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# ÉVALUATION DES MESURES DE RUINE DANS LE CADRE DE MODÈLES AVANCÉS DE RISQUE

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# RÉSUMÉ


La théorie du risque consiste en l'étude de modèles décrivant le processus de surplus d'une compagnie d'assurance. L'évaluation de différentes mesures de ruine dans le cadre de ces modèles permet d'obtenir une idée générale de la santé financière de la compagnie d'assurance et du risque assumé par celle-ci. Le modèle classique de risque pour décrire les arrivées et les coûts des sinistres est le modèle Poisson composé. Ce modèle est basé sur une hypothèse d'indépendance entre le montant des sinistres et le temps écoulé entre chacun. Cette hypothèse facilite le calcul des mesures de ruine mais peut s'avérer trop restrictive dans différents contextes.

L'objectif principal de cette thèse est l'étude d'extensions du modèle classique dans lesquelles sont introduites une structure de dépendance entre la sévérité et la fréquence des sinistres. La copule de Farlie-Gumbel-Morgenstern et une extension de cette copule sont utilisées pour définir cette structure. En raison de la forme et de la flexibilité de ces copules, il est possible d'adapter les outils développés récemment en théorie du risque dans l'évaluation et l'analyse des mesures de ruine. La fonction de Gerber-Shiu et certains cas particuliers de cette fonction, comme la transformée de Laplace du temps de la ruine et l'espérance de la valeur actualisée du déficit à la ruine, sont étudiées dans le cadre de ces extensions. On s'intéresse également à l'évolution du processus de surplus en présence d'une barrière horizontale. Les mesures de ruine citées plus haut, ainsi que le montant total actualisé des dividendes distribués sont évaluées.

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# CHAPITRE I

# INTRODUCTION

L'assurance est une opération par laquelle l'assureur offre à un assuré une protection en cas de réalisation d'un événement incertain prédéfini moyennant le paiement d'une prime par cet assuré. En contrepartie, la compagnie d'assurance s'engage à payer une partie ou la totalité de la perte encourue à la suite d'un sinistre résultant d'accidents, d'incendies, d'inondations, de catastrophes naturelles, etc. Ce sont ces deux engagements (paiement de la prime contre paiement, le cas échéant, de la prestation) qui constituent le contrat d'assurance.

La théorie de la ruine est un des domaines importants de la recherche en actuariat. Elle consiste en l'étude de modèles décrivant le processus de surplus d'une compagnie d'assurance et en l'évaluation de la probabilité que le surplus prenne au moins une fois une valeur négative. Le surplus d'un portefeuille d'une compagnie d'assurance est la différence entre le total des primes et des montants remboursés pour les sinistres. On définit la ruine comme étant le premier moment où le surplus devient négatif. Le modèle classique pour décrire les arrivées et les coûts des sinistres pour un portefeuille de contrats d'assurance est le modèle Poisson composé.

L'évolution du surplus est décrite par le processus $\underline{U} = \{U(t), t \geq 0\}$, où $U(t)$ représente le niveau de surplus à l'instant $t$ et est défini par

$$U(t) = u + pt - S(t), \quad t \geq 0, \tag{1.1}$$

où $p > 0$ est le taux de prime et $U(0) = u$ est le surplus initial. On suppose que le taux de prime est désigné par

$$p = (1+\mu)E\{S(1)\}, \tag{1.2}$$

où $\mu > 0$ est la marge relative de sécurité. L'évolution du processus de surplus $\underline{U}$ est représentée sur le graphique 1.2. Le processus $\underline{S} = \{S(t), \quad t \geq 0\}$, désigne le processus du montant total des sinistres où

$$S(t) = \sum_{j=1}^{N(t)} X_j, \qquad N(t) > 0, \tag{1.3}$$

et $S(t) = 0$ si $N(t) = 0$.

Dans le modèle classique de risque, le processus $\underline{N} = \{N(t), t \geq 0\}$, avec $(N(0) = 0)$ est un processus de Poisson homogène de paramètre $\lambda > 0$. Le montant du $i^{\text{ème}}$ sinistre est représenté par la variable aléatoire $X_i$ $(i = 1, 2, 3, \cdots)$. Soit $W_i$ $(i = 1, 2, 3, \cdots)$, une suite de variables aléatoires positives de loi exponentielle qui représentent le temps séparant le $(i-1)^{\text{ème}}$ et le $i^{\text{ème}}$ sinistre. On désigne par $T_n = W_1 + ... + W_n$ le temps d'attente jusqu'au $n^{\text{ème}}$ sinistre (voir figure 1.1). Le processus du nombre des sinistres jusqu'à l'instant $t$ est donc $N(t) = \sup\{n : T_n = W_1 + ... + W_n \leq t\}$.

On désigne par

$$T = \inf\{t > 0, U(t) < 0\}, \tag{1.4}$$

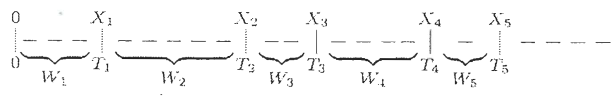


FIG. 1.1 – Le vecteur $(X, W)$ des montants des sinistres et des temps séparant deux sinistres

le premier moment où le surplus devient négatif.

De nombreux auteurs se sont intéressés à l'étude du modèle classique de risque. Des études sur ce sujet ont été présentées par Gerber (1979), Grandell (1991) et Rolski et al. (1999). Les probabilités de ruine, le déficit à la ruine, l'excédent juste avant la ruine ont été étudiés dans le modèle classique de risque.

On définit la fonction de Gerber-Shiu (1998) par

$$m_\delta(u) = E\Big[e^{-\delta T} w\Big\{U(T^-), |U(T)|\Big\} I(T < \infty)\Big|U(0) = u\Big], \quad u \geq 0, \qquad (1.5)$$

où $w(x, y)$ est une fonction positive pour $x, y \geq 0$, $\delta$ est un paramètre non négatif (la force d'intérêt) et $I$ est la fonction indicatrice, telle que $I(A) = 1$ si l'événement $A$ survient et égale à 0 sinon. La fonction (1.5) offre une généralisation d'un certain nombre de mesures de risque. En particulier, si $w(x, y) = 1$,

$$\phi_T(u) \quad = \quad E\left[e^{-\delta T} I(T < \infty)|U(0) = u\right], \quad u \geq 0, \qquad (1.6)$$

est la transformée de Laplace du temps de la ruine. Si $\delta = 0$ et $w\,(x, y) = 1$ pour $x, y \in \mathbb{R}^+$, (1.5) correspond à la probabilité de ruine $\psi\,(u) = \Pr\,(T < \infty)$.

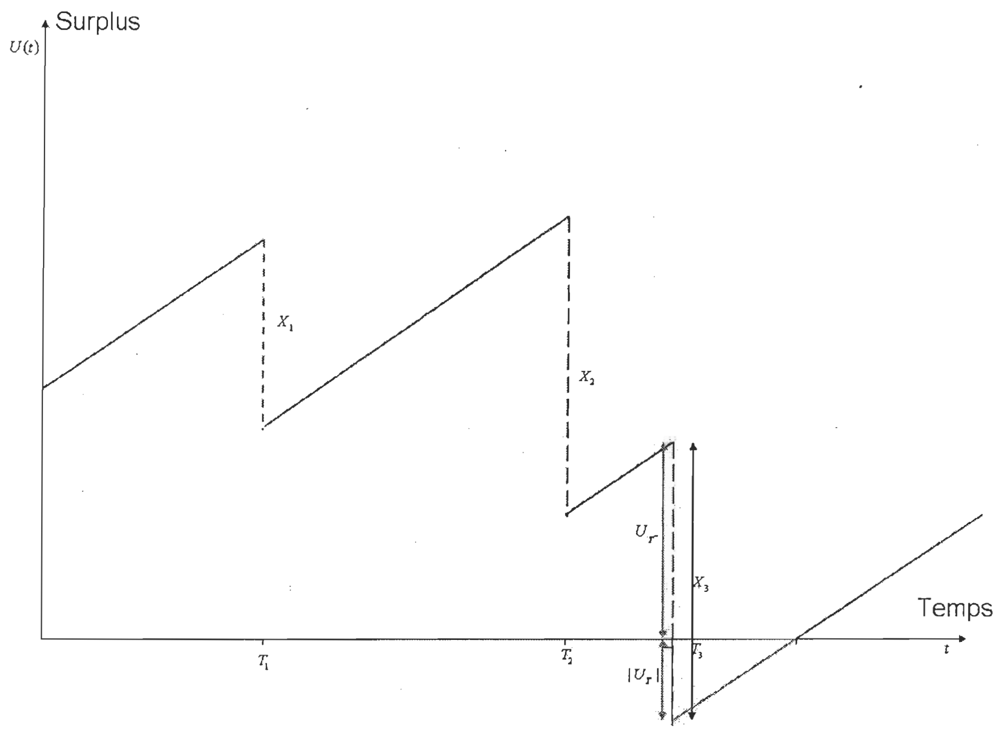


FIG. 1.2 – Évolution du surplus U(t)

La distribution de dividendes, en espèces ou en actions, constitue une composante fondamentale de la rentabilité d'une compagnie en croissance et un signal important d'une bonne santé financière. Soit $\underline{U_{\tilde{b}}} = \{U_{\tilde{b}}(t), t \geq 0\}$ le processus représentant le surplus de la compagnie au temps $t$ avec le versement instantanément aux actionnaires d'un dividende $p$ au-delà d'une barrière horizontale de niveau $\tilde{b} \geq u$, avec $u$ le surplus initial. Le surplus stagne à la valeur $\tilde{b}$ jusqu'à la prochaine réclamation. Ceci est illustré à la figure 1.3.

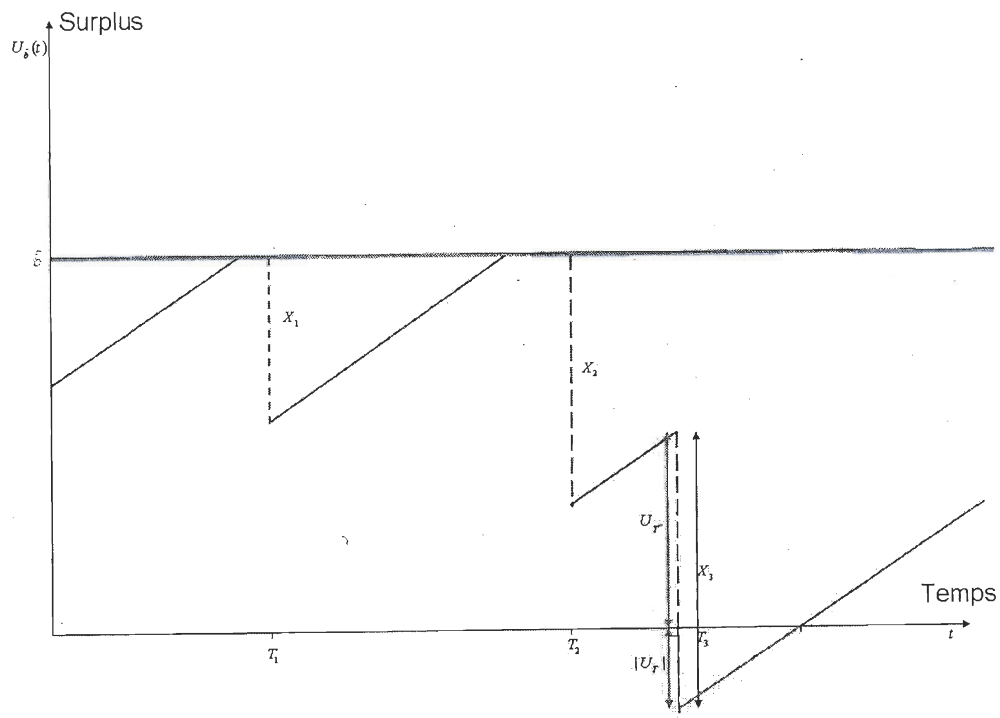


FIG. 1.3 – Évolution du surplus $U_{\tilde{b}}(t)$ avec une barrière $\tilde{b}$

Le surplus $U_{\tilde{b}}(t)$ est défini par

$$dU_{\tilde{b}}(t) = \begin{cases} pdt - dS(t), & U_{\tilde{b}}(t) < \tilde{b}, \\ -dS(t), & U_{\tilde{b}}(t) = \tilde{b}. \end{cases}$$

Le processus de surplus avec distribution de dividendes au-delà d'une barrière a été étudié par de nombreux auteurs, De Finetti (1957), Gerber (1979) et Lin et al. (2003). On définit par

$$T_{\tilde{b}} = \inf\{t > 0, U_{\tilde{b}}(t) < 0\},$$

le temps de la ruine en présence d'une barrière $\tilde{b}$. La transformée de Laplace de $T_{\tilde{b}}$, $\phi_{T_{\tilde{b}}}(u)$, est définie par

$$\phi_{T_{\tilde{b}}}(u) = E\left[e^{-\delta T_{\tilde{b}}} I(T_{\tilde{b}} < \infty) \Big| U_{\tilde{b}}(0) = u\right], \quad 0 \leq u \leq \tilde{b}.$$

On définit la fonction de Gerber-Shiu avec versement de dividende par

$$m_{\delta,\tilde{b}}(u) = E\left[e^{-\delta T_{\tilde{b}}} w\left(U(T_{\tilde{b}}^{-}), |U(T_{\tilde{b}})|\right) I(T_{\tilde{b}} < \infty) \Big| U_{\tilde{b}}(0) = u\right], \quad (1.7)$$

pour $0 \leq u \leq \tilde{b}$.

Dans le modèle classique de la théorie du risque, on suppose que les temps d'arrivées des sinistres et les montants des sinistres sont indépendants. Cependant, dans plusieurs contextes pratiques, cette hypothèse s'avère inadéquate. Pour un contrat d'assurance de dommages offrant une protection pour un tremblement de terre, le temps écoulé depuis la dernière catastrophe peut influencer la distribution des coûts de la présente catastrophe. Dans cette perspective, la dépendance entre les montants des sinistres et les temps d'arrivées des sinistres peut être modélisée à l'aide de copules qui sont un outil pertinent de l'étude de la dépendance.

Selon Sklar (1959), une copule est une fonction de répartition dont les marges sont uniformes. Les copules constituent un outil intéressant permettant de construire des distributions multivariées à partir des lois marginales. Nelsen (1999) et Joe (1997) sont des ouvrages de réference incontournables sur le sujet. Dans le cadre de l'actuariat et de la gestion quantitative des risques, citons notamment McNeil et al. (2005) et Denuit et al. (2005) qui présentent différentes applications de copules.

Dans cette thèse, nous nous intéressons à l'évaluation de la fonction de Gerber-Shiu dans le cadre d'un modèle de risque dont la structure de dépendance est basée sur une copule. La classe des copules choisie est celle de Farlie-Gumbel-Morgenstern généralisée. Cette copule flexible est définie comme suit :

$$C(u,v) \quad = \quad uv + \theta h(u)g(v), \quad 0 \leq u, v \leq 1.$$

où $h(u) = u^a(1-u)^b$ et $g(u) = u^c(1-u)^d$ sont des fonctions réelles définies sur $[0,1]$, $a, b, d$ sont des réels strictement positifs et $c$ un entier strictement positif. Rodríguez-Lallena et Úbena-Flores (2004) obtiennent des conditions sur $h$ et $g$ et des bornes sur le paramètre $\theta$ afin que $C$ soit une copule. Dans le cas où $a = b = c = d = 1$, $C$ est dite la copule de Farlie-Gumbel-Morgenstern classique.

Nous proposons une extension du modèle classique de risque dans laquelle la distribution du montant du $i^{\text{ème}}$ sinistre, $X_i$, dépend du temps écoulé entre le $(i-1)^{\text{ème}}$ et le $(i)^{\text{ème}}$ sinistre. On suppose que $\{(X_j, W_j), j \in \mathbb{N}^+\}$ forment une suite de vecteurs aléatoires indépendants et identiquement distribués de même loi que $(X, W)$ et que le montant du $i^{\text{ème}}$ sinistre $X_i$ et le temps $W_i$ sont dépendants. On introduit la classe de copules de Farlie-Gumbel-Morgenstern. Cette classe de copules qui est une perturbation de la copule d'indépendance, offre plus de flexibilité pour l'étude du processus de surplus. Sa forme permet d'adapter les outils développés récemment en théorie du risque dans l'évaluation et l'analyse des mesures de ruine.

Cette thèse se compose de cinq chapitres, en incluant le présent chapitre. Dans le chapitre 2, nous dérivons une équation intégro-différentielle de la fonction de Gerber-Shiu $m_\delta$ par l'entremise de la copule de Farlie-Gumbel-Morgenstern classique. Des expressions explicites sont obtenues pour la transformée de Laplace du temps de la ruine et du déficit à la ruine. Dans le chapitre 3, nous considérons une extension du modèle classique de risque où la distribution conjointe du temps entre deux sinistres et du montant des sinistres est introduite à l'aide de la copule de Farlie-Gumbel-Morgenstern généralisée. Des expressions analytiques exactes pour la transformée de Laplace du temps de la ruine et du déficit à la ruine sont évaluées. Le chapitre 4 est consacré à l'évaluation d'une équation intégro-différentielle de la fonction de Gerber-Shiu $m_{\delta,\tilde{b}}$ en présence d'une barrière constante $\tilde{b}$ et en supposant que la structure de dépendance est la copule classique de Farlie-Gumbel-Morgenstern. Sa solution peut être exprimée comme la somme de la fonction de Gerber-Shiu en l'absence de barrière plus une combinaison linéaire de deux solutions particulières indépendantes. Dans le chapitre 5, en présence d'une barrière constante, on évalue une équation intégro-différentielle de la fonction de Gerber-Shiu avec la copule de Farlie-Gumbel-Morgenstern généralisée. Une courte conclusion résume le travail accompli et propose de nouvelles perspectives de recherche.

# CHAPITRE II

## Analysis of ruin measures for the classical compound Poisson risk model with dependence


### Résumé

Dans ce papier, nous considérons une extension du modèle classique de risque. Dans la littérature, on suppose que les montants de sinistres et les temps entre les arrivées des sinistres sont indépendants. Dans cette contribution, une structure de dépendance entre les montants des sinistres et les temps entre les sinistres est introduite par la copule de Farlie-Gumbel-Morgenstern. Dans ce cadre, nous dérivons une équation intégro-différentielle pour la fonction de Gerber-Shiu. Des expressions explicites pour les transformées de Laplace du temps de la ruine et du déficit à la ruine sont évaluées pour des montants de sinistres qui obéissent à une loi exponentielle.

## Abstract

In this paper we consider an extension to the classical compound Poisson risk model. Historically, it has been assumed that the claim amounts and claim inter-arrival times are independent. In this contribution, a dependence structure between the claim amount and the interclaim time is introduced through a Farlie-Gumbel-Morgenstern copula. In this framework, we derive the integro-differential equation and the Laplace transform of the Gerber-Shiu discounted penalty function. An explicit expression for the Laplace transform of the discounted value of a general function of the deficit at ruin is obtained for claim amounts having an exponential distribution.

## 2.1 Introduction

The classical risk model describes the surplus process $\underline{U} = \{U(t), t \geq 0\}$ of a portfolio of insurance contracts as

$$U(t) = u + pt - S(t),$$

where $u$ is the initial surplus and $p$ is the premium rate. The total claim amount process, denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ ($\sum_a^b$ equals 0 if $b < a$), is a compound Poisson process (see e.g. Gerber (1979), Grandell (1991) and Rolski et al. (1999)). The claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a Poisson process where the interclaim times $\{W_j, j \in \mathbb{N}^+\}$ are a sequence of independent and strictly positive random variables (r.v.). The r.v.'s $\{W_j, j \in \mathbb{N}^+\}$, identically distributed as the canonical r.v. $W$, have an exponential distribution with expectation $\frac{1}{\lambda}$ with probability density function (p.d.f.) $f_W$, cumulative distribution function (c.d.f.) $F_W$ and Laplace transform (L.T.) $f_W^*$ with

$$f_W(t) = \lambda e^{-\lambda t}, \quad t > 0, \tag{2.1}$$

$$F_W(t) = 1 - e^{-\lambda t}, \quad t > 0, \tag{2.2}$$

$$f_W^*(s) = E\left[e^{-sW}\right] = \frac{\lambda}{\lambda + s}, \quad s > -\lambda. \tag{2.3}$$

The claim amount r.v.'s $\{X_j, j \in \mathbb{N}^+\}$, where $X_j$ corresponds to the amount of the $j$th claim, are assumed to be a sequence of strictly positive, independent and identically distributed (i.i.d.) r.v.'s with p.d.f. $f_X$, c.d.f. $F_X$ and p.g.f. $f_X^*$.

The classical risk model relies on the assumption that, on the occurrence of the $j$th claim, the claim amount $X_j$ and the interclaim time $W_j$ are independent.

This hypothesis simplifies the study of many risk quantities of interest but has proven to be inadequate and too restrictive in different contexts. In our paper, we assume that $\{(X_j, W_j), j \in \mathbb{N}^+\}$ form a sequence of i.i.d. random vectors distributed as the canonical r.v. $(X, W)$, in which the components may be dependent. The joint p.d.f. of $(X, W)$ is denoted by $f_{X,W}(x,t)$ with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$. When $X$ and $W$ are continuous, the associated L.T. is given by

$$f^*_{X,W}(s_1, s_2) = E\left[e^{-s_1 X} e^{-s_2 W}\right] = \int_0^\infty \int_0^\infty e^{-s_1 x} e^{-s_2 t} f_{X,W}(x,t)\, dx dt. \quad (2.4)$$

In this paper, the joint distribution of $(X, W)$ is defined with a Farlie-Gumbel-Morgenstern (FGM) copula.

Some recent papers consider extensions to the classical risk model by allowing dependence between the claim amount r.v. $X$ and the interclaim time r.v. $W$. Among them, Albrecher and Teugels (2006) consider a dependence structure for $(X, W)$ based on a copula. By employing the underlying random walk structure of the risk model, they derive exponential estimates for finite and infinite time ruin probabilities in the case of light-tailed claim sizes. Boudreault et al. (2006) propose an extension to the classical compound Poisson risk model assuming a dependence structure for $(X, W)$ in which the distribution of the next claim amount is defined in terms of the time elapsed since the last claim. They derive the defective renewal equation satisfied by the expected discounted penalty function. They also obtain an explicit expression for the Laplace transform of the time of ruin asssuming that the claim amount belongs to a large class of distributions.

The present paper is organized as follows. In Section 2.2 we briefly recall basic notions on copulas and present properties of the FGM copula. Basic definitions for ruin measures are given in Section 2.3. We derive the generalized Lundberg equation and we analyze its properties in Section 2.4. In Section 2.5, we obtain an integro-differential equation for the expected discounted penalty function and, in the following section, we derive the Laplace transform of the expected discounted penalty function. In Section 2.7, we derive the defective renewal equation for the expected discounted penalty function. An explicit expression for the Laplace transform of the discounted value of a general function of the deficit at ruin is obtained for claim amounts having an exponential distribution in Section 2.8. Numerical examples are also provided in Section 2.8.

## 2.2 Dependence structure based on FGM copula

As mentioned in the introduction, the joint distribution of $(X, W)$ is defined in this paper with the FGM copula.

A copula $C$ is the distribution function of a bivariate distribution with uniform marginals. By the theorem of Sklar (see e.g. Nelsen (2006)) any bivariate distribution function $F$ with marginals $F_1$ and $F_2$ can be written as $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, for some copula $C$. This copula is unique if $F$ is continuous. Otherwise it is only unique on the range of the marginals. We refer the reader to Nelsen (2006) or Joe (1997) for further details on copulas.

Modelling the dependence structure between r.v.'s using copulas has become popular in actuarial science and financial risk management. The reader may consult e.g. Frees and Valdez (1998), Wang (1998), Bouye et al. (2000), Denuit et al (2005) and McNeil et al. (2005) for applications of copulas in actuarial science and financial risk management. As in the present paper, Albrecher and Teugels (2006) use copulas to define the joint distribution for the interclaim time r.v. and the claim amount r.v.

Assume a bivariate random vector $(U, V)$ with continuous uniform marginals and with a dependence structure defined by a copula $F_{U,V}(u,v) = C(u,v)$ with $(u,v) \in [0,1] \times [0,1]$. Important copulas are the independence copula with $C^{\perp}(u,v) = uv$ ; the comonotonic copula with $C^{+}(u,v) = \min(u,v)$ ; the countermonotonic copula with $C^{-}(u,v) = \max(u+v-1;0)$. It is important to mention that all copulas satisfy the inequalities $C^{-}(u,v) \leq C(u,v) \leq C^{+}(u,v)$, for $(u,v) \in [0,1] \times [0,1]$.

The joint p.d.f. associated to a copula $C$ is defined by

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2). \tag{2.5}$$

Let the bivariate distribution function $F_{X,W}$ of $(X, W)$ with marginals $F_W$ and $F_X$ be defined as $F_{W,X}(t,x) = C(F_W(t), F_X(x))$ for $(t,x) \in \mathbb{R}^{+} \times \mathbb{R}^{+}$. The joint p.d.f. of $(X, W)$ is given by

$$f_{X,W}(x,t) = c(F_X(x), F_W(t)) f_X(x) f_W(t), \tag{2.6}$$

for $(t,x) \in \mathbb{R}^{+} \times \mathbb{R}^{+}$.

The Farlie-Gumbel-Morgenstern (FGM) copula is given by

$$C_\theta^{FGM}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2),$$

$-1 \leq \theta \leq 1$, where $C_0^{FGM} = C^{\perp}$. The FGM copula allows negative and positive dependence, includes the independence copula ($\theta = 0$), but does not include the comonotonic or countermonotonic copulas as limit cases.

For the FGM copula, the expression for (2.5) is given by

$$c_\theta^{FGM}(u_1, u_2) = 1 + \theta (1 - 2u_1)(1 - 2u_2). \tag{2.7}$$

The bivariate distribution function $F_{X,W}$ of $(X, W)$ with marginals $F_X$ and $F_W$ and defined with the FGM copula is given by

$$F_{X,W}(x,t) = F_X(x) F_W(t) + \theta F_X(x) F_W(t) (1 - F_X(x))(1 - F_W(t)),$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$. Combining (2.6) and (2.7), we obtain the expression for the joint p.d.f. of $(X, W)$,

$$f_{X,W}(x,t) = f_X(x) f_W(t) + \theta f_X(x) f_W(t) (1 - 2F_X(x))(1 - 2F_W(t)), \tag{2.8}$$

and, given (2.1), we have

$$f_{X,W}(x,t) = f_X(x) \lambda e^{-\lambda t} + \theta f_X(x) \lambda e^{-\lambda t} (1 - 2F_X(x)) \left(2e^{-\lambda t} - 1\right). \tag{2.9}$$

Defining $h_X(x) = (1 - 2F_X(x)) f_X(x)$ and denoting by $h_X^*(s)$ its associated Laplace transform, (2.9) can be written as

$$f_{X,W}(x,t) = f_X(x) \lambda e^{-\lambda t} + \theta h_X(x) \left(2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t}\right). \tag{2.10}$$

We recall the definitions of two frequently used measures of association. We assume that the r.v. $X$ and $W$ are continuous. Kendall's tau, denoted by

$\tau(X,W)$, is defined by

$$\tau(X,W) = P[(X_1 - X_2)(W_1 - W_2) > 0] - P[(X_1 - X_2)(W_1 - W_2) < 0],$$

where $(X_1, W_1)$ and $(X_2, W_2)$ are independent and identically distributed random vectors, with common joint distribution function $F_{X,W}$ (with marginals $F_X$ and $F_W$). Spearman's rho, denoted by $\rho(X,W)$, is defined by

$$\rho(X,W) = 3\Big\{P[(X_1 - X_2^{\perp})(W_1 - W_2^{\perp}) > 0] - P[(X_1 - X_2^{\perp})(W_1 - W_2^{\perp}) < 0]\Big\},$$

where $(X_1, W_1)$ and $(X_2^{\perp}, W_2^{\perp})$ are independent random vectors, the joint distribution function of $(X_1, W_1)$ is $F_{X,W}$ (with marginals $F_X$ and $F_W$), and the components of $(X_2^{\perp}, W_2^{\perp})$ are independent with marginals $F_X$ and $F_W$.

If the dependence structure of $(X, W)$ is defined with a copula $C$, $\tau(X,W)$ and $\rho(X,W)$ can be expressed in terms of $C$, i.e., we have

$$\tau(X,W) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$$

and

$$\rho(X,W) = 12\int_0^1\int_0^1 C(u,v)dudv - 3.$$

For the FGM copula, we have $\tau(X,W) = \frac{2\theta}{9}$ and $\rho(X,W) = \frac{\theta}{3}$ which implies that $-\frac{2}{9} \leq \tau(X,W) \leq \frac{2}{9}$ and $-\frac{1}{3} \leq \rho(X,W) \leq \frac{1}{3}$ in this case.

## 2.3 Ruin measures

We define the time of ruin as the r.v. $T$ where $T = \{t \geq 0, U(t) < 0\}$ with $T = \infty$ if $U(t) \geq 0$ for all $t \geq 0$ (i.e. ruin does not occur). To avoid almost sure occurence of ruin, the premium rate $p$ is such that

$$E\left[pW_i - X_i\right] > 0,\ i = 1, 2, ..., \tag{2.11}$$

providing a positive safety loading. The deficit at ruin and the surplus just prior to ruin are respectively denoted by $|U(T)|$ and $U(T^-)$. In recent years, a fair amount of research in ruin theory has been devoted to the analysis of the expected value of the discounted penalty function. Introduced by Gerber and Shiu (1998), this function is given by

$$m_\delta(u) = E\Big[e^{-\delta T} w\left\{U(T^-), |U(T)|\right\} I(T < \infty)\Big|U(0) = u\Big], \quad u \geq 0, \tag{2.12}$$

where $w(x, y)$, for $x, y \geq 0$, is the penalty function at the time of ruin for the surplus prior to ruin and the deficit at ruin, $\delta$ is a non negative parameter (the force of interest) and $I$ is the indicator function, such that $I(A) = 1$ if the event $A$ occurs and equals 0 otherwise. A special case of the Gerber-Shiu penalty function is when $w(x, y) = 1$, for all $x, y \geq 0$. Then $m_\delta(u)$ becomes the Laplace transform of the time of ruin, denoted by $\phi_T(u)$. If $\delta = 0$ in addition to $w(x, y) = 1$ for all $x, y \in \mathbb{R}^+$, (2.12) corresponds to the infinite-time ruin probability $\psi(u) = \Pr(T < \infty|U(0) = u)$.

## 2.4 Lundberg's generalized equation

One important step in the analysis of the ruin measures is the derivation of the so-called Lundberg's generalized equation and the examination of its properties. An analysis of Lundberg's generalized equation is required to find the defective renewal equation for $m_\delta(u)$. More precisely, we need to identify the number of roots to Lundberg's generalized equation in the right-half complex plane, i.e. with $\Re(s) \geq 0$. These roots are useful to derive the defective renewal equation for $m_\delta(u)$ as we shall see in the next sections.

To derive Lundberg's generalized equation, we consider the discrete time process embedded in the continuous-time surplus process $\underline{U}$. Let us define the discrete-time process $\underline{\widetilde{U}} = \left\{\widetilde{U}_k, k = 0, 1, 2, ...\right\}$, where $\widetilde{U}_0 = u$ and $\widetilde{U}_k = U(T_k)$ denotes the surplus immediately after the $k$th claim, i.e.

$$\widetilde{U}_k = u + \sum_{j=1}^{k} (pW_j - X_j) \quad \text{for} \quad k \in \mathbb{N}^+. \tag{2.13}$$

The process $\underline{\widetilde{V}} = \left\{e^{-\delta \sum_{j=1}^{k} W_j + s\widetilde{U}_k}, k = 0, 1, \cdots\right\}$, for $s > 0$ is a martingale if and only if

$$E\left(e^{-\delta W} e^{s(pW-X)}\right) = 1, \tag{2.14}$$

which corresponds to the so-called Lundberg generalized equation.

Given (2.10), the left-hand side of (2.14) can be written as

$$\begin{aligned} E\left(e^{-\delta W} e^{s(pW-X)}\right) &= \int_0^\infty \int_0^\infty e^{t(sp-\delta)} e^{-sx} f_{X,W}(x,t) dx dt \\ &= \int_0^\infty \int_0^\infty e^{t(sp-\delta)} e^{-sx} f_X(x) \lambda e^{-\lambda t} dx dt \\ &+ \theta \int_0^\infty \int_0^\infty e^{t(sp-\delta)} e^{-sx} h_X(x) (2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t}) dx dt. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp} + \theta h_X^\star(s) \left(\frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)}\right) = 1. \tag{2.16}$$

In the following proposition, we use Rouché's Theorem to show the number of roots of the generalized Lundberg equation.

**Proposition 1.** *For $\delta > 0$ and $\theta \neq 0$, Lundberg's equation in (2.16) has exactly 2 roots, say $\rho_1, \rho_2$, that have a positive real part $\Re(\rho_j) > 0$, $j = 1, 2$.*

**Proof.** We apply Rouché's theorem on the contour $C_r$, consisting of the imaginary axis running from $-ir$ to $ir$ and a semi-circle with radius $r$ running clockwise from $ir$ to $-ir$. We let $r \to \infty$ and denote by $C$ the limiting contour.

We want to show that

$$\left| f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp} + \theta h_X^\star(s) \left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right) \right| < 1. \tag{2.17}$$

The two terms in the left-side of (2.17),

$$f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp}$$

and

$$\theta h_X^\star(s) \left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right),$$

are ratios of polynomials with a strictly higher degree at the denominator which leads to

$$\left| f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp} + \theta h_X^\star(s) \left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right) \right| \to 0$$

on $C$ (excluding $s = 0$).

For $s = 0$, we observe that

$$\frac{\lambda}{(\lambda + \delta - sp)} > 0$$

and

$$\left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right) > 0$$

Also, at $s = 0$ and for $\delta > 0$, we have

$$\frac{\lambda}{(\lambda + \delta)} + \frac{\theta\lambda\delta}{(2\lambda + \delta)(\lambda + \delta)} = \frac{(2\lambda + \delta)\lambda + \theta\lambda\delta}{(2\lambda + \delta)(\lambda + \delta)} < 1,$$

since $(2\lambda + \delta)\lambda + \theta\lambda\delta < (2\lambda + \delta)(\lambda + \delta)$.

Finally, we have

$$\begin{aligned}
& \left| f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp} + \theta h_X^\star(s) \left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right) \right| \\
\leq & \left| f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp} \right| + \left| \theta h_X^\star(s) \left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right) \right| \\
= & \left| f_X^\star(s) \right| \left| \frac{\lambda}{\lambda + \delta - sp} \right| + \left| \theta h_X^\star(s) \right| \left| \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right| \\
\leq & \left| \frac{\lambda}{\lambda + \delta - sp} \right| + \left| \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right| \\
\leq & \frac{\lambda}{(\lambda + \delta)} + \frac{\theta\lambda\delta}{(2\lambda + \delta)(\lambda + \delta)} = \frac{(2\lambda + \delta)\lambda + \theta\lambda\delta}{(2\lambda + \delta)(\lambda + \delta)} < 1.
\end{aligned}$$

□

For $\delta = 0$, the conditions of Rouché's theorem are not satisfied (since

$$\left| f_X^\star(s) \frac{\lambda}{\lambda + \delta - sp} + \theta h_X^\star(s) \left( \frac{\lambda(\delta - sp)}{(2\lambda + \delta - sp)(\lambda + \delta - sp)} \right) \right| = 1$$

for $s = 0$). We apply an extension of Rouché's theorem, due to Klimenok (2001) to determine the number of roots to Lundberg's generalized equation with a positive real part.

**Proposition 2.** *For $\delta = 0$ and $\theta \neq 0$, Lundberg's equation in (2.16) has exactly 1 root, say $\rho_1(0)$, with $\Re(\rho_i(0)) > 0$ and a second root $\rho_2(0) = 0$.*

**Proof.** We define the contour $D_\kappa = \{s : |z| = 1\}$ where $z = \frac{\kappa - s}{\kappa}$. In terms of $s$, the contour $D_\kappa$ is a circle of radius $\kappa$ and origin $\kappa$. Similarly as in Proposition 1, we let $\kappa \to \infty$ and denote by $D$ the limiting contour. We use the same arguments (for $\delta = 0$) as the ones provided in the proof of Proposition 1 and we can deduce

$$|f_X^\star(s)\lambda(2\lambda - sp) + \theta\lambda(-sp)\,h_X^\star(s)| \leq |(\lambda - sp)(2\lambda - sp)|,$$

on $D$ (excluding $s = 0$ or equivalently $z = 1$). We also note that the functions

$$f_X^\star(s)\lambda(2\lambda - sp) + \theta\lambda(-sp)\,h_X^\star(s)$$

and

$$(\lambda - sp)(2\lambda - sp)$$

are continuous on $D$.

It remains to prove that

$$\frac{d}{dz}\left(1 - \frac{\lambda f_X^\star(\kappa - \kappa z)}{\lambda - (\kappa - \kappa z)\,p} + \frac{\theta h_X^\star(\kappa - \kappa z)\lambda(-(\kappa - \kappa z)\,p)}{(2\lambda - (\kappa - \kappa z)\,p)(\lambda - (\kappa - \kappa z)\,p)}\right)\Bigg|_{z=1} > 0 \qquad (2.18)$$

in order to apply Theorem 1 of Klimenok (2001). We observe that (2.18) is equal to

$$\frac{d}{dz}\left(1 - E\left[e^{-(\kappa-\kappa z)(X-pW)}\right]\right)\Bigg|_{z=1} = E\left[(X - pW)\right],$$

where $E\left[(X - pW)\right] > 0$ given the solvability condition (2.11). Based on Klimenok (2001), we conclude that the number of solutions to (2.14) inside $D$ is equal to 1. Moreover, a trivial root to Lundberg's generalized equation (2.14) (with $\delta = 0$) is $\rho_2(0) = 0$. □

## 2.5 An integro-differential equation

The main purpose of this section is to derive an integro-differential equation for the expected discounted penalty function $m_\delta(u)$. This equation will be useful to derive an explicit solution for $m_\delta(u)$ in the next section.

