa\Big)}{1-\kappa}. \tag{1.7}$$

Elle prend évidemment en compte les valeurs de la distribution au delà de la VaR au niveau de confiance κ et nous informe donc sur l'épaisseur de la queue de distribution. La définition de la TVaR à l'Equation (1.7) met en exergue sa relation avec la mesure CTE. En effet, si l'on considère une v.a. des coûts de sinistre X continue, alors $Pr\left(X \leq VaR_\kappa(X)\right) = \kappa$. L'Equation (1.7) devient ainsi $TVaR_\kappa(X) = E\left[X|X > VaR_\kappa(X)\right]$ qui est exactement la définition de la CTE. Pour une v.a. X non continue, la probabilité $Pr\left(X \leq VaR_\kappa(X)\right)$ n'est en général pas égal à κ et la TVaR n'est pas égale à la CTE. De plus, dans ce même contexte de non-continuité, la CTE ne respecte pas toutes les propriétés d'une mesure de risque cohérente alors que la TVaR sera cohérente aussi bien pour des v.a. continues que non continues.

Pour répondre à la nécessité de principes théoriques et pratiques, Artzner *et al.* (1999) ont introduit la notion de mesure de risque cohérente dont nous donnons la définition ci-dessous.

Définition 1.2. *Une mesure de risque ρ est dite cohérente si, pour deux v.a. X et Y, elle satisfait les propriétés suivantes :*

i. *monotonie : si $Pr(X \leq Y) = 1$, alors $\rho(X) \leq \rho(Y)$,*

ii. *sous-additivité : $\rho(X + Y) \leq \rho(X) + \rho(Y)$,*

iii. *homogénéité positive : $\rho(aX) = a\rho(X)$, pour tout $a > 0$,*

iv. *invariance par translation : $\rho(X + b) = \rho(X) + b$, pour tout $b > 0$.*

La monotonie permet de s'assurer que si le risque dû à Y est presque sûrement plus grand que celui dû à X alors Y est plus dangereux au sens de la mesure de risque que X. La sous-additivité traduit le fait que considérer deux risques simultanément est moins risqué que traiter les risques séparément. Cela intègre donc l'idée de diversification. L'homogénéité positive signifie que le fait de mesurer une proportion d'un risque revient à considérer la proportion de la mesure du risque seul. De même, ajouter un montant certain au risque implique l'ajout de ce même montant à la mesure du risque. Ces deux dernières propriétés entraînent le fait qu'une mesure de risque cohérente apprécie la nature aléatoire du risque.

Comme avancé précédemment, la *Value-at-Risk* ne respecte en général pas la propriété de sous de sous-additivité. Pour des v.a. non continues, la *Conditional-Tail-Expectation* n'est également en général pas sous-additive. Ainsi, ni la VaR ni la CTE ne sont pas des mesures de risque cohérentes. En revanche, la *Tail-Value-at-Risk* satisfait toujours les quatre axiomes et est donc une mesure de risque cohérente.

D'autres propriétés pourraient également être souhaitées pour une mesure de risque. Marceau (2009) suggère les propriétés suivantes : 1) si $X \leq x_{max}$, alors $\rho(X) \leq x_{max}$ ce qui signifie que la mesure d'un risque ne peut dépasser le montant maximal d'un sinistre dû à ce risque ; 2) $\rho(X) \geq E[X]$ ce qui signifie que la mesure d'un risque doit être plus grande que son espérance ; 3) $\rho(c) = c$, pour c scalaire, ce qui signifie que la mesure d'un montant certain est ce montant lui même.

Les mesures de risque peuvent notamment être employées dans le cadre de l'allocation de capital d'une compagnie d'assurance ou de réassurance pour l'ensemble de son portefeuille mais également pour chacune de ses branches de risque.

1.1.3 L'allocation de capital

Les compagnies d'assurance et de réassurance ont la nécessité de déterminer le montant de capital à allouer à l'ensemble de leurs risques mais surtout quelle part de ce montant total doit être allouée à chacun des risques. Il existe plusieurs raisons à ce recours au partage du capital économique global dans les différentes branches d'une compagnie. Cette allocation de capital peut tout d'abord aider une compagnie à déterminer quelles sont ses branches les plus ou les moins profitables. Le capital d'une compagnie étant défini par une mesure de risque qui tient compte des queues de distribution des différents risques, l'allocation de capital permet un partage équitable en fonction du niveau de risque de chaque branche. Cette nécessité d'une allocation de capital cohérente s'explique également comme une réponse aux exigences des organisations de contrôle comme nous l'avons vu dans le cadre de la réforme réglementaire européenne Solvabilité II.

Il existe dans la littérature actuarielle deux principales approches d'allocation de capital. La première peut être vue comme une approche basée sur les sinistres. Elle consiste, à l'aide d'une mesure de risque, à déterminer le capital total d'une compagnie et ensuite de le partager dans ses différentes branches. Cette distribution de capital peut être déduite de la mesure de risque utilisée comme dans Tasche (1999), Panjer (2002), Wang (2002), Dhaene *et al.* (2008), ou elle peut être déterminée indépendamment de la mesure de risque, voir Hesselager et Andersson (2002), Valdez et Chernih (2003) ou Goovaerts *et al.* (2005). La seconde approche consiste à considérer l'allocation de capital tel un problème de détermination de prix d'options de défaut de la compagnie ou de ses branches comme dans Myers et Read Jr (2001), Sherris et Australia (2006) et Kim et Hardy (2009). Une approche générale est avancée dans Dhaene *et al.* (2009) dans lequel l'allocation de capital est vue comme un problème d'optimisation visant à minimiser la somme pondérée des déviations entre les pertes et le capital alloué pour chaque risque. Ainsi, il est possible de déduire des allocations aussi bien basées sur les mesures de risque que des allocations prenant en compte l'option de défaut de l'assureur.

Dans cette thèse, nous utilisons la première approche basée sur les sinistres. Cette approche dite *top-down* consiste donc à déterminer tout d'abord le capital global puis à définir les parts à attribuer à chacun des risques en fonction de leur dangerosité. Nous avons vu auparavant que Solvabilité II préconise une évaluation du SCR par une approche que l'on qualifie de *bottom-up* dans laquelle on détermine dans un premier temps les SCR par risque. Ces montants sont ensuite agrégés pour former le SCR global. Les modèles d'allocation de capital que l'on décrit ci-dessous peuvent donc constituer des solutions de remplacement à la formule standard de Solvabilité II pour le calcul du SCR et pourraient ainsi faire office de propositions pour un modèle interne.

Considérons une période fixe de temps où le portefeuille d'une compagnie est composé de n risques. La v.a. X_i représente les coûts associés au risque i, $i = 1, ..., n$. Différents risques pouvant être liés les uns aux autres, il existe entre les X_i une structure de dépendance que l'on supposera indéfinie dans cette introduction. On note S le montant total des coûts du portefeuille tel que

$$S = \sum_{i=1}^{n} X_i.$$

On détermine le capital économique pour l'ensemble du portefeuille noté $EC_\kappa(S)$ à partir d'une mesure de risque ρ_κ pour un niveau de confiance κ, $0 \leq \kappa < 1$. Ce capital économique est défini comme l'excès du montant évalué par la mesure de risque au delà de la valeur espérée des coûts totaux du portefeuille et s'exprime par

$$EC_\kappa(S) = \rho_\kappa(S) - E[S]. \tag{1.8}$$

Dans la littérature, on parle principalement en termes de mesure de risque pour désigner le montant de capital à allouer. La mesure $\rho_\kappa(S)$ correspond au montant de capital requis pour couvrir l'ensemble du portefeuille. Ce montant total évalué, on en détermine la part à allouer à chacune des branches de risque telle que la condition d'allocation totale suivante soit respectée :

$$\rho_\kappa(S) = \sum_{i=1}^{n} AC(X_i; S), \tag{1.9}$$

où $AC(X_i; S)$ est la part du montant total de capital attribuée au risque i. Différents principes d'allocation ont été proposés, souvent basés sur une mesure de risque et pouvant être différentes de celle utilisée pour l'allocation sur l'ensemble du portefeuille. Un premier principe d'allocation populaire dans la pratique s'appelle le principe d'allocation relative où le capital alloué à la branche i est proportionnel à la mesure de risque $\rho(X_i)$ évaluée sur ce même risque :

$$AC(X_i; S)^{ar} = \rho(S) \frac{\rho(X_i)}{\rho(X_1) + \cdots + \rho(X_n)}. \tag{1.10}$$

On peut également utilser le principe de covariance qui définit la contribution de la branche i au capital global par

$$AC(X_i; S)^{cov} = \rho(S) \frac{cov(X_i, S)}{var(S)}. \tag{1.11}$$

Un principe basé sur la VaR peut être vu dans Albrecht (2004) qui propose d'avoir

$$AC(X_i; S)^{VaR} = E\left[X_i | S = VaR_\kappa(S)\right]. \tag{1.12}$$

L'une des méthodes d'allocation les plus utilisée dans la littérature introduite par Tasche (1999) est basée sur la CTE. La contribution de la branche i en est donnée par

$$AC(X_i; S)^{CTE} = E\left[X_i | S > VaR_\kappa(S)\right]. \tag{1.13}$$

Enfin, le principe d'allocation basée sur la TVaR que l'on peut retrouver dans Schmock et Straumann (1999), Kalkbrener (2005) ou encore Schmock (2006) s'écrit de la manière suivante :

$$AC(X_i; S)^{TVaR} = \frac{E\left[X_i \times \mathbf{1}_{\{S > VaR_\kappa(S)\}}\right] + \beta_S E\left[X_i \times \mathbf{1}_{\{S = VaR_\kappa(S)\}}\right]}{1-\kappa}, \tag{1.14}$$

avec

$$\beta_S = \begin{cases} \frac{Pr(S \leq VaR_\kappa(S)) - \kappa}{Pr(S = VaR_\kappa(S))}, & \text{si } Pr\left(S = VaR_\kappa(S)\right) > 0, \\ 0, & \text{sinon.} \end{cases} \tag{1.15}$$

Lorsque l'on travaille avec des variables aléatoires continues, β_S vaut 0 et on obtient

$$\begin{aligned} AC(X_i; S)^{TVaR} &= \frac{1}{1-\kappa} E\left[X_i \times \mathbf{1}_{\{S > VaR_\kappa(S)\}}\right] \\ &= E[X_i \mid S > VaR_\kappa(S)] \\ &= AC(X_i; S)^{CTE}. \end{aligned}$$

Comme pour la cohérence des mesures de risque, le sujet d'une allocation dite juste a été discuté et est encore discuté aujourd'hui. Différents axiomes ont été avancés mais à ce jour il n'y a pas de consensus autour de la définition d'une allocation de capital juste. C'est pourquoi nous préférons ne pas développer ce sujet. Pour plus d'informations, nous proposons au lecteur de se référer aux articles suivants : Denault (1999), Hesselager et Andersson (2002), Valdez et Chernih (2003), Kim (2007), Kalkbrener (2005) ou encore Kim et Hardy (2009). Au Chapitre 2, nous proposons d'utiliser le principe d'allocation de capital basé sur la TVaR.

1.1.4 La valeur présente du montant total des sinistres

Le deuxième aspect de la théorie du risque que nous abordons dans cette thèse est l'étude de l'évolution du montant total des coûts des sinistres d'un portefeuille d'assurance en fonction du temps. On note par $S(t)$ le processus stochastique représentant ce montant total à l'instant $t \geq 0$. Dans la modélisation classique, le taux d'intérêt net instantané n'est pas pris en compte, et $S(t)$ se définit par

$$S(t) = \begin{cases} \sum_{j=1}^{N(t)} X_j, & N(t) > 0, \\ 0, & N(t) = 0. \end{cases} \tag{1.16}$$

Ici, la v.a. positive ou nulle X_j représente le montant du $j^{\text{ème}}$ sinistre du portefeuille. Le processus $N(t)$ indiquant le nombre de sinistres au temps t est un processus de dénombrement, dont on donne la définition ci-dessous.

Définition 1.3. *On appelle* $\underline{N} = \{N(t), t \geq 0\}$ *un processus de dénombrement à valeurs dans l'espace des entiers naturels s'il respecte les propriétés suivantes :*

i. $N(0) = 0$,

ii. $N(t) \geq 0$,

iii. Si $t > s$, *alors* $N(t) \geq N(s)$,

iv. Si $t > s$, *alors* $N(t) - N(s)$ *correspond au nombre de sinistres sur l'intervalle de temps* $(s, t]$.

Le nombre $N(t)$ de sinistres apparus à l'instant t s'exprime par $N(t) = \sup\{k \geq 0 : T_k \leq t\}$ où la v.a. positive T_j représente le temps d'arrivée du $j^{\text{ème}}$ sinistre. On définit également les v.a. positives W_k, $k = 1, 2, \ldots$, comme les temps écoulés entre chaque sinistre tel que

$$\begin{aligned} W_1 &= T_1, \\ W_k &= T_k - T_{k-1}, \quad \text{pour } k = 2, 3, \ldots \end{aligned}$$

On les appelle également les temps inter-sinistre. Le temps d'arrivée du $j^{\text{ème}}$ sinistre peut ainsi s'écrire comme

$$T_j = W_1 + \cdots + W_j,$$

soit la somme des temps inter-sinistre écoulés jusqu'au $j^{\text{ème}}$ sinistre. Les hypothèses classiques pour le modèle de risque en temps continu sont les suivantes : les v.a. X_j, $j = 1, 2, \ldots$, sont indépendantes et identiquement distribuées (i.i.d.), indépendantes du processus de dénombrement de sinistres $\underline{N}$ et donc également indépendantes des v.a. T_j, $j = 1, 2, \ldots$, et W_k, $k = 1, 2, \ldots$

Nous introduisons ici le concept de valeur présente des coûts totaux associés à un portefeuille d'assurance ou de réassurance à l'aide d'un taux d'intérêt instantané constant. Le taux d'intérêt instantané $\overline{\delta} \geq 0$ peut être vu comme le cas limite d'un taux d'intérêt composé un nombre infini de fois et est associée à la fonction d'actualisation $e^{-\overline{\delta}t}$, pour $t > 0$. Ainsi, on peut définir le capital

$$C = c \times e^{-\overline{\delta}t}$$

à investir aujourd'hui ($t = 0$) afin d'obtenir le capital accumulé c à l'instant futur $t > 0$. On dit également que C est la valeur présente du capital valant c au moment $t > 0$.

Afin de prendre en compte l'effet de l'inflation, nous définissons également le taux d'intérêt net réel instantané qui est la différence entre le taux d'intérêt instantané $\overline{\delta}$ et le taux d'inflation γ telle que

$$\delta = \overline{\delta} - \gamma.$$

On observe généralement dans la réalité financière un taux d'intérêt plus fort que le taux d'inflation. Il arrive malgré tout, dans des situations de crise d'inflation, que le taux d'inflation devienne plus élevé que le taux d'intérêt et ainsi que le taux réel d'intérêt prenne des valeurs négatives.

Considérant la présence de l'intérêt et de l'inflation, nous pouvons définir la valeur présente du montant total des sinistres du portefeuille d'assurance ou de réassurance en intégrant le taux d'intérêt net réel δ à la définition du montant total des sinistres du portefeuille (1.16) tel que

$$Z(t) = \begin{cases} \sum_{j=1}^{N(t)} e^{-\delta T_j} X_j, & N(t) > 0 \\ 0, & N(t) = 0, \end{cases} \tag{1.17}$$

où les v.a. T_j, $j = 1, 2, \ldots$, sont les temps d'arrivée des sinistres.

De nombreux auteurs ont utilisé ce modèle de risque pour l'étude de la probabilité de ruine et d'autres mesures de risque qui lui sont associées telles que Gerber (1971), Taylor (1979), Waters (1983), Delbaen et Haezendonck (1987) ou encore Willmot (1989). Plus récemment, Sundt et Teugels (1995) ont obtenu des bornes inférieure et supérieure pour la probabilité de ruine en temps infini pour le modèle Poisson composé en utilisant un taux d'intérêt constante. La distribution du surplus immédiatement après la ruine pour le même modèle a été approchée par Yang et Zhang (2001). Kalashnikov et Konstantinides (2000) ont donné des résultats asymptotiques pour la même probabilité de ruine en temps fini mais en présence de montants de sinistre à queue lourde. En prenant en compte le taux d'intérêt, Tang (2005), Tang (2007) et Wang (2008) ont obtenu, pour les modèles classiques Poisson composé et de renouvellement, des approximations asymptotiques de la probabilité de ruine en temps fini pour des montants de sinistre à queue lourde. Dans Asimit et Badescu (2009) une structure de dépendance entre les temps inter-sinistre et les montants de sinistre comprenant différentes copules est introduite. Ils obtiennent ainsi des résultats asymptotiques pour des mesures de risque de $Z(t)$ ainsi que pour la probabilité de ruine en temps fini.

Il n'existe qu'assez peu de recherche consacrée à l'étude de la distribution même de $Z(t)$. Il est pourtant possible d'en obtenir, selon le modèle considéré, les moments ou encore la transformée de Laplace. En effet, Léveillé et Garrido (2001a) ont obtenu dans le cadre d'un modèle de renouvellement les deux premiers moments de la valeur présente du montant total des sinistres. Ce travail a été étendu à tous les moments dans Léveillé et Garrido (2001b). Dans le context d'un processus d'arrivée de sinistres Poisson, les deux premiers moments de $Z(t)$ ainsi que la transformée de Laplace de sa distribution ont été étudiés dans Jang (2004) et Jang (2007). Kim et Kim (2007) ont obtenu les deux premiers moments de $Z(t)$ lorsque les sinistres apparaissent selon un processus de Poisson et en supposant, à l'aide d'un environnement markovien, de la dépendance possible entre les montants de sinistre, entre les temps d'arrivée de sinistre et également entre les montants et les temps d'arrivée de sinistre. Les mêmes éléments ont été trouvés dans Ren (2008) lorsque les sinistres suivent un processus d'arrivée de Markov. La fonction génératrice des moments de $Z(t)$ a été étudiée dans Léveillé *et al.* (2009) pour le modèle de renouvellement. Asimit et Badescu (2009) donnent également des résultats asymptotiques sur les probabilités de queue de $Z(t)$ pour des distributions de montants de sinistre à queues lourdes et ce en supposant une structure de dépendance par copule entre les temps inter-sinistre et les montants de sinistre. En supposant une même dépendance entre les temps inter-sinistre et les montants de sinistre définie par une copule Farlie-Gumbel-Morgenstern, nous montrons au Chapitre 3 comment obtenir n'importe quel moment de la valeur présente du montant total des sinistres $Z(t)$.

1.1.5 La théorie de la ruine

Solvabilité II propose d'évaluer le SCR et donc d'examiner l'exposition aux risques des assureurs et réassureurs sur un horizon de temps de 1 an. Or une partie du passif des compagnies d'assurance et de réassurance est constitué de risques de long terme. La théorie de la ruine suggère d'étudier l'évolution du surplus de capital d'une compagnie dans le temps, ce qui permet d'avoir une appréhension des risques sur un horizon de temps plus large. Nous introduisons ici la probabilité de ruine qui est étudiée au Chapitre 4.

La probabilité de ruine permet d'évaluer le risque pour une compagnie d'être en état de ruine à au moins un instant sur une certaine période de temps future. Le terme ruine ne désigne pas seulement la ruine réelle d'insolvabilité d'une compagnie d'assurance ou de réassurance. Il est défini ici comme le passage du surplus de la compagnie en dessous d'un certain seuil permettant d'alerter la compagnie sur son état financier déficient. De manière générale, ce seuil est fixé à 0 et la ruine intervient donc dès lors que le surplus de la compagnie est négatif. Le processus de surplus $\underline{R} = \{R_t, t \geq 0\}$ d'une compagnie d'assurance ou de réassurance se définit au temps t par

$$R(t) = u + ct - S(t), \tag{1.18}$$

où $u \geq 0$ représente le capital initial que détient la compagnie au temps $t = 0$ et $c > 0$ est le taux de prime qui correspond au volume des cotisations par unité de temps. Le processus $S(t)$ est le processus du montant total des sinistres de la compagnie comme défini à l'Equation (1.16) et dont on rappelle l'expression pour $t \geq 0$:

$$S(t) = \begin{cases} \sum_{j=1}^{N(t)} X_j, & N(t) > 0, \\ 0, & N(t) = 0, \end{cases} \tag{1.19}$$

où $N(t)$ est le processus de dénombrement des sinistres indépendant des montants de sinistre X_j, $j = 1, 2, \ldots$, qui sont positifs ou nuls et i.i.d.. La Figure 1.4 donne une illustration d'une trajectoire potentielle pour le surplus $R(t)$ en fonction du temps.

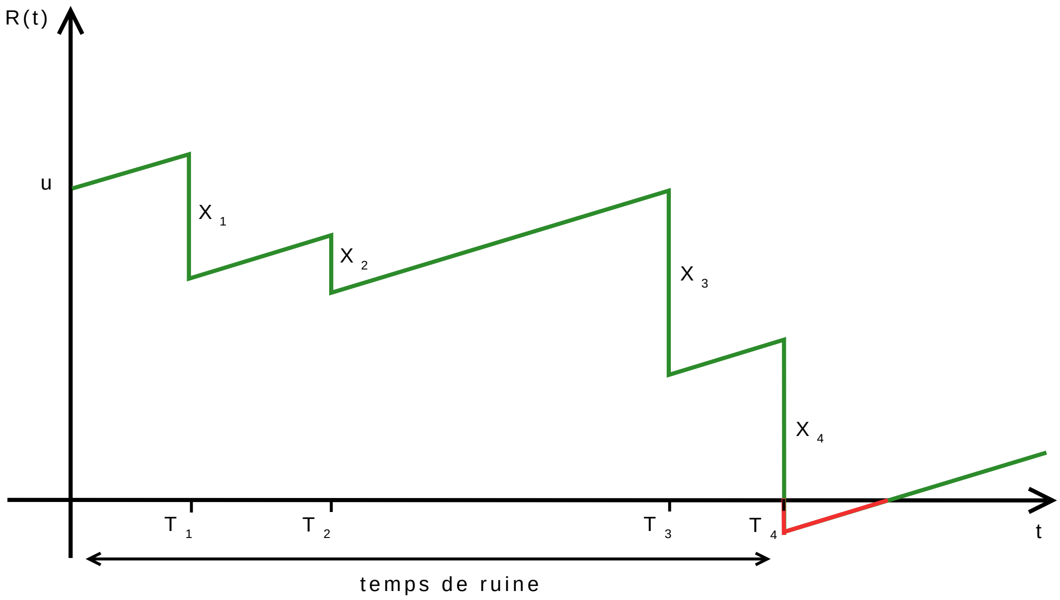

FIGURE 1.4 – Trajectoire du processus de surplus en fonction du temps

Le processus de dénombrement $N(t)$ peut être de différentes natures. Nous donnons ci-dessous la définition du processus de Poisson homogène, puis celle d'un processus de renouvellement.

Définition 1.4. *Le processus de dénombrement $\underline{N} = \{N(t), t \geq 0\}$ est un processus de Poisson homogène d'intensité $\lambda > 0$ si, pour $0 \leq t \leq \infty$, il vérifie les conditions suivantes :*

i. $N(0) = 0$,

ii. les accroissements de $\{N(t), t \geq 0\}$ sont indépendants et stationnaires,

iii. $N(t) \sim Pois(\lambda t)$.

Lorsque $N(t)$ est un processus de Poisson homogène de paramètre $\lambda > 0$, alors les temps inter-sinistre W_k, $k = 1, 2, \ldots$, sont des v.a. i.i.d suivant une loi exponentielle de paramètre λ. De plus, il en découle que les temps d'arrivée de sinistre sont distribués selon une loi Erlang tels que $T_j \sim Erlang(j, \lambda)$ pour $j = 1, 2, \ldots$

Définition 1.5. *Soit* $\{W_k, k = 1, 2, \ldots\}$ *une suite de v.a. i.i.d. à valeurs dans* $\mathbb{R}_+^*$. *On définit également les v.a.*

$$T_0 = 0 \text{ et } T_j = W_1 + \cdots + W_j, \text{ pour j=1,2,...}$$

Alors, on appelle $\underline{N} = \{N(t), t \geq 0\}$ *un processus de renouvellement le processus de dénombrement défini, pour* $t \geq 0$, *par*

$$N(t) = \sum_{i=1}^{+\infty} \mathbf{1}_{\{T_i \leq t\}}.$$

Ainsi, le processus de Poisson est un cas particulier du processus plus général de renouvellement. Lorsque $N(t)$ est un processus de Poisson, alors le modèle considéré pour le processus de surplus $R(t)$ est appelé modèle de Cramér-Lundberg. Lorsque l'on utilise un processus de renouvellement pour $N(t)$, alors le modèle est dit de Sparre-Andersen.

On définit le temps de ruine τ, v.a. correspondant au premier instant $t \geq 0$ auquel le surplus devient négatif, par

$$\tau = \inf\{t \geq 0 : R(t) < 0\} \tag{1.20}$$

avec comme convention que $\tau = \infty$ si $R(t) > 0$ pour tout $t > 0$. On peut ensuite également définir la probabilité de ruine en horizon de temps infini (ou en temps infini) avec comme capital initial $u \geq 0$ comme

$$\psi(u) = Pr\left(\exists t \geq 0, R(t) < 0 | R(0) = u\right) \tag{1.21}$$

$$= Pr\left(\inf_{t \geq 0}\{R(t)\} < 0 | R(0) = u\right) \tag{1.22}$$

$$= Pr\left(\tau < \infty | R(0) = u\right). \tag{1.23}$$

En restreignant l'observation de la probabilité de ruine à un horizon de temps fini et borné à l'instant $T \geq 0$, on définit la probabilité de ruine en temps fini par

$$\psi(u, T) = Pr\left(\exists t \in [0, T], R(t) < 0 | R(0) = u\right) \tag{1.24}$$

$$= Pr\left(\inf_{0 \leq t \leq T}\{R(t)\} < 0 | R(0) = u\right) \tag{1.25}$$

$$= Pr\left(\tau \leq T | R(0) = u\right). \tag{1.26}$$

La relation entre ces deux probabilités de ruine peut s'écrire par

$$\psi(u) = \lim_{T \to \infty} \psi(u, T). \tag{1.27}$$

Pour s'assurer que la compagnie ne se dirige pas vers une ruine certaine, le taux de prime doit comporter un chargement relatif de sécurité positif afin que $ct > E[S(t)]$ et donc que $c > E[S(1)]$. On a donc

$$c = (1 + \theta)E[S(1)], \tag{1.28}$$

où $\theta > 0$ désigne le chargement relatif de sécurité.

La probabilité de ruine, qui est un sujet grandement abordé dans la littérature, a initialement été étudiée dans le cadre du modèle de Cramér-Lundberg notamment par Dufresne et Gerber (1988). De nombreuses extensions à ce travail ont été proposées pour calculer de manière exacte ou approchée la probabilité de ruine en temps infini ou en temps fini. Nous donnons ici quelques exemples non exhaustifs d'extension du modèle classique de risque. On peut ainsi rencontrer des modèles utilisant des taux de prime variables (Taylor (1980), Michaud (1996), Jasiulewicz (2001)), intégrant des composantes d'investissement telles que le taux d'intérêt (Tang (2005), Wang (2008), Sundt et Teugels (1995)), le paiement de taxes (Albrecher et Hipp (2007) Albrecher *et al.* (2008)), des investissements sur le marché financier (Hipp et Plum (2000), Gaier *et al.* (2003)), le paiement de dividendes aux actionnaires (Gerber (1981), Albrecher et Kainhofer (2002), Gerber *et al.* (2008)) ou encore le recours à la réassurance (Dickson et Waters (1996), Schmidli (2002)). Il existe également des modèles permettant de prendre en compte les liens qui peuvent intervenir entre les différents éléments en jeu dans le processus de risque que sont les montants de sinistres, les temps inter-sinistre et les taux de prime. On peut observer par exemple l'utilisation de chocs communs (Frostig (2003)), de dépassements de seuils (Albrecher et Boxma (2004), Boudreault *et al.* (2006), Biard *et al.* (2009)), d'un environnement markovien (Reinhard (1984), Asmussen (1989), Lu et Li (2005)), ou encore de copules (Albrecher et Teugels (2006), Cossette *et al.* (2008), Biard *et al.* (2009)). De manière plus générale, de nombreux ouvrages peuvent servir de référence dans le domaine de la théorie de la ruine, notamment Gerber (1979), Grandell (1991), Rolski *et al.* (1999), Asmussen (2000), Kaas *et al.* (2001) ou encore Dickson (2005). Au Chapitre 4, nous étudions la probabilité de ruine en temps fini et continu d'une compagnie de réassurance.

1.2 L'introduction de dépendance dans les modèles de risque

Initialement, la théorie du risque a été développée autour de l'hypothèse d'indépendance entre les différentes variables aléatoires intervenant dans la modélisation.

L'évolution continuelle des risques et des produits d'assurance mène à des réflexions visant à relâcher cette hypothèse. Bien que dans la théorie l'hypothèse d'indépendance permette une plus grande facilité de modélisation, l'indépendance ne représente, en pratique, qu'un cas d'exception. Si l'indépendance ne s'exprime que d'une seule façon, il existe de nombreuses possibilités de définir la dépendance.

Dans la modélisation des risques d'assurance et de réassurance, on peut relever différentes natures de dépendance. Il existe notamment des dépendances intervenant entre les contrats d'un même portefeuille de risque, se traduisant par une corrélation entre les montants de sinistres. Par exemple, si l'on considère un portefeuille de risques d'incendie de biens immobiliers pour des immeubles d'une même rue, il existe un risque de propagation d'un incendie frappant l'un des bâtiments vers les autres bâtiments. On peut également mettre en avant un lien de dépendance, non plus entre les sinistres pour un même type de risque, mais entre les branches de risques comme on peut le constater notamment entre les risques de l'assurance vie et l'assurance santé. Les montants de sinistre et les temps inter-sinistre peuvent également présenter un lien de dépendance. Les risques, par exemple, de type tremblement de terre justifient effectivement cette hypothèse. En effet, le temps écoulé depuis le dernier grand séisme peut influer sur l'arrivée d'un autre fort séisme entraînant des sinistres à coûts élevés. Enfin, nous notons la possibilité de prendre en compte une influence d'un environnement extérieur décrivant par exemple les conditions économiques ou climatiques sur les différentes variables du modèle de risque.

Parmi les nombreux outils mathématiques possibles pour prendre en compte de telles dépendances, nous proposons dans cette thèse l'utilisation des copules pour les dépendances directes entre variables aléatoires et le recours à un environnement markovien pour une dépendance liée au context environnemental.

1.2.1 La théorie des copules

Les copules permettent l'introduction et la caractérisation d'une forme très flexible de dépendance entre différentes variables aléatoires. Elles servent à établir une fonction de répartition multivariée à partir des fonctions de répartition marginales. Il existe un grand nombre de familles de copules qui conduisent à des structures de dépendance positive ou négative très variées. Certaines permettent, par exemple, de la dépendance dans les queues de distributions ce qui est particulièrement intéressant dans les domaines de l'assurance et de la finance. Des ouvrages de référence sur la théorie des copules sont Joe (1997) et Nelsen (2006). Des applications en actuariat et gestion des risques sont proposées dans les livres de Denuit *et al.* (2005), McNeil *et al.* (2005) ou Marceau (2009).

Nous introduisons ici la théorie des copules dans un cadre bivarié. Cette théorie s'étend évidemment à des modèles multivariés comme nous le voyons au chapitre 2.

Une copule bivariée se définit tout d'abord de la manière suivante.

Définition 1.6. *Une copule bivariée* $C(u_1, u_2)$ *est une application non décroissante et continue à droite de* $[0,1]^2$ *dans* $[0,1]$ *satisfaisant les propriétés suivantes :*

i. $\lim_{u_i \to 0} C(u_1, u_2) = 0$ *pour* $i = 1, 2$,

ii. $\lim_{u_i \to 1} C(u_1, u_2) = u_{3-i}$ *pour* $i = 1, 2$,

iii. pour tout $u_1 \leq v_1$ *et* $u_2 \leq v_2$*, C vérifie*

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0. \tag{1.29}$$

Le théorème de Sklar (1959) définit le rôle de la copule entre une fonction de répartition bivariée et ses marginales.

