UNIVERSITÉ LAVAL

ÉVALUATION ET ALLOCATION DU RISQUE DANS LE CADRE DE MODÈLES AVANCÉS EN ACTUARIAT

Thèse

Khouzeima Moutanabbir

Doctorat en actuariat
Philosophiæ doctor (Ph.D.)

Québec, Canada

Résumé

Dans cette thèse, on s'intéresse à l'évaluation et l'allocation du risque dans le cadre de modèles avancés en actuariat. Dans le premier chapitre, on présente le contexte général de la thèse et on introduit les différents outils et modèles utilisés dans les autres chapitres. Dans le deuxième chapitre, on s'intéresse à un portefeuille d'assurance dont les composantes sont dépendantes. Ces composantes sont distribuées selon une loi mélange d'Erlang multivariée définie à l'aide de la copule Farlie-Gumbel-Morgenstern (FGM). On évalue le risque global de ce portefeuille ainsi que l'allocation du capital. En utilisant certaines propriétés de la copule FGM et la famille de distributions mélange d'Erlang, on obtient des formules explicites de la covariance entre les risques et de la Tail-Value-at-Risk du risque global. On détermine aussi la contribution de chacun des risques au risque global à l'aide de la régle d'allocation de capital basée sur la Tail-Value-at-Risk et celle basée sur la covariance. Dans le troisième chapitre, on évalue le risque pour un portefeuille sur plusieurs périodes en utilisant le modèle de Sparre Andersen. Pour cette fin, on étudie la distribution de la somme escomptée des *ladder heights* sur un horizon de temps fini ou infini. En particulier, on trouve une expression ferme des moments de cette distribution dans le cas du modèle classique Poisson-composé et le modèle de Sparre Andersen avec des montants de sinistres distribués selon une loi exponentielle. L'élaboration d'une expression exacte de ces moments nous permet d'approximer la distribution de la somme escomptée des *ladder heights* par une distribution mélange d'Erlang. Pour établir cette approximation, nous utilisons une méthode basée sur les moments. À l'aide de cette approximation, on calcule les mesures de risque VaR et TVaR associées à la somme escomptée des *ladder heights*. Dans le quatrième chapitre de cette thèse, on étudie la quantification des risques liés aux investissements. On élabore un modèle d'investissement qui est constitué de quatre modules dans le cas de deux économies : l'économie canadienne et l'économie américaine. On applique ce modèle dans le cadre de la quantification et l'allocation des risques. Pour cette fin, on génère des scénarios en utilisant notre modèle d'investissement puis on détermine une allocation du risque à l'aide de la règle d'allocation TVaR. Cette technique est très flexible ce qui nous permet de donner une quantification à la fois du risque d'investissement, risque d'inflation et le risque du taux de change.

Abstract

In this thesis, we are interested in risk evaluation and risk allocation problems using advanced actuarial models. First, we investigate risk aggregation and capital allocation problems for a portfolio of possibly dependent risks whose multivariate distribution is defined with the Farlie-Gumbel-Morgenstern copula and with mixed Erlang distributions for the marginals. In such a context, we first show that the aggregate claim amount has a mixed Erlang distribution. Based on a top-down approach, closed-form expressions for the contribution of each risk are derived using the TVaR and covariance rules. These findings are illustrated with numerical examples. Then, we propose to investigate the distribution of the discounted sum of ascending ladder heights over finite- or infinite-time intervals within the Sparre Andersen risk model. In particular, the moments of the discounted sum of ascending ladder heights over a finite- and an infinite-time intervals are derived in both the classical compound Poisson risk model and the Sparre Andersen risk model with exponential claims. The application of a particular Gerber-Shiu functional is central to the derivation of these results, as is the mixed Erlang distributional assumption. Finally, we define VaR and TVaR risk measures in terms of the discounted sum of ascending ladder heights. We use a moment-matching method to approximate the distribution of the discounted sum of ascending ladder heights allowing the computation of the VaR and TVaR risk measures. In the last chapter, we present a stochastic investment model (SIM) for international investors. We assume that investors are allowed to hold assets in two different economies. This SIM includes four components: interest rates, stocks, inflation and exchange rate models. First, we give a full description of the model and we detail the parameter estimation. The model is estimated using a state-space formulation and an extended Kalman filter. Based on scenarios generated from this SIM, we study the risk allocation to different background risks: asset, inflation and exchange rate risks. The risk allocation is based on the TVaR-based rule.

Table des matières

Liste des tableaux

Liste des figures

À mon père...

À mon père qui un jour a cru en moi

À mon père qui un jour a eu confiance en moi

À mon père qui aujourd'hui n'est plus là

À mon père, à mon père je voudrais dire merci

À Omar Moutanabbir....

Remerciements

Tout d'abord, je tiens à remercier mes directeurs de thèse, Mme. Hélène Cossette, M. Étienne Marceau et M. Patrice Gaillardetz, pour leurs encadrements, Leurs conseils et leurs encouragements dont j'ai bénéficiés tout au long de ma thèse. Je désire témoigner ma reconnaissance pour l'aide qu'ils m'ont apporté pour réaliser mes travaux de thèse. Je tiens à remercier plus spécialement Mme Hélène Cossette pour l'aide qu'elle m'a apporté pour améliorer la qualité de la rédaction de mes publications. Je tiens également à les remercier du financement qu'ils m'ont accordé par l'intermédiaire des subventions individuelles et collectives octroyées par la Chaire d'actuariat et le Conseil de recherche en science naturelles et en génie du Canada. Je leur suis extrêmement reconnaissant.

J'adresse mes sincères remerciements à MM. Gordon E. Willmot, Manuel Morales et Ilie Radu Mitric pour avoir accepté de faire partie du jury de ma thèse. Leurs remarques, suggestions et commentaires judicieux m'ont permis d'améliorer la version finale de cette thèse. Je remercie également le directeur de l'école d'actuariat M. Denis Latulippe pour avoir présidé la soutenance de ma thèse.

Merci à tous les membres de l'École d'actuariat de l'université Laval pour leur sympathie et leur disponibilité. Je voudrais aussi remercier les anciens directeurs de l'école d'actuariat M. Michel Giguère, M. Vincent Goulet, M. Guy Gendron et M. Robert Rousseau qui m'ont fait confiance et m'ont donné la chance d'enseigner durant mes études doctorales. Je tiens à remercier plus spécialement M. Robert Rousseau pour son support et ses encouragements.

J'exprime toute ma reconnaissance à Mme Marie-Pier Côté et M. David Landriault pour avoir accepté de collaborer avec moi. Ces collaborations m'ont beaucoup aidé pour réaliser une bonne partie de mes travaux de thèse.

Durant ces années de thèse, j'ai eu la chance de travailler, en collaboration avec mon directeur de thèse M. Étienne Marceau, sur un projet avec Aon-Consulting. Ce projet m'a beaucoup aidé à développer mon experience en modélisation financières et a beaucoup contribué à l'élaboration de mes travaux de thèse. J'aimerais bien remercier Ivor Krol, Étienne Dubé et Jocelyn Guérin de AON-Montréal et Mark Jeavons et Andrew Claringbold de AON-Londres. Je tiens à remercier plus spécialement M. Étienne Marceau de m'avoir donné cette chance pour participer à ce projet. Le travail en sa compagnie fut un véritable plaisir.

Un grand merci du fond du cœur à ma mère, mes sœurs et mes frères d'avoir toujours été présents à mes côtés. Je voudrais adresser un remerciement à mes amis et mes proches qui m'ont soutenu et encouragé au cours de ces dernières années.

J'aimerais aussi rendre hommage à mon ami et mon collègue Florent Toureille décédé le 15 Décembre 2010.

Avant-propos

Cette thèse étudie l'évaluation et l'allocation du risque en utilisant des modèles avancés en actuariat. Elle est constituée de cinq chapitres dont l'introduction et la conclusion générale. Les résultats de cette thèse sont présentés dans trois articles scientifiques. Le deuxième chapitre est constitué d'un article coécrit avec mon directeur de thèse à l'Université Laval Etienne Marceau, ma co-directrice de thèse de l'Université Laval Hélène Cossette et l'étudiante à la maitrise à Université McGill Marie-Pier Côté. Cet article est intitulé *Multivariate Mixed Erlang Distributions Defined with the Farlie-Gumbel-Morgenstern Coupula : Aggregation and Capital Allocation* et accepté pour publication dans la revue Insurance : Mathematics and Economics. Le troisième chapitre est basé sur un article publié dans la revue Insurance : Mathematics and Economics et intitulé *Analysis of the Discounted Sum of Ascending Ladder Heights.* Cet article est co-écrit avec Hélène Cossette, Etienne Marceau et David Landriault professeur de University of Waterloo. Le Quatrième chapitre repose sur un article s'intitulant *A Stochastic International Investment Model and Risk Allocation* et soumis pour publication. Cet article est réalisé en collaboration avec Hélène Cossette, Etienne Marceau et Patrice Gaillardetz mon co-directeur de thèse à Concordia University.

Chapitre 1

Introduction

1.1 Introduction

La gestion quantitative du risque constitue une bonne base d'une saine gestion des compagnies d'assurance et des institutions financières. Suite aux récents développements des marchés financiers et aux nouvelles réglementations, les compagnies d'assurance sont amenées à adopter une gestion du risque qui s'appuie à la fois sur des méthodes qualitatives ainsi que des méthodes quantitatives. Cette gestion quantitative des risques permet aux assureurs de bien mesurer et quantifier les risques auxquels ils s'exposent. Un bon plan de la gestion quantitative du risque doit réduire l'exposition aux risques et aussi répondre aux contraintes réglementaires. Mais, il doit aussi tenir en compte de la rentabilité des capitaux mobilisés termes de gestion du risque. Afin de bien optimiser l'utilisation de ces capitaux, la compagnie d'assurance a recours à une allocation des capitaux en fonction de l'allocation des risques.

Dans cette thèse, on s'intéresse à trois différents aspects de l'évaluation et de l'allocation des risques en actuariat. D'abord, on s'intéresse à l'agrégation des risques dépendants et l'allocation du capital pour un portefeuille de contrats d'assurance. Puis, on étudie l'évaluation des mesures de risque et de ruine dans le cadre de la théorie de la ruine. Ces mesures sont définies à l'aide d'un modèle dynamique de risque utilisé pour décrire l'evolution dans le temps du surplus associé au portefeuille. Pour finir, on présente un modèle d'investissement dans lequel il est possible d'investir dans deux économies. À l'aide de ce modèle, on étudie le problème d'allocation et d'évaluation des différents risques liés aux investissements. Au cours des dernières années, on a vu un intérêt croissant pour l'agrégation de risques dépendants, pour des modèles dynamiques de risque plus complexes et pour des modèles d'investissement. L'agrégation de risques dépendants est importante dans le contexte de la modélisation du risque global d'un portefeuille comportant plusieurs risques ou plusieurs lignes d'affaires. Les modèles dynamiques en théorie du risque sont utilisés pour décrire l'évolution des coûts totaux pour un portefeuille de risques. Les modèles d'investissement sont utilisés dans différents domaines de l'actuariat.

Dans cette introduction, on décrit le contexte général des différents projets présentés dans cette thèse. On présente aussi les différents outils utilisés tout au long des chapitres suivants.

1.2 Mesures de risque

Les mesures de risque jouent un rôle essentiel dans la quantification du risque. Elles sont utilisées dans l'évaluation et la comparaison des risques ainsi que dans la détermination des primes et des marges de solvabilité. De plus, les mesures de risque sont très utiles dans l'agrégation des risques et l'allocation du capital. En effet, les capitaux économiques sont définis à partir d'une mesure de risque. En plus, l'allocation du capital est établie en utilisant une mesure de risque qui quantifie la contribution de chaque risque individuel au risque global. Ci-dessous, on commence par définir la notion de mesure de risque et la mesure de risque cohérente, pour ensuite présenter les spécificités de chacune des mesures de risque d'intérêt : la mesure de risque VaR et la mesure de risque TVaR.

On considère l'espace de probabilité $(\Omega, \mathfrak{F}, \mathbf{P})$ et Ξ l'espace des variables aléatoires réelles définies sur cet espace.

Définition 1.2.1. *Une mesure du risque ρ est une application réelle définie sur Ξ telle que*

$$\begin{aligned} \rho : \Xi &\to \mathbb{R} \\ X &\mapsto \rho(X). \end{aligned}$$

Cette définition introduit la mesure de risque comme une application ou une relation fonctionnelle qui transforme la comparaison des risques, les variables aléatoires X, en une comparaison de scalaires réels, $\rho(X)$. Pour une variable aléatoire X, la quantité $\rho(X)$ représente un capital défini par la mesure de risque ρ.

Artzner et coll. (1999) introduit une caractérisation axiomatique des mesures de risque. Selon cette caractérisation, l'application ρ doit satisfaire certaines conditions pour être une mesure de risque cohérente.

Définition 1.2.2. *Soient X_1 et X_2 deux éléments dans Ξ l'espace des variables aléatoires réelles. Une mesure de risque ρ est dite cohérente si et seulement si elle satisfait les propriétés suivantes :*

1. *monotonie : si $Pr(X_1 \leq X_2) = 1$, alors $\rho(X_1) \leq \rho(X_2)$;*
2. *sous-additivité : $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$;*
3. *homogénéité positive : pour tout $a \geq 0$, $\rho(aX_1) = a\rho(X_1)$;*
4. *invariance par translation : pour tout $a \in \mathbb{R}$, $\rho(a + X_1) = a + \rho(X_1)$.*

Le premier axiome (i.e. monotonie) signifie que le risque X_2, qui est plus dangereux que le risque X_1, a besoin de mettre de côté plus de capital selon la mesure ρ monotone. La sous-addivité implique que le capital requis pour l'agrégat de deux risques est inférieur à la somme des capitaux nécessaires pour chaque risque, i.e. une mesure sous-additive engendre nécessairement une diversification des risques. L'homogénéité positive traduit le fait qu'un changement de numéraire n'induit pas de risque supplémentaire. De la même manière, ajouter un montant certain au risque implique l'ajout de ce même montant à une mesure du risque invariante par translation.

En sciences actuarielles, il est souhaitable que les mesures de risque satisfassent d'autres propriétés. On suggère de consulter par exemple Marceau (2012) ou Denuit et coll. (2005) pour plus de détails.

Une mesure de risque fort populaire est la Valeur-à-Risque (VaR).

Définition 1.2.3. *On définit la mesure de risque VaR, d'une variable aléatoire X à un niveau de confiance donné $0 < \kappa < 1$ par*

$$VaR_{\kappa}(X) = \inf\{x \epsilon \mathbb{R} : F_X(x) \geq \kappa\}.$$

La VaR est une mesure de risque monotone, positivement homogène, invariante par translation mais pas sous-additive. Ainsi, la mesure de risque VaR n'est pas une mesure de cohérente selon Artzner et coll. (1999). En plus, la VaR ne donne pas d'information sur la structure de la queue de distribution. D'autres mesures de risque peuvent être envisagées pour quantifier le risque auquel un assureur est exposé.

Parmi ces mesures alternatives de risque, on trouve la mesure Tail Value-at-Risk (TVaR). La mesure de risque TVaR présentée ensuite permet de surmonter les différents inconvénients de la VaR.

Définition 1.2.4. *À un niveau de confiance $0 < \kappa < 1$, on définit la $TVaR_{\kappa}$ de X par*

$$\begin{aligned} TVaR_{\kappa}(X) &= \frac{1}{1-\kappa}\int_{\kappa}^{1} VaR_s(X)\,ds \\ &= \frac{E\left[X \times 1_{\{X > VaR_{\kappa}(X)\}}\right] + VaR_{\kappa}(X)\left(F_X\left(VaR_{\kappa}(X)\right) - \kappa\right)}{1-\kappa}, \end{aligned}$$

où 1_A est la fonction indicatrice telle que $1_A(X) = 1$, si $X \in A$, et $1_A(X) = 0$, si $X \notin A$.

Remark 1.2.1. *Dans le cas de variable aléatoire continue, la mesure TVaR est égale à*

$$TVaR_{\kappa}(X) = \frac{E\left[X 1_{\{X > VaR_{\kappa}(X)\}}\right]}{1-\kappa}.$$

Cette mesure de risque, contrairement à la VaR, donne des informations sur la distribution de X au-delà de la VaR et donc sur l'épaisseur de la queue de la distribution. En plus, la mesure de risque TVaR est une mesure cohérente. On trouve dans Acerbi (2002), Acerbi et Tasche (2002) et McNeil et coll. (2005) plus de détails concernant les mesures VaR et TVaR.

Ces mesures de risque sont notamment utilisées dans le cadre de l'allocation du capital. Elles permettent de déterminer le montant global du capital dont la compagnie a besoin de mettre de côté pour l'ensemble du portefeuille. Ce capital économique doit être réparti sur les différentes composantes du portefeuille. Cette répartition du capital doit être effectuée en prenant en considération la contribution de chaque composante du portefeuille dans le risque global du portefeuille. Pour ceci, on présente dans la section suivante deux méthodes d'allocation du capital basées sur des mesures de risque.

1.3 Agrégation et allocation du risque

Les compagnies d'assurance sont amenées à déterminer le montant de capital (i.e. fonds propres) à allouer à un portefeuille. Afin de lui permettre de faire face à des situations exceptionnelles. Par la suite, une allocation optimale du capital doit être effectuée en fonction de l'ensemble des risques composants chaque portefeuille. Cette allocation doit tenir compte de la contribution de chaque risque au risque global à l'aide d'une méthode d'allocation de risque. Cette méthode permet une redistribution des ressources vers les composantes les plus viables du portefeuille de la compagnie.

Il existe dans la littérature actuarielle deux principales approches d'allocation de capital. La première est l'approche top-down et elle consiste à déterminer le capital total d'un portefeuille à l'aide d'une mesure de risque. Ensuite, ce capital est partagé entre les différents risques. Cette distribution du capital peut être déduite de la mesure de risque utilisée comme dans Tasche (1999), Panjer (2002), Wang (2002), Dhaene et coll. (2008), ou elle peut être déterminée indépendamment de la mesure de risque, voir Hesselager et Andersson (2002), Valdez et Chernih (2003) ou Goovaerts et coll. (2005). La seconde approche consiste à considérer l'allocation de capital tel un problème de détermination de prix d'options de défaut de la compagnie ou de ses branches comme dans Myers et Read Jr (2001), Sherris et Australia (2006) et Kim et Hardy (2009). Une approche générale est avancée dans Dhaene et coll. (2009) dans lequel l'allocation de capital est vue comme un problème d'optimisation visant à minimiser la somme pondérée des déviations entre les pertes et le capital alloué pour chaque risque. Dans la suite, on s'intéresse aux méthodes de type top-down. Selon cette approche, le processus d'allocation du capital comporte les étapes suivantes :

1. Modéliser le comportement des différentes composantes du portefeuille à l'aide d'une distribution multivariée.
2. Déterminer le capital économique global pour tout le portefeuille.

3. Répartir le capital économique entre les différentes composantes du portefeuille.

Pour une période fixée on suppose que le portefeuille de la compagnie est composé de n risques. La variable aléatoire X_i représente le coût associé au risque i, pour $i = 1, 2, ..., n$. Le montant total des coûts pour l'ensemble du portefeuille est défini par la variable aléatoire

$$S_n = X_1 + X_2 + ... + X_n.$$

On doit dans un premier temps établir la distribution conjointe du vecteur $\underline{X} = (X_1, \ldots, X_n)$ en utilisant une loi multivariée. Ensuite, on doit déterminer la distribution de S_n qui sert à évaluer le capital économique à l'aide d'une mesure de risque ρ. Pour un seuil de confiance $0 \leq \kappa \leq 1$, ce capital économique, noté CE_κ, est défini comme la différence entre la valeur du capital total déterminé par la mesure ρ_κ et la valeur espérée de la perte, i.e.

$$CE_\kappa (S_n) = \rho_\kappa (S_n) - E (S_n).$$

Par la sous-section qui suit, on présente quelques méthodes d'allocation du capital.

1.3.1 Allocation du capital

Le capital total $\rho_\kappa (S_n)$ est calculé pour l'ensemble du portefeuille et une méthode d'allocation de capital permet de déterminer le montant de capital que l'on contribue à chaque risque individuel du portefeuille. Le montant alloué au risque i, $i = 1, ..., n$, est fonction de la contribution au risque global de ce risque. Le risque global est défini à partir de la mesure de risque $\rho_\kappa (S_n)$. On obtient les différentes allocations en décomposant cette mesure de risque globale comme suit

$$\rho_\kappa (S_n) = \sum_{i=1}^{n} AC_\kappa(X_i, S_n),$$

où $AC_\kappa(X_i, S_n)$ est la part du montant total de capital attribuée au risque i.

Différentes règles d'allocation ont été proposées où les montant $AC_\kappa(X_i, S_n)$ sont souvent déterminés à l'aide de mesures de risque. La mesure de risque choisie pour définir les contributions au risque peut être différente de celle utilisée pour déterminer le risque global. Pour plus d'informations, on propose de consulter : McNeil et coll. (2005), Albrecht (2005) et Tasche (1999).

Dans la suite, on suppose que le capital économique au niveau du portefeuille est déterminé à l'aide de la mesure de risque TVaR. On s'intéresse à deux règles d'allocation : la règle de la TVaR et la règle de la Covariance.

Les montants alloués selon ces deux règles sont fournis dans les deux prochaines définitions.

$$TVaR_\kappa (S_n) = TVaR_\kappa (X_1, S_n) + TVaR_\kappa (X_2, S_n) + \ldots + TVaR_\kappa (X_n, S_n).$$

Définition 1.3.1. *La contribution du i^{ieme} risque, $AC_\kappa(X_i, S_n) = TVaR_\kappa(X_i, S_n)$, représente la part du capital à allouer au risque i qu'on définit par*

$$TVaR_\kappa(X_i, S_n) = \frac{E\left[X_i 1_{\{S_n > VaR_\kappa(S_n)\}}\right] + \beta_{S_n} E\left[X_i 1_{\{S_n = VaR_\kappa(S_n)\}}\right]}{1-\kappa},$$

où

$$\beta_{S_n} = \begin{cases} \frac{Pr(S_n \leq VaR_\kappa(S_n)) - \kappa}{Pr(S_n = VaR_\kappa(S_n))}, & si\ Pr(S_n = VaR_\kappa(S_n)) \neq 0 \\ 0, & sinon. \end{cases}$$

Remarque 1.3.1. *Pour les variables aléatoires continues, les allocations du capital selon la règle de la TVaR sont données par*

$$TVaR_\kappa(X_i, S_n) = \frac{E\left[X_i 1_{\{S_n > VaR_\kappa(S_n)\}}\right]}{1-\kappa}.$$

McNeil et coll. (2005) donne plus de détails concernant l'allocation selon la règle TVaR.

Définition 1.3.2. *La contribution du i^{ieme} risque selon la règle d'allocation basée sur la covariance, que l'on note $C_\kappa(X_i, S_n)$, est égale à*

$$C_\kappa(X_i, S_n) = E[X_i] + \frac{Cov(X_i, S_n)}{Var(S_n)}\left(TVaR_\kappa(S_n) - E[S_n]\right), \tag{1.1}$$

pour $i = 1, 2, \ldots, n$.

On trouve dans Hesselager et Anderson (2002) plus de détails à propos de l'allocation du capital selon le principe de la covariance.

1.3.2 Dépendance et théorie des copules

On retrouve dans la littérature sur l'agrégation des risques plusieurs articles scientifiques faisant référence aux copules. La théorie des copules est un outil mathématique puissant permettant d'introduire des dépendances entre les risques. Cette théorie permet de construire des distributions multivariées et offre une façon flexible pour décrire la structure de dépendance. Des ouvrages de référence sur la théorie des copules sont Joe (1997) et Nelsen (2006). Des applications en actuariat et gestion quantitative des risques sont proposées dans les livres de Denuit et coll. (2005), McNeil et coll. (2005) ou Marceau (2012).

Les copules permettent de modéliser la dépendance entre différentes variables aléatoires quelles que soient les lois marginales suivies par chacune d'entre elles. Une copule est une fonction de répartition multivariée C définie sur $[0;1]^n$ et dont les marginales sont uniformes sur $[0;1]$.

Définition 1.3.3. *On appelle une copule C de dimension n une application de $[0;1]^n$ vers $[0;1]$ ayant les propriétés suivantes :*

(a) $C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_n) = 0$ *pour tout* $i = 1, 2, ..., n$ *;*

(b) $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ *pour tout* $i = 1, 2, ..., n$ *;*

(c) *Pour tout* $a_1 \leq b_1$*,...,*$a_n \leq b_n$*, on a*

$$\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+...+i_n} C(u_{1,i_1}, ..., u_{n,i_n}) \geq 0,$$

où $u_{k,1} = a_k$ *et* $u_{k,2} = b_k$ *pour* $k = 1, ..., n$.