Throughout this paper we denote by $\mathcal{I}$ and $\mathcal{D}$ the identity and the differentiation operators, respectively.

**Proposition 3.** *The expected discounted penalty function $m_\delta(u)$ satisfies the following equation for $u \geq 0$ and -1 $\leq \theta \leq 1$ :*

$$\begin{aligned} \Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_\delta(u) &= \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_1(u) \\ &+ \lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u), \qquad (2.19) \end{aligned}$$

*where*

$$\begin{aligned} \sigma_1(u) &= \int_0^u m_\delta(u-x)f_X(x)dx + w_1(u), \qquad (2.20) \\ \sigma_2(u) &= \int_0^u m_\delta(u-x)h_X(x)dx + w_2(u), \qquad (2.21) \\ w_1(u) &= \int_u^\infty w(u,x-u)f_X(x)dx, \qquad (2.22) \\ w_2(u) &= \int_u^\infty w(u,x-u)h_X(x)dx. \qquad (2.23) \end{aligned}$$

**Proof.** By conditioning on the time and the amount of the first claim, we have

$$\begin{aligned} m_\delta(u) &= \lambda\int_0^\infty\int_0^{u+pt} e^{-\delta t}m_\delta(u+pt-x)f_{X,W}(x,t)dxdt \\ &+\lambda\int_0^\infty\int_{u+pt}^\infty e^{-\delta t}w(u+pt,x-u-pt)f_{X,W}(x,t)dxdt. \qquad (2.24) \end{aligned}$$

Given (2.10), (2.24) becomes

$$\begin{aligned} m_\delta(u) &= \lambda \int_0^\infty \int_0^{u+pt} e^{-\delta t} m_\delta(u+pt-x) f_X(x) e^{-\lambda t} dx dt \\ &+ \lambda \int_0^\infty \int_{u+pt}^\infty e^{-\delta t} w(u+pt, x-u-pt) f_X(x) e^{-\lambda t} dx dt \\ &+ \lambda\theta \int_0^\infty \int_0^{u+pt} e^{-\delta t} m_\delta(u+pt-x) h_X(x)(2e^{-2\lambda t} - e^{-\lambda t}) dx dt \\ &+ \lambda\theta \int_0^\infty \int_{u+pt}^\infty e^{-\delta t} w(u+pt, x-u-pt) h_X(x)(2e^{-2\lambda t} - e^{-\lambda t}) dx dt. \end{aligned} \tag{2.25}$$

We can rewrite equation (2.25) as

$$\begin{aligned} m_\delta(u) &= \lambda \int_0^\infty e^{-\delta t} \sigma_1(u+pt) e^{-\lambda t} dt + 2\theta\lambda \int_0^\infty e^{-\delta t} \sigma_2(u+pt) e^{-2\lambda t} dt \\ &- \theta\lambda \int_0^\infty e^{-\delta t} \sigma_2(u+pt) e^{-\lambda t} dt, \end{aligned} \tag{2.26}$$

where the functions $\sigma_1(u)$ and $\sigma_2(u)$ are as given in (2.20) and (2.21) respectively.

We substitute $u + pt = s$ in (2.26) which becomes

$$\begin{aligned} m_\delta(u) &= \frac{\lambda}{p} \int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})} \sigma_1(s) ds - \frac{\theta}{p}\lambda \int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})} \sigma_2(s) ds \\ &+ 2\frac{\theta}{p}\lambda \int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})} \sigma_2(s) ds. \end{aligned} \tag{2.27}$$

Differentiating (2.27) w.r.t. $u$ leads to

$$\begin{aligned} m'_\delta(u) &= \frac{\lambda}{p}\Big(\frac{\lambda+\delta}{p}\Big) \int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})} \sigma_1(s) ds - \lambda\frac{\theta}{p}\Big(\frac{\lambda+\delta}{p}\Big) \int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})} \sigma_2(s) ds \\ &+ 2\lambda\frac{\theta}{p}\Big(\frac{2\lambda+\delta}{p}\Big) \int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})} \sigma_2(s) ds - \lambda\frac{\theta}{p}\sigma_2(u) - \frac{\lambda}{p}\sigma_1(u). \end{aligned} \tag{2.28}$$

Multiplying (2.27) by $\frac{\lambda+\delta}{p}$, subtracting (2.28) to the result, and using the identity and differentiation operators, we obtain

$$\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_\delta(u)=\frac{\lambda}{p}\sigma_1(u)-\theta\frac{2\lambda^2}{p^2}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_2(s)ds+\theta\frac{\lambda}{p}\sigma_2(u). \tag{2.29}$$

We define

$$g_\delta(u)=\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_\delta(u). \tag{2.30}$$

Differentiating (2.30) w.r.t. $u$ and using (2.29), we find

$$\begin{aligned} g'_\delta(u) &= \frac{\lambda}{p}\sigma'_1(u)-\theta\frac{2\lambda^2(2\lambda+\delta)}{p^3}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_2(s)ds+\theta\frac{\lambda}{p}\sigma'_2(u)\\ &+ \theta\frac{2\lambda^2}{p^2}\sigma_2(u). \end{aligned} \tag{2.31}$$

Multiplying (2.29) by $\frac{2\lambda+\delta}{p}$, subtracting (2.31) and using the identity and differentiation operators, we obtain

$$\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_\delta(u)=\frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_1(u)+\lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u),$$

which is equivalent to (2.19). □

**Remark 4.** *If $\theta=0$, (2.19) corresponds to the integro-differential equation for $m_\delta$ when $X$ and $W$ are independent, as in the classical compound Poisson risk model.*

## 2.6 Laplace transform of the expected discounted penalty function

We use the integro-differential equation (2.19) to derive the Laplace transform of $m_\delta(u)$ which is stated in the next proposition.

**Proposition 5.** *The Laplace transform of* $m_\delta$ *is given by*

$$m_\delta^\star(s) = \frac{\beta_{1,\delta}^\star(s) + \beta_{2,\delta}^\star(s)}{h_{1,\delta}^\star(s) - h_{2,\delta}^\star(s)}, \tag{2.32}$$

*where* $\beta_{2,\delta}^\star(s)$ *is a polynomial of degree* 1, *with*

$$\beta_{2,\delta}^\star(s) = -\sum_{j=1}^{2} \beta_{1,\delta}^\star(\rho_j) \prod_{k=1,k\neq j}^{2} \frac{s-\rho_k}{\rho_j - \rho_k},$$

*and*

$$\beta_{1,\delta}^\star(s) = \frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p} - s\Big) w_1^\star(s) + \theta\frac{\lambda}{p}\Big(\frac{\delta}{p} - s\Big) w_2^\star(s). \tag{2.33}$$

*We also have*

$$h_{1,\delta}^\star(s) = \Big(\frac{\delta+\lambda}{p} - s\Big)\Big(\frac{\delta+2\lambda}{p} - s\Big), \tag{2.34}$$

*and*

$$h_{2,\delta}^\star(s) = \frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p} - s\Big) f_X^\star(s) + \theta\frac{\lambda}{p}\Big(\frac{\delta}{p} - s\Big) h_X^\star(s). \tag{2.35}$$

**Proof.** We define

$$\begin{aligned} d(u) &= \Big(\frac{2\lambda+\delta}{p}\mathcal{I} - \mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I} - \mathcal{D}\Big)m_\delta(u) - \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I} - \mathcal{D}\Big)\sigma_1(u) \\ &- \theta\frac{\lambda}{p}\Big(\frac{\delta}{p}\mathcal{I} - \mathcal{D}\Big)\sigma_2(u). \end{aligned} \tag{2.36}$$

We take the Laplace transform on both sides of (2.36). Then, using standard properties of Laplace transforms and using (2.20), (2.21), (2.22), and (2.23), we obtain

$$\begin{aligned} d^\star(s) &= \Big(\frac{2\lambda+\delta}{p} - s\Big)\Big(\frac{\lambda+\delta}{p} - s\Big)m_\delta^\star(s) - s m_\delta(0) - m_\delta'(0) \\ &+ \frac{2\delta+3\lambda}{p} m_\delta(0) - \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p} - s\Big)m_\delta^\star(s) f_X^\star(s) - \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p} - s\Big)w_1^\star(s) \\ &- \frac{\lambda}{p}w_1(0) - \theta\frac{\lambda}{p}\Big(\frac{\delta}{p} - s\Big)m_\delta^\star(s)h_X^\star(s) - \theta\frac{\lambda}{p}\Big(\frac{\delta}{p} - s\Big)w_2^\star(s) - \theta\frac{\lambda}{p}w_2(0), \end{aligned} \tag{2.37}$$

where $w_i^*(s)$ is the Laplace transform corresponding to $w_i(x)$, for $i = 1, 2$. Letting (2.37) equal to 0, we isolate $m_\delta^*(s)$

$$m_\delta^*(s) = \frac{\beta_{1,\delta}^*(s) + \beta_{2,\delta}^*(s)}{h_{1,\delta}^*(s) - h_{2,\delta}^*(s)}, \tag{2.38}$$

where $h_{1,\delta}^*(s)$, $h_{2,\delta}^*(s)$ and $\beta_{1,\delta}^*(s)$ are given in (2.34), (2.35), and (2.33) respectively. The term

$$\beta_{2,\delta}^*(s) = \Big(s - \frac{2\delta + 3\lambda}{p}\Big) m_\delta(0) + m_\delta'(0) + \frac{\lambda}{p} w_1(0) + \theta \frac{\lambda}{p} w_2(0) \tag{2.39}$$

in (2.38) is a polynomial of degree 1 or less. Given that (2.38) is analytic, the two roots $\rho_j, j = 1, 2$, of the denominator of (2.38) must also be roots of the numerator. By the Lagrange interpolating formula, (2.39) can be rewritten as

$$\beta_{2,\delta}^*(s) = -\sum_{j=1}^{2} \beta_{1,\delta}^*(\rho_j) \prod_{k=1, k\neq j}^{2} \frac{s - \rho_k}{\rho_j - \rho_k}.$$

This completes the proof. □

In the following corollary, we provide an explicit expression of the expected discounted penalty function when the initial surplus is zero.

**Corollary 6.** *We can write $m_\delta(0)$ in term of $w_1^*$ and $w_2^*$ as*

$$\begin{aligned} m_\delta(0) &= \frac{\lambda}{p} \left\{ \frac{(\frac{\delta+2\lambda}{p} - \rho_1) w_1^*(\rho_1) - (\frac{\delta+2\lambda}{p} - \rho_2) w_1^*(\rho_2)}{\rho_2 - \rho_1} \right\} \\ &+ \theta \frac{\lambda}{p} \left\{ \frac{(\frac{\delta}{p} - \rho_1) w_2^*(\rho_1) - (\frac{\delta}{p} - \rho_2) w_2^*(\rho_2)}{\rho_2 - \rho_1} \right\}. \end{aligned} \tag{2.40}$$

**Proof.** Since $\rho_1$ and $\rho_2$ are the roots of the denominator in (2.32), they have to be the roots of the numerator. We obtain the two following linear equations for $m_\delta(0)$ and $m'_\delta(0)$

$$-\frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p}-\rho_i\Big)w_1^\star(\rho_i)-\theta\frac{\lambda}{p}\Big(\frac{\delta}{p}-\rho_i\Big)w_2^\star(\rho_i) = \Big(\rho_i-\frac{2\delta+3\lambda}{p}\Big)m_\delta(0)+m'_\delta(0) + \frac{\lambda}{p}w_1(0)+\theta\frac{\lambda}{p}w_2(0), \qquad (2.41)$$

for $i=1,2$. Solving the two linear equations in (2.41) leads to (2.40). □

Note that a software package such as Maple can be applied to invert the Laplace transform of $m_\delta$.

## 2.7 Defective renewal equation for the expected discounted penalty function

In the present section, we derive the defective renewal equation for $m_\delta$. For that purpose, we use the Dickson-Hipp operator $T_r$ for an integrable real-valued function $f$ (introduced by Dickson and Hipp (2001)) defined by

$$T_rf(x)=\int_x^\infty e^{-r(u-x)}f(u)du, r\in C.$$

The properties of the operator $T_r$ are closely examined in Li and Garrido (2004).

We first provide an alternative expression for the Laplace transform of $m_\delta$ in terms of the Dickson-Hipp operator which is useful to derive the defective renewal equation for $m_\delta$.

**Proposition 7.** *The Laplace transform of $m_\delta$ is given by*

$$m_\delta^\star(s)=\frac{T_sT_{\rho_1}T_{\rho_2}\beta_{1,\delta}(0)}{1-T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0)}. \qquad (2.42)$$

**Proof.** We note that the $\rho_j, j = 1,2$ are the roots of the denominator of (2.32). They must also be the roots of the numerator. By the Lagrange Interpolating Formula and using the operator $T_r$, we obtain an alternative expression for the numerator $\beta^*_{1,\delta}(s) + \beta^*_{2,\delta}(s)$ of (2.32) :

$$\begin{aligned}
\beta^*_{1,\delta}(s) + \beta^*_{2,\delta}(s) &= \tau(s)\left\{\frac{\beta^*_{1,\delta}(s)}{\tau(s)} - \sum_{j=1}^{2}\frac{\beta^*_{1,\delta}(\rho_j)}{(s-\rho_j)\tau'(\rho_j)}\right\} \\
&= \tau(s)T_sT_{\rho_1}T_{\rho_2}\beta_{1,\delta}(0), \qquad (2.43)
\end{aligned}$$

where $\tau(s) = (s-\rho_1)(s-\rho_2)$.

Similarly, we can derive an alternative expression for the denominator $h^*_{1,\delta}(s) - h^*_{2,\delta}(s)$ of (2.32). We know that

$$h^*_{1,\delta}(\rho_j) = h^*_{2,\delta}(\rho_j)$$

for $j = 1, 2$. From (2.34), $h^*_{1,\delta}$ is a polynomial of degree 2 in $s$. Using again the Lagrangian Interpolating Formula, we have

$$h^*_{1,\delta}(s) = h^*_{1,\delta}(0)\prod_{k=1}^{2}\frac{(s-\rho_k)}{(-\rho_k)} + s\sum_{j=1}^{2}\frac{h^*_{1,\delta}(\rho_j)}{\rho_j}\prod_{k=1,k\neq j}^{2}\frac{s-\rho_k}{\rho_j-\rho_k},$$

which implies

$$
\begin{aligned}
h^*_{1,\delta}(s) - h^*_{2,\delta}(s) &= h^*_{1,\delta}(0)\frac{\tau(s)}{\tau(0)} + s\sum_{j=1}^{2}\frac{h^*_{2,\delta}(\rho_j)\tau(s)}{\rho_j(s-\rho_j)\tau'(\rho_j)} - h^*_{2,\delta}(s) \\
&= \tau(s)\left\{\frac{h^*_{1,\delta}(0)}{\tau(0)} + \sum_{j=1}^{2}\frac{(s-\rho_j+\rho_j)h^*_{2,\delta}(\rho_j)}{\rho_j(s-\rho_j)\tau'(\rho_j)} - \frac{h^*_{2,\delta}(s)}{\tau(s)}\right\} \\
&= \tau(s)\left\{\frac{h^*_{1,\delta}(0)}{\tau(0)} - \sum_{j=1}^{2}\frac{h^*_{1,\delta}(\rho_j)}{(-\rho_j)\tau'(\rho_j)}\right\} \\
&+ \tau(s)\left\{\sum_{j=1}^{2}\frac{h^*_{2,\delta}(\rho_j)}{(s-\rho_j)\tau'(\rho_j)} - \frac{h^*_{2,\delta}(s)}{\tau(s)}\right\} \\
&= \tau(s)\left\{\frac{h^*_{1,\delta}(0)}{\tau(0)} - \sum_{j=1}^{2}\frac{h^*_{1,\delta}(\rho_j)}{(-\rho_j)\tau'(\rho_j)} - T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0)\right\}, \qquad (2.44)
\end{aligned}
$$

using the Dickson-Hipp operator and its properties.

Also, we have

$$
\begin{aligned}
\frac{h^*_{1,\delta}(0)}{\tau(0)} + \sum_{j=1}^{2}\frac{h^*_{1,\delta}(\rho_j)}{\rho_j\tau'(\rho_j)} &= \frac{(\frac{\delta+\lambda}{p})(\frac{\delta+2\lambda}{p})}{\rho_1\rho_2} + \sum_{j=1}^{2}\frac{(\frac{\delta+\lambda}{p}-\rho_j)(\frac{\delta+2\lambda}{p}-\rho_j)}{\rho_j\tau'(\rho_j)} \\
&= \frac{(\frac{\delta+\lambda}{p})(\frac{\delta+2\lambda}{p})}{\rho_1\rho_2} + \frac{\rho_1(\frac{\delta+\lambda}{p}-\rho_2)(\frac{\delta+2\lambda}{p}-\rho_2) - \rho_2(\frac{\delta+\lambda}{p}-\rho_1)(\frac{\delta+2\lambda}{p}-\rho_1)}{\rho_1\rho_2(\rho_2-\rho_1)} \\
&= 1. \qquad (2.45)
\end{aligned}
$$

Substituting (2.45) into (2.44) leads to

$$
h^*_{1,\delta}(s) - h^*_{2,\delta}(s) = \tau(s)\left\{1 - T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0)\right\}. \qquad (2.46)
$$

Finally, replacing (2.43) and (2.46) into (2.32), we obtain (2.42). □

In the next proposition, we derive a defective renewal equation for $m_\delta$.

**Proposition 8.** *The Gerber-Shiu discounted penalty function $m_\delta$ admits a de-*

*fective renewal equation representation*

$$m_\delta(u) = \int_0^u m_\delta(u-y)\mu_\delta(y)dy + \eta_\delta(u), \tag{2.47}$$

*where*

$$\mu_\delta(u) = T_{\rho_1}T_{\rho_2}h_{2,\delta}(u), \tag{2.48}$$

$$\eta_\delta(u) = T_{\rho_1}T_{\rho_2}\beta_{1,\delta}(u). \tag{2.49}$$

*We can also express (2.47) as follows :*

$$m_\delta(u) = \frac{1}{1+\kappa_\delta}\int_0^u m_\delta(u-y)\vartheta_\delta(y)dy + \frac{1}{1+\kappa_\delta}G_\delta(u),$$

*where $\kappa_\delta$ is defined such that*

$$\frac{1}{(1+\kappa_\delta)} = T_0T_{\rho_1}T_{\rho_2}h_{2,\delta}(0) = 1 - \frac{\delta}{p}\frac{\left(\frac{\delta+2\lambda}{p}\right)}{\rho_1\rho_2} < 1.$$

*Also, we have*

$$G_\delta(u) = (1+\kappa_\delta)\eta_\delta(u) \tag{2.50}$$

*and*

$$\vartheta_\delta(y) = (1+\kappa_\delta)\mu_\delta(y), \tag{2.51}$$

*which is a proper density function.*

**Proof.** Defining $\mu_\delta(y)$ and $\eta_\delta(u)$ as in (2.48) and (2.49) respectively, we derive (2.47) from (2.42). We have

$$\begin{aligned}\int_0^\infty \mu_\delta(y)dy &= T_0T_{\rho_1}T_{\rho_2}h_{2,\delta}(0)\\ &= \frac{h^*_{2,\delta}(0)}{\rho_1\rho_2} + \sum_{j=1}^{2}\frac{h^*_{1,\delta}(\rho_j)}{\rho_j\tau'(\rho_j)}.\end{aligned}$$

Using (2.45), we have

$$\int_0^\infty \mu_\delta(y)dy = 1 - \frac{h_{1,\delta}^*(0)}{\tau(0)} + \frac{h_{2,\delta}^*(0)}{\rho_1\rho_2}. \tag{2.52}$$

Since $\tau(0) = \rho_1\rho_2$, (2.52) becomes

$$\int_0^\infty \mu_\delta(y)dy = 1 - \frac{h_{1,\delta}^*(0) - h_{2,\delta}^*(0)}{\rho_1\rho_2} = 1 - \frac{\delta}{p}\frac{\left(\frac{\delta+2\lambda}{p}\right)}{\rho_1\rho_2}. \tag{2.53}$$

Since $\frac{\delta}{p}\frac{\left(\frac{\delta+2\lambda}{p}\right)}{\rho_1\rho_2} > 0$, we have

$$\frac{1}{1+\kappa_\delta} = \int_0^\infty \mu_\delta(y)dy < 1.$$

Then, $\vartheta_\delta(y)$ defined as in (2.51) is a proper density function. □

In the following proposition, we consider the case where $w(x_1, x_2) = 1$. More precisely, we derive a defective renewal equation for the Laplace transform of the time of ruin $\phi_T$ defined in section 2.3.

**Proposition 9.** *The Laplace transform of the time of ruin $\phi_T$ admits a defective renewal equation*

$$\phi_T(u) = \frac{1}{1+\kappa_\delta}\int_0^u \phi_T(u-y)\vartheta_\delta(y)dy + \frac{1}{1+\kappa_\delta}\int_u^\infty \vartheta_\delta(y)dy, \tag{2.54}$$

*which has the following compound geometric representation :*

$$\phi_T(u) = \frac{\kappa_\delta}{1+\kappa_\delta}\sum_{j=1}^{\infty}\left(\frac{1}{1+\kappa_\delta}\right)^j \bar{V}_\delta^{\star j}(u) \quad u \geq 0,$$

*where $\bar{V}_\delta^{\star j}(u)$ is the survival distribution of the $j$-fold convolution of the p.d.f. $\vartheta_\delta$.*

**Proof.** If $w(x_1, x_2) = 1$, then the Laplace transforms of $w_1$ and $w_2$ defined in (2.22) and (2.23) are given by

$$w_1^*(s) = \frac{1 - f_X^*(s)}{s} \tag{2.55}$$

and

$$w_2^*(s) = \frac{-h_X^*(s)}{s}. \tag{2.56}$$

Substituting (2.55) and (2.56) in (2.33) and, after simplifications, we obtain

$$\beta_{1,\delta}^*(s) = \frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p} - s) - h_{2,\delta}^*(s)}{s}. \tag{2.57}$$

From (2.49) and using properties of the Dickson-Hipp operator, we obtain

$$\begin{aligned} s\eta_\delta^*(s) &= sT_sT_{\rho_1}T_{\rho_2}\beta_{1,\delta}(0) \\ &= s\left\{\frac{\beta_{1,\delta}^*(s)}{\tau(s)} - \sum_{j=1}^{2}\frac{\beta_{1,\delta}^*(\rho_j)}{(s-\rho_j)\tau'(\rho_j)}\right\}. \end{aligned} \tag{2.58}$$

Given (2.57), we rewrite (2.58) as

$$\begin{aligned} s\eta_\delta^*(s) &= \frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p} - s)}{\tau(s)} - \frac{h_{2,\delta}^*(s)}{\tau(s)} - s\sum_{j=1}^{2}\frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p} - \rho_j) - h_{2,\delta}^*(\rho_j)}{\rho_j(s-\rho_j)\tau'(\rho_j)} \\ &= \frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p} - s)}{\tau(s)} - s\sum_{j=1}^{2}\frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p} - \rho_j)}{\rho_j(s-\rho_j)\tau'(\rho_j)} + \sum_{j=1}^{2}\frac{h_{2,\delta}^*(\rho_j)}{\rho_j\tau'(\rho_j)} \\ &- T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0). \end{aligned} \tag{2.59}$$

The latter follows from property 6 of the Dickson-Hipp operator on page 394 of Li and Garrido (2004). Furthermore, the second term of (2.59) can be expressed as

$$s\sum_{j=1}^{2}\frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p}-\rho_j)}{\rho_j(s-\rho_j)\tau'(\rho_j)}$$

$$= \frac{\lambda}{p}\left\{\sum_{j=1}^{2}\frac{(\frac{\delta+2\lambda}{p}-\rho_j)}{\rho_j\tau'(\rho_j)}+\sum_{j=1}^{2}\frac{(\frac{\delta+2\lambda}{p}-\rho_j)}{(s-\rho_j)\tau'(\rho_j)}\right\}$$

$$= \frac{\lambda}{p}\left\{\begin{array}{c}\frac{\delta+2\lambda}{p}\sum_{j=1}^{2}\frac{1}{\rho_j\tau'(\rho_j)}-\sum_{j=1}^{2}\frac{1}{\tau'(\rho_j)}\\ +\frac{\delta+2\lambda}{p}\sum_{j=1}^{2}\frac{1}{(s-\rho_j)\tau'(\rho_j)}\\ +\sum_{j=1}^{2}\frac{1}{\tau'(\rho_j)}-s\sum_{j=1}^{2}\frac{1}{(s-\rho_j)\tau'(\rho_j)}\end{array}\right\}, \quad (2.60)$$

given the equality $\frac{s}{\rho_j(s-\rho_j)}=\frac{1}{s-\rho_j}-\frac{1}{\rho_j}$. Using Lemma 1 for the Dickson-Hipp operator on page 395 of Li and Garrido (2004), and after simplifications, (2.60) becomes

$$s\sum_{j=1}^{2}\frac{\frac{\lambda}{p}(\frac{\delta+2\lambda}{p}-\rho_j)}{\rho_j(s-\rho_j)\tau'(\rho_j)}=\frac{\lambda}{p}\Big(-\frac{\delta+2\lambda}{p}\frac{1}{\tau(0)}+\frac{\delta+2\lambda}{p}\frac{1}{\tau(s)}-\frac{s}{\tau(s)}\Big). \quad (2.61)$$

Inserting (2.61) in (2.59) and given (2.53), we get

$$\begin{aligned} s\eta_\delta^*(s) &= 1-\frac{\delta}{p}\frac{(\frac{\delta+2\lambda}{p})}{\rho_1\rho_2}-\mu_\delta^*(s)\\ &= \frac{1}{1+\kappa_\delta}-\mu_\delta^*(s). \end{aligned} \quad (2.62)$$

The Laplace transform of (2.50) is given by

$$G_\delta^*(s)=(1+\kappa_\delta)\eta_\delta^*(s). \quad (2.63)$$

We replace (2.62) in (2.63) and we obtain

$$\begin{aligned} sG_\delta^*(s) &= (1+\kappa_\delta)\left(\frac{1}{1+\kappa_\delta}-\mu_\delta^*(s)\right)\\ &= 1-(1+\kappa_\delta)\mu_\delta^*(s)\\ &= 1-\vartheta_\delta^*(s). \end{aligned} \quad (2.64)$$

From (2.47), we have

$$\phi_T^*(s) = \frac{G_\delta^*(s)}{1+\kappa_\delta - \vartheta_\delta^*(s)}. \tag{2.65}$$

Inserting (2.64) in (2.65), we get

$$\phi_T^*(s) = \frac{\frac{1-\vartheta_\delta^*(s)}{s}}{1+\kappa_\delta - \vartheta_\delta^*(s)},$$

and, after rearrangement, we have

$$(1+\kappa_\delta)\,\phi_T^*(s) = \vartheta_\delta^*(s)\phi_T^*(s) + \frac{1-\vartheta_\delta^*(s)}{s},$$

from which we obtain (2.54) by inverting. □

An expression for the Laplace transform of the time of ruin with an initial surplus equal to zero is given in the next corollary.

**Corollary 10.** *A closed-form expression for* $\phi_T(0)$ *is*

$$\phi_T(0) = 1 - \frac{\delta}{p}\frac{\left(\frac{\delta+2\lambda}{p}\right)}{\rho_1\rho_2}.$$

**Proof.** This follows by setting $u=0$ in (2.54). □

## 2.8 Exponentially distributed claims

We derive an analytic expression for the expected discounted penalty function $m_\delta$ assuming that the penalty function $w(x,y)$ is equal to a function $w(y)$ of the deficit at ruin. For example, when $w(y) = y$ and $\delta > 0$, $m_\delta(u)$ corresponds to the expectation of the present value of the deficit at ruin. We also suppose that the individual claim amounts follow an exponential distribution, with $F_X(x) = 1 - e^{-\alpha x}$, $f_X(x) = \alpha e^{-\alpha x}$, and $f_X^\star(s) = \alpha(\alpha + s)^{-1}$. It follows that

$$h_X(x) = 2\alpha e^{-2\alpha x} - \alpha e^{-\alpha x}, \qquad x \geq 0,$$

and

$$h_X^\star(s) = \frac{2\alpha}{s + 2\alpha} - \frac{\alpha}{s + \alpha}, \qquad s > -\alpha.$$

Hence, (2.22) and (2.23) become

$$w_1(u) \;=\; \alpha \int_u^\infty w(x-u)e^{-\alpha x}dx = \alpha e^{-\alpha u} \int_0^\infty w(v)e^{-\alpha v}dv = \alpha e^{-\alpha u} w^\star(\alpha) \qquad (2.66)$$

and

$$\begin{aligned} w_2(u) &= \alpha \int_u^\infty w(x-u)(2e^{-2\alpha x} - e^{-\alpha x})dx \\ &= \alpha \int_0^\infty w(v)(2e^{-2\alpha(u+v)} - e^{-\alpha(u+v)})dv \\ &= 2\alpha e^{-2\alpha u} w^\star(2\alpha) - \alpha e^{-\alpha u} w^\star(\alpha), \end{aligned} \qquad (2.67)$$

where $w^*(s) = \int_0^\infty w(v)e^{-\alpha v}dv$ is the Laplace transform associated to the function $w(u)$.

Taking the Laplace transform of (2.66) and (2.67) leads to

$$w_1^\star(s) = \frac{\alpha}{\alpha + s} w^\star(\alpha), \quad s > -\alpha, \qquad (2.68)$$

and

$$w_2^\star(s) = \frac{2\alpha}{2\alpha + s} w^\star(2\alpha) - \frac{\alpha}{\alpha + s} w^\star(\alpha). \qquad (2.69)$$

We know that the denominator of (2.32) has 2 roots, $\rho_1$ and $\rho_2$, with positive real parts. Given the assumption on the distribution of $X$, the denominator of (2.32) also has 2 roots $-R_1$ and $-R_2$ where $\Re(R_1), \Re(R_2) > 0$.

In the next proposition, an expression for $m_\delta$ is derived.

**Proposition 11.** *Assuming the roots $\{-R_j(\delta), j = 1, 2\}$ distinct and for $w(x, y) = w(y)$, a closed-form expression for $m_\delta(u)$, $u \geq 0$, is given by*

$$m_\delta(u) = \varsigma_1 e^{-R_1 u} + \varsigma_2 e^{-R_2 u}, \tag{2.70}$$

*where*

$$\varsigma_j = \frac{(\xi_{1,\delta}(-R_j) + \xi_{2,\delta}(-R_j))}{(\ell_{1,\delta}(0) - \xi_{2,\delta}(0))} \frac{\prod_{i=1}^{2} R_i}{\prod_{i=1, i \neq j}^{2} (R_i - R_j)} \prod_{i=1}^{2} \left( \frac{\rho_i}{R_j + \rho_i} \right), j = 1, 2. \tag{2.71}$$

**Proof.** Substituting (2.68) and (2.69) into (2.32), and multiplying both the numerator and denominator of (2.32) by $(\alpha + s)(2\alpha + s)$ gives

$$m_\delta^\star(s) = \frac{\xi_{1,\delta}(s) + \xi_{2,\delta}(s)}{\ell_{1,\delta}(s) - \ell_{2,\delta}(s)}, \tag{2.72}$$

where

$$\xi_{1,\delta}(s) = \frac{\alpha\lambda}{p}(2\alpha+s)\left(\frac{\delta + 2\lambda}{p} - s\right) w^\star(\alpha) + \theta \frac{\lambda}{p}\left(\frac{\delta}{p} - s\right)\{2\alpha w^\star(2\alpha)(\alpha+s) - \alpha w^\star(\alpha)(2\alpha+s)\},$$

$$\xi_{2,\delta}(s) = (\alpha + s)(2\alpha + s)\beta_2(s),$$

$$\ell_{2,\delta}(s) = \frac{\alpha\lambda}{p}\left(\frac{\delta + 2\lambda}{p} - s\right)(s + 2\alpha) + \theta \frac{\lambda\alpha}{p} s \left(\frac{\delta}{p} - s\right),$$

and

$$\ell_{1,\delta}(s) = (\alpha + s)(2\alpha + s)\Big(\frac{\delta+\lambda}{p} - s\Big)\Big(\frac{\delta+2\lambda}{p} - s\Big).$$

We have that $\ell_{1,\delta}(s) - \xi_{2,\delta}(s)$ is a polynomial of degree 4 which has 4 roots, $\rho_j$, with $\Re(\rho_j) > 0$, for $j = 1, 2$ and $-R_j$, with $\Re(R_j) > 0$, for $j = 1, 2$.

Using the Lagrange interpolating polynomial on the denominator and the numerator in (2.72), one finds

$$\xi_{1,\delta}(s) + \xi_{2,\delta}(s) = \sum_{j=1}^{2}(\xi_{1,\delta}(-R_j) + \xi_{2,\delta}(-R_j)) \prod_{k=1}^{2}\Big(\frac{s-\rho_k}{R_j+\rho_k}\Big) \prod_{k=1,k\neq j}^{2}\Big(\frac{s+R_k}{-R_j+R_k}\Big) \quad (2.73)$$

and

$$\ell_{1,\delta}(s) - \ell_{2,\delta}(s) = (\ell_{1,\delta}(0) - \ell_{2,\delta}(0)) \prod_{j=1}^{2}\Big(\frac{s-\rho_j}{\rho_j}\Big) \prod_{j=1}^{2}\Big(\frac{s+R_j}{R_j}\Big). \quad (2.74)$$

Combining (2.73) and (2.74) to (2.72), one concludes

$$m_\delta^\star(s) = \sum_{j=1}^{2} \frac{\varsigma_j}{s+R_j},$$

where $\varsigma_1$ and $\varsigma_2$ are defined in 2.71 □

Finally, if we consider the special case of $w(y) = 1$ (with $w^\star(s) = \frac{1}{s}$), the expression for

$$\phi_T(u) = E\left[e^{-\delta T} I(T < \infty) | U(0) = u\right]$$

can be found from (2.70), with $\xi_{1,\delta}(s) = \frac{\lambda}{p}(2\alpha + s)(\frac{\delta+2\lambda}{p} - s) - \theta\alpha\frac{\lambda}{p}\Big(\frac{\delta}{p} - s\Big)$.

We consider the following numerical example.

**Example 12.** *For the numerical results, we assume that both the claim amount r.v. and the interclaim time r.v. have an exponential distribution with mean 1 (i.e.* $X \sim Exp(1)$ *and* $W \sim Exp(1)$*). The premium rate is* $p = 1.5$*, which implies that the relative risk margin is 50%.*

*We provide below the analytic expressions for the probability of ultimate ruin* $\psi(u)$ *(derived with Maple) in function of the initial surplus* $u$ *(*$u \geq 0$*) and for different values of the dependence parameter* $\theta$ *:*

- *with* $\theta = -1$ :

$$\psi(u) = 0.7201508967e^{-0.2687389645u} - 0.01854637723e^{-2.220708719u};$$

- *with* $\theta = -0.5$ :

$$\psi(u) = 0.6957948813e^{-0.2976043940u} - 0.01047590296e^{-2.114760590u};$$

- *with* $\theta = 0$ :

$$\psi(u) = \frac{2}{3}e^{-\frac{1}{3}u};$$

- *with* $\theta = 0.5$ :

$$\psi(u) = 0.6311261756e^{-0.3788264025u} + 0.01399640216e^{-1.873562242u};$$

- *with* $\theta = 1$ :

$$\psi(u) = 0.5865437312e^{-0.4391578659u} + 0.03347620593e^{-1.730494168u}.$$

*As can be seen from Figure 2.4, the dependence parameter* $\theta$ *has a clear impact on the ruin probabilities.*

*We may interpret the impact of the dependence relation between the r.v.'s $W$ and $X$ on the ruin probabilities as follows. When the dependence relation is positive (negative), the probability of having an important claim increases as the time elapsed since the last claim increases (decreases). It implies that the probability that the insurance company has enough premium income to pay the claim is higher (lower) and the ruin probability is lower when a positive (negative) dependence relation is assumed. The impact on the ruin probabilities is more significant when the positive (negative) relation becomes stronger.*

*The analytic expressions for the expected discounted value of the deficit at ruin,*

$$m_\delta(u) = E\left[e^{-\delta T}|U(T)|I(T<\infty)|U(0)=u\right],$$

*assuming $\delta = 5\%$ are obtained for various values of the dependence parameter $\theta$ :*

- *with $\theta = -1$ :*

$$m_\delta(u) = 0.6997443091e^{-0.3206647526u} + 0.03664826823e^{-2.219872885u};$$

- *with $\theta = -0.5$ :*

$$m_\delta(u) = 0.6599098319e^{-0.3500911630u} + 0.01714786296e^{-2.114343900u};$$

- *with $\theta = 0$ :*

$$m_\delta(u) = 0.6137092190e^{-0.3862907812u};$$

- *with $\theta = 0.5$ :*

$$m_\delta(u) = 0.5591165404e^{-0.4321500210u} - 0.01344088192e^{-1.873928948u};$$

- *with* $\theta = 1$ :

$$m_\delta(u) = 0.4928389831e^{-0.4927941702u} - 0.02070699597e^{-1.731037829u}.$$

## 2.9 Acknowledgements

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## 2.10 References

- Albrecher, H. and J. Teugels (2006). Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability* 43 (1), 265-285.
- Boudreault, M., Cossette, H., Landriault, D. and E. Marceau (2006). On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, 301–323.
- Bouye, E., Nikeghbali, A., Riboulet, G., and T. Roncalli (2000). Copulas for Finance. A reading guide and some applications. Rapport technique du Groupe de recherche opérationnelle, Crédit Lyonnais.
- Denuit, M., Dhaene, J., Goovaerts, M.J., and R. Kaas (2005). *Actuarial Theory for Dependent Risks - Measures, Orders and Models.* Wiley. New York.