Théorème 1.1 (Théorème de Sklar)**.** *Soit* F *une fonction de répartition conjointe avec* F_1 *et* F_2 *comme marginales. Alors il existe une copule* $C : [0,1]^2 \to [0,1]$ *telle que, pour tous* x_1 *et* x_2 *dans* $\overline{\mathbb{R}} = [-\infty, +\infty]$,

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)). \tag{1.30}$$

Si F_1 *et* F_2 *sont continues alors* C *est unique. Sinon* C *est déterminée sur* $RanF_1 \times RanF_2$*. Inversement, si* C *est une copule et* F_1 *et* F_2 *sont des fonctions de répartition univariées, alors la fonction* F *définie en* (1.30) est une fonction de répartition bivariée avec F_1 et F_2 comme fonctions de répartition marginales.

Si la copule est différentiable et les marginales sont continues, on peut déduire de (1.30) la fonction de densité conjointe de (X_1, X_2) telle que

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) c(F_{X_1}(x_1), F_{X_2}(x_2)), \quad x_1, x_2 \in \mathbb{R}, \tag{1.31}$$

où c est la densité de la copule définie, pour u_1 et u_2 dans $[0,1]$, par

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2). \tag{1.32}$$

Le recours aux copules pour la modélisation de la dépendance en théorie du risque commence essentiellement à partir de la fin des années 1990. L'article de Frees et Valdez (1998), visant à introduire le concept des copules en actuariat, en présente quelques applications en analyse de survie. Wang (1998) a ensuite utilisé différentes copules afin d'étudier la distribution de la somme de risques corrélés. Une relation de dépendance entre les temps inter-sinistre et les montants de sinistres à l'aide d'une copule arbitraire a été introduite dans Albrecher et Teugels (2006) pour l'étude de la probabilité de ruine en temps fini et infini dans le cadre d'un modèle de renouvellement. En théorie de la ruine également, Cossette *et al.* (2009) et Cossette *et al.* (2008) ont obtenu l'expression de la transformée de Laplace de la fonction de pénalité de Gerber-Shiu avec une dépendance entre les temps inter-sinistre et les montants de sinistres structurée par une copule de Farlie-Gumbel-Morgenstern et une copule de Farlie-Gumbel-Morgenstern généralisée. On peut également noter l'utilisation de différentes copules dans un travail récent de Biard *et al.* (2009). Ceux-ci obtiennent une approximation asymptotique de la probabilité de ruine en temps fini prenant en compte de possibles crises de corrélation. Lors de ces crises, les montants de sinistre peuvent être dépendants à travers l'utilisation de certaines copules.

Dans les deux premiers chapitres de cette thèse, nous utilisons les copules afin de modéliser deux types de relation de dépendance. La première relation de dépendance est celle intervenant entre les différents types de risque comme nous le voyons au Chapitre 2 dans la cadre d'un problème d'allocation de capital. Au Chapitre 3, nous les utilisons pour décrire la dépendance entre les montants de sinistre et les temps inter-sinistre pour le modèle de la valeur présente du montant total des sinistres.

1.2.2 L'environnement markovien

Il peut non seulement exister de la dépendance directe entre les différentes variables aléatoires intervenant dans la modélisation actuarielle pouvant être prise en compte par l'utilisation des copules mais les paramètres d'un modèle d'assurance peuvent également être influencés par un environnement extérieur. Cet environnement peut représenter, par exemple, des conditions économiques, financières ou encore climatiques pouvant modifier simultanément le comportement de plusieurs variables impliquées dans le modèle de risque. Par exemple, les états du climat (sec, brumeux, pluvieux, neigeux,...) peuvent influencer le nombre de sinistres automobiles ainsi que leurs montants. Pour prendre en compte cette dépendance à un environnement extérieur, un processus de Markov, défini comme suit, est employé :

Définition 1.7. *Un processus aléatoire* $\{J(t), t \geq 0\}$ *est dit processus de Markov si pour tout ensemble de* $n+1$ *instants* $t_1 < \cdots < t_{n+1}$*, et pour tout ensemble d'états*

$\{x_1, \cdots, x_{n+1}\}$, *on a*

$$Pr(J(t_{n+1}) = x_{n+1}|J(t_1) = x_1, \ldots, J(t_n) = x_n) = Pr(J(t_{n+1}) = x_{n+1}|J(t_n) = x_n). \tag{1.33}$$

Cette définition est aussi désignée comme la propriété de Markov. Elle signifie que la probabilité de l'état futur d'un processus ne dépend du passé qu'à travers l'état présent de ce même processus. Cela se traduit également comme une absence de mémoire du processus.

Lorsque qu'un processus de Markov possède un nombre d'états discret, c'est-à-dire fini ou dénombrable, on l'appelle chaîne de Markov. Un chaîne de Markov à temps discret (*Discrete-Time Markov Chain*) est une chaîne de Markov dont le support de temps est discret. On appelle chaîne de Markov à temps continu (*Continuous-Time Markov Chain*) une chaîne de Markov ayant des temps d'attente exponentiels entre les changements d'état du processus. Ce caractère exponentiel garantit l'absence de mémoire du processus dans le sens où le futur dépend de l'état présent seulement. Lorsque le temps d'attente des changements d'état possède une distribution continue quelconque dépendante de l'état présent, alors le processus est dit semi-markovien et le futur dépend de l'état présent mais également du temps passé dans cet état. On parlera ici d'environnement markovien, qu'il s'agisse de processus de Markov ou semi-markovien.

Un processus de Markov ou semi-markovien de nombre d'états fini peut être utilisé pour décrire les conditions d'environnement d'un autre processus stochastique en fonction de l'état dans lequel il se trouve. Ainsi, différents paramètres du second processus stochastique étudié peuvent dépendre de l'état du processus d'environnement. Le processus bivarié associé au processus stochastique et au processus d'environnement markovien forme ce que l'on appelle un processus de Markov additif (*Markovian Additive Process*) tel que défini dans Asmussen (2000) :

Définition 1.8. *Un processus de Markov additif est un processus bivarié* $\{J(t), S(t); t \geq 0\}$ *où* $\{J(t)\}$ *est un processus de Markov (ou semi-markovien) et les accroissements de* $\{S(t)\}$ *sont gouvernés par* $\{J(t)\}$ *de telle sorte que pour toutes fonctions* f *et* g *et pour tous* $t \geq 0$ *et* $s > 0$*, on a*

$$E[f(S_{t+s} - S_t)g(J_{t+s})|\mathcal{F}_t] = E[f(S_s)g(J_s)|J_t, S_t = 0], \tag{1.34}$$

où $\{\mathcal{F}_t\}$ *est la filtration naturelle associée à* $\{J(t), S(t); t \geq 0\}$.

Dans le cadre de l'étude du processus de risque, un environnement markovien peut influer sur les paramètres des distributions des montants de sinistre, des temps inter-sinistre mais peut aussi moduler les taux de prime, d'intérêt, d'inflation, etc. Cette influence d'un environnement markovien a été considérée dans bon nombre d'articles. Notamment, Janssen et Reinhard (1985) ont étudié un modèle de risque où les montants et temps inter-sinistre sont dépendants d'un environnement markovien. Ils ont ainsi obtenu une expression formelle des probabilités de survie en termes de séries infinies de convolutions de matrices. Ce travail a été généralisé dans Albrecher *et al.* (2005) par l'analyse de la fonction de pénalité de Gerber-Shiu pour un modèle de risque semi-markovien en temps continu comprenant les modèles Poisson composé et de Sparre-Andersen avec des temps inter-sinistres phase-types. Différents résultats pour la probabilité de ruine lorsque les temps de sinistre sont décrits par un processus de Cox dont le processus d'intensité est markovien sont obtenus dans Asmussen (1989). Kim et Kim (2007) obtiennent l'expression des deux premiers moments de la valeur présente du montant total des sinistres pour des distributions de la fréquence et des montants de sinistre fluctuant selon une chaîne de Markov en temps continu. La distribution du surplus avant la ruine et la sévérité de la ruine dans un modèle de ruine discret markovien où les montants de sinistres dépendent d'une chaîne de Markov ont été étudiées par Reinhard et Snoussi (2001, 2002). Jasiulewicz (2001) obtient la probabilité de ruine en temps infini et continu avec un taux de prime variant en fonction de la réserve et où seuls les temps d'occurrence de sinistre sont influencés par un processus de Markov. La probabilité de ruine en temps fini et en temps infini dans un modèle de risque discret où les taux d'intérêt suivent une chaîne de Markov a été examinée dans Cai et Dickson (2004). Li et Dickson (2006) s'intéressent quant à eux aux paiements de dividende avant la ruine dans un modèle de risque en temps continu où les temps inter-sinistre, les montants de sinistre et les primes sont modulés par un processus de Markov. On notera aussi le travail de Lu et Tsai (2007) qui ont obtenu une expression de la fonction de pénalité de Gerber-Shiu pour un processus de risque en temps continu perturbé par une diffusion dans lequel les temps-inter-sinistre, les montants de sinistre mais également les variances du processus de Wiener sont influencés par un environnement markovien. Nous citerons enfin Ren (2008) ,qui parvient à une forme explicite de la transformée de Laplace pour la distribution de la valeur présente du montant total des sinistres lorsque les temps inter-sinistres sont influencés par un processus de Markov.

Plusieurs auteurs ont également étudié le processus de risque en le transposant en un modèle fluide markovien ou les états du processus de fluide dépendent d'un environnement markovien. Il est également possible de permettre à ce même processus de Markov d'influencer les temps inter-sinistre, les montants de sinistre ou les taux de prime. Ce sujet, que nous développons au Chapitre 2, a notamment été traité par Asmussen *et al.* (2002), Stanford *et al.* (2005a), Ramaswami (2006), Ramaswami *et al.* (2008), Badescu *et al.* (2007) et Badescu *et al.* (2005), entre autres.

Chapitre 2

Allocation de capital basée sur la mesure TVaR avec copules

Résumé

En raison des projets de régulation des organisations de contrôle tels que la réforme européenne solvabilité II et des événements économiques récents, les compagnies d'assurance ont le besoin de consolider leurs réserves en capital par une allocation cohérente aussi bien pour l'ensemble de la compagnie que pour chacune de ses branches d'activité. Nous considérons ainsi un portefeuille d'assurance composé de plusieurs types de risque liés par une copule. La mesure de risque employée est la *Tail-Value-at-Risk* (TVaR). Une expression exacte de la TVaR pour la somme des risques et des contributions à cette mesure de chacune des branches de risque est obtenue en utilisant la copule Farlie-Gumbel-Morgenstern et des montants de sinistre exponentiels puis distribués selon un mélange de distributions exponentielles. Nous examinons un modèle bivarié puis le cas multivarié. Nous montrons également comment approcher la TVaR sur l'ensemble du portefeuille et les différentes contributions en utilisant une copule quelconque.

2.1 Introduction

In recent years, a lot of research has focused on insurance capital allocation. Indeed the European Solvency II project and the recent events encourage insurance companies to consolidate their financial reserves and investments. Risk measures are well-known tools to determine the capital amount that has to be allocated to a risk portfolio. Artzner et al. (1999) proposed an axiomal definition of a coherent risk measure that can be used for allocation issues. This coherence property has also been discussed in Wang (2002). Using their definition, Artzner et al. (1999) proposed the tail value at risk (TVaR), also called expected shortfall (ES), as a coherent alternative to the non-coherent risk measure value at risk (VaR). Applied to continuous random variables, the TVaR can identically be defined as the conditional tail expectation (CTE). But these two risk measures differ in discrete contexts where the CTE is no longer coherent. The differences between these definitions and properties have been highlighted in Acerbi et al. (2001) and Acerbi and Tasche (2002).

In the literature on capital allocation, continuous situations are widely studied in contrast with discrete cases. That is why most of the references speak in terms of CTE. The capital allocation principle has first been introduced by Tasche (1999) where the capital allocated to each risk is expressed in terms of the CTE of the aggregate risk. This top down allocation method has then been used to provide several closed formulae and approximations of the CTE and the CTE-based allocations for different types of multivariate continuous distributions. The first multivariate top down model was considered by Panjer (2002) where the risks have a multivariate normal distribution. This work has been extended to a multivariate elliptical distribution in Landsman and Valdez (2003) and in Dhaene et al. (2008). A multivariate gamma distribution for risks has been studied in Furman and Landsman (2005) as well as a multivariate Tweedie distribution in Furman and Landsman (2007). In these papers, explicit expressions for the CTE and the CTE-based allocation are derived. Other closed form expressions for the CTE of the sum of multivariate phase-type distributed risks and the contribution of one risk to the portfolio have been given in Cai and Li (2005). More recently, Chiragiev and Landsman (2007) found a CTE and CTE-based allocation for multivariate Pareto risks. Further information on the CTE-based allocation of risk capital can be found in Kim (2007).

In most papers mentioned above, the dependence between the different lines of business of the insurance company is due to the construction of a multivariate distribution. In the present paper, we propose introducing dependence with a copula. Copulas are currently seen as effective and flexible tools to represent dependence between random variables. Furthermore, in order to have the risk measure coherence property in every continuous and discrete situation, we propose using the TVaR as defined in Acerbi et al. (2001) and Acerbi and Tasche (2002) to develop a top down approach of the capital allocation. Indeed, the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) advises in the Solvency II context the use of the TVaR for the evaluation of the Solvency Capital Requirement (SCR) ; see CEIOPS (2006) and CEIOPS (2007).

First, closed form expressions for the TVaR and then the TVaR-based contribution of one risk over the aggregation of all risks are obtained when the Farlie-Gumbel-Morgenstern copula describes the dependence between the risk marginals. With most copulas introducing dependence between different risks however, we are not able to reach closed form expressions. Consequently, we also present approximation methods to evaluate the TVaR and the TVaR-based allocation by the use of different discretization methods of continuous random variables which are applicable with any copula and any marginals.

In the first section, we give the general definitions for the tail value at risk of the aggregate risk and the contribution of one of the risks. The second section deals with the application of the TVaR-based allocation rule using the FGM copula and exponential distributed risks. We first consider two lines of business and then pursue to a multivariate context. We widen our results to risks that are distributed as mixture of exponentials in section 3. For these two last sections, we are able to have closed form expressions for both the TVaR and the individual risk contribution based on it. Then, we expose approximation methods for the TVaR and TVaR-based allocation when the dependence structure is defined by any copula. The results are illustrated with numerical applications.

2.2 Definition of the TVaR and the TVaR-based allocation

In this section, we define the tail value at risk (TVaR) for the aggregate risk and the TVaR-based allocation rule. We consider the aggregate claim amount (or loss) S of a portfolio of n risks. The claim amount (or loss) for risk i is denoted by X_i. Thus we have $S = X_1 + \cdots + X_n$ where all X_i's are non-negative random variables.

The value at risk at level κ, $0 < \kappa < 1$, of S is defined by

$$VaR_\kappa(S) = inf(x \in \mathbb{R}, F_S(x) \geq \kappa).$$

It is well known that the VaR is a risk measure that is not coherent. Thus we choose to work with the tail value at risk of S as introduced in Acerbi and Tasche (2002), Schmock and Straumann (1999) and Schmock (2006) at level κ, for $\kappa \in (0, 1)$. Its definition is

$$\begin{aligned} TVaR_\kappa(S) &= \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(S)du \\ &= \frac{E\left[S\mathbf{1}_{\{S>VaR_\kappa(S)\}}\right] + VaR_\kappa(S)\Big(Pr\left(S \leq VaR_\kappa(S)\right) - \kappa\Big)}{1-\kappa}, \end{aligned}$$

which is a coherent risk measure. When S is continuous, $Pr(S \leq VaR_\kappa(S)) = \kappa$ which implies that the TVaR is exactly the conditional tail expectation (CTE) meaning $TVaR_\kappa(S) = E[S|S > VaR_\kappa(S)] = CTE(S)$. In financial risk management, the TVaR is called the expected shortfall.

The additivity of the expectation allows the decomposition of the TVaR (CTE) into the sum of TVaR contributions as follows

$$TVaR_\kappa(S) = \sum_{i=1}^{n} TVaR_\kappa(X_i; S),$$

where the TVaR contribution of the ith risk to the total risk represents the part of the capital that is allocated to risk i. For $\kappa \in (0, 1)$, it can be expressed as

$$TVaR_\kappa(X_i; S) = \frac{E\left[X_i \times \mathbf{1}_{\{S>VaR_\kappa(S)\}}\right] + \beta_S E\left[X_i \times \mathbf{1}_{\{S=VaR_\kappa(S)\}}\right]}{1-\kappa},$$

with

$$\beta_S = \begin{cases} \frac{Pr(S \leq VaR_\kappa(S)) - \kappa}{Pr(S=VaR_\kappa(S))}, & \text{if } Pr\left(S = VaR_\kappa(S)\right) > 0, \\ 0, \text{otherwise}. \end{cases}$$

For continuous distributions, we have

$$\begin{aligned} TVaR_\kappa(X_i; S) &= \frac{1}{1-\kappa} E\left[X_i \times \mathbf{1}_{\{S>VaR_\kappa(S)\}}\right] \\ &= E[X_i \mid S > VaR_\kappa(S)] \\ &= CTE_\kappa(X_i|S). \end{aligned}$$

That means that the TVaR-based contribution of one risk is equal to the CTE-based contribution of the same risk when all the marginals are continuous.

Note that

$$\begin{aligned} E[X_i \mid S = s] &= E[X_i \mid X_1 + \cdots + X_n = s] \\ &= \int_0^s x f_{X_i|S}(x \mid S = s) dx \\ &= \int_0^s x \frac{f_{X_i,S}(x, s)}{f_S(s)} dx. \end{aligned}$$

Then, the CTE-based contribution can be expressed as

$$\begin{aligned} CTE_\kappa(X_i|S) &= \int_{VaR_\kappa(S)}^{\infty} E[X_i \mid S = s] f_{S|S>VaR_\kappa(S)}(s) ds \\ &= \frac{1}{Pr(S > VaR_\kappa(S))} \int_{VaR_\kappa(S)}^{\infty} E[X_i \mid S = s] f_S(s) ds \\ &= \frac{1}{1-\kappa} \int_{VaR_\kappa(S)}^{\infty} \int_0^s x f_{X_i,S}(x, s) dx ds. \end{aligned} \tag{2.1}$$

2.3 TVaR and the TVaR-based allocation with exponential marginals and the FGM copula

In this section, we derive the expression for the TVaR and the TVaR-based allocation for two exponentially distributed risks joined by a FGM copula. We also extend the results obtained with this bivariate model to a multivariate model. The exponential distribution is a classical distribution for the risk random variables. Its convenient and practical mathematical properties permit to develop explicit results. We are aware that the FGM copula introduces only light dependence. However, it admits positive as well as negative dependence between a set of random variables. As said in Yeo and Valdez (2006) where the FGM copula is used to link claim variables in a credibility model, even if it can model only weak dependence, the FGM copula permits to assign a unique dependence parameter for each pair or group of risks and allows a more complex dependence structure than most of the copulas which use only one or few parameters. Furthermore, its handy form allows explicit calculus and thus exact results. This copula was also used to describe different correlation relations on the financial markets in Gatfaoui (2005) and Gatfaoui (2007).

2.3.1 The bivariate case

Let X_1 and X_2 be two exponentially distributed random variables representing the claim amounts of two insurance risks. Their cumulative distribution functions (cdf) and probability density functions (pdf) are given by

$$F_{X_i}(x_i) = 1 - e^{-\lambda_i x_i},$$
$$f_{X_i}(x_i) = \lambda_i e^{-\lambda_i x_i}, \text{ for } i = 1, 2.$$

In order to simplify our presentation, we restrain our study to the constraints $\lambda_1 \neq \lambda_2, \lambda_1 \neq 2\lambda_2, \lambda_2 \neq 2\lambda_1$. It is possible to find adjusted results without these constraints by applying a similar method as the one exposed below.

A dependence structure for (X_1, X_2) based on the bivariate FGM copula is introduced. The FGM copula is defined by

$$C_\theta^{FGM}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2)$$

for $u_i \in [0, 1]$, $i = 1, 2$, and dependence parameter $\theta \in [-1, 1]$.

The density of the bivariate FGM copula is

$$\begin{aligned} c_\theta^{FGM}(u_1, u_2) &= \frac{\partial^2 C_\theta^{FGM}(u_1, u_2)}{\partial u_1 \partial u_2} \\ &= 1 + \theta(1 - 2u_1)(1 - 2u_2) \\ &= 1 + \theta(2\overline{u}_1 - 1)(2\overline{u}_2 - 1), \end{aligned}$$

where $\overline{u}_i = 1 - u_i$, $i = 1, 2$.

The TVaR of the aggregate risk $S = X_1 + X_2$ is given in the following proposition.

Proposition 2.1. *Let X_1 and X_2 be two exponentially distributed random variables with joint cdf defined by a bivariate FGM copula as follows*

$$F_{X_1,X_2}(x_1, x_2) = C_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2)),$$

with $\theta \in [-1, 1]$. Then, the TVaR of the aggregate risk $S = X_1 + X_2$ at level κ, $0 < \kappa < 1$, is

$$\begin{aligned} TVaR_\kappa(S) = \frac{1}{1-\kappa} \Big[&(1+\theta)\zeta(VaR_\kappa(S); \lambda_1; \lambda_2) - \theta\zeta(VaR_\kappa(S); 2\lambda_1; \lambda_2) \\ &- \theta\zeta(VaR_\kappa(S); \lambda_1; 2\lambda_2) + \theta\zeta(VaR_\kappa(S); 2\lambda_1; 2\lambda_2) \Big], \end{aligned} \tag{2.2}$$

where $\zeta(x; \gamma_1, \gamma_2) = \frac{\gamma_2}{\gamma_2 - \gamma_1} e^{-\gamma_1 x} \left(x + \frac{1}{\gamma_1}\right) + \frac{\gamma_1}{\gamma_1 - \gamma_2} e^{-\gamma_2 x} \left(x + \frac{1}{\gamma_2}\right)$.

Proof. The joint pdf of (X_1, X_2) is given by

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= c_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2))f_{X_1}(x_1)f_{X_2}(x_2) \\ &= f_{X_1}(x_1)f_{X_2}(x_2) + \theta f_{X_1}(x_1)f_{X_2}(x_2)(1-2F_{X_1}(x_1))(1-2F_{X_2}(x_2)) \\ &= (1+\theta)\lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2} - \theta 2\lambda_1 e^{-2\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2} \\ &\quad - \theta\lambda_1 e^{-\lambda_1 x_1} 2\lambda_2 e^{-2\lambda_2 x_2} + \theta 2\lambda_1 e^{-2\lambda_1 x_1} 2\lambda_2 e^{-2\lambda_2 x_2}. \end{aligned}$$

Let $h(x, \lambda_1, \lambda_2)$ be the distribution function of a generalized Erlang random variable X

$$h(x,\lambda_1,\lambda_2) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 x} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 x}.$$

Then, the pdf of S can be expressed as a combination of generalized Erlang pdf's

$$\begin{aligned} f_S(s) &= \int_0^s f_{X_1,S}(x,s)dx \\ &= \int_0^s f_{X_1,X_2}(x, s-x)dx \\ &= (1+\theta)h(s;\lambda_1;\lambda_2) - \theta h(s;2\lambda_1;\lambda_2) - \theta h(s;\lambda_1;2\lambda_2) + \theta h(s;2\lambda_1;2\lambda_2), \end{aligned} \tag{2.3}$$

where the first component of $f_S(s)$ is

$$\begin{aligned} \int_0^s (1+\theta)\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2(s-x)}dx &= (1+\theta)\lambda_1\lambda_2 e^{-\lambda_2 s}\int_0^s e^{x(\lambda_2-\lambda_1)}dx \\ &= (1+\theta)\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 s} + (1+\theta)\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 s} \\ &= (1+\theta)h(s;\lambda_1;\lambda_2). \end{aligned}$$

The three other components can be found similarly.

Thus, the TVaR of S takes the form

$$\begin{aligned} TVaR_\kappa(S) &= E[S \mid S > VaR_\kappa(S)] \\ &= \int_{VaR_\kappa}^\infty s\frac{f_S(s)}{Pr(S > VaR_\kappa(S))}ds \\ &= \frac{1}{1-F_S(VaR_\kappa(S))}\int_{VaR_\kappa(S)}^\infty s f_S(s)ds \\ &= \frac{1}{1-\kappa}\int_{VaR_\kappa(S)}^\infty s\Big[(1+\theta)h(s;\lambda_1;\lambda_2) - \theta h(s;2\lambda_1;\lambda_2) \\ &\quad - \theta h(s;\lambda_1;2\lambda_2) + \theta h(s;2\lambda_1;2\lambda_2)\Big]ds. \end{aligned} \tag{2.4}$$

Define

$$\int_{VaR_\kappa(S)}^{\infty} sh(s;\lambda_1;\lambda_2)ds = \int_{VaR_\kappa(S)}^{\infty} s\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 s}+\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 s}\right)ds$$
$$= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left\{\left[-s\frac{e^{-\lambda_1 s}}{\lambda_1}\right]_{VaR_\kappa(S)}^{\infty}+\int_{VaR_\kappa(S)}^{\infty}\frac{e^{-\lambda_1 s}}{\lambda_1}ds\right\}$$
$$+\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left\{\left[-s\frac{e^{-\lambda_2 s}}{\lambda_2}\right]_{VaR_\kappa(S)}^{\infty}+\int_{VaR_\kappa(S)}^{\infty}\frac{e^{-\lambda_2 s}}{\lambda_2}ds\right\}$$
$$= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(VaR_\kappa(S)\frac{e^{-\lambda_1 VaR_\kappa(S)}}{\lambda_1}+\frac{e^{-\lambda_1 VaR_\kappa(S)}}{\lambda_1^2}\right)$$
$$+\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(VaR_\kappa(S)\frac{e^{-\lambda_2 VaR_\kappa(S)}}{\lambda_2}+\frac{e^{-\lambda_2 VaR_\kappa(S)}}{\lambda_2^2}\right)$$
$$= \frac{\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 VaR_\kappa(S)}\left(VaR_\kappa(S)+\frac{1}{\lambda_1}\right) \tag{2.5}$$
$$+\frac{\lambda_1}{\lambda_1-\lambda_2}e^{-\lambda_2 VaR_\kappa(S)}\left(VaR_\kappa(S)+\frac{1}{\lambda_2}\right)$$
$$= \zeta(VaR_\kappa(S);\lambda_1,\lambda_2). \tag{2.6}$$

Inserting (2.6) in (2.4), we obtain

$$TVaR_\kappa(S) = \frac{1}{1-\kappa}\Big[(1+\theta)\zeta(VaR_\kappa(S);\lambda_1;\lambda_2)-\theta\zeta(VaR_\kappa(S);2\lambda_1;\lambda_2)$$
$$-\theta\zeta(VaR_\kappa(S);\lambda_1;2\lambda_2)+\theta\zeta(VaR_\kappa(S);2\lambda_1;2\lambda_2)\Big],$$

with

$$\zeta(x;\gamma_1,\gamma_2)=\frac{\gamma_2}{\gamma_2-\gamma_1}e^{-\gamma_1 x}\left(x+\frac{1}{\gamma_1}\right)+\frac{\gamma_1}{\gamma_1-\gamma_2}e^{-\gamma_2 x}\left(x+\frac{1}{\gamma_2}\right).$$

□

A closed form expression for the TVaR-based capital attributed to risk i, $i=1,2$, is given in the next proposition.

Proposition 2.2. *Let X_1 and X_2 be two exponentially distributed random variables with joint cdf defined by a bivariate FGM copula. Then, the TVaR-based contribution of risk i, $i=1,2$, to the aggregate risk $S=X_1+X_2$ at level κ, $0<\kappa<1$, is*

$$TVaR_\kappa(X_i;S)=\frac{1}{1-\kappa}\Big[(1+\theta)\xi\left(VaR_\kappa(S);\lambda_i;\lambda_j\right)-\theta\xi\left(VaR_\kappa(S);2\lambda_i;\lambda_j\right)$$
$$-\theta\xi\left(VaR_\kappa(S);\lambda_i;2\lambda_j\right)+\theta\xi\left(VaR_\kappa(S);2\lambda_i;2\lambda_j\right)\Big], \tag{2.7}$$

where $\xi(x;\gamma_i;\gamma_j) = \frac{\gamma_j e^{-\gamma_i x}(x+\frac{1}{\gamma_i})}{\gamma_j-\gamma_i} - \frac{\gamma_j e^{-\gamma_i x}-\gamma_i e^{-\gamma_j x}}{(\gamma_j-\gamma_i)^2}$ *and* $i \neq j$.