L'utilité de cette théorie est de pouvoir décomposer la loi conjointe d'un vecteur aléatoire $\underline{X} = (X_1, X_2, ..., X_n)$ à l'aide du théorème de Sklar. Le théorème de Sklar définit la fonction de répartition multivariée de dimension n, F, en terme d'une copule C décrivant la structure de dépendance et les fonctions de répartition marginales F_i, $i = 1, \ldots, n$. On a les relations suivantes :

$$F_{\underline{X}}(x_1, \ldots, x_n) = C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)), \tag{1.2}$$

où $\underline{X} = (X_1, \ldots, X_n)$. La fonction de densité conjointe de $\underline{X}$ est donnée par

$$f_{\underline{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \ldots f_{X_n}(x_n)\, c(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)), \tag{1.3}$$

où c est la fonction de densité de la copule C que l'on définit par

$$c(u_1, \ldots, u_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}.$$

L'application de la théorie des copules a connu une grande évolution avec l'apparition du livre de Nelsen en 1999. En actuariat, on a commencé à intégrer des structures de dépendance à l'aide des copules à partir de la fin des années 1990 avec la publication de Frees et Valdez (1998) qui introduit plusieurs applications des copules. Wang (1998) analyse la distribution de la somme de risques dépendants en utilisant des copules. En théorie du risque, les copules ont été utilisées pour analyser la probabilité de ruine dans le contexte de risques dépendants dans Albrecher et Teugels (2006), Cossette et coll. (2009) et Cossette et coll. (2008). Bargès et coll. (2009) traite le problème d'allocation de capital dans le cas d'un portefeuille à composantes dépendantes. Il existe plusieurs familles de copules et Nelsen (2006) donne une liste exhaustive des familles de copules les plus importantes.

Dans le troisième chapitre de cette thèse, on utilise la copule Farlie-Gumbel-Morgenstern (FGM). Dans la suite de cette sous-section, on présente celle-ci

Dans le cas bivarié, la copule FGM est définie par la fonction de répartition conjointe C :

$$C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2),$$

où la constante θ est le paramètre de dépendance C avec $\theta \in [-1, 1]$. Pour cette famille de copule, on trouve le cas d'indépendance lorsque l'on pose $\theta = 0$, i.e. $C^I(u_1, u_2) = u_1 u_2$. La fonction de probabilité conjointe c est égale à :

$$c(u_1, u_2) = (1+\theta) - \theta 2\overline{u}_1 - \theta 2\overline{u}_2 + \theta 2\overline{u}_1 2\overline{u}_2, \tag{1.4}$$

où $\overline{u}_i = 1 - u_i$.

La copule FGM est une perturbation de la copule indépendance. Elle est aussi une approximation de premier ordre des copules Ali Mikhail Haq, Frank et Placket (voir Nelsen (2006)). Pour cette famille de copules, les mesures d'association tau de Kendall et rho de Spearman sont données par $\tau = \frac{2\theta}{9}$ et $\rho = \frac{\theta}{3}$. Étant donné $\theta \in [-1, 1]$, cette famille de copule permet d'introduire une dépendance modérée positive ou négative. Dans la littérature financière et actuarielle, cette copule représente un outil mathématique simple qui permet de développer des expressions exactes et explicites. Par exemple, Bargès et coll. (2009) utilise la copule FGM et donne des expressions exactes d'allocation du capital dans le cas d'un portefeuille d'assurance dont les risques sont dépendants et exponentiellement distribués. Prieger (2002) souligne l'utilité de la copule FGM dans la modélisation d'assurance maladie. En théorie du risque, cette copule a été utilisée pour introduire une dépendance entre les montants de sinistres et les durées inter-sinistres dans le cadre de modèles dynamiques du risque (voir Cossette et coll. (2008), Zhang et Yang (2011) et Chadjiconstantinidis et Vrontos (2012)). Dans Yeo et Valdez (2009), la copule FGM permet d'introduire une dépendance entre les variables en crédibilité. En finance, cette famille de copule est utilisée dans Cherubini et coll. (2011), Gatfaoui (2005) et Gatfaoui (2007).

Dans le cas multivarié, pour $n \geq 3$, la n-copule FGM qui a $2^n - n - 1$ paramètres, est définie comme suit :

$$C(\underline{u}) = u_1 u_2 \dots u_n \left[1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \dots \overline{u}_{j_k} \right].$$

Ceci est équivalent à

$$C(\underline{u}) = u_1 u_2 \dots u_n P(u_1, u_2, \dots, u_n),$$

où P est une fonction polynomiale dont l'expression est :

$$P(u_1, u_2, \dots, u_n) = 1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \dots \overline{u}_{j_k}. \tag{1.5}$$

On peut réécrire la fonction de densité conjointe de la n-copule FGM en fonction de P et on trouve

$$c(\underline{u}) = P(2u_1, 2u_2, \dots, 2u_n). \tag{1.6}$$

On note que la fonction polynomiale P est une fonction linéaire pour chaque argument u_i, pour $i = 1, \ldots, n$. Selon Nelsen (2006), les $2^n - n - 1$ paramètres de la n-copule FGM doivent respecter certaines conditions pour que la copule C existe, soient les contraintes suivantes

$$1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \varepsilon_{j_1} \varepsilon_{j_2} \ldots \varepsilon_{j_k} \geq 0, \quad \varepsilon_{j_1} \varepsilon_{j_2} \ldots \varepsilon_{j_k} \epsilon \{-1, 1\}. \tag{1.7}$$

1.4 Théorie de la ruine

1.4.1 Modèle dynamique du surplus

En actuariat, la théorie de la ruine a pour objectif la modélisation de la dynamique du surplus d'un portefeuille d'une compagnie d'assurance en vue d'évaluer des mesures d'insolvabilité. On est intéressé par la ruine éventuelle du portefeuille, qui survient lorsque le surplus devient négatif. Le surplus d'un portefeuille d'une compagnie d'assurance est la différence entre le total des primes et des montants des sinistres. Le terme ruine signifie d'avantage une situation critique où une compagnie ne peut satisfaire ses engagements et mettre en péril sa situation financière. La principale mesure d'insolvabilité est la probabilité de ruine sur un horizon de temps fini ou infini.

On décrit l'évolution dans le temps du surplus par le processus $\underline{U} = \{U(t), t \geq 0\}$. Le surplus correspond au capital initial auquel on ajoute les primes accumulées jusqu'à l'instant t et on soustrait les montants de sinistres enregistrés jusqu'au moment t, i.e.

$$U(t) = u + ct - S(t),$$

où $u \geq 0$ est le surplus initial, $c > 0$ est le taux de prime par unité du temps et $S(t)$ représente le montant cumulé des sinistres au temps $t \geq 0$.

Définition 1.4.1. *Le montant cumulé des sinistres au temps $t \geq 0$ est représenté par le processus $\{S(t), t \geq 0\}$, avec $S(0) = 0$ et*

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) > 0, \\ 0, & N(t) = 0. \end{cases} \tag{1.8}$$

Dans cette définition, la variable aléatoire positive X_j représente le montant du $j^{\text{ème}}$ sinistre du portefeuille. Pour plus de détails concernant le processus du montant total des sinistres défini en (1.8), on peut consulter e.g. Rolski et coll. (1999) et Asmussen (2000).

Le processus $\underline{N} = \{N(t), t \geq 0\}$ est un processus de comptage où la variable aléatoire $N(t)$ correspond au nombre de sinistres au temps t.

Définition 1.4.2. *Le processus $\underline{N} = \{N(t), t \geq 0\}$ est un processus de comptage si et seulement si :*

– *Pour tout* $t \geq 0$*,* $N(t) \in \mathbb{N}$ *;*

– $N(0) = 0$ *;*

– *Si* $t \geq s \geq 0$*, alors* $N(t) \geq N(s)$*.*

– *La trajectoire de* $\underline{N}$ *est continu à droite.*

On suppose que le $j^{\text{ème}}$ sinistre du portefeuille se produit à l'instant T_j. Les variables aléatoires T_j $(j = 1, 2, ...)$ représentent les temps d'arrivée du processus de comptage $\underline{N}$. Pour un $t \geq 0$, le nombre de sinistre $N(t)$ est donné par $N(t) = \sup\{j \in \mathbb{N} | T_j \leq t\}$. Les durées inter-sinistres sont représentées par le processus $\underline{W} = \{W_j, j \in \mathbb{N}^+\}$ avec

$$W_1 = T_1,$$

et

$$W_j = T_j - T_{j-1},$$

pour $j = 2, 3, \ldots$

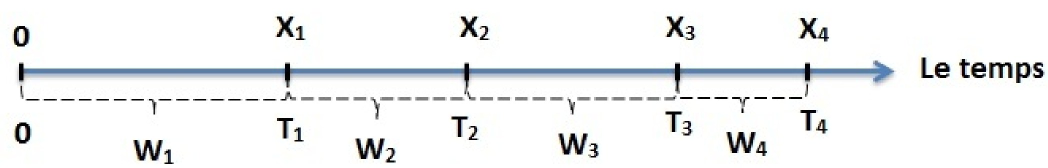


Figure 1.1: Montants des sinistres et durées inter-sinistres.

Si le processus $\underline{N} = \{N(t), t \geq 0\}$ est un processus à accroissements indépendants et identiquement distribués, i.e. les v.a. W_j sont i.i.d., alors le processus $\underline{N}$ est dit un processus de renouvellement. Si les v.a. W_j sont i.i.d. et suivent une loi exponentielle, alors $\underline{N}$ est dit un processus de Poisson homogène.

Initialement, le surplus des assureurs a été étudié en utilisant le modèle classique proposé par Filip Lundberg en 1903 et formulé dans Lundberg (1926) et Cramer (1930). Ce modèle classique, connu aussi sous le nom du modèle Cramer-Lundberg, représente l'évolution du surplus d'une compagnie d'assurance par un processus Poisson composé, soit le cas où le processus de comptage $\underline{N}$ est un processus de Poisson homogène. Le modèle classique est exhaustivement étudié dans la littérature. Le lecteur intéressé peut se référer à DeVylder (1996), Gerber (1979) ou Kass et coll. (2001).

Plusieurs extensions du modèle classique ont été proposées durant les trois dernières décennies. Dans Taylor (1980), Michaud (1996) et Jasiulewicz (2001), on introduit des taux de prime variables. Le versement de dividendes au-delà d'une barrière a été considéré dans Gerber

(1981), Albrecher et Kainhofer (2002), Gerber et coll. (2008), ainsi que dans Albrecher et coll. (2010). Albrecher et Hipp (2007) et Albrecher et coll. (2008) ont introduit un modèle avec taxes. L'impact des investissements sur la probabilité de la ruine a été traité dans Hipp et Plum (2000), Gaier et coll. (2003) de même que l'impact de la réassurance dans Dickson et Waters (1996), Schmidli (2002).

Dans le deuxième chapitre de cette thèse, on travaille avec le modèle de Sparre Andersen de la théorie de la ruine. Ce modèle, introduit dans Andersen (1957), généralise le modèle classique en considérant un processus de renouvellement comme processus de dénombrement des sinistres. Dans ce modèle, on travaille sous une hypothèse d'indépendance entre les montants de sinistres $\{X_j, j \in \mathbb{N}^+\}$ et les durées entre les sinistres $\{W_j, j \in \mathbb{N}^+\}$.

1.4.2 Mesures de la ruine

De façon générale, on définit le temps de la ruine par la variable aléatoire τ, correspondant au premier instant $t \geq 0$ auquel le surplus devient négatif i.e.

$$\tau = \inf\{t > 0 | U(t) < 0\}.$$

Si le suplus ne devient jamais négatif, on convient que $\tau = \infty$.

On désigne par $\psi(u, T)$ la probabilité de ruine sur un horizon de temps fini T avec un surplus initial u. Celle-ci correspond à la probabilité que le surplus devienne strictement négatif avant l'instant T, i.e.

$$\psi(u, T) = \Pr\left(\tau \leq T | U(0) = u\right).$$

La probabilité de ruine sur un horizon de temps infini est donnée par

$$\begin{aligned} \psi(u) &= \lim_{t \to \infty} \psi(u, t) \\ &= \Pr\left(\tau < \infty | U(0) = u\right). \end{aligned}$$

Les probabilités de survie correspondantes sont représentées par

$$\begin{aligned} \phi(u, T) &= 1 - \psi(u, T) \\ \phi(u) &= 1 - \psi(u). \end{aligned}$$

Pour éviter la ruine avec certitude, on suppose que la prime est calculé de telle sorte que

$$E\left[cW_i - X_i\right] > 0.$$

Outre le temps de la ruine, on s'intéresse aussi au déficit au moment de la ruine $|U(\tau)|$ et au surplus juste avant la ruine $U(\tau^-)$. À cette fin, on analyse l'espérance de la valeur actualisée de la fonction de pénalité de Gerber-Shiu apparue dans Gerber et Shiu (1998) qui est définie comme suit

$$\Upsilon_\delta(u) = E\left[e^{-\delta\tau} w\left(U(\tau^-), |U(\tau)|\right) I(\tau < \infty) | U(0) = u\right], \qquad u \geq 0, \tag{1.9}$$

où $\delta \geq 0$ est un taux d'actualisation et $w(x,y) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction de pénalité positive et I est la fonction indicatrice.

Cette fonction de pénalité est utilisée dans le cadre du modèle classique de la ruine dans Gerber et Shiu (1998) et à l'aide de processus de diffusion dans Powers (1995).

La fonction de Gerber-Shui offre une généralisation de certaines mesures de ruine. Dans le cas $\delta = 0$ et $w(x,y) = 1$, on trouve la probabilité de ruine sur un horizon de temps infini. La transformé de Laplace du temps de la ruine τ est obtenue si l'on pose $w(x,y) = 1$ et (1.9) devient

$$L_\delta(u) = E\left[e^{-\delta\tau} I(\tau < \infty) | U(0) = u\right], \qquad u \geq 0.$$

L'introduction de cette fonction de pénalité escomptée par Gerber et Shiu a constitué une grande contribution dans la littérature de la théorie de la ruine. Asmussen et Albrecher (2010, Chapitre XII) détaille davantage ce point. Dans le troisième chapitre de cette thèse, on utilise un cas particulier de la fonction Gerber-Shiu.

La Figure 2 montre une réalisation du processus de surplus $U(t)$ avec des v.a. d'intérêt de théorie de la ruine : le temps de la ruine, le surplus juste avant la ruine et le déficit au moment de la ruine.

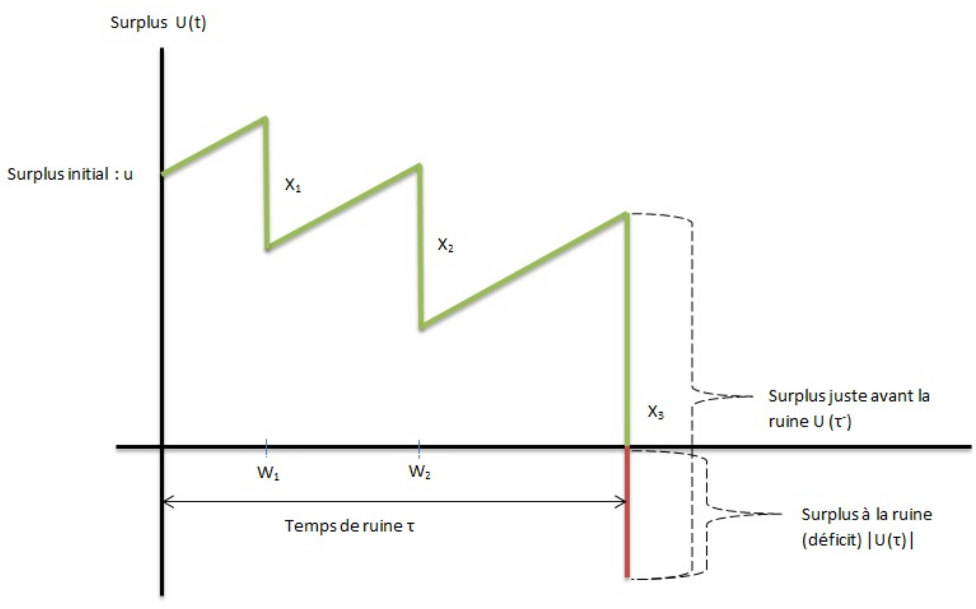


Figure 1.2: Une trajectoire du surplus $U(t)$.

En théorie de la ruine, d'autres quantités sont étudiées pour élaborer des mesures de ruine. Parmi ces quantités d'intérêt, on définit le processus de la perte totale $\underline{Y} = \{Y(t), t \geq 0\}$ tel que $Y(0) = 0$ et

$$Y(t) = S(t) - ct,$$

où $Y(t)$ représente la perte totale à l'instant t.

On peut écrire $Y(t) = u - U(t)$ et l'on exprime le moment de la ruine τ en fonction de $Y(t)$ comme suit

$$\tau = \inf\{t > 0 | Y(t) > u\}.$$

En théorie de la ruine, on s'intéresse aussi au processus de la perte maximale totale définie par

$$Z = \sup_{t \leq \tau} \{Y(t), t \geq 0\}.$$

On peut redéfinir la probabilité de ruine sur un horizon de temps infini avec un surplus initial u comme suit

$$\psi(u) = \overline{F}_Z(u),$$

où $\overline{F}_Z$ est la fonction de survie de Z.

La distribution de Z permet d'analyser le risque associé au portefeuille. En effet, le risque d'insolvabilité peut aussi être mesuré à l'aide des mesures du risque $VaR(Z)$ et $TVaR(Z)$. Dans le cadre du modèle de Sparre-Anderson et selon e.g. Rolski et coll. (1999, Chapitre 6), la variable aléatoire Z suit une distribution géométrique composée, i.e.

$$Z = \begin{cases} \sum_{i=1}^{M} L_i, & M > 0, \\ 0, & M = 0, \end{cases},$$

où les variables aléatoires L_i, $i = 1, , ...$, représentent les *ladder heights.* et M est une variable aléatoire géométrique dont la fonction de masse de probabilité est donnée par

$$\Pr(M = m) = (1 - \psi(0))(\psi(0))^m, \qquad m = 0, 1, ...$$

En plus de la perte maximale totale Z, on s'intéresse aussi aux maxima relatifs de la perte totale, i.e. $Z(t) = max_{s \leq \tau}\{Y(s) : 0 \leq s \leq t\}$, en regardant les moments où le processus atteint un nouveau pic record dans le temps. On définit la suite de variables aléatoires $\{\upsilon_i\}_{i=1}^{\infty}$ telle que

$$\upsilon_1 = \inf\{t \geq 0 : Y(t) > 0\},$$

et

$$\upsilon_i = \inf\{t \geq \upsilon_{i-1} : Y(t) - Y(\upsilon_{i-1}) > 0\}.$$

Alors, le montant de la perte maximale totale $Z(t)$ suit une distribution géométrique composée, i.e. on peut exprimer $Z(t)$ comme suit

$$Z(t) = \begin{cases} \sum_{i=1}^{M} L_i 1(\upsilon_i \leq t), & M > 0, \\ 0, & M = 0. \end{cases},$$

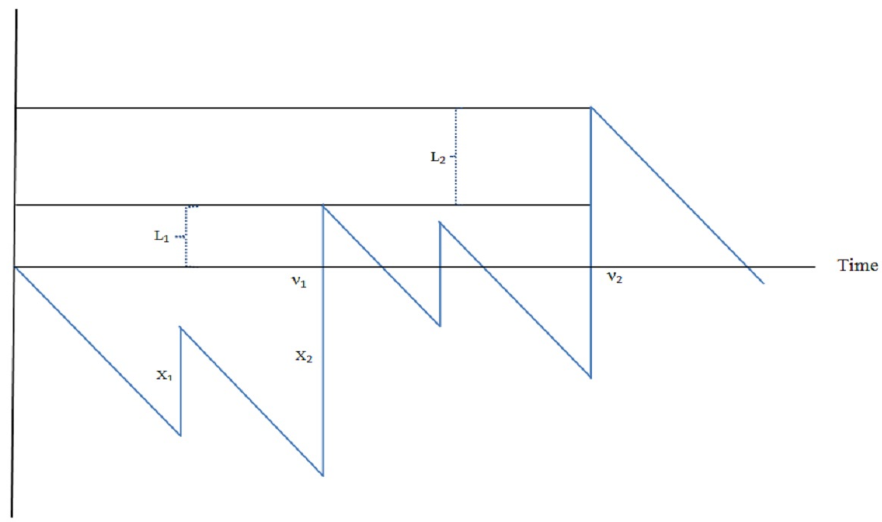


Figure 1.3: Une trajectoire de $\underline{Y}$.

La Figure 3 présente une illustration d'une trajectoire du processus $\underline{Y}$ et toutes les quantités associées.

Dans le deuxième chapitre de cette thèse, on s'intéresse au processus de la valeur actualisée de la perte maximale totale notée Z_δ où la constante δ est la force d'intérêt. Le processus Z_δ est défini par

$$Z_\delta(t) = \begin{cases} \sum_{i=1}^{M} e^{-\delta v_i} L_i 1(v_i \leq t), & M > 0, \\ 0, & M = 0. \end{cases},$$

On calcule les moments non-centrés, i.e. $m_{\delta,n}(t) = E[(Z_\delta(t))^n]$, pour un horizon de temps t dans deux contextes. Dans le premier cas, on suppose que les temps inter-sinistres suivent une loi exponentielle et les montants de sinistre ont une distribution mélange d'Erlang. Dans le second contexte, on fait l'hypothèse que les temps inter-sinistre suivent une loi mélange d'Erlang et les montants de sinistre ont une distribution exponentielle.

Dans les deux cas, on développe des formules récursives pour calculer les moments $m_{\delta,n}(t)$. Pour analyser la distribution de $Z_\delta(t)$, on propose une approximation basée sur la distribution mélange Erlang. Cette approximation utilise les moments exacts que l'on calcule à l'aide des formule récursives.

1.5 Modèles d'investissement

Les modèles stochastiques d'investissement (MSI) jouent un rôle très important en sciences actuarielles et gestion quantitative des risques. Ces modèles sont très utiles pour des applications actuarielles basées sur des générateurs de scénarios économiques et financiers. En assurance, les MSIs constituent la base de l'analyse dynamique et l'évaluation des risques. Ces modèles sont aussi importants pour les institutions de réglementation et les agences de notation qui évaluent la santé financière des compagnies d'assurance. En outre, les MSIs forment un excellent outil de gestion des risques dans le cadre de l'allocation d'actif et la gestion de portefeuille des compagnies. Les fonds de pension et les compagnies d'assurance vie utilisent les MSIs pour évaluer leurs passifs et aussi dans la gestion actif-passif. Les MSIs aident aussi à déterminer les réserves et les primes pour certains contrats d'assurance. Ahlgrim et coll. (2004) et Ahlgrim et coll. (2008) proposent une excellente revue de la littérature des générateurs de scénarios économiques développés à l'aide des MSIs et appliqués en assurance.

Le développement des MSIs a été initialisé avec la publication en 1980 du rapport Maturity Guarantees Working Party (MGWP) (1980). MGWP (1980) propose un modèle de séries chronologiques pour une variété de quantités financières. Ce modèle a été utilisé pour calculer les réserves des contrats d'assurance vie en unités de compte avec une garantie. La publication de ce rapport a conduit au développement du modèle de Wilkie dont la première version est décrite dans Wilkie (1986). Ce MSI proposé par Wilkie est un modèle de séries temporelles multivariées à quatre composantes : l'inflation, les rendements d'action, les dividendes sur les actions et les taux d'intérêt à long terme. Le modèle de Wilkie considère l'inflation comme variable indépendante et la relie aux autres variables selon une structure de dépendance en cascade. Ce modèle a été mis à jour dans Wilkie (1995). Cette mis-à-jour consiste en l'introduction de nouvelles composantes pour les salaires, les taux d'intérêt à court-terme, les biens immobiliers et le taux de change. Sahin et coll. (2008) discute les performances empiriques du modèle de Wilkie et y étudie aussi la stabilité des paramètres. On trouve plus de commentaires et discussions du modèle de Wilkie dans Sahin et coll. (2008). D'autres MSIs ont été développé à la lumière du modèle de Wilkie dont Yakoubov et al (1999) et Whitten et Thomas (1999).