- Dickson, D.C.M., and C. Hipp (2001). On the time to ruin for Erlang(2) risk processes. *Insurance : Mathematics and Economics* 29, 333–344.
- Frees, E. W., and E. A. Valdez (1998). Understanding relationships using copulas. *North American Actuarial Journal* 2, 1-25.
- Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory.* S.S. Huebner Foundation. University of Pennsylvania. Philadelphie.
- Gerber, H.U. and E.S.W. Shiu (1998). On the time value of ruin. *North American Actuarial Journal 2*, 48-78.
- Grandell, J. (1991). *Aspects of risk theory.* Springer-Verlag, New York.
- Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman and Hall, London.
- Klimenok, V. (2001). On the modification of Rouche's theorem for the queuing theory problems. *Queuing Systems* 38, 431-434.
- Li, S., and J. Garrido (2004). On ruin for the Erlang(n) risk process. *Insurance : Mathematics and Economics* 34, 391-408.
- McNeil, A., Frey, R., and P. Embrechts (2005). *Quantitative Risk Management.* Princeton Press, Princeton.
- Nelsen, R.B. (2006). *An Introduction to Copulas* (Second Edition). Springer Series in Statistics. Springer-Verlag, New York.
- Rolski, T., Schmidli, H., Schmidt, V. and J. Teugels (1999). *Stochastic Processes for Insurance and Finance.* Wiley, New York.
- Wang, S. (1998). Aggregation of correlated risk portfolios : Models and Algorithms. 1998 CAS Proceedings, 848-939.

Ruin Probabilities

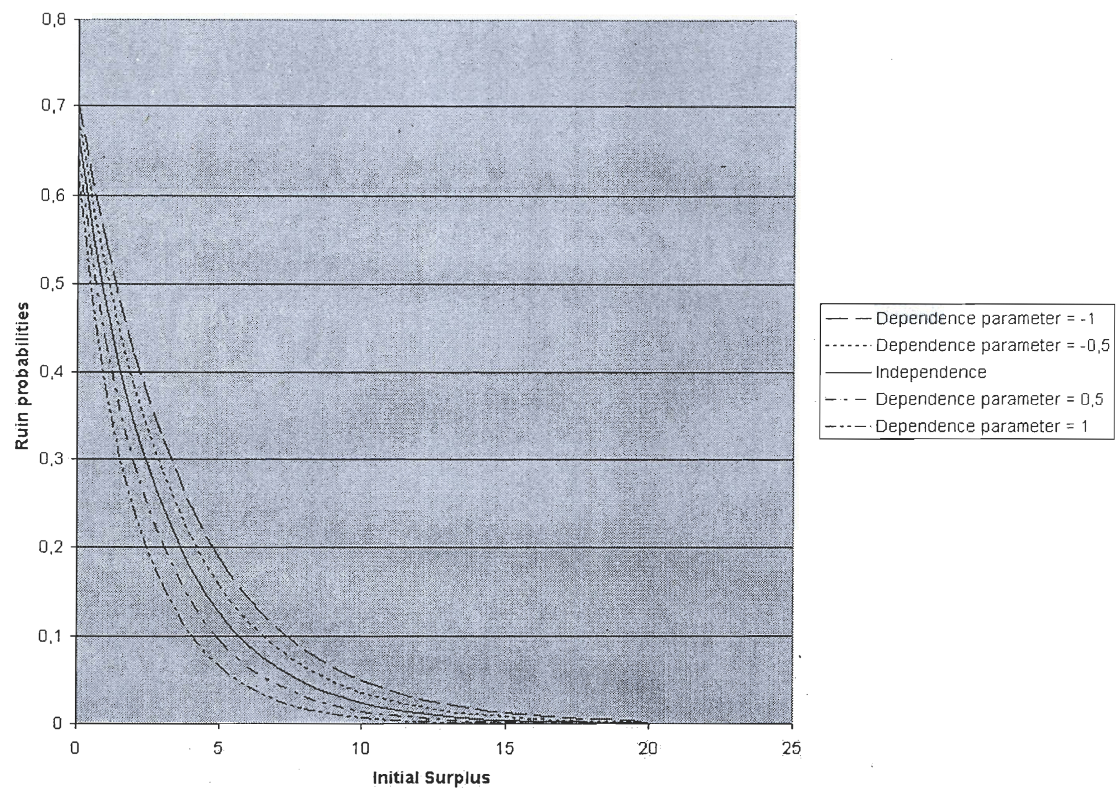


FIG. 2.4 – Ruin probabilities for $\theta$ (dependence parameter) equal to -1, -0.5, 0 (independence), 0.5, and 1.

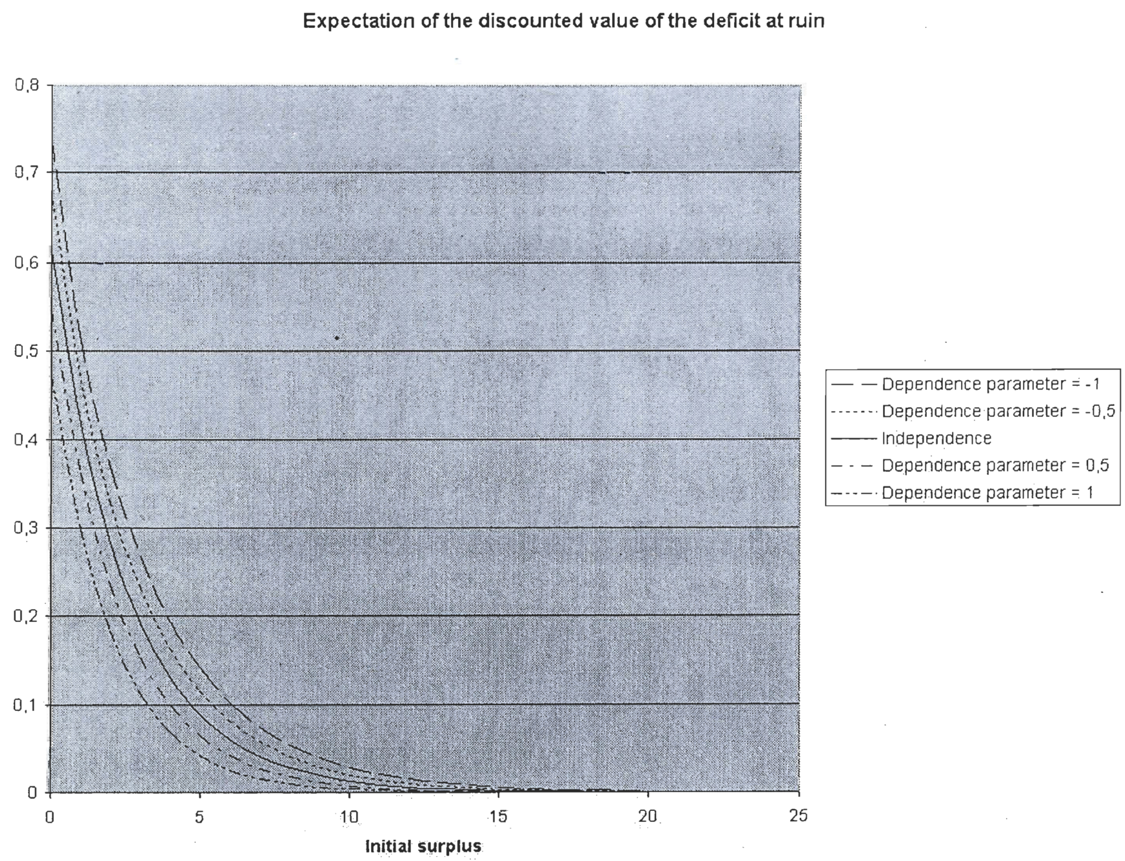


FIG. 2.5 – Values for the expectation of the discounted value of the deficit at ruin for $\theta$ (dependence parameter) equal to -1, -0.5, 0 (independence), 0.5, and 1.

# CHAPITRE III

## On the compound Poisson risk model with dependence based on a generalized Farlie-Gumbel-Morgenstern copula


## Résumé

Dans cet article, nous considérons une extension du modèle classique de risque pour lequel les montants de sinistre et les temps séparant deux sinistres sont dépendants. Nous considérons une structure de dépendance entre les montants des sinistres et les temps séparant deux sinistres par l'intermédiaire de la copule de Farlie-Gumbel-Morgenstern généralisée. Nous dérivons une transformée de Laplace de la fonction de Gerber-Shiu. Des expressions explicites pour les transformées de Laplace du temps de la ruine et du déficit à la ruine sont évaluées. Certains résultats concernant le calcul de la fonction de Gerber-Shiu dans le cas où les montants des sinistres suivent une loi exponentielle sont présentés. Une étude de l'effet de la dépendance sur les mesures de ruine conclut cet article.

## Abstract

In this paper we consider an extension to the classical compound Poisson risk model in which we introduce a dependence structure between the claim amounts and the interclaim time. This structure is embedded via a generalized Farlie-Gumbel-Morgenstern copula. In this framework, we derive the Laplace transform of the Gerber-Shiu discounted penalty function. An explicit expression for the Laplace transform of the time of ruin is given for exponential claim sizes.

## 3.1 Introduction

Classic risk models rely on the assumption of independence between the claim amounts and the interclaim times. This hypothesis simplifies the study of many risk quantities of interest under such a framework but has proven to be inadequate and too restrictive in different contexts. The need for generalizations of the classical risk models has led to several papers on the modelling of dependence. Among them, Albrecher and Teugels (2006) consider an arbitrary dependence structure based on a copula for the interclaim time and the subsequent claim size. They derive asymptotic results for both the finite and infinite-time ruin probabilities. Boudreault et al. (2006) examine several properties of an extension to the classical compound Poisson risk model assuming a dependence structure where the distribution of the next claim amount is defined in terms of the time elapsed since the last claim.

In this paper, we consider the family of risk models proposed by Albrecher and Teugels (2006) with a dependence structure defined with a generalized Farlie-Gumbel-Morgenstern (FGM) copula.

The claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a renewal process with interclaim times $\{W_j, j \in \mathbb{N}^+\}$ where $\{W_j, j \in \mathbb{N}^+\}$ is a sequence of independent and strictly positive random variables (r.v.). The r.v.'s $\{W_j, j \in \mathbb{N}^+\}$ are identically distributed as the canonical r.v. $W$ with probability density function (p.d.f.) $f_W$, cumulative distribution function (c.d.f.) $F_W$ and Laplace transform (L.T.) $f_W^*$. Throughout the paper, it is assumed that $W$ has an exponential distribution with expectation $\frac{1}{\lambda}$ with

$$f_W(t) = \lambda e^{-\lambda t}, \tag{3.1}$$

$$F_W(t) = 1 - e^{-\lambda t}, \tag{3.2}$$

$$f_W^*(s) = E\left[e^{-sW}\right] = \frac{\lambda}{\lambda + s}.$$

The claim amount r.v.'s $\{X_j, j \in \mathbb{N}^+\}$, where $X_j$ corresponds to the amount of the $j$th claim, are assumed to be a sequence of strictly positive, independent and identically distributed (i.i.d.) r.v.'s with p.d.f. $f_X$, c.d.f. $F_X$ and L.T. $f_X^*$.

We assume that $\{(X_j, W_j), j \in \mathbb{N}^+\}$ form a sequence of i.i.d. random vectors, where the components may be dependent. The joint p.d.f. of $(X_j, W_j)$ is denoted by $f_{X,W}(x,t)$ with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$. When $X$ and $W$ are continuous, the associated bivariate L.T. is given by

$$f_{X,W}^*(s_1, s_2) = E\left[e^{-s_1 X} e^{-s_2 W}\right] = \int_0^\infty \int_0^\infty e^{-s_1 x} e^{-s_2 t} f_{X,W}(x,t)\, dx dt. \tag{3.3}$$

The total claim amount process is denoted by $\underline{S} = \{S(t), t \geq 0\}$ where $S(t) = \sum_{j=1}^{N(t)} X_j$ and $\sum_a^b$ equals 0 if $b < a$.

We define the surplus process $\underline{U} = \{U(t), t \geq 0\}$ with $U(t) = u + pt - S(t)$ where $u$ ($u \in \mathbb{R}^+ = [0, \infty)$) is the initial surplus level and $p$ ($p \geq 0$) is the premium rate. Let $T = \inf_{t \geq 0} \{t, U(t) < 0\}$ be the time of ruin with $T = \infty$ if $U(t) \geq 0$ for all $t \geq 0$ (i.e. ruin does not occur). To ensure that ruin will not occur almost surely, the premium rate $p$ is such that

$$E[pW_i - X_i] > 0, \; i = 1, 2, ..., \tag{3.4}$$

providing a positive safety loading. The deficit at ruin and the surplus just prior to ruin are respectively denoted by $|U(T)|$ and $U(T^-)$.

In recent years, a fair amount of research in ruin theory has been devoted to the analysis of the expected value of the discounted penalty function. Introduced by Gerber and Shiu (1998), this function is given by

$$m_\delta(u) = E\left[e^{-\delta T} w\left(U(T^-), |U(T)|\right) I(T < \infty)|U(0) = u\right], \quad u \geq 0, \tag{3.5}$$

where $w(x, y)$, for $x, y \geq 0$, is the penalty function at the time of ruin for the surplus prior to ruin and the deficit at ruin, $\delta$ is a non-negative parameter (the force of interest) and $I$ is the indicator function, such that $I(A) = 1$ if the event $A$ occurs and equals 0 otherwise. A special case of the Gerber-Shiu penalty function with $w(x, y) = 1$, for all $x, y \geq 0$, is the Laplace transform of the time of ruin, denoted by $\phi_T(u)$. If $\delta = 0$ and $w(x, y) = 1$ for all $x, y \in \mathbb{R}^+$, (3.5) corresponds to the infinite-time ruin probability $\psi(u) = \Pr(T < \infty|U(0) = u)$.

In this risk model, it is clear that the increments $\{(X_j - pW_j), j \in \mathbb{N}^+\}$ of the surplus process are still independent. This allows us, as was done in an independent setting, to obtain with a martingale argument an exponential upper bound for the ruin probability

$$\psi(u) \leq e^{-\rho u},$$

where the adjustment coefficient $\rho$ (if it exists) is the strictly positive solution to $E\left[e^{-r(X_1 - pW_1)}\right] = 1$. Albrecher and Teugels (2006) provide an expression for $E\left[e^{-r(X_1-pW_1)}\right]$ for several copulas. The independence assumption between the increments of the surplus process also implies that the infinite-time ruin probability is a compound geometric tail in our risk model, as proven further in the paper.

In the present paper we examine the properties of a risk model assuming a dependence structure between the claim sizes and the claim interoccurence times based on a generalized FGM copula. In Section 3.2 we briefly recall basic properties of the generalized FGM copula considered in the model. In the framework of the proposed model, we discuss in Section 3.3 the Lundberg's generalized equation which is used in the following section to derive the Laplace transform of the expected discounted penalty function. Explicit expressions for special cases of the Laplace transform of the time of ruin are given in Section 3.5. Numerical examples are also provided in Section 3.6.

## 3.2 Dependence structure

Modelling the dependence structure between r.v.'s using copulas has become popular in actuarial science and financial risk management. The reader may consult e.g. Frees and Valdez (1998), Wang (1998), Bouye et al. (2000), Denuit et al. (2005) and McNeil et al. (2005) for applications of copulas in actuarial science and financial risk management. As in the present paper, Albrecher and Teugels (2006) use copulas to define the joint distribution for the interclaim time r.v. and the claim amount r.v.

We assume that the joint distribution of $(X, W)$ is defined with a generalized FGM copula. This copula which belongs to the family of copulas introduced and studied by Rodríguez-Lallena and Úbeda-Flores (2004), is defined by

$$C(u,v) = uv + \theta h(u) g(v), \tag{3.6}$$

where $h$ and $g$ are two non-zero real functions with support $[0,1]$. In this paper, we consider the special case mentioned in Example 4.1 of their paper where

$$h(u) = u^a(1-u)^b \tag{3.7}$$

$$g(v) = v^c(1-v)^d, \tag{3.8}$$

with $a$, $b$, $c$, $d \geq 1$. Combining (3.7) and (3.8) in (3.6), the expression of the copula is given by

$$C(u,v) = uv + \theta u^a(1-u)^b v^c(1-v)^d, \tag{3.9}$$

and it is an extension to the classical FGM copula

$$C(u,v) = uv + \theta uv(1-u)(1-v),$$

$-1 \leq \theta \leq 1$. Note that other extensions of the classical FGM copula are proposed in the literature (see e.g. Drouet Mari and Kotz (2001) for a review). One motivation of these extensions is to improve the range of dependence association (as measured by either Kendall's tau or Spearman's rho) between the components of $(X, W)$. Rodríguez-Lallena and Úbeda-Flores (2004) show that the admissible range for $\theta$ depends on the parameters $a$, $b$, $c$, and $d$ of the copula defined in (3.9). They also provide explicit expressions for Kendall's tau and Spearman's rho. For example, if $a = b = c = d = 2$, then $-27 \leq \theta \leq 27$ and Kendall's tau (Sperman's rho) goes from -0.24 to 0.24 (-0.36 to 0.36) while, for the classical FGM copula, Kendall's tau (Sperman's rho) takes values in $\left[-\frac{2}{9}, \frac{2}{9}\right]$ $\left(\left[-\frac{1}{3}, \frac{1}{3}\right]\right)$.

The p.d.f. associated to (3.9) is given by

$$c(u, v) = 1 + \theta h'(u) g'(v). \tag{3.10}$$

Given (3.9), the joint c.d.f. $F_{X,W}$ is defined by

$$\begin{aligned} F_{X,W}(x,t) &= C(F_X(x), F_W(t)) \\ &= F_X(x)F_W(t) + \theta F_X(x)^a \Big\{1 - F_X(x)\Big\}^b F_W(t)^c \Big\{1 - F_W(t)\Big\}^d. \end{aligned}$$

With (3.10), the joint p.d.f. $f_{X,W}$ of $(X, W)$ is

$$\begin{aligned} f_{X,W}(x,t) &= c(F_X(x), F_W(t)) f_X(x) f_W(t) \\ &= f_X(x)f_W(t) + \theta h'\Big\{(F_X(x)\Big\} g'\Big\{(F_W(t)\Big\} f_X(x) f_W(t). \end{aligned} \tag{3.11}$$

Throughout this paper, we suppose that $a$, $b \geq 1$, $c \in \{2, 3, ...\}$, and $d > 1$.

## 3.3 Lundberg's generalized equation

One important step in the analysis of the ruin measures is to develop the so-called Lundberg generalized equation and to examine its properties. An analysis of this equation is required to find the defective renewal equation for $m_\delta(u)$. More precisely, we need to identify the number of roots to the Lundberg's generalized equation in the right-half complex plane, i.e. with $\Re(s) \geq 0$. These roots are useful to derive the defective renewal equation for $m_\delta(u)$ as we shall see in the next sections.

To derive Lundberg's generalized equation, we consider the discrete time process embedded in the continuous-time process surplus $\underline{U}$. Let us define the discrete-time process $\underline{\widetilde{U}} = \left\{\widetilde{U}_k, k = 0, 1, 2, ...\right\}$, where $\widetilde{U}_0 = u$ and $\widetilde{U}_k = U_{T_k}$ denotes the surplus immediately after the $k$th claim, viz.

$$\widetilde{U}_k = u + \sum_{j=1}^{k} (pW_j - X_j) \quad \text{for} \quad k \in \mathbb{N}^+. \tag{3.12}$$

The process $\underline{\widetilde{V}} = \left\{ \exp\left\{ -\delta \sum_{j=1}^{k} W_j + s\widetilde{U}_k \right\}, k = 0, 1, \cdots \right\}$, for $s > 0$ is a martingale if and only if

$$E\left(e^{-\delta W} e^{s(pW - X)}\right) = 1, \tag{3.13}$$

which corresponds to Lundberg's generalized equation. Due to (3.11), the left-hand side of (3.13) can be written as

$$\begin{aligned} E\left(e^{-\delta W} e^{s(pW-X)}\right) &= \int_0^\infty \int_0^\infty e^{t(sp-\delta)} e^{-sx} f_{X,W}(x,t)\,dx\,dt \\ &= \int_0^\infty \int_0^\infty e^{t(sp-\delta)} e^{-sx} f_X(x) f_W(t)\,dx\,dt \\ &+ \theta \int_0^\infty \int_0^\infty e^{t(sp-\delta)} e^{-sx} h'(F_X(x)) g'(F_W(t)) f_X(x) f_W(t)\,dx\,dt, \end{aligned} \tag{3.14}$$

where the functions $g$ and $h$ are defined in (3.7) and (3.8) respectively. In order to rewrite (3.14), let us define the r.v. $Z$ with p.d.f. given by

$$g_Z(x) = f_X(x) - f_X(x) h'(F_X(x)) \tag{3.15}$$

and corresponding Laplace transform $g_Z^*(s)$. We also define the function $k_W$ by

$$k_W(t) = f_W(t) g'(F_W(t)), \tag{3.16}$$

with a Laplace transform

$$k_W^\star(s) = \int_0^\infty e^{-ts} f_W(t) g'(F_W(t)) dt. \tag{3.17}$$

With these newly defined functions, (3.14) can be written as

$$E\left(e^{-\delta W} e^{s(pW-X)}\right) = f_X^\star(s) f_W^*(\delta - sp) + \theta k_W^*(\delta - sp)\left\{f_X^\star(s) - g_Z^\star(s)\right\}. \tag{3.18}$$

Furthermore, by integration by parts and with simple rearrangements, (3.17) becomes (due to (3.8))

$$\begin{aligned} k_W^\star(s) &= \frac{s}{\lambda}\int_0^1 u^{(c+1)-1}(1-u)^{(d+s/\lambda)-1} du \\ &= \frac{\Gamma(c+1)\Gamma(d+s/\lambda)}{\Gamma(c+1+d+s/\lambda)}\frac{s}{\lambda} \\ &= \frac{c!\lambda^c s}{\prod_{i=1}^{c+1}(s+\lambda_i)}, \end{aligned} \tag{3.19}$$

where

$$\lambda_i = \lambda(d+i-1).$$

Using partial fractions, it follows that

$$k_W^\star(s) = \sum_{i=1}^{c+1}\frac{a_i}{s+\lambda_i} = \sum_{i=1}^{c+1}\alpha_i\frac{\lambda_i}{s+\lambda_i}$$

with

$$a_i = \frac{c!\lambda^c(-\lambda_i)}{\prod_{j=1,j\neq i}^{c+1}(-\lambda_i+\lambda_j)}$$

or

$$\alpha_i = \frac{c!\lambda^c(-1)}{\prod\limits_{j=1,j\neq i}^{c+1} (-\lambda_i + \lambda_j)}.$$

Consequently, we obtain

$$\begin{aligned} k_W(t) &= \sum_{i=1}^{c+1} a_i e^{-\lambda_i t} \\ &= \sum_{i=1}^{c+1} \alpha_i \lambda_i e^{-\lambda_i t} \qquad t \geq 0. \end{aligned}$$

If we insert (3.18) and (3.19) in (3.13) and since $f_W^\star(s) = \frac{\lambda}{\lambda+s}$, we obtain

$$f_X^\star(s)\frac{\lambda}{\lambda+\delta-sp} + \theta\frac{c!\lambda^c(\delta-sp)}{\prod\limits_{i=1}^{c+1}(\lambda_i+\delta-sp)}\{f_X^\star(s) - g_Z^\star(s)\} = 1,$$

which becomes

$$\begin{aligned} (\lambda+\delta-sp)\prod_{i=1}^{c+1}(\lambda_i+\delta-sp) &= f_X^\star(s)\lambda\prod_{i=1}^{c+1}(\lambda_i+\delta-sp) \\ &+ \theta c!\lambda^c(\delta-sp)(\lambda+\delta-sp)\{f_X^\star(s) - g_Z^\star(s)\}. \end{aligned} \tag{3.20}$$

This last equality will be used in the following proposition in which we apply Rouché's Theorem to identify the number of roots in the right half-plane of the generalized Lundberg equation (3.13).

**Proposition 13.** *For $\delta > 0$ and $\theta \neq 0$, Lundberg's generalized equation in (3.13) has exactly $c+2$ roots, say $\rho_1, \cdots, \rho_{c+2}$, in the right-half complex plane, i.e. with $\Re(\rho_i(\delta)) > 0$.*

**Proof.** Let us apply Rouché's theorem on the contour $C_r$, consisting of the imaginary axis running from $-ir$ to $ir$ and a semi-circle with radius $r$ running clockwise from $ir$ to $-ir$. We let $r \to \infty$ and denote by $C$ the limiting contour.

Working with the rearranged expression (3.20) of Lundberg's generalized equation (3.13), we want to show

$$\left| f_X^\star(s)\lambda \prod_{i=1}^{c+1}(\lambda_i + \delta - sp) + \theta c!\lambda^c (\delta - sp)(\lambda + \delta - sp)\{f_X^\star(s) - g_Z^\star(s)\} \right| \leq$$
$$\left| (\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp) \right|.$$

This is equivalent to showing that

$$\left| \frac{\lambda f_X^\star(s) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)} + \frac{\theta c!\lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)} \{f_X^\star(s) - g_Z^\star(s)\} \right| \leq 1.$$

The terms

$$\frac{\lambda \left\{ \prod_{i=1}^{c+1}(\lambda_i + \delta - sp) \right\}}{(\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)} = \frac{\lambda}{(\lambda + \delta - sp)}$$

and

$$\frac{\theta c!\lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)} = \frac{\theta c!\lambda^c (\delta - sp)}{\prod_{i=1}^{c+1}(\lambda_i + \delta - sp)}$$

are ratios of polynomials with a strictly higher degree at the denominator. Hence,

$$\left| \frac{\lambda f_X^\star(s) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)} + \frac{\theta c!\lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1}(\lambda_i + \delta - sp)} \{f_X^\star(s) - g_Z^\star(s)\} \right| \to 0$$

on $C$ (excluding $s = 0$).

At $s = 0$, it is clear that

$$\frac{\lambda}{(\lambda + \delta - sp)} > 0,$$

and

$$\frac{\theta c! \lambda^c (\delta - sp)}{\prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} > 0.$$

Also, at $s = 0$ and for $\delta > 0$, we have

$$\frac{\lambda}{(\lambda + \delta)} + \frac{\theta c! \lambda^c (\delta)}{\left\{ \prod_{i=1}^{c+1} (\lambda_i + \delta) \right\}} < 1.$$

Finally, we have

$$\left| \frac{\lambda f_X^\star(s) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} + \frac{\theta c! \lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \{ f_X^\star(s) - g_Z^\star(s) \} \right|$$

$$\leq \left| \frac{\lambda f_X^\star(s) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \right| + \left| \frac{\theta c! \lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \{ f_X^\star(s) - g_Z^\star(s) \} \right|$$

$$= |f_X^\star(s)| \left| \frac{\lambda \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \right| + |f_X^\star(s) - g_Z^\star(s)| \left| \frac{\theta c! \lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \right|$$

$$\leq \left| \frac{\lambda \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \right| + \left| \frac{\theta c! \lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \right|$$

$$\leq \frac{\lambda}{(\lambda + \delta)} + \frac{\theta c! \lambda^c \delta}{\prod_{i=1}^{c+1} (\lambda_i + \delta)} = \frac{\lambda \prod_{i=1}^{c+1} (\lambda_i + \delta) + (\lambda + \delta) \theta c! \lambda^c \delta}{(\lambda + \delta) \prod_{i=1}^{c+1} (\lambda_i + \delta)} < 1. \square$$

For $\delta = 0$, the conditions to Rouché's theorem are not satisfied since

$$\left| \frac{\lambda f_X^\star(s) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} + \frac{\theta c! \lambda^c (\delta - sp)(\lambda + \delta - sp)}{(\lambda + \delta - sp) \prod_{i=1}^{c+1} (\lambda_i + \delta - sp)} \left\{ f_X^\star(s) - g_Z^\star(s) \right\} \right| = 1$$

for $s = 0$. Based on a result in the queuing theory literature, we rely on an extension of Rouché's theorem to determine the number of roots to Lundberg's generalized equation with a positive real part.

**Proposition 14.** *For $\delta = 0$ and $\theta \neq 0$ Lundberg's generalized equation (3.13) has exactly $c + 1$ roots, say $\{\rho_i(0), i = 1, ..., c + 1\}$, with $\Re(\rho_i(0)) > 0$ and a $(c + 2)$-th root $\rho_{c+2}(0) = 0$.*

**Proof :** Let $z = \frac{\kappa - s}{\kappa}$ and define the contour $D_\kappa = \{s : |z| = 1\}$. In terms of $s$, the contour $D_\kappa$ is a circle of radius $\kappa$ and origin $\kappa$. Similarly as in Proposition 13, we let $\kappa \to \infty$ and denote by $D$ the limiting contour. Using identical arguments (for $\delta = 0$) as in the proof of Proposition 13, we can deduce

$$\left| f_X^\star(s) \lambda \prod_{i=1}^{c+1} (\lambda_i - sp) + \theta c! \lambda^c (-sp)(\lambda - sp) \left\{ f_X^\star(s) - g_Z^\star(s) \right\} \right| \leq \left| (\lambda - sp) \prod_{i=1}^{c+1} (\lambda_i - sp) \right|,$$

on $D$ (excluding $s = 0$ or equivalently $z = 1$). Moreover, we observe that the functions

$$f_X^\star(s) \lambda \prod_{i=1}^{c+1} (\lambda_i - sp) + \theta c! \lambda^c (-sp)(\lambda - sp) f_X^\star(s) - g_Z^\star(s)$$

and

$$(\lambda - sp) \prod_{i=1}^{c+1} (\lambda_i - sp)$$

are continuous on $D$.

In order to apply Theorem 1 of Klimenok (2001), we must prove

$$\frac{d}{dz}\left(1-\frac{f_X^\star(\kappa-\kappa z)\lambda}{(\lambda-(\kappa-\kappa z)p)}+\frac{\theta c!\lambda^c\left(-(\kappa-\kappa z)p\right)\left\{f_X^\star(\kappa-\kappa z)-g_Z^\star(\kappa-\kappa z)\right\}}{\prod_{i=1}^{c+1}(\lambda_i-(\kappa-\kappa z)p)}\right)\Bigg|_{z=1}>0. \quad (3.21)$$

However, (3.21) is equivalent to

$$\frac{d}{dz}\left(1-E\left[e^{-\kappa(1-z)(X-pW)}\right]\right)\Big|_{z=1}>0,$$

which is verified under (3.4). From Klimenok (2001), one concludes that the number of solutions to (*3.13*) inside $D$ is equal to $c+1$, i.e. the number of roots of $(\lambda-sp)\left\{\prod_{i=1}^{c+1}(\lambda_i-sp)\right\}$ inside $D$ minus 1. Moreover, a trivial root to the Lundberg's generalized equation (3.13) (with $\delta=0$) is $\rho_{c+2}(0)=0$. □

**Remark 15.** *As mentioned at the end of Section 3.2, we assume in the paper that* $d>1$. *When* $d=1$ *(and for* $\delta>0$ *and* $\theta\neq 0$*), Lundberg's generalized equation in (3.13) has exactly* $c+1$ *roots, say* $\rho_1,\cdots,\rho_{c+1}$*, in the right-half complex plane, i.e. with* $\Re(\rho_i(\delta))>0$. *For* $\delta=0$ *and* $\theta\neq 0$ *Lundberg's generalized equation (3.13) has exactly c roots, say* $\{\rho_i(0), i=1,...,c\}$*, with* $\Re(\rho_i(0))>0$ *and a* $(c+1)$*-th root* $\rho_{c+1}(0)=0$. *These results can be demonstrated in a similar way as in the proofs of Propositions 13 and 14.*

In the next sections, we only consider the cases when the roots are distinct.

## 3.4 Laplace transform of $m_\delta$

In this section, we find the Laplace transform of the Gerber-Shiu discounted penalty function $m_\delta(u)$ defined in (3.5).

By conditioning on the time and the amount of the first claim, we have

$$
\begin{aligned}
& m_\delta(u) \\
= & \int_0^\infty \int_0^\infty e^{-\delta t} m_\delta(u+pt-x) f_{X,W}(x,t)\,dx\,dt \\
= & \int_0^\infty \int_0^\infty e^{-\delta t} m_\delta(u+pt-x) f_X(x) f_W(t)\,dx\,dt \\
& +\theta \int_0^\infty \int_0^\infty e^{-\delta t} m_\delta(u+pt-x) h'(F_X(x)) g'(F_W(t)) f_X(x) f_W(t)\,dx\,dt \\
= & \int_0^\infty \int_0^\infty e^{-\delta t} m_\delta(u+pt-x) f_X(x) f_W(t)\,dx\,dt \\
& +\theta \int_0^\infty \int_0^\infty e^{-\delta t} m_\delta(u+pt-x) \left(g'(F_W(t)) f_W(t) f_X(x) - g'(F_W(t)) f_W(t) g_Z(x)\right) dx\,dt \qquad (3.22)
\end{aligned}
$$

where $h(u) = u^a(1-u)^b$, $g(u) = u^c(1-u)^d$. Using (3.15) and (3.16), we can rewrite (3.22) as

$$
\begin{aligned}
m_\delta(u) &= \int_0^\infty E\{m_\delta(u+pt-X)\} e^{-\delta t} f_W(t)\,dt \\
&+ \theta \int_0^\infty E\{m_\delta(u+pt-X)\} e^{-\delta t} k_W(t)\,dt \\
&- \theta \int_0^\infty E\{m_\delta(u+pt-Z)\} e^{-\delta t} k_W(t)\,dt. \qquad (3.23)
\end{aligned}
$$

With $y = u + pt$ in (3.23), we have

$$
\begin{aligned}
pm_\delta(u) &= \lambda \int_u^\infty E\{m_\delta(y-X)\} e^{-y(\frac{\delta+\lambda}{p})} e^{u(\frac{\delta+\lambda}{p})}\,dy \\
&+ \theta \sum_{i=1}^{c+1} \int_u^\infty a_i E\{m_\delta(y-X)\} e^{-y(\frac{\delta+\lambda_i}{p})} e^{u(\frac{\delta+\lambda_i}{p})}\,dy \\
&- \theta \sum_{i=1}^{c+1} \int_u^\infty a_i E\{m_\delta(y-Z)\} e^{-y(\frac{\delta+\lambda_i}{p})} e^{u(\frac{\delta+\lambda_i}{p})}\,dy. \qquad (3.24)
\end{aligned}
$$

In what follows, we use the Dickson-Hipp operator $T_r$ for an integrable real-valued function $f$ introduced by Dickson and Hipp (2001) :

$$
T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u)\,du, \qquad r \in \mathbb{C}. \qquad (3.25)
$$

From Li and Garrido (2004), we recall Properties 1, 2, and 6 of the operator $T_r$ that we use in the sequel :

- Property 1. $T_r f(0) = \int_0^\infty e^{-ru} f(u)\, du = f^*(u)$, for $r \in \mathbb{C}$, is the Laplace transform of $f$.
- Property 2. $T_{r_1} T_{r_2} f(x) = T_{r_2} T_{r_1} f(x) = \frac{(T_{r_1} f(x) - T_{r_2} f(x))}{r_2 - r_1}$, $r_1 \neq r_2 \in \mathbb{C}$, $x \geq 0$.
- Property 6. If $r_1, ..., r_k$ are distinct complex numbers, then

$$T_{r_k} ... T_{r_2} T_{r_1} f(x) = (-1)^{k-1} \sum_{l=1}^{k} \frac{T_{r_l} f(x)}{\tau_k'(r_l)}, \qquad x \geq 0,$$

where $\tau_k(r) = \prod_{l=1}^{k} (r - r_j)$. The corresponding Laplace transform is

$$T_s T_{r_k} ... T_{r_2} T_{r_1} f(0) = (-1)^k \left( \frac{f^*(s)}{\tau_k(s)} - \sum_{l=1}^{k} \frac{f^*(r_l)}{(s - r_l)\, \tau_k'(r_l)} \right), \qquad s \in \mathbb{C}.$$

Applying the Dickson-Hipp operator with Property 1 allows us to rewrite (3.24) as

$$pm_\delta(u) = \lambda T_{\frac{\delta+\lambda}{p}} \mu_1(u) + \theta \sum_{i=1}^{c+1} a_i T_{\frac{\delta+\lambda_i}{p}} \mu_1(u) - \theta \sum_{i=1}^{c+1} a_i T_{\frac{\delta+\lambda_i}{p}} \mu_2(u), \tag{3.26}$$

where

$$\begin{aligned}
\mu_1(u) &= E\{m_\delta(u - X)\} = \int_0^u m_\delta(u - x) f_X(x) dx + w_1(u) \\
\mu_2(u) &= E\{m_\delta(u - Z)\} = \int_0^u m_\delta(u - x) g_Z(x) dx + w_2(u) \\
w_1(u) &= \int_u^\infty w(u, x - u) f_X(x) dx,
\end{aligned}$$

and

$$w_2(u) = \int_u^\infty w(u, x - u) g_Z(x) dx.$$

Taking the Laplace transform of (3.26) and using Properties 1 and 2 of the Dickson-Hipp operator $T_r$ leads to

$$
\begin{aligned}
pm_\delta^\star(s) &= \lambda\left\{\frac{\mu_1^\star(s)-\mu_1^\star(\frac{\delta+\lambda}{p})}{(\frac{\delta+\lambda}{c}-s)}\right\}+\theta\sum_{i=1}^{c+1}a_i\left\{\frac{\mu_1^\star(s)-\mu_1^\star(\frac{(\delta+\lambda_i)}{p})}{(\frac{\delta+\lambda_i}{p}-s)}\right\}\\
&- \theta\sum_{i=1}^{c+1}a_i\left\{\frac{\mu_2^\star(s)-\mu_2^\star(\frac{(\delta+\lambda_i)}{p})}{(\frac{\delta+\lambda_i}{p}-s)}\right\},
\end{aligned}
$$

where

$$\mu_1^\star(s) = m_\delta^\star(s)f_X^\star(s)+w_1^\star(s),$$

and

$$\mu_2^\star(s) = m_\delta^\star(s)g_Z^\star(s)+w_2^\star(s).$$

The above equality simplifies to

$$m_\delta^\star(s) = \frac{w_1^\star(s)\left\{f_W^\star(\delta-sp)+\theta k_W^\star(\delta-sp)\right\}-\theta w_2^\star(s)k_W^\star(\delta-sp)+v^\star(s)}{1-f_X^\star(s)f_W^\star(\delta-sp)-\theta f_X^\star(s)k_W^\star(\delta-sp)+\theta g_Z^\star(s)k_W^\star(\delta-sp)}, \tag{3.27}$$

where

$$v^\star(s) = -\lambda\frac{\mu_1^\star(\frac{\delta+\lambda}{p})}{(\delta+\lambda-sp)}-\theta\sum_{i=1}^{c+1}a_i\frac{\mu_1^\star(\frac{\delta+\lambda_i}{p})}{(\delta+\lambda_i-sp)}+\theta\sum_{i=1}^{c+1}a_i\frac{\mu_2^\star(\frac{\delta+\lambda_i}{p})}{(\delta+\lambda_i-sp)}.$$

In the following theorem, an expression for the Laplace transform $m_\delta^\star$ is derived based on equation (3.27).