Proof. Let $i = 1$ and $j = 2$. Recall that for continuous random variables the TVaR-based allocation is equal to the CTE-based allocation. From (2.1), we have

$$\begin{aligned}
TVaR_\kappa(X_1;S) &= E[X_1 \mid S > VaR_\kappa(S)] \\
&= \frac{1}{Pr(S > VaR_\kappa(S))} \int_{VaR_\kappa(S)}^{\infty} \int_0^s x f_{X_1,S}(x,s)\,dx\,ds,
\end{aligned}$$

where

$$\begin{aligned}
\int_0^s x f_{X_1,S}(x,s)dx &= \int_0^s x f_{X_1,X_2}(x,s-x)dx \\
&= \int_0^s x\bigg((1+\theta)\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2(s-x)} - \theta 2\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2(s-x)} \\
&\quad - \theta\lambda_1 e^{-\lambda_1 x} 2\lambda_2 e^{-\lambda_2(s-x)} + \theta 2\lambda_1 e^{-\lambda_1 x} 2\lambda_2 e^{-\lambda_2(s-x)}\bigg)dx \\
&= (1+\theta)\lambda_1\lambda_2\left(\frac{se^{-\lambda_1 s}}{\lambda_2-\lambda_1} - \frac{e^{-\lambda_1 s}-e^{-\lambda_2 s}}{(\lambda_2-\lambda_1)^2}\right) \\
&\quad - \theta 2\lambda_1\lambda_2\left(\frac{se^{-2\lambda_1 s}}{\lambda_2-2\lambda_1} - \frac{e^{-2\lambda_1 s}-e^{-\lambda_2 s}}{(\lambda_2-2\lambda_1)^2}\right) \\
&\quad - \theta\lambda_1 2\lambda_2\left(\frac{se^{-\lambda_1 s}}{2\lambda_2-\lambda_1} - \frac{e^{-\lambda_1 s}-e^{-2\lambda_2 s}}{(2\lambda_2-\lambda_1)^2}\right) \\
&\quad + \theta 2\lambda_1 2\lambda_2\left(\frac{se^{-2\lambda_1 s}}{2\lambda_2-2\lambda_1} - \frac{e^{-2\lambda_1 s}-e^{-2\lambda_2 s}}{(2\lambda_2-2\lambda_1)^2}\right).
\end{aligned}$$

The first component on the right-hand side of the last equality is given by

$$\begin{aligned}
\int_0^s x(1+\theta)\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2(s-x)}dx &= (1+\theta)\lambda_1\lambda_2 e^{-\lambda_2 s}\int_0^s x e^{x(\lambda_2-\lambda_1)}dx \\
&= (1+\theta)\lambda_1\lambda_2 e^{-\lambda_2 s}\left(\left[x\frac{e^{x(\lambda_2-\lambda_1)}}{\lambda_2-\lambda_1}\right]_0^s - \int_0^s \frac{e^{x(\lambda_2-\lambda_1)}}{\lambda_2-\lambda_1}dx\right) \\
&= (1+\theta)\lambda_1\lambda_2 e^{-\lambda_2 s}\left(s\frac{e^{s(\lambda_2-\lambda_1)}}{\lambda_2-\lambda_1} - \left[\frac{e^{x(\lambda_2-\lambda_1)}}{(\lambda_2-\lambda_1)^2}\right]_0^s\right) \\
&= (1+\theta)\lambda_1\lambda_2 e^{-\lambda_2 s}\left(s\frac{e^{s\lambda_2}e^{-s\lambda_1}}{\lambda_2-\lambda_1} - \frac{e^{s\lambda_2}e^{-s\lambda_1}-1}{(\lambda_2-\lambda_1)^2}\right) \\
&= (1+\theta)\lambda_1\lambda_2\left(\frac{se^{-\lambda_1 s}}{\lambda_2-\lambda_1} - \frac{e^{-\lambda_1 s}-e^{-\lambda_2 s}}{(\lambda_2-\lambda_1)^2}\right).
\end{aligned}$$

Denoting $V = VaR_\kappa(S)$, we have

$$\begin{aligned}
TVaR_\kappa(X_1;S) &= \frac{1}{1-F_S(V)}\int_V^\infty \Bigg\{(1+\theta)\lambda_1\lambda_2\left(\frac{se^{-\lambda_1 s}}{\lambda_2-\lambda_1}-\frac{e^{-\lambda_1 s}-e^{-\lambda_2 s}}{(\lambda_2-\lambda_1)^2}\right)\\
&\quad -\theta 2\lambda_1\lambda_2\left(\frac{se^{-2\lambda_1 s}}{\lambda_2-2\lambda_1}-\frac{e^{-2\lambda_1 s}-e^{-\lambda_2 s}}{(\lambda_2-2\lambda_1)^2}\right)\\
&\quad -\theta\lambda_1 2\lambda_2\left(\frac{se^{-\lambda_1 s}}{2\lambda_2-\lambda_1}-\frac{e^{-\lambda_1 s}-e^{-2\lambda_2 s}}{(2\lambda_2-\lambda_1)^2}\right)\\
&\quad +\theta 2\lambda_1 2\lambda_2\left(\frac{se^{-2\lambda_1 s}}{2\lambda_2-2\lambda_1}-\frac{e^{-2\lambda_1 s}-e^{-2\lambda_2 s}}{(2\lambda_2-2\lambda_1)^2}\right)\Bigg\}ds\\
&= \frac{1}{1-F_S(V)}\Bigg[(1+\theta)\left(\frac{\lambda_2 e^{-\lambda_1 V}(V+\frac{1}{\lambda_1})}{\lambda_2-\lambda_1}-\frac{\lambda_2 e^{-\lambda_1 V}-\lambda_1 e^{-\lambda_2 V}}{(\lambda_2-\lambda_1)^2}\right)\\
&\quad -\theta\left(\frac{\lambda_2 e^{-2\lambda_1 V}(V+\frac{1}{2\lambda_1})}{\lambda_2-2\lambda_1}-\frac{\lambda_2 e^{-2\lambda_1 V}-2\lambda_1 e^{-\lambda_2 V}}{(\lambda_2-2\lambda_1)^2}\right)\\
&\quad -\theta\left(\frac{2\lambda_2 e^{-\lambda_1 V}(V+\frac{1}{\lambda_1})}{2\lambda_2-\lambda_1}-\frac{2\lambda_2 e^{-\lambda_1 V}-\lambda_1 e^{-2\lambda_2 V}}{(2\lambda_2-\lambda_1)^2}\right)\\
&\quad +\theta\left(\frac{2\lambda_2 e^{-2\lambda_1 V}(V+\frac{1}{2\lambda_1})}{2\lambda_2-2\lambda_1}-\frac{2\lambda_2 e^{-2\lambda_1 V}-2\lambda_1 e^{-2\lambda_2 V}}{(2\lambda_2-2\lambda_1)^2}\right)\Bigg].
\end{aligned}$$

We finally obtain

$$\begin{aligned}
TVaR_\kappa(X_1;S) = \frac{1}{1-\kappa}\Big[&(1+\theta)\xi\left(VaR_\kappa(S);\lambda_1;\lambda_2\right)-\theta\xi\left(VaR_\kappa(S);2\lambda_1;\lambda_2\right)\\
&-\theta\xi\left(VaR_\kappa(S);\lambda_1;2\lambda_2\right)+\theta\xi\left(VaR_\kappa(S);2\lambda_1;2\lambda_2\right)\Big],
\end{aligned}$$

where

$$\xi\left(x;\gamma_1;\gamma_2\right) = \frac{\gamma_2 e^{-\gamma_1 x}(x+\frac{1}{\gamma_1})}{\gamma_2-\gamma_1}-\frac{\gamma_2 e^{-\gamma_1 x}-\gamma_1 e^{-\gamma_2 x}}{(\gamma_2-\gamma_1)^2}.$$

The TVaR-based allocation for the second risk is symmetrically given by

$$\begin{aligned}
TVaR_\kappa(X_2;S) = \frac{1}{1-\kappa}\Big[&(1+\theta)\xi\left(VaR_\kappa(S);\lambda_2;\lambda_1\right)-\theta\xi\left(VaR_\kappa(S);2\lambda_2;\lambda_1\right)\\
&-\theta\xi\left(VaR_\kappa(S);\lambda_2;2\lambda_1\right)+\theta\xi\left(VaR_\kappa(S);2\lambda_2;2\lambda_1\right)\Big],
\end{aligned}$$

where

$$\xi\left(x;\gamma_2;\gamma_1\right) = \frac{\gamma_1 e^{-\gamma_2 x}(x+\frac{1}{\gamma_2})}{\gamma_1-\gamma_2}-\frac{\gamma_1 e^{-\gamma_2 x}-\gamma_2 e^{-\gamma_1 x}}{(\gamma_1-\gamma_2)^2}.$$

□

Remark 2.1. *It can be verified that the TVaR of S is the sum of the risk contributions*

$$TVaR_\kappa(S) = \sum_{i=1}^{2} TVaR_\kappa(X_i; S).$$

Explicit expressions for the TVaR and the one risk TVaR-based contribution cannot only be obtained in the bivariate case but even for an undefined number of risks.

2.3.2 The multivariate case

Suppose now that there are n different exponential risks joined by a multivariate FGM copula characterized by

$$C(u_1, u_2, \ldots, u_n) = u_1 u_2 \cdots u_n \left(1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} \theta_{j_1 j_2 \cdots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \cdots \overline{u}_{j_k}\right),$$

where $\overline{u}_{(.)} = 1 - u_{(.)}$ (see Nelsen (2006) p.108). We have here $2^n - n - 1$ copula parameters to describe the dependence between each pair or group of risks.
Its density can be written as

$$c(u_1, u_2, \ldots, u_n) = 1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} \theta_{j_1 j_2 \cdots j_k} (2\overline{u}_{j_1} - 1)(2\overline{u}_{j_2} - 1) \cdots (2\overline{u}_{j_k} - 1).$$

As in the bivariate case, we suppose that the parameters of the n exponential risks satisfy the conditions below

$$\lambda_i \neq \lambda_j \text{ and } \lambda_i \neq 2\lambda_j \text{ for } i \neq j. \tag{2.8}$$

Then, the following proposition holds.

Proposition 2.3. *Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum of n dependent exponential random variables with joint cdf defined by a multivariate FGM copula as follows*

$$F_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = C_\theta^{FGM}(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))$$

with $\theta \in [-1, 1]$. Then, the TVaR of S_n at level κ, $0 < \kappa < 1$, is

$$TVaR_\kappa(S_n) = \frac{1}{1-\kappa} \times \Bigg[1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} \theta_{j_1 j_2 \cdots j_k} \times \left(\sum_{l=0}^{k} \sum_{(a_1, \ldots, a_k) \in A_{l,k}} (-1)^l \zeta\left(VaR_\kappa(S_n); 2^{i_1}\lambda_1, \ldots, 2^{i_k}\lambda_k, \lambda_{i_{k+1}}, \ldots, \lambda_{i_n}\right)\right)\Bigg], \tag{2.9}$$

where $\zeta(x; \gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\gamma_j}{\gamma_j - \gamma_i}\right) e^{-\gamma_i x} \left(x + \frac{1}{\gamma_i}\right)$, $i_{k+1}, \ldots, i_n$ are the missing indexes of $j_1, \ldots, j_k$ to complete $1, \ldots, n$ and $A_{l,k}$ are the sets of k-tuples composed of l zeros and $(k-l)$ ones, for $l = 0, 1, \ldots, k$ and $k = 2, \ldots, n$.

In fact, the $A_{l,k}$ are defined by $A_{0,k} = \Big\{(1,1,\dots,1)_{1\times k}\Big\}$,
$A_{1,k} = \Big\{(1,1,\dots,0)_{1\times k},\dots,(0,1,\dots,1)_{1\times k}\Big\}$, $A_{2,k} = \Big\{(1,1,\dots,0,0)_{1\times k},\dots,(0,0,\dots,1)_{1\times k}\Big\}$, $\dots$, $A_{k,k} = \Big\{(0,0,\dots,0)_{1\times k}\Big\}$.

Proof. The joint pdf of $(X_1, X_2, \dots, X_n)$ is

$$\begin{aligned}
f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) &= c(F_{X_1}(x_1),F_{X_2}(x_2),\dots,F_{X_n}(x_n))f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n)\\
&= f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n)\times\Bigg[1+\sum_{k=2}^{n}\sum_{1\le j_1<\dots<j_k\le n}\theta_{j_1j_2\cdots j_k}\\
&\quad\times(1-F_{X_{j_1}}(x_{j_1}))(1-F_{X_{j_2}}(x_{j_2}))\cdots(1-F_{X_{j_k}}(x_{j_k}))\Bigg].
\end{aligned}$$

As the n risk random variables are exponentially distributed with parameters λ_i, $i = 1,\dots,n$ with constraint (2.8), we have

$$\begin{aligned}
f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) &= \lambda_1 e^{-\lambda_1x_1}\lambda_2e^{-\lambda_2x_2}\cdots\lambda_ne^{-\lambda_nx_n}\\
&\quad\times\Bigg[1+\sum_{k=2}^{n}\sum_{1\le j_1<\dots<j_k\le n}\theta_{j_1j_2\cdots j_k}(2e^{-\lambda_{j_1}x_{j_1}}-1)(2e^{-\lambda_{j_2}x_{j_2}}-1)\cdots(2e^{-\lambda_{j_n}x_{j_n}}-1)\Bigg]\\
&= \nu(x_1,x_2,\dots,x_n;\lambda_1,\lambda_2,\dots,\lambda_n)+\sum_{k=2}^{n}\sum_{1\le j_1<\dots<j_k\le n}\theta_{j_1j_2\cdots j_k}\times\Bigg[\sum_{l=0}^{k}\sum_{(a_1,\dots,a_k)\in A_{l,k}}(-1)^l\\
&\quad\times\nu\left(x_{j_1},x_{j_2},\dots,x_{j_k},x_{i_{k+1}},\dots,x_{i_n};2^{a_1}\lambda_{j_1},2^{a_2}\lambda_{j_2},\dots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\dots,\lambda_{i_n}\right)\Bigg],
\end{aligned} \tag{2.10}$$

where $i_{k+1},\dots,i_n$ and $A_{l,k}$ are defined as in the proposition and

$$\nu(x_1,\dots,x_n;\gamma_1,\dots,\gamma_n) = \gamma_1e^{-\gamma_1x_1}\times\gamma_2e^{-\gamma_2x_2}\times\cdots\times\gamma_ne^{-\gamma_nx_n}.$$

Given that

$$\begin{aligned}
f_{S_n}(s) &= \int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\dots-x_{n-1}} f_{X_1,X_2,\dots,X_{n-1},X_n}(x_1,x_2,\dots,x_{n-1},s-x_1-x_2-\dots-x_{n-1})\\
&\quad\times dx_1dx_2\cdots dx_{n-1}\\
&= \int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\dots-x_{n-2}} f_{X_1,X_2,\dots,X_{n-2},X_{n-1}+X_n}(x_1,x_2,\dots,x_{n-2},s-x_1-x_2-\dots-x_{n-2})\\
&\quad\times dx_1dx_2\cdots dx_{n-2}\\
&= \cdots\\
&= \int_0^s f_{X_1,X_2+\dots+X_n}(x_1,s-x_1)dx_1
\end{aligned}$$

and that

$$\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{n-1}} \nu(x_1, x_2, \ldots, x_n; \gamma_1, \gamma_2, \ldots, \gamma_n) dx_1 dx_2 \cdots dx_{n-1}$$
$$= \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\gamma_j}{\gamma_j - \gamma_i} \right) \gamma_i e^{-\gamma_i s}$$
$$= h(s; \gamma_1, \gamma_2, \ldots, \gamma_n),$$

one can write

$$f_{S_n}(s) = h(s; \lambda_1, \ldots, \lambda_n) + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \theta_{j_1 j_2 \cdots j_k}$$
$$\times \left(\sum_{l=0}^{k} \sum_{(a_1, \ldots, a_k) \in A_{l,k}} (-1)^l h\left(s; 2^{a_1}\lambda_{j_1}, 2^{a_2}\lambda_{j_2}, \ldots, 2^{a_k}\lambda_{j_k}, \lambda_{i_{k+1}}, \ldots, \lambda_{i_n}\right) \right).$$

Then, the TVaR of S_n for $n \geq 2$ and $0 < \kappa < 1$ is

$$TVaR_\kappa(S_n) = E[S_n \mid S_n > VaR_\kappa(S_n)]$$
$$= \int_{VaR_\kappa}^{\infty} s \frac{f_{S_n}(s)}{Pr(S_n > VaR_\kappa(S_n))} ds$$
$$= \frac{1}{1-\kappa} \int_{VaR_\kappa(S_n)}^{\infty} s f_{S_n}(s) ds$$
$$= \frac{1}{1-\kappa} \times \left[1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \theta_{j_1 j_2 \cdots j_k} \times \right.$$
$$\left. \left(\sum_{l=0}^{k} \sum_{(a_1, \ldots, a_k) \in A_{l,k}} (-1)^l \zeta\left(VaR_\kappa(S_n); 2^{i_1}\lambda_1, \ldots, 2^{i_k}\lambda_k, \lambda_{i_{k+1}}, \ldots, \lambda_{i_n}\right) \right) \right],$$

where

$$\zeta(x; \gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\gamma_j}{\gamma_j - \gamma_i} \right) e^{-\gamma_i x} \left(x + \frac{1}{\gamma_i} \right).$$

□

As in the bivariate case, the capital allocation for risk i can also be explicitly given. In order to find that expression, we first need to introduce the nth order divided difference of a function f as in Chiragiev and Landsman (2007). Consider $x_1, x_2, \ldots, x_n, x_{n+1}$ arbitrary points such that $x_i \neq x_j$ for $i \neq j$. The nth order divided difference of f on $x_1, x_2, \ldots, x_n, x_{n+1}$ is defined by

$$f(x_1, x_2, \ldots, x_n, x_{n+1}) = \sum_{i=1}^{n+1} \frac{f(x_i)}{\prod_{j\neq i}(x_i - x_j)}.$$

The following proposition gives the expression of the TVaR-based allocation.

Proposition 2.4. *Let $X_1, \ldots, X_n$ be n exponentially distributed random variables with joint cdf defined by a multivariate FGM copula. Then, the TVaR-based contribution of risk i, $i = 1, \ldots, n$, to the sum $S_n = X_1 + \cdots + X_n$ at level κ, $0 < \kappa < 1$, is*

$$
\begin{aligned}
TVaR_\kappa(X_i; S_n) = \frac{(-1)^{n-1}\Lambda}{1-\kappa} \Bigg[& \overline{H}_i(VaR_\kappa(S_n); \lambda_1; \ldots; \lambda_n) + \sum_{k=2}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n \bigcap i \in \{j_1, \ldots, j_k\}} \theta_{j_1 \cdots j_k} \\
& \times \Bigg(\sum_{l=0}^{k-1} \sum_{(a1, \ldots, a_k)_{-i} \in A_{l,k-1}} \Big\{ (-1)^{l+1} 2^{k-1-l} \overline{H}_i(VaR_\kappa(S_n); 2^{a_1}\lambda_{j_1}; \ldots; 2^{a_k}\lambda_{j_k}; \lambda_{i_{k+1}}; \ldots; \lambda_{i_n}) \\
& + (-1)^l 2^{k-1-l} \overline{G}_i(VaR_\kappa(S_n); 2^{a_1}\lambda_{j_1}; \ldots; 2^{a_k}\lambda_{j_k}; \lambda_{i_{k+1}}; \ldots; \lambda_{i_n}) \Big\} \Bigg) \\
& + \sum_{k=2}^{n-1} \sum_{1 \le j_1 < \cdots < j_k \le n \bigcap i \notin \{j_1, \ldots, j_k\}} \theta_{j_1 \cdots j_k} \\
& \times \Bigg(\sum_{l=0}^{k} \sum_{(a1, \ldots, a_k) \in A_{l,k}} (-1)^l 2^{k-l} \overline{H}_i(VaR_\kappa(S_n); 2^{a_1}\lambda_{j_1}; \ldots; 2^{a_k}\lambda_{j_k}; \lambda_{i_{k+1}}; \ldots; \lambda_{i_n}) \Bigg) \Bigg],
\end{aligned}
\tag{2.11}
$$

where $\overline{H}_i(x; \gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n)$ and $\overline{G}_i(x; \gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n)$ are the $(n-2)$-th order divided differences of respectively $\overline{H}_i(x; \gamma) = \frac{xe^{-\gamma_i x}}{\gamma_i(\gamma_i - \gamma)} + \frac{e^{-\gamma_i x}}{\gamma_i^2(\gamma_i - \gamma)} + \frac{\gamma e^{-\gamma_i x} - \gamma_i e^{-\gamma x}}{\gamma\gamma_i(\gamma_i - \gamma)^2}$ and $\overline{G}_i(x; \gamma) = 2\left(\frac{xe^{-2\gamma_i x}}{2\gamma_i(2\gamma_i - \gamma)} + \frac{e^{-2\gamma_i x}}{(2\gamma_i)^2(2\gamma_i - \gamma)} + \frac{\gamma e^{-2\gamma_i x} - 2\gamma_i e^{-\gamma x}}{\gamma 2\gamma_i(2\gamma_i - \gamma)^2} \right)$, $\Lambda = \lambda_1 \times \cdots \times \lambda_n$ is the product of the parameters of exponential distributions, $i_{k+1}, \ldots, i_n$ are the missing indexes of $j_1, \ldots, j_k$ to complete $1, \ldots, n$ and $A_{l,k}$ are the sets of k-tuples composed of l zeros and $(k-l)$ ones, for $l = 0, 1, \ldots, k$ and $k = 2, \ldots, n$.

Proof. The capital attributed to the continuous distributed risk i can be expressed as

$$
\begin{aligned}
TVaR_\kappa(X_i; S_n) &= CTE_\kappa(X_i|S_n) \\
&= \frac{1}{\overline{F}_{S_n}(VaR_\kappa(S_n))} \int_{VaR_\kappa(S_n)}^{\infty} \int_0^s x_i f_{X_i,S_n}(x_i, s) dx ds.
\end{aligned}
\tag{2.12}
$$

A recursive formula for $f_{X_i,S_n}(x_i, s) = f_{X_i,S_n - X_i}(x_i, s - x_i)$ is needed to evaluate this expression. Given that the risk random variables here are not independent, we cannot directly separate $f_{X_i,S_n-X_i}(x_i, s - x_i)$ into the product of $f_{X_i}(x_i)$ and $f_{S_{n-1}}(s - x_i)$.

First, we have

$$
\begin{aligned}
f_{X_i,S_n-X_i}(x_i, s - x_i) &= f_{X_i, X_1 + X_2 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_n}(x_i, s - x_i) \\
&= \int_0^{s-x_i} \int_0^{s-x_i-x_1} \int_0^{s-x_i-x_1-x_2} \cdots \int_0^{s-x_1-\cdots-x_{n-1}} \\
&\quad \times f_{X_1, X_2, \ldots, X_{n-1}, X_n}(x_1, x_2, \ldots, x_{n-1}, s - x_1 - \cdots - x_{n-1}) dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_{n-1}
\end{aligned}
$$

with $f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n)$ defined as in (2.10) and which can be extended to

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = f_{X_i}(x_i)\Bigg(\nu_{-i}(x_1;x_2;\ldots;x_n;\lambda_1;\lambda_2;\ldots;\lambda_n)$$
$$+\sum_{k=2}^{n}\sum_{1\leq j_1<\cdots<j_k\leq n\bigcap i\in\{j_1,\ldots,j_k\}}\theta_{j_1j_2\cdots j_k}(1-2F_{X_i}(x_i))\times\Big[\sum_{l=0}^{k}\sum_{(a_1,\ldots,a_k)_{-i}\in A_{l,k-1}}(-1)^l$$
$$\times\,\nu_{-i}(x_{j_1};x_{j_2};\ldots;x_{j_k};x_{i_{k+1}};\ldots;x_{i_n};2^{a_1}\lambda_{j_1};2^{a_2}\lambda_{j_2};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})\Big]$$
$$+\sum_{k=2}^{n}\sum_{1\leq j_1<\cdots<j_k\leq n\bigcap i\notin\{j_1,\ldots,j_k\}}\theta_{j_1j_2\cdots j_k}\times\Big[\sum_{l=0}^{k}\sum_{(a_1,\ldots,a_k)_{-i}\in A_{l,k-1}}(-1)^l$$
$$\times\,\nu_{-i}(x_{j_1},x_{j_2};\ldots;x_{j_k};x_{i_{k+1}};\ldots;x_{i_n};2^{a_1}\lambda_{j_1};2^{a_2}\lambda_{j_2};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})\Big],$$

where $i_{k+1},\ldots,i_n$ are the missing indexes of $j_1,\ldots,j_k$ to complete $1,\ldots,n$, $A_{0,k}=\Big\{(1,1,\ldots,1)_{1\times k}\Big\}$, $A_{1,k}=\Big\{(1,1,\ldots,0)_{1\times k},\ldots,(0,1,\ldots,1)_{1\times k}\Big\}$, $A_{2,k}=\Big\{(1,1,\ldots,0,0)_{1\times k},\ldots,(0,0,\ldots,1)_{1\times k}\Big\}$, $\ldots$, $A_{k,k}=\Big\{(0,0,\ldots,0)_{1\times k}\Big\}$ and

$$\nu_{-i}(x_1;x_2;\ldots;x_n;\gamma_1;\gamma_2;\ldots;\gamma_n)=\gamma_1e^{-\gamma_1x_1}\times\gamma_2e^{-\gamma_2x_2}$$
$$\times\cdots\times\gamma_{i-1}e^{-\gamma_{i-1}x_{i-1}}\times\gamma_{i+1}e^{-\gamma_{i+1}x_{i+1}}\times\cdots\times\gamma_ne^{-\gamma_nx_n}.$$

As in the proof of the Proposition 4, we use the fact that

$$\int_0^{s-x_i}\int_0^{s-x_i-x_1}\int_0^{s-x_i-x_1-x_2}\cdots\int_0^{s-x_1-\cdots-x_{n-1}}\nu_{-i}(x_1;x_2;\ldots;x_n;\gamma_1;\gamma_2;\ldots;\gamma_n)$$
$$\times\,dx_1dx_2\cdots dx_{i-1}dx_{i+1}\cdots dx_{n-1}$$
$$=\sum_{\substack{j=1\\j\neq i}}^{n}\left(\prod_{\substack{k=1,k\neq j\\k\neq i}}^{n}\frac{\gamma_k}{\gamma_k-\gamma_j}\right)\gamma_je^{-\gamma_j(s-x_i)}$$
$$=h_{-i}(s-x_i;\gamma_1,\gamma_2,\ldots,\gamma_n),$$

and we obtain

$$f_{X_i,S_n-X_i}(x_i,s-x_i)=f_{X_i}(x_i)\Big[h_{-i}(s-x_i;\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_n)$$
$$+\sum_{k=2}^{n}\sum_{1\leq j_1<\cdots<j_k\leq n\bigcap i\in\{j_1,\ldots,j_k\}}\theta_{j_1j_2\cdots j_k}(1-2F_{X_i}(x))$$
$$\times\left(\sum_{l=0}^{k}\sum_{(a_1,\ldots,a_k)_{-i}\in A_{l,k-1}}(-1)^lh_{-i}(s-x_i;2^{a_1}\lambda_{j_1},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_n})\right)$$
$$+\sum_{k=2}^{n}\sum_{1\leq j_1<\cdots<j_k\leq n\bigcap i\notin\{j_1,\ldots,j_k\}}\theta_{j_1j_2\cdots j_k}$$
$$\times\left(\sum_{l=0}^{k}\sum_{(a_1,\ldots,a_k)\in A_{l,k}}(-1)^lh_{-i}(s-x_i;2^{a_1}\lambda_{j_1},\ldots,2^{a_k}\lambda_{j_k},\lambda_{i_{k+1}},\ldots,\lambda_{i_n})\right)\Big].\qquad(2.13)$$

Using the divided difference as in Chiragiev and Landsman (2007), notice that

$$h_{-i}(x;\gamma_1,\ldots,\gamma_{i-1},\gamma_{i+1},\ldots,\gamma_n) = (-1)^{n-2}\times\gamma_1\times\cdots\times\gamma_{i-1}\times\gamma_{i+1}\times\cdots\times\gamma_n \times\overline{F}(x;\gamma_1;\ldots;\gamma_{i-1};\gamma_{i+1};\ldots;\gamma_n),$$

where $\overline{F}(x;\gamma_1;\ldots;\gamma_{i-1};\gamma_{i+1};\ldots;\gamma_n)$ is the $(n-2)$-th order divided difference of $\overline{F}(x;\gamma) = e^{-\gamma x}$.

Then, we have

$$\begin{aligned}
&\int_0^s x_i f_{X_i}(x_i) h_{-i}(s-x_i;\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_n)dx_i \\
&= \int_0^s x_i f_{X_i}(x_i)(-1)^{n-2}\Lambda_{-i}\overline{F}(x_i;\lambda_1;\ldots;\lambda_{i-1};\lambda_{i+1};\ldots;\lambda_n)dx_i \qquad (2.14)\\
&= (-1)^{n-1}\Lambda H_i(s;\lambda_1;\ldots;\lambda_{i-1};\lambda_{i+1};\ldots;\lambda_n), \qquad (2.15)
\end{aligned}$$

where $\Lambda_{-i} = \lambda_1\times\cdots\times\lambda_{i-1}\times\lambda_{i+1}\times\cdots\times\lambda_n$, $\Lambda = \lambda_1\times\lambda_2\times\cdots\times\lambda_n$ and

$$H_i(s;\lambda) = -\int_0^s x_i e^{-\lambda_i x_i}\overline{F}(s-x_i;\lambda)dx_i.$$

Let also

$$G_i(s;\lambda) = -\int_0^s 2x_i e^{-2\lambda_i x_i}\overline{F}(s-x_i;\lambda)dx_i.$$

Given that the risks here are exponentially distributed, H_i and G_i take the form

$$\begin{aligned}
H_i(s;\lambda) &= -\int_0^s x_i e^{-\lambda_i x_i}\overline{F}(s-x_i;\lambda)dx_i \\
&= -e^{-\lambda s}\int_0^s x_i e^{-(\lambda_i-\lambda)x_i}ds \\
&= \frac{se^{-\lambda_i s}}{\lambda_i-\lambda} + \frac{e^{-\lambda_i s}}{(\lambda_i-\lambda)^2} - \frac{e^{-\lambda s}}{(\lambda_i-\lambda)^2}
\end{aligned}$$

and

$$\begin{aligned}
G_i(s;\lambda) &= -\int_0^s 2x_i e^{-2\lambda_i x_i}\overline{F}(s-x_i;\lambda)dx_i \\
&= -2e^{-\lambda s}\int_0^s x_i e^{-(2\lambda_i-\lambda)x_i}ds \\
&= 2\left(\frac{se^{-2\lambda_i s}}{2\lambda_i-\lambda} + \frac{e^{-2\lambda_i s}}{(2\lambda_i-\lambda)^2} - \frac{e^{-\lambda s}}{(2\lambda_i-\lambda)^2}\right).
\end{aligned}$$

Using (2.15), we can express $\int_0^s x_i f_{X_i,S_n}(x_i,s)dx_i$ as

$$\int_0^s x f_{X_i,S_n}(x,s)dx = (-1)^{n-1}\Lambda\Bigg[H_i(s;\lambda_1;\ldots;\lambda_n) + \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n\bigcap i\in\{j_1,\ldots,j_k\}}\theta_{j_1\cdots j_k}$$
$$\times\Bigg\{\sum_{l=0}^{k-1}\sum_{(a1,\ldots,a_k)_{-i}\in A_{l,k-1}}\Big((-1)^{l+1}2^{k-1-l}H_i(s;2^{a_1}\lambda_{j_1};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})$$
$$+(-1)^{l}2^{k-1-l}G_i(s;2^{a_1}\lambda_{j_1};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})\Big)\Bigg\}$$
$$+\sum_{k=2}^{n-1}\sum_{1\le j_1<\cdots<j_k\le n\bigcap i\notin\{j_1,\ldots,j_k\}}\theta_{j_1\cdots j_k}$$
$$\times\Bigg\{\sum_{l=0}^{k}\sum_{(a1,\ldots,a_k)\in A_{l,k}}(-1)^{l}2^{k-l}H_i(s;2^{a_1}\lambda_{j_1};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})\Bigg\}\Bigg]. \tag{2.16}$$

To calculate the risk contribution as in (2.12), the $H_i(s)$ and $G_i(s)$ terms in (2.16) must be integrated on s as follows

$$\begin{aligned}\overline{H}_i(V;\lambda) &= \int_V^\infty H_i(s;\lambda)ds\\ &= \int_V^\infty\left(\frac{se^{-\lambda_i s}}{\lambda_i-\lambda}+\frac{e^{-\lambda_i s}}{(\lambda_i-\lambda)^2}-\frac{e^{-\lambda s}}{(\lambda_i-\lambda)^2}\right)ds\\ &= \frac{Ve^{-\lambda_i V}}{\lambda_i(\lambda_i-\lambda)}+\frac{e^{-\lambda_i V}}{\lambda_i^2(\lambda_i-\lambda)}+\frac{\lambda e^{-\lambda_i V}-\lambda_i e^{-\lambda V}}{\lambda\lambda_i(\lambda_i-\lambda)^2}\end{aligned}$$

and

$$\begin{aligned}\overline{G}_i(V;\lambda) &= \int_V^\infty G_i(s;\lambda)ds\\ &= \int_V^\infty 2\left(\frac{se^{-2\lambda_i s}}{2\lambda_i-\lambda}+\frac{e^{-2\lambda_i s}}{(2\lambda_i-\lambda)^2}-\frac{e^{-\lambda s}}{(2\lambda_i-\lambda)^2}\right)ds\\ &= 2\left(\frac{Ve^{-2\lambda_i V}}{2\lambda_i(2\lambda_i-\lambda)}+\frac{e^{-2\lambda_i V}}{(2\lambda_i)^2(2\lambda_i-\lambda)}+\frac{\lambda e^{-2\lambda_i V}-2\lambda_i e^{-\lambda V}}{\lambda 2\lambda_i(2\lambda_i-\lambda)^2}\right).\end{aligned}$$

Finally, expression (2.12) for $TVaR_\kappa(X_i; S_n)$ is obtained

$$TVaR_\kappa(X_i;S_n) = \frac{(-1)^{n-1}\Lambda}{1-\kappa}\Bigg[\overline{H}_i(VaR_\kappa(S_n);\lambda_1;\ldots;\lambda_n) + \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n\bigcap i\in\{j_1,\ldots,j_k\}}\theta_{j_1\cdots j_k}$$
$$\times\Bigg(\sum_{l=0}^{k-1}\sum_{(a1,\ldots,a_k)_{-i}\in A_{l,k-1}}\Big\{(-1)^{l+1}2^{k-1-l}\overline{H}_i(VaR_\kappa(S_n);2^{a_1}\lambda_{j_1};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})$$
$$+(-1)^l 2^{k-1-l}\overline{G}_i(VaR_\kappa(S_n);2^{a_1}\lambda_{j_1};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})\Big\}\Bigg)$$
$$+\sum_{k=2}^{n-1}\sum_{1\le j_1<\cdots<j_k\le n\bigcap i\notin\{j_1,\ldots,j_k\}}\theta_{j_1\cdots j_k}$$
$$\times\Bigg(\sum_{l=0}^{k}\sum_{(a1,\ldots,a_k)\in A_{l,k}}(-1)^l 2^{k-l}\overline{H}_i(VaR_\kappa(S_n);2^{a_1}\lambda_{j_1};\ldots;2^{a_k}\lambda_{j_k};\lambda_{i_{k+1}};\ldots;\lambda_{i_n})\Bigg)\Bigg].$$

□

Remark 2.2. *To obtain the TVaR-based contribution* (2.7) in the bivariate case from (2.11), we just have to use the equalities

$$\overline{H_1}(x;\lambda_2) = -\frac{1}{\lambda_1\lambda_2}\xi(x;\lambda_1;\lambda_2),$$
$$\overline{H_1}(x;2\lambda_2) = -\frac{1}{\lambda_1 2\lambda_2}\xi(x;\lambda_1;2\lambda_2),$$
$$\overline{G_1}(x;\lambda_2) = -2\frac{1}{2\lambda_1\lambda_2}\xi(x;2\lambda_1;\lambda_2),$$
$$\overline{G_1}(x;2\lambda_2) = -2\frac{1}{2\lambda_1 2\lambda_2}\xi(x;2\lambda_1;2\lambda_2).$$

2.3.3 Numerical application

We illustrate here our results with a numerical example for the bivariate exponential case. Suppose that the parameters of the distributions of X_1 and X_2 are respectively $\lambda_1 = 1/2$ and $\lambda_2 = 1/3$. Let us calculate the VaR, TVaR and TVaR-based allocations for X_1 and X_2 for different risk levels κ and different FGM copula parameters θ. We write bellow the cumulative distribution function of S which can be expressed in the current case as a combination of generalized Erlang cdf's

$$F_S(s) = (1+\theta)H(s;\lambda_1;\lambda_2) - \theta H(s;2\lambda_1;\lambda_2) - \theta H(s;\lambda_1;2\lambda_2) + \theta H(s;2\lambda_1;2\lambda_2),$$

where $H(s;\lambda_1;\lambda_2) = \frac{\lambda_2}{\lambda_2-\lambda_1}\left(1-e^{-\lambda_1 s}\right) + \frac{\lambda_1}{\lambda_1-\lambda_2}\left(1-e^{-\lambda_2 s}\right)$ is the cdf of a 2-generalized Erlang distribution with parameters (λ_1,λ_2). The numerical results for the VaR, TVaR and TVaR-based allocations are displayed in Tables 2.1, 2.2 and 2.3.