Un autre MSI a été proposé par Hibbert et coll. (2001) dans lequel on génère des scenarios pour la structure par terme de taux d'intérêt (nominale et réelle), des taux d'inflations, ainsi que des rendements et les dividendes d'actions. Selon ce modèle, le taux réel court-terme suit un modèle à deux facteurs Hull-White. La dynamique de l'inflation est supposée suivre un processus avec retour à la moyenne à deux facteurs. La structure par terme des taux d'intérêt réels est déduite de celles des taux d'intérêt et des taux d'inflation. On détermine les rendements d'actions en supposant un modèle à changement de régime markovien et on introduit des excès de rendement par rapport au taux d'intérêt nominal à court-terme. Le modèle des dividendes sur les actions est un modèle autorégressif de premier ordre.

D'autres modèles d'investissement ont été élaborés dans en temps continu (Battocchio et coll. (2004) et Munk et coll. (2004)). Battocchio et coll. (2004) propose un MIS qu'on applique pour analyser la dynamique d'un régime de retraite et l'allocation optimale d'actif d'un régime de retraite à cotisations déterminées. Ce modèle comporte trois principales composantes. On détermine les taux d'intérêt nominaux à l'aide un processus Ornstein-Uhlenbeck comme dans Vasicek (1977). Le modèle des rendements d'action contient une prime du risque qui décrit l'excès de ces rendements par rapport au taux d'intérêt à court terme. Le MSI, décrit dans Battocchio et coll. (2004), intègre l'inflation à l'aide d'un processus stochastique pour le niveau de prix à la consommation.

Munk et coll. (2004) propose un MSI pour analyser la rationalité des décisions d'allocation d'actif. Dans ce modèle, le taux d'intérêt court-terme est décrit par un modèle de Vasicek et les rendements sur les actions sont déterminés par un modèle de retour à la moyenne. Munk et coll. (2004) introduit une dynamique d'inflation à l'aide d'un modèle à deux facteurs de l'indice des prix à la consommation.

Les récents développements des marchés financiers ont contribué aux développements de la finance internationale et encouragé les investisseurs à chercher des opportunités d'investissement à l'étranger. Ceci a conduit au besoin de développer des modèles d'investissement pour plusieurs économies. Dans cette thèse, on propose un MSI dans le contexte deux économies. L'objectif de ce modèle est de représenter conjointement l'incertitude dans les deux économies. On propose un modèle à quatre composantes : une structure conjointe de taux d'intérêt, un modèle conjoint de rendements d'action, un modèle des taux d'inflation dans les deux économies et un modèle de la dynamique du taux de change. Dans les prochaines sous-sections, on présente un résumé des différents modèles proposés dans la littérature financières et économétriques pour chaque composante de notre MIS.

1.5.1 Modèles affine de structure par terme des taux d'intérêts

La structure par terme des taux d'intérêt joue un rôle très important en finance. C'est l'outil le plus important en gestion quantitative des risques et dans la détermination des coûts de capitaux. Par conséquent, une part importante de la littérature financière a été consacrée à développer des modèles de taux d'intérêt. La plupart de ces modèles décrivent l'évolution du taux d'intérêt à court terme en utilisant des processus stochastiques. Les premiers modèles stochastiques de taux d'intérêt ont été proposés par Vasicek (1977) et Cox et coll. (1985). Depuis la publication de ces deux articles, on trouve dans la littérature sur les taux d'intérêt beaucoup d'articles scientifiques portant sur l'expression des rendements des obligations sous forme d'une fonction affine de certaines variables d'état. Ceci a conduit à des extensions des modèles Vasicek et CIR. Duffie et Kan (1996) propose une caractérisation générale de ces modèles affines. Cet article est considéré comme l'un des piliers de la théorie moderne des taux d'intérêt. Dai et Singleton (2000) décrit ces modèles affines en introduisant une classification canonique des processus stochastiques suivis par les variables d'état. On suggère de consulter Piazzesi (2003) pour plus de détails concernant les modèles affines de taux d'intérêt. Piazzesi (2003) présente les caractéristiques de ces modèles qui les rendent très utiles dans l'évaluation des actifs à revenu fixe. Récemment, on s'intéresse au comportement conjoint de plusieurs marchés d'obligations. Des extensions des modèles affines pour un seul marché ont été introduites pour déterminer la structure par terme conjointe des taux d'intérêt. On trouve dans Ahn (2004) et Dewachter et Maes (2001) des modèles affines à trois facteurs pour plusieurs économies et Brennan et Xia (2004) introduit un modèle affine gaussien pour déterminer la structure par terme des taux d'intérêt dans deux économies. La performance des modèles affines des taux d'intérêts pour plusieurs économies a été analysée dans Mosburger et Schneider (2005) et Egorov et coll. (2008).

1.5.2 Modèle avec changement de régime Markovien des rendements d'action

La modélisation des rendements des actions ou des indices boursiers est aussi essentielle dans la pratique et la recherche financière. Prédire le comportement de ces rendements est la base de la gestion des risques financiers, de la gestion de portefeuille et d'allocation d'actif. Plusieurs modèles ont été développés pour capter le comportement aléatoire des rendements observés sur le marché financier. En observant les rendements d'action, on peut caractériser le comportement de ces rendements par : une queue de distribution épaisse, un *clustering* de la volatilité, une mémoire longue et persistante de la volatilité. On observe que les rendements ont des distributions leptokurtiques qui s'expliquent par la présence d'hétérosdacité conditionnelle. Le *clustering* de la volatilité consiste en présence de nombreuses périodes de forte volatilité et d'autres périodes de faible volatilité. La théorie financière introduit cette volatilité sous forme d'un processus stochastique non-linéaire. Pour tenir compte de ce caractère non linéaire, plusieurs approches sont proposées. Parmi celles-ci, on retrouve les modèles markoviens à changement de régime, selon lesquels une série temporelle financière est modélisée en supposant la coexistence de plusieurs régimes qui affectent la dynamique du processus. La transitions entre les différents régimes dans le temps est représentée par une chaine de Markov.

Les modèles à changements de régimes markoviens ont été introduits par Lindgren (1978) en économie. Ces modèles ont été fortement popularisés par Hamilton (1989), qui a employé ces modèles afin de décrire les cycles conjoncturels. Les travaux qui s'en suivent sont des travaux visant à employer ces modèles pour modéliser les séries de taux d'intérêt (Gray, 1996, Ang et Bekaert, 2001). D'autres travaux ont introduit les modèles à changements de régimes markovien pour les actions et on suggère de consulter Huntley et van Norden (1997), Nielsen et Olesen (2001) et Hardy (2001) pour plus de détails.

1.5.3 Modélisation de l'inflation

L'inflation est une composante clé dans la modèlisation des coûts en assurance et en gestion quantitative des risques financiers. Aussi bien en théorie qu'en pratique, l'inflation est considérée comme une force conductrice des autres variables macro-économiques et financières par exemple les taux d'intérêt ou l'indice des salaires. Elle constitue aussi un facteur de risque pour les actifs à revenu fixe et les dettes liées à l'inflation (inflation-linked liabilities). On suggère de consulter Daykin et coll. (1994) qui souligne l'importance de l'inflation pour l'industrie de l'assurance.

Depuis la publication du modèle de Wilkie, le développement des modèles stochastiques des taux d'inflation pour des applications en actuariat et en investissement est rendu un sujet très intéressant pour les actuaires. Dans la littérature actuarielle, la modélisation des taux d'inflation a été initialisée dans Wilkie (1986). Dans cette première version du modèle, Wilkie estime un modèle linéaire autorégressif d'ordre 1 en utilisant une série de données annuelles des taux d'inflation au Royaume Uni. Ce modèle d'inflation a été amélioré dans la deuxième version du modèle de Wilkie en 1995 en ajoutant des termes Auto-Régressifs Conditionnellement Hétéroscédastiques (ARCH) aux résidus. Cette composante ARCH est apte à capter le comportement de la volatilité dans le temps. La révision du modèle de Wilkie proposée dans Whitten et Thomas (1999) critique l'utilisation des modèles de type ARCH pour l'inflation. Comme alternative, Whitten et Thomas (1999) défend l'utilisation des modèles non-linéaires en se basant sur 'self-exciting threshold autoregressive models' (SETAR). Les modèles SETAR cherchent à capter le comportement non-linéaire des taux d'inflation en permettant un changement de régime. Ce changement de régime dépend des valeurs prises par les taux d'inflation d'où le terme 'self' dans le nom SETAR.

Les modèles d'inflation développés dans Wilkie (1986), Wilkie (1995) et Whitten et Thomas (1999) sont à temps discret et basés sur les séries temporelles. D'autres modèles ont été proposés en temps continu, ces modèles sont généralement sous forme de modèles de diffusions définis à l'aide des équations différentielles stochastiques. Parmi ces modèles en temps continu, on trouve la composante inflation dans le MSI introduit par Hibbert (1995) où l'on cherche à générer la structure par terme de l'inflation anticipée. Le taux d'inflation est supposé suivre un modèle gaussien à deux facteurs avec retour à la moyenne. L'utilisation d'un modèle à deux facteurs permet de contrôler la volatilité de l'inflation et ajoute une flexibilité à la corrélation de la structure par terme d'inflation. Le modèle stochastique pour inflation proposé par Munk et coll. (2004) est aussi un modèle à deux facteurs dont un seul facteur possède un retour à la moyenne. Dans Battochio et coll. (2004), on propose un processus d'Itô pour le niveau général des prix. Dans ce processus, on introduit des corrélations avec les rendements des actions et le taux d'intérêt court-terme.

1.5.4 Modèles dynamiques de taux de change

Tous les investisseurs ou les assureurs qui détiennent des actifs ou des passif en plusieurs devises sont appelés à gérer le risque du taux de change. Ultimement, une gestion du risque lié au taux de change nécessite une modélisation de la dynamique du taux de change. Il existe plusieurs approches pour prédire les taux de change. Dans un premier temps, on peut utiliser d'autres variables macro-économiques pour expliquer les variations du taux de change. Cette approche est généralement adoptée par les banques centrales pour établir leurs politiques monétaires.

Depuis l'introduction des taux de change flexible au début des années 1970, les économistes ont essayé de développer des théories qui expliquent le comportement des taux de change et pour prédire son évolution. Comme il le souligne De Grauwe et coll. (1993) et après plusieurs essaies, le succès de ces théories était très limité. Selon Rosenberg (1996), les données empiriques confirment que le modèle de marche aléatoire performe mieux que ces modèles basés sur des théories économiques fondamentales. Une autre approche intéressante a été introduite par De Grauwe et coll. (1993) où l'on fait appel à la théorie du chaos qui est très utile pour présenter des processus à volatilité importante. La théorie de la structure conjointe par terme des taux d'intérêt propose aussi de déterminer la dynamique du taux de change. Selon cette théorie, le taux de change est un facteur qui assure l'absence d'opportunité d'arbitrage entre le marché obligataire de chaque économie. On utilise cette approche dans Dewachter et Maes (2001) et dans Mosburger (2005) où l'on dérive la dynamique de taux de change à l'aide de modèles affines des taux d'intérêt.

1.5.5 Notre modèle d'investissement

Dans le quatrième chapitre de cette thèse, on présente un modèle d'investissement qui est constitué de quatre modules dans le cas de deux économies : l'économie canadienne et l'économie américaine. Le premier module décrit le comportement du taux d'intérêt et propose une modélisation conjointe des structures par terme dans les deux économies. Ce module sert à évaluer tous les produits à revenus fixes tels que les obligations et les fonds obligataires dans les deux pays. Le second module décrit le comportement conjoint des indices boursiers canadien et américain. Le troisième module propose une modélisation du comportement de l'inflation dans les deux pays. Ce modèle d'investissement décrit aussi la dynamique du taux de change canadien/américain

On donne aussi quelques applications de notre modèle dans le cadre de la quantification et l'allocation des risques. Pour cette fin, on génère des scénarios en utilisant notre modèle d'investissement puis on détermine une allocation du risque à l'aide de la règle d'allocation TVaR. Cette technique est très flexible ce qui nous permet de donner une quantification à la fois du risque d'investissement, du risque d'inflation et du risque du taux de change. Dans ce chapitre, on étudie aussi le comportement de l'allocation du risque en présence d'un passif.

Chapitre 2

Multivariate Mixed Erlang Distributions defined with the Farlie-Gumbel-Morgenstern Copula : Aggregation and Capital Allocation

Résumé

Dans ce chapitre, on étudie le problème d'agrégation des risques et d'allocation du capital pour des risques dépendants. On est interessé par un portefeuille de n contrats d'assurance. La distribution du montant à payer pour chaque contrat (risque), X_i , suit une loi mélange d'Erlang et on note $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, pour $i = 1, ..., n$.

Le montant total des sinistres à payer pour tout le portefeuille est défini par la somme

$$S_n = \sum_{i=1}^{n} X_i.$$

On suppose que les risques X_i , $i = 1, ..., n$, sont dépendants et la structure de dépendance est basée sur la copule FGM définie dans le premier chapitre.

En se basant sur certaines propriétés des distributions mélange d'Erlang et de la famille de copules FGM, on prouve que le montant total des sinistres S_n a une distribution mélange d'Erlang dont on détermine les caractéristiques. En plus, on trouve des expressions exactes de la covariance entre les risques X_i et X_j, pour $i \neq j = 1, \ldots, n$. On donne aussi l'expression de la mesure du risque TVaR et de la prime stop-loss associées à la variable aléatoire S_n .

Dans ce chapitre, on s'intéresse aussi à l'allocation du capital. Deux règles d'allocation du

capital sont étudiées : la règle basée sur la TVaR et la règle basée sur la covariance. Dans les deux cas, on trouve des expressions exactes du montant alloué à chaque risque X_i , pour $i = 1, \ldots, n$. Les résultats élaborés dans ce chapitre sont présentés dans un premier temps pour le cas bivarié puis ensuite généralisés au cas multivarié. Pour des fins d'illustration, on présente deux exemples numériques dans le cas bivarié et trivarié.

2.1 Introduction

In light of the new regulation requirements, insurance companies are required to determine their capital allocation according to the risks to which they are exposed. In such a context, risk management raises some issues about risk aggregation and capital allocation. A risk capital must be held by the institution for the whole business portfolio to insure a safety financial level and also to be allocated adequately to each risk. The different capitals are commonly determined using an adequate risk measure which is a mapping from the random variable space into the real numbers that allows risk ordering. Artzner et al. (1999) give an axiomatic definition of a risk measure and introduce the concept of coherent measures of risk. Artzner (1999) examines the implication of using coherent risk measures on capital requirements in an insurance context. As for Wang (2002), he notably discusses coherent methods to determine the aggregate capital requirement for a firm and the capital allocation to individual business units. These methods for enterprise risk management can be used for asset/loss portfolio optimization. Both Artzner (1999) and Wang (2002) suggest using the Tail Value at Risk (TVaR), also called the Expected Shortfall (ES), to replace the usual Value at Risk given that it does not meet the subadditivity criterion. The TVaR is a coherent risk measure and it is equal to the Conditional Tail Expectation (CTE) in the continuous case. In a discrete setting, as explained in Acerbi (2001) and Acerbi and Tasche (2002), the TVaR remains a coherent risk measure while the CTE is no longer coherent. See also McNeil et al. (2005) for details on risk measures and their application in a quantitative risk management context.

To allocate capital to different lines of business, Denault (2001) suggests a set of desirable properties for a fair risk capital allocation principle. More precisely, his axioms define the coherence of risk capital allocation principles, in a similar way as Artzner et al. (1999) in the context of risk measures. The top down allocation method introduced by Tasche (2000) has been used to provide several closed-form formulae and approximations of the TVaR and the TVaR-based allocations for different types of multivariate continuous distributions. For example, Panjer (2002) shows that the TVaR-based allocation principle is identical to the covariance-based principle when multivariate normal distributions are considered. Dhaene et al. (2008) develop a closed-form expression for the TVaR allocation under multivariate elliptical distributions. Barges et al. (2009) give a closed-form expression for the TVaR-based allocation when lines of business of an insurance portfolio are linked with a Farlie-Gumbel-Morgenstern (FGM) copula and when marginal risks are distributed as mixtures of exponentials. Other applications of the TVaR-based allocation principle are also provided in Cossette et al (2012a). Buch and Dorfleitner (2008) discuss the gradient allocation principle which generalizes well known allocation principles including the TVaR and covariance rules.

In this paper, we address risk aggregation and capital allocation problems for a portfolio of dependent risks whose multivariate distribution is defined with an FGM copula and mixed Erlang distributed marginals. The class of mixed Erlang distributions has many interesting features which are discussed in detail notably in Willmot and Lin (2010) (see also references therein). These authors show the usefulness of the class of mixed Erlang distributions. With several examples, they illustrate the versatility of this distribution to model claim amounts and the feasibility to obtain closed-form expressions for various quantities of interest in risk theory. We capitalize on the tractability of the FGM copula and the mixed Erlang distribution class to analyze the stochastic behavior of the aggregate claim amount for a portfolio of dependent risks in order to determine the amount of economic capital needed for the whole portfolio. More precisely, we show that under these assumptions the aggregate claim amount follows a mixed Erlang distribution which will then allow us to find, based on a top-down approach, explicit expressions for the amount of capital to be allocated to each risk based on the TVaR and covariance rules.

The paper is organized as follows. The first section fixes some notations, definitions and provides some preliminary results. The results for the bivariate case are given in Section 3 together with essential theorems and an illustrative example. Section 4 shows how to extend the results to the multivariate case and provides a numerical example for the trivariate case.

2.2 Definitions

In this section, we briefly recall the definition and characteristics of the FGM copula. We also give the definitions of the risk measures Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) as well as the allocation rules based on the TVaR and the covariance (see e.g. McNeil et al. (2005)). We end this section by presenting the mixed Erlang distribution and its properties.

2.2.1 Farlie-Gumbel-Morgenstern copula

Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n continuous random variables (rvs) with joint cumulative distribution function (cdf) denoted by $F_{\underline{X}}$ and univariate marginals F_{X_i}, $i = 1, \ldots, n$. According to Sklar's theorem, see e.g. Sklar (1959) and Nelsen (2006), $F_{\underline{X}}$ can be written as a function of the univariate marginals F_{X_i}, $i = 1, \ldots, n$, and the copula C describing the dependence structure as follows

$$F_{\underline{X}}(x_1, \ldots, x_n) = C\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right).$$

The joint probability density function (pdf) of $\underline{X}$ is given by

$$f_{\underline{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \ldots f_{X_n}(x_n)\, c\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right), \tag{2.1}$$

where c is the corresponding pdf of the copula C defined by

$$c(u_1, \ldots, u_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}.$$

In this paper, we are interested in the FGM copula. The bivariate FGM copula is defined by the joint cdf

$$C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \tag{2.2}$$

where the scalar θ is the dependence parameter with $\theta \in [-1, 1]$. The independence structure is reached when $\theta = 0$, i.e. $C^I(u_1, u_2) = u_1 u_2$. The pdf of the bivariate FGM copula is given by

$$c(u_1, u_2) = (1 + \theta) - \theta 2\overline{u}_1 - \theta 2\overline{u}_2 + \theta 2\overline{u}_1 2\overline{u}_2, \tag{2.3}$$

where $\overline{u}_i = 1 - u_i$.

The FGM copula is a perturbation of the product copula and, as mentioned in Nelsen (2006), is a first order approximation to the Ali Mikhail Haq, Frank and Placket copulas. With $\theta \in [-1, 1]$ and association measures such as Kendall's tau and Spearman's rho respectively given by $\tau = \frac{2\theta}{9}$ and $\rho = \frac{\theta}{3}$, moderate positive and negative dependence can be modelled with the FGM copula. This copula is attractive due to its simplicity and its form which allows explicit calculus and exact results. For example, Barges et al. (2009) investigate aggregation and capital allocation problems for an insurance company with several lines of business with dependence structure based on the FGM copula and with exponentially distributed risks. Prieger (2002) highlights its usefulness in model selection into health insurance plans. The FGM copula was also used to link claim variables in a credibility model in Yeo and Valdez (2006). The FGM copula with exponential margins was proposed by Jang and Fu (2011) to measure tail dependence between collateral losses. In finance, Cherubini et al. (2011) use the FGM copula for the analysis of financial time series and suggest a new technique to construct first order Markov processes using this copula. The FGM copula was also used to describe different correlation relations on the financial markets in Gatfaoui (2005) and Gatfaoui (2007). In risk theory, Cossette et al. (2008), Zhang and Yang (2011) and Chadjiconstantinidis and Vrontos (2012) consider risk models with a dependence structure between claim sizes and interclaim times based on the FGM copula.

In the multivariate case, for $n \geq 3$, the FGM n-copula, which has $2^n - n - 1$ parameters, is defined as follows

$$C(\underline{u}) = u_1 u_2 \dots u_n \left(1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \dots \overline{u}_{j_k}\right),$$

which is equivalent to

$$C(\underline{u}) = u_1 u_2 \dots u_n P(u_1, u_2, \dots, u_n), \tag{2.4}$$

where P denotes the polynomial

$$P(u_1, u_2, \dots, u_n) = 1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \dots < j_k \leq n} \theta_{j_1 j_2 \dots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \dots \overline{u}_{j_k}. \tag{2.5}$$

The pdf of the FGM n-copula is given by

$$c(\underline{u}) = P(2u_1, 2u_2, \ldots, 2u_n), \tag{2.6}$$

where the polynomial P is linear in argument u_i, for $i = 1, \ldots, n$.

According to Nelsen (2006), the FGM n-copula exists if the following constraints hold

$$1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \varepsilon_{j_1} \varepsilon_{j_2} \ldots \varepsilon_{j_k} \geq 0, \quad \varepsilon_{j_1} \varepsilon_{j_2} \ldots \varepsilon_{j_k} \in \{-1, 1\}. \tag{2.7}$$

We also mention that, for $k \in \{2, 3, ..., n\}$, each k-margin of an FGM n-copula is an FGM k-copula.

2.2.2 Risk measures and risk allocation

Let us consider the aggregate claim amount rv $S_n = \sum_{i=1}^{n} X_i$ with cdf F_{S_n}. The Value-at-Risk at level κ, $0 \leq \kappa < 1$, of S_n is defined by

$$VaR_\kappa(S_n) = \inf\{x \in \mathbb{R}, F_{S_n}(x) \geq \kappa\},$$

and the Tail-Value-at-Risk at level κ, $0 \leq \kappa < 1$, is defined by

$$\begin{aligned} TVaR_\kappa(S_n) &= \frac{1}{1-\kappa} \int_\kappa^1 VaR_u(S_n)\, du \\ &= \frac{E\left[S_n \times 1_{\{S_n > VaR_\kappa(S_n)\}}\right] + VaR_\kappa(S_n)\left(F_{S_n}(VaR_\kappa(S_n)) - \kappa\right)}{1-\kappa}, \end{aligned} \tag{2.8}$$

where 1_A is the indicator function such that $1_A(S_n) = 1$, if $S_n \in A$, and $1_A(S_n) = 0$, if $S_n \notin A$. Note that the truncated expectation of S_n, denoted by $E\left[S_n \times 1_{\{S_n > b\}}\right]$, can be expressed as $E[S_n] - E\left[S_n \times 1_{\{S_n \leq b\}}\right]$. See e.g. Acerbi (2002), Acerbi and Tasche (2002) and McNeil et al. (2005) for details on the risk measures VaR and TVaR.

When the rv S_n is continuous, we have $F_{S_n}(VaR_\kappa(S_n)) - \kappa = 0$ and (3.2) becomes

$$TVaR_\kappa(S_n) = \frac{E\left[S_n \times 1_{\{S_n > VaR_\kappa(S_n)\}}\right]}{1-\kappa} = E[S_n | S_n > VaR_\kappa(S_n)], \tag{2.9}$$

where $E[S_n | S_n > VaR_\kappa(S_n)] = CTE_\kappa(S_n)$ which means that the Tail-Value-at-Risk of a continuous rv is equal to its conditional tail expectation (which is not the case generally). In the present work, we prefer to use the term TVaR which is always coherent rather than the term CTE.

Risk measures allow to determine the amount of capital needed for the whole portfolio but one also needs to find the part of the capital that is allocated to each risk. For that purpose, we consider two capital allocation rules : the TVaR-based allocation rule and the Cov-based allocation rule.