**Corollary 16.** *We assume the compound Poisson risk model with a dependence structure based on the generalized FGM copula defined in (3.9) with parameters* $a$, $b \geq 1$, $c \in \{2, 3, ...\}$, *and* $d > 1$. *The Laplace transform of the Gerber-Shiu discounted penalty function* $m_\delta$ *is given by*

$$m_\delta^\star(s) = \frac{\beta_{1,\delta}^\star(s) + \beta_{2,\delta}^\star(s)}{h_{1,\delta}^\star(s) - h_{2,\delta}^\star(s)}, \tag{3.28}$$

*where*

$$h_{1,\delta}^\star(s) = \Big(\frac{\delta+\lambda}{p} - s\Big)\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p} - s\Big), \tag{3.29}$$

$$h_{2,\delta}^\star(s) = h_{1,\delta}^\star(s)\left\{f_X^\star(s)f_W^\star(\delta - sp) + \theta f_X^\star(s)k_W^\star(\delta - sp) - \theta g_Z^\star(s)k_W^\star(\delta - sp)\right\}, \tag{3.30}$$

$$\beta_{1,\delta}^\star(s) = h_{1,\delta}^\star(s)\left\{w_1^\star(s)\left(f_W^\star(\delta - sp) + \theta k_W^\star(\delta - sp)\right) - \theta w_2^\star(s)k_W^\star(\delta - sp)\right\}, \tag{3.31}$$

*and* $\beta_{2,\delta}^\star(s)$ *is a polynomial of degree* $c+1$, *or less given by*

$$\begin{aligned}\beta_{2,\delta}^\star(s) &= -\sum_{j=1}^{c+2}\beta_{1,\delta}^\star(\rho_j)\prod_{k=1,k\neq j}^{c+2}\frac{s-\rho_k}{\rho_j-\rho_k} \\ &= -\tau_{c+2}(s)\sum_{j=1}^{c+2}\frac{\beta_{1,\delta}^\star(\rho_j)}{(s-\rho_j)\tau_{c+2}'(\rho_j)},\end{aligned}$$

*with* $\tau_{c+2}(s) = \prod_{j=1}^{c+2}(s-\rho_j)$ *and* $\tau_{c+2}'(\rho_j) = \prod_{j=1,j\neq k}^{c+2}(\rho_j - \rho_k)$.

**Proof.** Multiplying both the numerator and denominator of (3.27) by $\big(\frac{\delta+\lambda}{p} -$

$s\Big)\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)$ yields (3.28), with

$$\begin{aligned}
\beta^*_{2,\delta}(s) &= h^*_{1,\delta}(s)v^*(s)\\
&= \Big(\frac{\delta+\lambda}{p}-s\Big)\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)v^*(s)\\
&= -\frac{\lambda}{p}\mu^*_1\Big(\frac{\delta+\lambda}{p}\Big)\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)\\
&\quad -\theta\Big(\frac{\delta+\lambda-sp}{p}\Big)\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j-sp}{p}\Big)\sum_{i=1}^{c+1}a_i\frac{\mu^*_1\Big(\frac{\delta+\lambda_i}{p}\Big)}{\Big(\delta+\lambda_i-sp\Big)}\\
&\quad +\theta\Big(\frac{\delta+\lambda-sp}{p}\Big)\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j-sp}{p}\Big)\sum_{i=1}^{c+1}a_i\frac{\mu^*_2\Big(\frac{\delta+\lambda_i}{p}\Big)}{\Big(\delta+\lambda_i-sp\Big)}\\
&= -\frac{\lambda}{p}\mu^*_1\Big(\frac{\delta+\lambda}{p}\Big)\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)\\
&\quad -\frac{\theta}{p}\Big(\frac{\delta+\lambda-sp}{p}\Big)\sum_{i=1}^{c+1}a_i\left(\prod_{\substack{j=1\\ j\neq i}}^{c+1}\Big(\frac{\delta+\lambda_j-sp}{p}\Big)\right)\mu^*_1\Big(\frac{\delta+\lambda_i}{p}\Big)\\
&\quad +\frac{\theta}{p}\Big(\frac{\delta+\lambda-sp}{p}\Big)\sum_{i=1}^{c+1}a_i\left(\prod_{\substack{j=1\\ j\neq i}}^{c+1}\Big(\frac{\delta+\lambda_j-sp}{p}\Big)\right)\mu^*_2\Big(\frac{\delta+\lambda_i}{p}\Big)
\end{aligned}$$

which is a polynomial of degree $c+1$ or less.

Note that $m^*_\delta(s)$ is analytic for $\Re(s)\geq 0$ which means that the roots $\rho_j, j=1,\cdots,c+2$ of the denominator in (3.28) are also the roots of the numerator. Given the denominator in (3.27), these roots are also the solutions to Lundberg's generalized equation

$$\begin{aligned}
E\left(e^{-\delta W}e^{s(pW-X)}\right) &= f^*_X(s)f^*_W(\delta-sp)+\theta f^*_X(s)k^*_W(\delta-sp)-\theta g^*_Z(s)k^*_W(\delta-sp)\\
&= 1.
\end{aligned}$$

By the Lagrange interpolation formula with the $(c+2)$ points $\rho_1,\ldots,\rho_{c+2}$ (since

$\beta^*_{2,\delta}(s)$ is a polynomial of degree $(c+1)$) at which $\beta^*_{2,\delta}(s) = -\beta^*_{1,\delta}(s)$, we have

$$\beta^*_{2,\delta}(s) = -\sum_{j=1}^{c+2} \beta^*_{1,\delta}(\rho_j) \prod_{k=1,k\neq j}^{c+2} \frac{s-\rho_k}{\rho_j - \rho_k}$$

which completes the proof of the corollary. □

## 3.5 Defective renewal equation for $m_\delta$

In order to find the defective renewal equation for $m_\delta$, we need the following lemma.

**Lemma 17.** *In the compound Poisson risk model with a dependence structure based on the generalized FGM copula defined in (3.9) with parameters $a$, $b \geq 1$, $c \in \{2, 3, ...\}$, and $d > 1$, the Laplace transform $m^*_\delta(s)$ of the Gerber-Shiu discounted penalty function $m_\delta$ satisfies*

$$m^*_\delta(s) = \frac{T_s T_{\rho_1} \cdots T_{\rho_{c+2}} \beta_{1,\delta}(0)}{1 - T_s T_{\rho_1} \cdots T_{\rho_{c+2}} h_{2,\delta}(0)}. \tag{3.32}$$

**Proof.** Given property 6 of the Dickson-Hipp operator, we have the following equality

$$\begin{aligned} \beta^*_{1,\delta}(s) + \beta^*_{2,\delta}(s) &= \beta^*_{1,\delta}(s) - \tau_{c+2}(s) \sum_{j=1}^{c+2} \frac{\beta^*_{1,\delta}(\rho_j)}{(s-\rho_j)\tau'_{c+2}(\rho_j)} \\ &= \tau_{c+2}(s) \left\{ \frac{\beta^*_{1,\delta}(s)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2} \frac{\beta^*_{1,\delta}(\rho_j)}{(s-\rho_j)\tau'_{c+2}(\rho_j)} \right\} \\ &= \tau_{c+2}(s)(-1)^{c+2} T_s T_{\rho_1} \cdots T_{\rho_{c+2}} \beta_{1,\delta}(0). \end{aligned} \tag{3.33}$$

Similarly, we can derive an alternative expression for the denominator $h^*_{1,\delta}(s) - h^*_{2,\delta}(s)$ in (3.28). We know that $h^*_{1,\delta}$ is a polynomial of degree $c+2$ in $s$ (due to (3.29)) and that

$$h^*_{1,\delta}(\rho_j) = h^*_{2,\delta}(\rho_j)$$

for $j = 1 \cdots c+2$ from Propositions 13 and 14. Given that a Lagrange interpolating polynomial of degree $(c+2)$ passes through $(c+3)$ points, we choose the point $(0,h^*_{1,\delta}(0))$ in addition to the points $(\rho_1,h^*_{1,\delta}(\rho_1)),\ldots,(\rho_{c+2},h^*_{1,\delta}(\rho_{c+2}))$ to write

$$h^*_{1,\delta}(s) = h^*_{1,\delta}(0)\prod_{k=1}^{c+2}\frac{(s-\rho_k)}{(-\rho_k)} + s\sum_{j=1}^{c+2}\frac{h^*_{1,\delta}(\rho_j)}{\rho_j}\prod_{k=1,k\neq j}^{c+2}\frac{s-\rho_k}{\rho_j-\rho_k}$$

which implies that

$$\begin{aligned}
& h^*_{1,\delta}(s) - h^*_{2,\delta}(s) \\
= \; & h^*_{1,\delta}(0)\frac{\tau_{c+2}(s)}{\prod\limits_{k=1}^{c+2}(-\rho_k)} + s\sum_{j=1}^{c+2}\frac{h^*_{2,\delta}(\rho_j)\tau_{c+2}(s)}{\rho_j(s-\rho_j)\tau'_{c+2}(\rho_j)} - h^*_{2,\delta}(s) \\
= \; & \tau_{c+2}(s)\left\{\frac{h^*_{1,\delta}(0)}{\prod\limits_{k=1}^{c+2}(-\rho_k)} + \sum_{j=1}^{c+2}\frac{sh^*_{2,\delta}(\rho_j)}{\rho_j(s-\rho_j)\tau'_{c+2}(\rho_j)} - \frac{h^*_{2,\delta}(s)}{\tau_{c+2}(s)}\right\} \\
= \; & \tau_{c+2}(s)\left\{\frac{h^*_{1,\delta}(0)}{\prod\limits_{k=1}^{c+2}(-\rho_k)} + \sum_{j=1}^{c+2}\frac{(s-\rho_j+\rho_j)h^*_{2,\delta}(\rho_j)}{\rho_j(s-\rho_j)\tau'_{c+2}(\rho_j)} - \frac{h^*_{2,\delta}(s)}{\tau_{c+2}(s)}\right\} \\
= \; & \tau_{c+2}(s)\left\{\frac{h^*_{1,\delta}(0)}{\tau_{c+2}(0)} - \sum_{j=1}^{c+2}\frac{h^*_{1,\delta}(\rho_j)}{(-\rho_j)\tau'_{c+2}(\rho_j)} + \sum_{j=1}^{c+2}\frac{h^*_{2,\delta}(\rho_j)}{(s-\rho_j)\tau'_{c+2}(\rho_j)} - \frac{h^*_{2,\delta}(s)}{\tau_{c+2}(s)}\right\} \\
= \; & \tau_{c+2}(s)\left\{\frac{h^*_{1,\delta}(0)}{\tau_{c+2}(0)} - \sum_{j=1}^{c+2}\frac{h^*_{1,\delta}(\rho_j)}{(-\rho_j)\tau'_{c+2}(\rho_j)} - (-1)^{c+2}T_sT_{\rho_1}\cdots T_{\rho_{c+2}}h_{2,\delta}(0)\right\}. \qquad (3.34)
\end{aligned}$$

Since $h^*_{1,\delta}(\rho_j)$ is a polynomial of degree $(c+2)$ in $\rho_j$, we have

$$h^*_{1,\delta}(\rho_j) = \Big(\frac{\delta+\lambda}{p} - \rho_j\Big)\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p} - \rho_j\Big) = \sum_{l=0}^{c+2}\sigma_l \rho_j^{c+2-l}$$

with $\sigma_0 = (-1)^{c+2}$ and $\sigma_{c+2} = \left(\frac{\delta+\lambda}{p}\right)\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)$. We do not give the expressions for $\sigma_1, ..., \sigma_{c+1}$ because they cancel out in the developments that follow. This leads to

$$\begin{aligned}
\sum_{j=1}^{c+2}\frac{h^*_{1,\delta}(\rho_j)}{\rho_j \tau'_{c+2}(\rho_j)} &= \sum_{j=1}^{c+2}\frac{\left(\frac{\delta+\lambda}{p} - \rho_j\right)\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - \rho_j\right)}{\rho_j \tau'_{c+2}(\rho_j)} \\
&= \sum_{l=0}^{c+2}\sigma_l \sum_{j=1}^{c+2}\frac{\rho_j^{(c+1-l)}}{\tau'_{c+2}(\rho_j)}. \qquad (3.35)
\end{aligned}$$

Based on Lemma 1 on page 395 (and also on the relations on page 400) of Li and Garrido (2004), the terms summed over the indices $j$ in (3.35) are equal to zero except for $l = 0$, for which sums to $\sigma_0$ and the last term, for $l = c+2$, for which we obtain $\frac{-\sigma_{c+2}}{\tau_{c+2}(0)}$. Hence, we have

$$
\begin{aligned}
\frac{h_{1,\delta}^{*}(0)}{\tau_{c+2}(0)}+\sum_{j=1}^{c+2}\frac{h_{1,\delta}^{*}(\rho_j)}{\rho_j\tau_{c+2}'(\rho_j)} &= \frac{h_{1,\delta}^{*}(0)}{\tau_{c+2}(0)}+\sum_{l=0}^{c+2}\sigma_l\sum_{j=1}^{c+2}\frac{\rho_j^{(c+1-l)}}{\tau_{c+2}'(\rho_j)} \\
&= \frac{h_{1,\delta}^{*}(0)}{\tau_{c+2}(0)}+\sigma_0\sum_{j=1}^{c+2}\frac{\rho_j^{(c+1)}}{\tau_{c+2}'(\rho_j)}+\sum_{l=1}^{c+1}\sigma_l\sum_{j=1}^{c+2}\frac{\rho_j^{(c+1-l)}}{\tau_{c+2}'(\rho_j)} \\
&+ \sigma_{c+2}\sum_{j=1}^{c+2}\frac{\rho_j^{(-1)}}{\tau_{c+2}'(\rho_j)} \\
&= \frac{h_{1,\delta}^{*}(0)}{\tau_{c+2}(0)}+\sigma_0+\sigma_{c+2}\frac{(-1)^{c+1}}{\prod_{j=1}^{c+2}\rho_j} \\
&= \frac{h_{1,\delta}^{*}(0)}{\tau_{c+2}(0)}+(-1)^{c+2}+\left(\frac{\delta+\lambda}{p}\right)\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)\frac{(-1)^{c+1}}{\prod_{j=1}^{c+2}\rho_j} \\
&= (-1)^{c+2}. \qquad (3.36)
\end{aligned}
$$

Substituting (3.36) into (3.34) yields

$$
h_{1,\delta}^{*}(s)-h_{2,\delta}^{*}(s)=\tau_{c+2}(s)(-1)^{c+2}\left\{1-T_sT_{\rho_1}\cdots T_{\rho_{b+2}}h_{2,\delta}(0)\right\}. \qquad (3.37)
$$

Finally, substituting (3.33) and (3.37) into (3.28) completes the proof. □

In the following proposition, we give a defective renewal equation for the Gerber-Shiu discounted penalty function.

**Proposition 18.** *In the compound Poisson risk model with a dependence structure based on the generalized FGM copula defined in (3.9) with parameters* $a$, $b \geq 1$, $c \in \{2, 3, ...\}$, *and* $d > 1$, *the Gerber-Shiu discounted penalty function* $m_\delta$

*admits a defective renewal equation representation*

$$m_\delta(u) = \int_0^u m_\delta(u-y)\mu_\delta(y)dy + \eta_\delta(u), \tag{3.38}$$

$$= \frac{1}{1+\kappa_\delta}\int_0^u m_\delta(u-y)\vartheta_\delta(y)dy + \frac{1}{1+\kappa_\delta}G_\delta(u), \tag{3.39}$$

*where* $\mu_\delta(y) = T_{\rho_1}\cdots T_{\rho_{c+2}}h_{2,\delta}(u)$, $\eta_\delta(u) = T_{\rho_1}\cdots T_{\rho_{c+2}}\beta_{1,\delta}(u)$, $k_\delta$ *is such that*

$$\frac{1}{(1+\kappa_\delta)} = 1 - \frac{\delta}{p}\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\prod_{j=1}^{c+2}\rho_j} = T_0T_{\rho_1}\cdots T_{\rho_{c+2}}h_{2,\delta}(0) < 1, G_\delta(u) = (1+\kappa_\delta)\eta_\delta(u),$$

*and* $\vartheta_\delta(y) = (1+\kappa_\delta)\mu_\delta(y)$ *is a proper density function.*

**Proof.** The renewal equation (3.38) is obtained by inverting the Laplace transform in (3.32). It remains to be proven that $\int_0^\infty \mu_\delta(y)dy < 1$ for (3.38) to be a defective renewal equation. Due to Property 6 of the Dickson-Hipp operator, we have

$$\begin{aligned}
&\int_0^\infty \mu_\delta(y)dy \\
&= T_0T_{\rho_1}\cdots T_{\rho_{c+2}}h_{2,\delta}(0) \\
&= (-1)^{c+2}\left[\frac{h^*_{2,\delta}(0)}{\tau_{c+2}(0)} + \sum_{j=1}^{c+2}\frac{\left(\frac{\delta+\lambda}{p}-\rho_j\right)\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{\rho_j\tau'_{c+2}(\rho_j)}\right].
\end{aligned} \tag{3.40}$$

Since $\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)$ is a polynomial in $\rho_j$ of degree $(c+1)$, we can write $\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right) = \sum_{l=0}^{c+1}\sigma_l\rho_j^{c+1-l}$ with $\sigma_0 = (-1)^{c+1}$ and $\sigma_{c+1} = \prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)$.

Hence,

$$\int_0^\infty \mu_\delta(y)dy = (-1)^{c+2}\left[\frac{\lambda}{p}\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\tau_{c+2}(0)} + \sum_{j=1}^{c+2}\frac{\left(\frac{\delta+\lambda}{p}-\rho_j\right)\sum_{l=0}^{c+1}\sigma_l\rho_j^{c+1-l}}{\rho_j\tau'_{c+2}(\rho_j)}\right]$$

$$= (-1)^{c+2}\left[\frac{\lambda}{p}\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\tau_{c+2}(0)} + \left(\frac{\delta+\lambda}{p}\right)\sum_{l=0}^{c+1}\sigma_l\sum_{j=1}^{c+2}\frac{\rho_j^{c-l}}{\tau'_{c+2}(\rho_j)}\right]$$

$$- (-1)^{c+2}\sum_{l=0}^{c+1}\sigma_l\sum_{j=1}^{c+2}\frac{\rho_j^{c+1-l}}{\tau'_{c+2}(\rho_j)}. \quad (3.41)$$

Using Lemma 1 of the Dickson-Hipp operator on page 395 of Li and Garrido (2004), the terms summed in (3.41) are equal to zero except for $l = 0$ and $l = c + 1$, meaning

$$\sum_{l=0}^{c+1}\sigma_l\sum_{j=1}^{c+2}\frac{\rho_j^{c-l}}{\tau'_{c+2}(\rho_j)} = -\frac{\sigma_{c+1}}{\tau_{c+2}(0)},$$

and

$$\sum_{l=0}^{c+1}\sigma_l\sum_{j=1}^{c+2}\frac{\rho_j^{c+1-l}}{\tau'_{c+2}(\rho_j)} = \sigma_0.$$

Therefore,

$$\int_0^\infty \mu_\delta(y)dy = (-1)^{c+2}\left[\frac{\lambda}{p}\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\tau_{c+2}(0)} - \left(\frac{\delta+\lambda}{p}\right)\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\tau_{c+2}(0)} - (-1)^{c+1}\right]$$

$$= 1 - \frac{\delta}{p}\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\prod_{j=1}^{c+2}\rho_j} < 1. \quad (3.42)$$

□

## 3.6 Laplace transform of the time of ruin

In the following proposition, we consider a special case of the Gerber-Shiu penalty function with $w(x, y) = 1$ for all $x, y \geq 0$. We show that the Laplace transform of the time of ruin, denoted by $\phi_T(u)$, satisfies a defective renewal equation which also has a compound geometric representation.

**Proposition 19.** *In the compound Poisson risk model with a dependence structure based on the generalized FGM copula defined in (3.9) with parameters a, $b \geq 1$, $c \in \{2, 3, ...\}$, and $d > 1$, the Laplace transform of the time of ruin $\phi_T$ admits a defective renewal equation representation*

$$\phi_T(u) = \frac{1}{1+\kappa_\delta}\int_0^u \phi_T(u-y)\vartheta_\delta(y)dy + \frac{1}{1+\kappa_\delta}\int_u^\infty \vartheta_\delta(y)dy, \qquad (3.43)$$

*which has a compound geometric representation :*
$\phi_T(u) = \frac{\kappa_\delta}{1+\kappa_\delta}\sum_{j=1}^{\infty}\left(\frac{1}{1+\kappa_\delta}\right)^j \bar{V}_\delta^{\star j}(u)$, $u \geq 0$ *where $\bar{V}_\delta^{\star j}(u)$ is the survival distribution of the n-fold convolution of the p.d.f. $\vartheta_\delta$.*

**Proof.** If we let $w(x, y) = 1$ for all $x, y \geq 0$ in (3.39), then the terms $k_\delta$ and $\vartheta_\delta(y)$ remain unchanged and $m_\delta(u)$ becomes $\phi_T(u)$. For $G_\delta(u)$ we must simplify look at $\eta_\delta(u)$ since

$$\begin{aligned} G_\delta(u) &= (1+\kappa_\delta)\eta_\delta(u) \\ &= (1+\kappa_\delta)T_{\rho_1}\cdots T_{\rho_{c+2}}\beta_{1,\delta}(u). \end{aligned}$$

We first obtain the Laplace transform $\eta_\delta^*(u)$ when $w(x, y) = 1$ and then invert it. We find

$$
\begin{aligned}
s\eta_\delta^*(s) &= sT_sT_{\rho_1}\cdots T_{\rho_{c+2}}\beta_{1,\delta}(0) \\
&= s(-1)^{c+2}\left\{\frac{\beta_{1,\delta}^*(s)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2}\frac{\beta_{1,\delta}^*(\rho_j)}{(s-\rho_j)\tau_{c+2}'(\rho_j)}\right\}
\end{aligned}
$$

with

$$
\beta_{1,\delta}^*(s) = \frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-s\right) - h_{2,\delta}^*(s)}{s}, \tag{3.44}
$$

given that $w_1^*(s)$ and $w_2^*(s)$ respectively simplify to $\frac{1-f_X^*(s)}{s}$ and $\frac{1-g_Z^*(s)}{s}$.

Given that $\frac{s}{\rho_j(s-\rho_j)} = \frac{1}{s-\rho_j} + \frac{1}{\rho_j}$, we have

$$
\begin{aligned}
s\eta_\delta^*(s) &= (-1)^{c+2}\left\{\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-s\right)}{\tau_{c+2}(s)} - \frac{h_{2,\delta}^*(s)}{\tau_{c+2}(s)} - s\sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{\rho_j(s-\rho_j)\tau_{c+2}'(\rho_j)} + s\sum_{j=1}^{c+2}\frac{h_{2,\delta}^*(\rho_j)}{\rho_j(s-\rho_j)\tau_{c+2}'(\rho_j)}\right\} \\
&= (-1)^{c+2}\left\{\begin{array}{l}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-s\right)}{\tau_{c+2}(s)} - \frac{h_{2,\delta}^*(s)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{(s-\rho_j)\tau_{c+2}'(\rho_j)} - \sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{\rho_j\tau_{c+2}'(\rho_j)} \\ \qquad + \sum_{j=1}^{c+2}\frac{h_{2,\delta}^*(\rho_j)}{(s-\rho_j)\tau_{c+2}'(\rho_j)} + \sum_{j=1}^{c+2}\frac{h_{2,\delta}^*(\rho_j)}{\rho_j\tau_{c+2}'(\rho_j)}\end{array}\right\} \\
&= (-1)^{c+2}\left\{\begin{array}{l}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-s\right)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{(s-\rho_j)\tau_{c+2}'(\rho_j)} - \sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{\rho_j\tau_{c+2}'(\rho_j)} \\ \qquad - \left(\frac{h_{2,\delta}^*(s)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2}\frac{h_{2,\delta}^*(\rho_j)}{(s-\rho_j)\tau_{c+2}'(\rho_j)}\right) + \sum_{j=1}^{c+2}\frac{h_{2,\delta}^*(\rho_j)}{\rho_j\tau_{c+2}'(\rho_j)}\end{array}\right\} \\
&= (-1)^{c+2}\left\{\begin{array}{l}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-s\right)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{(s-\rho_j)\tau_{c+2}'(\rho_j)} \\ -\sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}-\rho_j\right)}{\rho_j\tau_{c+2}'(\rho_j)} + \sum_{j=1}^{c+2}\frac{h_{2,\delta}^*(\rho_j)}{\rho_j\tau_{c+2}'(\rho_j)}\end{array}\right\} - T_sT_{\rho_1}...T_{\rho_{c+2}}h_{2,\delta}(0).
\end{aligned}
$$

Due to Property 6 of the Dickson-Hipp operator at $s = 0$, we can write

$$s\eta_\delta^*(s) = (-1)^{c+2}\left\{\begin{array}{l} \dfrac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - s\right)}{\tau_{c+2}(s)} - \sum_{j=1}^{c+2}\dfrac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - \rho_j\right)}{(s-\rho_j)\tau'_{c+2}(\rho_j)} \\ -\sum_{j=1}^{c+2}\dfrac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - \rho_j\right)}{\rho_j\tau'_{c+2}(\rho_j)} + (-1)^{c+2}T_0T_{\rho_1}\ldots T_{\rho_{c+2}}h_{2,\delta}(0) \\ -\frac{h_{2,\delta}^*(0)}{\tau_{c+2}(0)} \end{array}\right\}$$
$$-T_sT_{\rho_1}\ldots T_{\rho_{c+2}}h_{2,\delta}(0), \tag{3.45}$$

where

$$h_{2,\delta}^*(0) = \frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right). \tag{3.46}$$

Since

$$\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - \rho_j\right) = \sum_{l=0}^{c+1}\sigma_l\rho_j^{c+1-l} = \sum_{i=0}^{c+1}\sigma'_i(s-\rho_j)^{c+1-i},$$

where

$$\sigma_0 = (-1)^{c+1}, \cdots, \sigma_{c+1} = \prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right), \sigma'_0 = 1, \cdots, \sigma'_{c+1} = \prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - s\right)$$

and based on Lemma 1 of Li and Garrido (2004), the second and third term summed in (3.45) simplify to

$$\sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - \rho_j\right)}{(s-\rho_j)\tau'_{c+2}(\rho_j)} + \sum_{j=1}^{c+2}\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - \rho_j\right)}{\rho_j\tau'_{c+2}(\rho_j)} = -\frac{\sigma_{c+1}}{\tau_{c+2}(0)} + \frac{\sigma'_{c+1}}{\tau_{c+2}(s)}$$
$$= -\frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p}\right)}{\tau_{c+2}(0)} + \frac{\frac{\lambda}{p}\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda_i}{p} - s\right)}{\tau_{c+2}(s)}. \tag{3.47}$$

Substituting (3.47) and (3.46) in (3.45) and with simple rearrangements, we get

$$\begin{aligned} s\eta_\delta^*(s) &= T_0T_{\rho_1}...T_{\rho_{c+2}}h_{2,\delta}(0) - T_sT_{\rho_1}...T_{\rho_{c+2}}h_{2,\delta}(0) \\ &= \frac{1}{1+\kappa_\delta} - \mu_\delta^*(s). \end{aligned}$$

Hence, we obtain for the Laplace transform $G_\delta^*(s)$ when $w(x,y) = 1$ for all $x, y \geq 0$

$$\begin{aligned} G_\delta^*(s) &= \frac{1}{s}(1+\kappa_\delta)\eta_\delta^*(s) \\ &= \frac{1}{s}(1+\kappa_\delta)(\frac{1}{1+\kappa_\delta} - \mu_\delta^*(s)) \\ &= \frac{1}{s}(1-\vartheta_\delta^*(s)). \end{aligned}$$

Recall that for a general function $w(x,y)$ we have due to (3.32)

$$m_\delta^*(s) = \frac{G_\delta^*(s)}{(1+\kappa_\delta) - \vartheta_\delta^*(s)}$$

which becomes for $w(x,y) = 1$ for all $x, y \geq 0$

$$\phi_T^*(s) = \frac{\frac{1-\vartheta_\delta^*(s)}{s}}{(1+\kappa_\delta) - \vartheta_\delta^*(s)}. \tag{3.48}$$

An inversion of the Laplace transform in (3.48) leads to the desired result.□

We give the expression for the Laplace transform of the time of ruin when the initial surplus is zero in the next corollary.

**Corollary 20.** *We can write $\phi_T(0)$ in terms of $\lambda, \delta, p, \rho_j, c$ and $d$ as*

$$\phi_T(0) = 1 - \frac{\delta}{p}\frac{\prod_{i=1}^{c+1}\left(\frac{\delta+\lambda(d+i-1)}{p}\right)}{\prod_{j=1}^{c+2}\rho_j}.$$

**Proof.** This follows from (3.42) and (3.43). □

## 3.7 Exponential claim amounts

In this section, we assume that the individual claim amount follows an exponential distribution with p.d.f. $f_X(x) = \gamma e^{-\gamma x}$ and Laplace transform $f_X^*(s) = \frac{\gamma}{\gamma+s}$. It implies that $(X, W)$ has a bivariate exponential distribution based on the generalized FGM copula (3.9) with a covariance given by

$$\begin{aligned} Cov(X,W) &= \int_0^1 \int_0^1 \{C(u,v) - uv\}\, dF_X^{-1}(u) dF_W^{-1}(v) \\ &= \frac{\theta}{\gamma\lambda} B(a+1,b)B(c+1,d), \end{aligned} \tag{3.49}$$

where $B(x,y)$ is the beta function $\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$. In the following proposition, we find the explicit expression for the Laplace transform $\phi_T$ of the time of ruin in this special case.

**Proposition 21.** *Assuming the roots $\{-R_j(\delta), j = 1, a+2\}$ distinct, a closed-form expression for $\{\phi_T(u), u \geq 0\}$ is given by*

$$\phi_T(u) = \sum_{j=1}^{a+2} \varsigma_j e^{-R_j u}, \tag{3.50}$$

*where*

$$\varsigma_j = \frac{(\xi_{1,\delta}(-R_j) - \xi_{2,\delta}(-R_j))}{(\ell_{1,\delta}(0) - \xi_{2,\delta}(0))} \prod_{i=1, i\neq j}^{a+2} \frac{R_j}{-R_i + R_j} \prod_{j=1}^{c+2} \Big( \frac{-\rho_j}{-R_i + \rho_j} \Big),$$

*and $a$ is an integer. The functions $\xi_{1,\delta}(s)$, $\xi_{2,\delta}(s)$, and $\ell_{1,\delta}(s)$ are defined in (3.52), (3.53), and (3.54) respectively.*

**Proof.** When claim amounts are exponentially distributed, the Laplace transform of $g_Z(x)$, defined in (3.15), becomes

$$g_Z^\star(s) = \frac{\gamma}{s+\gamma} - \sum_{i=1}^{a+1} \frac{b_i}{s+\gamma_i},$$

which leads to

$$g_Z(x) = \gamma e^{-\gamma x} - \sum_{i=1}^{a+1} b_i e^{-\gamma_i x}, \qquad x \geq 0$$

with

$$\begin{aligned} \gamma_i &= \gamma(b+i-1) \\ b_i &= \frac{a!\gamma^a(-\gamma_i)}{\prod_{j=1, j\neq i}^{a+1} (-\gamma_i + \gamma_j)}. \end{aligned}$$

We must first modify the term $\beta^*_{1,\delta}(s)$ in (3.44). After inserting the result obtained for $\beta^*_{1,\delta}(s)$ into (3.28), and multiplying both the numerator and denominator by

$$(\gamma + s) \prod_{j=1}^{a+1} (s+\gamma_i)$$

we have

$$s\phi^*_T(s) = \frac{\xi_{1,\delta}(s) - \xi_{2,\delta}(s)}{\ell_{1,\delta}(s) - \xi_{2,\delta}(s)}, \tag{3.51}$$

where

$$\xi_{1,\delta}(s) = \frac{\lambda}{p}(\gamma+s) \prod_{i=1}^{a+1}(s+\gamma_i) \prod_{i=1}^{c+1} \Big(\frac{\delta+\lambda_i}{p} - s\Big) + s(\gamma+s)\beta^*_{2,\delta}(s) \prod_{i=1}^{a+1}(s+\gamma_i), \tag{3.52}$$

$$\xi_{2,\delta}(s) = \frac{\gamma\lambda}{p} \prod_{i=1}^{c+1} \Big(\frac{\delta+\lambda_i}{p} - s\Big) \prod_{i=1}^{a+1}(s+\gamma_i) + \Big(\theta c!\lambda^c a!\gamma^a \frac{1}{p^{c+1}} s(\frac{\delta+\lambda}{p} - s)(\delta - sp)(s+\gamma)\Big), \tag{3.53}$$

and

$$\ell_{1,\delta}(s) = (\gamma + s)\Big(\frac{\delta+\lambda}{p} - s\Big)\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p} - s\Big)\prod_{i=1}^{a+1}(s+\gamma_i). \tag{3.54}$$

We have that $\ell_{1,\delta}(s) - \xi_{2,\delta}(s)$ is a polynomial of degree $a+c+4$. It has $a+c+4$ roots, $\rho_j$, with $\Re(\rho_j) > 0$, for $j = 1, \cdots, c+2$ and $-R_j$, with $\Re(R_j) > 0$, for $j = 1, \cdots, a+2$.

Using the Lagrange interpolating polynomial on the denominator and the numerator in (3.51), one finds

$$\xi_{1,\delta}(s) - \xi_{2,\delta}(s) = \sum_{j=1}^{a+2}(\xi_{1,\delta}(-R_j) - \xi_{2,\delta}(-R_j))\frac{s}{-R_j}\prod_{k=1}^{c+2}\Big(\frac{s-\rho_k}{-R_j-\rho_k}\Big)\prod_{k=1,k\neq j}^{a+2}\Big(\frac{s+R_k}{-R_j+R_k}\Big) \tag{3.55}$$

and

$$\ell_{1,\delta}(s) - \xi_{2,\delta}(s) = (\ell_{1,\delta}(0) - \xi_{2,\delta}(0))\prod_{j=1}^{c+2}\Big(\frac{s-\rho_j}{-\rho_j}\Big)\prod_{j=1}^{a+2}\Big(\frac{s+R_j}{R_j}\Big). \tag{3.56}$$

Combining (3.55) and (3.56) to (3.51), one concludes

$$s\phi_T^*(s) = \sum_{j=1}^{a+2}\frac{\gamma_j}{s+R_j}. \tag{3.57}$$

The inversion of the Laplace transform in (3.57) leads to (3.50).□

We illustrate the above results in the next example.