$\theta = -1$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S)$	4.3188	6.5053	11.0436	15.5235	17.4860
$TVaR_\kappa(S)$	7.3270	9.3394	13.8369	18.3810	20.3716
$TVaR_\kappa(X_1; S)$	2.7244	3.1489	3.5085	3.2649	3.0613
$TVaR_\kappa(X_2; S)$	4.6026	6.1905	10.3283	15.1161	17.3103

TABLE 2.1 – Bivariate exponential example with $\theta = -1$.

$\theta = 0$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S)$	4.1589	6.7187	11.9994	16.9914	19.1073
$TVaR_\kappa(S)$	7.6589	9.9967	15.0984	20.0310	22.1324
$TVaR_\kappa(X_1; S)$	2.9206	3.5756	4.6115	5.2234	5.4002
$TVaR_\kappa(X_2; S)$	4.7383	6.4211	10.4869	14.8075	16.7323

TABLE 2.2 – Bivariate exponential example with $\theta = 0$.

$\theta = 1$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S)$	3.9328	6.9975	12.8673	18.0635	20.2236
$TVaR_\kappa(S)$	7.9817	10.6369	16.0906	21.1529	23.2818
$TVaR_\kappa(X_1; S)$	3.1066	3.9947	5.4022	6.2662	6.5272
$TVaR_\kappa(X_2; S)$	4.8750	6.6422	10.6883	14.8867	16.7546

TABLE 2.3 – Bivariate exponential example with $\theta = 1$.

2.4 TVaR and the TVaR-based allocation with mixtures of exponential marginals and the FGM copula

Let us consider now that we have two risks X_1 and X_2 which are distributed as a mixture of exponentials. Their cdf's and pdf's can be written as

$$\begin{aligned} F_{X_1}(x) &= \alpha_{11}(1-e^{-\lambda_{11}x}) + \alpha_{12}(1-e^{-\lambda_{12}x}) \\ F_{X_2}(x) &= \alpha_{21}(1-e^{-\lambda_{21}x}) + \alpha_{22}(1-e^{-\lambda_{22}x}) \\ f_{X_1}(x) &= \alpha_{11}\lambda_{11}e^{-\lambda_{11}x} + \alpha_{12}\lambda_{12}e^{-\lambda_{12}x} \\ f_{X_2}(x) &= \alpha_{21}\lambda_{21}e^{-\lambda_{21}x} + \alpha_{22}\lambda_{22}e^{-\lambda_{22}x}, \end{aligned}$$

where we restrict our model to $\lambda_{1i} \neq \lambda_{2j}$, $\lambda_{1i} \neq 2\lambda_{2j}$, $\lambda_{11}+\lambda_{12} \neq \lambda_{2j}$, $\lambda_{11}+\lambda_{12} \neq 2\lambda_{2j}$ and $\lambda_{11}+\lambda_{12} \neq \lambda_{21}+\lambda_{22}$. As for the exponential distribution case, the calculations can be done without these constraints but the results are not presented here. Mixtures of exponential distributions, also called hyper-exponential distributions, belong to the phase-type distribution family, see Neuts (1981) and Asmussen (2000). They can be used to approximate light- or heavy-tailed distributions with completely monotone pdf's and decreasing failure rates as shown in Feldmann and Whitt (1998), Keatinge (1999) and Khayari et al. (2003). The practical form of the mixture of exponential distributions also permits explicit results.

The following two propositions can be proven similarly as Proposition 2.1 and Proposition 2.2 in the previous section given that a mixture of exponentials is just an extension of the exponential distribution.

Proposition 2.5. *Let X_1 and X_2 be two random variables with mixture of exponential distributions and a joint cdf defined by a bivariate FGM copula as follows*

$$F_{X_1,X_2}(x_1,x_2) = C_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2))$$

with $\theta \in [-1,1]$. Then, the TVaR of the aggregate risk $S = X_1 + X_2$ at level κ, $0 < \kappa < 1$, is

$$TVaR_\kappa(S) = (I+J+K) \times \frac{1}{1-\kappa},$$

where

$$\begin{aligned} I = \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\{ \Big[& \alpha_{1i}\alpha_{2j} + \theta(\alpha_{1i}\alpha_{2j} - 2\alpha_{1i}^2\alpha_{2j} - 2\alpha_{11}\alpha_{12}\alpha_{2j} - 2\alpha_{1i}\alpha_{2j}^2 - 2\alpha_{1i}\alpha_{21}\alpha_{22} \\ & + 4\alpha_{1i}^2\alpha_{2j}^2 + 4\alpha_{1i}^2\alpha_{21}\alpha_{22} + 4\alpha_{11}\alpha_{12}\alpha_{2j}^2 + 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22})\Big]\zeta(VaR_\kappa(S);\lambda_{1i};\lambda_{2j})\Big\}, \end{aligned}$$

$$J = \theta \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg\{ \left[\alpha_{1i}^2\alpha_{2j} - 2\alpha_{1i}^2\alpha_{2j}^2 - 2\alpha_{1i}^2\alpha_{21}\alpha_{22}\right]\zeta(VaR_\kappa(S); 2\lambda_{1i}; \lambda_{2j})$$
$$+ \left[2\alpha_{11}\alpha_{12}\alpha_{2j} - 4\alpha_{11}\alpha_{12}\alpha_{2j}^2 - 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\right]\frac{\lambda_{1i}}{\lambda_{11}+\lambda_{12}}\zeta(VaR_\kappa(S); \lambda_{11}+\lambda_{12}; \lambda_{2j})\Bigg\}$$

and

$$K = \theta \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg\{ \left[\alpha_{1i}\alpha_{2j}^2 - 2\alpha_{1i}^2\alpha_{2j}^2 - 2\alpha_{11}\alpha_{12}\alpha_{2j}^2\right]\zeta(VaR_\kappa(S); \lambda_{1i}; 2\lambda_{2j})$$
$$+ \alpha_{1i}^2\alpha_{2j}^2\zeta(VaR_\kappa(S); 2\lambda_{1i}; 2\lambda_{2j})$$
$$+ \left[2\alpha_{1i}\alpha_{21}\alpha_{22} - 4\alpha_{1i}^2\alpha_{21}\alpha_{22} - 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\right]\frac{\lambda_{2j}}{\lambda_{21}+\lambda_{22}}\zeta(VaR_\kappa(S); \lambda_{1i}; \lambda_{21}+\lambda_{22})$$
$$+ 2\alpha_{1i}^2\alpha_{21}\alpha_{22}\frac{\lambda_{2j}}{\lambda_{21}+\lambda_{22}}\zeta(VaR_\kappa(S); 2\lambda_{11}; \lambda_{21}+\lambda_{22})$$
$$+ 2\alpha_{11}\alpha_{12}\alpha_{2j}^2\frac{\lambda_{1i}}{\lambda_{11}+\lambda_{12}}\zeta(VaR_\kappa(S); \lambda_{11}+\lambda_{12}; +2\lambda_{2j})$$
$$+ 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\frac{\lambda_{1i}\lambda_{2j}}{(\lambda_{11}+\lambda_{12})(\lambda_{21}+\lambda_{22})}\zeta(VaR_\kappa(S); \lambda_{11}+\lambda_{12}; \lambda_{21}+\lambda_{22})\Bigg\}.$$

Proposition 2.6. *Let X_1 and X_2 be two mixture of exponentials distributed random variables with joint cdf defined by a bivariate FGM copula. Then, the TVaR-based contribution of risk i, $i = 1, 2$, to the aggregate risk $S = X_1 + X_2$ at level κ, $0 < \kappa < 1$, is*

$$TVaR_\kappa(X_i; S) = (L + M + N) \times \frac{1}{1-\kappa},$$

where

$$L = \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg\{\Big[\alpha_{1i}\alpha_{2j} + \theta(\alpha_{1i}\alpha_{2j} - 2\alpha_{1i}^2\alpha_{2j} - 2\alpha_{11}\alpha_{12}\alpha_{2j} - 2\alpha_{1i}\alpha_{2j}^2 - 2\alpha_{1i}\alpha_{21}\alpha_{22}$$
$$+ 4\alpha_{1i}^2\alpha_{2j}^2 + 4\alpha_{1i}^2\alpha_{21}\alpha_{22} + 4\alpha_{11}\alpha_{12}\alpha_{2j}^2 + 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22})\Big]\xi(VaR_\kappa(S); \lambda_{1i}; \lambda_{2j})\Bigg\},$$

$$M = \theta \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg\{ \left[\alpha_{1i}^2\alpha_{2j} - 2\alpha_{1i}^2\alpha_{2j}^2 - 2\alpha_{1i}^2\alpha_{21}\alpha_{22}\right]\xi(VaR_\kappa(S); 2\lambda_{1i}; \lambda_{2j})$$
$$+ \left[2\alpha_{11}\alpha_{12}\alpha_{2j} - 4\alpha_{11}\alpha_{12}\alpha_{2j}^2 - 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\right]\frac{\lambda_{1i}}{\lambda_{11}+\lambda_{12}}\xi(VaR_\kappa(S); \lambda_{11}+\lambda_{12}; \lambda_{2j})\Bigg\},$$

$$
\begin{aligned}
N = \theta \sum_{i=1}^{2} \sum_{j=1}^{2} \Bigg\{ & \left[\alpha_{1i}\alpha_{2j}^2 - 2\alpha_{1i}^2\alpha_{2j}^2 - 2\alpha_{11}\alpha_{12}\alpha_{2j}^2\right] \xi(VaR_\kappa(S); \lambda_{1i}; 2\lambda_{2j}) \\
& + \alpha_{1i}^2\alpha_{2j}^2 \xi(VaR_\kappa(S); 2\lambda_{1i}; 2\lambda_{2j}) \\
& + \left[2\alpha_{1i}\alpha_{21}\alpha_{22} - 4\alpha_{1i}^2\alpha_{21}\alpha_{22} - 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\right] \frac{\lambda_{2j}}{\lambda_{21}+\lambda_{22}} \xi(VaR_\kappa(S); \lambda_{1i}; \lambda_{21}+\lambda_{22}) \\
& + 2\alpha_{1i}^2\alpha_{21}\alpha_{22} \frac{\lambda_{2j}}{\lambda_{21}+\lambda_{22}} \xi(s; 2\lambda_{1i}; \lambda_{21}+\lambda_{22}) \\
& + 2\alpha_{11}\alpha_{12}\alpha_{2j}^2 \frac{\lambda_{1i}}{\lambda_{11}+\lambda_{12}} \xi(VaR_\kappa(S); \lambda_{11}+\lambda_{12}; +2\lambda_{2j}) \\
& + 4\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22} \frac{\lambda_{1i}\lambda_{2j}}{(\lambda_{11}+\lambda_{12})(\lambda_{21}+\lambda_{22})} \xi(VaR_\kappa(S); \lambda_{11}+\lambda_{12}; \lambda_{21}+\lambda_{22}) \Bigg\}.
\end{aligned}
$$

2.5 Approximation methods for TVaR-based allocation

We have seen in the previous sections that it is possible to have an exact expression for the TVaR of a sum of several dependent random variables and the contribution of each random variable to the aggregate TVaR for some specific situations, in particular when using the FGM copula. For most copulas, it is more complicated to directly calculate this risk measure. Embrechts and Puccetti (2007) proposed an algorithm to compute numerically the cdf of the sum of two random variables joined by a copula. They used an approximation of the set $\{(x_1, x_2) \in [0, +\infty)^2 : x_1 + x_2 \leq s\}$ by a countable union of disjoint rectangles to obtain an evaluation of $F_S(s)$ with $S = X_1 + X_2$. In the present paper, we expose a simple alternative to approximate this cumulative distribution function with the use of common discretization methods that can be found in Klugman et al. (2008). Then we evaluate the TVaR and its contributions when the random variables are linked by any copula. The method is here exposed for two random variables but can be expanded to more random variables as shown in the numerical applications.

2.5.1 Discretization methods

We use three discretization methods in our study that are defined just below. For these three methods, we suppose that X is a continuous random variable with cdf F_X and that h is the discretization span.

Definition 2.1 (Lower method)**.** *The lower method provides a probability mass function of* *the discretized random variable* $\tilde{X}$ *given by*

$$\begin{cases} f_{\tilde{X}}(0) = 0 \\ f_{\tilde{X}}(jh) = F_X(jh) - F_X((j-1)h), \quad \textit{for } j = 1, 2, \ldots \end{cases}$$

Definition 2.2 (Upper method)**.** *The upper method provides a probability mass function of the discretized random variable* $\tilde{X}$ *given by*

$$\begin{cases} f_{\tilde{X}}(0) = F_X(h) \\ f_{\tilde{X}}(jh) = F_X((j+1)h) - F_X(jh), \quad \textit{for } j = 1, 2, \ldots \end{cases}$$

Definition 2.3 (Mean preserving method)**.** *The mean preserving method provides a probability mass function of the discretized random variable* $\tilde{X}$ *given by*

$$\begin{cases} f_{\tilde{X}}(0) = 1 - \frac{E[X \wedge h]}{h} \\ f_{\tilde{X}}(jh) = \frac{2E[X \wedge jh] - E[X \wedge (j-1)h] - E[X \wedge (j+1)h]}{h}, \quad \textit{for } j = 1, 2, \ldots \end{cases}$$

This method ensures that the mean of the discretized distribution is the same as the original distribution.

Remark 2.3. *It is shown e.g. in Müller and Stoyan (2002) and Denuit et al. (2005) that* $X \leq_{sd} \tilde{X}$ *under the lower method,* $\tilde{X} \leq_{sd} X$ *under the upper method, and* $X \leq_{icx} \tilde{X}$ *under the mean preserving method, where* $\leq_{sd}$ *and* $\leq_{icx}$ *designate the stochastic dominance order and the increasing convex order respectively (see the same references for the definitions).*

2.5.2 The bivariate case

Suppose that we have two continuous distributed risks X_1 and X_2. The joint cdf F_{X_1,X_2} is defined by a fixed copula C which introduces a dependence structure between the risks. We discretize X_1 and X_2 with one of the three methods described before. We denote by $\tilde{X}_1$, $\tilde{X}_2$ the discretized random variables obtained and keep the same dependence relation between these two new random variables with the copula C. Then, we define $\tilde{S} = \tilde{X}_1 + \tilde{X}_2$ that we use two approximate $S = X_1 + X_2$.

For a constant discretization span h, the cdf of $(\tilde{X}_1, \tilde{X}_2)$ for $k \geq 0$ and $l \geq 0$ is

$$F_{\tilde{X}_1,\tilde{X}_2}(kh, lh) = \sum_{i=0}^{k} \sum_{j=0}^{l} Pr(\tilde{X}_1 = ih, \tilde{X}_2 = jh).$$

The cdf of $\tilde{S}$ for $j \geq 0$ is given by

$$F_{\tilde{S}}(jh) = \sum_{i=0}^{j} Pr(\tilde{S} = ih),$$

where the probability mass function (pmf) of $\tilde{S}$ is

$$Pr(\tilde{S}=0)=Pr(\tilde{X}_1=0;\tilde{X}_2=0),\ \text{and}$$
$$Pr(\tilde{S}=jh)=\sum_{i=0}^{j}Pr(\tilde{X}_1=ih;\tilde{X}_2=(j-i)h),\ \text{for } j=1,2,\ldots$$

The joint pmf of $(\tilde{X}_1,\tilde{X}_2)$ is obtained with the copula as follows

$$\begin{aligned}
Pr(\tilde{X}_1=0,\tilde{X}_2=0) &= C\Big(F_{\tilde{X}_1}(0),F_{\tilde{X}_2}(0)\Big),\\
Pr(\tilde{X}_1=0,\tilde{X}_2=jh) &= C\Big(F_{\tilde{X}_1}(0),F_{\tilde{X}_2}(jh)\Big)-C\Big(F_{\tilde{X}_1}(0),F_{\tilde{X}_2}((j-1)h)\Big),\\
Pr(\tilde{X}_1=ih,\tilde{X}_2=0) &= C\Big(F_{\tilde{X}_1}(ih),F_{\tilde{X}_2}(0)\Big)-C\Big(F_{\tilde{X}_1}((i-1)h),F_{\tilde{X}_2}(0)\Big),\\
Pr(\tilde{X}_1=ih,\tilde{X}_2=jh) &= C\Big(F_{\tilde{X}_1}(ih),F_{\tilde{X}_2}(jh)\Big)-C\Big(F_{\tilde{X}_1}((i-1)h),F_{\tilde{X}_2}(jh)\Big)\\
&\quad -C\Big(F_{\tilde{X}_1}(ih),F_{\tilde{X}_2}((j-1)h)\Big)+C\Big(F_{\tilde{X}_1}((i-1)h),F_{\tilde{X}_2}((j-1)h)\Big).
\end{aligned}$$

Then, the TVaR of S can be approximated by the TVaR of $\tilde{S}$ which is given by

$$\begin{aligned}
TVaR_\kappa(\tilde{S}) &= \frac{E[\tilde{S}\times 1_{\{\tilde{S}>VaR_\kappa(\tilde{S})\}}]+VaR_\kappa(\tilde{S})\Big(Pr(\tilde{S}\le VaR_\kappa(\tilde{S}))-\kappa\Big)}{1-\kappa}\\
&= \frac{E[\tilde{S}\times 1_{\{\tilde{S}>k_0h\}}]+k_0h\Big(Pr(\tilde{S}\le k_0h)-\kappa\Big)}{1-\kappa},
\end{aligned}$$

where $VaR_\kappa(\tilde{S})=k_0h$.

The TVaR-based allocation of risk X_i over the global risk S can be approximated by the TVaR-based allocation of risk $\tilde{X}_i$ over the global discretized risk $\tilde{S}$ where

$$\begin{aligned}
TVaR_\kappa(\tilde{X}_i;\tilde{S}) &= \frac{E\left[\tilde{X}_i\times 1_{\{\tilde{S}>VaR_\kappa(\tilde{S})\}}\right]+\beta_{\tilde{S}}E\left[\tilde{X}_i\times 1_{\{\tilde{S}=VaR_\kappa(\tilde{S})\}}\right]}{1-\kappa}\\
&= \frac{E\left[\tilde{X}_i\times 1_{\{\tilde{S}>k_0h\}}\right]+\beta_{\tilde{S}}E\left[\tilde{X}_i\times 1_{\{\tilde{S}=k_0h\}}\right]}{1-\kappa},
\end{aligned}$$

$$\text{where } \beta_{\tilde{S}}=\begin{cases}\frac{Pr(\tilde{S}\le k_0h)-\kappa}{Pr(\tilde{S}=k_0h)}, & \text{if } Pr\left(\tilde{S}=k_0h\right)>0,\\ 0, \text{otherwise.}\end{cases}$$

Remark 2.4. *In corollary 4.6 of Müller and Scarsini (2001), it is shown that if* (X_1,X_2) *and* $\left(\widetilde{X}_1,\widetilde{X}_2\right)$ *are random vectors with a common conditionally increasing copula and if* $X_i\le_{cx}\widetilde{X}_i$ *for* $i=1,2$*, then for all non-negative scalars* a_1 *and* a_2 *we have*

$$a_1X_1+a_2X_2\le_{cx}a_1\widetilde{X}_1+a_2\widetilde{X}_2.$$

This result also holds for the increasing convex order since it is implied by the convex order. From Bäuerle and Müller (2006), it follows that

$$TVaR_{\kappa}\left(a_1X_1 + a_2X_2\right) \leq TVaR_{\kappa}\left(a_1\widetilde{X}_1 + a_2\widetilde{X}_2\right)$$

for $\kappa \in (0,1)$.

2.5.3 Numerical applications

Bivariate case

Suppose that X_1 and X_2 are exponentially distributed with parameters $\lambda_1 = 1/2$ and $\lambda_2 = 1/3$ respectively. The dependence between these two risks is defined by the bivariate FGM copula with parameter $\theta^{FGM} = 0.8$. This value of θ^{FGM} leads to a correlation of 0.2 between X_1 and X_2. Figure 2.1 in the appendix illustrates the accuracy of the approximations of the cdf of S_2 when using the three discretization methods. Indeed, the decrease of the discretization span h implicates a convergence of the discretized cdf to the real one.

The discretization methods allow an approximation of the cdf of S_2 with any copula. We illustrate the use of the FGM, the Clayton, the Frank and the Gumbel copula. The copulas' parameters are chosen such that we have the same coefficient of correlation between X_1 and X_2. Figure 2.2 in the appendix shows the impact of the copulas on the cdf of S_2 which is discretized with the mean preserving method when the correlation coefficient is fixed at 0.2. The first graph displays the drawing of the approximated cdf's of S_2. We then do the difference between a dependent cdf and the independent cdf and trace this difference for the FGM, the Frank, the Clayton and the Gumbel copula in the second graph of Figure 2.2. These two graphs highlight the fact that the Clayton copula introduces dependence in lower values and that the Gumbel copula permits dependence in the upper queue.

In Figure 2.3 in the appendix, we trace the differences between the TVaR of S_2 using dependent risks with one of the copulas discussed before and the TVaR of S_2 using the independent copula against the risk level κ. This is done for three increasing values of correlation between X_1 and X_2. Given that the FGM copula only allows weak dependence it just appears on the first graph. The graphs confirm the fact that the Gumbel copula introduces dependence in high values.

Tables 2.4 and 2.5 expose the numerical results for the TVaR and the TVaR-based allocation for X_1 and X_2 with the four copulas discussed above and confidence levels equal to 0.99 and 0.995. The calculations are done with the mean preserving discretization method with span $h = 0.05$ for the four copulas and also with the exact expression for the FGM copula. The tables attest the good precision of the approximation method and confirm the high dependence values inserted by the Gumbel copula.

Copula	$TVaR_{0.99}(S_2)$	$TVaR_{0.99}(X_1; S_2)$	% of $TVaR_{0.99}(S_2)$	$TVaR_{0.99}(X_2; S_2)$	% of $TVaR_{0.99}(S_2)$
FGM exact	20.9561	6.0998	29.1%	14.8563	70.9%
FGM M.P.	20.9574	6.1003	29.1%	14.8571	70.9%
Clayton M.P.	20.7918	5.9419	28.6%	14.8499	71.4%
Frank M.P.	21.0612	6.2158	29.5%	14.8454	70.5%
Gumbel M.P.	22.9669	7.7988	34.0%	15.1682	66%

TABLE 2.4 – TVaR and TVaR-based allocation for S_2 with $\kappa = 0.99$.

Copula	$TVaR_{0.995}(S_2)$	$TVaR_{0.995}(X_1; S_2)$	% of $TVaR_{0.995}(S_2)$	$TVaR_{0.995}(X_2; S_2)$	% of $TVaR_{0.995}(S_2)$
FGM exact	23.0839	6.3523	27.5%	16.7316	72.5%
FGM M.P.	23.0859	6.3530	27.5%	16.7329	72.5%
Clayton M.P.	22.9135	6.1776	27.0%	16.7359	73.0%
Frank M.P.	23.2014	6.4953	28.0%	16.7061	72.0%
Gumbel M.P.	26.0088	8.9850	34.5%	17.0237	65.5%

TABLE 2.5 – TVaR and TVaR-based allocation for S_2 with $\kappa = 0.995$.

Trivariate case

We illustrate here the difference between the exact and the approximated methods with three risk variables dependent through a trivariate FGM copula. Suppose that X_1, X_2 and X_3 are exponentially distributed with parameters $\lambda_1 = 1/2$, $\lambda_2 = 1/3$ and $\lambda_3 = 1/5$ respectively. The copula parameters θ_i^{FGM}, θ_{ij}^{FGM} and θ_{ijk}^{FGM}, for $i = 1, 2, 3$, $j = 1, 2, 3$, $k = 1, 2, 3$, $j \neq i$, $k \neq i$ and $k \neq j$ are all fixed to 1. As for the bivariate case, we show in Figure 2.4 of the appendix the convergence to the real cdf of S_3 of the discretized cdf's when using the three discretization methods.

Tables 2.6 and 2.7 compare the numerical results for the TVaR and the TVaR-based allocation for the three risks with confidence levels equal to 0.99 and 0.995 between exact expressions and approximated results using the mean preserving discretization method with span $h = 0.3$. As for the bivariate case, they show a satisfying accuracy of the approximation method.

Method	$TVaR_{0.99}(S_3)$	$TVaR_{0.99}(X_1; S_3)$	$TVaR_{0.99}(X_2; S_3)$	$TVaR_{0.99}(X_3; S_3)$
Exact	37.5988	4.1726	8.4033	25.0230
Mean P.	37.6062	4.1760	8.4060	25.0243

TABLE 2.6 – TVaR and TVaR-based allocation for S_3 with $\kappa = 0.99$.

Method	$TVaR_{0.995}(S_3)$	$TVaR_{0.995}(X_1; S_3)$	$TVaR_{0.995}(X_2; S_3)$	$TVaR_{0.995}(X_3; S_3)$
Exact	41.1177	4.2044	8.6536	28.2597
Mean P.	41.1262	4.2080	8.6567	28.2616

TABLE 2.7 – TVaR and TVaR-based allocation for S_3 with $\kappa = 0.995$.

2.6 Conclusion

This paper introduces the use of copulas in TVaR-based capital allocation. We obtain explicit expressions for the TVaR and TVaR-based allocation for risks that have exponential and mixture of exponentials distributions linked by a FGM copula. The handy form of this copula permits a direct calculation of the coherent risk measure and its decomposition when we suppose only two different risks. In the multivariate situation, we use divided differences as in Chiragiev and Landsman (2007). For other copulas, we present approximations for the TVaR and the TVaR-based allocation using three discretization methods for continuous distributions.