The aim of capital allocation is to determine the amount of contribution C_i that is allocated to risk i $(i = 1, ..., n)$ such that

$$TVaR_\kappa\left(S_n\right) = \sum_{i=1}^{n} C_i.$$

Under the TVaR-based allocation rule, the amount of contribution C_i, denoted by $TVaR_\kappa\left(X_i, S_n\right)$, is given by

$$\begin{aligned} C_i &= TVaR_\kappa\left(X_i, S_n\right) \\ &= \frac{E\left[X_i 1_{\{S_n > VaR_\kappa(S_n)\}}\right] + \beta_{S_n} E\left[X_i 1_{\{S_n = VaR_\kappa(S_n)\}}\right]}{1-\kappa}, \; i = 1, ..., n, \end{aligned}$$

where

$$\beta_{S_n} = \begin{cases} \frac{Pr(S_n \le VaR_\kappa(S_n)) - \kappa}{Pr(S_n = VaR_\kappa(S_n))}, & \text{if } Pr\left(S_n = VaR_\kappa\left(S_n\right)\right) \neq 0 \\ 0, & \text{otherwise.} \end{cases}.$$

For continuous rvs, the TVaR-based allocations are equal to

$$TVaR_\kappa\left(X_i, S_n\right) = \frac{E\left[X_i 1_{\{S_n > VaR_\kappa(S_n)\}}\right]}{1-\kappa}.$$

Under the Covariance-based allocation rule, the contribution amount C_i, denoted $C_\kappa\left(X_i, S_n\right)$, is given by

$$C_\kappa\left(X_i, S_n\right) = E\left[X_i\right] + \frac{Cov(X_i, S_n)}{Var(S_n)}\left(TVaR_\kappa\left(S_n\right) - E\left[S_n\right]\right), \; i = 1, ..., n. \tag{2.10}$$

For both allocation rules, the sum of the allocations $\sum_{i=1}^{n} C_i$ is equal to $TVaR_\kappa\left(S_n\right)$, the amount of capital needed for the entire portfolio. Note also that both allocation rules satisfy Euler's principle. See e.g. Tasche (1999) and McNeil et al. (2005) for details on both allocation rules and Hesselager and Andersson (2002) for further information on the Cov-based allocation rule.

2.2.3 Mixed Erlang distribution

The mixed Erlang class has many useful analytic properties for actuarial and risk management applications. Closed-form expressions for important quantities in actuarial science are derived using this class of distributions (e.g. stop-loss premium, finite time ruin probabilities among others). Willmot and Lin (2010) and Lee and Lin (2010) give several illustrative examples. Tijms (1994) shows that the mixed Erlang class is dense in the set of positive continuous probability distributions. This is a significant advantage of this class since any positive continuous distribution can be approximated by a member of this class. In this subsection, we define the mixed Erlang distribution and present a list of its useful properties.

Let the pdf and cdf of an Erlang distribution of order $k \in \mathbb{N}^*$ and scale factor $\beta \in \mathbb{R}^+$ be defined by

$$h(x;k,\beta) = \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x},\ x > 0,$$

and

$$H(x;k,\beta) = 1 - e^{-\beta x} \sum_{j=0}^{n-1} \frac{(\beta x)^j}{j!},\ x > 0.$$

Let Y be a mixed Erlang rv with scale parameter β. The pdf and cdf of Y are respectively given by

$$\begin{aligned} f_Y(x) &= \sum_{k=1}^{\infty} p_k h(x;k,\beta), \\ F_Y(x) &= \sum_{k=1}^{\infty} p_k H(x;k,\beta), \end{aligned}$$

where p_k is the probability mass associated with the k^{th} Erlang distribution in the mixture and β is the scale parameter. We use the notation $Y \sim MixErl\left(\underline{p},\beta\right)$ with $\underline{p} = \{p_1, p_2, \ldots\}$.

One of the main advantages of the mixed Erlang distribution is that, most of the time, risk measures or other risk related quantities, such as the stop-loss premium, have either an exact expression or can be easily computed (see e.g. Willmot and Lin (2010) and references therein). For instance, since the expression for F_Y is analytic, the value of $VaR_\kappa(Y)$ can be easily obtained with any optimization tool (e.g. `optimize` in R and `solver` in Excel). The expression for $TVaR_\kappa(Y)$ is given by

$$TVaR_\kappa(Y) = \frac{1}{1-\kappa} \sum_{k=1}^{\infty} p_k \frac{k}{\beta} \overline{H}\left(VaR_\kappa(Y); k+1, \beta\right), \tag{2.11}$$

where $\overline{H}(x;k,\beta) = 1 - H(x;k,\beta)$ is the survival function of an Erlang distribution. For the expression of the stop-loss premium for a given retention $d \geq 0$, defined by $\pi_Y(d) = E\left[\max(Y-d;0)\right]$, we have

$$\pi_Y(d) = \sum_{k=1}^{\infty} p_k e^{-\beta d} \frac{(\beta d)^k}{k!}. \tag{2.12}$$

In the rest of this subsection, we present some useful properties of the mixed Erlang distribution. First, we show that the equilibrium density of Y, defined by $f_Y^{\varepsilon}(x) = \frac{\overline{F}_Y(x)}{E[Y]}$ with $\overline{F}_Y(x)$ the survival function of Y, can be written as the pdf of a mixed Erlang distribution.

Lemma 2.2.1. *The equilibrium distribution associated with the rv Y is a mixed Erlang distribution with parameters $(\underline{\varepsilon}(\underline{p}) = \{\varepsilon(k, \underline{p}), k \in \mathbb{N}^*\}, \beta)$ and pdf given by*

$$f_Y^{\varepsilon}(x) = \sum_{k=1}^{\infty} \varepsilon(k, \underline{p}) h(x; k, \beta),$$

where

$$\varepsilon(k, \underline{p}) = \frac{\sum_{j=k}^{\infty} p_j}{\sum_{j=1}^{\infty} j p_j}, \text{ for } k = 1, 2, \dots \ .$$

Proof 2.2.1. *See Section 3 of Willmot and Lin (2010).*

Let Y_1 and Y_2 be two independent rvs with $Y_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$ and $\beta_i = \beta$, $i = 1, 2$. The next lemma shows that the sum $Y_1 + Y_2$ also has a mixed Erlang distribution.

Lemma 2.2.2. *Under the condition that $\beta_1 = \beta_2 = \beta$, the rv $T_2 = Y_1 + Y_2$ has a mixed Erlang distribution with parameters $(\underline{\sigma}^{(2)}(\underline{p}_1, \underline{p}_2) = \{\sigma^{(2)}(k, \underline{p}_1, \underline{p}_2), k \in \mathbb{N}^*\}, \beta)$ and pdf given by*

$$f_{T_2}(s) = \sum_{k=1}^{\infty} \sigma^{(2)}(k, \underline{p}_1, \underline{p}_2) h(s; k, \beta),$$

where

$$\sigma^{(2)}(k, \underline{p}_1, \underline{p}_2) = \begin{cases} 0, & \text{for } k = 1, \\ \sum_{j=1}^{k-1} p_{1,j} p_{2,k-j}, & \text{for } k = 2, 3, \dots \ . \end{cases}$$

Proof 2.2.2. *This result is a special case of Proposition 5 in Cossette et al. (2012a).*

Remark 2.2.1. *The result of Lemma 2.2.2 can be generalized to a portfolio of n independent risks $Y_1, ..., Y_n$, where $Y_i \sim MixErl\left(\underline{p}_i, \beta\right)$ and $\beta_i = \beta$ for $i = 1, 2, ..., n$. More precisely, for the aggregate claim amount $T_n = \sum_{i=1}^{n} Y_i$, we have*

$$T_n \sim MixErl\left(\underline{\sigma}^{(n)}(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n), \beta\right),$$

where the probabilities $\underline{\sigma}^{(n)}(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n) = \left(\sigma^{(n)}(k, \underline{p}_1, \underline{p}_2, \dots, \underline{p}_n), k \in \mathbb{N}^*\right)$ are computed with the following recursive expression

$$\sigma^{(n+1)}(k, \underline{p}_1, \underline{p}_2, \dots, \underline{p}_{n+1}) = \begin{cases} 0, & \text{for } k = 1, ..., n \\ \sum_{j=n}^{k-1} \sigma^{(n)}(j, \underline{p}_1, \underline{p}_2, \dots, \underline{p}_n) p_{n+1,k-j}, & \text{for } k = n+1, n+2, \dots \end{cases}$$

for $n = 2, 3, ...$.

Let Y be a positive rv with $Y \sim MixErl\left(\underline{p}, \beta\right)$. In the lemma that follows, we state that $2 f_Y(x) \overline{F}_Y(x)$ is the pdf of a mixed Erlang distribution.

Lemma 2.2.3. *Let* $f_Y^\pi(x) = 2f_Y(x)\overline{F}_Y(x)$ *where* $Y \sim MixErl\left(\underline{p},\beta\right)$. *Then,* $f_Y^\pi(x)$ *is the pdf of a mixed Erlang distribution with parameters* $(\underline{\pi}(\underline{p}) = \{\pi(j,\underline{p}), j \in \mathbb{N}^*\}, 2\beta)$ *and defined as*

$$f_Y^\pi(x) = \sum_{j=1}^{\infty} \pi(j,\underline{p}) h(x; j, 2\beta),$$

where

$$\pi(j,\underline{p}) = \frac{1}{2^{j-1}} \sum_{k=1}^{j} \binom{j-1}{k-1} p_k \sum_{l=j-k+1}^{\infty} p_l, \ for\ j = 1, 2, \dots \ . \tag{2.13}$$

Proof 2.2.3. *To identify the expression of* $\pi(j,\underline{p})$, *we begin by writing* $f_Y^\pi(x)$ *as*

$$f_Y^\pi(x) = 2E[Y] f_Y(x) f_Y^\varepsilon(x),$$

where

$$\begin{aligned} f_Y(x) f_Y^\varepsilon(x) &= \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} p_k \varepsilon(j-k+1,\underline{p}) \beta^{j+1} e^{-2\beta x} \frac{x^{j-1}}{(k-1)!(j-k)!} \qquad (2.14) \\ &= \sum_{j=1}^{\infty} \frac{\beta}{2^j} \sum_{k=1}^{j} \binom{j-1}{k-1} p_k \varepsilon(j-k+1,\underline{p}) h(x; j, 2\beta). \end{aligned}$$

Using Lemma 2.2.1, we multiply both sides of (2.14) by $2E[Y]$ *and we obtain*

$$\begin{aligned} f_Y^\pi(x) &= \sum_{j=1}^{\infty} \frac{\beta E[Y]}{2^{j-1}} \sum_{k=1}^{j} \binom{j-1}{k-1} p_{1,k} \varepsilon(j-k+1,\underline{p}) h(x; j, 2\beta) \\ &= \sum_{j=1}^{\infty} \left(\frac{1}{2^{j-1}} \sum_{k=1}^{j} \binom{j-1}{k-1} p_{1,k} \sum_{l=j-k+1}^{\infty} p_l \right) h(x; j, 2\beta) \\ &= \sum_{j=1}^{\infty} \pi(j,\underline{p}) h(x; j, 2\beta), \end{aligned}$$

where the expression for $\pi(j,\underline{p})$ *is as given in (2.13).*

From (2.13), we observe that $\pi(j,\underline{p}) \geq 0$, *for* $j = 1, 2, \dots$ *and since* $\int_0^\infty f_\pi(x)\,dx = 2E[1 - F_Y(Y)] = 1$, *we have* $\sum_{j=1}^\infty \pi(j,\underline{p}) = 1$. *We conclude that* f_Y^π *can be written as the pdf of a mixed Erlang distribution.*

We may write the pdf of a mixed Erlang rv with scale parameter β_1 in terms of the pdf of a mixed Erlang rv with scale parameter β_2. This is shown below.

Lemma 2.2.4. *Let* $Y \sim MixErl\left(\underline{p},\beta_1\right)$ *and* $\beta_1 \leq \beta_2$. *Then, the pdf of* Y *can be expressed as the pdf of a mixed Erlang distribution with parameters* $(\underline{\omega}(\underline{p},\beta_1,\beta_2) = \{\omega(k,\underline{p},\beta_1,\beta_2), k \in \mathbb{N}^*\}, \beta_2)$ *as*

$$f_Y(x) = \sum_{k=1}^{\infty} \omega(k,\underline{p},\beta_1,\beta_2) h(x; k, \beta_2),$$

with

$$\omega(k,\underline{p},\beta_1,\beta_2)=\sum_{j=1}^{k}p_j\binom{k-1}{k-j}\left(\frac{\beta_1}{\beta_2}\right)^j\left(1-\frac{\beta_1}{\beta_2}\right)^{k-j},\ for\ k=1,2,\dots\ .$$

Proof 2.2.4. *See Proposition 2 in Cossette et al. (2012b).*

Remark 2.2.2. *Note that for a given positive rv* $Y\sim MixErl\left(\underline{p},\beta\right)$*, we have* $\underline{\omega}(\underline{p},\beta,\beta)=\underline{p}$*.*

For a given rv $Y\sim MixErl\left(\underline{p},\beta\right)$, we define $f_Y^{\alpha}(x)=\frac{xf_Y(x)}{E[Y]}$. The following lemma shows that f_Y^{α} is the pdf of a mixed Erlang distribution.

Lemma 2.2.5. *Let* $f_Y^{\alpha}(x)=\frac{xf_Y(x)}{E[Y]}$ *where* $Y\sim MixErl\left(\underline{p},\beta\right)$*. Then,* $f_Y^{\alpha}(x)$ *can be expressed as the pdf of a mixed Erlang distribution with parameters* $(\underline{\alpha}(\underline{p})=\{\alpha(k,\underline{p}),k\in\mathbb{N}^*\},\beta)$ *i.e.*

$$f_Y^{\alpha}(x)=\sum_{k=1}^{\infty}\alpha(k,\underline{p})h\left(x;k,\beta\right),$$

with

$$\alpha(k,\underline{p})=\begin{cases}0, & if\ k=1,\\ \frac{(k-1)p_{k-1}}{\sum_{j=1}^{\infty}jp_j}, & if\ k=2,3,\dots\ .\end{cases}$$

Proof 2.2.5. *We have*

$$\begin{aligned}xf_Y(x) &= x\sum_{k=1}^{\infty}p_kh\left(x;k,\beta\right)\\ &= \sum_{k=1}^{\infty}p_ke^{-\beta x}\beta^k\frac{x^k}{(k-1)!}\\ &= \sum_{k=2}^{\infty}p_{k-1}e^{-\beta x}\beta^{k-1}\frac{x^{k-1}}{(k-2)!}\\ &= \sum_{k=2}^{\infty}\frac{k-1}{\beta}p_{k-1}h(x;k,\beta).\end{aligned}$$

Since $E[Y]=\sum_{j=1}^{\infty}\frac{j}{\beta}p_j$ *it follows that* f_Y^{α} *is a mixed Erlang pdf with the specified parameters.*

2.3 Bivariate mixed Erlang distribution defined with the FGM copula

In this section, we assume that the couple (X_1,X_2) has a bivariate distribution defined with the FGM copula given in (2.2) and $X_i\sim MixErl\left(\underline{p}_i,\beta_i\right)$, $i=1,2$. Without loss of generality, we assume that $\beta_1\leq\beta_2$.

In the following proposition, we provide a closed-from expression for the covariance between X_1 and X_2.

Proposition 2.3.1. *Let (X_1, X_2) have a bivariate mixed Erlang distribution defined with the FGM copula and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. We have the following expression for the covariance between X_1 and X_2*

$$Cov(X_1, X_2) = \frac{\theta}{\beta_1\beta_2} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij \left(\pi(i, \underline{p}_1) - p_{1,i}\right) \left(\pi(j, \underline{p}_2) - p_{2,j}\right). \tag{2.15}$$

Proof 2.3.1. *Assuming that the dependence structure for (X_1, X_2) is given by the FGM copula with dependence parameter θ, Armstrong and Galli (2002) obtain the following expression for the covariance $Cov\left(X_1, X_2\right)$:*

$$Cov\left(X_1, X_2\right) = \theta\gamma_1\gamma_2, \tag{2.16}$$

where

$$\gamma_i = 2E\left[X_i\overline{F}_{X_i}(X_i)\right] - E\left[X_i\right],$$

for $i = 1, 2$.

Using Lemma 2.2.3, we have

$$2E\left[X_i\overline{F}_{X_i}(X_i)\right] = \sum_{j=1}^{\infty} \frac{j}{\beta_i}\pi(j, \underline{p}_i)$$

for $i = 1, 2$, which leads to the expression given in (2.15).

Remark 2.3.1. *According to Lemma 2.2.4, the covariance formula given in (2.15) can also be written as*

$$Cov(X_1, X_2) = \frac{\theta}{\beta_2^2} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij \left(\pi(i, \underline{\omega}(\underline{p}_1, \beta_1, \beta_2)) - \omega(i, \underline{p}_1, \beta_1, \beta_2)\right) \left(\pi(j, \underline{p}_2) - p_{2,j}\right).$$

2.3.1 Distribution of S_2

In the next proposition, we show that $S_2 = X_1 + X_2$ follows a mixed Erlang distribution.

Proposition 2.3.2. *Let (X_1, X_2) have a bivariate mixed Erlang distribution defined with the FGM copula and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. Then, $S_2 = X_1 + X_2 \sim MixErl\left(\underline{p}^{(2)}, 2\beta_2\right)$, where $\underline{p}^{(2)} = \{p_j^{(2)}, j \in \mathbb{N}^*\}$ with*

$$\begin{aligned} p_j^{(2)} \quad = \quad & (1+\theta)\sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) \\ & -\theta\sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) \\ & -\theta\sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2)) \\ & +\theta\sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2)), \end{aligned} \tag{2.17}$$

for j=2,3,... and $p_1^{(2)} = 0$. The pdf of S_2 is given by

$$f_{S_2}\left(s\right) = \sum_{j=1}^{\infty} p_j^{(2)} h\left(s; j, 2\beta_2\right).$$

Proof 2.3.2. *Given (2.1) and (2.3), the pdf of* S_2 *is obtained from the joint pdf of* (X_1, X_2) *as*

$$f_{S_2}(s) = \int_0^s f_{\underline{X}}(x, s-x)\,dx,$$

where

$$\begin{aligned} f_{\underline{X}}(x, s-x) &= \Big((1+\theta) - 2\theta\overline{F}_{X_1}(x) - 2\theta\overline{F}_{X_2}(s-x) + 4\theta\overline{F}_{X_1}(x)\overline{F}_{X_2}(s-x)\Big) \\ &\quad \times f_{X_1}(x) f_{X_2}(s-x). \end{aligned}$$

This leads to

$$f_{S_2}(s) = I_1(s) + I_2(s) + I_3(s) + I_4(s),$$

where

$$\begin{aligned} I_1(s) &= (1+\theta)\int_0^s f_{X_1}(x) f_{X_2}(s-x)dx, \\ I_2(s) &= -2\theta\int_0^s \overline{F}_{X_1}(x) f_{X_1}(x) f_{X_2}(s-x)dx, \\ I_3(s) &= -2\theta\int_0^s f_{X_1}(x)\overline{F}_{X_2}(s-x) f_{X_2}(s-x)dx, \\ I_4(s) &= 4\theta\int_0^s \overline{F}_{X_1}(x) f_{X_1}(x)\overline{F}_{X_2}(s-x) f_{X_2}(s-x)dx. \end{aligned}$$

Each term $I_i(s)$, $i = 1, 2, 3, 4$, *can be expressed as a convolution of two mixed Erlang distributions. In the sequel, we determine the closed-form expression for each* $I_i(s)$, $i = 1, 2, 3, 4$.

For the first term $I_1(s)$, *changing the scale parameter of the distribution of* X_1 *and* X_2 *using Lemma 2.2.4 leads to*

$$f_{X_1}(s) = \sum_{j=1}^{\infty} \omega(j, \underline{p}_1, \beta_1, 2\beta_2) h(s; j, 2\beta_2),$$

and

$$f_{X_2}(s) = \sum_{j=1}^{\infty} \omega(j, \underline{p}_2, \beta_2, 2\beta_2) h(s; j, 2\beta_2).$$

According to Lemma 2.2.2, one can write

$$\begin{aligned} I_1(s) &= (1+\theta)\int_0^s f_{X_1}(x) f_{X_2}(s-x)dx \\ &= (1+\theta)\sum_{j=2}^{\infty} \sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) h(s; j, 2\beta_2). \end{aligned}$$

For the second term $I_2(s)$, *one must first use Lemmas 2.2.3 and 2.2.4 to write*

$$2\overline{F}_{X_1}(x) f_{X_1}(x) = \sum_{k=1}^{\infty} \omega(k, \underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2) h(x; k, 2\beta_2).$$

To perform the convolution in $I_2(s)$, *we write* $f_{X_2}(x)$ *as a pdf of a mixed Erlang rv with scale parameter* $2\beta_2$ *using Lemma 2.2.4*

$$f_{X_2}(x) = \sum_{k=1}^{\infty} \omega(k, \underline{p}_2, \beta_2, 2\beta_2) h(x; k, 2\beta_2).$$

Then, the convolution formula leads to

$$I_2(s) = -\theta \sum_{j=2}^{\infty} \sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2))h(s; j, 2\beta_2).$$

For $I_3(s)$ and $I_4(s)$, we obtain with similar calculations

$$I_3(s) = -\theta \sum_{j=2}^{\infty} \sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2))h(s; j, 2\beta_2),$$

and

$$I_4(s) = \theta \sum_{j=2}^{\infty} \sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2))h(s; j, 2\beta_2).$$

Since S_2 follows a mixed Erlang distribution, we have closed-form expressions for the TVaR and the stop-loss premium associated with S_2.

Corollary 2.3.1. *Let (X_1, X_2) have a bivariate mixed Erlang distribution defined with the FGM copula and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. At a given level $\kappa \in [0, 1]$, the closed-form expression for the TVaR risk measure is given by*

$$TVaR_\kappa(S_2) = \frac{1}{1-\kappa} \sum_{j=1}^{\infty} p_j^{(2)} \frac{j}{2\beta_2} \overline{H}\left(VaR_\kappa(S_2); j+1, 2\beta_2\right),$$

and for a given retention $d \in \mathbb{R}^+$, the stop-loss premium is

$$\pi_{S_2}(d) = \sum_{j=1}^{\infty} p_j^{(2)} e^{-2\beta_2 d} \frac{(2\beta_2 d)^j}{j!},$$

where probabilities $\underline{p}^{(2)}$ are as given in (3.3).

Proof 2.3.3. *Based on Proposition 2.3.2, the expressions for the TVaR risk measure and the stop-loss premium follow from (2.11) and (2.12).*

2.3.2 TVaR-based Capital Allocation

In the following proposition, we provide the expression for the amount allocated to the risks X_1 and X_2 under the TVaR-based allocation rule.