**Example 22.** *For the numerical results, we assume that $X \sim Exp(1)$, $W \sim Exp(1)$, and $a = b = c = d = 2$. The premium rate is $p = 1.5$, which implies a relative risk margin of 50%. In Table 1, we first provide the analytic expressions of the ruin probability $\psi(u)$ for different dependence parameters :*

$\theta = -20, -5, 0, 5, 20$ *(derived with Maple).*

| $\theta$ | *Expressions for the ruin probability* $\psi(u)$ |
|---|---|
| *-20* | $0.712e^{-0.296u} + 0.020e^{-1.598u} - 0.038e^{-3.798u}cos(0.985u) - 0.057e^{-3.798u}sin(0.985u)$ |
| *-5* | $0.677e^{-0.323u} + 0.012e^{-1.835u} - 0.017e^{-3.607u}cos(0.369u) - 0.065e^{-3.607u}sin(0.369u)$ |
| *0* | $\frac{2}{3}e^{-\frac{1}{3}u}$ |
| *5* | $0.652e^{-0.344u} - 0.012e^{-4.230u} + 0.017e^{-2.358u}cos(0.284u) - 0.096e^{-2.358u}sin(0.284u)$ |
| *20* | $0.603e^{-0.379u} - 0.025e^{-4.609u} + 0.048e^{-2.090u}cos(0.872u) - 0.109e^{-2.090u}sin(0.872u)$ |

*Table 1. Analytic expressions of* $\psi(u)$ *for* $\theta$ *equal to -20, -5, 0, 5, and 20.*

*The resulting ruin probabilites are depicted in Figure 3.6*

*We also derive the analytic expressions for the Laplace transform of the time of ruin* $\phi_T(u)$ *with* $\delta = 0.05$ *for* $\theta = -20, -5, 0, 5, 20$ *(derived with Maple) which are given in Table 2.*

| $\theta$ | *Expressions for the Laplace transform* $\phi_T(u)$ |
|---|---|
| *-20* | $0.663e^{-0.348u} + 0.024e^{-1.598u} - 0.045e^{-3.795u}cos(0.981u) - 0.068e^{-3.795u}sin(0.981u)$ |
| *-5* | $0.627e^{-0.376u} + 0.014e^{-1.836u} - 0.020e^{-3.606u}cos(0.365u) - 0.077e^{-3.606u}sin(0.365u)$ |
| *0* | $0.613e^{-0.386u}$ |
| *5* | $0.599e^{-0.396u} - 0.013e^{-4.230u} + 0.019e^{-2.359u}cos(0.280u) - 0.114e^{-2.359u}sin(0.280u)$ |
| *20* | $0.549e^{-0.430u} - 0.029e^{-4.608u} + 0.054e^{-2.094u}cos(0.868u) - 0.127e^{-2.094u}sin(0.868u)$ |

*Table 2. Analytic expressions of* $\phi_T(u)$ *for* $\theta$ *equal to -20, -5, 0, 5, and 20.*

*The resulting* $\phi_T(u)$ *are depicted in Figure 3.7. The impact of the dependence structure on* $\phi_T(u)$ *can be clearly observed.*

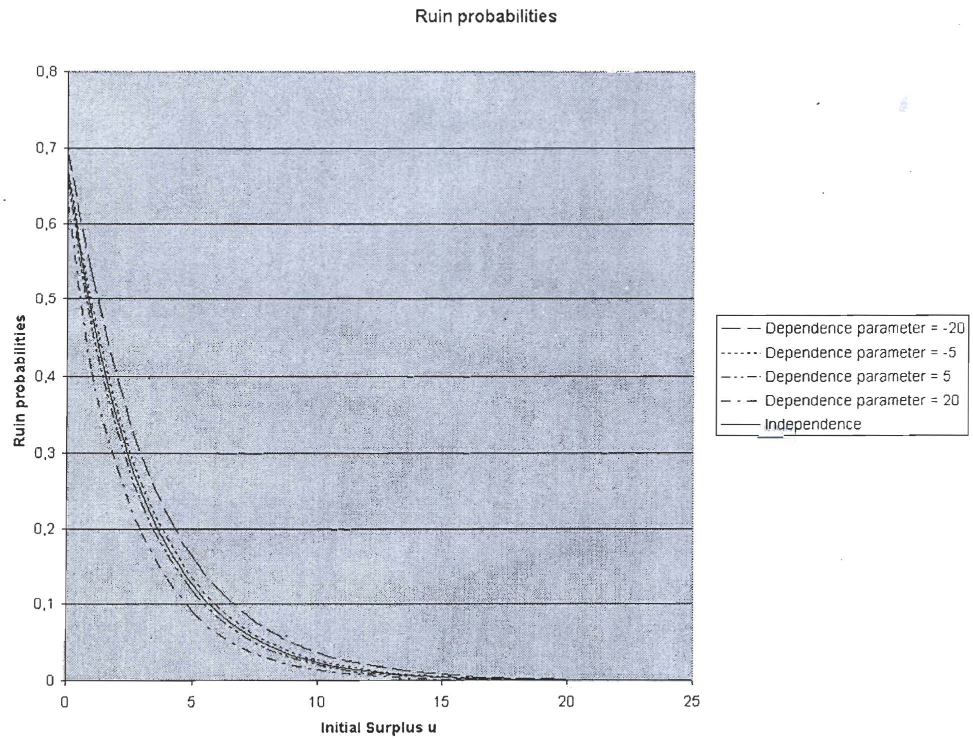


FIG. 3.6 – Ruin probabilities for $\theta$ (dependence parameter) equal to -20, -5, 0 (independence), 5, and 20.

## 3.8 Acknowledgements

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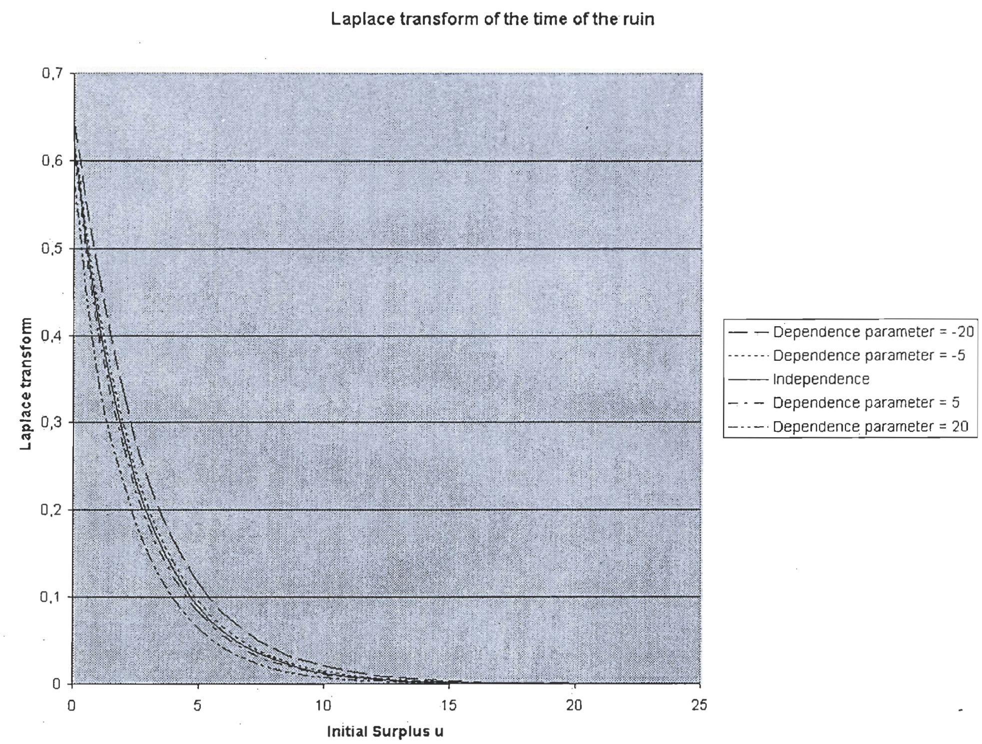


FIG. 3.7 – Values of $\phi_T(u)$ for $\theta$ (dependence parameter) equal to -20, -5, 0 (independence), 5, and 20.

## 3.9 References


- Albrecher, H. and J. Teugels (2006). Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability* 43 (1), 265-285.
- Boudreault, M., Cossette, H., Landriault, D. and E. Marceau (2006). On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, 301–323.
- Bouye, E., Nikeghbali, A., Riboulet, G. and T. Roncalli (2000). Copulas for Finance. A reading guide and some applications. Rapport technique du Groupe de recherche opérationnelle, Crédit Lyonnais.
- Denuit, M., Dhaene, J., Goovaerts, M.J., and R. Kaas (2005). *Actuarial Theory for Dependent Risks – Measures, Orders and Models.* Wiley. New York.
- Dickson, D.C.M. and C. Hipp (2001). On the time to ruin for Erlang(2) risk process, *Insurance : Mathematics and Economics* 29, 333–344.
- Drouet Mari, D., and S. Kotz (2001). *Correlation and Dependence.* Imperial College Press.
- Frees, E. W. and E. A. Valdez (1998). Understanding relationships using copulas. *North American Actuarial Journal* 2, 1-25.
- Gerber, H.U. and E.S.W Shiu (1998). On the time value of ruin. *North American Actuarial Journal 2*, 48-78.
- Klimenok, V. (2001). On the modification of Rouche's theorem for the queuing theory problems. *Queuing Systems* 38, 431-434.

- Li, S. and J. Garrido (2004). On ruin for the Erlang(n) risk process. *Insurance : Mathematics and Economics* 34, 391-408.
- McNeil, A., Frey, R., and P. Embrechts. (2005). *Quantitative Risk Management.* Princeton Press, Princeton.
- Rodríguez-Lallena, J.A., and M. Ubena-Flores (2004). A new class of bivariate copulas. *Statistics and Probability Letters* 66, 315- 325.
- Wang, S. (1998). Aggregation of correlated risk portfolios : Models and Algorithms. CAS Proceedings, 848-939.

# CHAPITRE IV

## On a compound Poisson risk model with dependence and in the presence of a constant dividend barrier

### Résumé

Nous considérons, dans ce papier, le processus classique du surplus en présence d'une barrière de dividendes constante. Une structure de dépendance entre les montants des sinistres et les temps séparant deux sinistres est introduite par la copule classique de Farlie-Gumbel-Morgenstern. Une équation intégro-différentielle avec des bornes aux frontières est obtenue. Sa solution est exprimée comme la somme de la fonction de Gerber-Shiu en l'absence de la barrière de dividendes plus une combinaison linéaire de deux solutions particulières indépendantes. Pour terminer, nous trouvons une solution explicite quand les montants des sinistres sont exponentiellement distribués et nous étudions les effets de la dépendance sur les mesures de ruine.

# Abstract

In this paper, we consider a classical risk process with dependence and in the presence of a constant dividend barrier. The dependence structure between the claim amounts and the inter-claim times is introduced through a Farlie-Gumbel-Morgenstern copula. We analyze the expectation of the discounted penalty function and the expectation of the present value of the distributed dividends. For each function, an integro-differential equation with boundary conditions is derived and the solution is provided. Finally, we find an explicit solution for each function when the claim amounts are exponentially distributed. We illustrate the impact of the dependence on these two quantities.

## 4.1 Introduction

In this paper, we consider a compound Poisson risk model with dependence for an insurance portfolio. As in the classical Poisson risk model, the evolution of the claim number for an insurance portfolio is assumed to follow a Poisson process $\underline{N} = \{N(t), \ t \in \mathbb{R}^+\}$. The interclaim times $\{W_j, j \in \mathbb{N}^+\}$ form a sequence of independent random variables (r.v.) identically distributed as the canonical r.v. $W$. The r.v. $W$ has an exponential distribution with expectation $\frac{1}{\lambda}$ with probability density function (p.d.f.) $f_W$, cumulative distribution function (c.d.f.) $F_W$ and Laplace transform (L.T.) $f_W^*$ with

$$f_W(t) = \lambda e^{-\lambda t}, \tag{4.1}$$

$$F_W(t) = 1 - e^{-\lambda t}, \tag{4.2}$$

$$f_W^*(s) = E\left[e^{-sW}\right] = \frac{\lambda}{\lambda + s}.$$

The r.v. $X_j$ $(j = 1, 2, ...)$ corresponds to the amount of the $j$th claim. The claim amounts $\{X_j, j \in \mathbb{N}^+\}$ form a sequence of i.i.d. r.v.'s distributed as the r.v. $X$ with p.d.f. $f_X$, c.d.f. $F_X$ and L.T. $f_X^*$.

The classic compound Poisson risk model relies on the assumption that, on the occurrence of the $j$th claim, the claim amount $X_j$ and the interclaim time $W_j$ are independent (see e.g. Gerber (1979), Grandell (1991) and Rolski et al. (1999)). In the compound Poisson risk model with dependence, this assumption is relaxed and we assume that $\{(X_j, W_j), j \in \mathbb{N}^+\}$ form a sequence of i.i.d. random vectors distributed as the canonical random vector $(X, W)$, in which the components may be dependent. The joint p.d.f. of $(X, W)$ is denoted by

$f_{X,W}(x,t)$ with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$. The associated L.T. is given by

$$f^*_{X,W}(s_1, s_2) = E\left[e^{-s_1 X} e^{-s_2 W}\right] = \int_0^\infty \int_0^\infty e^{-s_1 x} e^{-s_2 t} f_{X,W}(x,t)\, dx dt.$$

Different models can be assumed for the joint distribution of $(X, W)$. In this paper, the joint distribution for $(X, W)$ is defined as in Cossette et al. (2008) with the Farlie-Gumbel-Morgenstern (FGM) copula. For other types of dependence structure for $(X, W)$, the reader can consult Albrecher and Boxma (2004) and Boudreault et al. (2006). Albrecher and Teugels (2006) consider a dependence structure for $(X, W)$ based on a copula. They derive exponential estimates for finite and infinite time ruin probabilities in the case of light-tailed claim sizes by employing the underlying random walk structure of the risk model. Note that Landriault (2008) studies the model proposed by Boudreault et al. (2006) in the presence of a constant dividend barrier.

The FGM copula is given by

$$C_\theta^{FGM}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2),$$

for $(u_1, u_2) \in [0,1] \times [0,1]$. The dependence parameter $\theta$ takes value in $[-1, 1]$, where $\theta > 0$ $(< 0)$ corresponds to a positive (negative) dependence relation. The joint p.d.f. associated to the FGM copula is given by

$$c_\theta^{FGM}(u_1, u_2) = 1 + \theta (1 - 2u_1)(1 - 2u_2).$$

The joint c.d.f. $F_{X,W}$ of $(X, W)$ with marginals $F_X$ and $F_W$ and defined with the FGM copula is given by

$$F_{X,W}(x,t) = C_\theta^{FGM}(F_X(x), F_W(t)) = F_X(x) F_W(t) + \theta F_X(x) F_W(t) (1 - F_X(x))(1 - F_W(t)),$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$. The joint p.d.f. of $(X,W)$ is

$$\begin{aligned} f_{X,W}(x,t) &= c_\theta^{FGM}(F_X(x), F_W(t)) f_X(x) f_W(t) \\ &= f_X(x) f_W(t) + \theta f_X(x) f_W(t)(1-2F_X(x))(1-2F_W(t)). \end{aligned} \tag{4.3}$$

Since $W$ has an exponential distribution, we replace (4.1) and (4.2) in (4.3) which becomes

$$f_{X,W}(x,t) = \lambda e^{-\lambda t} f_X(x) + \theta h_X(x)(2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t}), \quad -1 \leq \theta \leq 1, \tag{4.4}$$

where

$$h_X(x) = (1-2F_X(x))f_X(x). \tag{4.5}$$

For a general survey on the theory of copulas, the reader may consult e.g. Nelsen (2006). A review of applications in actuarial science and risk management can be found in e.g. Denuit et al. (2005) and McNeil et al. (2005) respectively. Using copulas provides a more flexible approach to the modelling procedure. While there are a large number of copula families, we choose the FGM copula because it offers the advantage to be mathematically tractable, as illustrated in Cossette et al. (2008), and has a simple analytical form. It is also known that the FGM copula is a Taylor approximation of order one to the Frank copula and the Ali-Milkhail-Haq copula (see Nelsen (2006)).

The aggregate claim amount process is denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ ($\sum_a^b$ equals 0 if $b < a$). In the absence of a dividend barrier, the dynamic of the surplus process $\underline{U} = \{U(t), t \geq 0\}$ for the insurance portfolio is defined by

$$U(t) = u + pt - S(t), \tag{4.6}$$

where $u$ is the initial surplus and the premium rate is $p$. The premium rate $p$ satisfies the solvency condition $E\left[(X - pW)\right] < 0$ which implies that $p = (1+\eta)\frac{E[X]}{E[W]}$, where $\eta > 0$ is the relative security margin.

In this paper, we consider a barrier strategy which assumes a constant barrier at level $\tilde{b} \geq u$. The introduction of a constant barrier $\tilde{b}$ leads to a modification of the surplus process defined in (4.6). When the surplus reaches the barrier $\tilde{b}$, dividends are paid continuously at a constant rate $p$, which means that the surplus does not exceed the barrier and it remains at the barrier level until the next claim occurs. We denote by $\underline{U_{\tilde{b}}} = \left\{U_{\tilde{b}}\left(t\right),\ t \geq 0\right\}$, the surplus process in the presence of a constant dividend barrier $\tilde{b}$ with $U_{\tilde{b}}(0) = u$ being the initial surplus and with the following dynamic

$$dU_{\tilde{b}}(t) = \begin{cases} pdt - dS(t), & U_{\tilde{b}}(t) < \tilde{b} \\ -dS(t), & U_{\tilde{b}}(t) = \tilde{b}. \end{cases}$$

The time of ruin associated to the modified surplus process is defined by the r.v. $T_{\tilde{b}} = \inf\left\{t > 0, U_{\tilde{b}}(t) < 0\right\}$, which is the first time the surplus becomes negative. When $\tilde{b} = \infty$, $T_{\infty} = T$ corresponds to the time of ruin in the dependent risk model without a dividend barrier. The expected value of the discounted penalty function with a constant dividend barrier $\tilde{b}$ is defined as

$$m_{\delta,\tilde{b}}(u) = E\left[e^{-\delta T_{\tilde{b}}} w\left(U(T_{\tilde{b}}^{-}), |U(T_{\tilde{b}})|\right) I(T_{\tilde{b}} < \infty) \middle| U(0) = u\right], \quad 0 \leq u \leq \tilde{b},$$

where $w(x, y)$, for $x, y \geq 0$, is the penalty function at the time of ruin for the surplus prior to ruin and the deficit at ruin, $\delta$ is a non negative parameter and $I$ is the indicator function such that $I(A) = 1$ if the event $A$ occurs and equals 0 otherwise. In particular, if $w(x, y) = 1$, $m_{\delta,\tilde{b}}(u)$ corresponds to the Laplace transform of the time of ruin $T_{\tilde{b}}$ with respect to $\delta$, which we denote by

$$\phi_{T_{\tilde{b}}}(u) = E\left[e^{-\delta T_{\tilde{b}}} I(T_{\tilde{b}} < \infty) \middle| U_{\tilde{b}}(0) = u\right], \quad 0 \leq u \leq \tilde{b}.$$

We mention that $m_{\delta,\infty}(u)$ is the expected value of the discounted penalty function introduced by Gerber and Shiu (1998) when there is no barrier. An analysis of $m_{\delta,\infty}(u)$ in the present risk model is carried out in Cossette et al. (2008). Note that, in the presence of a constant dividend barrier, the time of ruin $T_{\tilde{b}}$ is a finite-valued r.v. This implies that it is not necessary to examine the ruin probability which is certain in this case.

Another quantity of interest in the assessment of the quality of a dividend barrier strategy is $V_{\delta,\tilde{b}}(u)$, which denotes the expectation of the present value of all dividends distributed until ruin, given an initial surplus $u$, for $0 \leq u \leq \tilde{b}$.

In the actuarial literature, the barrier strategy has been the subject of several research works since it has been initially considered by De Finetti (1957). We can find a review of the most important works in this field in Lin et al. (2003). Among them, we mention Bühlmann (1970) and Gerber (1979) who consider the problem of optimal dividend strategy in the classical compound Poisson risk model. Recently, Lin et al. (2003) examine the derivation of $m_{\delta,\tilde{b}}(u)$, analyze several of its properties and discuss its computation for special cases of the classical risk model. Li and Garrido (2004b) find the expression for $m_{\delta,\tilde{b}}(u)$ for a class of compound renewal risk models. Albrecher et al. (2005) present some results on the distribution of dividend payments until ruin under a compound renewal risk model with generalized Erlang($n$)-distributed inter-claim times and a constant dividend barrier. Gerber and Shiu (2006) examine optimal dividend strategies in the compound Poisson risk model. Landriault (2008) derives the expressions for $m_{\delta,\tilde{b}}(u)$ and $V_{\delta,\tilde{b}}(u)$ in the context of the compound Poisson risk model with dependence proposed by Boudreault et al. (2006).

In this paper, we mainly focus on finding some results related to $m_{\delta,\tilde{b}}(u)$ and $V_{\delta,\tilde{b}}(u)$ in the context of the compound Poisson risk model with dependence described above. It will be interesting to see how these quantities behave in the presence of positive and negative dependence.

This paper is organized as follows. In Section 4.2, an integro-differential equation with boundary conditions and its general solution are obtained for the expected discounted penalty function. In Section 4.3, an integro-differential equation with boundary conditions and its general solution are derived for the expectation of the discounted dividend payments. Explicit solutions for both functions are developed when the claim amount follows an exponential distribution in Section 4.4. This last section is completed with a numerical example to illustrate the effect of the dependence on both functions.

## 4.2 Expected discounted penalty function

In this section, we begin our analysis of $m_{\delta,\tilde{b}}(u)$ with the derivation of an integro-differential equation for $m_{\delta,\tilde{b}}(u)$ $(0 \leq u \leq \tilde{b})$ which we later solve to obtain its solution.

### 4.2.1 An integro-differential equation

To determine the integro-differential equation for $m_{\delta,\tilde{b}}(u)$ in our context, we need to go through several steps as explained in Lin et al. (2003). We first use a renewal argument to establish an integral equation for $m_{\delta,\tilde{b}}(u)$. The first claim occurs at time $t$ and its amount is $x$. Heuristically speaking, the "joint probability" that the first claim occurs at $t$ and that its amount is $x$ is given by $f_{X,W}(x,t)\,dxdt$. Conditioning on the time and the amount of the first claim, there are 4 different possibilities :

- $t \leq \frac{\tilde{b}-u}{p}$ and $x \leq u+pt$ : the first claim occurs before the surplus reaches the barrier $\tilde{b}$ and the claim amount is such that the surplus is still positive with a value of $u+pt-x$ ;
- $t > \frac{\tilde{b}-u}{p}$ and $x \leq \tilde{b}$ : the first claim occurs after the surplus reaches the barrier $\tilde{b}$ and the claim amount is such that the surplus is still positive with a value of $\tilde{b}-x$ ;
- $t \leq \frac{\tilde{b}-u}{p}$ and $x > u+pt$ : the first claim occurs before the surplus reaches the barrier $\tilde{b}$ and the claim amount is such that the surplus becomes negative which means that ruin occurs, the surplus before ruin is $u+pt$, and the deficit at ruin is $x-u-pt$ ; and,
- $t > \frac{\tilde{b}-u}{p}$ and $x > \tilde{b}$ : the first claim occurs after the surplus reaches the barrier $\tilde{b}$ and the claim amount is such that the surplus becomes negative which also implies that ruin occurs, the surplus before ruin is $\tilde{b}$, and the deficit at ruin is $x-\tilde{b}$.

Considering these 4 cases, we have

$$\begin{aligned}
m_{\delta,\tilde{b}}(u) &= \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} m_{\delta,\tilde{b}}(u+pt-x) f_{X,W}(x,t)\, dxdt \\
&+ \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} m_{\delta,\tilde{b}}(\tilde{b}-x) f_{X,W}(x,t)\, dxdt \\
&+ \int_0^{\frac{\tilde{b}-u}{p}} \int_{u+pt}^{\infty} e^{-\delta t} w(u+pt, x-u-pt) f_{X,W}(x,t)\, dxdt \\
&+ \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta t} w(\tilde{b}, x-\tilde{b}) f_{X,W}(x,t)\, dxdt. \qquad (4.7)
\end{aligned}$$

We replace the joint pdf $f_{X,W}$ for $(X, W)$ in (4.7) by (4.4) and we obtain

$$\begin{aligned}
m_{\delta,\tilde{b}}(u) &= \lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} m_{\delta,\tilde{b}}(u+pt-x) f_X(x) e^{-\lambda t} dxdt \\
&+ \lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} m_{\delta,\tilde{b}}(\tilde{b}-x) f_X(x) e^{-\lambda t} dxdt \\
&+ \lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_{u+pt}^{\infty} e^{-\delta t} w(u+pt, x-u-pt) f_X(x) e^{-\lambda t} dxdt \\
&+ \lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta t} w(\tilde{b}, x-\tilde{b}) f_X(x) e^{-\lambda t} dxdt \\
&+ \theta\lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} m_{\delta,\tilde{b}}(u+pt-x) h_X(x) (2e^{-2\lambda t} - e^{-\lambda t}) dxdt \\
&+ \theta\lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} m_{\delta,\tilde{b}}(\tilde{b}-x) h_X(x) (2e^{-2\lambda t} - e^{-\lambda t}) dxdt \\
&+ \theta\lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_{u+pt}^{\infty} e^{-\delta t} w(u+pt, x-u-pt) h_X(x) (2e^{-2\lambda t} - e^{-\lambda t}) dxdt \\
&+ \theta\lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta t} w(\tilde{b}, x-\tilde{b}) h_X(x) (2e^{-2\lambda t} - e^{-\lambda t}) dxdt. \qquad (4.8)
\end{aligned}$$

To simplify the presentation, we introduce the following functions :

$$\sigma_{1,\delta,\tilde{b}}(u) = \int_0^u m_{\delta,\tilde{b}}(u-x)f_X(x)dx + w_1(u), \tag{4.9}$$

$$\sigma_{2,\delta,\tilde{b}}(u) = \int_0^u m_{\delta,\tilde{b}}(u-x)h_X(x)dx + w_2(u), \tag{4.10}$$

$$w_1(u) = \int_u^\infty w(u, x-u)f_X(x)dx,$$

$$w_2(u) = \int_u^\infty w(u, x-u)h_X(x)dx.$$

Using (4.9) and (4.10), the integral equation in (4.8) becomes

$$\begin{aligned} m_{\delta,\tilde{b}}(u) &= \lambda \int_0^{\frac{\tilde{b}-u}{p}} e^{-\delta t}\sigma_{1,\delta,\tilde{b}}(u+pt)e^{-\lambda t}dt + \lambda \int_{\frac{\tilde{b}-u}{p}}^\infty e^{-\delta t}\sigma_{1,\delta,\tilde{b}}(\tilde{b})e^{-\lambda t}dt \\ &+\theta\lambda \int_0^{\frac{\tilde{b}-u}{p}} e^{-\delta t}\sigma_{2,\delta,\tilde{b}}(u+pt)(2e^{-2\lambda t} - e^{-\lambda t})dt \\ &+\theta\lambda \int_{\frac{\tilde{b}-u}{p}}^\infty e^{-\delta t}\sigma_{2,\delta,\tilde{b}}(\tilde{b})(2e^{-2\lambda t} - e^{-\lambda t})dt. \end{aligned} \tag{4.11}$$

Substituting $u + pt = s$ in (4.11) yields

$$\begin{aligned} m_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\int_u^{\tilde{b}} e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{1,\delta,\tilde{b}}(s)ds + \frac{\lambda}{p}\int_{\tilde{b}}^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{1,\delta,\tilde{b}}(\tilde{b})ds \\ &-\frac{\theta}{p}\lambda \int_u^{\tilde{b}} e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s)ds + 2\frac{\theta}{p}\lambda \int_u^{\tilde{b}} e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s)ds \\ &-\frac{\theta}{p}\lambda \int_{\tilde{b}}^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(\tilde{b})ds + 2\frac{\theta}{p}\lambda \int_{\tilde{b}}^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(\tilde{b})ds, \end{aligned}$$

which can also be written as

$$\begin{aligned} m_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{1,\delta,\tilde{b}}(s\wedge\tilde{b})ds - \frac{\theta\lambda}{p}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s\wedge\tilde{b})ds \\ &+2\frac{\theta\lambda}{p}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s\wedge\tilde{b})ds, \end{aligned} \tag{4.12}$$

where $s \wedge \tilde{b} = \min(s, \tilde{b})$.

We are now in a position to derive an integro-differential equation for $m_{\delta,\tilde{b}}(u)$. We denote the differentiation and the identity operators by $\mathcal{D}$ and $\mathcal{I}$ where $\mathcal{D}^k f(u) = \frac{d^k}{du^k} f(u)$ $(k = 1, 2, ...)$ and $\mathcal{I} f(u) = f(u)$.

**Proposition 23.** *For $-1 \leq \theta \leq 1$, the expected discounted penalty function with a constant dividend barrier $\tilde{b}$, $m_{\delta,\tilde{b}}(u)$, satisfies the following integro-differential equation for $0 \leq u \leq \tilde{b}$,*

$$\begin{aligned}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_{1,\delta,\tilde{b}}(u) \\ &+ \lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_{2,\delta,\tilde{b}}(u), \quad (4.13)\end{aligned}$$

*with boundary conditions*

$$m'_{\tilde{b}}(\tilde{b}) = 0, \quad (4.14)$$

$$m''_{\tilde{b}}(\tilde{b}) = -\frac{\lambda}{p}\sigma'_{1,\delta,\tilde{b}}(\tilde{b}) - \theta\frac{\lambda}{p}\sigma'_{2,\delta,\tilde{b}}(\tilde{b}). \quad (4.15)$$

**Proof.** We first differentiate (4.12) with respect to (w.r.t.) $u$ and we obtain

$$\begin{aligned}m'_{\delta,\tilde{b}}(u) &= -\frac{\lambda}{p}\sigma_{1,\delta,\tilde{b}}(u) + \frac{\lambda}{p}\Big(\frac{\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{1,\delta,\tilde{b}}(s\wedge\tilde{b})ds \\ &-\frac{\theta\lambda}{p}\Big(\frac{\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s\wedge\tilde{b})ds - \frac{\theta\lambda}{p}\sigma_{2,\delta,\tilde{b}}(u) \\ &+2\frac{\theta\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s\wedge\tilde{b})ds. \quad (4.16)\end{aligned}$$

Multiplying (4.12) by $\frac{\lambda+\delta}{p}$, subtracting (4.16) and using the identity and differentiation operators $\mathcal{I}$ and $\mathcal{D}$, we get

$$\begin{aligned}\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\sigma_{1,\delta,\tilde{b}}(u) - \theta\frac{2\lambda^2}{p^2}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s)ds \\ &+\theta\frac{\lambda}{p}\sigma_{2,\delta,\tilde{b}}(u). \quad (4.17)\end{aligned}$$

Let us define

$$g_{\delta,\tilde{b}}(u) = \Big(\frac{\lambda+\delta}{p}\mathcal{I} - \mathcal{D}\Big)m_{\delta,\tilde{b}}(u). \tag{4.18}$$

By differentiating (4.18) w.r.t. $u$ and using the right-hand side of (4.17), we deduce

$$\begin{aligned} g'_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\sigma'_{1,\delta,\tilde{b}}(u) - \theta\frac{2\lambda^2(2\lambda+\delta)}{p^3}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\sigma_{2,\delta,\tilde{b}}(s)ds + \theta\frac{\lambda}{p}\sigma'_{2,\delta,\tilde{b}}(u) \\ &+\theta\frac{2\lambda^2}{p^2}\sigma_{2,\delta,\tilde{b}}(u). \end{aligned} \tag{4.19}$$

Multiplying (4.18) by $\frac{2\lambda+\delta}{p}$, subtracting (4.19), and using (4.17), we obtain

$$\begin{aligned} \Big(\frac{2\lambda+\delta}{p}\mathcal{I} - \mathcal{D}\Big)g_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I} - \mathcal{D}\Big)\sigma_{1,\delta,\tilde{b}}(u) \\ &+ \lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I} - \mathcal{D}\Big)\sigma_{2,\delta,\tilde{b}}(u), \end{aligned} \tag{4.20}$$

which is equivalent to the desired integro-differential equation in (4.13).

To obtain the boundary conditions (4.14) and (4.15), we consider $u = \tilde{b}$ in (4.12) and (4.16), which leads to

$$m_{\tilde{b}}(\tilde{b}) = \frac{\lambda}{\delta+\lambda}\sigma_{1,\delta,\tilde{b}}(\tilde{b}) + \theta\frac{\lambda\delta}{(\delta+\lambda)(\delta+2\lambda)}\sigma_{2,\delta,\tilde{b}}(\tilde{b}), \tag{4.21}$$

$$m'_{\tilde{b}}(\tilde{b}) = 0. \tag{4.22}$$

Letting $u = \tilde{b}$ in (4.13), we obtain

$$\begin{aligned} \Big(\frac{2\lambda+\delta}{p}\Big)\Big(\frac{\lambda+\delta}{p}\Big)m_{\delta,\tilde{b}}(\tilde{b}) &- \Big(\frac{3\lambda+2\delta}{p^2}\Big)m'_{\delta,\tilde{b}}(\tilde{b}) + m''_{\delta,\tilde{b}}(\tilde{b}) = \frac{\lambda}{p}\frac{2\lambda+\delta}{p}\sigma_{1,\delta,\tilde{b}}(\tilde{b}) \\ &- \frac{\lambda}{p}\sigma'_{1,\delta,\tilde{b}}(\tilde{b}) + \lambda\frac{\theta}{p}\frac{\delta}{p}\sigma_{2,\delta,\tilde{b}}(\tilde{b}) - \lambda\frac{\theta}{p}\sigma'_{2,\delta,\tilde{b}}(\tilde{b}). \end{aligned} \tag{4.23}$$

Using (4.21), (4.22) and (4.23) yields (4.15). □

### 4.2.2 A general solution

In the classical compound Poisson risk model in the presence of a constant dividend barrier, Lin et al. (2003) show that the solution to the integro-differential equation for the Gerber-Shiu discounted penalty function is a linear combination of the Gerber-Shiu function with no barrier and the solution to the associated homogeneous integro-differential equation. When a dependence structure for $(X, W)$ is defined via the FGM copula, the solution to the integro-differential equation (4.13) for the expected discounted penalty function with a constant dividend barrier $\tilde{b}$, $m_{\delta,\tilde{b}}(u)$, has a similar form. However, the solution to the associated homogeneous integro-differential equation is a linear combination of two independent solutions. This is shown in the proposition below.

**Proposition 24.** *For the expected discounted penalty function, a closed form expression for $m_{\delta,\tilde{b}}(u)$ is given by*

$$m_{\delta,\tilde{b}}(u) = m_{\delta,\infty}(u) + \eta_1 v_{1,\delta}(u) + \eta_2 v_{2,\delta}(u), \quad 0 \leq u \leq \tilde{b}, \tag{4.24}$$

*where the constants $\eta_1, \eta_2$ are the solutions to the following system of linear equations :*

$$\eta_1 v'_{1,\delta}(\tilde{b}) + \eta_2 v'_{2,\delta}(\tilde{b}) = -m'_{\delta,\infty}(\tilde{b}), \tag{4.25}$$

*and*

$$
\begin{aligned}
& \eta_1 \left( v''_{1,\delta}(\tilde{b}) + \left( \frac{\lambda}{p}\mathcal{D}\int_0^u v_{1,\delta}(u-x)f_X(x)dx + \theta\frac{\lambda}{p}\mathcal{D}\int_0^u v_{1,\delta}(u-x)h_X(x)dx \right)\Big|_{u=\tilde{b}} \right) \\
+ \; & \eta_2 \left( v''_{2,\delta}(\tilde{b}) + \left( \frac{\lambda}{p}\mathcal{D}\int_0^u v_{2,\delta}(u-x)f_X(x)dx + \theta\frac{\lambda}{p}\mathcal{D}\int_0^u v_{2,\delta}(u-x)h_X(x)dx \right)\Big|_{u=\tilde{b}} \right) \\
= \; & -\frac{\lambda}{p}\left(\mathcal{D}\int_0^u m_{\delta,\infty}(u-x)f_X(x)dx\right)\Big|_{u=\tilde{b}} - \theta\frac{\lambda}{p}\left(\mathcal{D}\int_0^u m_{\delta,\infty}(u-x)h_X(x)dx\right)\Big|_{u=\tilde{b}} \\
- \; & m''_{\delta,\infty}(\tilde{b}) - \frac{\lambda}{p}w'_1(\tilde{b}) - \theta\frac{\lambda}{p}w'_2(\tilde{b}). \qquad (4.26)
\end{aligned}
$$

*The L.T. of $v_{1,\delta}(u)$ and $v_{2,\delta}(u)$ are provided in (4.30) and (4.31).*

**Proof.** It follows from the theory on differential equations that the general solution to the second order non-homogeneous integro-differential equation (4.13) for $m_{\delta,\tilde{b}}(u)$ with boundary conditions (4.14) and (4.15), can be expressed as the sum of a particular solution $m_{\delta,\infty}(u)$ (the Gerber-Shiu function in the absence of a dividend barrier) and a linear combination of 2 linearly independent solutions to the associated homogeneous integro-differential equation. Hence, the general solution to the integro-differential equation (4.13) is of the form

$$
m_{\delta,\tilde{b}}(u) = m_{\delta,\infty}(u) + \eta_1 v_{1,\delta}(u) + \eta_2 v_{2,\delta}(u), \quad 0 \le u \le \tilde{b}, \qquad (4.27)
$$

where the expected discounted penalty function without a barrier $m_{\delta,\infty}(u)$ is the solution to a defective renewal equation, as shown in Cossette et al. (2008) and satisfies the following integro-differential equation

$$
\begin{aligned}
\left(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\right)\left(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\right)m_{\delta,\infty}(u) \; &= \; \frac{\lambda}{p}\left(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\right)\left(\int_0^u m_{\delta,\infty}(u-x)f_X(x)dx + w_1(u)\right) \\
&+ \; \lambda\frac{\theta}{p}\left(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\right)\left(\int_0^u m_{\delta,\infty}(u-x)h_X(x)dx + w_2(u)\right).
\end{aligned}
$$

The quantities $\{v_{i,\delta}(u), u \ge 0\}$ $i = 1, 2$ are two linearly independent solutions of the following homogeneous integro-differential equation associated to (4.13) :

$$\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)v_\delta(u) = \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\left(\int_0^u v_\delta(u-x)f_X(x)dx\right) + \lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\left(\int_0^u v_\delta(u-x)h_X(x)dx\right). \quad (4.28)$$

Taking the L.T. on both sides of (4.28), we get

$$v_\delta^\star(s) = \frac{(s-\frac{2\delta+3\lambda}{p})v_\delta(0)+v_\delta'(0)}{(\frac{\delta+\lambda}{p}-s)(\frac{\delta+2\lambda}{p}-s)-\frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p}-s\Big)f_X^\star(s)-\theta\frac{\lambda}{p}\Big(\frac{\delta}{p}-s\Big)h_X^\star(s)}. \quad (4.29)$$

From (4.29), we deduce that a solution to (4.28) is a combination of the two linearly independent solutions $\{v_{1,\delta}(u), u\geq 0\}$ and $\{v_{2,\delta}(u), u\geq 0\}$, where

$$v_{1,\delta}^\star(s) = \frac{(s-\frac{2\delta+3\lambda}{p})}{(\frac{\delta+\lambda}{p}-s)\Big(\frac{\delta+2\lambda}{p}-s\Big)-\frac{\lambda}{p}(\frac{\delta+2\lambda}{p}-s)f_X^\star(s)-\theta\frac{\lambda}{p}(\frac{\delta}{p}-s)h_X^\star(s)}, \quad (4.30)$$

and

$$v_{2,\delta}^\star(s) = \frac{1}{\Big(\frac{\delta+\lambda}{p}-s\Big)(\frac{\delta+2\lambda}{p}-s)-\frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p}-s\Big)f_X^\star(s)-\theta\frac{\lambda}{p}\Big(\frac{\delta}{p}-s\Big)h_X^\star(s)}. \quad (4.31)$$

By using the Initial Value Theorem in (4.30) and (4.31), we have $v_{1,\delta}(0) = \lim_{s\to\infty} sv_{1,\delta}^\star(s) = 1$ and $v_{1,\delta}'(0) = \lim_{s\to\infty} s(sv_{1,\delta}^\star(s)-1) = 0$. Similarly, we have $v_{2,\delta}(0)=0$ and $v_{2,\delta}'(0)=1$.