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APPENDIX

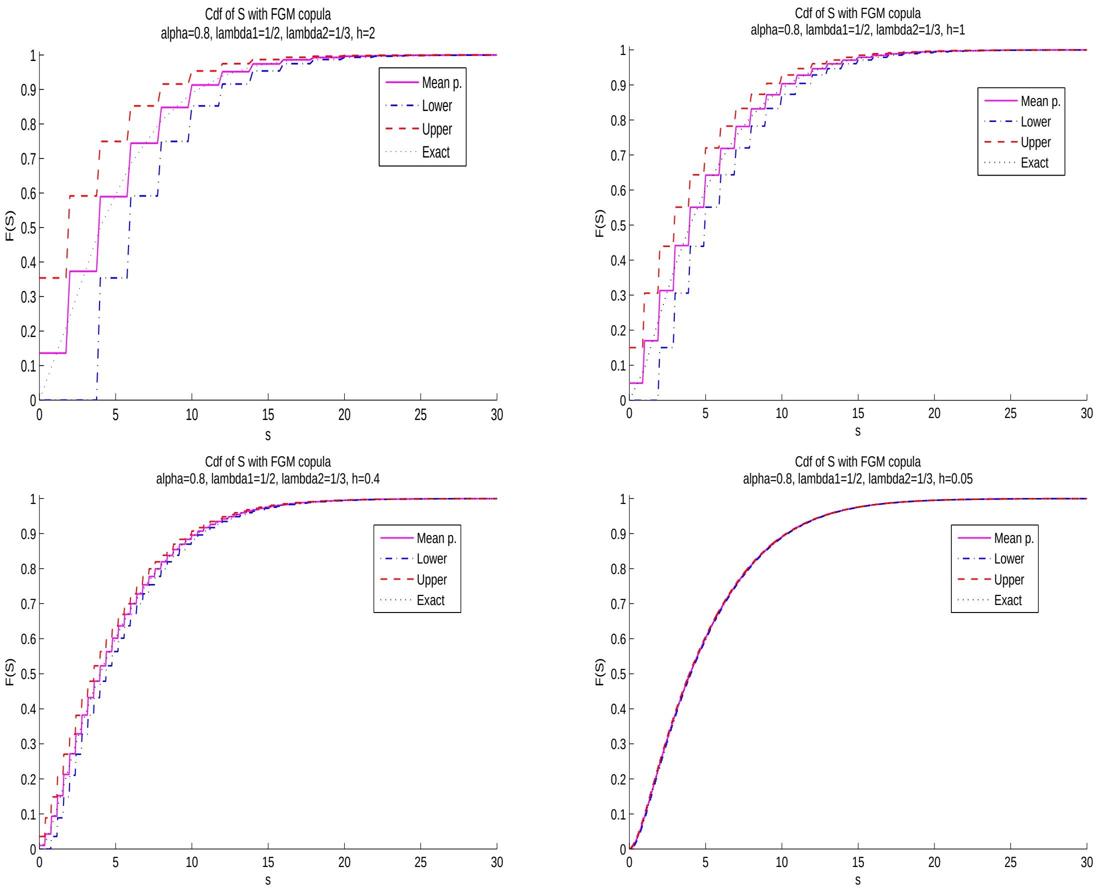

FIGURE 2.1 – Discretized cdf's vs exact cdf for 2 risks

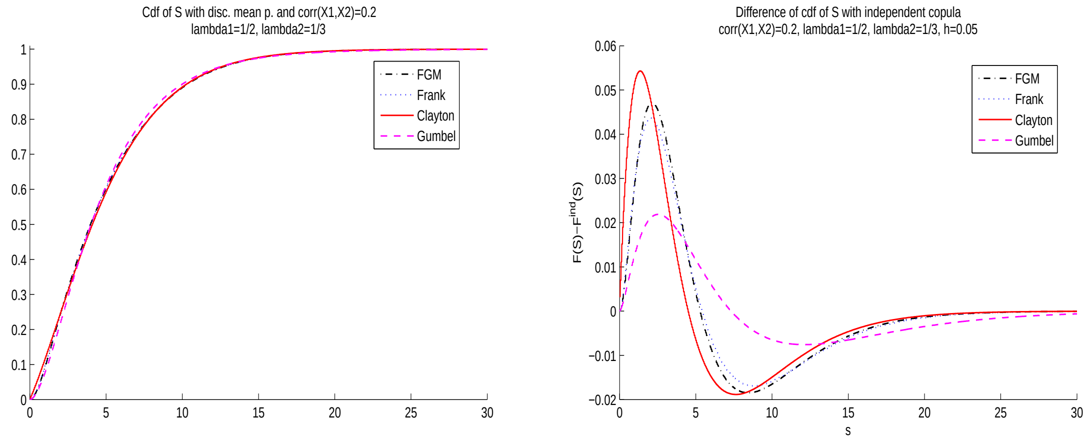

FIGURE 2.2 – Comparison of copulas with $Corr(X_1, X_2) = 0.2$

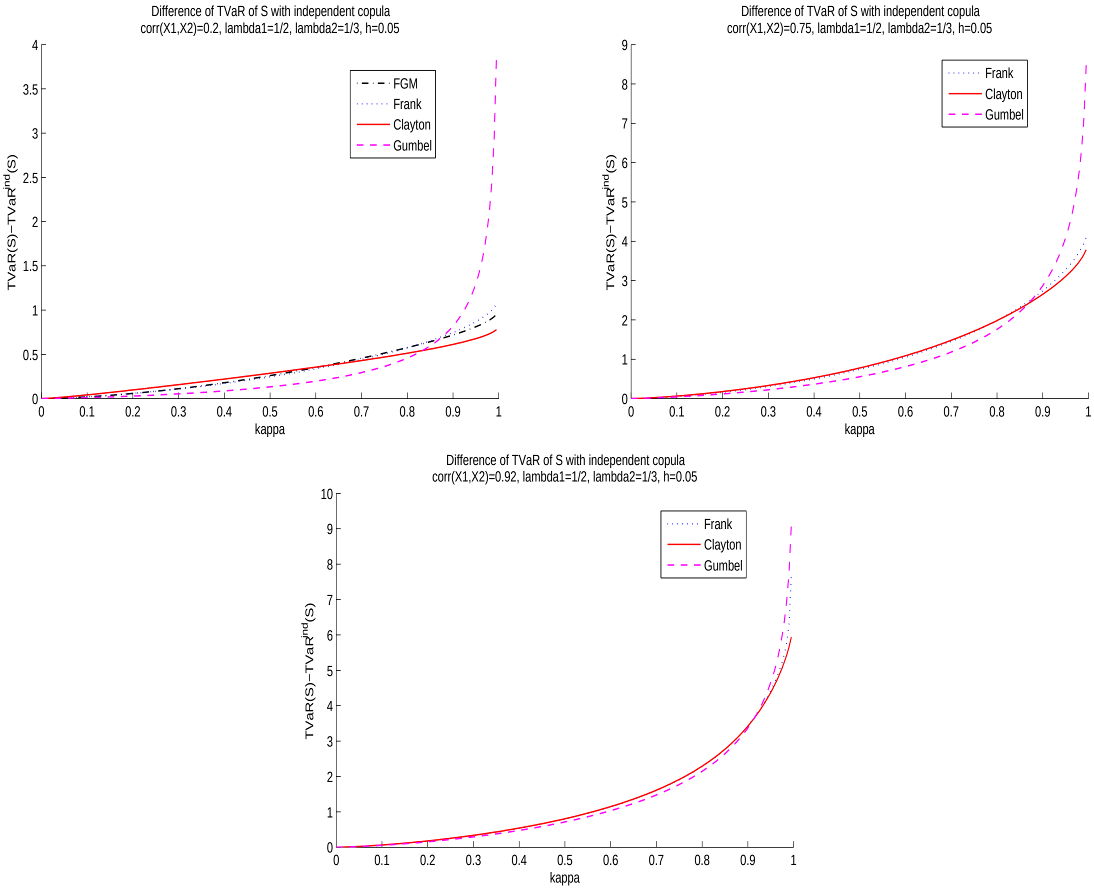

FIGURE 2.3 – TVaR of S with different copulas and correlation coefficients

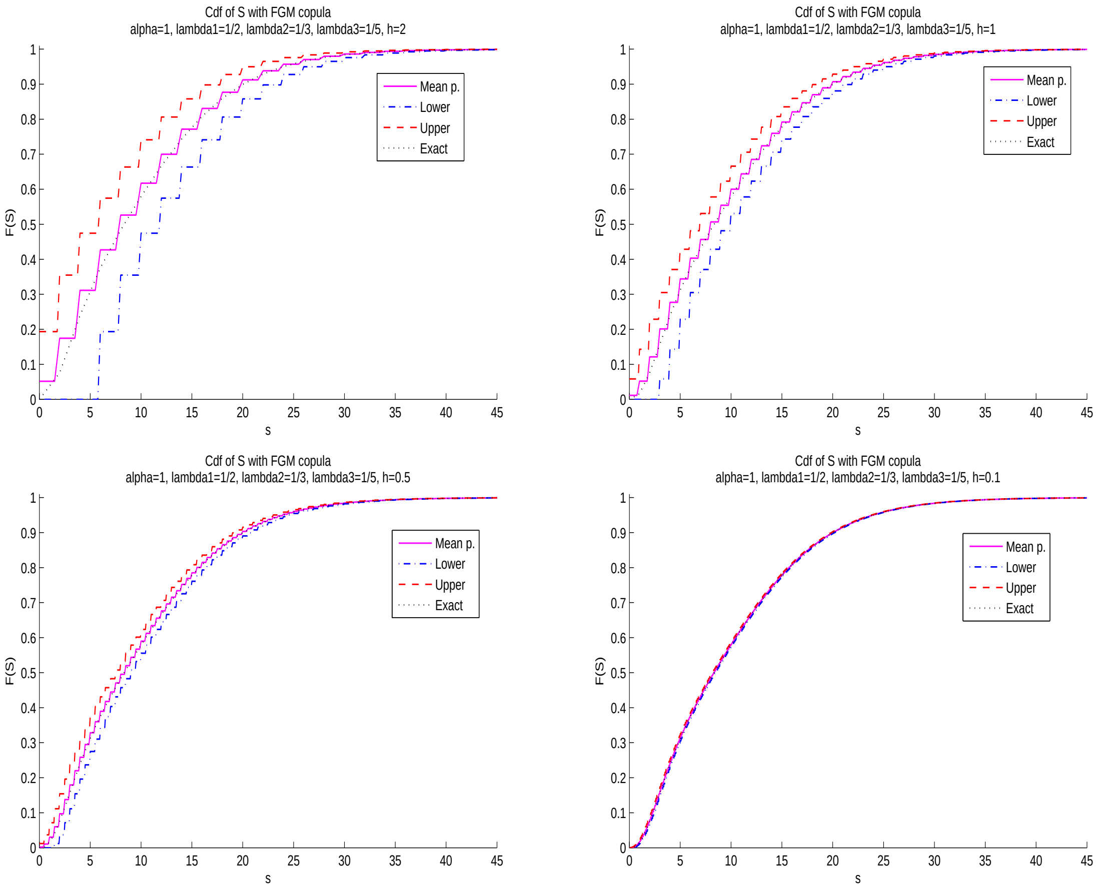

FIGURE 2.4 – Discretized cdf's vs exact cdf for 3 risks

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Chapitre 3

Moments de la somme des valeurs présentes des sinistres avec dépendance introduite par la copule FGM

Résumé

Dans ce chapitre, nous nous intéressons à la distribution de la somme des valeurs présentes des sinistres pour un processus de dénombrement de sinistres Poisson homogène. Nous introduisons une relation de dépendance entre le montant d'un sinistre et le temps depuis le sinistre précédent par une copule de Farlie-Gumbel-Morgenstern. Sous certaines conditions pour les valeurs du taux d'intérêt net instantané, nous donnons une expression pour le premier et le second moment du processus étudié puis nous obtenons une forme générale pour le moment d'ordre m. Ces résultats sont illustrés par des applications au calcul de prime, à une méthode d'ajustement par les moments de la distribution du processus ainsi qu'à l'étude de scenarios de crise d'inflation dans le contexte de Solvabilité II.

3.1 Introduction

We consider a continuous-time compound renewal risk model for an insurance portfolio and we define the compound process of the discounted claims $e^{-\delta T_i} X_i$, $i = 1, 2, \ldots$ occurring at time T_i, $i = 1, 2, \ldots$ by $\underline{Z} = \{Z(t), t \geq 0\}$ with

$$Z(t) = \begin{cases} \sum_{i=1}^{N(t)} e^{-\delta T_i} X_i, & N(t) > 0 \\ 0, & N(t) = 0, \end{cases}$$

where $\underline{N} = \{N(t), t \geq 0\}$ is an homogeneous Poisson counting process and δ is the instantaneous rate of net interest. In actuarial risk theory, it has been assumed that the claim amounts X_i, $i = 1, 2, \ldots$ are independent and identically distributed (i.i.d.) random variables (r.v.'s) and the interclaim times $W_1 = T_1$ and $W_j = T_j - T_{j-1}$, $j = 2, 3, \ldots$ are also i.i.d. r.v.'s. The r.v.'s X_i and W_i, $i = 1, 2, \ldots$ are classically supposed independent. This last assumption also implies that $X_i, i = 1, 2, \ldots$ are independent from $\underline{N}$. This risk process has been used in ruin theory by many authors such as Taylor (1979), Waters (1983), Delbaen and Haezendonck (1987), Willmot (1989), Sundt and Teugels (1995) and more recently Kalashnikov and Konstantinides (2000), Yang and Zhang (2001) and Tang (2005). They mainly focused on the ruin probability and related ruin measures.

Only a few recent works deal with the distribution of the aggregate discounted claims $Z(t)$. Léveillé and Garrido (2001a) provide the first two moments of this process. These first two moments were also obtained in Jang (2004) using martingale theory. This result has since been generalized by relaxing some of the classical assumptions presented above. Léveillé and Garrido (2001b) and Léveillé et al. (2009) derived recursive formulas for all the moments of the aggregate discounted claims considering a compound renewal process where $\underline{N}$ is not necessarily a Poisson process. In Jang (2007), the Laplace transform of the distribution of a jump diffusion process and its integrated process is derived and used to obtain the moments of the compound Poisson process $Z(t)$. Kim and Kim (2007) and Ren (2008) studied the discounted aggregate claims in a Markovian environment which modulates the distributions of the interclaim times and claim sizes for the former and the distribution of the interclaim times for the latter. They both provided the Laplace transform of the distribution of the discounted aggregate claims and then gave expressions for its first two moments.

The aggregation of discounted random variables is also used in many other fields of application. For example, it can be used in warranty cost modeling, see Duchesne and Marri (2009), or in reliability in civil engineering, see van Noortwijk and Frangopol (2004) or Porter et al. (2004).

In this paper, we want to introduce some dependence between the interclaim times and the subsequent claim amounts. There exists different ways to take this dependence assumption into account as using copulas or a Markovian environment. In risk theory, this dependence between interclaim times and claim amounts has already been explored as in Albrecher and

Boxma (2004) where it is supposed that if a claim amount exceeds a certain threshold, then the parameters of the distribution of the next interclaim time is modified. In Albrecher and Teugels (2006) the dependence is introduced with the use of an arbitrary copula. Conversely to Albrecher and Boxma (2004), Boudreault et al. (2006) assumed that if an interclaim time is greater than a certain threshold then the parameters of the distribution of the next claim amount is modified. In a similar dependence model, but with more freedom in the choice of the copula between each interclaim time and the subsequent claim amount, Asimit and Badescu (2009) consider a constant force of interest and heavy-tailed claim amounts. Dependence concepts used in Boudreault et al. (2006) were then extended in Biard et al. (2009) where they suppose that the distribution of a claim amount has its parameters modified when several preceding interclaim times are all greater or all lower than a certain threshold. All these papers were interested in finding exact expressions or approximations for some ruin measures such as the ruin probability or the Gerber-Shiu function.

In our study, the assumption of independence between the claim amount X_j and the interclaim time W_j is relaxed to allow $\{(X_j, W_j), j \in \mathbb{N}^+\}$ to form a sequence of i.i.d. random vectors distributed as the canonical random vector (X, W) in which the components may be dependent. We follow the idea of Albrecher and Teugels (2006) supposing that dependence between an interclaim time and its subsequent claim amount is modelized by a copula. More specifically, we use the Farlie-Gumbel-Morgenstern (FGM) copula which is defined by

$$C_\theta^{FGM}(u, v) = uv + \theta uv (1-u)(1-v), \tag{3.1}$$

for $(u, v) \in [0, 1] \times [0, 1]$ and where the dependence parameter θ takes value in $[-1, 1]$. While there are a large number of copula families, we choose the FGM copula because it offers the advantage of being mathematically tractable as illustrated in Cossette et al. (2009). Even if the FGM copula introduces only light dependence, it admits positive as well as negative dependence between a set of random variables and includes the independence copula when $\theta = 0$. It is also known that the FGM copula is a Taylor approximation of order one of the Frank copula (see Nelsen (2006), page 133), Ali-Milkhail-Haq copula and Plackett copula (see Nelsen (2006), page 100).

The paper is structured as follows. In the second section, we present the model of the continuous time compound Poisson risk process that we use and give some notations. The first moment, the second moment and then a generalization to the mth moment are derived in Section 3. Applications to premium calculation, moment matching methods and Solvency II are given in the fourth section. In particular, we show how our method may be used to determine Solvency Capital Requirements and to perform part of Own Risk and Solvency Assessment (ORSA) analysis in Solvency II for some cat risks and inflation risk.

3.2 The model

Assuming that $E[X_i^m] < \infty$ for $i = 1, 2, \ldots$ and $m \in \mathbb{N}$, we introduce a specific structure of dependence based on the Farlie-Gumbel-Morgenstern copula between the ith claim amount and the ith interclaim time. Using (3.1), the joint cumulative distribution function (c.d.f.) for the canonical random vector (X, W) is

$$\begin{aligned} F_{X,W}(x,t) &= C\left(F_X(x), F_W(t)\right) \\ &= F_X(x) F_W(t) + \theta F_X(x) F_W(t) \left(1 - F_X(x)\right)\left(1 - F_W(t)\right), \end{aligned}$$

for $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and where F_X and F_W are the marginals of respectively X and W. This dependence relation implies that $X_1, X_2, X_3, \ldots$ are no more independent of $\underline{N}$. Recalling the density of the FGM copula

$$c_\theta^{FGM}(u,v) = 1 + \theta (1 - 2u)(1 - 2v),$$

for $(u, v) \in [0, 1] \times [0, 1]$, the joint probability density function (p.d.f.) of (X, W) is

$$\begin{aligned} f_{X,W}(x,t) &= c_\theta^{FGM}\left(F_X(x), F_W(t)\right) f_X(x) f_W(t) \\ &= f_X(x) f_W(t) + \theta f_X(x) f_W(t) \left(1 - 2F_X(x)\right)\left(1 - 2F_W(t)\right), \end{aligned}$$

where f_X and f_W are the p.d.f.'s of respectively X and W.

The mth moment of $Z(t)$ is denoted by $\mu_Z^{(m)}(t) = E\left[Z^{(m)}(t)\right]$ and its Laplace transform by $\widetilde{\mu}_Z^{(m)}(r)$. We see in the next section how to derive explicit formulas for these moments.

3.3 Moments of the aggregate discounted claims

3.3.1 First moment

To derive the expression for the first moment $\mu_Z(t)$ of $Z(t)$, we assume that $E[X] < \infty$, that $W \sim Exp(\beta)$ and that $\delta > -\beta$. Conditioning on the arrival of the first claim gives

$$\begin{aligned} \mu_Z(t) &= E\left[Z(t)\right] \\ &= E\left[E\left[e^{-\delta s} X_1 + e^{-\delta s} Z(t-s) | W_1 = s\right]\right] \\ &= \int_0^t f_W(s) e^{-\delta s} E\left[X | W = s\right] ds + \int_0^t f_W(s) e^{-\delta s} \mu_Z(t-s)\, ds, \end{aligned}$$

where

$$\begin{aligned}
E\left[X|W=s\right] &= \int_0^\infty x f_{X|W=s}\left(x\right)dx \\
&= \int_0^\infty x\left\{\left(1+\theta\left(1-2F_X\left(x\right)\right)\left(1-2F_W\left(s\right)\right)\right)\right\} f_X\left(x\right)dx \\
&= E\left[X\right]+\theta\int_0^\infty x\left(2-2F_X\left(x\right)\right)\left(1-2F_W\left(s\right)\right) f_X\left(x\right)dx \\
&\quad -\theta\int_0^\infty x\left(1-2F_W\left(s\right)\right) f_X\left(x\right)dx \\
&= E\left[X\right]\left(1-\theta\left(1-2F_W\left(s\right)\right)\right)+\theta\left(1-2F_W\left(s\right)\right)\int_0^\infty \left(1-F_X\left(x\right)\right)^2 dx. \qquad (3.2)
\end{aligned}$$

Let X' be the random variable having $(1-F_X(x))^2$ as survival function such that

$$E\left[X'\right] = \int_0^\infty \left(1-F_X\left(x\right)\right)^2 dx < \int_0^\infty \left(1-F_X\left(x\right)\right)dx = E\left[X\right].$$

Then (3.2) becomes

$$E\left[X\right]+\left(E\left[X'\right]-E\left[X\right]\right)\theta\left(1-2F_W\left(s\right)\right). \qquad (3.3)$$

From (3.3), we can derive the following remarks. If $\theta>0$ ($\theta<0$) and $s<F_W^{-1}\left(0.5\right)$ ($s>F_W^{-1}\left(0.5\right)$, respectively), then $E\left[X|W=s\right]<E\left[X\right]$. Conversely, if $\theta>0$ ($\theta<0$) and $s>F_W^{-1}\left(0.5\right)$ ($s<F_W^{-1}\left(0.5\right)$, respectively), then $E\left[X|W=s\right]>E\left[X\right]$.

Considering that W has an exponential distribution with mean $\frac{1}{\beta}$, we have

$$\begin{aligned}
h\left(t;\beta\right) &= f_W\left(t\right)=\beta e^{-\beta t}, \qquad (3.4)\\
F_W\left(t\right) &= 1-e^{-\beta t}, \qquad (3.5)\\
\widetilde{h}\left(t;\beta\right) &= E\left[e^{-tW}\right]=\frac{\beta}{\beta+t},
\end{aligned}$$

where the notation $h\left(t;\beta\right)$ is introduced for simplification purposes in order to derive the moments of $Z(t)$.

We obtain the following expression for $\mu_Z(t)$

$$\begin{aligned}
\mu_Z(t) &= \int_0^t f_W(s)\, e^{-\delta s} E[X]\, ds + \theta\left(E[X'] - E[X]\right) \int_0^t f_W(s)\, e^{-\delta s} \left(1 - 2F_W(s)\right) ds \\
&\quad + \int_0^t f_W(s)\, e^{-\delta s} \mu_Z(t-s)\, ds \\
&= \int_0^t \beta e^{-\beta s} e^{-\delta s} E[X]\, ds + \theta\left(E[X'] - E[X]\right) \int_0^t \beta e^{-\beta s} e^{-\delta s} \left(2e^{-\beta s} - 1\right) ds \\
&\quad + \int_0^t \beta e^{-\beta s} e^{-\delta s} \mu_Z(t-s)\, ds \\
&= \int_0^t \frac{\beta}{\beta+\delta} h(s;\beta+\delta)\, E[X]\, ds \\
&\quad + \theta\left(E[X'] - E[X]\right) \int_0^t \frac{2\beta}{2\beta+\delta} h(s;2\beta+\delta)\, ds \\
&\quad - \theta\left(E[X'] - E[X]\right) \int_0^t \frac{\beta}{\beta+\delta} h(s;\beta+\delta)\, ds \\
&\quad + \int_0^t \frac{\beta}{\beta+\delta} h(s;\beta+\delta)\, \mu_Z(t-s)\, ds. \qquad (3.6)
\end{aligned}$$

We take the Laplace transform on both sides of (3.6) and after some rearrangements, we obtain

$$\widetilde{\mu}_Z(r) = \frac{\frac{\widetilde{h}(r;\beta+\delta)}{r} \frac{\beta}{\beta+\delta} E[X] + \theta\left(E[X'] - E[X]\right)\left(\frac{2\beta}{2\beta+\delta} \frac{\widetilde{h}(r;\beta+\delta)}{r} - \frac{\beta}{\beta+\delta} \frac{\widetilde{h}(r;\beta+\delta)}{r}\right)}{1 - \frac{\beta}{\beta+\delta} \widetilde{h}(r;\beta+\delta)}, \qquad (3.7)$$

which is equivalent to

$$\widetilde{\mu}_Z(r) = \frac{\frac{1}{r} \frac{\beta+\delta}{\beta+\delta+r} \frac{\beta}{\beta+\delta} E[X] + \theta\left(E[X'] - E[X]\right)\left(\frac{2\beta}{2\beta+\delta} \frac{1}{r} \frac{2\beta+\delta}{2\beta+\delta+r} - \frac{\beta}{\beta+\delta} \frac{1}{r} \frac{\beta+\delta}{\beta+\delta+r}\right)}{1 - \frac{\beta}{\beta+\delta} \frac{\beta+\delta}{\beta+\delta+r}}. \qquad (3.8)$$

Rearranging (3.8), we deduce

$$\widetilde{\mu}_Z(r) = \frac{\beta E[X]}{r(\delta+r)} + \theta \frac{\beta\left(E[X'] - E[X]\right)}{r(2\beta+\delta+r)}. \qquad (3.9)$$

Inverting (3.9), we obtain

$$\mu_Z(t) = \beta E[X] \frac{1 - e^{-\delta t}}{\delta} + \theta\beta\left(E[X'] - E[X]\right) \frac{1 - e^{-(2\beta+\delta)t}}{2\beta+\delta}. \qquad (3.10)$$

Notice that when the r.v.'s X and W are independent which corresponds to $\theta = 0$, the expected value of the compound process of the discounted claims, denoted $Z_{ind}(t)$, becomes

$$\mu_{Z_{ind}}(t) = \beta E[X] \frac{1 - e^{-\delta t}}{\delta}.$$

3.3.2 Second moment

We suppose now that $E[X^i] < \infty$, for $i = 1, 2$, that $\delta > -\beta/2$ and that $\delta \neq 2\beta$. The following method can be used to find the second moment of $Z(t)$ when $\delta = 2\beta$ but we focus here on the more general case where $\delta \neq 2\beta$. As for the first moment of the discounted total claim amount, we condition on the arrival of the first claim to obtain the second moment of $Z(t)$

$$\begin{aligned}\mu_Z^{(2)}(t) &= E\left[E\left[(e^{-\delta s}X_1 + e^{-\delta s}Z(t-s))^2 | W_1 = s\right]\right] \\ &= \int_0^t f_W(s)\, e^{-2\delta s} E\left[X^2 | W = s\right] ds + 2\int_0^t f_W(s)\, e^{-2\delta s} E\left[X | W = s\right] \mu_Z(t-s)\, ds \\ &\quad + \int_0^t f_W(s)\, e^{-2\delta s} \mu_Z^{(2)}(t-s)\, ds.\end{aligned}$$

Similarly as in (3.2), we have

$$\begin{aligned}E\left[X^2 | W = s\right] &= E\left[X^2\right](1 - \theta(1 - 2F_W(s))) + \theta(1 - 2F_W(s)) \int_0^\infty 2x(1 - F_X(x))^2\, dx \\ &= E\left[X^2\right] + \left(E\left[\left(X'\right)^2\right] - E\left[X^2\right]\right)\theta(1 - 2F_W(s)),\end{aligned}$$

where

$$E\left[(X')^2\right] = \int_0^\infty 2x(1 - F_X(x))^2\, dx < \int_0^\infty 2x(1 - F_X(x))\, dx = E\left[X^2\right].$$

We find the following expression for $\mu_Z^{(2)}(t)$

$$\begin{aligned}\mu_Z^{(2)}(t) &= \int_0^t f_W(s)\, e^{-2\delta s} E\left[X^2\right] ds + \theta\left(E\left[(X')^2\right] - E\left[X^2\right]\right) \int_0^t f_W(s)\, e^{-2\delta s}(1 - 2F_W(s))\, ds \\ &\quad + 2\int_0^t f_W(s)\, e^{-2\delta s} E[X]\, \mu_Z(t-s)\, dsds \\ &\quad + 2\theta\left(E[(X')] - E[X]\right) \int_0^t f_W(s)\, e^{-2\delta s}(1 - 2F_W(s))\, \mu_Z(t-s)\, ds \\ &\quad + \int_0^t f_W(s)\, e^{-2\delta s} \mu_Z^{(2)}(t-s)\, ds \\ &= \int_0^t \frac{\beta}{\beta + 2\delta} h(s; \beta + 2\delta)\, E\left[X^2\right] ds \\ &\quad + \theta\left(E\left[(X')^2\right] - E\left[X^2\right]\right) \int_0^t \left(\frac{2\beta}{2\beta + 2\delta} h(s; 2\beta + 2\delta) - \frac{\beta}{\beta + 2\delta} h(s; \beta + 2\delta)\right) ds \\ &\quad + 2\int_0^t \frac{\beta}{\beta + 2\delta} h(s; \beta + 2\delta)\, E[X]\, \mu_Z(t-s)\, ds \\ &\quad + 2\theta\left(E[(X')] - E[X]\right) \int_0^t \left(\frac{2\beta}{2\beta + 2\delta} h(s; 2\beta + 2\delta) - \frac{\beta}{\beta + 2\delta} h(s; \beta + 2\delta)\right) \mu_Z(t-s)\, ds \\ &\quad + \int_0^t \frac{\beta}{\beta + 2\delta} h(s; \beta + 2\delta)\, \mu_Z^{(2)}(t-s)\, ds. \qquad (3.11)\end{aligned}$$

We take the Laplace transform on both sides of (3.11) and after some rearrangements, we obtain

$$\tilde{\mu}_Z^{(2)}(r) = \frac{1}{1-\frac{\beta}{\beta+2\delta}\widetilde{h}(r;\beta+2\delta)}\Bigg[\frac{\widetilde{h}(r;\beta+2\delta)}{r}\frac{\beta}{\beta+2\delta}E[X^2]$$
$$+\theta\left(E[(X'^2]-E[X^2]\right)\left(\frac{2\beta}{2\beta+2\delta}\frac{\widetilde{h}(r;2\beta+2\delta)}{r}-\frac{\beta}{\beta+2\delta}\frac{\widetilde{h}(r;\beta+2\delta)}{r}\right)$$
$$+2E[X]\frac{\beta}{\beta+2\delta}\widetilde{h}(r;\beta+2\delta)\tilde{\mu}_Z(r)$$
$$+2\theta\left(E\left[X'\right]-E[X]\right)\left(\frac{2\beta}{2\beta+2\delta}\widetilde{h}(r;2\beta+2\delta)-\frac{\beta}{\beta+2\delta}\widetilde{h}(r;\beta+2\delta)\right)\tilde{\mu}_Z(r)\Bigg],$$

which becomes

$$\begin{aligned}
\tilde{\mu}_Z^{(2)}(r) &= \frac{\beta E[X^2]}{r(2\delta+r)}+\theta\frac{\beta\left(E\left[(X'^2\right]-E[X^2]\right)}{r(2\beta+2\delta+r)}+2\frac{\beta E[X]}{2\delta+r}\tilde{\mu}_Z(r)+2\theta\frac{\beta\left(E[X']-E[X]\right)}{2\beta+2\delta+r}\tilde{\mu}_Z(r)\\
&= \frac{\beta E[X^2]}{r(2\delta+r)}+\theta\frac{\beta\left(E\left[(X'^2\right]-E[X^2]\right)}{r(2\beta+2\delta+r)}+2\frac{\beta E[X]}{2\delta+r}\left(\frac{\beta E[X]}{r(\delta+r)}+\theta\frac{\beta\left(E[X']-E[X]\right)}{r(2\beta+\delta+r)}\right)\\
&\quad+2\theta\frac{\beta\left(E[X']-E[X]\right)}{2\beta+2\delta+r}\left(\frac{\beta E[X]}{r(\delta+r)}+\theta\frac{\beta\left(E[X']-E[X]\right)}{r(2\beta+\delta+r)}\right)\\
&= \frac{\beta E[X^2]}{r(2\delta+r)}+\theta\frac{\beta\left(E\left[(X'^2\right]-E[X^2]\right)}{r(2\beta+2\delta+r)}+2\frac{\beta^2E[X]^2}{r(\delta+r)(2\delta+r)}+2\theta\frac{\beta^2E[X]\left(E\left[X'\right]-E[X]\right)}{r(2\beta+\delta+r)(2\delta+r)}\\
&\quad+2\theta\frac{\beta^2E[X]\left(E\left[X'\right]-E[X]\right)}{r(\delta+r)(2\beta+2\delta+r)}+2\theta^2\frac{\beta^2\left(E\left[X'\right]-E[X]\right)^2}{r(2\beta+\delta+r)(2\beta+2\delta+r)}.
\end{aligned} \tag{3.12}$$

This last Laplace transform is a combination of terms of the form

$$\tilde{f}(r) = \frac{1}{r(\alpha_1+r)(\alpha_2+r)\cdots(\alpha_n+r)},$$

with f a function defined for all non-negative real numbers. As described in the proof of Theorem 1.1 in Baeumer (2003), each of these terms can be expressed as a combination of partial fractions such as

$$\tilde{f}(r) = \gamma_0\frac{1}{r}+\gamma_1\frac{1}{\alpha_1+r}+\gamma_2\frac{1}{\alpha_2+r}+\cdots+\gamma_n\frac{1}{\alpha_n+r}, \tag{3.13}$$

where $\gamma_0 = \frac{1}{\alpha_1\cdots\alpha_n}$ and , for $i = 1,\ldots,n$,

$$\gamma_i = -\frac{1}{\alpha_i}\prod_{j=1;j\neq i}^{n}\frac{1}{\alpha_j-\alpha_i}. \tag{3.14}$$

Since the inverse Laplace transform of $\frac{1}{\alpha_i+r}$ is $e^{-\alpha_i t}$, it is easy to inverse $\tilde{f}$ and obtain

$$f(t) = \gamma_0+\gamma_1e^{-\alpha_1 t}+\gamma_2e^{-\alpha_2 t}+\cdots+\gamma_ne^{-\alpha_2 t}. \tag{3.15}$$

Using (3.15) in (3.12), it follows that

$$\mu^{(2)}(t) = \beta E[X^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right) + \theta\beta\left(E\left[(X'^2\right] - E[X^2]\right)\left(\frac{1-e^{-(2\beta+2\delta)t}}{2\beta+2\delta}\right)$$
$$+ 2\beta^2 E[X]^2\left(\frac{1}{2\delta^2} - \frac{e^{-\delta t}}{\delta^2} + \frac{e^{-2\delta t}}{2\delta^2}\right)$$
$$+ 2\theta\beta^2 E[X]\left(E[X'] - E[X]\right)\left(\frac{1}{2\delta(2\beta+\delta)} - \frac{e^{-(2\beta+\delta)t}}{(2\beta+\delta)(-2\beta+\delta)} + \frac{e^{-2\delta t}}{2\delta(-2\beta+\delta)}\right)$$
$$+ 2\theta\beta^2 E[X]\left(E[X'] - E[X]\right)\left(\frac{1}{\delta(2\beta+2\delta)} - \frac{e^{-\delta t}}{\delta(2\beta+\delta)} + \frac{e^{-(2\beta+2\delta)t}}{(2\beta+2\delta)(2\beta+\delta)}\right)$$
$$+ 2\theta^2\beta^2\left(E[X'] - E[X]\right)^2\left(\frac{1}{(2\beta+\delta)(2\beta+2\delta)} - \frac{e^{-(2\beta+\delta)t}}{\delta(2\beta+\delta)} + \frac{e^{-(2\beta+2\delta)}}{\delta(2\beta+2\delta)}\right). \tag{3.16}$$

3.3.3 mth moment

We now generalize the previous results to the mth moment of the discounted total claim amount. We suppose that $E[X^i] < \infty$ for $i = 1, \cdots, m$, that $\delta > -\beta/m$ and that $\delta \neq 2\beta/n$ for $n = 1, \cdots, m-1$. As for the second moment, we deal with the more general situation but the following method can be applied when considering some equalities in the last assumptions. Conditioning on the arrival of the first claim leads to

$$\mu_Z^{(m)}(t) = \int_0^t f_W(s)\, e^{-m\delta s} E\left[X^m | W = s\right] ds$$
$$+ \sum_{j=1}^{m-1}\binom{m}{j}\int_0^t f_W(s)\, e^{-m\delta s} E\left[X^j | W = s\right]\mu_Z^{(m-j)}(t-s)\, ds$$
$$+ \int_0^t f_W(s)\, e^{-m\delta s}\mu_Z^{(m)}(t-s)\, ds.$$