Proposition 2.3.3. *Let (X_1, X_2) have a bivariate mixed Erlang distribution defined with the FGM copula and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. Then, the expression for $TVaR_\kappa(X_i, S_2)$ at level κ, $0 < \kappa < 1$, is given by*

$$TVaR_\kappa(X_i, S_2) = \frac{1}{1-\kappa} \sum_{k=1}^{\infty} q_{i,k}^{(2)} \overline{H}\left(VaR_\kappa(S_2); k, 2\beta_2\right),$$

where

$$\begin{aligned} q_{1,k}^{(2)} &= (1+\theta)E\left[X_1\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_1),\beta_1,2\beta_2),\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2)) \\ &\quad -\theta\Pi_1\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)),2\beta_1,2\beta_2),\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2)) \\ &\quad -\theta E\left[X_1\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_1),\beta_1,2\beta_2),\underline{\pi}(\underline{p}_2)) \\ &\quad +\theta\Pi_1\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)),2\beta_1,2\beta_2),\underline{\pi}(\underline{p}_2)) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} q_{2,k}^{(2)} &= (1+\theta)E\left[X_2\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_2),\beta_2,2\beta_2),\underline{\omega}(\underline{p}_1,\beta_1,2\beta_2)) \\ &\quad -\theta\Pi_2\sigma^{(2)}(k,\underline{\alpha}(\underline{\pi}(\underline{p}_2)),\underline{\omega}(\underline{p}_1,\beta_1,2\beta_2)) \\ &\quad -\theta E\left[X_2\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_2),\beta_2,2\beta_2),\underline{\omega}(\underline{\pi}(\underline{p}_1),2\beta_1,2\beta_2)) \\ &\quad +\theta\Pi_2\sigma^{(2)}(k,\underline{\alpha}(\underline{\pi}(\underline{p}_2)),\underline{\omega}(\underline{\pi}(\underline{p}_1),2\beta_1,2\beta_2)) \end{aligned} \tag{2.19}$$

for k=3,4... and $q_{i,k}^{(2)}=0$, *for i=1,2, k=1,2 and*

$$\Pi_i = E\left[X_i\overline{F}_{X_i}(X_i)\right] = \sum_{j=1}^{\infty}\frac{j}{2\beta_i}\pi(j,\underline{p}_i).$$

Proof 2.3.4. *We know that*

$$\begin{aligned} TVaR_\kappa\left(X_1,S_2\right) &= \frac{E\left[X_1 1_{\{S_n>VaR_\kappa(S_n)\}}\right]}{1-\kappa} \\ &= \frac{1}{1-\kappa}\int_{VaR_\kappa(S_2)}^{\infty} E\left[X_1 1_{\{S_2=s\}}\right]ds. \end{aligned}$$

The expression for $E\left[X_1 1_{\{S_2=s\}}\right]$ *is given by*

$$\begin{aligned} E\left[X_1 1_{\{S_2=s\}}\right] &= \int_0^s x f_{\underline{X}}(x,s-x)dx \\ &= J_1\left(s\right)+J_2\left(s\right)+J_3\left(s\right)+J_4\left(s\right), \end{aligned}$$

where

$$\begin{aligned} J_1\left(s\right) &= (1+\theta)\int_0^s x f_{X_1}(x)f_{X_2}(s-x)dx, \\ J_2\left(s\right) &= -2\theta\int_0^s \overline{F}_{X_1}(x)x f_{X_1}(x)f_{X_2}(s-x)dx, \\ J_3\left(s\right) &= -2\theta\int_0^s x f_{X_1}(x)\overline{F}_{X_2}(s-x)f_{X_2}(s-x)dx, \\ J_4\left(s\right) &= 4\theta\int_0^s \overline{F}_{X_1}(x)x f_{X_1}(x)\overline{F}_{X_2}(s-x)f_{X_2}(s-x)dx. \end{aligned}$$

In the sequel, each term $J_i(s)$, *for* $i=1,2,3,4$, *is expressed as a pdf of a mixed Erlang distribution.*

Simple manipulations lead to

$$J_1(s) = (1+\theta)E[X_1] \int_0^s \frac{x f_{X_1}(x)}{E[X_1]} f_{X_2}(s-x)dx.$$

Lemma 2.2.5 allows to write $\frac{x f_{X_1}(x)}{E[X_1]}$ *as the pdf of a mixed Erlang rv, meaning*

$$\frac{x f_{X_1}(x)}{E[X_1]} = \sum_{k=2}^{\infty} \alpha(k, \underline{p}_1) h(x; k, \beta_1). \tag{2.20}$$

Using Lemma 2.2.4, we change the scale parameter of the pdf of the mixed Erlang distribution in (2.20) and we get

$$J_1(s) = (1+\theta)E[X_1] \sum_{k=3}^{\infty} \sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_1), \beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) h(s; k, 2\beta_2).$$

For the second term, we write

$$J_2(s) = -\theta \Pi_1 \int_0^s \frac{2x\overline{F}_{X_1}(x) f_{X_1}(x)}{\Pi_1} f_{X_2}(s-x)dx,$$

where

$$\begin{aligned} \Pi_1 &= E\left[X_1 \overline{F}_{X_1}(X_1)\right] \\ &= \sum_{j=1}^{\infty} \frac{j}{2\beta_1} \pi(j, \underline{p}_1). \end{aligned}$$

According to Lemmas 2.2.3 and 2.2.5, we can state that

$$\frac{2x\overline{F}_{X_1}(x) f_{X_1}(x)}{\Pi_1} = \sum_{j=2}^{\infty} \alpha(j, \underline{\pi}(\underline{p}_1)) h(x; j, 2\beta_1).$$

Then, one can write $J_2(s)$ *as follows*

$$J_2(s) = -\theta \Pi_1 \sum_{k=3}^{\infty} \sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)), 2\beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) h(s; k, 2\beta_2).$$

Using similar calculations and according to Lemmas 2.2.3, 2.2.4 and 2.2.5, we have the following expressions for $J_3(s)$ *and* $J_4(s)$

$$J_3(s) = -\theta E[X_1] \sum_{k=3}^{\infty} \sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_1), \beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2)) h(s; k, 2\beta_2),$$

and

$$J_4(s) = \theta \Pi_1 \sum_{k=3}^{\infty} \sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)), 2\beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2)) h(s; k, 2\beta_2).$$

One can write

$$\int_0^s x f_{X_1,S_2}(x,s)dx = \sum_{j=1}^{\infty} q_{1,j}^{(2)} h(s; j, 2\beta_2),$$

where the expressions for the probabilities $q_{1,j}^{(2)}$ *are as given in (2.18). Then, we find the desired result for* $TVaR_\kappa(X_1, S_2)$*. Similarly, one can derive the expression for* $TVaR_\kappa(X_2, S_2)$ *with* $q_{2,j}^{(2)}$ *as given in (2.19).*

In what follows, we verify that the relation $TVaR_\kappa(X_1,S_2)+TVaR_\kappa(X_2,S_2)=TVaR_\kappa(S_2)$ is satisfied. Simple manipulations lead to

$$\omega(k,E\left[X_1\right]\underline{\alpha}(\underline{p}_1),\beta_1,2\beta_2)=\frac{k-1}{2\beta_2}\omega(k-1,\underline{p}_1,\beta_1,2\beta_2),$$

and

$$\omega(k,E\left[X_2\right]\underline{\alpha}(\underline{p}_2),\beta_2,2\beta_2)=\frac{k-1}{2\beta_2}\omega(k-1,\underline{p}_2,\beta_2,2\beta_2).$$

We recall that the function $\underline{\sigma}^{(2)}(\underline{p}_1,\underline{p}_2)$ is linear in each variable $\underline{p}_1$. Then, we get

$$E\left[X_1\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_1),\beta_1,2\beta_2),\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2))=\sum_{j=1}^{k-2}\frac{j}{2\beta_2}\omega(j,\underline{p}_1,\beta_1,2\beta_2)\omega(k-j-1,\underline{p}_2,\beta_2,2\beta_2),$$

and

$$\begin{aligned}E\left[X_2\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_2),\beta_2,2\beta_2),\underline{\omega}(\underline{p}_1,\beta_1,2\beta_2)) &= \sum_{j=1}^{k-2}\frac{k-j-1}{2\beta_2}\omega(j,\underline{p}_1,\beta_1,2\beta_2)\\ &\quad\times\omega(k-j-1,\underline{p}_2,\beta_2,2\beta_2),\end{aligned}$$

and hence

$$\begin{aligned}\frac{k-1}{2\beta_2}\sigma^{(2)}(k-1,\underline{\omega}(\underline{p}_1,\beta_1,2\beta_2),\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2)) &= E\left[X_1\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_1),\beta_1,2\beta_2)\\ &\quad,\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2))\\ &\quad+E\left[X_2\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_2),\beta_2,2\beta_2)\\ &\quad,\underline{\omega}(\underline{p}_1,\beta_1,2\beta_2)).\end{aligned}$$

With similar calculations, we obtain

$$\begin{aligned}\frac{k-1}{2\beta_2}\sigma^{(2)}(k-1,\underline{\omega}(\underline{\pi}(\underline{p}_1),2\beta_1,2\beta_2),\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2)) &= \Pi_1\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)),2\beta_1,2\beta_2)\\ &\quad,\underline{\omega}(\underline{p}_2,\beta_2,2\beta_2))\\ &\quad+\Pi_2\sigma^{(2)}(k,\underline{\alpha}(\underline{\pi}(\underline{p}_2)),\underline{\omega}(\underline{p}_1,\beta_1,2\beta_2)),\end{aligned}$$

$$\begin{aligned}\frac{k-1}{2\beta_2}\sigma^{(2)}(k-1,\underline{\omega}(\underline{p}_1,2\beta_1,2\beta_2),\underline{\pi}(\underline{p}_2)) &= E\left[X_1\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_1),\beta_1,2\beta_2),\underline{\pi}(\underline{p}_2))\\ &\quad+E\left[X_2\right]\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{p}_2),\beta_2,2\beta_2)\\ &\quad,\underline{\omega}(\underline{\pi}(\underline{p}_1),2\beta_1,2\beta_2)),\end{aligned}$$

and

$$\begin{aligned}\frac{k-1}{2\beta_2}\sigma^{(2)}(k-1,\underline{\omega}(\underline{\pi}(\underline{p}_1),2\beta_1,2\beta_2),\underline{\pi}(\underline{p}_2)) &= \Pi_1\sigma^{(2)}(k,\underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)),2\beta_1,2\beta_2),\underline{\pi}(\underline{p}_2))\\ &\quad+\Pi_2\sigma^{(2)}(k,\underline{\alpha}(\underline{\pi}(\underline{p}_2)),\underline{\omega}(\underline{\pi}(\underline{p}_1),2\beta_1,2\beta_2)),\end{aligned}$$

which leads to

$$\begin{aligned} TVaR_\kappa(X_1,S_2)+TVaR_\kappa(X_2,S_2) &= \frac{1}{1-\kappa}\sum_{j=3}^{\infty}\left(q_{1,j}^{(2)}+q_{2,j}^{(2)}\right)\overline{H}\left(VaR_\kappa\left(S_2\right);j,2\beta_2\right) \\ &= \frac{1}{1-\kappa}\sum_{k=3}^{\infty}p_{k-1}^{(2)}\frac{k-1}{2\beta_2}\overline{H}\left(VaR_\kappa\left(S_2\right);k,2\beta_2\right) \\ &= TVaR_\kappa\left(S_2\right). \end{aligned}$$

2.3.3 Covariance-based Capital Allocation

A closed-form expression for the amount of capital attributed to risk $i = 1,2$ under the covariance allocation rule is given in the next proposition.

Proposition 2.3.4. *Let (X_1, X_2) have a bivariate distribution defined with the FGM copula and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1,2$ with $\beta_1 \leq \beta_2$. Then, the contribution $C_\kappa(X_i, S_2)$ at level κ, $0 < \kappa < 1$, is given by*

$$C_\kappa\left(X_i,S_2\right)=\sum_{k=1}^{\infty}c_{i,k}\frac{k}{\beta_2}, \tag{2.21}$$

where

$$c_{i,k}=\omega(k,\underline{p}_i,\beta_i,\beta_2)+2\rho_{i,k}p_k^{(2)}\left[\frac{\overline{H}(VaR_\kappa(S_2);k+1,2\beta_2)}{1-\kappa}-1\right],$$

with

$$\begin{aligned} \rho_{i,k} \;&=\; \frac{\sum_{l=1}^{\infty}l\omega(l,\underline{p}_i,\beta_i,\beta_2)-\left(\sum_{l=1}^{\infty}l\omega(l,\underline{p}_i,\beta_i,\beta_2)\right)^2}{\sum_{l=1}^{\infty}lp_l^{(2)}-\left(\sum_{l=1}^{\infty}lp_l^{(2)}\right)^2} \\ &+\frac{\theta\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}lm\left(\pi(l,\underline{\omega}(\underline{p}_1,\beta_1,\beta_2))-\omega(l,\underline{p}_1,\beta_1,\beta_2)\right)\left(\pi(m,\underline{p}_2)-p_{2,m}\right)}{\sum_{l=1}^{\infty}lp_l^{(2)}-\left(\sum_{l=1}^{\infty}lp_l^{(2)}\right)^2}, \end{aligned}$$

for $i = 1,2$.

Démonstration. The amount of capital attributed to risk i, for $i = 1,2$, is given in (2.10) by

$$C_\kappa\left(X_i,S_2\right)=E\left[X_i\right]+\frac{Cov(X_i,S_2)}{Var(S_2)}\left(TVaR_\kappa\left(S_2\right)-E\left[S_2\right]\right). \tag{2.22}$$

By Proposition 2.3.2, we have

$$E\left[S_2\right]=\sum_{k=1}^{\infty}p_k^{(2)}\frac{k}{2\beta_2}.$$

Using Lemma 2.2.4, we have

$$E\left[X_i\right]=\sum_{k=1}^{\infty}\omega(k,\underline{p}_i,\beta_i,\beta_2)\frac{k}{\beta_2}.$$

Given Corollary 2.3.1, (2.22) becomes

$$C_\kappa\left(X_i, S_2\right) = \sum_{k=1}^{\infty}\left[\omega(k, \underline{p}_i, \beta_i, \beta_2) + \frac{Cov(X_i, S_2)}{2Var(S_2)} p_k^{(2)}\left[\frac{\overline{H}\left(VaR_\kappa\left(S_2\right); k+1, 2\beta_2\right)}{1-\kappa} - 1\right]\right]\frac{k}{\beta_2}.$$

We also know that $S_2 \sim MixErl\left(\underline{p}^{(2)}, 2\beta_2\right)$, and hence

$$Var(S_2) = \frac{1}{4\beta_2}\sum_{l=1}^{\infty} l p_l^{(2)} - \left(\sum_{l=1}^{\infty} l p_l^{(2)}\right)^2.$$

Using the expression for $Cov(X_1, X_2)$ given in Remark 2.3.1, one may find the expression in (2.21).

2.3.4 Numerical Example

Let (X_1, X_2) have a bivariate mixed Erlang distribution defined by the FGM copula with $\theta = 0.5$. The pdf of X_1 and X_2 are given by

$$\begin{aligned} f_{X_1}(x) &= 0.6h(x; 1, 0.1) + 0.4h(x; 2, 0.1), \\ f_{X_2}(x) &= 0.3h(x; 1, 0.15) + 0.5h(x; 2, 0.15) + 0.2h(x; 3, 0.15). \end{aligned}$$

The values of the expectations, variances and covariances of X_1 and X_2 are displayed in Table 1.

$E[X_1]$	$E[X_2]$	$Var[X_1]$	$Var[X_2]$	$Cov(X_1, X_2)$
14	12.67	164	106.22	17.98

Table 2.1: Basic quantities for X_1 and X_2.

We calculate the VaR and TVaR for $S_2 = X_1 + X_2$ and also determine the amount allocated under the TVaR- and Cov-based allocation rules for X_1 and X_2.

By Proposition 2.3.2, we have $S_2 \sim MixErl\left(\underline{p}^{(2)}, 2\beta_2\right)$ and Table 2 gives the first 40 values of the probabilities $\underline{p}^{(2)} = \left(p_k^{(2)}, k = 1, 2, \ldots\right)$ for a dependence parameter $\theta = 0.5$. It is clear that the value of $p_k^{(2)}$ goes to zero for high values of k.

For different values of κ, we provide in Table 3 the exact values of $VaR_\kappa(S_2)$ and $TVaR_\kappa(S_2)$.

For $\kappa = 0.95$ and for different values of θ, we calculate the amount allocated under the TVaR-based allocation rule and the Cov-based allocation rule. We observe that these allocations show a different behavior according to the dependence relationship measured by the value of the copula parameter θ (see Table 4 and Figure 1). We also calculate values of the TVaR, TVaR-based capital allocation and Cov-based capital allocation for different values of θ and also at different levels κ. Figure 2 summarizes the behavior of each measure and illustrates

k	1	2	3	4	5	6	7	8	9	10
p_k^*	0	0.045	0.0895	0.1092	0.1060	0.0955	0.0867	0.0794	0.0719	0.0635
k	11	12	13	14	15	16	17	18	19	20
p_k^*	0.0544	0.0452	0.0366	0.0290	0.0225	0.0171	0.0129	0.0096	0.0070	0.0051
k	21	22	23	24	25	26	27	28	29	30
p_k^*	0.0037	0.0027	0.0019	0.0013	0.0009	0.0006	0.0004	0.0003	0.0002	0.0001
k	31	32	33	34	35	36	37	38	39	40
p_k^*	0.0001	8.1E-05	5.6E-05	3.9E-05	2.7E-05	1.8E-05	1.3E-05	8.9E-06	6.2E-06	4.2E-06

Table 2.2: Probabilities $p_k^{(2)}$ for $S_2 = X_1 + X_2$.

κ	0.05	0.10	0.50	0.75	0.90	0.95	0.99	0.995	0.999
$VaR_\kappa(S_2)$ exact	5.189	7.62	23.09	36.18	50.52	60.21	80.75	89.10	107.83
$TVaR_\kappa(S_2)$ exact	27.89	29.08	40.08	51.03	63.89	72.91	92.54	100.62	118.61

Table 2.3: Values of VaR and TVaR for $S_2 = X_1 + X_2$.

the impact of the copula parameter θ and κ on the ratio $\frac{TVaR_\kappa(X_1,S_2)}{C_\kappa(X_1,S_2)}$. It is clear that the TVaR increases as θ goes to 1 and one sees that the Cov-based allocation underestimates the capital that should be allocated to risk X_1 especially for high negative dependence.

θ	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$VaR_{0.95}(S_2)$	53.08	54.07	55.06	56.04	57.01	57.96	58.88	59.78	60.64	61.48	62.29
$TVaR_{0.95}(S_2)$	63.81	65.28	66.66	67.96	69.18	70.33	71.42	72.44	73.40	74.31	75.17
$TVaR_{0.95}(X_1.S_2)$	41.08	41.45	41.79	42.13	42.47	42.80	43.12	43.44	43.75	44.06	44.37
$C_{0.95}(X_1.S_2)$	37.98	38.55	39.08	39.59	40.06	40.50	40.92	41.31	41.67	42.02	42.35
$TVaR_{0.95}(X_2.S_2)$	22.72	23.83	24.86	25.82	26.71	27.54	28.30	29.00	29.65	30.24	30.80
$C_{0.95}(X_2.S_2)$	25.83	26.73	27.57	28.37	29.12	29.83	30.50	31.13	31.73	32.29	32.82

Table 2.4: Amounts allocated under the TVaR-based and the Covariance-based rules.

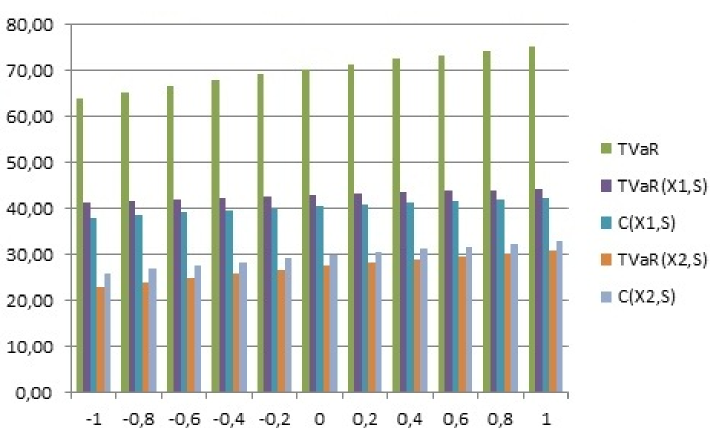
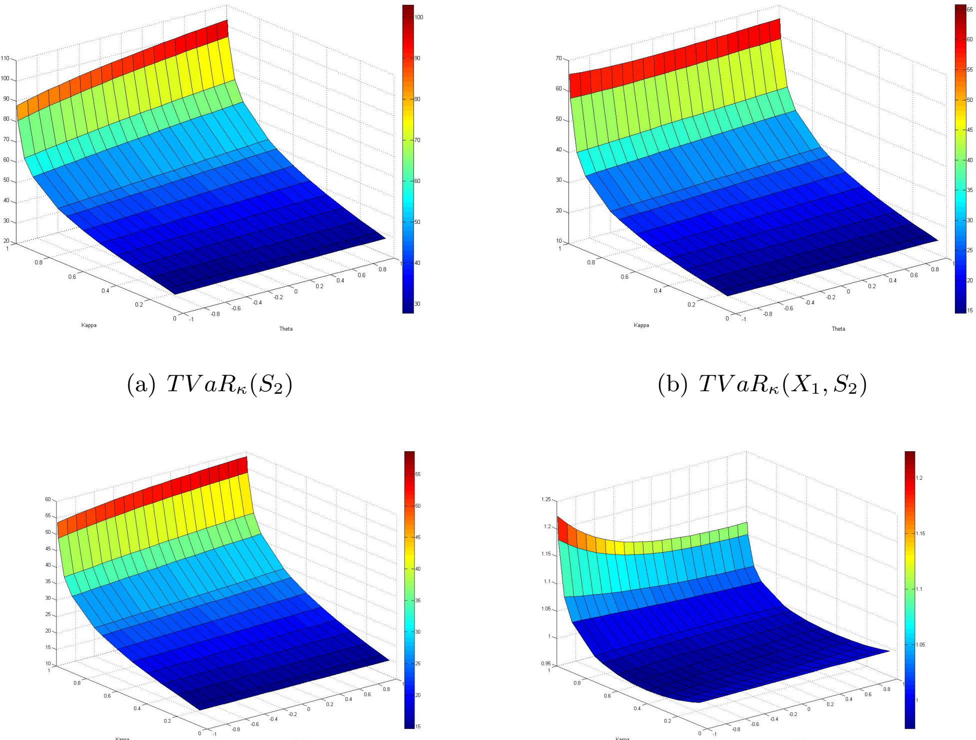


Figure 2.1: Capital allocation based on the TVaR and the Covariance rules.



(a) $TVaR_\kappa(S_2)$ (b) $TVaR_\kappa(X_1, S_2)$

(c) $C_\kappa(X_1, S_2)$ (d) $\frac{TVaR_\kappa(X_1,S_2)}{C_\kappa(X_1,S_2)}$

Figure 2.2: Comparison of the capital allocation based on the TVaR and the Covariance rules.

2.4 Multivariate mixed Erlang distribution defined with the FGM copula

In this section, we assume that $(X_1, \ldots, X_n)$ has a multivariate mixed Erlang distribution defined by the FGM copula given in (2.4) and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$. Without loss of generality, we assume that $\beta_i \leq \beta_n$, for $i = 1, \ldots, n-1$.

We provide, in the following proposition, a closed-from expression for the covariance between X_i and X_j, for $i \neq j = 1, \ldots, n$.

Proposition 2.4.1. *Let $(X_1, \ldots, X_n)$ have a multivariate mixed Erlang distribution defined with the FGM n-copula and $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$ with $\beta_i \leq \beta_n$, for $i = 1, 2, \ldots, n-1$. We have the following expression for the covariance $Cov(X_i, X_j)$, for $i \neq j = 1, \ldots, n$:*

$$Cov(X_i, X_j) = \frac{\theta_{ij}}{\beta_i \beta_j} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} lm \left(\pi(l, p_i) - p_{i,l}\right) \left(\pi(m, p_j) - p_{j,m}\right), \tag{2.23}$$

where $\theta_{ij} = \theta_{min(i,j)max(i,j)}$.

Proof 2.4.1. *We know that each k-margin of the FGM n-copula is an FGM k-copula, for* $k = 2, \ldots, n$*. Particularly, for* $i \neq j = 1, \ldots, n$*, the dependence structure for* (X_i, X_j) *is given by a bivariate FGM copula with a dependence parameter* $\theta_{ij} = \theta_{min(i,j)max(i,j)}$*. Using Proposition 2.3.1, we find the expression given in (2.23).*

Remark 2.4.1. *Using Lemma 2.2.4, the covariance formula given in (2.23) can be written as follows*

$$\begin{aligned} Cov(X_i, X_j) &= \frac{\theta}{\beta_n^2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} lm \left(\pi(l, \underline{\omega}(\underline{p}_i, \beta_i, \beta_n)) - \omega(l, \underline{p}_i, \beta_i, \beta_n) \right) \\ &\quad \times \left(\pi(m, \underline{\omega}(\underline{p}_j, \beta_j, \beta_n)) - \omega(m, \underline{p}_j, \beta_j, \beta_n) \right). \end{aligned}$$

2.4.1 Distribution of S_n

In the next proposition, we show that $S_n = \sum_{i=1}^n X_i$ follows a mixed Erlang distribution.