The constants $\eta_1$, $\eta_2$ are determined such that the two boundary conditions (4.14) and (4.15) to the integro-differential equation (4.13) are satisfied. They must therefore be the solutions to the following systems of linear equations :

$$\begin{aligned}\eta_1 v_{1,\delta}'(\tilde{b})+\eta_2 v_{2,\delta}'(\tilde{b}) &= -m_{\delta,\infty}'(\tilde{b}),\\ \eta_1 v_{1,\delta}''(\tilde{b})+\eta_2 v_{2,\delta}''(\tilde{b}) &= -\frac{\lambda}{p}\sigma_{1,\delta,\tilde{b}}'(\tilde{b})-\theta\frac{\lambda}{p}\sigma_{2,\delta,\tilde{b}}'(\tilde{b})-m_{\delta,\infty}''(\tilde{b}). \quad (4.32)\end{aligned}$$

We need to find the expressions for $\sigma_{1,\delta,\tilde{b}}'(\tilde{b})$ and $\sigma_{2,\delta,\tilde{b}}'(\tilde{b})$. For that purpose, we differentiate (4.9) and (4.10) w.r.t. $u$ and, with (4.27), we obtain

$$\begin{aligned}
\sigma'_{1,\delta,\bar{b}}(u) &= \mathcal{D}\int_0^u m_{\delta,\infty}(u-x)f_X(x)dx + \eta_1\mathcal{D}\int_0^u v_{1,\delta}(u-x)f_X(x)dx \\
&+ \eta_2\mathcal{D}\int_0^u v_{2,\delta}(u-x)f_X(x)dx + w_1'(u), \qquad (4.33) \\
\sigma'_{2,\delta,\bar{b}}(u) &= \mathcal{D}\int_0^u m_{\delta,\infty}(u-x)h_X(x)dx + \eta_1\mathcal{D}\int_0^u v_{1,\delta}(u-x)h_X(x)dx \\
&+ \eta_2\mathcal{D}\int_0^u v_{2,\delta}(u-x)h_X(x)dx + w_2'(u). \qquad (4.34)
\end{aligned}$$

Substituting (4.33) and (4.34) into (4.32) yields (4.26). □

### 4.2.3 Comments

In (4.24), the expression for $m_{\delta,\bar{b}}(u)$ is defined in terms of $m_{\delta,\infty}(u)$, $v_{1,\delta}(u)$ and $v_{2,\delta}(u)$. Cossette et al (2008) analyze $m_{\delta,\infty}(u)$ and show in particular that it satisfies a defective renewal equation representation.

It is possible to show that $v_{1,\delta}(u)$ and $v_{2,\delta}(u)$ can also be expressed as a defective renewal equations. In Proposition 7 of Cossette et al. (2008), it is demonstrated that the common denominator of (4.30) and (4.31)

$$\Big(\frac{\delta+\lambda}{p}-s\Big)\Big(\frac{\delta+2\lambda}{p}-s\Big)-\frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p}-s\Big)f_X^\star(s)-\theta\frac{\lambda}{p}\Big(\frac{\delta}{p}-s\Big)h_X^\star(s),$$

is equal to

$$\left(1-T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0)\right)(s-\rho_1)(s-\rho_2), \qquad (4.35)$$

where $T_r$ is the Dickson-Hipp operator for an integrable real-valued function $f$ (introduced by Dickson and Hipp (2001)) defined by

$$T_rf(x)=\int_x^\infty e^{-r(u-x)}f(u)du, \quad r\in C.$$

The properties of the operator $T_r$ are examined in details in Li and Garrido (2004a). Note that the common denominator also corresponds to Lundberg's generalized equation where $\rho_1$ and $\rho_2$ are its two distincts roots with real positive parts (see Cossette et al. (2008) for more details). Moreover, the definition of $h_{2,\delta}(u)$ is such that its L.T. is given by

$$h_{2,\delta}^{\star}(s)=\frac{\lambda}{p}\Big(\frac{\delta+2\lambda}{p}-s\Big)f_X^{\star}(s)+\theta\frac{\lambda}{p}\Big(\frac{\delta}{p}-s\Big)h_X^{\star}(s).$$

Using (4.35), (4.30) and (4.31) become

$$v_{1,\delta}^{\star}(s)=\frac{\frac{(s-\frac{2\delta+3\lambda}{p})}{(s-\rho_1)(s-\rho_2)}}{1-T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0)}, \tag{4.36}$$

and

$$v_{2,\delta}^{\star}(s)=\frac{\frac{1}{(s-\rho_1)(s-\rho_2)}}{1-T_sT_{\rho_1}T_{\rho_2}h_{2,\delta}(0)}. \tag{4.37}$$

Note that (4.36) and (4.37) are similar to (24) and (25) in Landriault (2008). Inverting (4.36) and (4.37) and proceeding as in the proof of Proposition 8 in Cossette et al. (2008), we find that (4.36) and (4.37) can be expressed as the following defective renewal equations :

$$v_{1,\delta}(u)=\frac{1}{1+\kappa_\delta}\int_0^u v_{1,\delta}(u-y)\vartheta_\delta(y)dy+\frac{\rho_1-\frac{2\delta+3\lambda}{p}}{\rho_2-\rho_1}e^{\rho_2 u}+\frac{\rho_2-\frac{2\delta+3\lambda}{p}}{\rho_1-\rho_2 1}e^{\rho_1 u}$$

and

$$v_{2,\delta}(u)=\frac{1}{1+\kappa_\delta}\int_0^u v_{2,\delta}(u-y)\vartheta_\delta(y)dy+\frac{e^{\rho_1 u}-e^{\rho_2 u}}{\rho_1-\rho_2},$$

where

$$\vartheta_\delta(y)=(1+\kappa_\delta)T_{\rho_1}T_{\rho_2}h_{2,\delta}(y)$$

and

$$\frac{1}{(1+\kappa_\delta)}=T_0T_{\rho_1}T_{\rho_2}h_{2,\delta}(0)=1-\frac{\delta}{p}\frac{\left(\frac{\delta+2\lambda}{p}\right)}{\rho_1\rho_2}<1.$$

We conclude that $m_{\delta,\tilde{b}}(u)$ is a linear combination of three components ($m_{\delta,\infty}(u)$, $v_{1,\delta}(u)$, and $v_{2,\delta}(u)$) where each one satisfies a defective renewal equation. Therefore, it is possible to apply the results in Willmot and Lin (2001) to these three components.

## 4.3 Expected discounted dividend payments

As mentioned in the introduction, we now consider a second quantity of interest, the expectation of the discounted dividend payments $V_{\delta,\tilde{b}}(u)$. In this section, we derive an integro-differential equation satisfied by $V_{\delta,\tilde{b}}(u)$ $(0 \leq u \leq \tilde{b})$ with certain boundary conditions. From that integro-differential equation, we obtain a general solution for $V_{\delta,\tilde{b}}(u)$.

### 4.3.1 An integro-differential equation

As in the previous section, we use a renewal argument to establish an integral equation for $V_{\delta,\tilde{b}}(u)$. The first claim occurs at time $t$ and its amount is $x$. We condition on the time and the amount of the first claim and, again, we have the 4 following situations :

- $t \leq \frac{\tilde{b}-u}{p}$ and $x \leq u+pt$ : no dividend payment;
- $t > \frac{\tilde{b}-u}{p}$ and $x \leq \tilde{b}$ : dividends are paid between the moment $(\frac{\tilde{b}-u}{p})$ when the surplus reaches the barrier and time $t$;
- $t \leq \frac{\tilde{b}-u}{p}$ and $x > u+pt$ : no dividend payment before ruin;
- $t > \frac{\tilde{b}-u}{p}$ and $x > \tilde{b}$ : dividends are paid before ruin, i.e. between the moment $(\frac{\tilde{b}-u}{p})$ at which the surplus reaches the barrier and time $t$.

Given the 4 possible cases described above, we have

$$\begin{aligned} V_{\delta,\tilde{b}}(u) &= \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} V_{\delta,\tilde{b}}(u+pt-x) f_{X,W}(x,t)\, dxdt \\ &+ \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} (e^{-\delta(\frac{\tilde{b}-u}{p})} p\bar{a}_{\overline{t-(\frac{\tilde{b}-u}{p})|}} + e^{-\delta t} V_{\delta,\tilde{b}}(\tilde{b}-x)) f_{X,W}(x,t)\, dxdt \\ &+ \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta(\frac{\tilde{b}-u}{p})} p\bar{a}_{\overline{t-(\frac{\tilde{b}-u}{p})|}} f_{X,W}(x,t)\, dxdt. \qquad (4.38) \end{aligned}$$

The terms

$$\int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta(\frac{\tilde{b}-u}{p})} p\bar{a}_{\overline{t-(\frac{\tilde{b}-u}{p})|}} f_{X,W}(x,t)\, dxdt + \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta(\frac{\tilde{b}-u}{p})} p\bar{a}_{\overline{t-(\frac{\tilde{b}-u}{p})|}} f_{X,W}(x,t)\, dxdt$$

in (4.38) can be simplified and expressed in the form

$$\begin{aligned} \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\infty} e^{-\delta(\frac{\tilde{b}-u}{p})} p\bar{a}_{\overline{t-(\frac{\tilde{b}-u}{p})|}} f_{X,W}(x,t)\, dxdt &= \int_{\frac{\tilde{b}-u}{p}}^{\infty} e^{-\delta(\frac{\tilde{b}-u}{p})} p\bar{a}_{\overline{t-(\frac{\tilde{b}-u}{p})|}} f_W(t)\, dt \\ &= \int_{\frac{\tilde{b}-u}{p}}^{\infty} p f_W(t) \int_{\frac{\tilde{b}-u}{p}}^{t} e^{-\delta y} dy dt \\ &= \int_{\frac{\tilde{b}-u}{p}}^{\infty} p e^{-\delta y} \int_y^{\infty} f_W(t)\, dtdy = \int_{\frac{\tilde{b}-u}{p}}^{\infty} p e^{-\delta y} \int_y^{\infty} \lambda e^{-\lambda t} dtdy \\ &= \frac{p}{\lambda+\delta} e^{-(\delta+\lambda)(\frac{\tilde{b}-u}{p})}, \qquad (4.39) \end{aligned}$$

since $f_W(t) = \lambda e^{-\lambda t}$. Replacing (4.39) and the joint p.d.f. (4.4) for $(X, W)$ in (4.38), we obtain

$$\begin{aligned} V_{\delta,\tilde{b}}(u) &= \lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} V_{\delta,\tilde{b}}(u+pt-x) e^{-\lambda t} f_X(x) dxdt \\ &+\lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} V_{\delta,\tilde{b}}(\tilde{b}-x) e^{-\lambda t} f_X(x) dxdt \\ &+\lambda\theta \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} V_{\tilde{b}}(u+pt-x) h_X(x) (2e^{-2\lambda t} - e^{-\lambda t}) dxdt \\ &+\lambda\theta \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} V_{\delta,\tilde{b}}(\tilde{b}-x) h_X(x) (2e^{-2\lambda t} - e^{-\lambda t}) dxdt \\ &+\frac{p}{\lambda+\delta} e^{-(\delta+\lambda)(\frac{\tilde{b}-u}{p})}. \qquad (4.40) \end{aligned}$$

Substituting $u + pt = s$ in (4.40), we have

$$\begin{aligned} V_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\int_u^{\tilde{b}} e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{1,\delta,\tilde{b}}(s)ds + \frac{\lambda}{p}\int_{\tilde{b}}^{\infty} e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{1,\delta,\tilde{b}}(\tilde{b})ds \\ &- \frac{\theta}{p}\lambda\int_u^{\tilde{b}} e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s)ds + 2\frac{\theta}{p}\lambda\int_u^{\tilde{b}} e^{-(\delta+2\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s)ds \\ &- \frac{\theta}{p}\lambda\int_{\tilde{b}}^{\infty} e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(\tilde{b})ds + 2\frac{\theta}{p}\lambda\int_{\tilde{b}}^{\infty} e^{-(\delta+2\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(\tilde{b})ds \\ &+ \frac{p}{\lambda+\delta}e^{-(\delta+\lambda)(\frac{\tilde{b}-u}{p})}, \end{aligned} \tag{4.41}$$

where

$$\omega_{1,\delta,\tilde{b}}(u) = \int_0^u V_{\delta,\tilde{b}}(u-x)f_X(x)dx \tag{4.42}$$

$$\omega_{2,\delta,\tilde{b}}(u) = \int_0^u V_{\delta,\tilde{b}}(u-x)h_X(x)dx. \tag{4.43}$$

We rewrite (4.41) as follows :

$$\begin{aligned} V_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\int_u^{\infty} e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{1,\delta,\tilde{b}}(s\wedge\tilde{b})ds - \frac{\theta\lambda}{p}\int_u^{\infty} e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s\wedge\tilde{b}))ds \\ &+ 2\frac{\theta\lambda}{p}\int_u^{\infty} e^{-(\delta+2\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s\wedge\tilde{b}))ds + \frac{p}{\lambda+\delta}e^{-(\delta+\lambda)(\frac{\tilde{b}-u}{p})}. \end{aligned} \tag{4.44}$$

The next proposition provides an integro-differential equation for the expected discounted dividend payments $V_{\delta,\tilde{b}}(u)$, derived from the integral equation (4.44).

**Proposition 25.** *For $-1 \leq \theta \leq 1$, the expected discounted dividend payment $V_{\delta,\tilde{b}}(u)$ satisfies the following integro-differential equation for $0 \leq u \leq \tilde{b}$ :*

$$\begin{aligned} \Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)V_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\omega_{1,\delta,\tilde{b}}(u) \\ &+ \lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\omega_{2,\delta,\tilde{b}}(u), \end{aligned} \tag{4.45}$$

*with boundary conditions*

$$V'_{\delta,\tilde{b}}(\tilde{b}) = 1, \tag{4.46}$$

$$V''_{\delta,\tilde{b}}(\tilde{b}) = -\frac{\lambda}{p}\omega'_{1,\delta,\tilde{b}}(\tilde{b}) - \theta\frac{\lambda}{p}\omega'_{2,\delta,\tilde{b}}(\tilde{b}) + \frac{\lambda+\delta}{p}. \tag{4.47}$$

**Proof.** To derive (4.45), we begin by differentiating (4.44) w.r.t. $u$

$$\begin{aligned} V'_{\delta,\tilde{b}}(u) = & -\frac{\lambda}{p}\omega_{1,\delta,\tilde{b}}(u) + \frac{\lambda}{p}\Big(\frac{\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{1,\delta,\tilde{b}}(s\wedge\tilde{b})ds \\ & -\frac{\theta\lambda}{p}\Big(\frac{\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s\wedge\tilde{b}))ds - \frac{\theta\lambda}{p}\omega_{2,\delta,\tilde{b}}(u) \\ & +2\frac{\theta\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s\wedge\tilde{b}))ds + e^{-(\delta+\lambda)(\frac{\tilde{b}-u}{p})}. \end{aligned} \tag{4.48}$$

We apply the operator $(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D})$ on (4.44) and, using (4.48), it follows that

$$\begin{aligned} \Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)V_{\delta,\tilde{b}}(u) = & \frac{\lambda}{p}\omega_{1,\delta,\tilde{b}}(u) - \frac{2\theta\lambda^2}{p^2}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s\wedge\tilde{b})ds \\ & +\frac{\theta\lambda}{p}\omega_{2,\delta,\tilde{b}}(u). \end{aligned} \tag{4.49}$$

Let us define

$$K_{\delta,\tilde{b}}(u) = \Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)V_{\delta,\tilde{b}}(u). \tag{4.50}$$

We differentiate (4.50) w.r.t. $u$ and, using (4.49), we obtain

$$\begin{aligned} K'_{\delta,\tilde{b}}(u) = & \frac{\lambda}{p}\omega'_{1,\delta,\tilde{b}}(u) - \theta\frac{2\lambda^2(2\lambda+\delta)}{p^3}\int_u^\infty e^{-(\delta+2\lambda)(\frac{s-u}{p})}\omega_{2,\delta,\tilde{b}}(s\wedge\tilde{b})ds \\ & + \theta\frac{\lambda}{p}\omega'_{2,\delta,\tilde{b}}(u) + \theta\frac{2\lambda^2}{p^2}\omega_{2,\delta,\tilde{b}}(u). \end{aligned} \tag{4.51}$$

Multiplying (4.50) by $\frac{2\lambda+\delta}{p}$, subtracting (4.51) and using (4.49), it follows that

$$\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)K_{\delta,\tilde{b}}(u) = \frac{\lambda}{p}\Big(\frac{2\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\omega_{1,\delta,\tilde{b}}(u) + \lambda\frac{\theta}{p}\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\omega_{2,\delta,\tilde{b}}(u)$$

which is equivalent to (4.45).

We now derive the boundary conditions (4.46) and (4.47). Letting $u = \tilde{b}$ in (4.41), we obtain

$$V_{\delta,\tilde{b}}(\tilde{b}) = \frac{\lambda}{\lambda+\delta}\omega_{1,\delta,\tilde{b}}(\tilde{b}) + \theta\frac{\lambda\delta}{(\lambda+\delta)(2\lambda+\delta)}\omega_{2,\delta,\tilde{b}}(\tilde{b}) + \frac{p}{\lambda+\delta}. \tag{4.52}$$

Setting $u = \tilde{b}$ into (4.48) yields (4.46). Substituting (4.46) and (4.52) into (4.45) with $u = \tilde{b}$, we obtain (4.47). □

### 4.3.2 A general solution

From the integro-differential equation in (4.45) for the expected discounted dividend payments $V_{\delta,\tilde{b}}(u)$, we find the following result.

**Proposition 26.** *A closed form expression for the expected discounted dividend payments, $V_{\delta,\tilde{b}}(u)$, is given by*

$$V_{\delta,\tilde{b}}(u) = \eta_1(\tilde{b})v_{1,\delta}(u) + \eta_2(\tilde{b})v_{2,\delta}(u), \quad 0 \le u \le \tilde{b}, \tag{4.53}$$

*where $v_{1,\delta}(u)$ and $v_{2,\delta}(u)$ also appear in (4.24). The coefficients $\eta_1(\tilde{b})$ and $\eta_2(\tilde{b})$ are determined by solving the following system of linear equations :*

$$\eta_1(\tilde{b})v'_{1,\delta}(\tilde{b}) + \eta_2(\tilde{b})v'_{2,\delta}(\tilde{b}) = 1, \tag{4.54}$$

*and*

$$\begin{aligned} & \eta_1(\tilde{b})\left(v''_{1,\delta}(\tilde{b}) + \left(\frac{\lambda}{p}\mathcal{D}\int_0^u v_{1,\delta}(u-x)f_X(x)dx + \theta\frac{\lambda}{p}\mathcal{D}\int_0^u v_{1,\delta}(u-x)h_X(x)dx\right)\bigg|_{u=\tilde{b}}\right) \\ + \;& \eta_2(\tilde{b})\left(v''_{2,\delta}(\tilde{b}) + \left(\frac{\lambda}{p}\mathcal{D}\int_0^u v_{2,\delta}(u-x)f_X(x)dx + \theta\frac{\lambda}{p}\mathcal{D}\int_0^u v_{2,\delta}(u-x)h_X(x)dx\right)\bigg|_{u=\tilde{b}}\right) \\ = \;& \frac{\lambda+\delta}{p}. \end{aligned} \tag{4.55}$$

**Proof.** From the theory of differential equations, the solution to (4.45) with boundary conditions (4.46) and (4.47) is of the form (4.53) where the constants $\eta_1(\tilde{b})$ and $\eta_2(\tilde{b})$ are determined by the following system of linear equations :

$$\eta_1(\tilde{b})v'_{1,\delta}(\tilde{b}) + \eta_2(\tilde{b})v'_{2,\delta}(\tilde{b}) = 1, \tag{4.56}$$

$$\eta_1(\tilde{b})v''_{1,\delta}(\tilde{b}) + \eta_2(\tilde{b})v''_{2,\delta}(\tilde{b}) = -\frac{\lambda}{p}\omega'_{1,\delta,\tilde{b}}(\tilde{b}) - \theta\frac{\lambda}{p}\omega'_{2,\delta,\tilde{b}}(\tilde{b}) + \frac{\lambda+\delta}{p}. \tag{4.57}$$

Differentiating (4.42) and (4.43) w.r.t. $u$ and using (4.53) we obtain

$$\omega'_{1,\delta,\tilde{b}}(u) = \eta_1(\tilde{b})\mathcal{D}\int_0^u v_{1,\delta}(u-x)f_X(x)dx + \eta_2(\tilde{b})\mathcal{D}\int_0^u v_{2,\delta}(u-x)f_X(x)dx, \tag{4.58}$$

$$\omega'_{2,\delta,\tilde{b}}(u) = \eta_1(\tilde{b})\mathcal{D}\int_0^u v_{1,\delta}(u-x)h_X(x)dx + \eta_2(\tilde{b})\mathcal{D}\int_0^u v_{2,\delta}(u-x)h_X(x)dx. \tag{4.59}$$

Substituting $\omega'_{1,\delta,\tilde{b}}(\tilde{b})$ and $\omega'_{2,\delta,\tilde{b}}(\tilde{b})$ in (4.58) and (4.59) at $u = \tilde{b}$ into (4.57) yields (4.55). □

## 4.4 Explicit solutions

In this section, we assume that the claim amounts follow an exponential distribution with c.d.f $F_X(x) = 1 - e^{-\alpha x}$, p.d.f. $f_X(x) = \alpha e^{-\alpha x}$, and L.T.

$$f_X^\star(s) = \frac{\alpha}{s+\alpha}, \quad s > -\alpha. \tag{4.60}$$

From (4.5), we have

$$h_X(x) = 2\alpha e^{-2\alpha x} - \alpha e^{-\alpha x}, \qquad x \geq 0,$$

and

$$h_X^\star(s) = \frac{2\alpha}{s+2\alpha} - \frac{\alpha}{s+\alpha}, \quad s > -\alpha. \tag{4.61}$$

We derive explicit solutions for $m_{\delta,\bar{b}}(u)$ when the penalty function $w(x,y)$ is equal to 1 for any values of $x$ and $y$ (denoted by $\phi_{T_{\bar{b}}}(u)$) and for $V_{\delta,\bar{b}}(u)$. For both $\phi_{T_{\bar{b}}}(u)$ and $V_{\delta,\bar{b}}(u)$, we find the expressions for $v_{1,\delta}(s)$ and $v_{2,\delta}(s)$ by inverting their L.T., $v_{1,\delta}^\star(s)$ and $v_{2,\delta}^\star(s)$. Substituting the expressions (4.60) and (4.61) for $f_X^\star(s)$ and $h_X^\star(s)$ in (4.30) and (4.31), we obtain

$$v_{1,\delta}^\star(s) = \frac{(s-\frac{2\delta+3\lambda}{p})(\alpha+s)(2\alpha+s)}{\left(\frac{\delta+\lambda}{p}-s\right)\left(\frac{\delta+2\lambda}{p}-s\right)(\alpha+s)(2\alpha+s) - \frac{\lambda}{p}\alpha(\frac{\delta+2\lambda}{p}-s)(2\alpha+s) - \theta\frac{\lambda}{p}\alpha\left(\frac{\delta}{p}-s\right)s}, \tag{4.62}$$

and

$$v_{2,\delta}^\star(s) = \frac{(\alpha+s)(2\alpha+s)}{(\frac{\delta+\lambda}{p}-s)\left(\frac{\delta+2\lambda}{p}-s\right)(\alpha+s)(2\alpha+s) - \frac{\lambda}{p}\alpha\left(\frac{\delta+2\lambda}{p}-s\right)(2\alpha+s) - \theta\frac{\lambda}{p}\alpha\left(\frac{\delta}{p}-s\right)s}. \tag{4.63}$$

The denominator on the right-hand side of (4.62) can be factored as

$$\begin{aligned}
&\left(\frac{\delta+\lambda}{p}-s\right)\left(\frac{\delta+2\lambda}{p}-s\right)(\alpha+s)(2\alpha+s) - \frac{\lambda}{p}\alpha\left(\frac{\delta+2\lambda}{p}-s\right)(2\alpha+s) - \theta\frac{\lambda}{p}\alpha\left(\frac{\delta}{p}-s\right)s \\
= \;& (s-\rho_1)(s-\rho_2)(s+R_1)(s+R_2),
\end{aligned}$$

where $\rho_1, \rho_2$ are the roots with positive real parts to Lundberg's equation

$$\left(\frac{\delta+\lambda}{p}-s\right)\left(\frac{\delta+2\lambda}{p}-s\right) = \frac{\lambda}{p}\left(\frac{\delta+2\lambda}{p}-s\right)f_X^\star(s) + \theta\frac{\lambda}{p}\left(\frac{\delta}{p}-s\right)h_X^\star(s),$$

and $-R_1$, $-R_2$ denote the roots with negative real parts. If $\rho_1, \rho_2$, $-R_1$, and $-R_2$ are all distinct, by partial fractions, (4.62) can be rewritten as

$$v_{1,\delta}^\star(s) = \sum_{j=1}^{2}\frac{\varsigma_j}{s-\rho_j} + \sum_{j=1}^{2}\frac{\beta_j}{s+R_j}, \tag{4.64}$$

where

$$\varsigma_j = \frac{(\rho_j - \frac{2\delta+3\lambda}{p})(\alpha+\rho_j)(2\alpha+\rho_j)}{\prod_{i=1}^{2}(R_i+\rho_j)\prod_{i=1,i\neq j}^{2}(-\rho_i+\rho_j)}, \quad j=1,2,$$

and

$$\beta_j = \frac{(-R_j - \frac{2\delta+3\lambda}{p})(\alpha-R_j)(2\alpha-R_j)}{\prod_{i=1}^{2}(R_j+\rho_i)\prod_{i=1,i\neq j}^{2}(R_i-R_j)}, \quad j=1,2.$$

Inverting (4.64) yields

$$v_{1,\delta}(u) = \sum_{j=1}^{2}\varsigma_j e^{\rho_j u} + \sum_{j=1}^{2}\beta_j e^{-R_j u}.$$

From (4.63), we proceed similarly and we find

$$v_{2,\delta}^{\star}(s) = \sum_{j=1}^{2}\frac{\xi_j}{s-\rho_j} + \sum_{j=1}^{2}\frac{\nu_j}{s+R_j}, \tag{4.65}$$

where

$$\xi_j = \frac{(\alpha+\rho_j)(2\alpha+\rho_j)}{\prod_{i=1}^{2}(R_i+\rho_j)\prod_{i=1,i\neq j}^{2}(-\rho_i+\rho_j)}, \quad j=1,2,$$

and

$$\nu_j = \frac{(\alpha-R_j)(2\alpha-R_j)}{\prod_{i=1}^{2}(R_j+\rho_i)\prod_{i=1,i\neq j}^{2}(R_i-R_j)}, \quad j=1,2.$$

Inverting (4.65) yields

$$v_{2,\delta}(u) = \sum_{j=1}^{2}\xi_j e^{\rho_j u} + \sum_{j=1}^{2}\nu_j e^{-R_j u}.$$

An explicit expression for $m_{\delta,\infty}(u)$ is derived in Cossette et al. (2008). We use Maple to find the coefficients $\eta_1$ (or $\eta_1(\tilde{b})$) and $\eta_2$ (or $\eta_2(\tilde{b})$) in the expression (4.24) (or (4.53)) for $\phi_{T_{\tilde{b}}}(u)$ (or $V_{\delta,\tilde{b}}(u)$) as the solutions to the linear system of two equations given in (4.25) and (4.26) (or (4.54) and (4.55)). The above procedure is illustrated in the following example.

**Example 27.** *Let $X \sim Exp(1)$, $W \sim Exp(1)$, and let $p = 1.5$. From Cossette et al. (2008), we recall in Table 1 the analytic expressions for the expected discounted value of the deficit at ruin*

$$m_{\delta,\infty}(u) = E\left[e^{-\delta T}|U(T)|I(T < \infty)|U(0) = u\right]$$

*that are obtained for $\delta = 0.05$ and for various values of the dependence parameter $\theta$.*

| $\theta$ | $m_{\delta,\infty}(u)$ |
|---|---|
| -1 | $0.699744e^{-0.320665u} + 0.036648e^{-2.219873u}$ |
| -0.5 | $0.659910e^{-0.350091u} + 0.017148e^{-2.114344u}$ |
| 0 | $0.613709e^{-0.386291u}$ |
| 0.5 | $0.559117e^{-0.432150u} - 0.013441e^{-1.873929u}$ |
| 1 | $0.492839e^{-0.492794u} - 0.020707e^{-1.731038u}$ |

*We fix the constant dividend barrier $\tilde{b}$ at 10. The analytic expressions for the expected discounted deficit at ruin in terms of the initial surplus $u$ with different* *values of the dependence parameter $\theta$ are provided in Table 2.*

| $\theta$ | $m_{\delta,\bar{b}}(u)$ |
|---|---|
| *-1* | $0.676864e^{-0.320665u} + 0.037593e^{-2.219873u} + 0.037351e^{0.084032u}$ |
| *-0.5* | $0.641740e^{-0.350091u} + 0.017575e^{-2.114344u} + 0.030769e^{0.085124u}$ |
| *0* | $0.600291e^{-0.386291u} + 0.023751e^{0.086291u}$ |
| *0.5* | $0.550009e^{-0.432150u} - 0.013739e^{-1.873929u} + 0.017071e^{0.087541u}$ |
| *1* | $0.487522e^{-0.492794u} - 0.021144e^{-1.731038u} + 0.010763e^{0.088886u}$ |

*The five functions are depicted in Figure 4.8..*

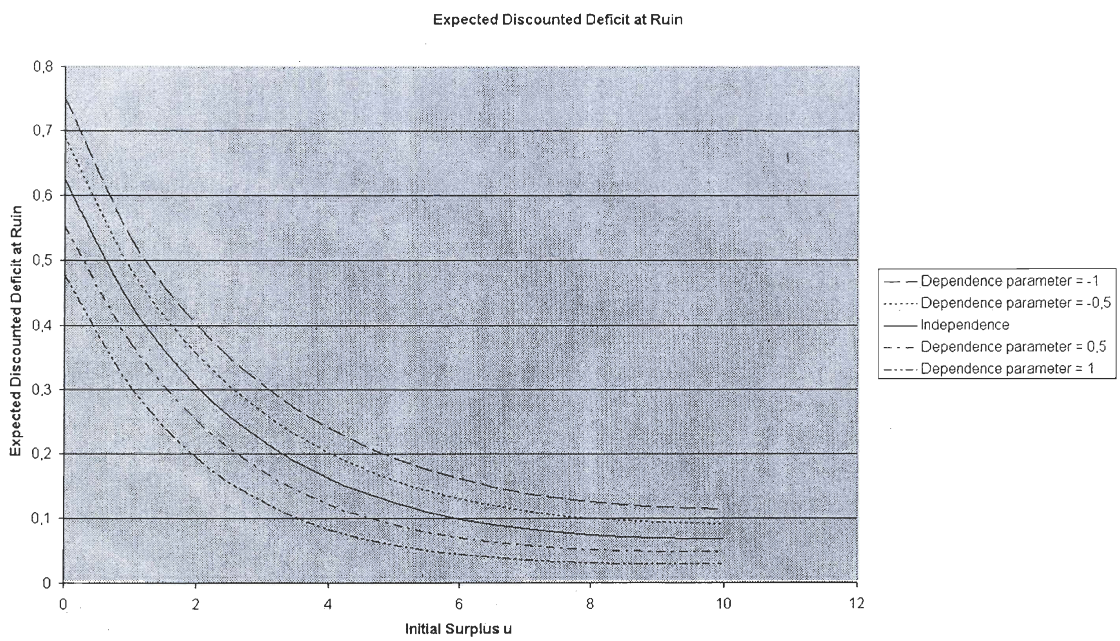


FIG. 4.8 – Expected discounted deficit at ruin for values of the dependence parameter $\theta$ equal to -1, -0.5, 0 (independence), 0.5, and 1.

*In Table 3 below, we provide the analytic expressions for the expected dis-*

*counted dividend payments $V_{\delta,\tilde{b}}(u)$ with initial surplus u for different parameters $\theta$.*

| $\theta$ | $V_{\delta,\tilde{b}}(u)$ |
|---|---|
| -1 | $-2.946450e^{-0.320665u} + 0.121683e^{-2.219873u} + 4.809853e^{0.084032u}$ |
| -0.5 | $-2.845440e^{-0.350091u} + 0.066905e^{-2.114344u} + 4.816056e^{0.085124u}$ |
| 0 | $-2.699233e^{-0.386291u} + 4.777755e^{0.086291u}$ |
| 0.5 | $-2.538068e^{-0.432150u} - 0.083219e^{-1.873929u} + 4.757324e^{0.087541u}$ |
| 1 | $-2.328715e^{-0.492794u} - 0.191435e^{-1.731038u} + 4.714218e^{0.088886u}$ |

*The five functions are shown in Figure 4.9.*

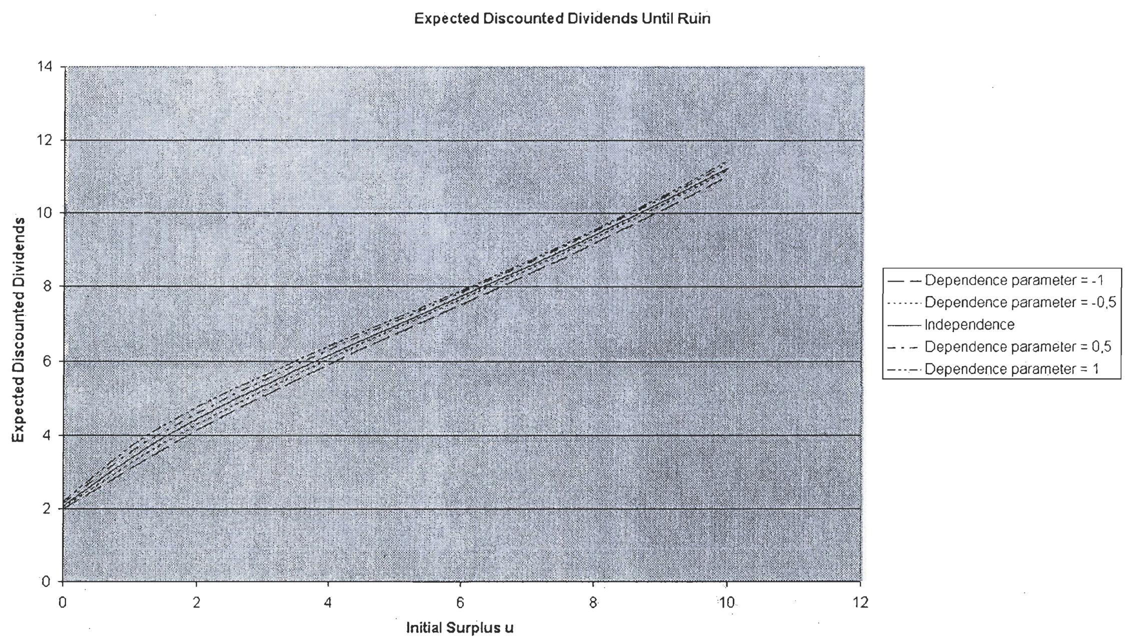


FIG. 4.9 – Expected discounted dividend payments for values of the dependence parameter $\theta$ equal to -1, -0.5, 0 (independence), 0.5, and 1.

*We observe that, for a fixed dependence parameter $\theta$, the expected discounted deficit at ruin decreases as the initial surplus increases while the expectation of the discounted dividend payments increases. Also, for a given initial surplus $u$, the expected discounted deficit at ruin decreases as the dependence parameter $\theta$ goes from -1 to 1 while the expectation of the discounted dividend payments increases. Comparing two cases with $\theta > \theta'$ (with identical initial surplus $u$), the probability that ruin occurs sooner is higher when the dependence parameter is $\theta$ rather than $\theta'$ which implies that the dividends are expected to be paid on a shorter period in the first case than in the second one.*

## 4.5 Acknowledgements

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## 4.6 References


- Albrecher, H. and O. Boxma (2004). A ruin model with dependence between claim sizes and claim intervals. *Insurance : Mathematics and Economics* 35, 245-254.
- Albrecher, H., Claramunt, M.M., and M. Marmol (2005). On the distribution of the dividend payments in a Sparre Andersen model with generalized Erlang(n) interclaim times. *Insurance : Mathematics and Economics* 37, 324-334.
- Albrecher, H. and J. Teugels (2006). Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability* 43 (1), 265-285.
- Boudreault, M., Cossette, H., Landriault, D. and, E. Marceau (2006). On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, 265-285.
- Bühlmann, H. (1970). *Mathematical Methods in Risk Theory*, Springer Verlag, New York.
- Cossette, H., Marceau, E., and F. Marri (2008). On the analysis of ruin for the compound Poisson risk model with dependence. Submitted for publication.
- De Finetti, B. (1957). Su un impostazione alternativa della teoria collectiva del rischio. *Trans. XVth Int. Congr. Actuaries* 2, 433–443.
- Denuit, M., Dhaene, J., Goovaerts, M.J. and R. Kaas (2005). *Actuarial Theory for Dependent Risks – Measures, Orders and Models.* Wiley. New

York.

- Dickson, D.C.M. and C. Hipp. (2001). On the time to ruin for Erlang(2) risk process, *Insurance : Mathematics and Economics* 29, 333–344.
- Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory.* S.S. Huebner Foundation. University of Pennsylvania. Philadelphie.
- Gerber, H.U. and E.S.W. Shiu (1998). On the time value of ruin. *North American Actuarial Journal* 2, 48-78.
- Gerber, H.U. and E.S.W. Shiu (2006). On optimal dividend strategies in the compound Poisson risk model. *North American Actuarial Journal* 10, 76-93.
- Grandell, J. (1991). *Aspects of risk theory.* Springer-Verlag, New York.
- Landriault, D. (2008). Constant dividend barrier in a risk model with interclaim-dependent claim sizes. *Insurance : Mathematics and Economics* 42, 31-38.
- Li, S. and J. Garrido (2004a). On ruin for the Erlang(n) risk process. *Insurance : Mathematics and Economics* 34, 391-408.
- Li, S. and J. Garrido (2004b). On a class of renewal risk models with a constant dividend barrier. *Insurance : Mathematics and Economics* 35, 691–701.
- Lin, X.S., Willmot, G.E. and S. Drekic (2003). The classical risk model with a dividend barrier : analysis of the Gerber-Shiu discounted penalty function. *Insurance : Mathematics and Economics* 33, 551-566.
- McNeil, A., Frey, R., and P. Embrechts (2005). *Quantitative Risk Mana-*

*gement.* Princeton Press, Princeton.