With $E\left[(X')^j\right] = \int_0^\infty jx^{j-1}\left(1-F_X(x)\right)^2 dx$ where for applications we need to have

$$\lim_{x\to 0} jx^{j-1}\left(1-F_X(x)\right)^2 < \infty \text{ and} \tag{3.17}$$
$$\lim_{x\to \infty} jx^{j-1}\left(1-F_X(x)\right)^2 < \infty, \tag{3.18}$$

the Laplace transform of $\mu_Z^{(m)}(t)$ is given by

$$\tilde{\mu}_Z^{(m)}(r) = \frac{1}{1-\frac{\beta}{\beta+m\delta}\widetilde{h}(r;\beta+m\delta)}\left[\frac{\widetilde{h}(r;\beta+m\delta)}{r}\frac{\beta}{\beta+m\delta}E\left[X^m\right]\right.$$
$$+\theta\left(E\left[(X')^m\right]-E\left[X^m\right]\right)\left(\frac{2\beta}{2\beta+m\delta}\frac{\widetilde{h}(r;2\beta+m\delta)}{r}-\frac{\beta}{\beta+m\delta}\frac{\widetilde{h}(r;\beta+m\delta)}{r}\right)$$
$$+\sum_{j=1}^{m-1}\binom{m}{j}E\left[X^j\right]\frac{\beta}{\beta+m\delta}\widetilde{h}(r;\beta+m\delta)\,\tilde{\mu}_Z^{(m-j)}(r)+\theta\sum_{j=1}^{m-1}\binom{m}{j}\left(E\left[(X')^j\right]-E\left[X^j\right]\right)$$
$$\left.\times\left(\frac{2\beta}{2\beta+m\delta}\widetilde{h}(r;2\beta+m\delta)-\frac{\beta}{\beta+m\delta}\widetilde{h}(r;\beta+m\delta)\right)\tilde{\mu}_Z^{(m-j)}(r)\right], \tag{3.19}$$

which can also be expressed as follows

$$\tilde{\mu}_Z^{(m)}(r) = \binom{m}{m}\frac{\beta E[X^m]}{r(m\delta+r)}+\binom{m}{m}\theta\frac{\beta\left(E\left[(X'^m\right]-E[X^m]\right)}{r(2\beta+m\delta+r)}+\sum_{j=1}^{m-1}\binom{m}{j}\frac{\beta E[X^j]}{m\delta+r}\tilde{\mu}_Z^{(m-j)}(r)$$
$$+\theta\sum_{j=1}^{m-1}\binom{m}{j}\frac{\beta\left(E[(X'^j]-E[X^j]\right)}{2\beta+m\delta+r}\tilde{\mu}_Z^{(m-j)}(r).$$

Noting for $i=1,\ldots,m$, $j=1,\ldots,m$ and $k=0,1$

$$\zeta(i;j;k) = \binom{i}{j}\theta^k\frac{\beta\left(E[X^j]\right)^{1-k}\left(E[X'^j]-E[X^j]\right)^k}{k\times 2\beta+i\delta+r}, \tag{3.20}$$

we can rewrite $\tilde{\mu}_Z(r)$ and $\tilde{\mu}_Z^{(2)}(r)$ as

$$\tilde{\mu}_Z(r) = \frac{1}{r}\Big[\zeta(1,1,0)+\zeta(1,1,1)\Big],$$

$$\tilde{\mu}_Z^{(2)}(r) = \frac{1}{r}\Big[\zeta(2,2,0)+\zeta(2,2,1)+[\zeta(2,1,0)+\zeta(2,1,1)]\,[\zeta(1,1,0)+\zeta(1,1,1)]\Big]$$
$$= \frac{1}{r}\Big[\zeta(2,2,0)+\zeta(2,2,1)+\zeta(2,1,0)\zeta(1,1,0)+\zeta(2,1,0)\zeta(1,1,1)+\zeta(2,1,1)\zeta(1,1,0)$$
$$+\zeta(2,1,1)\zeta(1,1,1)\Big].$$

The term $\tilde{\mu}_Z^{(m)}(r)$ can also be expressed using (3.20)

$$\tilde{\mu}_Z^{(m)}(r) = \frac{1}{r}\sum_{n=1}^{m}\sum_{((i_1,j_1,k_1),\ldots,(i_n,j_n,k_n))\in\mathcal{A}_{mn}}\zeta(i_1,j_1,k_1)\times\cdots\times\zeta(i_n,j_n,k_n), \tag{3.21}$$

where $A_{mn} = \Big\{(i_1,j_1,k_1),\ldots,(i_n,j_n,k_n)\,;\, i_1=m,\, i_1>\cdots>i_n,\, j_n=i_n,\, j_1+\cdots+j_n=m,$
$0<j_.\leq n,\, k_.\in\{0,1\}\Big\}.$

To invert (3.21), let $I\left(\zeta(i_1;j_1;k_1);\ldots;\zeta(i_n;j_n;k_n)\right)$ be the inverse Laplace transform of $\frac{1}{r}\zeta(i_1;j_1;k_1)\times\cdots\times\zeta(i_n;j_n;k_n)$, for $n=1,\ldots,m$. Using (3.13) and (3.15), we have

$$I\left(\zeta(i_1;j_1;k_1);\ldots;\zeta(i_n;j_n;k_n)\right)=\Lambda(i_1;j_1;k_1)\times\cdots\times\Lambda(i_n;j_n;k_n)\\ \times\left(\gamma_0+\gamma_1 e^{-\alpha(i_1;k_1)t}+\cdots+\gamma_n e^{-\alpha(i_n;k_n)t}\right)$$

where $\Lambda(i;j;k)=\binom{i}{j}\theta^k\beta\left(E[X^j]\right)^{1-k}\left(E[(X')^j]-E[X^j]\right)^k$ and $\alpha(i;k)=k\times 2\beta+i\delta$ with, referring to (3.14), $\gamma_0=\frac{1}{\alpha(i_1;k_1)\cdots\alpha(i_n;k_n)}$ and $\gamma_u=-\frac{1}{\alpha(i_u;k_u)}\prod_{v=1;v\neq u}^{u}\frac{1}{\alpha(i_v;k_v)-\alpha(i_u;k_u)}$, $u=1,,\ldots,n$.

It finally follows that

$$\mu^{(m)}(t)=\sum_{n=1}^{m}\sum_{((i_1,j_1,k_1),\ldots,(i_n,j_n,k_n))\in\mathcal{A}_{mn}} I\left(\zeta(i_1;j_1;k_1);\ldots;\zeta(i_n;j_n;k_n)\right). \tag{3.22}$$

Remark 3.1. *We give in the appendix the expression of the third moment of $Z(t)$ that is useful for the following application section.*

3.4 Applications

As we have already discussed in the introduction, several scientific domains have recourse to discounted aggregations. We present here some applications of our results in actuarial sciences where the claim distributions are assumed to be positive and continuous.

3.4.1 Premium calculation

Now that we are able to compute the moments of $Z(t)$, it is possible to compute the premium related to the risk of an insurance portfolio represented by $Z(t)$. We propose here to study several premium calculation principles. The loaded premium $\Pi(t)$ consists in the sum of the pure premium $P(t)$, which is the expected value of the costs related to the portfolio, and a loading for the risk $L(t)$ as

$$\Pi(t)=P(t)+L(t)=E[Z(t)]+L(t).$$

The loading for the risk differs according to the premium calculation principles.

Denote by $\kappa\geq 0$ the safety loading. The expected value principle defines the loaded premium as

$$\Pi(t)=E[Z(t)]+\kappa E[Z(t)],$$

where $L(t) = \kappa E[Z(t)]$.

The variance principle gives

$$\Pi(t) = E[Z(t)] + \kappa Var(Z(t)),$$

where $L(t) = \kappa Var(Z(t))$.

And finally, one can use the standard deviation principle which is determined by

$$\Pi(t) = E[Z(t)] + \kappa\sqrt{Var(Z(t))},$$

where $L(t) = \kappa\sqrt{Var(Z(t))}$.

As we only need the first two moments for these examples, we can use the equations (3.10) and (3.16) to determine the loading for the risk and then the loaded premium (see e.g. Rolski et al. (1999) for details on premium principles).

3.4.2 First three moments based approximation for the distribution of $Z(t)$

Here, we suggest to use a moment matching method in order to approximate the distribution of $Z(t)$. Several moment matching methods exist and can be used according to the queue of the distribution that one want to approximate as the translated-gamma, the translated F, the inverse gamma or the generalized Pareto approximations as referred in McNeil et al. (2005). For this illustration, we choose to work with exponentially distributed claims and then we propose to approximate the distribution of $Z(t)$ with a moment matching method based on a mixture of two Erlang distributions with a common shape parameter. This method that is due to Johnson and Taaffe (1989) is well adapted for light tailed distributions. Furthermore, as said in Tijms (1994), the class of mixture of Erlang distributions is dense in the space of positive continuous distributions. The distribution function of a mixture of two Erlang distributions with respective rate parameters λ_1 and λ_2 and common shape parameter n is given by

$$F_Y(y) = p_1F_1(y) + p_2F_2(y),$$

where F_1 and F_2 are two Erlang c.d.f.'s and p_1 and p_2 their respective weight in the mixture. The p.d.f. Y is

$$f_Y(y) = p_1f_1(y) + p_2f_2(y),$$

where f_1 and f_2 are two Erlang p.d.f.'s. The m-th moment of the mixture of two Erlang distributions is

$$\mu^{(m)} = E[Y^n] = p_1\mu_1^{(m)} + p_2\mu_2^{(m)},$$

where $\mu_1^{(m)}$ and $\mu_2^{(m)}$ are the respective m-th moment of two Erlang distributions. Under the conditions $x \geq 0$ and $y \geq 0$ which are fulfilled in the following numerical example where x and y are defined just below, Theorem 3 of Johnson and Taaffe (1989) gives the parameters of the mixture of two Erlang distributions with the same shape parameter n in terms of its first three moments as follows

$$\lambda_i^{-1} = \left(-B + (-1)^i \sqrt{B^2 - 4AC}\right) / (2A), \quad i = 1, 2,$$

and

$$p_1 = 1 - p_2 = \left(\frac{\mu^{(1)}}{n} - \lambda_2^{-1}\right) / \left(\lambda_1^{-1} - \lambda_2^{-1}\right),$$

where $A = n(n+2)\mu^{(1)}y$, $B = -\left(nx + \frac{n(n+2)}{n+1}y^2 + (n+2)(\mu^{(1)})^2 y\right)$, $C = \mu^{(1)}x$, $y = \mu^{(2)} - \left(\frac{n+1}{n}\right)(\mu^{(1)})^2$ and $x = \mu^{(1)}\mu^{(3)} - \left(\frac{n+2}{n+1}\right)(\mu^{(2)})^2$.

For the numerical illustration, suppose that $X \sim Exp(\lambda = 1/100)$, the inter-claim time distribution parameters $\beta = 1, 5$ and 10, the interest rate $\delta = 4\%$. We use three different values for the copula parameter $\theta = -1, 0, 1$ and fix the time $t = 5$. The m-th moment of X is

$$E[X^m] = \frac{1}{\lambda^m} m! \tag{3.23}$$

As $E[(X')^m] = \int_0^\infty m x^{m-1}(1 - F_X(x))^2 dx$, we have that

$$E[(X')^m] = \frac{1}{(2\lambda)^m} m! \tag{3.24}$$

The first three moments of $Z(t)$ and the matched parameters for the mixture of Erlang distributions are presented in Tables 3.1, 3.2 and 3.3.

θ	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	n	λ_1	λ_2	p_1	p_2
-1	477.682	3.346×10^5	2.967×10^8	3	0.0442	0.00563	0.119	0.881
0	453.173	2.878×10^5	2.277×10^8	4	0.0263	0.00747	0.215	0.785
1	428.664	2.434×10^5	1.679×10^8	4	0.0430	0.00867	0.088	0.911

TABLE 3.1 – Moments of $Z(5)$ and parameters of the mixture of Erlang distributions for $\beta = 1$.

For this last case, Figures 3.1, 3.2 and 3.3 in the appendix show the drawings of the simulated p.d.f. of $Z(5)$ versus the approximated p.d.f. of $Z(5)$ with a mixture of Erlang distributions moment matching. We see on our illustration that the fit of the approximations is satisfying.

θ	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	n	λ_1	λ_2	p_1	p_2
-1	2290.766	5.766×10^6	1.576×10^{10}	11	0.0146	0.00475	0.0159	0.984
0	2265.866	5.546×10^6	1.455×10^{10}	13	0.135	0.00572	0.00337	0.997
1	2240.965	5.329×10^6	1.338×10^{10}	17	0.0454	0.00757	0.00308	0.997

TABLE 3.2 – Moments of $Z(5)$ and parameters of the mixture of Erlang distributions for $\beta = 5$.

θ	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	n	λ_1	λ_2	p_1	p_2
-1	4556.681	2.180×10^7	1.091×10^{11}	21	0.0118	0.00459	0.00557	0.994
0	4531.731	2.136×10^7	1.045×10^{11}	26	0.0118	0.00572	0.00605	0.994
1	4506.781	2.093×10^7	9.999×10^{10}	34	0.0157	0.00753	0.00326	0.998

TABLE 3.3 – Moments of $Z(5)$ and parameters of the mixture of Erlang distributions for $\beta = 10$.

θ	MC $\beta = 1$	MM $\beta = 1$	MC $\beta = 5$	MM $\beta = 5$	MC $\beta = 10$	MM $\beta = 10$
-1	1606.311	1620.153	4451.252	4498.420	7486.069	7545.406
0	1434.566	1426.921	4168.524	4220.984	7121.053	7166.169
1	1244.871	1251.674	3859.026	3895.557	6718.142	6755.696

TABLE 3.4 – VaR calculated from the Monte Carlo (MC) simulations and the moment matching (MM).

In Tables 3.4, we compare the Value at Risk (VaR) obtained from Monte Carlo simulations of $Z(5)$ against the VaR for the mixture of Erlang distributions approximation for a confidence level $\alpha = 99.5\%$. Once again, the approximated VaR's are satisfying.

Remark 3.2. *Let $S(t) = Z(t)$ when the instantaneous rate of interest $\delta = 0$. As, in general for $\delta \geq 0$, we have $E[\varphi(Z(t))] \leq E[\varphi(S(t))]$ for every non-decreasing function φ, we have that $Z(t) \leq_{sd} S(t)$ where $\leq_{sd}$ designate the stochastic dominance order. Furthermore, this implies that $VaR_\alpha(Z(t)) \leq VaR_\alpha(S(t))$ for every $\alpha \in [0, 1]$.*

3.4.3 Solvency II internal model

The European Solvency II project is going to lay down some new regulatory requirements that every insurance company inside the European Union will have to fulfill. In addition, several other countries outside the European Union (e.g. Canada, Columbia or Mexico) are likely to use similar principles. The directive has been adopted in April 2009 and the implementation measures are in progress in order to have the new system in force on October 31st, 2012. Determination of Solvency Capital Requirement (SCR) is one of the main points of the quantitative pillar of this reform. Indeed, in addition to the best estimate (which is defined as the expected present value of all potential future cash flows that would be incurred in meeting policyholders' liabilities) of liabilities and a risk margin, insurance companies and reinsurers will have to own an extra capital to cope with unfavorable events. The computation of the Solvency II standard formula for SCR is based on the 1-year 99.5%-Value-at-risk (VaR). Most often in the standard formula, it is assumed that the heaviness of the tail of the distribution of random loss X is quite moderate, and so the SCR, defined as the difference $VaR_{99.5\%}(X) - E(X)$, is replaced by a proxy $q\sigma_X$, where σ_X denotes the standard error coefficient of X and q is a quantile factor which should be set at $q = 3$. Seldom, if appropriate, factor $q = 3$ may be replaced by a larger value, close to 5 for example, to take into account potential heavier tails. This is the case in particular in the current version of the Counterparty Risk module (see Consultation Paper 51 of CEIOPS). Although quantile factors may vary from one line of business to the other, it has become classical to compute the SCR in the standard formula as a multiple of the standard error coefficient of the random loss, or with stress scenarios. Even if internal models or partial internal models are being encouraged, companies will anyway have to provide the SCR computations with the standard formula as complement. Some of those partial internal models are based on a different time horizon, up to 5 or 10 years for some reinsurers. Besides, all insurers have to provide an Own Risk and Solvency Assessment (ORSA) which aims to study risks that may affect the long-term solvency of the company. Either for ORSA or for SCR computations, it may be useful to determine the first two moments of the discounted aggregate claim amount, both with constant interest rate and inflation, and in a stress scenario where inflation increases. Inflation is very low currently and, in an ORSA analysis, it would be interesting to study the impact of inflation on Best Estimate (BE) and on the SCR if inflation increases after the current crisis. This is what we investigate in Tables 3.6 and 3.7. Solvency II standard formula often uses the independence between claim amounts and the claim arrival process. In practice, for risks like earthquake risk or flood and drought risks, the next claim amount is not independent from the time elapsed before the previous claim, and this must be taken into account in partial internal models. The advantage of our method is that it remains valid for negative values of δ (as long as they are not too negative), which can be seen as the difference between the interest rate and the inflation rate. If inflation becomes larger than the interest rate, then δ becomes negative, and our method still applies for small enough values of $|\delta|$. Some other approaches are possible as cat risk is sometimes addressed directly by the means of extreme scenarios.

Here we compute the SCR in the standard formula approach and in the internal model approach for a 5-year horizon for exponentially distributed inter-claim times and Exponential and Pareto claim amount distributions. For the internal model approach, we use Equations (3.23) and (3.24) from the previous example to compute the mth moment of $Z(t)$ when the claim amounts are exponentially distributed. If the claim amount r.v. X is Pareto with c.d.f.

$$F_X(x) = 1 - \left(\frac{\gamma}{\gamma + x}\right)^{\kappa}, \ x > 0,$$

and mth moment

$$E[X^m] = \frac{\gamma^m m!}{\prod_{i=1}^{m}(\kappa - i)} \tag{3.25}$$

for $\gamma > 0$ and $\kappa > m$ then $E[(X')^m]$ according to (3.18) becomes

$$E[(X')^m] = \frac{\gamma^m m!}{\prod_{i=1}^{m}(2\kappa - i)} \tag{3.26}$$

for $\kappa \geq m/2$.

Thus the mth moment of $Z(t)$ can be explicitly expressed using (3.10) and (3.16) for the first and second moments, or using (3.22) for greater moments. The SCR for the internal model is obtained from the first moment of $Z(t)$ and a simulated VaR with Monte Carlo method.

Let the FGM dependence parameter be -1, 0 or 1, and $\delta = 3\%$. The parameter for the inter-claim time distribution is $\beta = 2$. Assume that the claim amount r.v. $X \sim Exp(\lambda = 1/10)$ for the Exponential case and that $X \sim Pareto(\kappa = 2.5, \gamma = 15)$ for the Pareto case with the same expected value 10 but with variances respectively equal to 100 and 500. As discussed above, we set the quantile factor q for the standard formula approach at 3 for the Exponential case and at 5 for the Pareto case. The SCR's for the standard formula and the internal model approaches are presented in Table 3.5. Using the internal model approach, we also compute the SCR (and the Best Estimate (BE)) with inflation crises ($\delta = 1.5\%$, 0.5% or -5%) in comparison to $\delta = 3\%$ for the Pareto case. The results are shown in Table 3.6.

We also provide some results for the same values for δ when the time horizon is equal to 10 years and the copula parameter $\theta = 1$ in Table 3.7.

	Exponential case		Pareto case	
Copula parameter	Standard formula, $q = 3$	Internal model	Standard formula, $q = 5$	Internal model
$\theta = -1$	140.508	151.075	385.760	314.362
$\theta = 0$	124.703	132.149	359.987	295.574
$\theta = 1$	107.091	111.254	332.933	276.368

Table 3.5 – Comparison between the standard formula and the internal model approaches for the SCR, 5-year time horizon.

Copula parameter	$\delta = 3\%$	$\delta = 1.5\%$	$\delta = 0.5\%$	$\delta = -5\%$
$\theta = -1$				
BE	95.963	99.455	101.881	116.775
SCR	314.362	325.107	331.891	383.146
$\theta = 0$				
BE	92.861	96.342	98.760	113.610
SCR	295.574	306.034	313.842	362.760
$\theta = 1$				
BE	89.760	93.229	95.639	110.446
SCR	276.368	287.600	295.391	342.066

Table 3.6 – Effect of inflation crisis for Pareto claim amounts, 5-year time horizon.

Finally, we also provide in Table 3.8 a few results with $\theta = 1$ and $\beta = 0.5$ to see the influence

	$\delta = 3\%$	$\delta = 1.5\%$	$\delta = 0.5\%$	$\delta = -5\%$
$\theta = 1$				
BE	169.686	182.609	191.961	256.324
SCR	356.386	383.095	402.398	543.695

Table 3.7 – Effect of inflation crisis for Pareto claim amounts, 10-year time horizon.

of parameter β and to illustrate the case where large claims occur in average every k years, with $k > 1$.

First, we see that both SCR and BE are decreasing as the copula parameter θ increases from

	$\delta = 3\%$	$\delta = 1.5\%$	$\delta = 0.5\%$	$\delta = -5\%$
$\theta = 1$				
BE	40.163	43.352	45.661	61.583
SCR	182.448	197.233	207.688	284.735

Table 3.8 – Effect of inflation crisis for Pareto claim amounts, 10-year time horizon, $\beta = 0.5$.

-1 to 1. This is logical as positive dependence between inter-claim times and claim amounts is a form of diversification effect. SCR are larger for Pareto claim amounts than for Exponential claim amounts, as usual. Nevertheless, Table 3.5 shows that the so-called internal model approach leads to higher values of SCR than the ones obtained by the standard formula for Exponentially distributed claim amounts, while it is the opposite for Pareto distributed claim amounts. Finally, regardless of θ, both SCR and BE increase as δ decreases. The impact of inflation cannot be neglected : in Table 3.8, the case where $\delta = -5\%$ (which corresponds to scenarios where the inflation rate becomes 5% larger than the interest rate) leads to more than a 50%-increase in Best Estimate and SCR, in the most favorable case where $\theta = 1$.

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APPENDIX

The first two moments of the distribution of $Z(t)$ which are given by (3.10) and (3.16) respectively. The third moment is

$$\mu^{(3)}(t) = \sum_{n=1}^{3} \sum_{((i_1,j_1,k_1),\dots,(i_n,j_n,k_n))\in\mathcal{A}_{3n}} I\left(\zeta(i_1;j_1;k_1);\dots;\zeta(i_n;j_n;k_n)\right),$$

where $A_{3n} = \Big\{(i_1,j_1,k_1),\dots,(i_n,j_n,k_n)\,;\; i_1 = 3,\; i_1 > \dots > i_n,\; j_n = i_n,\; j_1+\dots+j_n = 3,$ $0 < j_{\cdot} \le n,\; k_{\cdot} \in \{0,1\}\Big\}$. It can be developed as

$$\begin{aligned}
\mu_3(t) = {} & \binom{3}{3}\beta E[X^3]\left(\frac{1}{3\delta} - \frac{e^{-3\delta t}}{3\delta}\right) + \binom{3}{3}\theta\beta\left(E[X'^3] - E[X^3]\right)\left(\frac{1}{2\beta+3\delta} - \frac{e^{-(2\beta+3\delta)t}}{2\beta+3\delta}\right)\\
&+ \binom{3}{2}\binom{1}{1}\beta^2 E[X]E[X^2]\left(\frac{1}{3\delta^2} - \frac{e^{-\delta t}}{2\delta^2} + \frac{e^{-3\delta t}}{6\delta^2}\right)\\
&+ \binom{3}{2}\binom{1}{1}\theta\beta^2 E[X^2]\left(E[X'] - E[X]\right)\left(\frac{1}{3\delta(2\beta+\delta)} - \frac{e^{-(2\beta+\delta)t}}{(2\beta+\delta)(-2\beta+2\delta)} + \frac{e^{-3\delta t}}{3\delta(-2\beta+2\delta)}\right)\\
&+ \binom{3}{2}\binom{1}{1}\theta\beta^2 E[X]\left(E[X'^2] - E[X^2]\right)\left(\frac{1}{\delta(2\beta+3\delta)} - \frac{e^{-\delta t}}{\delta(2\beta+2\delta)} + \frac{e^{-(2\beta+3\delta)t}}{(2\beta+3\delta)(2\beta+2\delta)}\right)\\
&+ \binom{3}{2}\binom{1}{1}\theta^2\beta^2\left(E[X'] - E[X]\right)\left(E[X'^2] - E[X^2]\right)\left(\frac{1}{(2\beta+\delta)(2\beta+3\delta)} - \frac{e^{-(2\beta+\delta)t}}{2\delta(2\beta+\delta)} + \frac{e^{-(2\beta+3\delta)t}}{2\delta(2\beta+3\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{2}\beta^2 E[X]E[X^2]\left(\frac{1}{6\delta^2} - \frac{e^{-2\delta t}}{2\delta^2} + \frac{e^{-3\delta t}}{3\delta^2}\right)\\
&+ \binom{3}{1}\binom{2}{2}\theta\beta^2 E[X]\left(E[X'^2] - E[X^2]\right)\left(\frac{1}{3\delta(2\beta+2\delta)} - \frac{e^{-(2\beta+2\delta)t}}{(2\beta+2\delta)(-2\beta+\delta)} + \frac{e^{-3\delta t}}{3\delta(-2\beta+\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{2}\theta\beta^2 E[X^2]\left(E[X'] - E[X]\right)\left(\frac{1}{2\delta(2\beta+3\delta)} - \frac{e^{-2\delta t}}{2\delta(2\beta+\delta)} + \frac{e^{-(2\beta+3\delta)t}}{(2\beta+\delta)(2\beta+3\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{2}\theta^2\beta^2\left(E[X'] - E[X]\right)\left(E[X'^2] - E[X^2]\right)\left(\frac{1}{(2\beta+2\delta)(2\beta+3\delta)} - \frac{e^{-(2\beta+2\delta)t}}{\delta(2\beta+2\delta)} + \frac{e^{-(2\beta+3\delta)t}}{\delta(2\beta+3\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\beta^3 E[X]^3\left(\frac{1}{6\delta^3} - \frac{e^{-\delta t}}{2\delta^3} + \frac{e^{-2\delta t}}{2\delta^3} - \frac{e^{-3\delta t}}{6\delta^3}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta\beta^3 E[X]^2\left(E[X'] - E[X]\right)\left(\frac{1}{6\delta^2(2\beta+\delta)} - \frac{e^{-(2\beta+\delta)t}}{(2\beta+\delta)(-2\beta+\delta)(-2\beta+2\delta)} + \frac{e^{-2\delta t}}{2\delta^2(-2\beta+\delta)} - \frac{e^{-3\delta t}}{3\delta^2(-2\beta+2\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta\beta^3 E[X]^2\left(E[X'] - E[X]\right)\left(\frac{1}{3\delta^2(2\beta+2\delta)} - \frac{e^{-\delta t}}{2\delta^2(2\beta+\delta)} + \frac{e^{-(2\beta+2\delta)t}}{(2\beta+2\delta)(2\beta+\delta)(-2\beta+\delta)} - \frac{e^{-3\delta t}}{6\delta^2(-2\beta+\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta\beta^3 E[X]^2\left(E[X'] - E[X]\right)\left(\frac{1}{2\delta^2(2\beta+3\delta)} - \frac{e^{-\delta t}}{\delta^2(2\beta+2\delta)} + \frac{e^{-2\delta t}}{2\delta^2(2\beta+\delta)} - \frac{e^{-(2\beta+3\delta)t}}{(2\beta+3\delta)(2\beta+2\delta)(2\beta+\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta^2\beta^3 E[X]\left(E[X'] - E[X]\right)^2\left(\frac{1}{3\delta(2\beta+\delta)(2\beta+2\delta)} - \frac{e^{-(2\beta+\delta)t}}{\delta(2\beta+\delta)(-2\beta+2\delta)} + \frac{e^{-(2\beta+2\delta)t}}{\delta(2\beta+2\delta)(-2\beta+\delta)} - \frac{e^{-3\delta t}}{3\delta(-2\beta+2\delta)(-2\beta+\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta^2\beta^3 E[X]\left(E[X'] - E[X]\right)^2\left(\frac{1}{2\delta(2\beta+\delta)(2\beta+3\delta)} - \frac{e^{-(2\beta+\delta)t}}{2\delta(2\beta+\delta)(-2\beta+\delta)} + \frac{e^{-2\delta t}}{2\delta(-2\beta+\delta)(2\beta+\delta)} - \frac{e^{-(2\beta+3\delta)t}}{2\delta(2\beta+3\delta)(2\beta+\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta^2\beta^3 E[X]\left(E[X'] - E[X]\right)^2\left(\frac{1}{\delta(2\beta+2\delta)(2\beta+3\delta)} - \frac{e^{-\delta t}}{\delta(2\beta+\delta)(2\beta+2\delta)} + \frac{e^{-(2\beta+2\delta)t}}{\delta(2\beta+2\delta)(2\beta+\delta)} - \frac{e^{-(2\beta+3\delta)t}}{\delta(2\beta+3\delta)(2\beta+2\delta)}\right)\\
&+ \binom{3}{1}\binom{2}{1}\binom{1}{1}\theta^3\beta^3\left(E[X'] - E[X]\right)^3\left(\frac{1}{(2\beta+\delta)(2\beta+2\delta)(2\beta+3\delta)} - \frac{e^{-(2\beta+\delta)t}}{2\delta^2(2\beta+\delta)} + \frac{e^{-(2\beta+2\delta)t}}{\delta^2(2\beta+2\delta)} - \frac{e^{-(2\beta+3\delta)t}}{2\delta^2(2\beta+3\delta)}\right)
\end{aligned}$$

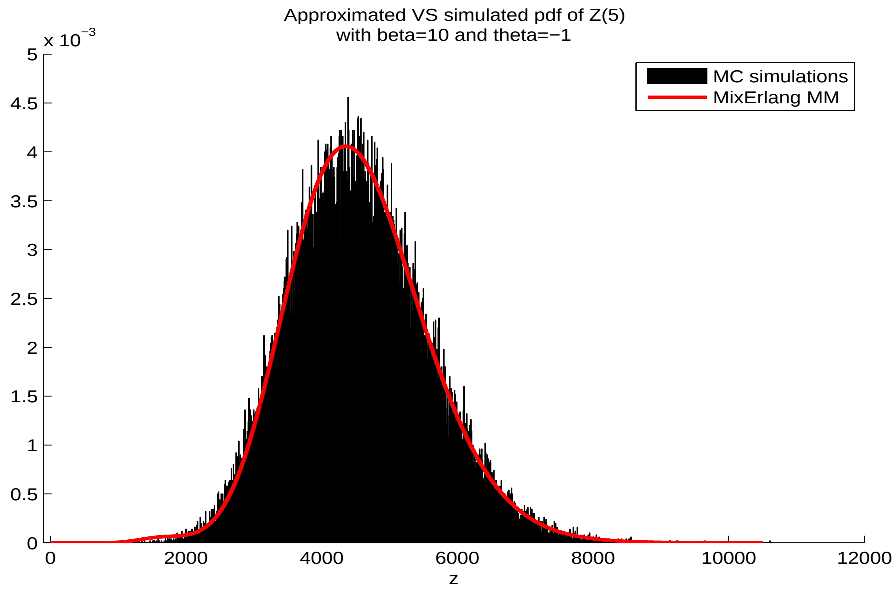

FIGURE 3.1 – Comparison between the simulated pdf and the approximated pdf with $\theta = -1$

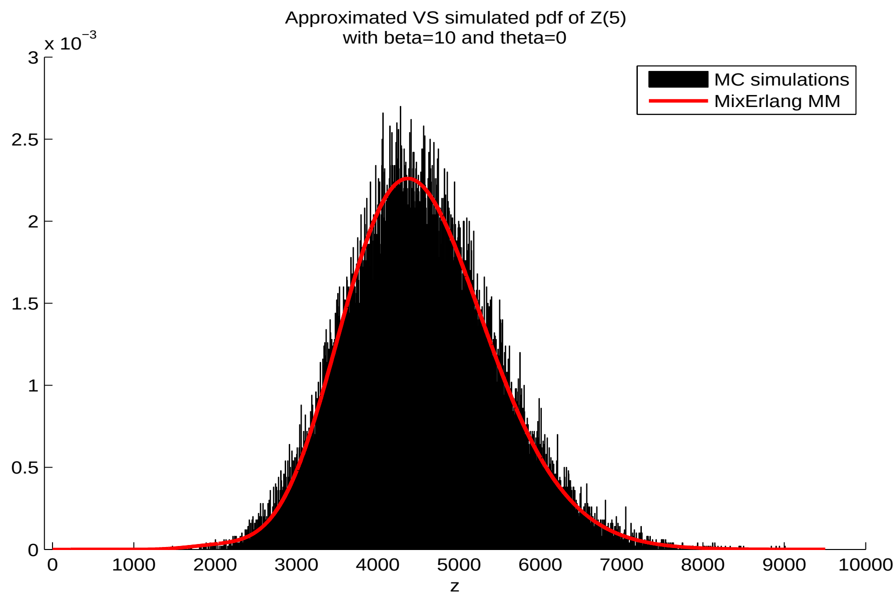

FIGURE 3.2 – Comparison between the simulated pdf and the approximated pdf with $\theta = 0$

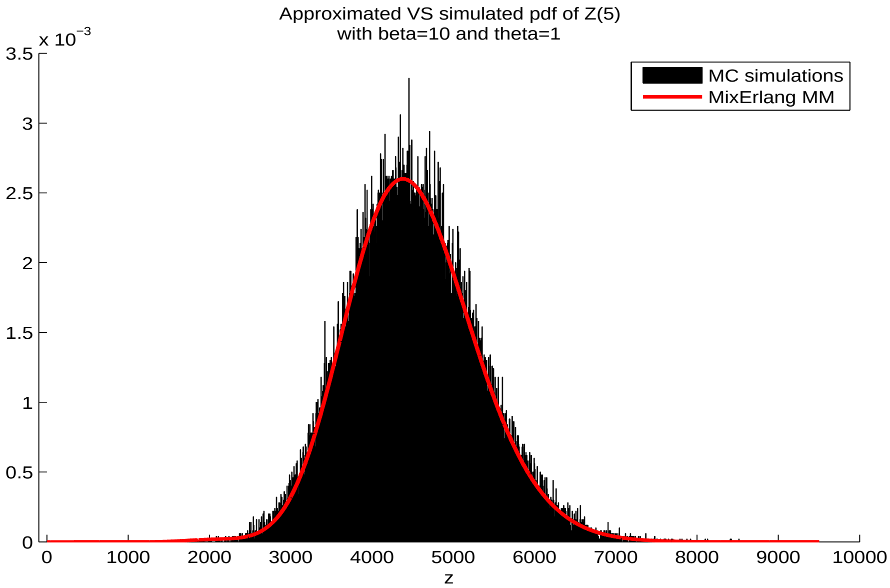

FIGURE 3.3 – Comparison between the simulated pdf and the approximated pdf with $\theta = 1$

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Chapitre 4

Théorie de la ruine avec cycles de réassurance influencés par les larges sinistres

Résumé

Les cycles de marché jouent un grand rôle dans la réassurance de par leur influence sur les primes exigées par les réassureurs aux compagnies d'assurance. Dans ce contexte, un grand sinistre peut inciter les compagnies de réassurance à augmenter leurs demandes en primes aux cédantes. Pour prendre cela en compte, un modèle de risque semi-markovien est proposé et analysé. Une méthode d'erlangisation adaptée est développée afin d'évaluer la probabilité de ruine d'une compagnie de réassurance. Comme ce modèle nécessite d'avoir des montants de sinistres distribués selon une loi *phase-type*, nous indiquons dans une application numérique comment ajuster un mélange de distributions Erlang sur une distribution à queue épaisse. Nous comparons les résultats obtenus avec notre modèle et ceux obtenus avec des simulations. L'impact de la dépendance entre les montants de sinistre et les changements de phase est également étudié.