Proposition 2.4.2. *Let* $(X_1, \ldots, X_n)$ *have a multivariate mixed Erlang distribution defined with the FGM n-copula and* $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$*,* $i = 1, \ldots, n$ *with* $\beta_i \leq \beta_n$*, for* $i = 1, 2, \ldots, n-1$*. Then,* $S_n = \sum_{i=1}^n X_i \sim MixErl\left(\underline{p}^{(n)}, 2\beta_n\right)$*, where* $\underline{p}^{(n)} = \{p_j^{(n)}, j \in \mathbb{N}^*\}$ *with*

$$\begin{aligned} p_j^{(n)} &= (1+\zeta)\sigma^{(n)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_n), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_n, \beta_n, 2\beta_n)) \\ &\quad + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \ldots, i_{k-l}\} \in C_{J_k}^{k-l}} (-1)^l \\ &\quad \sigma^{(n)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_{i_1}), 2\beta_{i_1}, 2\beta_n), \ldots, \underline{\omega}(\underline{\pi}(\underline{p}_{i_{k-l}}), 2\beta_{i_{k-l}}, 2\beta_n), \underline{\omega}(\underline{p}_{i_{k-l+1}}, \beta_{i_{k-l+1}}, 2\beta_n), \ldots \\ &\quad , \underline{\omega}(\underline{p}_{i_n}, \beta_{i_n}, 2\beta_n)), \end{aligned} \tag{2.24}$$

for $j = n, n+1, \ldots$*, and with* $p_j^{(n)} = 0$ *for* $j = 1, \ldots, n-1$*.*

Also, $\zeta = \sum_{k=2}^n \sum_{1 \leq j_1 < \ldots < j_k \leq n} (-1)^k \theta_{j_1 j_2 \ldots j_k}$ *and* $C_{J_k}^{k-l}$ *is the set of all combinations of* $(k-l)$ *elements from* $J_k = \{1 \leq j_1 < \ldots < j_k \leq n\}$*. The pdf of* S_n *is given by*

$$f_{S_n}(s) = \sum_{j=1}^{\infty} p_j^{(n)} h(s; j, 2\beta_n).$$

Proof 2.4.2. *To simplify the presentation, it is assumed that* $n > 2$*. Using (2.1) and (3.34), the joint pdf of* $\underline{X} = (X_1, \ldots, X_n)$ *is given by*

$$f_{\underline{X}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) P(2F_{X_1}(x_1), \ldots, 2F_{X_n}(x_n)) \tag{2.25}$$

and the expression for the pdf of S_n *is obtained with*

$$f_{S_n}(s) = \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} f_{\underline{X}}(x_1, x_2, \ldots, s-x_1-\ldots-x_{n-1}) dx_{n-1} \ldots dx_2 dx_1. \tag{2.26}$$

Let us denote by J_k the set $\{1 \leq j_1 < \ldots < j_k \leq n\}t$ and $C_{J_k}^{k-l}$ the sets of all the $(k-l)$-combinations of J_k, for $l = 0, 1, \ldots, k-1$ and $k = 2, \ldots, n$. Using (2.5), we have

$$\begin{aligned} P(2F_{X_1}(x_1), 2F_{X_2}(x_2), \ldots, 2F_{X_n}(x_n)) &= 1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} (2\overline{F}_{X_{j_1}}(x_{j_1}) - 1) \ldots \\ &\quad (2\overline{F}_{X_{j_k}}(x_{j_k}) - 1) \\ &= 1 + \zeta + Q(x_1, x_2, \ldots, x_n), \end{aligned} \tag{2.27}$$

where $\zeta = \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} (-1)^k \theta_{j_1 j_2 \ldots j_k}$ and

$$Q(x_1, \ldots, x_n) = \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \ldots, i_{k-l}) \in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}). \tag{2.28}$$

The joint pdf in (2.25) hence becomes

$$\begin{aligned} f_{\underline{X}}(x_1, \ldots, x_n) &= (1+\zeta) \prod_{i=1}^{n} f_{X_i}(x_i) + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \ldots, i_{k-l}\} \in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \\ &\quad \times \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \prod_{i=1}^{n} f_{X_i}(x_i). \end{aligned} \tag{2.29}$$

For calculation purposes, we decompose (2.28) as follows

$$\begin{aligned} Q(x_1, \ldots, x_n) &= \sum_{k=2}^{n-1} \sum_{1 \leq j_1 < \ldots < j_k \leq n-1} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \ldots, i_{k-l}) \in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \\ &\quad + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_{k-1} \leq n-1} \theta_{j_1 j_2 \ldots j_n} \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \ldots, i_{k-l}) \in C_{J_k - \{n\}}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \\ &\quad + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \ldots < j_{k-1} \leq n-1} \theta_{j_1 j_2 \ldots j_n} \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \ldots, i_{k-l-1}) \in C_{J_k - \{n\}}^{k-l-1}} (-1)^l 2^{k-l} \overline{F}_{X_n}(x_n) \\ &\quad \times \prod_{m=1}^{k-l-1} \overline{F}_{X_{i_m}}(x_{i_m}), \end{aligned} \tag{2.30}$$

where $C_{J_k - \{n\}}^{k-l}$ are the sets of $(k-l)$-combinations of $J_k - \{n\}$.

With (2.27) and (2.30), the expression for the pdf of S_n in (2.26) becomes

$$\begin{aligned} f_{S_n}(s) &= (1+\zeta) \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} \prod_{i=1}^{n-1} f_{X_i}(x_i) f_{X_n}(s - x_1 - \ldots - x_{n-1}) dx_{n-1} \\ &\quad \ldots dx_2 dx_1 + I(s), \end{aligned} \tag{2.31}$$

where

$$
\begin{aligned}
I(s) = & \sum_{k=2}^{n-1} \sum_{1\leq j_1<\ldots<j_k\leq n-1} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{(i_1,i_2,\ldots,i_{k-l})\in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} \\
& \times \prod_{i=1}^{n-1} f_{X_i}(x_i) \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) f_{X_n}(s-x_1-\ldots-x_{n-1}) dx_{n-1}\ldots dx_2 dx_1 \\
& + \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_{k-1}\leq n-1} \theta_{j_1 j_2 \ldots j_n} \sum_{l=0}^{k-1} \sum_{(i_1,i_2,\ldots,i_{k-l})\in C_{J_k-\{n\}}^{k-l}} (-1)^l 2^{k-l} \int_0^s \int_0^{s-x_1} \ldots \\
& \int_0^{s-x_1-\ldots-x_{n-2}} \prod_{i=1}^{n-1} f_{X_i}(x_i) \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) f_{X_n}(s-x_1-\ldots-x_{n-1}) dx_{n-1}\ldots dx_2 dx_1 \\
& + \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_{k-1}\leq n-1} \theta_{j_1 j_2 \ldots j_n} \sum_{l=0}^{k-1} \sum_{(i_1,i_2,\ldots,i_{k-l-1})\in C_{J_k-\{n\}}^{k-l-1}} (-1)^l 2^{k-l} \int_0^s \int_0^{s-x_1} \ldots \\
& \int_0^{s-x_1-\ldots-x_{n-2}} \prod_{i=1}^{n-1} f_{X_i}(x_i) \prod_{m=1}^{k-l-1} \overline{F}_{X_{j_m}}(x_{j_m}) f_{X_n}(s-x_1-\ldots-x_{n-1}) \\
& \times \overline{F}_{X_n}(s-x_1-\ldots-x_{n-1}) dx_{n-1}\ldots dx_2 dx_1. \qquad (2.32)
\end{aligned}
$$

After some rearrangements, the expression for $I(s)$ becomes

$$
\begin{aligned}
I(s) = & \sum_{k=2}^{n-1} \sum_{1\leq j_1<\ldots<j_k\leq n-1} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1,i_2,\ldots,i_{k-l}\}\in C_{J_k}^{k-l}} (-1)^l \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} \\
& \prod_{m=k-l+1}^{n-1} f_{X_{i_m}}(x_{i_m}) \prod_{m=1}^{k-l} \left(2\overline{F}_{X_{i_m}}(x_{i_m}) f_{X_{i_m}}(x_{i_m})\right) f_{X_n}(s-x_1-\ldots-x_{n-1}) dx_{n-1} \\
& \ldots dx_2 dx_1 \\
& + \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_{k-1}\leq n-1} \theta_{j_1 j_2 \ldots j_n} \sum_{l=0}^{k-1} \sum_{(i_1,i_2,\ldots,i_{k-l})\in C_{J_k-\{n\}}^{k-l}} (-1)^l \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} \\
& \times \prod_{m=k-l+1}^{n-1} f_{X_{i_m}}(x_{i_m}) \prod_{m=1}^{k-l} \left(2\overline{F}_{X_{i_m}}(x_{i_m}) f_{X_{i_m}}(x_{i_m})\right) f_{X_n}(s-x_1-\ldots-x_{n-1} \\
& dx_{n-1}\ldots dx_2 dx_1 \\
& + \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_{k-1}\leq n-1} \theta_{j_1 j_2 \ldots j_n} \sum_{l=0}^{k-1} \sum_{(i_1,i_2,\ldots,i_{k-l-1})\in C_{J_k-\{n\}}^{k-l-1}} (-1)^l \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} \\
& \times \prod_{m=k-l+1}^{n-1} f_{X_{i_m}}(x_{i_m}) \prod_{m=1}^{k-l-1} \left(2\overline{F}_{X_{j_m}}(x_{j_m}) f_{X_{i_m}}(x_{i_m})\right) \\
& \times \left(2\overline{F}_{X_n}(s-x_1-\ldots-x_{n-1}) f_{X_n}(s-x_1-\ldots-x_{n-1})\right) dx_{n-1}\ldots dx_2 dx_1. \qquad (2.33)
\end{aligned}
$$

By replacing (2.33) in (2.31), we observe that the expression for the pdf of S_n can be seen as a sum of convolutions of mixed Erlang distributions as in the bivariate case. Hence, with

Lemmas 2.2.1, 2.2.2, 2.2.3 and 2.2.4, one can write

$$f_{S_n}(s) = \sum_{j=n}^{\infty} p_j^{(n)} h(s; j, 2\beta_n),$$

with $p_j^{(n)}$ *as given in (2.24)*

Since S_n follows a mixed Erlang distribution, we have closed-form expressions for the TVaR and the stop-loss premium associated to S_n.

Corollary 2.4.1. *Let* $X_1, \ldots, X_n$ *be n mixed Erlang distributed rvs with* $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$. *Assuming* $\beta_i \leq \beta_n$, *for* $i = 1, 2, \ldots, n-1$, *and a dependence structure for* $(X_1, \ldots, X_n)$ *based on the FGM n-copula, the closed form expression for the TVaR risk measure, at a given level* $\kappa \in [0,1]$, *is given by*

$$TVaR_\kappa(S_n) = \frac{1}{1-\kappa} \sum_{j=1}^{\infty} p_j^{(n)} \frac{j}{2\beta_n} \overline{H}\left(VaR_\kappa(S_n); j+1, 2\beta_n\right). \tag{2.34}$$

For a given retention $d \in \mathbb{R}^+$, *the stop-loss premium is given by*

$$\pi_{S_n}(d) = \sum_{j=1}^{\infty} p_j^{(n)} e^{-2\beta_n d} \frac{(2\beta_n d)^j}{j!}, \tag{2.35}$$

where the probabilities $\underline{p}^{(n)}$ *are as given in (2.24).*

Proof 2.4.3. *Applying the result of Proposition 2.4.2 with (2.11) and (2.5) leads to (2.34) and (2.35).*

2.4.2 Capital allocation

In the following proposition, we provide the expression for the amount allocated to the risks X_i, $i = 1, \ldots, n$, under both TVaR- and Cov-based allocation rule.

TVaR-based Capital Allocation

Proposition 2.4.3. *Let* $(X_1, \ldots, X_n)$ *have a multivariate mixed Erlang distribution defined with the FGM n-copula and* $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$ *with* $\beta_i \leq \beta_n$, *for* $i = 1, 2, \ldots, n-1$. *Then, the expression for* $TVaR_\kappa(X_i, S_n)$ *at level* κ, $0 < \kappa < 1$, *is given by*

$$TVaR_\kappa(X_i, S_n) = \frac{1}{1-\kappa} \sum_{k=1}^{\infty} q_{i,k}^{(n)} \frac{k}{2\beta_n} \overline{H}\left(VaR_\kappa(S_n); k+1, 2\beta_n\right),$$

for $k = n+1, n+2, ...$ *with* $q_{i,k}^{(n)} = 0$ *for* $i = 1, ..., n$ *and*

$$\begin{aligned}
q_{i,k}^{(n)} &= (1+\zeta)E\left[X_i\right]\sigma^{(n)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_n), \ldots, \underline{\omega}(\alpha(\underline{p}_{i_n}), \beta_{i_n}, 2\beta_n, \ldots, \underline{\omega}(\underline{p}_n, \beta_n, 2\beta_n)) \\
&+\Pi_i \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_k\leq n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1,i_2,\ldots,i_{k-l}\}\in C_{J_k,i}^{k-l}} (-1)^l \\
&\sigma^{(n)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_{i_1}), 2\beta_{i_1}, 2\beta_n), \ldots, \underline{\omega}(\alpha(\underline{\pi}(\underline{p}_i)), 2\beta_i, 2\beta_n), \ldots \\
&, \underline{\omega}(\alpha(\underline{\pi}(\underline{p}_{i_{k-l}})), 2\beta_{i_{k-l}}, 2\beta_n), \underline{\omega}(\underline{p}_{i_{k-l+1}}, \beta_{i_{k-l+1}}, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_{i_n}, \beta_{i_n}, 2\beta_n)) \\
&+E\left[X_i\right] \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_k\leq n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1,i_2,\ldots,i_{k-l}\}\in C_{J_k,-i}^{k-l}} (-1)^l \\
&\sigma^{(n)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_{i_1}), 2\beta_{i_1}, 2\beta_n), \ldots, \underline{\omega}(\underline{\pi}(\underline{p}_{i_{k-l}}), 2\beta_{i_{k-l}}, 2\beta_n), \underline{\omega}(\underline{p}_{i_{k-l+1}}, \beta_{i_{k-l+1}}, 2\beta_n), \ldots \\
&, \underline{\omega}(\underline{p}_i, \beta_i, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_{i_n}, \beta_{i_n}, 2\beta_n)), \qquad (2.36)
\end{aligned}$$

where

$$\Pi_i = E\left[X_i \overline{F}_{X_i}(X_i)\right] = \sum_{j=1}^{\infty} \frac{j}{2\beta_i} \pi(j, \underline{p}_i),$$

and $\zeta = \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_k\leq n} (-1)^k \theta_{j_1 j_2 \ldots j_k}$. *Also,* $C_{J_k,i}^{k-l}$ *is the set of all combinations of* $(k-l)$ *elements from* $J_k = \{1 \leq j_1 < \ldots < j_k \leq n\}$ *which includes* i. *Finally,* $C_{J_k,-i}^{k-l}$ *is the set of all combinations of* $(k-l)$ *elements from* $J_k = \{1 \leq j_1 < \ldots < j_k \leq n\}$ *which does not include* i.

Proof 2.4.4. *For* $i = 1, 2, \ldots, n$, *the capital attributed to the risk* i *is expressed as*

$$\begin{aligned}
TVaR_\kappa\left(X_i, S_n\right) &= E[X_i 1_{\{S_n > VaR_\kappa(S_n)\}}] \qquad (2.37) \\
&= \frac{1}{1-\kappa} \int_{VaR_\kappa(S_n)}^{+\infty} E[X_i 1_{\{S_n = s\}}] ds,
\end{aligned}$$

where

$$E[X_i 1_{\{S_n=s\}}] = \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} x_i f_{\underline{X}}(x_1, \ldots, s-x_1-\ldots-x_{n-1}) dx_{n-1} \ldots dx_2 dx_1. \qquad (2.38)$$

From the joint pdf expression in (2.29), one can conclude that, for $i = 1, 2, \ldots, n$, *we have*

$$\begin{aligned}
x_i f_{\underline{X}}(x_1, x_2, \ldots, x_n) &= (1+\zeta) \prod_{i=1}^{n} f_{X_i}(x_i) + \sum_{k=2}^{n} \sum_{1\leq j_1<\ldots<j_k\leq n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1,i_2,\ldots,i_{k-l}\}\in C_{J_k}^{k-l}} \\
&(-1)^l 2^{k-l} x_i \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \prod_{i=1}^{n} f_{X_i}(x_i). \qquad (2.39)
\end{aligned}$$

After some rearrangements, (2.39) becomes

$$\begin{aligned} x_i f_{\underline{X}}(x_1, x_2, \ldots, x_n) &= (1+\zeta)\prod_{i=1}^{n} f_{X_i}(x_i) \\ &+ \sum_{k=2}^{n} \sum_{1\le j_1<\ldots<j_k\le n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1,i_2,\ldots,i_{k-l}\}\in C_{J_k,i}^{k-l}} (-1)^l (2x_i f_{X_i}(x_i)\overline{F}_{X_i}(x_i)) \\ &\prod_{m=1\neq i}^{k-l} 2f_{X_{i_m}}(x_{i_m})\overline{F}_{X_{i_m}}(x_{i_m}) \prod_{m=k+l+1}^{n} f_{X_{i_m}}(x_{i_m}) \\ &+ \sum_{k=2}^{n} \sum_{1\le j_1<\ldots<j_k\le n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1,i_2,\ldots,i_{k-l}\}\in C_{J_k ı}^{k-l}} (-1)^l (x_i f_{X_i}(x_i)) \\ &\prod_{m=1}^{k-l} 2f_{X_{i_m}}(x_{i_m})\overline{F}_{X_{i_m}}(x_{i_m}) \prod_{m=k+l+1\neq i}^{n} f_{X_{i_m}}(x_{i_m}). \end{aligned} \tag{2.40}$$

Substituting (2.40) into (2.38) and using Lemmas 2.2.3, 2.2.4 and 2.2.5 lead to the following expression for $E[X_i 1_{\{S_n=s\}}]$

$$E[X_i 1_{\{S_n=s\}}] = \sum_{k=n+1}^{\infty} q_{i,k}^{(n)} h(s; k, 2\beta_n),$$

where the probabilities $q_{i,k}$ *are as given in (2.36). The desired result follows.*

Covariance-based Capital Allocation

In the following proposition, we provide the expression for the amount allocated to the risks X_i, $i = 1, \ldots, n$, under the Cov-based allocation rule.

Proposition 2.4.4. *Let* $(X_1, \ldots, X_n)$ *have a multivariate mixed Erlang distribution defined with the FGM n-copula and* $X_i \sim MixErl\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$ *with* $\beta_i \le \beta_n$, *for* $i = 1, 2, \ldots, n-1$. *Then, the contribution* $C_\kappa(X_i, S_n)$ *at level* κ, $0 < \kappa < 1$, *is given by*

$$C_\kappa(X_i, S_n) = \sum_{k=1}^{\infty} c_{i,k} \frac{k}{\beta_n}, \tag{2.41}$$

where

$$c_{i,k} = \omega(k, \underline{p}_i, \beta_i, \beta_n) + 2\rho_{i,k} p_k^{(n)} \left[\frac{\overline{H}(VaR_\kappa(S_n); k+1, 2\beta_n)}{1-\kappa} - 1\right]$$

with

$$\begin{aligned} \rho_{i,k} &= \frac{\sum_{l=1}^{\infty} l\omega(l, \underline{p}_i, \beta_i, \beta_n) - \left(\sum_{l=1}^{\infty} l\omega(l, \underline{p}_i, \beta_i, \beta_n)\right)^2}{\sum_{l=1}^{\infty} l p_l^{(n)} - \left(\sum_{l=1}^{\infty} l p_l^{(n)}\right)^2} \\ &+ \frac{\sum_{j=1, j\neq i}^{n} \theta_{i,j} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} lm \left(\pi(l, \underline{\omega}(\underline{p}_j, \beta_i, \beta_n)) - \omega(l, \underline{p}_i, \beta_i, \beta_n)\right)\left(\pi(m, \underline{p}_j) - p_{j,m}\right)}{\sum_{l=1}^{\infty} l p_l^{(n)} - \left(\sum_{l=1}^{\infty} l p_l^{(n)}\right)^2}, \end{aligned}$$

for $i = 1, \ldots, n$, *and* $\theta_{i,j} = \theta_{\min(i,j)\max(i,j)}$.

Proof 2.4.5. *The amount of capital attributed to risk i, for $i = 1\dots,n$, is given in (2.10) by*

$$C_\kappa\left(X_i, S_n\right) = E\left[X_i\right] + \frac{Cov(X_i, S_n)}{Var(S_n)}\left(TVaR_\kappa\left(S_n\right) - E\left[S_n\right]\right). \quad (2.42)$$

Using Lemma 2.2.4, we have

$$E\left[X_i\right] = \sum_{k=1}^{\infty} \omega(k, \underline{p}_i, \beta_i, \beta_n)\frac{k}{\beta_n},$$

We also know that $S_n \sim MixErl\left(\underline{p}^{(n)}, 2\beta_n\right)$, hence

$$E\left[S_n\right] = \sum_{k=1}^{\infty} p_k^{(n)}\frac{k}{2\beta_n},$$

and

$$Var(S_n) = \frac{1}{4\beta_n}\sum_{l=1}^{\infty} lp_l^{(n)} - \left(\sum_{l=1}^{\infty} lp_l^{(n)}\right)^2.$$

Given Corollary 2.4.1, (2.42) becomes

$$C_\kappa\left(X_i, S_n\right) = \sum_{k=1}^{\infty}\left[\omega(k, \underline{p}_i, \beta_i, \beta_n) + \frac{Cov(X_i, S_n)}{2Var(S_n)}p_k^{(n)}\left[\frac{\overline{H}\left(VaR_\kappa\left(S_n\right); k+1, 2\beta_n\right)}{1-\kappa} - 1\right]\right]\frac{k}{\beta_n}.$$

Using the expression for the covariance given in Remark 2.4.1, one may find the expression in (2.41).

2.4.3 A numerical application : the trivariate case

We illustrate here our results for the trivariate case via a numerical example. We suppose that the pdfs of X_i, for $i = 1, 2, 3$ are given by

$$\begin{aligned}
f_{X_1}(x) &= 0.5h(x; 1, 0.1) + 0.5h(x; 2, 0.1),\\
f_{X_2}(x) &= 0.3h(x; 1, 0.15) + 0.7h(x; 2, 0.15),\\
f_{X_3}(x) &= 0.2h(x; 1, 0.2) + 0.4h(x; 2, 0.2) + 0.4h(x; 3, 0.2).
\end{aligned}$$

We assume that the dependence structure is given by the following FGM 3-copula

$$C\left(u_1, u_2, u_3\right) = u_1u_2u_3P\left(u_1, u_2, u_3\right),$$

where

$$\begin{aligned}
P\left(u_1, u_2, u_3\right) &= 1 + \theta_{12}(1-u_1)(1-u_2) + \theta_{13}(1-u_1)(1-u_3) + \theta_{23}(1-u_2)(1-u_3)\\
&\quad + \theta_{123}(1-u_1)(1-u_2)(1-u_3).
\end{aligned}$$

For $n = 3$, (2.29) becomes

$$\begin{aligned} f_{\underline{X}}(x_1, x_2, x_3) &= \prod_{i=1}^{3} f_{X_i}(x_i)[1 + \overline{\theta} + 2(\theta_{123} - \theta_{12} - \theta_{13}) \overline{F}_{X_1}(x_1) \\ &+2(\theta_{123} - \theta_{12} - \theta_{23}) \overline{F}_{X_2}(x_2) + 2(\theta_{123} - \theta_{13} - \theta_{23}) \overline{F}_{X_3}(x_3) \\ &-4(\theta_{123} - \theta_{12}) \overline{F}_{X_1}(x_1)\overline{F}_{X_2}(x_2) - 4(\theta_{123} - \theta_{13}) \overline{F}_{X_1}(x_1)\overline{F}_{X_3}(x_3) \\ &-4(\theta_{123} - \theta_{23}) \overline{F}_{X_2}(x_2)\overline{F}_{X_3}(x_3) + 8\theta_{123}\overline{F}_{X_1}(x_1)\overline{F}_{X_2}(x_2)\overline{F}_{X_3}(x_3)]. \end{aligned}$$

We set $\theta_{12} = 0.3$, $\theta_{13} = 0.2$, $\theta_{23} = -0.1$, $\theta_{123} = 0.15$, which respects conditions (2.7). Table 5 displays the values of the expectations, variances and covariances of X_1, X_2 and X_3. Table 6 gives the first 50 values of the probabilities $p_k^{(3)}, k = 1, 2, \ldots$ and Table 7 displays the results obtained with the closed-form formulas for the VaR and TVaR of S_3. In Table 8, we compare once again the capital allocation using the TVaR-based allocation rule and the Cov-based allocation rule at different levels κ. As in the bivariate case, important differences between the results obtained with the two different allocation rules. Note that the routine implemented in Matlab takes only a few seconds to produce these results.