- Nelsen, R.B. (2006). *An Introduction to Copulas* (Second Edition). Springer Series in Statistics. Springer-Verlag, New York.
- Rolski, T., Schmidli, H., Schmidt, V. and J. Teugels (1999). *Stochastic Processes for Insurance and Finance.* Wiley, New York.
- Willmot, G.E. and X.S. Lin (2001). *Lundberg Approximations for Compound Distributions with Insurance Applications.* Springer Series in Statistics. Springer-Verlag, New York.

# CHAPITRE V

## Constant dividend barrier in a risk model with a generalized Farlie-Gumbel-Morgenstern copula


## Résumé

Dans ce papier, nous considérons le processus classique du surplus en présence d'une barrière constante de dividende. Une structure de dépendance entre les montants de réclamation et le temps séparant deux sinistres est introduite par la copule de Farlie-Gumbel-Morgenstern généralisée. Une équation intégro-différentielle avec des bornes de frontière est dérivée. Sa solution est exprimée comme la somme de la fonction de pénalité Gerber-Shiu en l'absence de la barrière de dividende plus une combinaison linéaire de $(c+2)$ solutions particulières indépendantes. De plus, nous déterminons une solution explicite pour le cas où les montants de sinistres sont de loi exponentielle. Pour terminer, nous étudions l'impact de la dépendance sur le temps de la ruine et sur le déficit à la ruine.

# Abstract

In this paper, we consider the classical surplus process with a constant dividend barrier and a dependence structure between the claim amounts and the inter-claim times. We derive an integro-differential equation with boundary conditions. Its solution is expressed as the Gerber-Shiu discounted penalty function in the absence of a dividend barrier plus a linear combination of $(c+2)$ linearly independent particular solutions to the associated homogeneous integro-differential equation. Finally, we obtain an explicit solution when the claim amounts are exponentially distributed and we investigate the effects of dependence on ruin quantities.

## 5.1 Introduction

In the actuarial literature, risk models under a dividend strategy have been the subject of several research papers since they have been initially proposed by De Finetti (1957) for a binomial model. In the context of the classical risk model, we mention Bühlmann (1970) and Gerber (1979) who consider the problem of optimal dividend strategy in the classical compound Poisson risk model. More recently, Lin et al. (2003) carry an extensive study of the expected discounted penalty function in the context of the compound Poisson risk model with a constant dividend barrier strategy and Gerber and Shiu (2006) consider optimal dividend strategies. Within the same framework, Albrecher et al. (2005) consider a linear dividend barrier strategy while Lin and Pavlova (2006) and Lin and Sendova (2008) examine threshold dividend strategies. Li and Garrido (2004) find the expression for the expected discounted penalty function within a class of compound renewal risk models. Landriault (2008) consider a constant barrier strategy in the context of the compound Poisson risk model with dependence proposed by Boudreault et al. (2006).

In the classical risk model, the definition of the surplus process is based on the compound Poisson process which considers independent the interclaim times and the claim amounts (see e.g. Gerber (1979), Grandell (1991), and Rolski et al. (1999)). In different practical situations, this assumption may be too restrictive and more flexibility within the model is called for. In this paper, we consider the dependence structure defined with the generalized Farlie-Gumbel-Morgenstern (FGM) copula, as proposed in Cossette et al. (2008). Other types of dependence structures for the joint distribution of the interclaim time and the claim amount have previously been considered, notably in Albrecher and Boxma (2004) and Boudreault et al. (2006). Albrecher and Teugels (2006) consider a dependence structure for $(X, W)$ based on a copula.

The objective of the present paper is to study the expected discounted penalty function in an extension with dependence of the classical risk model and assuming a constant dividend barrier strategy. In Section 5.2, we present the risk model with a dependence structure based on the generalized Farlie-Gumbel-Morgenstern (FGM) copula including a constant dividend barrier strategy. In section 5.3, we derive the expected discounted penalty function with a constant dividend barrier as a solution to an integro-differential equation with boundary conditions. In section 5.4, the special case where the distribution of the claim amounts is exponential is considered in detail.

## 5.2 The risk model

For an insurance portfolio, the surplus process is $\underline{U} = \{U(t), t \geq 0\}$. The surplus level at time $t$, $U(t)$, is defined by

$$U(t) = u + pt - S(t),$$

where $U(0) = u$ is the initial surplus, $p$ is the premium rate, and $\underline{S} = \{S(t), \ t \in \mathbb{R}^{+}\}$ is the aggregate claim amount process. In the classical risk model, $\underline{S}$ is a compound Poisson process with $S(t) = \sum_{j=1}^{N(t)} X_j$ ($\sum_{a}^{b}$ equals 0 if $b < a$) where $\underline{N} = \{N(t), \ t \in \mathbb{R}^{+}\}$ is a Poisson process with parameter $\lambda$. The random variable (r.v.) $X_j$ $(j = 1, 2, ...)$ corresponds to the amount of the $j$th claim. The time between the $(i-1)$th and the $i$th claim $(i = 1, 2, ...)$ is defined by the r.v. $W_i$ (with $W_0 = 0$). The claim amounts $\{X_j, j \in \mathbb{N}^+\}$ form a sequence of i.i.d. r.v.'s distributed as the r.v. $X$ with probability density function (p.d.f.) $f_X$, cumulative distribution function (c.d.f.) $F_X$ and Laplace transform (L.T.) $f_X^*$. The interclaim times $\{W_j, j \in \mathbb{N}^+\}$ form a sequence of independent r.v.'s identically distributed as the canonical r.v. $W$. The r.v. $W$ has an exponential distribution with expectation $\frac{1}{\lambda}$ with p.d.f. $f_W$, c.d.f. $F_W$ and L.T. $f_W^*$ with

$$f_W(t) = \lambda e^{-\lambda t}, \tag{5.1}$$

$$F_W(t) = 1 - e^{-\lambda t}, \tag{5.2}$$

$$f_W^*(s) = E\left[e^{-sW}\right] = \frac{\lambda}{\lambda + s}.$$

In Cossette et al. (2008), $\{(X_j, W_j), j \in \mathbb{N}^+\}$ form a sequence of i.i.d. random vectors distributed as the canonical random vector $(X, W)$, in which the components may be dependent. The joint p.d.f. of $(X, W)$ is denoted by $f_{X,W}(x, t)$ with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$. The associated bivariate L.T. is given by

$$f_{X,W}^*(s_1, s_2) = E\left[e^{-s_1 X} e^{-s_2 W}\right] = \int_0^\infty \int_0^\infty e^{-s_1 x} e^{-s_2 t} f_{X,W}(x, t)\, dx dt. \tag{5.3}$$

The joint distribution of $(X, W)$ is based on the generalized Farlie-Gumbel-Morgenstern copula. Proposed by Rodríguez-Lallena and Úbeda-Flores (2004), this copula is defined by

$$C(u, v) = uv + \theta h(u)\, g(v), \tag{5.4}$$

for every $(u, v)$ in $[0,1]^2$. Conditions and examples for $h(u)$ and $g(v)$ are also presented in their paper. In the present paper, we consider the special case where $h(u) = u^a(1-u)^b$ and $g(u) = u^c(1-u)^d$ with $a$, $b$, $c \in \{2, 3, ...\}$, and $d > 1$, which means that

$$C(u, v) = uv + \theta u^a(1-u)^b v^c(1-v)^d. \tag{5.5}$$

The lower and upper bounds for the dependence parameter $\theta$ such that (5.5) is a proper copula depend on the values of $a$, $b$, $c$, $d$ (see Rodríguez-Lallena and Úbeda-Flores (2004)). For instance, if $a = b = c = d = 2$, then $-27 \leq \theta \leq 27$. The p.d.f. associated to (5.4) is given by

$$c(u, v) = 1 + \theta h'(u) g'(v). \tag{5.6}$$

The joint c.d.f. $F_{X,W}$ is defined by

$$\begin{aligned} F_{X,W}(x,t) &:= C(F_X(x), F_W(t)) \\ &= F_X(x)F_W(t) + \theta F_X(x)^a(1-F_X(x))^b F_W(t)^c(1-F_W(t))^d. \end{aligned}$$

With (5.6), the joint p.d.f. $f_{X,W}$ of $(X, W)$ is given by

$$\begin{aligned} f_{X,W}(x,t) &= c(F_X(x), F_W(t)) f_X(x) f_W(t) \\ &= f_X(x)f_W(t) + \theta h'((F_X(x))g'((F_W(t))f_X(x)f_W(t). \end{aligned}$$

To simplify the presentation, let

$$k_X(x) = f_X(t)h'(F_X(x)), \tag{5.7}$$

and

$$k_W(t) = f_W(t)g'(F_W(t)). \tag{5.8}$$

With this notation, the p.d.f. of $(X, W)$ can be written as

$$f_{X,W}(x,t) = f_X(x) f_W(t) + \theta k_X(x)\, k_W(t)\,. \tag{5.9}$$

The associated L.T. of $k_X(x)$ and $k_W(t)$ are denoted $k_X^*(s)$ and $k_W^*(s)$, respectively.

In the following lemma, we provide analytic expressions for $k_W(t)$ and $k_W^*(s)$.

**Lemma 28.** *The function $k_W(t)$ defined in (5.8) can be expressed as*

$$k_W(t) = \sum_{i=1}^{c+1} a_i e^{-\lambda_i t}$$

*and its associate L.T. is given by*

$$\begin{aligned} k_W^*(s) &= \frac{c!\lambda^c s}{\prod_{i=1}^{c+1}(s+\lambda_i)} && (5.10)\\ &= \sum_{i=1}^{c+1} \frac{a_i}{s+\lambda_i}, && (5.11)\end{aligned}$$

*where*

$$\lambda_i = \lambda(d+i-1), \tag{5.12}$$

$$a_i = \frac{c!\lambda^c(-\lambda_i)}{\prod_{j=1, j\neq i}^{c+1}(-\lambda_i+\lambda_j)} = \frac{c!\lambda^c(-1)^{c-1}\lambda_i}{\tau_{c+1}'(\lambda_i)}, \tag{5.13}$$

*and*

$$\tau_{c+1}(s) = \prod_{j=1}^{c+1}(s-\lambda_j). \tag{5.14}$$

**Proof.** See Cossette et al. (2008). □

Note that both $k_X(x)$ and $k_W(t)$ are not proper p.d.f. and that, given the Initial Value Theorem, we have

$$\sum_{i=1}^{c+1} a_i = k_W(0) = \lim_{s\to\infty} s k_W^*(s) = \lim_{s\to\infty} s\frac{c!\lambda^c s}{\prod_{i=1}^{c+1}(s+\lambda_i)} = 0 \tag{5.15}$$

for $c > 1$. This result can also be obtained with Lemma 1 of Li and Garrido (2004) with $s_0 = 0$ and $m = 1$.

Let $\underline{U}_{\tilde{b}} = \left\{U_{\tilde{b}}(t),\ t \geq 0\right\}$ be the surplus process under the constant barrier $\tilde{b}$ where $U_{\tilde{b}}(t)$ is the amount of the surplus at time $t$ and $U_{\tilde{b}}(0) = u$ is the initial surplus. Under a constant barrier strategy, there is a horizontal barrier of level $\tilde{b} \geq u$ such that when the surplus reaches $\tilde{b}$, dividends are paid continuously to shareholders at the premium rate until a new claim occurs. The surplus $U_{\tilde{b}}(t)$ can never therefore exceed $\tilde{b}$. The dynamic of the surplus process $\underline{U}_{\tilde{b}}$ is defined as

$$dU_{\tilde{b}}(t) = \begin{cases} pdt - dS(t), & U_{\tilde{b}}(t) < \tilde{b}, \\ -dS(t), & U_{\tilde{b}}(t) = \tilde{b}. \end{cases} \tag{5.16}$$

Let $T_{\tilde{b}} = \inf\left\{t > 0, U_{\tilde{b}}(t) < 0\right\}$ be the first time the surplus process (5.16) becomes negative. The expected discounted penalty function with a constant dividend barrier $\tilde{b}$ is given by

$$m_{\delta,\tilde{b}}(u) = E\left[e^{-\delta T_{\tilde{b}}} w\left(U(T_{\tilde{b}}^{-}), |U(T_{\tilde{b}})|\right) I(T_{\tilde{b}} < \infty) \middle| U_{\tilde{b}}(0) = u\right], \quad 0 \leq u \leq \tilde{b}, \tag{5.17}$$

where $w(x, y)$, for $x, y \geq 0$, is the penalty function at the time of ruin for the surplus prior to ruin and the deficit at ruin, $\delta \geq 0$ is the force of interest and $I$ is the indicator function such that $I(A) = 1$ if the event $A$ occurs and equals 0 otherwise. In particular, if $w(x, y) = 1$,

$$\phi_{T_{\tilde{b}}}(u) = E\left[e^{-\delta T_{\tilde{b}}} I(T_{\tilde{b}} < \infty) | U_{\tilde{b}}(0) = u\right], \quad 0 \leq u \leq \tilde{b} \tag{5.18}$$

is the L.T. of the time of ruin, $T_{\tilde{b}}$, with respect to $\delta$.

## 5.3 Expected discounted penalty function with a constant dividend barrier

The expected discounted penalty function with a constant dividend barrier, $m_{\delta,\bar{b}}(u)$, is obtained in three main steps. In the first step, we derive an integro-differential equation for $m_{\delta,\bar{b}}(u)$. In the second step, we identify the boundary conditions for this integro-differential equation and, finally, we find a solution to the integro-differential equation which provides the expression for $m_{\delta,\bar{b}}(u)$.

### 5.3.1 Integro-differential equation for $m_{\delta,\bar{b}}(u)$

The derivation of the integro-differential equation for $m_{\delta,\bar{b}}(u)$ is based on a renewal-type argument. We condition on the first occurrence of a claim $W_1 = t$ and its amount $X_1 = x$. There are 2 main scenarios : if the first claim occurs before the surplus reaches the constant dividend barrier, then no dividends are paid ; otherwise, dividends are distributed between the time $\frac{\bar{b}-u}{p}$ when the surplus reaches the barrier and the time of occurrence $t$ of the first claim. In the first scenario, the first claim is either greater or lower than $u + pt$ which means that the surplus becomes negative or remains positive and ruin occurs or not. Similarly, in the second scenario, the first claim is greater or lower than $\bar{b}$ which leads to ruin or no ruin. When ruin does not occur, the renewal argument can be applied in both scenarios, which means that the surplus process starts anew.

In summary, $m_{\delta,\tilde{b}}(u)$ can be expressed as follows :

$$\begin{aligned} m_{\delta,\tilde{b}}(u) &= \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} m_{\delta,\tilde{b}}(u+pt-x) f_{X,W}(x,t)\, dxdt \\ &+ \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} m_{\delta,\tilde{b}}(\tilde{b}-x) f_{X,W}(x,t)\, dxdt \\ &+ \int_0^{\frac{\tilde{b}-u}{p}} \int_{u+pt}^{\infty} e^{-\delta t} w(u+pt, x-u-pt) f_{X,W}(x,t)\, dxdt \\ &+ \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta t} w(\tilde{b}, x-\tilde{b}) f_{X,W}(x,t)\, dxdt. \end{aligned} \tag{5.19}$$

With the p.d.f. (5.9) of $(X, W)$, (5.19) becomes

$$\begin{aligned} m_{\delta,\tilde{b}}(u) &= \lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} m_{\delta,\tilde{b}}(u+pt-x) f_X(x) e^{-\lambda t} dxdt \\ &+ \lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} m_{\delta,\tilde{b}}(\tilde{b}-x) f_X(x) e^{-\lambda t} dxdt \\ &+ \lambda \int_0^{\frac{\tilde{b}-u}{p}} \int_{u+pt}^{\infty} e^{-\delta t} w(u+pt, x-u-pt) f_X(x) e^{-\lambda t} dxdt \\ &+ \lambda \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta t} w(\tilde{b}, x-\tilde{b}) f_X(x) e^{-\lambda t} dxdt \\ &+ \theta \int_0^{\frac{\tilde{b}-u}{p}} \int_0^{u+pt} e^{-\delta t} m_{\delta,\tilde{b}}(u+pt-x) k_X(x) k_W(t) dxdt \\ &+ \theta \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_0^{\tilde{b}} e^{-\delta t} m_{\delta,\tilde{b}}(\tilde{b}-x) k_X(x) k_W(t) dxdt \\ &+ \theta \int_0^{\frac{\tilde{b}-u}{p}} \int_{u+pt}^{\infty} e^{-\delta t} w(u+pt, x-u-pt) k_X(x) k_W(t) dxdt \\ &+ \theta \int_{\frac{\tilde{b}-u}{p}}^{\infty} \int_{\tilde{b}}^{\infty} e^{-\delta t} w(\tilde{b}, x-\tilde{b}) k_X(x) k_W(t) dxdt. \end{aligned} \tag{5.20}$$

For simplification purposes, we define the following functions that will be used for the remainder of the paper :

$$\sigma_1(z) = \int_0^z m_{\delta,\tilde{b}}(z-x) f_X(x) dx + w_1(z) \tag{5.21}$$

and

$$\sigma_2(z) = \int_0^z m_{\delta,\tilde{b}}(z-x)k_X(x)dx + w_2(z), \tag{5.22}$$

where

$$\begin{aligned} w_1(z) &= \int_z^\infty w(z,x-z)f_X(x)dx \\ w_2(z) &= \int_z^\infty w(z,x-z)k_X(x)dx. \end{aligned}$$

Given (5.21) and (5.22), (5.20) becomes

$$\begin{aligned} m_{\delta,\tilde{b}}(u) &= \int_0^{\frac{\tilde{b}-u}{p}} e^{-\delta t}\lambda e^{-\lambda t}\sigma_1\left(u+pt\right)dt + \int_{\frac{\tilde{b}-u}{p}}^\infty e^{-\delta t}\lambda e^{-\lambda t}\sigma_1(\tilde{b})dt \\ &\quad +\theta\int_0^{\frac{\tilde{b}-u}{p}} e^{-\delta t}k_W\left(t\right)\sigma_2\left(u+pt\right)dxdt + \theta\int_{\frac{\tilde{b}-u}{p}}^\infty e^{-\delta t}k_W\left(t\right)\sigma_2(\tilde{b})dt. \end{aligned} \tag{5.23}$$

Letting $s = u + pt$ in (5.23) and using Lemma 28, we get

$$\begin{aligned} pm_{\delta,\tilde{b}}(u) &= \lambda\int_u^{\tilde{b}} e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s)ds + \lambda\int_{\tilde{b}}^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(\tilde{b})ds \\ &\quad +\theta\sum_{i=1}^{c+1}a_i\int_u^{\tilde{b}} e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s)ds + \theta\sum_{i=1}^{c+1}a_i\int_{\tilde{b}}^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(\tilde{b})ds, \end{aligned}$$

which can also be written as

$$m_{\delta,\tilde{b}}(u) = \frac{\lambda}{p}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds + \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds, \tag{5.24}$$

where $s_1 \wedge s_2 = \min(s_1; s_2)$. In order to derive an integro-differential equation for $m_{\delta,\tilde{b}}(u)$, we need the following lemma.

**Lemma 29.** *For $\lambda_i$ $(i = 1, \cdots, c+1)$ defined by (5.12), we have*

$$\sum_{i=1}^{c+1}\lambda_i(\lambda-\lambda_i)\frac{\prod_{j=1}^{l}(\lambda_j-\lambda_i)}{\tau'_{c+1}(\lambda_i)} = \begin{cases} 0, & l = 0,\cdots,c-3, \\ (-1)^{c-1}, & l = c-2, \\ (-1)^{c-1}(\lambda-\lambda_c-\lambda_{c+1}), & l = c-1, \\ (-1)^{c-1}\lambda_{c+1}(\lambda_{c+1}-\lambda), & l = c. \end{cases}$$

**Proof.** Since $\lambda_i(\lambda - \lambda_i)\prod_{j=1}^{l}(\lambda_j - \lambda_i)$ is a polynomial of degree $(l+2)$ in $\lambda_i$, we have

$$\sum_{i=1}^{c+1} \lambda_i(\lambda - \lambda_i)\frac{\prod_{j=1}^{l}(\lambda_j - \lambda_i)}{\tau'_{c+1}(\lambda_i)} = \sum_{i=1}^{c+1} \frac{\sum_{k=0}^{l+2} \varrho_k \lambda_i^{l+2-k}}{\tau'_{c+1}(\lambda_i)},$$

with $\varrho_0 = (-1)^{l+1}$.

This leads to

$$\sum_{i=1}^{c+1} \lambda_i(\lambda - \lambda_i)\frac{\prod_{j=1}^{l}(\lambda_j - \lambda_i)}{\tau'_{c+1}(\lambda_i)} = \sum_{k=0}^{l+2} \varrho_k \sum_{i=1}^{c+1} \frac{\lambda_i^{l+2-k}}{\tau'_{c+1}(\lambda_i)}. \tag{5.25}$$

Based on Lemma 1 on page 395 (and also on the relations on page 400) of Li and Garrido (2004), the terms summed in (5.25) are equal to zero for $l = 0, \cdots, c-3$. For $l = c-2$, we have

$$\sum_{i=1}^{c+1} \lambda_i(\lambda - \lambda_i)\frac{\prod_{j=1}^{c-2}(\lambda_j - \lambda_i)}{\tau'_{c+1}(\lambda_i)} = \sum_{k=0}^{c} \varrho_k \sum_{i=1}^{c+1} \frac{\lambda_i^{c-k}}{\tau'_{c+1}(\lambda_i)} = \varrho_0 = (-1)^{c-1}.$$

If $l = c-1$, we obtain

$$\begin{aligned}
\sum_{i=1}^{c+1} \lambda_i(\lambda - \lambda_i)\frac{\prod_{j=1}^{c-1}(\lambda_j - \lambda_i)}{\tau'_{c+1}(\lambda_i)} &= \lambda_c(\lambda - \lambda_c)\frac{\prod_{j=1}^{c-1}(\lambda_j - \lambda_c)}{\tau'_{c+1}(\lambda_c)} + \lambda_{c+1}(\lambda - \lambda_{c+1})\frac{\prod_{j=1}^{c-1}(\lambda_j - \lambda_{c+1})}{\tau'_{c+1}(\lambda_{c+1})} \\
&= (-1)^{c-1}\left(\frac{\lambda_{c+1}(\lambda - \lambda_{c+1}) - \lambda_c(\lambda - \lambda_c)}{\lambda_{c+1} - \lambda_c}\right) \\
&= (-1)^{c-1}(\lambda - \lambda_c - \lambda_{c+1}).
\end{aligned}$$

Finally, if $l = c$, we have

$$\begin{aligned}
\sum_{i=1}^{c+1} \lambda_i(\lambda - \lambda_i)\frac{\prod_{j=1}^{c}(\lambda_j - \lambda_i)}{\tau'_{c+1}(\lambda_i)} &= \lambda_{c+1}(\lambda - \lambda_{c+1})\frac{\prod_{j=1}^{c}(\lambda_j - \lambda_{c+1})}{\tau'_{c+1}(\lambda_{c+1})} \\
&= \lambda_{c+1}(\lambda - \lambda_{c+1})(-1)^c
\end{aligned}$$

which completes the proof. □

Let us now derive an integro-differential equation for the expected discounted penalty function with a constant dividend barrier $\tilde{b}$, $m_{\delta,\tilde{b}}(u)$.

**Theorem 30.** *The expected discounted penalty function with a constant dividend barrier $\tilde{b}$, denoted $m_{\delta,\tilde{b}}(u)$, satisfies the following integro-differential equation for $0 \leq u \leq \tilde{b}$,*

$$\begin{aligned}\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_{\delta,\tilde{b}}(u) &= \frac{\lambda}{p}\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_1(u) \\ &+ \frac{\theta c!\lambda^c}{p^c}\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u), \qquad (5.26)\end{aligned}$$

*where $\mathcal{I}$ and $\mathcal{D}$ are the identity and differential operators.*

**Proof.** First, we take the derivative on both sides of (5.24) w.r.t. $u$ yielding

$$\begin{aligned}m'_{\delta,\tilde{b}}(u) &= -\frac{\lambda}{p}\sigma_1(u)+\frac{\lambda}{p}\Big(\frac{\lambda+\delta}{p}\Big)\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\ &+\frac{\theta}{p}c!\lambda^c(-1)^{c-1}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}\Big(\frac{\lambda_i+\delta}{p}\Big)\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds, \qquad (5.27)\end{aligned}$$

for $0 \leq u \leq \tilde{b}$, since $\sum_{i=1}^{c+1} a_i = 0$. Then, we multiply (5.24) by $\frac{\lambda+\delta}{p}$, to which we subtract (5.27) and obtain

$$\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_{\delta,\tilde{b}}(u) = \frac{\lambda}{p}\sigma_1(u)+\frac{\theta}{p}g_{\delta,\tilde{b}}(u), \qquad (5.28)$$

where

$$g_{\delta,\tilde{b}}(u) = c!\lambda^c(-1)^{c-1}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}\Big(\frac{\lambda-\lambda_i}{p}\Big)\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds. \qquad (5.29)$$

In order to apply the operator $\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)$ to $\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_{\delta,\tilde{b}}(u)$, we first take the derivative w.r.t. $u$ on both sides of (5.29). Given Lemma 28 with $l=0$, we find

$$g'_{\delta,\tilde{b}}(u) = (-1)^{c-1}\frac{c!\lambda^c}{p}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}(\lambda-\lambda_i)\frac{(\delta+\lambda_i)}{p}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds, \qquad (5.30)$$

which allows us to derive the following result :

$$\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_{\delta,\tilde{b}}(u)=(-1)^{c-1}\frac{c!\lambda^c}{p^2}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}(\lambda-\lambda_i)(\lambda_j-\lambda_i)\int_u^{\infty}e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds.$$

Due to Lemma 29, we must apply the operator $\prod_{j=1}^{c+1}(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D})$ successively. First, we apply the operator $\prod_{j=1}^{c-2}(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D})$ to both sides of (5.30) with Lemma 28 for $l=0,\cdots,c-3$ to obtain

$$\prod_{j=1}^{c-2}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_{\delta,\tilde{b}}(u)=(-1)^{c-1}\frac{c!\lambda^c}{p^{c-1}}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}(\lambda-\lambda_i)\prod_{j=1}^{c-2}(\lambda_j-\lambda_i)\int_u^{\infty}e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds. \tag{5.31}$$

We repeat the operation by applying the operator $\Big(\frac{\lambda_{c-1}+\delta}{p}\mathcal{I}-\mathcal{D}\Big)$ to both sides of (5.31) and, using Lemma 28 for $l=c-2$, we find

$$\begin{aligned}\prod_{j=1}^{c-1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_{\delta,\tilde{b}}(u) &= (-1)^{c-1}\frac{c!\lambda^c}{p^{c}}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}(\lambda-\lambda_i)\prod_{j=1}^{c-1}(\lambda_j-\lambda_i)\int_u^{\infty}e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds\\ &\quad+\frac{c!\lambda^c}{p^{c-1}}\sigma_2(u).\end{aligned} \tag{5.32}$$

With Lemma 28 for $l=c-1$, we apply the operator $\Big(\frac{\lambda_c+\delta}{p}\mathcal{I}-\mathcal{D}\Big)$ on both sides of (5.32) which leads to

$$\begin{aligned}\prod_{j=1}^{c}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_{\delta,\tilde{b}}(u) &= (-1)^{c-1}\frac{c!\lambda^c}{p^{c+1}}\sum_{i=1}^{c+1}\frac{\lambda_i}{\tau'_{c+1}(\lambda_i)}(\lambda-\lambda_i)\prod_{j=1}^{c}(\lambda_j-\lambda_i)\int_u^{\infty}e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds\\ &\quad+\frac{c!\lambda^c}{p^{c}}(\lambda-\lambda_c-\lambda_{c+1})\sigma_2(u)+\frac{c!\lambda^c}{p^{c-1}}\Big(\frac{\lambda_c+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u).\end{aligned} \tag{5.33}$$

Finally, the operator $\Big(\frac{\lambda_{c+1}+\delta}{p}\mathcal{I}-\mathcal{D}\Big)$ is applied to both sides of (5.33) with

Lemma 28 for $l = c$ :

$$\begin{aligned}\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_{\delta,\bar{b}}(u) &= \frac{c!\lambda^c}{p^c}(\lambda-\lambda_c-\lambda_{c+1})\Big(\frac{\lambda_{c+1}+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u)\\ &+ \frac{c!\lambda^c}{p^{c-1}}\Big(\frac{\lambda_{c+1}+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda_c+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u)\\ &+ \frac{c!\lambda^c}{p^{c+1}}\lambda_{c+1}(\lambda_{c+1}-\lambda)\sigma_2(u).\end{aligned}$$

which is equivalent to

$$\begin{aligned}\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)g_{\delta,\bar{b}}(u) &= \frac{c!\lambda^c}{p^{c+1}}\delta(\lambda+\delta)\sigma_2(u)-\frac{c!\lambda^c}{p^c}(2\delta+\lambda)\sigma_2'(u)+\frac{c!\lambda^c}{p^{c-1}}\sigma_2''(u)\\ &= \frac{c!\lambda^c}{p^{c-1}}\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\sigma_2(u). \qquad (5.34)\end{aligned}$$

Finally, applying the operator $\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)$ to both sides of (5.28) and using (5.34) yields (5.26). □

### 5.3.2 Boundary conditions

The next step in the derivation of an expression for $m_{\delta,\bar{b}}(u)$ is to find the boundary conditions to the integro-differential equation (5.26) for $m_{\delta,\bar{b}}(u)$ given in Theorem 30. Before stating these conditions, we need the two following lemmas.

**Lemma 31.** *For $\lambda_i$ and $a_i$ $(i = 1, \cdots, c+1)$ defined in (5.12) and (5.13), we have*

$$\sum_{i=1}^{c+1} a_i(\lambda_i+\delta)^l = \begin{cases} 0, & l = 0, \cdots, c-2, \\ (-1)^{c-1}c!\lambda^c, & l = c-1, \\ (-1)^{c-1}c!\lambda^c(c\delta+\sum_{i=1}^{c+1}\lambda_i), & l = c. \end{cases}$$

**Proof.** Define

$$k_\delta(t) = \sum_{i=1}^{c+1} a_i e^{-(\lambda_i+\delta)t} = e^{-\delta t} k_W(t)$$

and its associated L.T. $k_\delta^*(s) = k_W^*(s+\delta)$. By the Initial Value Theorem, we have

$$k_\delta(0) = k_W(0) = \sum_{i=1}^{c+1} a_i = \lim_{s\to\infty} s k_W^\star(s) = 0.$$

With the Initial Value Theorem and (5.10), the $l$th derivative of $k_\delta(t)$, denoted by $k_\delta^{(l)}(t)$ and for $l = 0, 1, 2, ..., c$ are the following when evaluated at $t = 0$ :

$$k_\delta^{(l)}(t)\Big|_{t=0} = \sum_{i=1}^{c+1} a_i(-1)^l(\lambda_i+\delta)^l = \lim_{s\to\infty} s^{l+1} k_\delta^\star(s) = \lim_{s\to\infty} s^{l+1} k_W^\star(s+\delta) = 0,$$

for $l = 0, 1, 2, ..., c-2$;

$$k_\delta^{(c-1)}(t)\Big|_{t=0} = \sum_{i=1}^{c+1} a_i(-1)^{c-1}(\lambda_i+\delta)^{c-1} = \lim_{s\to\infty} s^c \frac{c!\lambda^c(s+\delta)}{\prod_{i=1}^{c+1}(s+\delta+\lambda_i)} = c!\lambda^c,$$

and

$$\begin{aligned} k_\delta^{(c)}(t)\Big|_{t=0} &= \sum_{i=1}^{c+1} a_i(-1)^c(\lambda_i+\delta)^c \\ &= \lim_{s\to\infty} s(s^c k_W^\star(s+\delta) - c!\lambda^c) \\ &= \lim_{s\to\infty} s(s^c \frac{c!\lambda^c(s+\delta)}{\prod_{i=1}^{c+1}(s+\delta+\lambda_i)} - c!\lambda^c) \\ &= -c!\lambda^c(c\delta + \sum_{i=1}^{c+1}\lambda_i). \end{aligned}$$

□

**Lemma 32.** *For $u \le \tilde{b}$, the derivatives of order $n = 1, 2, ..., c+1$ of $m_{\delta,\tilde{b}}(u)$*

*are as follows :*

$$
\begin{aligned}
m_{\delta,\tilde{b}}^{(n)}(u) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{n-1}\left(\frac{\lambda+\delta}{p}\right)^{n-j-1}\sigma_1^{(j)}(u)\right)+\frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{n}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\
&+ \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{n}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds, \quad n=1,\cdots,c-1; \qquad (5.35)\\
m_{\delta,\tilde{b}}^{(c)}(u) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{c-1}\left(\frac{\lambda+\delta}{p}\right)^{c-j-1}\sigma_1^{(j)}(u)\right)+\frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{c}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\
&+ \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{c}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds-\frac{\theta}{p^c}(-1)^{c-1}c!\lambda^c\sigma_2(u); \qquad (5.36)\\
m_{\delta,\tilde{b}}^{(c+1)}(u) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{c}\left(\frac{\lambda+\delta}{p}\right)^{c-j}\sigma_1^{(j)}(u)\right)+\frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{c+1}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\
&+ \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{c+1}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds-\frac{\theta}{p^c}(-1)^{c-1}c!\lambda^c\sigma_2'(u)\\
&- \frac{\theta}{p^{c+1}}(-1)^{c-1}c!\lambda^c\left(c\delta+\sum_{j=1}^{c+1}\lambda_j\right)\sigma_2(u). \qquad (5.37)
\end{aligned}
$$

**Proof.** We use induction to prove the desired result. The case $n=1$ reduces to (5.27). We assume that (5.35) holds for $n=1,\cdots,c-2$. Taking the derivative of (5.35) w.r.t. $u$, we obtain

$$
\begin{aligned}
m_{\delta,\tilde{b}}^{(n+1)}(u) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{n-1}\left(\frac{\lambda+\delta}{p}\right)^{n-j-1}\sigma_1^{(j+1)}(u)\right)+\frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{n+1}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\
&+ \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{n+1}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds\\
&- \frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{n}\sigma_1(u)-\frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{n}\sigma_2(u).
\end{aligned}
$$

Using Lemma 31 with $l=0,\cdots,c-2$ and with simple rearrangements, we obtain

$$
\begin{aligned}
m_{\delta,\tilde{b}}^{(n+1)}(u) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{n}\left(\frac{\lambda+\delta}{p}\right)^{n-j}\sigma_1^{(j)}(u)\right)+\frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{n+1}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\
&+ \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{n+1}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds
\end{aligned}
$$

which corresponds to (5.35).

The derivative of (5.35) with $n=c-1$ w.r.t. $u$ is

$$
\begin{aligned}
m_{\delta,\tilde{b}}^{(c)}(u) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{c-1}\left(\frac{\lambda+\delta}{p}\right)^{c-j-1}\sigma_1^{(j)}(u)\right)+\frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{c}\int_u^\infty e^{-(\delta+\lambda)(\frac{s-u}{p})}\sigma_1(s\wedge\tilde{b})ds \\
&+ \frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{c}\int_u^\infty e^{-(\delta+\lambda_i)(\frac{s-u}{p})}\sigma_2(s\wedge\tilde{b})ds-\frac{\theta}{p}\sum_{i=1}^{c+1}a_i\left(\frac{\lambda_i+\delta}{p}\right)^{c-1}\sigma_2(u)
\end{aligned}
$$

which leads to (5.36) with $l=c-1$ in Lemma 31.

Similarly, we find (5.37) by differentiating (5.36) and using Lemma 31 with $l=c$. □

The boundary conditions to the integro-differential equation (5.26) for $m_{\delta,\tilde{b}}(u)$ are provided in the next theorem.

**Theorem 33.** *The integro-differential equation (5.26) for $m_{\delta,\tilde{b}}(u)$ satisfies the following $c+2$ boundary conditions :*

$$
\begin{aligned}
m_{\delta,\tilde{b}}(\tilde{b}) &= \frac{\lambda}{\delta+\lambda}\sigma_1(\tilde{b})+\theta\frac{c!\lambda^c\delta}{\prod_{i=1}^{c+1}(\delta+\lambda_i)}\sigma_2(\tilde{b}); && (5.38)\\
m_{\delta,\tilde{b}}^{(1)}(\tilde{b}) &= 0; && (5.39)\\
m_{\delta,\tilde{b}}^{(n)}(\tilde{b}) &= -\frac{\lambda}{p}\left(\sum_{j=1}^{n-1}\left(\frac{\lambda+\delta}{p}\right)^{n-j-1}\sigma_1^{(j)}(\tilde{b})\right),\quad n=2,\cdots,c; && (5.40)\\
m_{\delta,\tilde{b}}^{(c+1)}(\tilde{b}) &= -\frac{\lambda}{p}\left(\sum_{j=1}^{c}\left(\frac{\lambda+\delta}{p}\right)^{c-j}\sigma_1^{(j)}(\tilde{b})\right)-\frac{\theta}{p^c}(-1)^{c-1}c!\lambda^c\sigma_2'(\tilde{b}). && (5.41)
\end{aligned}
$$

**Proof.** Setting $u = \tilde{b}$ in (5.24) yields

$$m_{\delta,\tilde{b}}(\tilde{b}) = \frac{\lambda}{\delta+\lambda}\sigma_1(\tilde{b}) + \theta \sum_{i=1}^{c+1} \frac{a_i}{\delta+\lambda_i}\sigma_2(\tilde{b}).$$

With $s = \delta$ in (5.11) and (5.10), we obtain (5.38). The second condition is derived by letting $u = \tilde{b}$ in (5.27) and with $l = 0$ in Lemma 31. For the third condition, we let $u = \tilde{b}$ in (5.35) of Lemma 32 to find

$$\begin{aligned} m^{(n)}_{\delta,\tilde{b}}(\tilde{b}) &= -\frac{\lambda}{p}\left(\sum_{j=0}^{n-1}\left(\frac{\lambda+\delta}{p}\right)^{n-j-1}\sigma_1^{(j)}(\tilde{b})\right) + \frac{\lambda}{p}\left(\frac{\lambda+\delta}{p}\right)^{n-1}\sigma_1(\tilde{b}) \\ &+ \frac{\theta}{p}\sum_{i=1}^{c+1} a_i\left(\frac{\lambda_i+\delta}{p}\right)^{n-1}\sigma_2(\tilde{b}), \qquad n = 1,\cdots,c-1 \end{aligned}$$

which combined with $l = 0,\cdots,c-2$ in Lemma 31 leads to

$$m^{(n)}_{\delta,\tilde{b}}(\tilde{b}) = -\frac{\lambda}{p}\left(\sum_{j=1}^{n-1}\left(\frac{\lambda+\delta}{p}\right)^{n-j-1}\sigma_1^{(j)}(\tilde{b})\right) \qquad n = 2,\cdots,c-1.$$

When $n = c$, we use (5.36) with $u = \tilde{b}$ and Lemma 31 with $l = c-1$. Similarly, (5.41) is obtained with $u = \tilde{b}$ in (5.37) and Lemma 31 with $l = c$. □

### 5.3.3 A general solution

In the final step, we determine an expression for the expected discounted penalty function with a constant dividend barrier, $m_{\delta,\tilde{b}}(u)$.