4.1 Introduction

The classical Sparre-Anderson risk model describes the surplus process $\{R_t\}_{t\geq 0}$ for an insurance or a reinsurance company as $R_0 = u$ and

$$R_t = u + ct - S_t,$$

where u represents the non-negative initial capital and $c > 0$ is the premium income rate. The compound renewal process $\{S_t\}_{t\geq 0}$ designates the cumulated claim amount up to time t and is expressed as

$$S_t = \begin{cases} 0, \text{ if } N_t = 0, \\ \sum_{i=1}^{N_t} X_i \text{ otherwise,} \end{cases}$$

where the claim amounts X_i, $i \geq 1$, are non-negative and i.i.d. random variables. The number of claims up to time t is modeled by the renewal process $\{N_t\}_{t\geq 0}$ with i.i.d. inter-arrival times noted by W_k, $k \geq 1$. The X_i's, $i \geq 1$ are independent from the process $(N_t)_{t\geq 0}$. In this paper, we are interested in the evaluation of the finite-time ruin probability

$$\psi(u, T) = P\left[\exists t \in [0, T], \quad u + ct - S_t < 0\right], \quad u \geq 0, T \geq 0.$$

Different methods to compute this finite-time (non-)ruin have been proposed in the literature. Picard and Lefèvre (1997) derived an explicit formula for the finite-time non-ruin probability in a compound Poisson model where the claim amounts are integer-valued. This idea was then widely used and extended to more general modelizations as in De Vylder and Goovaerts (1999), Ignatov and Kaishev (2004), Rullière and Loisel (2004) or Lefèvre and Loisel (2009), among others. Asymptotic methods have also been used to evaluate the finite-time ruin probability as in Biard et al. (2009) where the claim sizes are heavy-tailed and stationarity and independence assumptions are relaxed in possible correlation crises. Previous general asymptotic results have been provided for sums of random variables as in Wüthrich (2003), Albrecher et al. (2006) or Kortschak and Albrecher (2009).

The method we focus on is called Erlangization. It was first introduced in Avram and Usabel (2003) but principally developed by Asmussen et al. (2002). The Erlangization method consists in replacing the deterministic time horizon T by an Erlang-distributed random variable $\mathcal{H}$ with mean T and independent from the claim process. Asmussen et al. (2002) have shown that the ruin probability before the independent Erlang time $\mathcal{H}$ coincides with the ruin probability in a certain Markovian fluid model with a matrix-exponential form. They considered the compound Poisson risk model with phase-type claim amounts. Stanford et al. (2005a) proposed an extension to the general Sparre-Anderson risk model. In Yu et al. (2007), the risk process with phase-type distributed inter-claim times and claim amounts is perturbed by a standard Brownian motion. Ramaswami et al. (2008) also extended the Erlangization method to Markovian modulated fluid flow models. Erlangization has also been used in some other domains as in Stanford et al. (2005b) where the forestry problem of the evolution of uncontrolled fire perimeter over time is treated. For a complete review on fluid flow matrix analytic methods applied in ruin theory, see Badescu and Landriault (2009).

We propose here to consider the reinsurance market. In the reinsurance business, the presence of underwriting cycles cannot be ignored. As said in Cummins et al. (2008), reinsurance cycles are composed of soft phases, during which reinsurance supply is extensive and reinsurance premium income rates are quite low and decreasing and hard phases where the supply is too low and the premium income rate becomes higher and higher. Phase transitions are hard to explain, and seem to happen due to both endogenous and exogenous events. In Weiss (2007), it is mentioned that these changes can come from institutional features as well as the effects of real phenomena. These phenomena can be asymmetric information in capital markets, capital surpluses and shortages in insurance or interest and/or loss shocks.

A simple generalization of the above-mentioned Erlangization model could be obtained with a Markovian environment process which would modulate the premium income rate, following the idea of Asmussen (1989) and of many others. Nevertheless, this first generalization leads to a model in which the cumulated claim amount process has no feedback control on the phases of the Markovian environment process. This is not the case in real world, as changes in cycle phases may be triggered by a large claim. In this paper, we propose a generalization of the above risk model that takes into account this possible trigger.

We consider a simplification of a multi-phases cycle as it is exposed in Subsection 4.3.1 into a 2-phases reinsurance cycle consisting of a soft phase with low premium rate and a hard phase with high premium rate. In financial insurance, it is usual to suppose only two environment states as in the model introduced by Hardy (2001) where the volatility of a stock is either high or low according to the state environment. Considering two environment phases permits of course not only model simplifications but also avoids many estimation problems which appear when more phases are assumed.

In the Solvency II framework, the capital required to cover risk is determined in such a way that a 1-year Value-at-Risk is controlled. Nevertheless, for reinsurance risk, and in particular for catastrophe risk, 1-year occurrence probabilities of extreme events are small, and risk must be controlled within a larger time period, 5 to 10 years, to determine not only solvency requirements but also long term economic capital in internal risk models. The average length of reinsurance cycles typically ranges from 4 to 10 years, see Venezian (1985), Cummins and Outreville (1987) and Weiss (2007). It is thus necessary to use a risk model that incorporates this cyclic behavior.

Even if the numerical application provided is Section 4.6 is detailed when only the premium income rates changes according to the cycle phases, our model is developed for the inter-claim time and claim amount distributions to be dependent from the environment cycle process. The feedback of the claim amounts on the environment process thus involves some dependence between inter-claim times and claim amounts. Other types of dependence have already been explored in the study of infinite- or finite-time ruin probability. Boudreault et al. (2006) and Albrecher and Boxma (2004) assumed that each claim amount depends on the previous inter-claim time or inversely that each inter-claim time depend on the previous claim amount. This has been extended to claim amounts depending on several preceding inter-claim times as can be seen in Biard et al. (2009). In Albrecher and Teugels (2006) the dependence between the inter-claim times and claim amounts is introduced by an arbitrary copula.

One of the contributions of the paper is to propose a more suitable model for long term economic capital management for reinsurance companies. Within this framework, the dependence of the premium income rate, the claim amount distribution and the inter-claim time distribution with the cycle environment is taken into account. The model also permits a feedback control from the claim amounts on the cycle phases. We extend the Erlangization method to this context in order to approximate the finite-time ruin probability. We use intensive calculation techniques to this aim and compare our results to Monte Carlo simulations. We evaluate the model error in several examples and quantify the impact of the claim amounts feedback over a 5-year time horizon. We show that for some reasonable values, the finite-time ruin probability decreases by 6% when this correlation is not considered.

This paper begins with a recall on the Erlangization method and related results. In Section 4.3 reinsurance cycles are discussed. General notations are given and a first naive model is proposed where we show how the Erlangization method can be adapted to deterministic phase changes. The main model incorporating the impact of the claim amounts on the change in cycle phases is presented in Section 4.4. Section 4.5 is dedicated to the control of the precisions of the proposed main model. Finally, an extended and practical illustration is provided in Section 4.6 where fitting methods and finite-time ruin probability approximations are given.

4.2 Erlangization method

We recall in this section the principle of the Erlangization method and the central theorems from Asmussen et al. (2002) which lead to an approximation for the finite-time ruin probability. We follow the presentation of Asmussen et al. (2002) and Ramaswami et al. (2008) and slightly adapt the notation for it to be consistent with our purpose. First, the surplus process R_t is embedded in an initial continuous Markovian additive fluid process $\{(\tilde{J}_t, \tilde{V}_t)\}$ where $\tilde{V}_t$ is a fluid flow and $\tilde{J}_t$ a Markovian environment process. The state space $\tilde{E}$ of $\tilde{J}_t$ consists of two subspaces $\tilde{E}_-$ and $\tilde{E}_+$ such that when $\tilde{J}_t$ is in state $i \in \tilde{E}_-$, $\tilde{V}_t$ is in a decreasing inter-claim time state (no jump) with fluid rate $r_i = -c_i$ and when $\tilde{J}_t$ is in state $i \in \tilde{E}_+$, $\tilde{V}_t$ is in an increasing claim state (jump) with fluid rate $r_i = 1$. The transition intensity matrix of $\tilde{J}_t$ is noted as

$$\Lambda = \begin{pmatrix} \Lambda_{--} & \Lambda_{-+} \\ \Lambda_{+-} & \Lambda_{++} \end{pmatrix},$$

where $\Lambda_{i_1 i_2}$ correspond to the transition from a state of subspace $\tilde{E}_{i_1}$ to a state of subspace $\tilde{E}_{i_2}$ with i_1 and $i_2 \in \{-,+\}$.

The aim of the Erlangization method is to evaluate the finite-time ruin probability before an Erlang horizon $\mathcal{H} \sim Erlang(a, L)$. The finite-time ruin probability can thus be seen as the probability that the fluid flow reaches the initial reserve u in an increasing state (claim state) conditioned on starting in a decreasing state (inter-claim time state) before the expiry of the Erlang horizon $\mathcal{H}$. For this purpose, we construct an expanded absorbing Markovian fluid process $\{(J_t, V_t)\}$. The state space E of the Markovian environment process J_t is now composed of three subsets E_0, E_- and E_+ where E_0 is the one state absorbing space, E_- is the decreasing inter-claim time state space and E_+ is the increasing claim state space.

The transition intensity matrix for the Markovian environment process J_t is

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 \\ \widehat{h}_- & H \oplus \Lambda_{--} & I_L \otimes \Lambda_{-+} \\ 0 & I_L \otimes \Lambda_{+-} & I_L \otimes \Lambda_{++} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \widehat{h}_- & \widehat{\Theta}_{--} & \widehat{\Theta}_{-+} \\ 0 & \widehat{\Theta}_{+-} & \widehat{\Theta}_{++} \end{pmatrix}, \tag{4.1}$$

where H is the phase generator of the Erlang horizon $\mathcal{H}$ and $\widehat{h}_-$ is the vector of absorbtion probabilities for the fluid process in a decreasing state. Operators $\otimes$ and $\oplus$ respectively denote the Kronecker product and the Kronecker sum.

The incorporation of the premium income rates which correspond to the fluid rates for the fluid process V_t is done by time scaling. These fluid rates are $r_i = 0$ for $i \in E_0$, $r_i = -c_i$ for $i \in E_-$ and $r_i = 1$ for $i \in E_+$. The matrix Θ which takes the premium income rates into account associated with $\widehat{\Theta}$ is

$$\begin{aligned} \Theta &= \begin{pmatrix} 0 & 0 & 0 \\ (I_L \otimes R_-)^{-1} \times \widehat{h}_- & (I_L \otimes R_-)^{-1} \times \widehat{\Theta}_{--} & (I_L \otimes R_-)^{-1} \times \widehat{\Theta}_{-+} \\ 0 & \widehat{\Theta}_{+-} & \widehat{\Theta}_{++} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ (C_-)^{-1} \times \widehat{h}_- & (C_-)^{-1} \times \widehat{\Theta}_{--} & (C_-)^{-1} \times \widehat{\Theta}_{-+} \\ 0 & \widehat{\Theta}_{+-} & \widehat{\Theta}_{++} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ h_- & \Theta_{--} & \Theta_{-+} \\ 0 & \Theta_{+-} & \Theta_{++} \end{pmatrix}, \end{aligned}$$

where R_- is a diagonal matrix formed with the decreasing fluid rates.

Denote by η_{ij} the probability that when $J_0 = i \in E_-$, then the first upcrossing of level 0 of $\{V_t\}$ occurs at time t with $J_t = j \in E_+$. The matrix η corresponds to the matrix of the upcrossing phase probabilities at the completion of a downwards excursion for $\{V_t\}$, conditioned on starting in a decreasing state. We now cite some results which were first derived in Asmussen et al. (2002) for a Poisson counting process and then generalized in Ramaswami et al. (2008) on the evaluation of the probability $\psi(u, \mathcal{H})$ of reaching u conditioned on starting in a decreasing state before the expiry of the Erlang horizon. Denoting by e the unit column vector, we first recall a general result on the upcrossing probabilities of fluid models.

Theorem 4.1. *Let $\{(\tilde{J}_t, \tilde{V}_t)\}$ be a Markovian fluid process and Λ the transition intensity matrix of the Markovian environment process $\{\tilde{J}_t\}$. Let also η denote the upcrossing phase probabilities at the completion of a downwards excursion for $\{\tilde{V}_t\}$, conditioned on starting in a decreasing state $i \in E_-$. Note by $\psi(u)$ the vector of the infinite horizon phase-distribution at u (the initial surplus, starting at 0),conditioned on starting in a decreasing state. Then, the matrix η satisfies the Riccati equation :*

$$\eta\Lambda_{++} + \eta\Lambda_{+-}\eta + \Lambda_{--}\eta + \Lambda_{-+} = 0 \tag{4.2}$$

and

$$\psi(u) = \eta \exp(Uu)e$$

where

$$U = \Lambda_{++} + \Lambda_{+-}\eta.$$

The proof of this theorem is based on fluid flow matrix analytic methods and can be found in the Asmussen et al. (2002) and Ramaswami et al. (2008). Focusing on the semi-Markovian fluid process $\{(J_t, V_t)\}$ before the expiry of the Erlang horizon time $\mathcal{H}$ with initial probability vector γ, we have the following corollary.

Corollary 4.1. *Including the Erlang time $\mathcal{H}$ with the initial probability vector γ, it follows that*

$$\eta\Theta_{++} + \eta\Theta_{+-}\eta + \Theta_{--}\eta + \Theta_{-+} = 0 \tag{4.3}$$

and

$$\psi(u, \mathcal{H}) = \gamma\eta \exp(Vu)e, \tag{4.4}$$

where $V = \Theta_{++} + \Theta_{+-}\eta$.

Finally, Asmussen et al. (2002) obtained the following convergence result.

Theorem 4.2. *For $T > 0$, $\mathcal{H}$ the Erlang distribution with L stages and mean T, then*

$$\psi(u, \mathcal{H}) \to \psi(u, T) \text{ as } L \to \infty, \text{ at rate } O(L^{-1}).$$

Matrix η in Equations (4.2) and (4.3) can be evaluated by a fixed-point method or using a quadratic algorithm as proposed in Ahn and Ramaswami (2003). Matrix inversions that are involved in the fixed point method proposed by Asmussen et al. (2002) are explained in the appendix.

4.3 A few words on reinsurance cycles modeling

4.3.1 Reinsurance cycles

In order to give an idea of a possible generalization of the model we propose in this paper, we first expose here a cycle modeling with several phases. Then we recall the reasons of our simplification choice to a 2-phase cycle.

Reinsurance cycles can be composed of several phases with soft phases where the premium rates are low and hard phases with high premium rates. Intermediate phases have premium rates varying according to their proximity with the soft or the hard phases. We suppose that a reinsurance cycle has, for example, 10 phases where $i = 1$ corresponds to the softest phase and $i = 10$ to the hardest phase. We note c_i, $i = 1, \ldots, 10$, the premium income rate for the 10 phases with $c_1 \leq c_2 \leq \cdots \leq c_{10}$ and assume that when the cycle is in phase i, it could only go to phase $i-1$, $i+1$, $i+2$ or stay at i depending on the claim amount. When a large claim occurs, the cycle has high probability to go to phase $i+1$ and if the claim is very large, the cycle can also directly jump to phase $i+2$. Figure 4.1 shows a possible cycle phase path using these ideas.

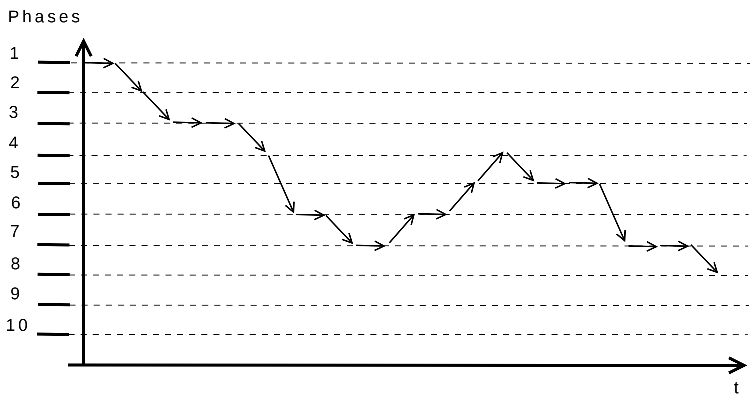

FIGURE 4.1 – Example of a cycle path with 10 phases

We could also double the phase states such as $1^{\downarrow}$, $1^{\uparrow}$, $2^{\downarrow}$, $2^{\uparrow}$, $3^{\downarrow}$, $3^{\uparrow}$, ... (or 1, $2^{\downarrow}$, $2^{\uparrow}$, $3^{\downarrow}$, $3^{\uparrow}$, ..., 10) in order to keep the general trend of the cycle. For example, we could set probabilities at state 3 according to the cycle trend as in Table 4.1.

But, in this paper, we focus our study on a reinsurance cycle that is composed of only a soft phase with low premium rate and a hard phase with high premium rate. Indeed, dealing with models with more than two phases in a cycle would require very large transition matrices and would highly increase computational issues. Moreover, it is conventional in finance insurance to consider environment processes with 2 states, as in Hardy (2001), which furthermore avoids many estimation problems.

Phase $3^{\downarrow}$ to	Probability	Phase $3^{\uparrow}$ to	Probability
Phase 2	0.001	Phase 2	0.09
Phase 3	0.9	Phase 3	0.9
Phase 4	0.09	Phase 4	0.009
Phase 5	0.009	Phase 5	0.001

TABLE 4.1 – Example of phase change probabilities at phase 3.

We assume that large claims have high probabilities to cause a change from the soft phase to the hard phase. This impact of the claim amounts on the environment process represents quite well the general reinsurance loss market as most of insurance and reinsurance companies hold high expositions in the most risky regions (Western Europe, Japan, Eastern United-States, South-East Asia, China,...). In consequence, a big event (catastrophe,...) impacts all the actors with a very large claim.

4.3.2 General notation

We fix now some notation for all the following sections. We suppose that in each cycle phase the claim amounts are i.i.d. and phase-type distributed such as $X_i \sim PH(G_i, \alpha_i)$ where G_i is the phase generator matrix and α_i the initial row vector in cycle phase i. The inter-claim times are also i.i.d. and phase-type distributed. The inter-claim time random variable for cycle phase i is noted as $W_i \sim PH(K_i, \beta_i)$ with K_i the phase generator matrix and β_i the initial row vector. We write by $g_i = -G_i e$ and $k_i = -K_i e$ the rates of absorption for X_i and W_i, respectively, where e is the unit column vector. Denote by T the fixed time horizon. The phase-type parametrization of the Erlang horizon time $\mathcal{H}$ with L stages, mean T and rate $a = L/T$ is $\mathcal{H} \sim PH(H, \gamma = (1, 0, \ldots, 0))$ where

$$H = \begin{pmatrix} -a & a & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -a & a \\ 0 & 0 & 0 & -a \end{pmatrix}.$$

4.3.3 Deterministic phase changes

We first give a naive model which assumes that only one change of phase happens and that this time of phase change is known in advance. We present here the case where the cycle environment process begins in the soft phase and switches to the hard phase at a deterministic time T_1 and stops at the horizon T. But this model could be adapted with the cycle beginning in the hard phase and going to the soft phase. The times spent in the soft phase and in the hard phase are T_1 and $T_2 = T - T_1$ respectively. Here, the Erlang horizon time $\mathcal{H}$ with mean T and L stages is the sum of two intermediate Erlang horizon times $\mathcal{H}_1$ and $\mathcal{H}_2$ with means T_1 and T_2, L_1 and L_2 stages respectively and a common rate parameter a. We can then write the transition intensity matrix $\widehat{\Theta}$ as

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 \\ \widehat{h}_- & \widehat{\Theta}_{--} & \widehat{\Theta}_{-+} \\ 0 & \widehat{\Theta}_{+-} & \widehat{\Theta}_{++} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & H_1 \oplus K_1 & M_{12} & Id_{L_1} \otimes k_1\alpha_1 & 0 \\ h_2 \otimes e_{k_2} & 0 & H_2 \oplus K_2 & 0 & Id_{L_2} \otimes k_2\alpha_2 \\ 0 & Id_{L_1} \otimes g_1\beta_1 & 0 & Id_{L_1} \otimes G_1 & 0 \\ 0 & 0 & Id_{L_2} \otimes g_2\beta_2 & 0 & Id_{L_2} \otimes G_2 \end{pmatrix},$$

where $M_{12} = ((h_1 \times \gamma_2) \times e_{k_1}) \times \beta_2$ is the bloc matrix of transitions between the two phases and $h_i = -H_i e$ the absorption rate vector for the phase-type parametrization of the Erlang horizon $\mathcal{H}_i$.

Remark 4.1. *The way to determine the numbers of stages L_1 and L_2 for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively is shown in the appendix.*

Example 4.1. *Consider that $H_1 \sim Erlang(1, a)$ and $H_2 \sim Erlang(2, a)$, then we have*

$$\widehat{\Theta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_1 - aId_{n_1} & (a \times e_{n_1}) \times \beta_2 & 0 & k_1\alpha_1 & 0 & 0 \\ 0 & 0 & K_2 - aId_{n_2} & aId_{n_2} & 0 & k_2\alpha_2 & 0 \\ ae_{n_2} & 0 & 0 & K_2 - aId_{n_2} & 0 & 0 & k_2\alpha_2 \\ 0 & g_1\beta_1 & 0 & 0 & G_1 & 0 & 0 \\ 0 & 0 & g_2\beta_2 & 0 & 0 & G_2 & 0 \\ 0 & 0 & 0 & g_2\beta_2 & 0 & 0 & G_2 \end{pmatrix},$$

where n_1 is the number of stages of W_1 and n_2 the number of stages of W_2.

Using the same idea, it is also possible to construct an intensity matrix $\widehat{\Theta}$ for the situation where the environment process begins in the hard phase and goes to the soft phase at a deterministic time. But in fact, cycle phase changes are random and may be influenced by large claims.

4.4 Phase change instants dependent from claim amounts

We now focus on the main model of the article where we let the cycle switch from the soft phase to the hard phase when a claim amount is greater than a threshold $x_0 \geq 0$. In what follows, we suppose that the cycle begins in the soft phase but using the idea of the previous simple model, one can add an initial hard phase with a deterministic length. As noted in the Introduction, the going back to the soft phase should also depend from the claim amount, for example according to the severity of the claim that caused the passing to the hard phase. But for simplicity, we suppose here that it happens after a deterministic time $t_0 \geq 0$. As the time spent in the previous soft phase is not known, we cannot proceed as in the deterministic phase change model developed above.

We require here that the claim amounts are distributed as mixtures of Erlang distributions. Mixtures of Erlang distributions include the exponential distribution, the Erlang distribution and mixtures of Exponential distributions. The canonical random variable for the claim amounts in cycle phase i, $i = 1, 2$, is then written as

$$X_i = \sum_{j=1}^{n_i} \alpha_{ij} Y_{ij},$$

where Y_{ij} are n_i independent and Erlang distributed random variables, $Y_{ij} \sim Erlang(m_{ij}, \lambda_{ij})$ with shape parameters $m_{ij} > 0$ and rate parameters $\lambda_{ij} > 0$. The scalars $\alpha_{ij} \geq 0$ are the mixture proportions and must satisfy $\alpha_{i1} + \cdots + \alpha_{in_i} = 1$. We assume that $E[Y_{i1}] = m_{i1}/\lambda_{i1} > \cdots > E[Y_{in_i}] = m_{in_i}/\lambda_{in_i}$.

This section consists of 2 subsections. The first one introduces the transition from the soft phase to the hard phase which depends on the claim amounts. The second subsection exposes the way the go back to the soft phase.

4.4.1 From the soft phase to the hard phase

We first introduce the transition from the soft phase to the hard phase which is dependent from the claim amounts. To model the fact that the environment process goes to the hard phase when a claim amount is greater than the threshold x_0, we use the probabilities

$$p_{1j} = P(Y_{1j} > x_0) = 1 - F_{Y_{1j}}(x_0) \tag{4.5}$$

where $Y_{1j} \sim Erlang(m_{1j}, \lambda_{1j})$, for $j = 1, \ldots, n_1$. As the random variables Y_{1j} are ordered by decreasing mean, we have $p_{11} > \cdots > p_{1n_1}$. Then we integrate these probabilities into the transition matrix Λ of the initial Markov process $\{I_t\}$ as follows. Construct line vectors $\widehat{p}_{1j} = (0, \ldots, 0, p_{1j})$ of length m_{1j} and concatenate them in $p_1 = (\widehat{p}_{11}, \ldots, \widehat{p}_{1n_1})$. Then, set

$$P_1 = (p_1' \otimes e_{k_1}),$$

the matrix containing the probabilities for the claim amounts to be greater than x_0 when the environment process is in the soft phase as needed to incorporate them into Λ. The latter becomes

$$\Lambda = \begin{pmatrix} K_1 & 0 & k_1\alpha_1 & 0 \\ 0 & K_2 & 0 & k_2\alpha_2 \\ P_1^- \odot g_1\beta_1 & P_1 \odot g_1\beta_2 & G_1 & 0 \\ 0 & g_2\beta_2 & 0 & G_2 \end{pmatrix},$$

where

$$P_1^- = ((e_{n_1} - p_1)' \otimes e_{k_1}),$$

and the operator $\odot$ is the Hadamard product (entrywise product). Thus the transition intensity matrix $\widehat{\Theta}$ of $\{J_t\}$ can be obtained as exposed in Section 4.2. The concept of dependent changes is here probabilistically respected to approximate the ruin probability.

4.4.2 From the hard phase to the soft phase

We now include the fact that the cycle process can go back to the soft phase from the hard phase. As said previously, we do not suppose that this transition is dependent from the claim amount but that the expected time spent in the hard phase is fixed at time t_0. This transition can happen at any state of the Erlang time $\mathcal{H}$ when the process is in the hard phase. The possibility to go from the hard phase to the soft phase is introduced by a transition probability q in the block $\widehat{\Theta}_{--}$ of the Markovian process $\{J_t\}$. It corresponds to the probability of changing from the hard phase to the soft phase at any state of the Erlang time $\mathcal{H}$ and is given as follows.

Note by Z the number of Bernoulli trials with success probability q needed to get one change from the hard phase to the soft phase. The random variable Z is indeed geometrically distributed with mean $1/q$. We set the expected value of Z equal to the number of stages of the Erlang time $\mathcal{H}$ at which we want the cycle to jump to the soft phase. We write this as

$$E[Z] = a \times t_0$$

which ensures that the time spent in the hard phase is t_0 on average. The rate parameter a of the Erlang time $\mathcal{H}$ corresponds to the number of states needed to have a time of length of one year. We thus deduce the probability q as

$$q = \frac{1}{a \times t_0} = \frac{T}{L \times t_0}.$$

For a better overview, suppose that $\mathcal{H} \sim Erlang(L, a)$ with $L = 2$ and that the inter-claim times W_1 and W_2 are identically distributed. Their canonical random variable W is distributed as a mixture of exponential distributions with intensity matrix K such as

$$K = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}.$$

The block $\widehat{\Theta}_{--}$ becomes

$$\begin{aligned}
\widehat{\Theta}_{--} &= \left(\begin{array}{cccc|cccc}
-\lambda_1 - a & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & -\lambda_2 - a & 0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & -\lambda_1 - a & 0 & aq & 0 & a(1-q) & 0 \\
0 & 0 & 0 & -\lambda_2 - a & 0 & aq & 0 & a(1-q) \\
\hline
0 & 0 & 0 & 0 & -\lambda_1 - a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda_2 - a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda_1 - a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 - a
\end{array}\right) \\
&= \begin{pmatrix} \Lambda_{--} - aI & M^* \\ 0 & \Lambda_{--} - aI \end{pmatrix}.
\end{aligned}$$

The environment process can go from the hard phase to the soft phase without affecting the run of the Erlang time $\mathcal{H}$. This can of course be extended for $L \geq 2$ leading to a block matrix

$$\widehat{\Theta}_{--} = \begin{pmatrix}
\Lambda_{--} - aI & M^* & 0 & 0 & \cdots & 0 & 0 \\
0 & \Lambda_{--} - aI & M^* & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \Lambda_{--} - aI & M^* \\
0 & 0 & 0 & 0 & \cdots & 0 & \Lambda_{--} - aI
\end{pmatrix},$$

where the block M^* contains the transition probabilities from the hard phase to the soft phase.