$E[X_1]$	$E[X_2]$	$E[X_3]$	$Var[X_1]$	$Var[X_2]$	$Var[X_3]$	$Cov(X_1, X_2)$	$Cov(X_1, X_3)$	$Cov(X_2, X_3)$
15	11.33	11	175	84.88	69	10.00	6.11	-2.15

Table 2.5: Descriptive statistics of X_1, X_2 and X_3.

k	1	2	3	4	5	6	7	8	9	10
p_k^*	0	0	0.0021	0.0081	0.0179	0.0299	0.0415	0.0512	0.0583	0.0628
k	11	12	13	14	15	16	17	18	19	20
p_k^*	0.0652	0.0658	0.0648	0.0625	0.0592	0.0551	0.0504	0.0454	0.0404	0.0354
k	21	22	23	24	25	26	27	28	29	30
p_k^*	0.0307	0.0264	0.0224	0.0189	0.0158	0.0131	0.0107	0.0088	0.0071	0.0058
k	31	32	33	34	35	36	37	38	39	40
p_k^*	0.0047	0.0037	0.0030	0.0024	0.0019	0.0015	0.0011	0.0009	0.0007	0.0005
k	41	42	43	44	45	46	47	48	49	50
p_k^*	0.0004	0.0003	0.0002	0.0002	0.0001	0.0001	0.0001	0.00007	0.00006	0.00004

Table 2.6: Probabilities p_k^* for $S_3 = X_1 + X_2 + X_3$.

κ	0.1	0.5	0.6	0.7	0.75	0.85	0.9	0.95	0.99	0.995	0.999
$VaR_\kappa(S_3)$	15.83	34.48	39.30	44.89	48.16	56.61	62.79	72.63	93.44	101.87	120.75
$TVaR_\kappa(S_3)$	40.28	52.06	55.86	60.48	63.28	70.73	76.33	85.45	105.15	113.10	129.70

Table 2.7: Value-at-Risk measures and Tail-Value-at-Risk measures for $S_3 = X_1 + X_2 + X_3$.

2.5 Acknowledgements

The authors acknowledge the Natural Sciences and Engineering Research Council of Canada and the Chaire d'actuariat de l'Université Laval for their support.

κ	0.1	0.5	0.6	0.7	0.75	0.85	0.9	0.95	0.99	0.995	0.999
$TVaR_{\kappa}(X_1,S_3)$	16.21	21.64	23.58	26.09	27.69	32.25	35.95	42.40	57.82	64.58	80.48
$C_{\kappa}(X_1,S_3)$	16.53	22.85	24.89	27.37	28.87	32.87	35.87	40.78	51.44	55.83	65.73
$TVaR_{\kappa}(X_2,S_3)$	12.13	14.97	15.82	16.82	17.42	18.97	20.09	21.81	25.04	26.17	28.28
$C_{\kappa}(X_2,S_3)$	12.07	15.14	16.13	17.33	18.06	20.00	21.46	23.84	29.01	31.14	35.95
$TVaR(X_3,S_3)$	11.93	15.45	16.46	17.57	18.17	19.51	20.30	21.28	22.51	22.81	23.28
$C(X_2,S_3)$	11.58	13.99	14.77	15.72	16.29	17.82	18.97	20.84	24.91	26.58	30.36
$TVaR(S_3)$	40.28	52.06	55.87	60.49	63.29	70.74	76.35	85.50	105.38	113.57	132.05

Table 2.8: Capital allocation based on the TVaR and the Covariance rules for the trivariate case.

2.6 References

1. Acerbi, C., Nordio, C., Sirtori, C., 2001. Expected shortfall as a tool for financial risk management. Working Paper.
2. Acerbi, C., Tasche, D., 2002. On the coherence of expected shortfall. Journal of Banking and Finance 26 (7), 1487–1503.
3. Armstrong, M., Galli, A., 2002. Sequential Nongaussian Simulations Using the FGM Copula. Working Paper.
4. Artzner, P., 1999. Application of Coherent Risk Measures to Capital Requirements in Insurance. North American Actuarial Journal 3 (2),83-109.
5. Artzner, P., Delbaen, F., Eber, J-M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9 (3), 203-228.
6. Bargès, M., Cossette, H., Marceau E., 2009. TVaR-based capital allocation with copulas. Insurance : Mathematics and Economics. 45, 348-361.
7. Buch, A., Dorfleitner, G., 2008. Coherent Risk Measures, Coherent Capital Allocations and the Gradient Allocation Principle. Insurance : Mathematics and Economics. 42 (1), 235-242.
8. Chadjiconstantinidis, S., and Vrontos, S., 2012. On a Renewal Risk Process with Dependence under a Farlie-Gumbel-Morgenstern Copula. Scandinavian Actuarial Journal. 1-34.
9. Cherubini, U., Mulinacci, S., Romagnoli, S., 2011. A Copula-based Model of Speculative Price Dynamics in Discrete Time. Journal of Multivariate Analysis. 102 (6), 1047-1063.
10. Cossette, H., Marceau, E., Marri, F., 2008. On the Compound Poisson Risk Model with Dependence based on a Generalized Farlie Gumbel Morgenstern Copula. Insurance : Mathematics and Economics 43, 444-455.
11. Cossette, H., Mailhot, M., Marceau, E., 2012a. TVaR-Based Capital Allocation for Multivariate Compound Distributions with Positive Continuous Claim Amounts. Insurance : Mathematics and Economics. 50 (2), 247–256.

12. Cossette, H., Landriault, D., Marceau, E., Moutanabbir, K., 2012b. Analysis of the Discounted Sum of Ascending Ladder Heights. Insurance : Mathematics and Economics.To appear.
13. Denault, M., 2001. Coherent Allocation of Risk Capital . The journal of Risk. 4 (1), 24-40.
14. Dhaene, J., Henrard, L., Landsman, Z., Vandendorpe, A., Vanduffel, S., 2008. Some results on the CTE-based capital allocation rule. Insurance : Mathematics and Economics 42 (2), 855–863.
15. Gatfaoui, H. 2005. How does systematic risk impact U.S. credit spreads ? A copula study. Bankers Markets and Investors, 77 :5–12.
16. Gatfaoui, H., 2007. Credit default swap spreads and US financial market : Investigating some dependence. 20th Australasian Finance and Banking Conference.
17. Hesselager, O., Andersson, U., 2002. Risk Sharing and Capital Allocation, Tryg Insurance DP.
18. Jang, J., Fu, G., 2011. Measuring tail dependence for aggregate collateral losses using bivariate compound Cox process with shot noise intensity. Working paper.
19. Lee, S.C.K., Lin, X.S. 2010. Modeling and Evaluating Insurance Losses via Mixtures of Erlang Distributions. North American Actuarial Journal. 14, 107-130.
20. McNeil, A.J., Frey, R., Embrechts, P., 2005. Quantitative Risk Management. Princeton University Press, Princeton.
21. Nelsen, R.B., 2006. An Introduction to Copulas, second ed. In : Springer Series in Statistics. Springer, New York.
22. Panjer, H.H., 2002. Measurement of risk, solvency requirements and allocation of capital within financial conglomerates. Research Report 01 15, Institute of Insurance and Pension Research, University of Waterloo.
23. Prieger J., 2002. A Flexible Parametric Selection Model for Non-Normal Data with an Application to Health Care Usage. Journal of Applied Econometrics 17, 367-392.
24. Sklar, A., 1959. Fonctions de Répartition à n Dimensions et leurs Marges. Publications de l'Institut de Statistique de l'Université de Paris 8, 229-231.
25. Tasche, D., 2000. Risk Contributions and Performance Measurement. Research paper, Zentrum Mathematik (SCA).
26. Tijms, H., 1994. Stochastic Models : An Algorithmic Approach. John Wiley, Chichester.
27. Wang, S.S., 2002. A Set of New Methods and Tools for Enterprise Risk Capital Management and Portfolio Optimization. Working paper.
28. Willmot, G.E., and X.S. Lin. 2011. Risk modelling with the mixed Erlang distribution. *Applied Stochastic Models in Business and Industry* 27(1) : 8-22.

29. Yeo, K.L., Valdez, E.A., 2006. Claim Dependence with Common Effects in Credibility Models. Insurance : Mathematics and Economics 38 (3), 609-629.

30. Zhang, Z. and Yang, H., 2011. Gerber-Shiu Analysis in a Perturbed Risk model with Dependence between Claim Sizes and Interclaim Times. Journal of Computational and Applied Mathematics. 235, 1189-1204.

Chapitre 3

Analysis of the Discounted Sum of Ascending Ladder Heights

Résumé

Dans le cadre du modèle de Sparre Andersen, la probabilité de la ruine sur un horizon de temps infini $\psi(u)$ correspond à la fonction de survie de la perte maximale totale Z. i.e.

$$\psi(u) = \Pr(\tau < \infty | U(0) = u) = \overline{F}_Z(u),$$

avec $Z = \sup_{t \leq \tau} \{S(t) - ct, t \geq 0\}$. Cette perte maximale totale obeit à une distribution géométrique composée et elle est définie par

$$Z = \begin{cases} \sum_{i=1}^{M} L_i, & M > 0, \\ 0, & M = 0, \end{cases},$$

où la variable aléatoire M obeit à une loi géométrique.

Les variables aléatoires L_i, $i = 1, , ...,$ représentent les *ladder heights.*

Dans ce chapitre, on étudie la distribution de la somme escomptée des *ladder heights* sur un horizon de temps fini ou infini donné par :

$$Z_\delta(t) = \begin{cases} \sum_{i=1}^{M} e^{-\delta v_i} L_i 1(v_i \leq t), & M > 0, \\ 0, & M = 0. \end{cases},$$

On calcule les moments non-centées, i.e. $m_{\delta,n}(t) = E[(Z_\delta(t))^n]$.

En particulier, on trouve une expression ferme de ces moments dans le cas du modèle classique Poisson-composé et le modèle de Sparre Andersen avec des montants de sinistres distribués selon la loi exponentielle. Pour cette fin, nous allons appliquer la fonction de Gerber-Shiu avec une fonction de pénalité spécifique et en se basant sur certaines caractéristiques de la

loi mélange d'Erlang. L'élaboration d'une expression exacte de ces moments nous permet d'approximer la distribution de la somme escomptée des *ladder heights* par une distribution mélange d'Erlang. Pour établir cette approximation, nous allons utiliser une méthode basée sur les moments. À l'aide de cette approximation, on calcule les mesures de risque VaR et TVaR associées à la somme escomptée des *ladder heights*. On conclut le chapitre par des exemples numériques.

3.1 Introduction

We consider the Sparre Andersen risk model in ruin theory. Let the claim number process $\underline{N} = \{N(t), t \geq 0\}$ be a renewal process where the (positive) interclaim times $\{W_j, j \in \mathbb{N}^+\}$ form a sequence of independent and identically distributed (iid) rvs with density k, cumulative distribution function (cdf) $K(t) = 1 - \overline{K}(t)$ and Laplace transform (LT) $\widetilde{k}$. Also, the claim amount rvs $\{X_j, j \in \mathbb{N}^+\}$ are an iid sequence of positive rvs with density p, cdf $P(x) = 1 - \overline{P}(x)$, LT $\widetilde{p}$ and mean μ. We further assume that any pair (W_j, X_i) consist of two independent rvs.

Based on the pioneer work of Andersen (1957), we describe the surplus process $\underline{U} = \{U(t), t \geq 0\}$ associated to a portfolio of insurance business as

$$U(t) = u + ct - S(t),$$

where $u \geq 0$ is the initial surplus level, $c > 0$ is the premium rate, and $\underline{S} = \{S(t), t \geq 0\}$ is the aggregate claim amount process defined as

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) > 0, \\ 0, & N(t) = 0, \end{cases}$$

(see, e.g., Gerber (1979), Grandell (1991), and Rolski et al. (1999)). We assume that the risk process has a positive security loading, i.e. $E[cW_i - X_i] > 0$. In ruin theory, the first passage time below level 0 (commonly referred to as the *time to ruin*), namely $\tau = \inf_{t \geq 0}\{t, U(t) < 0\}$ with $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$, is a central theme of interest. Various characteristics pertaining to this first passage time have been extensively examined to enhance the understanding of the events leading to/taking place at the time to ruin. The deficit at ruin $|U(\tau)|$ and the surplus just prior to ruin $U(\tau^-)$ are among those quantities which have attracted considerable attention through the analysis of the expected discounted penalty function proposed by Gerber and Shiu (1998) :

$$\Upsilon_\delta(u) = E\left[e^{-\delta\tau} w\left(U(\tau^-), |U(\tau)|\right) I(\tau < \infty) | U(0) = u\right], \qquad u \geq 0, \tag{3.1}$$

where $\delta \geq 0$, $w(x, y) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the so-called penalty function, and I is the indicator function (that is $I(A) = 1$ if the event A occurs and 0 otherwise). From the seminal contribution of Gerber and Shiu (1998), a significant body of ruin theory literature emerged on the analysis of these Gerber-Shiu type functionals in various risk models. The reader is referred to Asmussen and Albrecher (2010, Chapter XII) for a summary on the topic.

Inspired from Trufin et al. (2010), we analyze in Cossette et al. (2011a) the Value-at-Risk (VaR) and the Tail-Value-at-Risk (TVaR) risk measures which are defined in terms of the maximum aggregate loss $Z = \sup_{t \leq \tau}\{S(t) - ct, t \geq 0\}$, with c.d.f. $F_Z(x)$ and survival function $\overline{F}_Z(x) = 1 - F_Z(x)$. The infinite-time ruin probability with an initial surplus u is

$\psi(u) = \Pr(\tau < \infty | U(0) = u) = \overline{F}_Z(u)$. The VaR at level κ, $0 \leq \kappa < 1$, of Z is defined by

$$VaR_\kappa(Z) = F_Z^{-1}(\kappa),$$

where $F_Z^{-1}(\kappa) = \inf(x \in \mathbb{R}, F_X(x) \geq \kappa)$ is the usual definition of the inverse of F_Z. The TVaR at level κ, $0 \leq \kappa < 1$, is defined by

$$\begin{aligned} TVaR_\kappa(Z) &= \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(Z)\,du \\ &= \frac{E\left[Z \times 1_{\{Z > VaR_\kappa(Z)\}}\right] + VaR_\kappa(Z)\left(F_Z(VaR_\kappa(Z)) - \kappa\right)}{1-\kappa}, \end{aligned} \tag{3.2}$$

where $E\left[Z \times 1_{\{X>b\}}\right]$ is the truncated expectation of Z and 1_A is the indicator function such that $1_A(Z) = 1$, if $Z \in A$, and $1_A(Z) = 0$, if $Z \notin A$. Note that $E\left[Z \times 1_{\{Z>b\}}\right]$ can be expressed as $E[Z] - E\left[Z \times 1_{\{Z \leq b\}}\right]$. See e.g. Acerbi (2002), Acerbi and Tasche (2002) and McNeil et al. (2005) for details on the VaR and TVaR. As it is explained in Cossette et al. (2011a), either VaR or TVaR of Z can be used to determine the initial surplus u, which necessitate the knowledge of the distribution of Z.

It is well known that the maximum aggregate loss Z follows a compound geometric distribution, in which the summands consists of the ascending ladder heights. In the present paper, we propose to investigate the distribution of the discounted sum of ascending ladder heights over a finite or an infinite-time intervals. In particular, we derive the expressions for the moments of the discounted sum of ascending ladder heights. We make use of a particular Gerber-Shiu function, namely the discounted moments of the deficit at ruin

$$\Upsilon_{\delta,l}(u) = E\left[e^{-\delta\tau}\left(|U(\tau)|\right)^l I(\tau < \infty)|U(0) = u\right], \qquad u \geq 0,$$

to obtain the expressions for the moments of the discounted sum of ascending ladder heights, which provides an example of direct applications of Gerber-Shiu functionals as risk management tools. Finally, we define VaR and TVaR risk measures in terms of the discounted sum of ascending ladder heights which may be computed with a moment based approximation using the moments of the discounted sum of ascending ladder heights.

The paper is structured as follows : in Section 2, the discounted sum of the ascending ladder heigths is formally defined, and its connection to the discounted aggregate loss in risk theory is established. We also review some (known) results about mixed Erlang distribution which will be useful in the later part of this paper. Section 3 considers the analysis of the moments of the discounted sum of ascending ladder heights in both the classical compound Poisson risk model and the Sparre Andersen risk model with exponential claims. A closed-form expression which involves the tractable Erlang densities is identified. In Section 4, a moment-matching method is used to approximate the distribution of the discounted sum of ascending ladder heights over a given time horizon. We apply this approximation to compute the VaR and the TVaR defined in terms of the discounted sum of ascending ladder heights.

3.2 Preliminaries

We define the net aggregate loss process $\underline{Y} = \{Y(t), t \geq 0\}$ by $Y(t) = S(t) - ct$. Let $\{\upsilon_i\}_{i=1}^{\infty}$ be the sequence of ascending ladder heights epochs associated to the net aggregate loss process $\underline{Y}$, i.e.

$$\upsilon_1 = \inf\{t \geq 0 : Y(t) > 0\},$$

and recursively

$$\upsilon_i = \inf\{t \geq \upsilon_{i-1} : Y(t) - Y(\upsilon_{i-1}) > 0\}.$$

For convenience, let $\upsilon_0 = 0$. Also, define $\gamma_i = \upsilon_i - \upsilon_{i-1}$ to be the duration of the ith ladder epoch, and $L_i = Y(\upsilon_i) - Y(\upsilon_{i-1})$ to be its corresponding ascending ladder height (whenever $\upsilon_i < \infty$). Also, let M be the number of ladder epochs of $\underline{Y}$, i.e. $M = \sup\{i \in \mathbb{N} : \upsilon_i < \infty\}$. Due to the regenerative property of the process $\underline{Y}$ at claim instants, it is immediate that $\{(\gamma_i, L_i)\}_{i\geq 1}$ forms a sequence of iid random pairs, distributed as $(\tau, |U(\tau)|)$ (i.e. the time to ruin and the deficit at ruin) when the initial surplus level is 0.

It is well known that, in the Sparre Andersen risk model (see e.g., Rolski et al. (1999, Chapter 6)), the maximum aggregate loss Z follows a compound geometric distribution, e.g.

$$Z = \begin{cases} \sum_{i=1}^{M} L_i, & M > 0, \\ 0, & M = 0, \end{cases}$$

where M is a geometric rv with probability mass function (pmf)

$$\Pr(M = m) = (1 - \psi(0))(\psi(0))^m, \qquad m = 0, 1, ...$$

Also, let $\underline{Z} = \{Z(t), t \geq 0\}$ be the maximum aggregate loss process where $Z(t) = \sup_{s\leq\tau}\{Y(s) \ : 0 \leq s \leq t\}$ or alternatively

$$Z(t) = \begin{cases} \sum_{i=1}^{M} L_i 1(\upsilon_i \leq t), & M > 0, \\ 0, & M = 0. \end{cases}$$

An illustration of a sample path of the $\underline{Y}$ is provided in Figure 3.1. We introduce the concept of time value of money, and extend $\underline{Z}$ to its discounted version $\underline{Z}_\delta = \{Z_\delta(t), t \geq 0\}$ by discounting the ascending ladder heights at a constant force of interest $\delta \geq 0$, i.e.

$$Z_\delta(t) = \begin{cases} \sum_{i=1}^{M} e^{-\delta \upsilon_i} L_i 1(\upsilon_i \leq t), & M > 0, \\ 0, & M = 0. \end{cases}$$

Note that in risk theory, $Z_\delta(t)$ can be viewed as the discounted aggregate loss (see, e.g., Léveillé and Garrido (2001a,b) and Ren (2008)) where the interclaim times and claim sizes are distributed as an arbitrary γ_i and L_i respectively. In our context, the pairs $\{(\gamma_i, L_i)\}_{i\geq 1}$ are

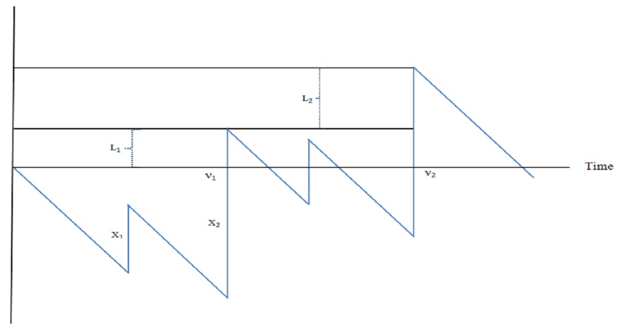


Figure 3.1: A sample path of $\underline{Y}$.

mutually independent, whereas the rvs γ_i and L_i are in general not independent (similarly as in Bargès et al. (2009)). This departs from the usual assumption in risk theory for the analysis of the (discounted) aggregate loss.

In the following sections, mixed Erlang distributions will play an important role. We recall definitions and known results on this family of distributions in the following proposition. The reader is primarily referred to Willmot and Woo (2007) and Willmot and Lin (2011) for more details.

Proposition 3.2.1. *Let X be mixed Erlang with LT $\widetilde{p}(s) \equiv \int_0^\infty e^{-sx} p(x)\, dx = Q\left(\frac{\beta}{\beta+s}\right)$ where $\beta > 0$, $Q(z) = \sum_{i=1}^\infty q_i z^i$ and $\{q_i\}_{i\geq 1}$ is a pmf. Its density is given by*

$$p(x) = \sum_{i=1}^{\infty} q_i \tau_{\beta,i}(x),$$

where $\tau_{\beta,i}(x) = \beta^i x^{i-1} e^{-\beta x}/(i-1)!$ for $x \geq 0$.

(i) The density of X admits the factorization

$$p(x+y) = \sum_{i=1}^{\infty} \kappa_i(y)\, \tau_{\beta,i}(x),$$

where

$$\kappa_i(y) = \sum_{j=0}^{\infty} q_{i+j} \tau_{\beta,j+1}(y).$$

(ii) The equilibrium distribution of X is also mixed Erlang with density

$$p_e(x) \equiv \frac{\int_x^\infty p(y)\, dy}{\mu} = \sum_{j=1}^{\infty} w_j \tau_{\beta,j}(x), \tag{3.3}$$

where $w_j = \frac{\overline{Q}_j}{\sum_{k=1}^\infty \overline{Q}_k}$ and $\overline{Q}_j = \sum\limits_{k=j}^{\infty} q_k$ $(j = 1, 2, ...)$.

We also state a special case of Eq.(2.26) in Willmot and Woo (2007) which will often be used in the sequel.

Proposition 3.2.2. *For* $0 < \beta_1 \leq \beta_2$,

$$\tau_{\beta_1,k}(t) = \sum_{i=k}^{\infty} q_{k,i}\left(\frac{\beta_1}{\beta_2}\right)\tau_{\beta_2,i}(t), \tag{3.4}$$

where

$$q_{k,j}(a) = \binom{j-1}{k-1} a^k (1-a)^{j-k}.$$

3.3 Discounted sum of ladder heights

3.3.1 General structure

We first examine the LT of the discounted of ascending ladder heights over the time interval $[0,t]$. By conditioning on the time and the amount of the first ladder height, one finds

$$E\left[e^{-sZ_\delta(t)}\right] = \Pr(\gamma_1 > t) + E\left[e^{-se^{-\delta\gamma_1}\left(L_1+Z'_\delta(t-\gamma_1)\right)};\gamma_1 < t\right], \tag{3.5}$$

where $Z'_\delta(t)$ is identically distributed as $Z_\delta(t)$ for all $t \geq 0$. The regenerative property of the renewal risk process at claim instants implies that $Z'_\delta(t)$ is mutually independent of either γ_1 and L_1.

By taking the n-th derivative ($n = 1, 2, ...$) with respect to (wrt) s on both sides of (3.5) and then letting $s \to 0$, one obtains that

$$E\left[(Z_\delta(t))^n\right] = E\left[e^{-n\delta\gamma_1}\left(L_1 + Z'_\delta(t-\gamma_1)\right)^n;\gamma_1 < t\right]. \tag{3.6}$$

Letting $m_{\delta,n}(t) = E\left[(Z_\delta(t))^n\right]$ ($n = 0, 1, ...$), simple manipulations of (3.6) lead to

$$\begin{aligned} m_{\delta,n}(t) &= \sum_{l=0}^{n}\binom{n}{l}E\left[e^{-n\delta\gamma_1}(L_1)^{n-l}\left(Z'_\delta(t-\gamma_1)\right)^l;\gamma_1 < t\right] \\ &= \sum_{l=0}^{n}\binom{n}{l}\int_0^t e^{-n\delta w} f_\gamma(w)\, E\left[(L_1)^{n-l}\,|\gamma_1 = w\right] m_{\delta,l}(t-w)\,dw, \end{aligned} \tag{3.7}$$

where f_γ is the density function of an arbitrary γ_i ($i = 1, 2, ...$). Define $\widetilde{m}_{\delta,n}(z) = \int_0^\infty e^{-zt} m_{\delta,n}(t)\,dt$. Taking Laplace transforms on both sides of (3.7) yields

$$\widetilde{m}_{\delta,n}(z) = \sum_{l=0}^{n}\binom{n}{l}E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l}\right]\widetilde{m}_{\delta,l}(z),$$

or alternatively

$$\widetilde{m}_{\delta,n}(z) = \sum_{l=0}^{n-1}\binom{n}{l}\frac{E\left[e^{-(z+n\delta)\gamma_1}(L_1)^{n-l}\right]}{1 - E\left[e^{-(z+n\delta)\gamma_1}\right]}\widetilde{m}_{\delta,l}(z). \tag{3.8}$$

For $\delta > 0$, (3.8) is also equivalent to

$$\widetilde{m}_{\delta,n}(z) = \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-(z+n\delta)\gamma_1} (L_1)^{n-l}; \gamma_1 < \infty\right]}{1 - E\left[e^{-(z+n\delta)\gamma_1}; \gamma_1 < \infty\right]} \widetilde{m}_{\delta,l}(z). \tag{3.9}$$

It is interesting to point out that $\widetilde{m}_{\delta,n}(z)$ is expressed in terms of the discounted moments of the deficit at ruin for an initial surplus of 0 which is known in a variety of risk models (thanks to the numerous advances on the Gerber-Shiu discounted penalty function - see, e.g., Lin and Willmot (2000), Tsai and Willmot (2002), and Ren (2007)).