**Theorem 34.** *A closed form expression for $m_{\delta,\tilde{b}}(u)$ is given by*

$$m_{\delta,\tilde{b}}(u) = m_{\delta,\infty}(u) + \sum_{l=1}^{c+2}\eta_l v_{l,\delta}(u), \quad 0 \le u \le \tilde{b}, \tag{5.42}$$

*where $m_{\delta,\infty}(u)$ corresponds to the expected discounted penalty function without a dividend barrier as defined in Cossette et al. (2008). The constants $\eta_1,\cdots\eta_{c+2}$*

*are such that the system of linear equations in Theorem 33 holds. The L.T. of $v_{i,\delta}(u)$ for i=1,2,...,c+2 are provided in (5.60).*

**Proof.** See the appendix. □

**Remark 35.** *In the proof of Theorem 34 , which has been moved to the appendix, the denominator of (5.58), given by*

$$\Big(\frac{\delta+\lambda}{p}-s\Big)\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p}-s\Big)-\frac{\lambda}{p}\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p}-s\Big)f_X^{\star}(s)-\theta\frac{c!\lambda^c}{p^c}\Big(\frac{\delta+\lambda}{p}-s\Big)\Big(\frac{\delta}{p}-s\Big)k_X^{\star}(s),$$

*corresponds to Lundberg's generalized equation where $\rho_j, j=1,\cdots c+2$ are the roots with real positive parts. We refer the reader to Cossette et al. (2008) for additional details. If the $\rho_j, j=1,\cdots c+2$ are distinct, the term $\zeta_{c+1,\delta}(s)$ defined in the proof of Theorem 34 can be expressed as*

$$\zeta_{c+1,\delta}(s)=\sum_{j=1}^{c+2}\zeta_{c+1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{c+2}\frac{s-\rho_k}{\rho_j-\rho_k},$$

*by the Lagrange interpolating formula. This will be useful in the computation of $m_{\delta,\bar{b}}(u)$ as illustrated in Section 5.4.*

**Remark 36.** *The roots $\rho_j, j=1,\cdots c+2$ are also useful to derive the defective renewal equations for $m_{\delta,\infty}(u)$ (see in Cossette et al. (2008)) and for $v_{1,\delta}(u)$, ..., $v_{c+2,\delta}(u)$. See Willmot and Lin (2001) for applications and results on defective renewal equations.*

## 5.4 Explicit solution with exponentially distributed claim amounts

In this section, we consider the case when the individual claim sizes are exponentially distributed. More precisely, we derive an explicit expression for the L.T. of the time of ruin $T_{\tilde{b}}$ with respect to $\delta$, $\phi_{T_{\tilde{b}}}(u)$, defined in (5.18). One can interpret $\phi_{T_{\tilde{b}}}(u)$ as the expected discounted value of one dollar paid at ruin.

We suppose that the parameter $a$ of the generalized FGM copula is a strictly positive integer and that the individual claim amount r.v. $X$ follows an exponential distribution with $f_X(x) = \alpha e^{-\alpha x}$ and

$$f_X^\star(s) = \frac{\alpha}{s+\alpha}. \tag{5.43}$$

Given the assumption for the distribution of $X$ and the definitions of the functions $h$ (with $a \in \{1, 2, ...\}$) and $g$ (with $c \in \{2, 3, ...\}$) in the generalized FGM copula, $k_X(x)$, defined in (5.7), has the same form as $k_W(t)$. We can therefore use Lemma 28 to find the following L.T. associated to $k_X(x)$ :

$$k_X^\star(s) = \sum_{i=1}^{a+1} \frac{b_i}{s+\alpha_i} = \frac{a!\alpha^a s}{\prod_{i=1}^{a+1}(s+\alpha_i)}, \tag{5.44}$$

where $\alpha_i = \alpha(b+i-1)$ $(i = 1, \cdots, a+1)$ and

$$b_i = \frac{a!\alpha^a(-\alpha_i)}{\prod_{j=1, j\neq i}^{a+1}(-\alpha_i+\alpha_j)}.$$

We replace $f_X^\star(s)$ and $k_X^\star(s)$ in (5.60) by (5.43) and (5.44) and then we multiply the numerator and the denominator of (5.60) by $(\alpha+s)\prod_{j=1}^{a+1}(s+\alpha_j)$ to obtain

$$v^{\star}_{i,\delta}(s) = \frac{\chi_{1,i,a+c+3,\delta}(s)}{\chi_{2,a+c+4,\delta}(s)}, \qquad i = 1, \cdots, c+2, \tag{5.45}$$

where

$$\chi_{1,i,a+c+3,\delta}(s) = (\alpha + s) \left\{ \prod_{j=1}^{a+1} (s + \alpha_j) \right\} \zeta_{i,c+1,\delta}(s),$$

and

$$\begin{aligned}
\chi_{2,a+c+4,\delta}(s) \quad = \quad & (\alpha + s) \prod_{j=1}^{a+1} (s + \alpha_j) \Big( \frac{\delta + \lambda}{p} - s \Big) \prod_{i=1}^{c+1} \Big( \frac{\delta + \lambda_i}{p} - s \Big) \\
& - \frac{\lambda \alpha}{p} \prod_{j=1}^{a+1} (s + \alpha_j) \prod_{i=1}^{c+1} \Big( \frac{\delta + \lambda_i}{p} - s \Big) \\
& - \theta \frac{c! \lambda^c}{p^c} \Big( \frac{\delta + \lambda}{p} - s \Big) \Big( \frac{\delta}{p} - s \Big) a! \alpha^a s(\alpha + s).
\end{aligned}$$

Note that $\chi_{1,i,a+c+3,\delta}(s)$ and $\chi_{2,a+c+4,\delta}(s)$ are polynomials of degree $a+c+3$ and $a+c+4$ respectively.

The polynomial $\chi_{2,i,a+c+4,\delta}(s)$ has $c+2$ roots $\rho_1, \ldots, \rho_{c+2}$, with $\Re(\rho_j) > 0$ for $j = 1, \cdots, c+2$ (see Remark 35), and $a+2$ roots $-R_1, \ldots, -R_{a+2}$ with $\Re(R_j) > 0$, for $j = 1, \cdots, a+2$. Assuming the roots $\{\rho_j, j = 1, \ldots, c+2\}$ and the roots $\{-R_j(\delta), j = 1, \ldots, a+2\}$ distinct, we use the Lagrange interpolating polynomial formula both on the numerator and the denominator of (5.45) to obtain

$$\begin{aligned}
\chi_{1,i,a+c+3,\delta}(s) \quad = \quad & \sum_{j=1}^{a+2} \chi_{1,i,a+c+3,\delta}(-R_j) \prod_{k=1}^{c+2} \Big( \frac{s - \rho_k}{-R_j - \rho_k} \Big) \prod_{k=1, k \neq j}^{a+2} \Big( \frac{s + R_k}{-R_j + R_k} \Big) \\
+ \quad & \sum_{j=1}^{c+2} \chi_{1,i,a+c+3,\delta}(\rho_j) \prod_{k=1}^{a+2} \Big( \frac{s + R_k}{\rho_j + R_k} \Big) \prod_{k=1, k \neq j}^{c+2} \Big( \frac{s - \rho_k}{\rho_j - \rho_k} \Big)
\end{aligned} \tag{5.46}$$

and

$$\chi_{2,a+c+4,\delta}(s) = \chi_{2,a+c+4,\delta}(0) \prod_{j=1}^{c+2} \Big( \frac{s - \rho_j}{-\rho_j} \Big) \prod_{j=1}^{a+2} \Big( \frac{s + R_j}{R_j} \Big). \tag{5.47}$$

Combining (5.46) and (5.47) to (5.45), one gets

$$v_{i,\delta}^{\star}(s)=\sum_{j=1}^{a+2}\frac{\alpha_{1,i,j}}{s+R_j}+\sum_{j=1}^{c+2}\frac{\alpha_{2,i,j}}{s-\rho_j}, \tag{5.48}$$

for $i=1,\cdots,c+2$, where

$$\alpha_{1,i,j}=\frac{\chi_{1,i,a+c+3,\delta}(-R_j)}{\chi_{2,a+c+4,\delta}(0)}\frac{\prod_{j=1}^{c+2}(-\rho_j)}{\prod_{k=1}^{c+2}(-R_j-\rho_k)}\frac{\prod_{j=1}^{a+2}R_j}{\prod_{k=1,k\neq j}^{a+2}(-R_j+R_k)},$$

and

$$\alpha_{2,i,j}=\frac{\chi_{1,i,a+c+3,\delta}(\rho_j)}{\chi_{2,a+c+4,\delta}(0)}\frac{\prod_{j=1}^{c+2}(-\rho_j)}{\prod_{k=1}^{a+2}(\rho_j+R_k)}\frac{\prod_{j=1}^{a+2}R_j}{\prod_{k=1,k\neq j}^{c+2}(\rho_j-\rho_k)}.$$

Taking the inverse of the L.T. in (5.48), for $i=1,\cdots,c+2$, leads to

$$v_{i,\delta}(u)=\sum_{j=1}^{a+2}\alpha_{1,i,j}e^{-R_ju}+\sum_{j=1}^{c+2}\alpha_{2,i,j}e^{\rho_ju},\qquad i=1,\cdots,c+2. \tag{5.49}$$

The closed form expression for the L.T. of the time of ruin $T_\infty$ with no barrier, denoted $\phi_{T_\infty}(u)$, is given by

$$\phi_{T_\infty}(u)=\sum_{j=1}^{a+2}\varsigma_je^{-R_ju}, \tag{5.50}$$

where the details on the derivation of $\phi_{T_\infty}(u)$ and the definition of the $\sigma_j$'s are provided in Cossette et al. (2008). Letting (5.49) and (5.50) in (5.42), the expression for $\phi_{T_b}(u)$ is

$$\begin{aligned}\phi_{T_b}(u) &= \sum_{j=1}^{a+2}\varsigma_je^{-R_ju}+\sum_{i=1}^{c+2}\eta_i\sum_{j=1}^{a+2}\alpha_{1,i,j}e^{-R_ju}+\sum_{i=1}^{c+2}\eta_i\sum_{j=1}^{c+2}\alpha_{2,i,j}e^{\rho_ju}\\ &= \sum_{j=1}^{a+2}e^{-R_ju}\left(\varsigma_j+\sum_{i=1}^{c+2}\eta_i\alpha_{1,i,j}\right)+\sum_{j=1}^{c+2}e^{\rho_ju}\sum_{i=1}^{c+2}\eta_i\alpha_{2,i,j}.\end{aligned}$$

Given that the inclusion of a dividend barrier does not impact Lundberg's generalized equation, the roots $-R_1,...,-R_{a+2}$ are the same whether a barrier is assumed or not.

**Example 37.** *Let $X \sim Exp(1)$, $W \sim Exp(1)$. Also, let all the parameters of the generalized FGM copula be fixed at 2. The force of interest $\delta = 0.05$, the premium rate $p = 1.5$, and the constant dividend barrier $\tilde{b}$ is fixed at 6 and 8. In Table 1, we provide the analytic expressions for the L.T. of the time of ultimate ruin (or the expected discounted value of 1 paid at ruin) $\phi_{T_{\tilde{b}}}(u)$ for an initial surplus $u$ with different values of the parameter $\theta$ which are derived by Maple. The values taken by $\phi_{T_{\tilde{b}}}(u)$ are depicted in Figure 5.10. In the table, values have been rounded to 3 decimal places.*

| $\theta$ | *Expressions for $\phi_{T_{\tilde{b}}}(u)$ with a dividend barrier $\tilde{b} = 6$* |
|---|---|
| -20 | $0.565e^{-0.348u} + 0.021e^{-1.598u} - 0.038e^{-3.795u}cos(0.981u) - 0.056e^{-3.795u}sin(0.981u) + 0.161e^{0.085u}$ |
| -5 | $0.543e^{-0.376u} + 0.013e^{-1.836u} - 0.017e^{-3.606u}cos(0.365u) - 0.065e^{-3.606u}sin(0.365u) + 0.145e^{0.086u}$ |
| 0 | $0.543e^{-0.386u} + 0.125e^{0.086u}$ |
| 5 | $0.537e^{-0.396u} - 0.012e^{-4.230u} + 0.017e^{-2.359u}cos(0.280u) - 0.101e^{-2.359u}sin(0.280u) + 0.113e^{0.087u}$ |
| 20 | $0.511e^{-0.430u} - 0.026e^{-4.608u} + 0.050e^{-2.094u}cos(0.868u) - 0.117e^{-2.094u}sin(0.868u) + 0.076e^{0.088u}$ |

*In Figure 5.10, we clearly observe the impact of the dependence parameter $\theta$ on both ruin quantities, $\phi_{T_{\tilde{b}}}(u)$ and $m_{\delta,\tilde{b}}(u)$. In order to compare the impact of a dividend barrier, we plot in Figure 5.11 the values of $\phi_{T_{\tilde{b}}}(u)$ ($\tilde{b} = 6$) and $\phi_{T_\infty}(u)$ (no dividend payments) for $\theta = -20$ and 20. For a fixed value of the dependence parameter $\theta$, we observe that the expected discounted value of 1 paid at ruin increases with the introduction of a barrier for any value of initial surplus $u$. We can find the expressions for $\phi_{T_\infty}(u)$ in Cossette et al. (2008).*

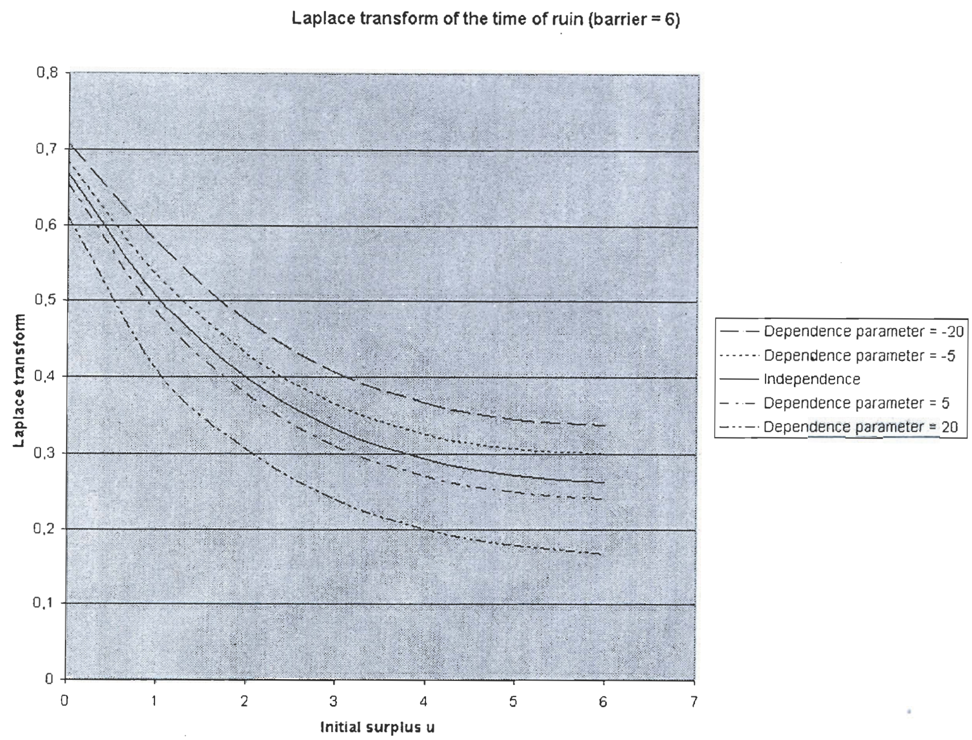


FIG. 5.10 – Laplace Transform of the time of ruin for values of the dependence parameter $\theta$ equal to -20, -5, 0 (independence), 5, and 20.

## 5.5 Appendix

**Proof of Theorem 34.** It follows from the general theory of differential equations that every solution to the $(c+2)$ order non-homogeneous integro-differential equation (5.26) for $m_{\delta,\bar{b}}(u)$ from Theorem 30 with boundary equations (5.38), (5.39), (5.40) and (5.41) provided in Theorem 33 can be expressed as a particular solution $m_{\delta,\infty}(u)$ (the Gerber-Shiu function in the absence of a

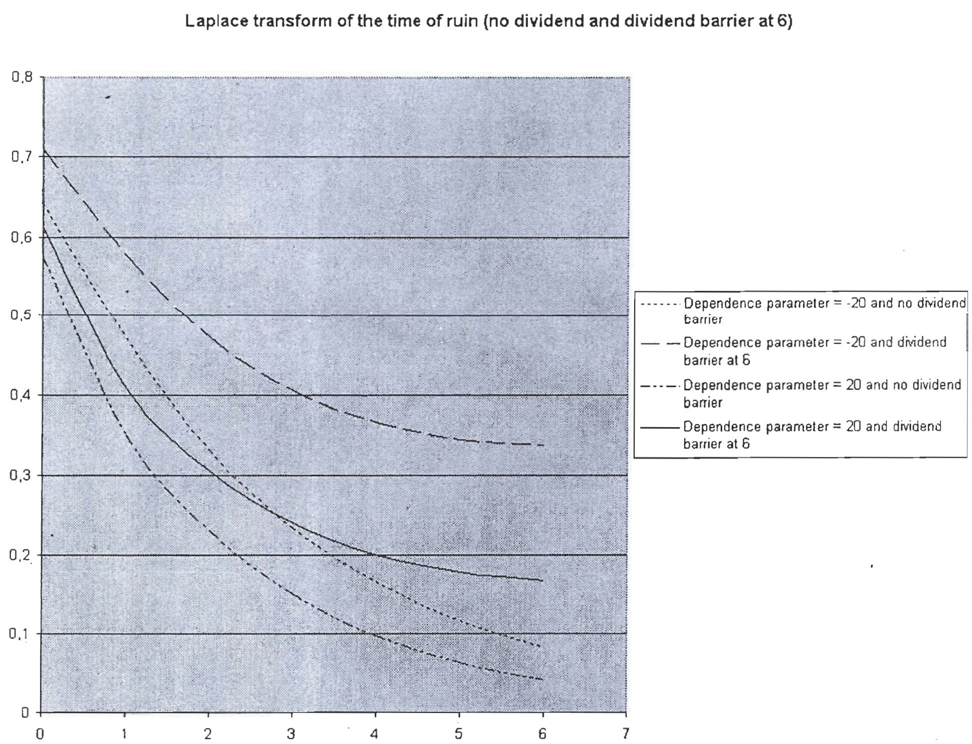


FIG. 5.11 – Values of $\phi_{T_{\tilde{b}}}(u)$ (dividend barrier at $\tilde{b} = 6$) and $\phi_{T_\infty}(u)$ (no dividend payments) for $\theta = -20$ and 20.

dividend barrier) plus a linear combination of $c+2$ linearly independent solutions to the associated homogeneous integro-differential equation.

The discounted penalty function without a barrier, $m_{\delta,\infty}(u)$, is a solution to a defective renewal equation, as shown in Cossette et al. (2008) and satisfies an integro-differential equation defined by

$$\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)m_{\delta,\infty}(u)=\frac{\lambda}{p}\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\left(\int_0^u m_{\delta,\infty}(u-x)f_X(x)dx+w_1(u)\right)$$
$$+\quad\frac{\theta c!\lambda^c}{p^c}\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\left(\int_0^u m_{\delta,\infty}(u-x)k_X(x)dx+w_2(u)\right).$$

The functions $\{v_{i,\delta}(u), u\geq 0\}$ $i=1,\cdots,c+2$ are $c+2$ linearly independent solutions to the homogeneous integro-differential equation

$$\begin{aligned}P(\mathcal{D})\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)v_\delta(u) &= \frac{\lambda}{p}P(\mathcal{D})\int_0^u v_\delta(u-z)f_X(z)dz\\ &\quad+\frac{\theta c!\lambda^c}{p^c}\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\int_0^u v_\delta(u-z)k_X(z)dz, \qquad (5.51)\end{aligned}$$

where

$$P(\mathcal{D})=\prod_{j=1}^{c+1}\Big(\frac{\lambda_j+\delta}{p}\mathcal{I}-\mathcal{D}\Big)=\sum_{k=0}^{c+1}P_k\mathcal{D}^k$$

is an $(c+1)$th-order linear differentiation operator. Note that (5.51) corresponds to the integro-differential equation (5.26) given in Theorem 30 from which the terms involving the functions $w_1$ and $w_2$ have been removed. For that purpose, we need to find the L.T. of $v_\delta(u)$ from (5.51), which can be expressed as

$$d(u)=\frac{\lambda}{p}q_1(u)+\frac{\theta c!\lambda^c}{p^c}q_2(u), \qquad (5.52)$$

where

$$\begin{aligned}d(u) &= P(\mathcal{D})\Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)v_\delta(u)\\ &= \frac{\lambda+\delta}{p}P_0v_\delta(u)+\frac{\lambda+\delta}{p}\sum_{k=1}^{c+1}P_k\mathcal{D}^k v_\delta(u)-\sum_{k=0}^{c+1}P_k\mathcal{D}^{k+1}v_\delta(u) \qquad (5.53)\\ q_1(u) &= P(\mathcal{D})\int_0^u v_\delta(u-z)f_X(z)dz\\ q_2(u) &= \Big(\frac{\lambda+\delta}{p}\mathcal{I}-\mathcal{D}\Big)\Big(\frac{\delta}{p}\mathcal{I}-\mathcal{D}\Big)\int_0^u v_\delta(u-z)k_X(z)dz.\end{aligned}$$

The corresponding L.T. to (5.52) is

$$d^\star(s) = \frac{\lambda}{p} q_1^\star(s) + \frac{\theta c! \lambda^c}{p^c} q_2^\star(s). \tag{5.54}$$

Consequently, we need to find the L.T. $d^\star(s)$, $q_1^\star(s)$, and $q_2^\star(s)$.

Taking the L.T. on both sides of (5.53) yields

$$\begin{aligned} d^\star(s) &= \frac{\lambda+\delta}{p} P_0 v_\delta^\star(s) + \frac{\lambda+\delta}{p} \sum_{k=1}^{c+1} P_k [s^k v_\delta^\star(s) - \sum_{j=1}^{k} s^{k-j} v_\delta^{j-1}(0)] \\ &- \sum_{k=0}^{c+1} P_k [s^{k+1} v_\delta^\star(s) - \sum_{j=1}^{k+1} s^{k+1-j} v_\delta^{j-1}(0)] \\ &= \Big(\frac{\delta+\lambda}{p} - s\Big) \prod_{i=1}^{c+1} \Big(\frac{\delta+\lambda_i}{p} - s\Big) v_\delta^\star(s) - \frac{\lambda+\delta}{p} \sum_{k=1}^{c+1} P_k \sum_{j=1}^{k} s^{k-j} v_\delta^{j-1}(0) \\ &+ \sum_{k=0}^{c+1} P_k \sum_{j=0}^{k} s^{k-j} v_\delta^{j}(0), \end{aligned}$$

which becomes

$$\begin{aligned} d^\star(s) &= \Big(\frac{\delta+\lambda}{p} - s\Big) \prod_{i=1}^{c+1} \Big(\frac{\delta+\lambda_i}{p} - s\Big) v_\delta^\star(s) - \sum_{k=1}^{c+1} P_k \sum_{j=1}^{k} s^{k-j} \Big(\frac{\lambda+\delta}{p} v_\delta^{j-1}(0) - v_\delta^{j}(0)\Big) \\ &+ P_0 v_\delta(0) + \sum_{k=1}^{c+1} P_k s^k v_\delta(0) \\ &= \Big(\frac{\delta+\lambda}{p} - s\Big) \prod_{i=1}^{c+1} \Big(\frac{\delta+\lambda_i}{p} - s\Big) v_\delta^\star(s) - \sum_{k=1}^{c+1} P_k \sum_{j=1}^{k} s^{k-j} \Big(\frac{\lambda+\delta}{p} v_\delta^{j-1}(0) - v_\delta^{j}(0)\Big) \\ &+ P(s) v_\delta(0). \end{aligned} \tag{5.55}$$

Similarly, the L.T. $q_1^\star(s)$ and $q_2^\star(s)$ are given by

$$q_1^\star(s) = \prod_{i=1}^{c+1} \Big(\frac{\delta+\lambda_i}{p} - s\Big) f_X^\star(s) v_\delta^\star(s) - \sum_{k=1}^{c+1} P_k \sum_{j=0}^{k-1} s^{k-1-j} \mathcal{D}^{(j)} \int_0^u v_\delta(u-x) f_X(x) dx \Big|_{u=0} \tag{5.56}$$

and

$$q_2^\star(s) = \Big(\frac{\delta+\lambda}{p} - s\Big)\Big(\frac{\delta}{p} - s\Big) k_X^\star(s) v_\delta^\star(s) - \mathcal{D} \int_0^u v_\delta(u-x) k_X(x) dx \Big|_{u=0}. \tag{5.57}$$

Now, we can bring together (5.55), (5.56), and (5.57) in (5.54) which becomes

$$v_\delta^\star(s) = \frac{\zeta_{c+1,\delta}(s)}{(\frac{\delta+\lambda}{p}-s)\prod_{i=1}^{c+1}(\frac{\delta+\lambda_i}{p}-s) - \frac{\lambda}{p}\prod_{i=1}^{c+1}(\frac{\delta+\lambda_i}{p}-s)f_X^\star(s) - \theta\frac{c!\lambda^c}{p^c}(\frac{\delta+\lambda}{p}-s)(\frac{\delta}{p}-s)k_X^\star(s)}. \tag{5.58}$$

The numerator of (5.58), $\zeta_{c+1,\delta}(s)$, is a $(c+1)$-th degree polynomial in $s$, defined by

$$\begin{aligned}
\zeta_{c+1,\delta}(s) &= -\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)v_\delta(0) + \sum_{k=1}^{c+1}P_k\sum_{j=0}^{k-1}s^{k-1-j}\Big(\frac{\delta+\lambda}{p}v_\delta^{(j)}(0) - v_\delta^{(j+1)}(0)\Big) \\
&\quad - \frac{\lambda}{p}\sum_{k=1}^{c+1}P_k\sum_{j=0}^{k-1}s^{k-1-j}\mathcal{D}^{(j)}\int_0^u v_\delta(u-x)f_X(x)dx\Big|_{u=0} - \theta\frac{c!\lambda^c}{p^c}\mathcal{D}\int_0^u v_\delta(u-x)k_X(x)dx\Big|_{u=0} \\
&= -\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)v_\delta(0) + \sum_{j=0}^{c}\sum_{k=j+1}^{c+1}P_k s^{k-1-j}\Big(\frac{\delta+\lambda}{p}v_\delta^{(j)}(0) - v_\delta^{(j+1)}(0)\Big) \\
&\quad - \frac{\lambda}{p}\sum_{k=1}^{c+1}\sum_{j=0}^{k-1}P_k s^{k-1-j}\mathcal{D}^{(j)}\int_0^u v_\delta(u-x)f_X(x)dx\Big|_{u=0} - \theta\frac{c!\lambda^c}{p^c}\mathcal{D}\int_0^u v_\delta(u-x)k_X(x)dx\Big|_{u=0}
\end{aligned} \tag{5.59}$$

After some rearrangements, (5.59) can be written as

$$\begin{aligned}
\zeta_{c+1,\delta}(s) &= -\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)v_\delta(0) + \sum_{j=0}^{c}\sum_{k=j+1}^{c+1}P_k s^{k-1-j}\Big(\frac{\delta+\lambda}{p}v_\delta^{(j)}(0) - v_\delta^{(j+1)}(0)\Big) \\
&\quad -\frac{\lambda}{p}\sum_{k=1}^{c+1}\sum_{j=0}^{k-1}P_k s^{k-1-j}\sum_{i=0}^{j-1}v_\delta^{(i)}(0)f_X^{(j-i-1)}(0) - \theta\frac{c!\lambda^c}{p^c}v_\delta(0)k_X(0) \\
&= -\prod_{j=1}^{c+1}\Big(\frac{\delta+\lambda_j}{p}-s\Big)v_\delta(0) + \sum_{j=0}^{c}\sum_{k=j+1}^{c+1}P_k s^{k-1-j}\Big(\frac{\delta+\lambda}{p}v_\delta^{(j)}(0) - v_\delta^{(j+1)}(0)\Big) \\
&\quad -\frac{\lambda}{p}\sum_{i=0}^{c-1}\sum_{j=i+1}^{c}\sum_{k=j+1}^{c+1}P_k s^{k-1-j}f_X^{(j-i-1)}(0)v_\delta^{(i)}(0) - \theta\frac{c!\lambda^c}{p^c}v_\delta(0)k_X(0).
\end{aligned}$$

In order to obtain the $c+2$ linearly independent solutions $v_{i,\delta}(s), i = 1,\cdots,c+2$ to the equation (5.58), we need to specify the initial condition as $v_{i,\delta}^{(j)}(0) = I_{\{j=i-1\}}$, $j = 0,\cdots,c+1$. It follows that for $i = 1,\cdots,c+2$

$$v_{i,\delta}^\star(s) = \frac{\zeta_{i,c+1,\delta}(s)}{\Big(\frac{\delta+\lambda}{p}-s\Big)\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p}-s\Big) - \frac{\lambda}{p}\prod_{i=1}^{c+1}\Big(\frac{\delta+\lambda_i}{p}-s\Big)f_X^\star(s) - \theta\frac{c!\lambda^c}{p^c}\Big(\frac{\delta+\lambda}{p}-s\Big)\Big(\frac{\delta}{p}-s\Big)k_X^\star(s)}, \tag{5.60}$$

where

- $\zeta_{1,c+1,\delta}(s) = -\prod_{j=1}^{c+1}\left(\frac{\delta+\lambda_j}{p}-s\right)+\frac{\delta+\lambda}{p}\sum_{k=1}^{c+1}P_k s^{k-1}-\frac{\lambda}{p}\sum_{j=1}^{c}\sum_{k=j+1}^{c+1}P_k s^{k-1-j} f_X^{(j-1)}(0)-$
$\theta\frac{c!\lambda^c}{p^c}k_X(0)$;
- $\zeta_{i,c+1,\delta}(s) = \frac{\delta+\lambda}{p}\sum_{k=i}^{c+1}P_k s^{k-i}-\sum_{k=i-1}^{c+1}P_k s^{k+1-i}-\frac{\lambda}{p}\sum_{j=i}^{c}\sum_{k=j+1}^{c+1}P_k s^{k-1-j} f_X^{(j-i)}(0)$,
$i = 2,\cdots,c$;
- $\zeta_{c+1,c+1,\delta}(s) = \frac{\delta+\lambda}{p}P_{c+1} - P_c - P_{c+1}s$;
- $\zeta_{c+2,c+1,\delta}(s) = -P_{c+1}$.

Using a similar argument as in Li and Garrido (2004), it is possible to show that the $v_{i,\delta}(s)$, for $i = 1, 2, ..., c+2$, are linearly independent. □

## 5.6 References


- Albrecher, H. and O. Boxma (2004). A ruin model with dependence between claim sizes and claim intervals. *Insurance : Mathematics and Economics* 35, 245-254.
- Albrecher, H., Hartinger, J. and R.F. Tichy (2005). On the distribution of dividend payments and the discounted penalty function in a risk model with linear dividend barrier. *Scandinavian Actuarial Journal*, 103-126.
- Albrecher, H. and J. Teugels (2006). Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability* 43 (1), 265-285.
- Boudreault, M., Cossette, H., Landriault, D. and E. Marceau (2006). On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, 301–323.
- Bühlmann, H. (1970). *Mathematical Methods in Risk Theory.* Springer-Verlag, Berlin.
- Cossette, H., Marceau, E. and F. Marri (2008). On a compound Poisson risk model with dependence based on the generalized Farlie-Gumbel-Morgenstern copula. Under revision.
- De Finetti, B. (1957). Su un'impostazione alternativa dell teoria colletiva del rischio. *Transactions of the XV International Congress of Actuaries* 2, 433–443.
- Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* S.S. Huebner Foundation Monographs, University of Pennsylvania.

- Gerber, H.U. and E.S.W. Shiu (2006). On optimal dividend strategies in the compound Poisson risk model. *North American Actuarial Journal* 10, 76-93.
- Grandell, J. (1991). *Aspects of risk theory.* Springer-Verlag, New York.
- Landriault, D. (2008). Constant dividend barrier in a risk model with interclaim-dependent claim sizes. *Insurance : Mathematics and Economics* 42, 31-38.
- Li, S. and J. Garrido (2004). On a class of renewal risk models with a constant dividend barrier. *Insurance : Mathematics and Economics* 35, 691–701.
- Lin, X.S., Willmot, G.E. and S. Drekic. (2003). The classical risk model with a dividend barrier : analysis of the Gerber-Shiu discounted penalty function. *Insurance : Mathematics and Economics* 33, 551-566.
- Lin, X.S. and K. Pavlova (2006). The compound Poisson risk model with a threshold dividend strategy. *Insurance : Mathematics and Economics* 38, 57-80.
- Lin, X.S. and K. Sendova (2008). The compound Poisson risk model with multiple thresholds. *Insurance : Mathematics and Economics* 42, 617-627.
- Rodríguez-Lallena, J.A. and M. Úbeda-Flores (2004). A new class of bivariate copulas. *Statistics and Probability Letters* 66, 315-325.
- Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance.* Wiley, New York.
- Willmot, G.E. and X.S. Lin (2001). *Lundberg Approximations for Com-*

*pound Distributions with Insurance Applications*. Springer Series in Statistics. Springer-Verlag, New York.

# CONCLUSION

Nous avons présenté une extension du modèle classique de la théorie du risque en supposant que les temps qui séparent deux sinistres et les montants des sinistres sont dépendants. Les structures de dépendance ont été construites à l'aides de deux copules, la copule de Farlie-Gumbel-Morgenstern classique et Farlie-Gumbel-Morgenstern généralisée. Ces deux copules possèdent une structure particulière qui nous permet d'adapter les techniques utilisées dans le modèle classique fondé sur l'indépendance entre les montants des sinistres et les temps d'arrivée des sinistres. Nous avons évalué la fonction de Gerber-Shiu dans le cadre des modèles avec dépendance définie à l'aide des deux différentes copules. En absence et en présence d'une barrière horizontale, des équations intégro-différentielles pour la fonction de Gerber-Shiu avec des bornes aux frontières ont également été traitées. En utilisant les transformées de Laplace, des résultats analytiques pour la fonction de Gerber-Shiu ont été obtenus. Ceci a permis d'étudier l'effet de la dépendance sur le temps de la ruine et le déficit à la ruine.

Il serait intéressant d'étudier le modèle de renouvellement (Sparre-Andersen) en supposant que les temps séparant deux sinistres et les montants des sinistres sont dépendants. L'étude du modèle classique et du modèle de Sparre-Andersen de risque avec une structure de dépendance basée sur les copules archimédiennes, les copules gaussiennes et les copules de valeurs extrêmes constituerait aussi un axe d'investigation à considérer.

# BIBLIOGRAPHIE

[1] De Finetti, B. (1957). Su un'impostazione alternativa dell teoria colletiva del rischio. *Transactions of the XV International Congress of Actuaries 2.* 433443.

[2] Denuit, M., Dhaene, J., Goovaerts, M.J., et R. Kaas (2005). *Actuarial Theory for Dependent Risks - Measures, Orders and Models.* Wiley. New York.

[3] Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* S.S. Huebner Foundation Monographs, University of Pennsylvania.

[4] Gerber, H.U. et E.S.W. Shiu (1998). On the time value of ruin. *North American Actuarial Journal.* 2, 48-78.

[5] Grandell, J. (1991). *Aspects of risk theory.* Springer-Verlag, New York.

[6] Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman and Hall, London.

[7] Lin, X.S., Willmot, G.E., et S. Drekic (2003). The classical risk model with a constant dividend barrier : Analysis of the Gerber-Shiu discounted penalty function. *Insurance : Mathematics and Economics.* 33, 551566.

[8] McNeil, A., Frey, R., et P. Embretchs (2005). *Quantitative Risk Management.* Princeton Press, Princeton.

[9] Nelsen, R.B. (2006). *An Introduction to Copulas* (Second Edition). Springer Series in Statistics. Springer-Verlag, New York.

[10] Rodríguez-Lallena, J.A., et M. Úbena-Flores (2004). A new class of bivariate copulas. *Statistics and Probability Letters*. 66, 315-325.

[11] Rolski, T., Schmidli, H., Schmidt, V., et J. Teugels (1999). *Stochastic Processes for Insurance and Finance*. Wiley, New York.

[12] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de statistique de l'Université de Paris 8, 229-231.*