4.4.3 Computational aspects

Ramaswami et al. (2008) proposed a recursion procedure to solve Equation (4.3) in η using its particular structure which permits to reduce the dimension of the matrices involved in Theorem 4.1 and gain significative computational time. Indeed, the upcrossing phase probability matrix η has the following upper triangular block Toeplitz structure

$$\eta = \begin{pmatrix}
\eta_0 & \eta_1 & \eta_2 & \cdots & \eta_{L-1} \\
0 & \eta_0 & \eta_1 & \cdots & \eta_{L-2} \\
0 & 0 & \eta_0 & \cdots & \eta_{L-3} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \eta_0
\end{pmatrix}.$$

Denote that in our model, the block $\widehat{\Theta}_{--}$ of the intensity matrix $\widehat{\Theta}$ can be expressed as

$$\widehat{\Theta}_{--} = \begin{pmatrix} \Lambda_{--} - aI & M^* & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda_{--} - aI & M^* & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \Lambda_{--} - aI & M^* \\ 0 & 0 & 0 & 0 & \cdots & 0 & \Lambda_{--} - aI \end{pmatrix},$$

where the block M^* contains the transition probabilities between the two phases as explained above. Following the idea of Ramaswami et al. (2008), writing out Equation (4.3) leads to the recursion equations for the blocks of η

$$\eta_0 \Lambda_{+-} \eta_0 + R_{--}^{-1} (\Lambda_{--} - aI) \eta_0 + \eta_0 \Lambda_{++} + R_{--}^{-1} \Lambda_{-+} = 0 \tag{4.6}$$

$$\eta_\ell \Lambda_{++} + \sum_{j=0}^{\ell} \eta_j \Lambda_{+-} \eta_{\ell-j} + R_{--}^{-1} (\Lambda_{--} - aI) \eta_\ell + R_{--}^{-1} M^* \eta_{\ell-1} = 0; \qquad \ell = 1, \ldots, L-1. \tag{4.7}$$

Define for a matrix A the vector vec(A) as the vector obtained by concatenating successive columns of A from a single matrix. From Graham (1981), we can write for a matrix product AXB that

$$\text{vec}(AXB) = (B^\top \otimes A)\text{vec}(X),$$

where $B^\top$ is the transpose of the matrix B. As in Ramaswami et al. (2008), we use this notation and Equations (4.7) to derive a recursive expression for η_ℓ, $\ell = 1, \ldots, L-1$ and obtain

$$\begin{aligned} \text{vec}(\eta_\ell) = &\left\{ [\Lambda_{++} + \Lambda_{+-}\eta_0]^\top \otimes I + I \otimes \left[R_{--}^{-1} (\Lambda_{--} - aI) + \eta_0 \Lambda_{+-} \right] \right\}^{-1} \\ &\times \text{vec} \left(- \left[R_{--}^{-1} M^* \eta_{\ell-1} + \sum_{j=1}^{\ell-1} \eta_j \Lambda_{+-} \eta_{\ell-j} \right] \right). \end{aligned} \tag{4.8}$$

Then, we only need to solve the nonlinear Equation (4.6) using a fixed point calculation or the quadratic algorithm from Ahn and Ramaswami (2003) to approximate η_0 and use the recursive expression (4.8) for η_ℓ, $\ell = 1, \ldots, L-1$ in order to derive an approximation of η.

4.5 Control of precisions

The control of precisions of our approximation method is based on the number L of stages of the Erlang horizon $\mathcal{H}$, the cycle change probabilities and the fitting algorithms that we use to obtain claim amounts distributed as mixtures of Erlang distributions.

4.5.1 Model errors

As shown is Theorem 6 of Asmussen et al. (2002), the following convergence result for the finite-time ruin probability stands:

$$\psi(u, H_L) = \psi(u, T) + \frac{\psi^{(2)}(u, T)}{L2!} + O(L^{-2}),$$

where $T > 0$ and H_L denote the Erlang distribution for $\mathcal{H}$ with L stages and mean T. The model error can be estimated using the numerical computation for the second derivative of $\psi(u, T)$ as

$$\psi^{(2)}(u, T) = \frac{\psi(u, T + h) + 2\psi(u, T) - \psi(u, T - h)}{h^2} + O(h^2).$$

The same authors also suggest to use the Richardson extrapolation leading to the improved estimate

$$\psi(u, T) \approx (L + 1)\psi(u, H_{L+1}) - L\psi(u, H_L) \tag{4.9}$$

with a refined convergence rate L^{-2}.

Using probabilities $p_{1j} = Pr(Y_{1j} > x_0)$, $j = 1, \ldots, n_1$, as probabilistic modeling for the cycle environment to change from the soft phase to the hard phase can provide some errors. Indeed, the closer to 0 or to 1 all the p_{1j}'s, the more efficient the approximation. So, the threshold x_0 must be appropriately fixed. Giving such phase change probabilities leads to consider the Y_{1j} as if they are separate random variables.

The change of the environment process from the hard phase to the soft phase as explained in Subsection 4.4.2 introduces a possible change of phase even when the process enters in the hard phase in the t_0 last years of the horizon time T. There, the process should not be able to change the phase but setting different probabilities for the phase change breaks the repeating block structure of η and then the recursion equation (4.8) could not be used anymore. We suggest then to use the possibility for the environment process to go back to the soft phase only for a large horizon time T and a large ratio T/t_0.

Another part of the potential error comes from the fitting algorithm which is explained in the next Section. To ensure a good fit, several statistical tools such as Chi-squared test, Kolmogorov-Smirnov test, QQ- and PP-plots can be used.

4.5.2 Numerical errors

Some numerical issues can also include some deviation as matrix inversions required to derive matrix η. These matrix inversions are done using the matrix right division operator

"/" in Matlab R2007a software. Furthermore, we use the software package *Expokit* from R. B. Sidje (see Sidje (1998)) which uses Krylov subspace projection methods in order two compute the matrix exponential of Equation (4.4). The resort to this numerical tool can also include a slight deviation but allows to save substantial computation time.

4.6 Numerical applications

We illustrate in this section the use of our model when it is considered that the phase change instants depend on the claim amounts. As discussed in the Introduction, the European Solvency II regulation project will impose the insurance and reinsurance companies to have a 1-year horizon time ruin probability less than a certain threshold which is fixed at 0.5%. For reinsurance companies dealing with large claims and catastrophic events, it seems more realistic to consider a 5- or 10-year horizon time with a ruin probability respectively around 2.5% or 5%. We propose for this application to deal with a 5-year period with a ruin probability around 2.5%. We also suppose that the environment process begins in the soft phase and does not go back to it. This last hypothesis of not going back to the soft phase could be seen as restrictive but is in fact quite realistic. Indeed, our studying time horizon is smaller than the average underwriting cycle length of 6 years as explained in Weiss (2007), Cummins and Outreville (1987) or Venezian (1985). The phases differ here only by their premium income rates, a low one for the soft phase and a high one for the hard phase.

4.6.1 Data simulation

We first simulate 5000 realizations from a Generalized Pareto Distribution (GPD) with cumulative distribution function (c.d.f.), for $x > 0$,

$$F_{GPD}(x) = 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{-1/\xi},$$

where $\sigma > 0$ is the scale parameter and $\xi > 0$ the shape parameter. This last parameter is also referred as the extreme value parameter in extreme value theory, where $\alpha = 1/\xi$ is called the index of the regular variation of the distribution. This distribution is heavy-tailed and allows to model large claims and catastrophic events as fire losses, wind storm, earthquake, etc. The smaller the index α and, consequently, the greater the shape parameter ξ, the heavier the right tail of the distribution. For this illustration, we set $\alpha = 2.5$ ($\xi = 1/2.5$) and $\sigma = 1$. A histogram for the simulated data is given in Figure 4.2. The empirical mean and variance are here respectively 1.63130 and 11.06983.

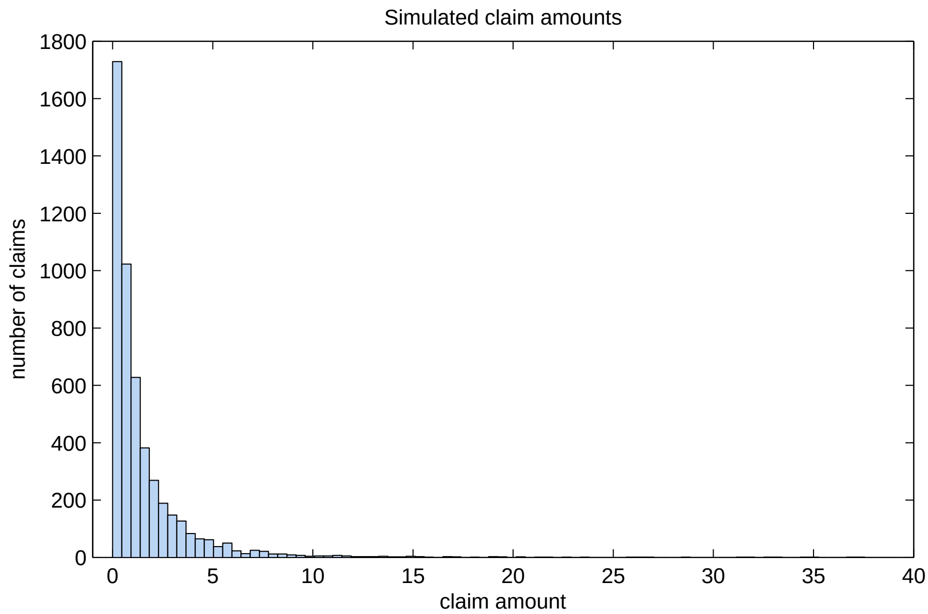

FIGURE 4.2 – Simulation of claim amounts from a GPD

4.6.2 Phase-type fitting

Our model requires the claim amounts to be phase-type distributed. There are different propositions that have been made in the literature to fit phase-type distributions to heavy-tailed distribution or data sets. One could object that this does not allow for heavy tails. However, as we are interested in ruin probabilities of order 2.5% or 5% (for time horizon 5 years or 10 years), we are not necessarily in the domain of asymptotic approximation and it is possible to have an appropriate fit of the claim size distribution up to the required level in our case. If we want to study ruin probabilities of magnitude 10^{-4}, this method would of course be inappropriate. For example, Feldmann and Whitt (1998) proposed a heuristic-based approach to fit heavy-tailed distribution functions. But this requires to first fit a heavy-tailed distribution to the data set which introduces additional errors. The popular expectation-maximization (EM) algorithm allows to fit directly data sets. Several variants of fitting methods using the EM algorithm to obtain a general or a particular phase-type distribution have been explored as in Asmussen and Bladt (1996), Khayari et al. (2003) or Riska et al. (2004). For this application, we use the EM algorithm developed in Lee and Lin (2009) to fit mixtures of Erlang distributions with common scale parameter to heavy-tailed data sets. As mentioned in this reference, this class of mixtures of Erlang distributions with common scale parameter is dense in the space of positive continuous distributions and then can approximate any positive absolutely continuous distribution to any accuracy. Moreover, this class is closed in mixing, convolution, compounding and aggregation of data which offers a good tractability for actuarial purposes. Wang et al. (2006) built an EM fitting algorithm for mixtures of Erlang distributions with common shape parameter and Thummler et al. (2006) with no required common parameter. But these two classes does not fulfill the closure properties mentioned above.

The density of a mixture of k Erlang distributions with common scale parameter $\theta > 0$ is

$$f(x) = \sum_{j=1}^{k} \alpha_j \frac{x^{r_j - 1} e^{-x/\theta}}{\theta^{r_j} (r_j - 1)!},$$

where $r_j > 0$ and $\alpha_j > 0$ are respectively the shape parameter and the weight of the jth Erlang distribution. The distribution weights must observe the following constraint $\sum_{j=1}^{k} \alpha_j = 1$.

Applying the Lee & Lin algorithm to our simulated data set, it results that a density using a mixture of 14 Erlangs fits the empirical density very satisfactorily. The common scale parameter is evaluated at $\theta = 0.81585$ and the shape parameters and weights for every Erlang distributions of the mixture are presented in Table 4.2. This table also displays the expected value associated for each Erlang r.v. Y_j of the distribution mixture.

j	r_j	α_j	$E[Y_j]$	j	r_j	α_j	$E[Y_j]$
1	75	0.00063	61.18865	8	22	0.00122	17.94867
2	59	0.00021	48.13507	9	16	0.00469	13.05358
3	48	0.00012	47.31922	10	15	0.00283	12.23773
4	40	0.00199	32.63395	11	14	0.00166	11.42188
5	39	0.00024	31.81810	12	8	0.03157	6.52679
6	25	0.00078	20.39622	13	4	0.14131	3.26339
7	24	0.00122	19.58037	14	1	0.81155	0.81585

Table 4.2 – Parameters of the fitted mixture of Erlang distributions.

To attest to the goodness of fit, a graphical comparison between a relative histogram for the simulated data and the fitted distribution can be seen in Figure 4.3, a PP-plot and a QQ-plot are drawn in Figure 4.4. All these graphical tools already show that the fit is very satisfying even in the right tail. A Kolmogorov-Smirnov test is used to confirm the high quality of the fit. Indeed, with a test statistic of 0.00578 and a cutoff value of 0.01917, the p-value is evaluated at 0.99612 and the hypothesis that the data sample comes from the fitted distribution is not rejected.

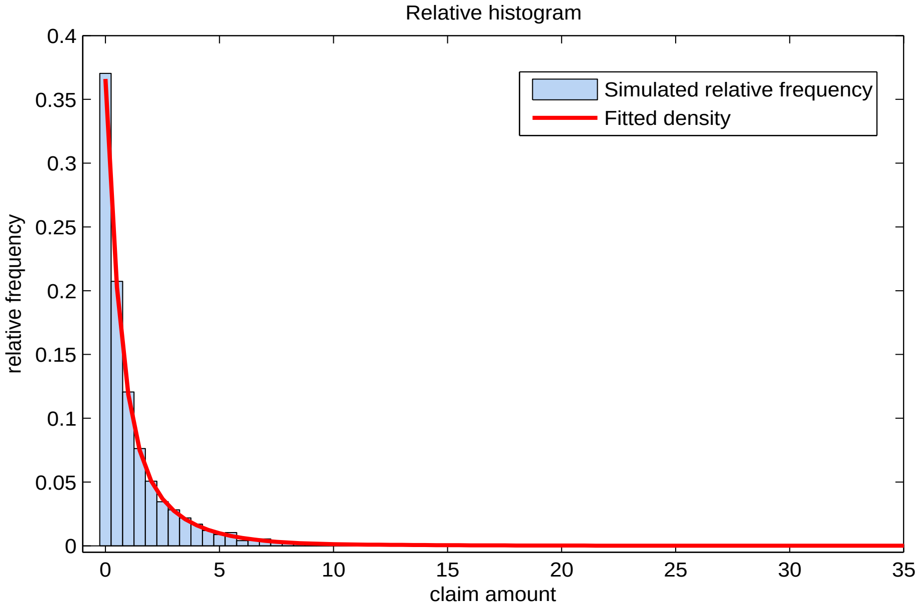

FIGURE 4.3 – Comparison between a relative histogram for the simulated data and the fitted density

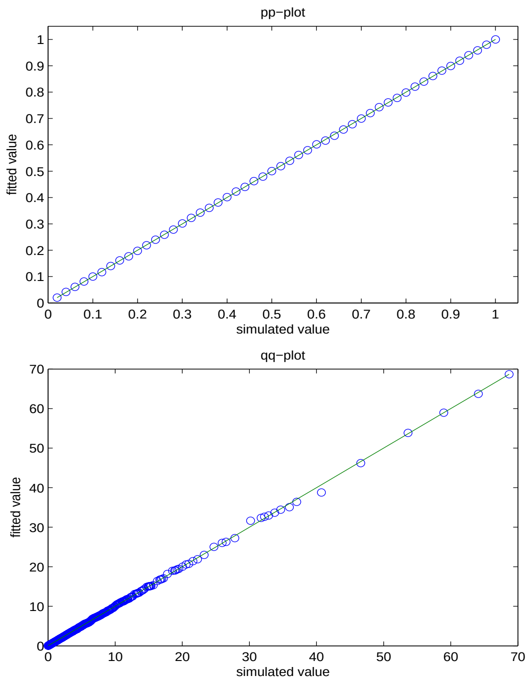

FIGURE 4.4 – PP-plot and QQ-plot for the simulated data versus the fitted distribution

4.6.3 Ruin probability approximation

We now use our model to evaluate the ruin probability for a horizon $T = 5$ years where the claim amounts are distributed according to the fitted mixture of Erlang distributions. Let the inter-claim time random variable W be exponentially distributed with parameter $\lambda = 2$. So that, the average time between two claim arrivals is 0.5 year and two claims per year are expected. As for the claim amounts, we suppose that the inter-claim time r.v.'s are identically distributed in the two phases of the cycle. Only the premium income rates differs between the two phases. Indeed, we define the premium rate c_i, with $i = 1$ for the soft phase and $i = 2$ for the hard phase, such as

$$c_i = (1 + \rho_i) \times E[\widehat{S}_1] = (1 + \rho_i) \times E[W]E[\widehat{X}_1],$$

where ρ_i is the security loading corresponding to the phase i, $\widehat{S}_t$ is the surplus process where the claim amount random variables $\widehat{X_k}$ are i.i.d. with the empirical simulated distribution. Froot (2001) shows that in the market of large claims and catastrophic events the reinsurance premiums can be more than seven times the expected loss during a period. Following this idea, we set $\rho_1 = 30\%$ and $\rho_2 = 300\%$. We fix the initial surplus at $u = 30$ and the claim amount threshold $x_0 > 0$ that provokes the passage from the soft phase to the hard phase at $x_0 = 25$. This last value represents the 99.663% quantile of the empirical cumulative distribution. Over the 5 years horizon with 2 claims per year, this does not seem to have an impact the ruin probability. In Table 4.3 is displayed the probability p_{1j} of causing a change from the soft cycle to the hard cycle for the jth Erlang distribution of the mixture, $j = 1, \ldots, 14$, as they are defined in Equation (4.5).

j	p_{1j}	j	p_{1j}
1	≈ 1	8	0.04326
2	≈ 1	9	0.00138
3	0.99999	10	0.00064
4	0.94060	11	0.00028
5	0.91826	12	≈ 0
6	0.13159	13	≈ 0
7	0.09433	14	≈ 0

TABLE 4.3 – Phase change probabilities of the Erlang distributions of the fitted mixture

We can now construct the required matrices Λ, R_{--} and M^* for our dependent model and use Equations (4.6), (4.8) and Theorem 4.1 to approximate the finite-time ruin probability $\psi(u, T)$. For comparison issues, Monte Carlo (MC) simulations for $\psi(u, T)$ are performed using both empirical and fitted distributions for the claim amounts. The results for the three approximations are given in Table 4.4 where we used for our method a number of Erlang stages $L = 50$.

Method	Approximation for $\psi(u,T)$
Dependent model	2.37199%
MC sim. with fitted distr.	2.35448%
MC sim. with empirical distr.	2.34998%

Table 4.4 – Approximated finite time ruin probabilities

We see on this table that our approximation is quite accuracy with a relative error of 0.73% from the Monte Carlo simulation with the fitted distribution and 0.92% from the Monte Carlo simulation with the empirical distribution. The error inserted from the distribution fitting is 0.19%. The computational time required for our model is here around 100 times less than than the one needed for the two simulation methods.

The previous example is the reference parametrization that is used in the next subsection. But, we also give now some results in Table 4.5 when modifying one parameter. We see that our approximations are sill very closed to the Monte Carlo simulations with the empirical distribution for the claim amounts.

Parameter modified	MC simulation	Dependent model	Relative error
$x_0 = 30$	2.42243%	2.42480%	0.098%
$\rho_2 = 200\%$	2.38310%	2.40007%	0.707%

Table 4.5 – Approximated finite time ruin probabilities with a modified parameter

4.6.4 Comparison between our dependent model and independent changes of cycle phase

We illustrate now the impact of the dependence of the change of phase from claim amounts. We compare our model to a so-called independent model in which the change of phase can occur without any dependence from the claim amounts. For this independent model, we construct an intensity matrix $\widehat{\Theta}$ for the environment Markov process J_t in which the change from the soft phase to the hard phase can appear at any state of the Erlang horizon time $\mathcal{H}$ with probability $p_0 \in [0,1]$. This is done by modifying the bloc matrix $\widehat{\Theta}_{--}$ such that at any state i, $i = 1, \ldots, L-1$, of the Erlang time in the soft cycle phase, the process can go to state $i+1$ of the Erlang time in the hard cycle phase with probability p_0. Therefore, the change of phase is independent from the claim amount.

For exponentially distributed inter-claim times with parameter λ and for an Erlang time horizon $\mathcal{H} \sim Erlang(2, a)$, the bloc $\widehat{\Theta}_{--}$ becomes

$$\widehat{\Theta}_{--} = \left(\begin{array}{cc|cc} -\lambda - a & 0 & a(1-p_0) & ap_0 \\ 0 & -\lambda - a & 0 & a \\ \hline 0 & 0 & -\lambda - a & 0 \\ 0 & 0 & 0 & -\lambda - a \end{array}\right)$$

and can also be written as

$$\widehat{\Theta}_{--} = \begin{pmatrix} \Lambda_{--} - aI & M^* \\ 0 & \Lambda_{--} - aI \end{pmatrix},$$

where, as in Subsection 4.4.2, M^* contains the direct transitions between the phases. This can, of course, also be extended to a number of Erlang stages $L \geq 2$.

In order to compare this independent model to our dependent model, we assign to these two models the same probability to change at least one time from the soft phase to the hard phase. For the independent model, this probability is $P_0 = 1 - (1 - p_0)^L$. In our dependent model, the probability to go to the hard phase after each claim is

$$\sum_{j=1}^{n} \alpha_j p_{1j}.$$

Then the probability to have no change from the soft phase to the hard phase during $[0, T]$ is

$$E[(1 - \sum_{j=1}^{n} \alpha_j p_{1j})^N]$$

where $N \sim Poisson(T\lambda)$. Finally, the probability to change from the soft phase to the hard phase at least one time during $[0, T]$ is

$$P = 1 - E[(1 - \sum_{i=1}^{n} \alpha_i p_i)^N] = P_0$$

and we obtain

$$p_0 = 1 - E[(1 - \sum_{i=1}^{n} \alpha_i p_i)^N]^{1/L}.$$

We also construct an intermediate model such that the probability to change from the soft phase to the hard phase at least one time is still the same where the cycle can change either independently from claim amounts or because of a large claim. This intermediate model permits to observe the slide between the dependent and independent models and to see the impact of the claim amount dependence on the finite-time ruin probability. The probability to change from the soft phase to the hard phase at least one time during $[0, T]$ for this intermediate model is

$$P(\theta) = 1 - \left((1 - p(\theta))^L\right)\left(E\left[(1 - \sum_{i=1}^{n} \theta \alpha_i p_i)^N\right]\right) = P_0,$$

where $\theta \in [0,1]$ and $p(\theta)$ represents the part of the phase change probability that is independent from the claim amounts. Then we obtain

$$p(\theta) = 1 - (1-p_0) \times \left(E\left[(1 - \sum_{i=1}^{n} \theta \alpha_i p_i)^N \right] \right)^{-1/L}.$$

The bounds $p(0) = p_0$ and $p(1) = 0$ correspond respectively to the independent model and the dependent model.

For our application, as we use the Richardson extrapolation recalled in Equation (4.9) for a number of Erlang stages $L = 50$, two probabilities p_0 are evaluated. For $L = 50$, we obtain $p_0 = 6.63365 \times 10^{-4}$ and for $L = 51$, $p_0 = 6.50362 \times 10^{-4}$. Thus, the ruin probability is approximated at 2.50960% with the independent model versus an evaluation at 2.37199% for our dependent model. We recall that the MC simulation with the empirical distribution provides a finite-time ruin probability at 2.34998%. The error from the independent model on our dependent model can be evaluated at 5.48% and the one on the MC-simulation at 6.36%. We can imagine that this difference would be much larger when looking at a greater time horizon but also when including more phases in a cycle. On Figure 4.5 are drawn the slide of the ruin probability with the intermediate model and the slide of the underlying error made comparing to our dependent model.

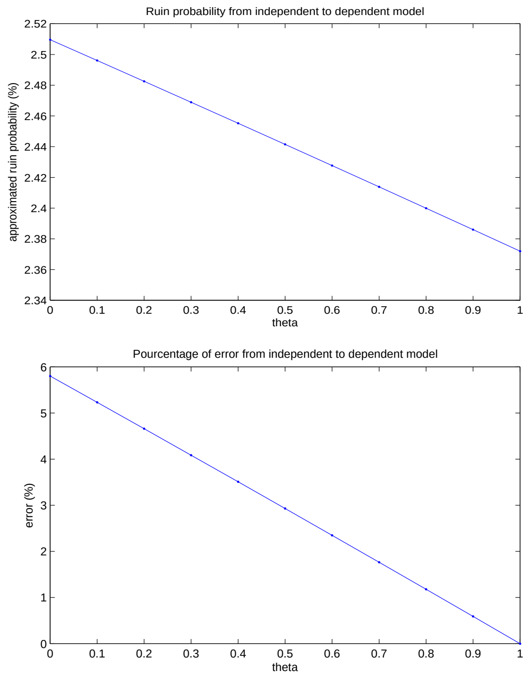

FIGURE 4.5 – Comparaison between the dependent model and the independent model

4.7 Conclusion

This paper provides an extension for the Erlangization method taking into account the reinsurance cycles in order to evaluate the finite-time ruin probability in the Sparre-Andersen model with phase-type distributed claim amounts and inter-claim times. We assume the presence of two phases in a reinsurance cycle, a soft phase and a hard phase. We first propose a naive model with deterministic changes between the two phases. Then, a more realistic model is developed where the change from the soft phase to the hard phase depends on the severity of the claim amounts. We illustrate our model by a consistent application and compare our results to Monte Carlo simulations and to the case where the claim amounts have no impact on the phase changes. We show thet this feedback control may change the ruin probability by 6% in our numerical example.

Acknowledgements

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APPENDIX

A Matrix inversion

The Erlangization method used to evaluate the ruin probability or equivalently the probability for the fluid model to reach u before the horizon $\mathcal{H}$ requires the inversion of several matrices. The matrix of interest η in Corollary 4.1 can be approximated by the iteration scheme $\eta = \lim_{n\to\infty} \eta^{(n)}$ where $\eta^{(0)} = 0$ and

$$\eta^{(n+1)} = ((\delta I + \Theta_{--})\eta^{(n)} + \Theta_{-+})(\delta I - (\Theta_{++} + \Theta_{+-}\eta^{(n)}))^{-1} \tag{4.10}$$

$$= A^{(n)} \times \left(B^{(n)}\right)^{-1}, \tag{4.11}$$

for a convergence improving parameter $\delta > 0$, satisfying $\delta > max(-(\Theta_{--})_{ii})$ for all state i of $\mathcal{H}$.

Thus, to determine $\eta^{(n+1)}$, we need to inverse the matrice $B^{(n)}$. The Erlangization method, according to the specific structure of the intensity matrix of the Erlang horizon $\mathcal{H}$, provides that the block matrices Θ_{--}, Θ_{-+}, Θ_{+-} and Θ_{++} have all an upper triangular block Toeplitz structure and that the matrix $B^{(n)}$ takes this form

$$B^{(n)} = \begin{pmatrix} B_1^{(n)} & B_2^{(n)} & B_3^{(n)} & \dots & B_L^{(n)} \\ 0 & B_1^{(n)} & B_2^{(n)} & \dots & B_{L-1}^{(n)} \\ 0 & 0 & B_1^{(n)} & \dots & B_{L-2}^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_1^{(n)} \end{pmatrix}.$$

.

Indeed, this structure is stable for addition, multiplication and inversion and $\left(B^{(n)}\right)^{-1}$, $A^{(n)}$ and $\eta^{(n)}$ also have the form.

Let M be a $mn \times mn$ matrix such as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where the $m \times m$ block matrix M_1 and the $n \times n$ block matrix M_4 are invertible. Then, we have

$$M^{-1} = \begin{pmatrix} (M_1 - M_2M_4^{-1}M_3)^{-1} & -M_1^{-1}M_2(M_4 - M_3M_1^{-1}M_2)^{-1} \\ -M_4^{-1}M_3(M_1 - M_2M_4^{-1}M_3)^{-1} & (M_4 - M_3M_1^{-1}M_2)^{-1} \end{pmatrix}.$$

To inverse the matrix $B^{(n)}$ described before, we can split it into four blocks as follows :

$$B^{(n)} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} B_1^{(n)} & B_2^{(n)} & \dots & B_{L-1}^{(n)} \\ 0 & B_1^{(n)} & \dots & B_{L-2}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_1^{(n)} \end{pmatrix}, \quad M_2 = \begin{pmatrix} B_L^{(n)} \\ B_{L-1}^{(n)} \\ \vdots \\ B_2^{(n)} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } M_4 = \begin{pmatrix} B_1^{(n)} \end{pmatrix}.$$

Thus we have

$$(B^{(n)})^{-1} = \begin{pmatrix} M_1^{-1} & -M_1^{-1} M_2 M_4^{-1} \\ 0 & M_4^{-1} \end{pmatrix}.$$

The remaining difficulty lies in the inversion of the block matrix M_1. To this end, we proceed algorithmically using the same procedure as for $B^{(n)}$.

B Initialization for the deterministic phase change instants model

For the computation of the approximation of the ruin probability in the case of deterministic phase change instants, we have some technical but not stopping constraints on the numbers of states L_1 for $\mathcal{H}_1$ and L_2 for $\mathcal{H}_2$. Indeed, we have

$$\mathcal{H}_1 \sim Erlang(L_1, a),$$
$$\mathcal{H}_2 \sim Erlang(L_2, a).$$

The final horizon time T can be decomposed as $T = T_1 + T_2$ such as

$$T_1 = E[\mathcal{H}_1] = \frac{L_1}{a},$$
$$T_2 = E[\mathcal{H}_2] = \frac{L_2}{a}.$$

Both T_1 and T_2 are fixed and then we have to find appropriate values for $a \in \mathbb{R}_+^*$, $L_1 \in \mathbb{N}$ and $L_2 \in \mathbb{N}$ such that

$$L_1 = aT_1,$$
$$L_2 = aT_2,$$
$$L = L_1 + L_2 = a(T_1 + T_2) = aT.$$

These equalities can be fulfilled by choosing reasonable values for L, L_1 and L_2 knowing that improving this values leads to better approximations.

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Chapitre 5

Conclusion

Nous avons présenté différents aspects de la théorie du risque dans lesquels nous introduisons des modèles de dépendance. Dans un premier temps, nous avons vu comment utiliser les copules pour lier les différents risques du portefeuille d'assurance afin d'obtenir des expressions exactes avec la copule FGM puis des approximations avec n'importe quelle copule pour l'allocation de capital basée sur la TVaR. Dans ce cadre, le recours à la copule FGM généralisée pour l'obtention d'expressions exactes pourrait être étudié, ce qui permettrait d'étendre le rang de dépendance entre les variables aléatoires par rapport à celui que propose la copule FGM.

Ensuite, nous avons considéré le processus aléatoire de la somme des valeurs présentes des sinistres dans lequel nous avons introduits une dépendance entre les temps inter-sinistres et les montants de sinistre. Pour ce faire, nous avons également considéré une copule FGM. Nous avons ainsi obtenu l'expression exacte du premier, du deuxième puis du moment d'ordre m du processus lorsque les temps inter-sinistre sont distribués selon une loi exponentielle. Comme précédemment, la généralisation à l'aide de la copule FGM généralisée pourrait être envisagée, de même que l'étude de la transformée de Laplace de la distribution de la somme des valeurs présentes des sinistres.

Enfin, nous avons abordé la théorie de la ruine en temps fini pour un portefeuille de compagnie de réassurance. Les différents éléments du modèle de surplus employé peuvent être influencés par les cycles de souscription. Nous avons ainsi proposé l'utilisation de la méthode d'erlangisation pour fournir une approximation de la probabilité de ruine sous cette influence des cycles, ceux-ci dépendant eux-même des montants de sinistre. Il serait également intéressant d'analyser sous ces hypothèses la fonction de pénalité de Gerber-Shiu permettant des mesures de l'instant de ruine, de la sévérité de la ruine et du niveau de surplus juste avant la ruine.

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