Remark 3.3.1. *(Infinite-time horizon) From the final value theorem (see Feller (1968)), it is known that*

$$\lim_{z \to 0} z\widetilde{m}_{\delta,n}(z) = E\left[(Z_\delta)^n\right],$$

where $Z_\delta \equiv Z_\delta(\infty)$. Hence, pre-multiplying both sides of (3.9) by z and then taking the limit when $z \to 0$, one arrives at

$$E\left[(Z_\delta)^n\right] = \sum_{l=0}^{n-1} \binom{n}{l} \frac{E\left[e^{-n\delta\gamma_1} (L_1)^{n-l}; \gamma_1 < \infty\right]}{1 - E\left[e^{-n\delta\gamma_1}; \gamma_1 < \infty\right]} E\left[(Z_\delta)^l\right]. \tag{3.10}$$

We point out that Eq. (3.10) allows for the recursive calculation of the moments of Z_δ, the discounted sum of ascending ladder heights, whenever a closed-form expression exists for the discounted moments of the deficit at ruin with an initial surplus of 0.

In a finite-time span, the structure of Eq. (3.9) indicates that a (recursive) expression for the moments of $Z_\delta(t)$ can be identified when the inversion of the right-hand side of (3.9) wrt z can be performed. By further examining (3.9), one notes that the argument z is present in the Laplace transform argument of the first passage time γ_1. Capitalizing on the recent advances of the analytic inversion of the Laplace transform of the time to ruin (see e.g. Dickson and Willmot (2005), Borovkov and Dickson (2008) and Landriault et al. (2011)), we will examine in more details two particular Sparre Andersen risk models, namely the compound Poisson risk model and the Sparre Andersen risk model with exponential claims.

3.3.2 Classical risk model

The discounted moments of the deficit at ruin has been thoroughly studied in the context of the classical compound Poisson risk model by Lin and Willmot (2000). Capitalizing on their results (notably, Theorem 4.1 on Page 28), we aim at analytically inverting (3.9) to obtain a closed-form expression for the n-th (discounted) moment of the ascending ladder heights in the time horizon $[0, t]$. The following lemma contains a result particularly relevant in this regard.

Lemma 3.3.1. *In the classical risk model with Poisson arrivals at rate* $\lambda > 0$*,*

$$\frac{E\left[e^{-\delta\gamma_1}(L_1)^l;\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]}=\int_0^\infty e^{-\delta t}g_l(t)\,dt, \tag{3.11}$$

where

$$g_l(t)=ce^{-\lambda t}f_l(ct)+\sum_{n=1}^\infty\frac{\lambda^n}{n!}t^{n-1}e^{-\lambda t}\int_0^{ct}yp^{*n}(ct-y)\,f_l(y)\,dy, \tag{3.12}$$

and f_l *is as defined in (3.14).*

Proof : Let ρ be the unique nonnegative solution of the generalized Lundberg equation

$$\frac{\lambda+\delta}{c}-s=\frac{\lambda}{c}\widetilde{p}(s).$$

From e.g. Landriault and Willmot (2009, Section 3), we have

$$E\left[e^{-\delta\gamma_1-zL_1};\gamma_1<\infty\right]=\frac{\lambda}{c}\mathcal{T}_{\rho_\delta}\mathcal{T}_z p(0),$$

where $\mathcal{T}_z$ is the Dickson-Hipp operator defined as

$$\mathcal{T}_z f(x)=\int_x^\infty e^{-z(y-x)}f(y)\,dy,$$

for $Re(z)\geq 0$. It is immediate that

$$\begin{aligned}\frac{E\left[e^{-\delta\gamma_1-zL_1};\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]} &= \frac{\frac{\lambda}{c}\mathcal{T}_{\rho_\delta}\mathcal{T}_z p(0)}{1-\frac{\lambda}{c}\mathcal{T}_{\rho_\delta}\mathcal{T}_0 p(0)}\\ &= \frac{\frac{\lambda}{c}\mathcal{T}_{\rho_\delta}\mathcal{T}_z p(0)}{1-\frac{\lambda\mu}{c}\mathcal{T}_{\rho_\delta}p_e(0)}.\end{aligned} \tag{3.13}$$

Letting $\widetilde{p}_e(s)=\int_0^\infty e^{-sx}p_e(x)\,dx$, simple manipulations of (3.13) lead to

$$\begin{aligned}\frac{E\left[e^{-\delta\gamma_1-zL_1};\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]} &= \frac{\lambda}{c}\mathcal{T}_{\rho_\delta}\mathcal{T}_z p(0)\sum_{k=0}^\infty\left(\frac{\lambda\mu}{c}\widetilde{p}_e(\rho_\delta)\right)^k\\ &= \frac{\lambda}{c}\sum_{k=0}^\infty\left(\frac{\lambda\mu}{c}\right)^k\left\{\mathcal{T}_{\rho_\delta}\mathcal{T}_z p(0)\right\}\left(\widetilde{p}_e(\rho_\delta)\right)^k\\ &= \frac{\lambda}{c}\sum_{k=0}^\infty\left(\frac{\lambda\mu}{c}\right)^k\left\{\int_0^\infty e^{-zy}T_{\rho_\delta}p(y)\,dy\right\}\left(\widetilde{p}_e(\rho_\delta)\right)^k.\end{aligned}$$

One concludes that

$$\begin{aligned}\frac{E\left[e^{-\delta\gamma_1-zL_1};\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]} &= \int_0^\infty\int_0^\infty e^{-\rho_\delta t-zy}\left\{\frac{\lambda}{c}p(t+y)\right\}dydt\\ &\quad+\int_0^\infty\int_0^\infty e^{-\rho_\delta t-zy}\left\{\frac{\lambda}{c}\sum_{k=1}^\infty\left(\frac{\lambda\mu}{c}\right)^k\int_0^t p(y+w)\,p_e^{*k}(t-w)\,dw\right\}dydt.\end{aligned}$$

As a by-product, one obtains

$$\frac{E\left[e^{-\delta\gamma_1}\left(L_1\right)^l;\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]} = \int_0^\infty e^{-\rho_\delta t}\left\{\frac{\lambda}{c}\mu_{l,t}\overline{P}\left(t\right)\right\}dt + \int_0^\infty e^{-\rho_\delta t}\left\{\frac{\lambda}{c}\sum_{k=1}^\infty\left(\frac{\lambda\mu}{c}\right)^k\int_0^t \mu_{l,w}\overline{P}\left(w\right)p_e^{*k}\left(t-w\right)dw\right\}dt,$$

where $\mu_{m,t}$ is the m-th moment of the residual lifetime density $p_t\left(y\right)=p\left(t+y\right)/\overline{P}\left(t\right)$ for $y\geq 0$. Letting

$$f_l\left(t\right)=\frac{\lambda}{c}\left\{\mu_{l,t}\overline{P}\left(t\right)+\sum_{k=1}^\infty\left(\frac{\lambda\mu}{c}\right)^k\int_0^t \mu_{l,w}\overline{P}\left(w\right)p_e^{*k}\left(t-w\right)dw\right\}, \tag{3.14}$$

it follows that

$$\frac{E\left[e^{-\delta\gamma_1}\left(L_1\right)^l;\gamma_1<\infty\right]}{1-E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]}=\int_0^\infty e^{-\rho_\delta t}f_l\left(t\right)dt. \tag{3.15}$$

Using the Lagrangian identity

$$e^{-\rho_\delta t}=e^{-\frac{\lambda+\delta}{c}t}+\sum_{n=1}^\infty\frac{\left(\frac{\lambda}{c}\right)^n}{n!}t\int_t^\infty x^{n-1}e^{-\frac{\lambda+\delta}{c}x}p^{*n}\left(x-t\right)dx,$$

(see, e.g., Lin and Willmot (1999) and Dickson and Willmot (2005)), one can substitute the LT (3.15) in ρ_δ into a LT in δ. Using Eq. (4) of Dickson and Willmot (2005), it is immediate that (3.15) becomes (3.11). □

Replacing δ by $z+n\delta$ in (3.11) yields

$$\frac{E\left[e^{-(z+n\delta)\gamma_1}\left(L_1\right)^l;\gamma_1<\infty\right]}{1-E\left[e^{-(z+n\delta)\gamma_1};\gamma_1<\infty\right]}=\int_0^\infty e^{-zt}\left\{e^{-n\delta t}g_l\left(t\right)\right\}dt. \tag{3.16}$$

which allows to rewrite (3.9) as

$$\widetilde{m}_{\delta,n}\left(z\right) = \int_0^\infty e^{-zt}\left\{\int_0^t e^{-n\delta w}g_n\left(w\right)dw\right\}dt + \int_0^\infty e^{-zt}\left\{\sum_{l=1}^{n-1}\binom{n}{l}\int_0^t e^{-n\delta w}g_{n-l}\left(w\right)m_{\delta,l}\left(t-w\right)dw\right\}dt. \tag{3.17}$$

Defining the functional $\{\varphi_{\delta,n}\left(t\right),t\geq 0\}$ through its LT $\widetilde{\varphi}_{\delta,n}\left(z\right)=z\widetilde{m}_{\delta,n}\left(z\right)$, it is clear that

$$m_{\delta,n}\left(t\right)=\int_0^t\varphi_{\delta,n}\left(y\right)dy. \tag{3.18}$$

For convenience, we propose to examine the moments of $Z_\delta\left(t\right)$ through the functional $\{\varphi_{n,\delta}\left(t\right),t\geq 0\}$. In this regard, one multiplies both sides of (3.17) by z to obtain

$$\widetilde{\varphi}_{\delta,n}\left(z\right)=\int_0^\infty e^{-zt}\left\{e^{-n\delta t}g_n\left(t\right)\right\}dt+\int_0^\infty e^{-zt}\left\{\sum_{l=1}^{n-1}\binom{n}{l}\int_0^t e^{-n\delta w}g_{n-l}\left(w\right)\varphi_{\delta,l}\left(t-w\right)dw\right\}dt$$

By the uniqueness property of Laplace transforms (see, e.g., Feller (1968)), one concludes that

$$\varphi_{\delta,n}(t) = e^{-n\delta t} g_n(t) + \sum_{l=1}^{n-1} \binom{n}{l} \int_0^t e^{-n\delta w} g_{n-l}(w) \varphi_{\delta,l}(t-w)\, dw. \tag{3.19}$$

Mixed Erlang claim sizes

In this subsection, we assume that claim sizes are mixed Erlang with LT

$$\widetilde{p}(s) = Q\left(\frac{\beta}{\beta+s}\right), \tag{3.20}$$

where $Q(z) = \sum_{i=1}^{\infty} q_i z^i$ is the probability generating function (pgf) of the pmf $\{q_i\}_{i\geq 1}$. We aim at first deriving a closed-form expression for $g_l(t)$.

Lemma 3.3.2. *In the classical risk model with claim sizes having LT (3.20),*

$$g_l(t) = \sum_{n=1}^{\infty} \varepsilon_{l,n} \tau_{\lambda+c\beta,n}(t), \tag{3.21}$$

where

$$\varepsilon_{l,n} = \alpha_{l,n} \left(\frac{c\beta}{\lambda+c\beta}\right)^n + \sum_{j=1}^{n} \frac{\beta}{n} \zeta_{l,j} \binom{n-1}{j-1} \left(\frac{\lambda}{\lambda+c\beta}\right)^{n-j+1} \left(\frac{c\beta}{\lambda+c\beta}\right)^{j-1},$$

with $\alpha_{l,n}$ and $\zeta_{l,j}$ defined in (3.24) and (3.27) respectively.

Proof : We start with the identification of a closed-form expression for f_l defined in (3.14). Using Proposition 3.2.1 (i), one easily derives that

$$\mu_{l,t}\overline{P}(t) = \sum_{i=1}^{\infty} \eta_{l,i} \tau_{\beta,i}(t), \tag{3.22}$$

where

$$\eta_{l,i} \equiv \int_0^{\infty} y^l \kappa_i(y)\, dy = \frac{1}{\beta^{l+1}} \sum_{j=0}^{\infty} q_{i+j} \frac{(j+l)!}{j!}.$$

Substituting (3.22) and (3.3) into (3.14), one finds that

$$\begin{aligned} f_l(t) &= \frac{\lambda}{c}\left\{ \sum_{i=1}^{\infty} \eta_{l,i}\tau_{\beta,i}(t) + \sum_{k=1}^{\infty}\left(\frac{\lambda\mu}{c}\right)^k \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \eta_{l,i} w_j^{*k} \int_0^t \tau_{\beta,i}(w)\tau_{\beta,j}(t-w)\, dw \right\} \\ &= \sum_{n=1}^{\infty} \alpha_{l,n}\tau_{\beta,n}(t), \end{aligned} \tag{3.23}$$

where

$$\alpha_{l,n} = \frac{\lambda}{c}\left(\eta_{l,n} + \sum_{i=1}^{n-1} \eta_{l,i} \sum_{k=1}^{\infty} \left(\frac{\lambda\mu}{c}\right)^k w_{n-i}^{*k}\right), \tag{3.24}$$

and w^{*k} is the k-fold convolution of the pmf $\{w_j\}_{j\geq 1}$.

We now turn our attention to $g_l(t)$, and show that it also admits a mixed Erlang form. Indeed, we have

$$g_l(t) = ce^{-\lambda t} f_l(ct) + \sum_{n=1}^{\infty} \frac{\tau_{\lambda,n}(t)}{n} \int_0^{ct} yp^{*n}(ct-y) f_l(y)\, dy, \tag{3.25}$$

where

$$\begin{aligned}
\int_0^{ct} yp^{*n}(ct-y) f_l(y)\, dy &= \sum_{i=1}^{\infty} q_i^{*n} \sum_{j=1}^{\infty} \alpha_{l,j} \int_0^{ct} y\tau_{\beta,i}(ct-y)\tau_{\beta,j}(y)\, dy \\
&= \sum_{i=1}^{\infty} q_i^{*n} \sum_{j=1}^{\infty} \frac{j}{\beta}\alpha_{l,j} \int_0^{ct} \tau_{\beta,i}(ct-y)\tau_{\beta,j+1}(y)\, dy \\
&= \sum_{i=1}^{\infty} q_i^{*n} \sum_{j=i+2}^{\infty} \frac{j}{\beta}\alpha_{l,j-i-1}\tau_{\beta,j}(ct) \\
&= \sum_{j=1}^{\infty} \zeta_{l,j}\tau_{\beta,j}(ct),
\end{aligned} \tag{3.26}$$

and

$$\zeta_{l,j} = \frac{1}{\beta}\sum_{i=1}^{j-2} i q_{j-1-i}^{*n}\alpha_{l,i}. \tag{3.27}$$

From (3.26) and (3.23), (3.25) becomes

$$\begin{aligned}
g_l(t) &= c\sum_{n=1}^{\infty}\alpha_{l,n}e^{-\lambda t}\tau_{\beta,n}(ct) + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{\infty}\zeta_{l,j}\tau_{\lambda,n}(t)\tau_{\beta,j}(ct) \\
&= \sum_{n=1}^{\infty}\alpha_{l,n}\left(\frac{c\beta}{\lambda+c\beta}\right)^n\tau_{\lambda+c\beta,n}(t) + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{\infty}\zeta_{l,j}\frac{\lambda^n\beta^j c^{j-1}}{(n-1)!\,(j-1)!}t^{n+j-2}e^{-(\lambda+c\beta)t} \\
&= \sum_{n=1}^{\infty}\alpha_{l,n}\left(\frac{c\beta}{\lambda+c\beta}\right)^n\tau_{\lambda+c\beta,n}(t) \\
&\quad + \sum_{n=1}^{\infty}\frac{\beta}{n}\sum_{j=1}^{\infty}\zeta_{l,j}\binom{n+j-2}{j-1}\left(\frac{\lambda}{\lambda+c\beta}\right)^n\left(\frac{c\beta}{\lambda+c\beta}\right)^{j-1}\tau_{\lambda+c\beta,n+j-1}(t) \\
&= \sum_{n=1}^{\infty}\alpha_{l,n}\left(\frac{c\beta}{\lambda+c\beta}\right)^n\tau_{\lambda+c\beta,n}(t) \\
&\quad + \sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{\beta}{n}\zeta_{l,j}\binom{n-1}{j-1}\left(\frac{\lambda}{\lambda+c\beta}\right)^{n-j+1}\left(\frac{c\beta}{\lambda+c\beta}\right)^{j-1}\tau_{\lambda+c\beta,n}(t).
\end{aligned}$$

Interchanging the order of summation yields (3.21). □

In the next proposition, we apply an inductive argument to identify the functional form of $\varphi_{\delta,n}(t)$.

Proposition 3.3.1. $\varphi_{\delta,n}$ *has a mixed Erlang representation of the form*

$$\varphi_{\delta,n}(t) = \sum_{j=1}^{\infty}\chi_{n,j}\tau_{\lambda+c\beta+n\delta,j}(t), \tag{3.28}$$

where the weights $\chi_{n,j}$ *are obtained recursively via*

$$\chi_{n,j}=\varepsilon_{n,j}\left(\frac{\lambda+c\beta}{\lambda+c\beta+n\delta}\right)^{j}+\sum_{l=1}^{j-1}\binom{j}{l}\sum_{i=1}^{n-1}\varepsilon_{j-l,n-i}\left(\frac{\lambda+c\beta}{\lambda+c\beta+j\delta}\right)^{n-i}\sum_{k=1}^{i}\chi_{l,k}q_{k,i}\left(\frac{\lambda+c\beta+l\delta}{\lambda+c\beta+j\delta}\right).$$

with $\chi_{1,j}=\varepsilon_{1,j}\left(\frac{\lambda+c\beta}{\lambda+c\beta+\delta}\right)^{j}$.

Proof : We shall first prove (3.28) at $n=1$. Using (3.21), it is clear that

$$\begin{aligned}\varphi_{\delta,l}(t) &= e^{-\delta t}g_1(t)\\ &= \sum_{j=1}^{\infty}\chi_{1,j}\tau_{\lambda+c\beta+\delta,j}(t),\end{aligned}$$

where $\chi_{1,j}=\varepsilon_{1,j}\left(\frac{\lambda+c\beta}{\lambda+c\beta+\delta}\right)^{j}$. Through an inductive argument, we assume that (3.28) holds for $n=1,2,...,k-1$, and subsequently prove that (3.28) is valid at $n=k$. Substituting (3.21) and (3.28) into (3.19) at $n=k$ yields

$$\begin{aligned}\varphi_{\delta,k}(t) &= \sum_{n=1}^{\infty}\varepsilon_{k,n}e^{-k\delta t}\tau_{\lambda+c\beta,n}(t)\\ &\quad+\sum_{l=1}^{k-1}\binom{k}{l}\sum_{n=1}^{\infty}\varepsilon_{k-l,n}\sum_{j=1}^{\infty}\chi_{l,j}\int_0^t e^{-k\delta w}\tau_{\lambda+c\beta,n}(w)\,\tau_{\lambda+c\beta+l\delta,j}(t-w)\,dw\\ &= \sum_{n=1}^{\infty}\varepsilon_{k,n}\left(\frac{\lambda+c\beta}{\lambda+c\beta+k\delta}\right)^{n}\tau_{\lambda+c\beta+k\delta,n}(t)\\ &\quad+\sum_{l=1}^{k-1}\binom{k}{l}\sum_{n=1}^{\infty}\varepsilon_{k-l,n}\left(\frac{\lambda+c\beta}{\lambda+c\beta+k\delta}\right)^{n}\sum_{j=1}^{\infty}\chi_{l,j}\\ &\quad\times\int_0^t\tau_{\lambda+c\beta+k\delta,n}(w)\,\tau_{\lambda+c\beta+l\delta,j}(t-w)\,dw. \qquad (3.29)\end{aligned}$$

Using (3.4) with $\beta_1=\lambda+c\beta+l\delta$ and $\beta_2=\lambda+c\beta+k\delta$ for the Erlang density $\tau_{\lambda+c\beta+l\delta,j}$ in (3.29), it follows that

$$\begin{aligned}\varphi_{\delta,k}(t) &= \sum_{n=1}^{\infty}\varepsilon_{k,n}\left(\frac{\lambda+c\beta}{\lambda+c\beta+k\delta}\right)^{n}\tau_{\lambda+c\beta+k\delta,n}(t)\\ &\quad+\sum_{l=1}^{k-1}\binom{k}{l}\sum_{n=1}^{\infty}\varepsilon_{k-l,n}\left(\frac{\lambda+c\beta}{\lambda+c\beta+k\delta}\right)^{n}\sum_{j=1}^{\infty}\chi_{l,j}\\ &\quad\times\sum_{i=j}^{\infty}q_{j,i}\left(\frac{\lambda+c\beta+l\delta}{\lambda+c\beta+k\delta}\right)\tau_{\lambda+c\beta+k\delta,n+i}(t). \qquad (3.30)\end{aligned}$$

Simple manipulations of (3.30) lead to allows to rewrite $\varphi_{k,\delta}(t)$ as

$$\varphi_{\delta,k}(t)=\sum_{n=1}^{\infty}\chi_{k,n}\tau_{\lambda+c\beta+k\delta,n}(t),$$

where

$$\begin{aligned}\chi_{k,n} &= \varepsilon_{k,n}\left(\frac{\lambda+c\beta}{\lambda+c\beta+n\delta}\right)^{i} \\ &\quad+\sum_{l=1}^{k-1}\binom{k}{l}\sum_{i=1}^{n-1}\varepsilon_{k-l,n-i}\left(\frac{\lambda+c\beta}{\lambda+c\beta+k\delta}\right)^{n-i}\sum_{j=1}^{i}\chi_{l,j}q_{j,i}\left(\frac{\lambda+c\beta+l\delta}{\lambda+c\beta+k\delta}\right).\end{aligned}$$

This completes the proof of this proposition. □

From Proposition 3.3.1 in conjunction with (3.18), a closed-form expression for the n-th moment of $Z_\delta(t)$ can easily be derived.

Proposition 3.3.2. *The n-th moment of $Z_\delta(t)$ is given by*

$$m_{\delta,n}(t)=\sum_{j=1}^{\infty}\chi_{n,j}\left(1-\sum_{i=0}^{j-1}\frac{e^{-(\lambda+c\beta+n\delta)t}\left((\lambda+c\beta+n\delta)t\right)^{i}}{i!}\right),$$

for $t\geq 0$ and $n=1,2,\ldots$

3.3.3 Sparre Andersen risk model with exponential claim sizes

When claim sizes are exponentially distributed with mean $1/\beta$, the time of the first ascending ladder height and its size (also exponential thanks to the lack-of-memory property) are mutually independent. As a result,

$$E\left[e^{-\delta\gamma_1}(L_1)^n;\gamma_1<\infty\right]=\frac{n!}{\beta^n}E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]. \tag{3.31}$$

Letting $\phi_\delta=E\left[e^{-\delta\gamma_1};\gamma_1<\infty\right]$, the substitution of (3.31) into (3.9) yields

$$\widetilde{m}_{\delta,n}(z)=\frac{\phi_{z+n\delta}}{1-\phi_{z+n\delta}}\sum_{l=0}^{n-1}\frac{n!}{l!}\frac{1}{\beta^{n-l}}\widetilde{m}_{\delta,l}(z).$$

By rearrangements, we obtain

$$\widetilde{m}_{\delta,1}(z)=\frac{1}{z}\frac{\phi_{z+\delta}}{1-\phi_{z+\delta}}\frac{1}{\beta}, \tag{3.32}$$

and

$$\begin{aligned}\widetilde{m}_{\delta,n}(z) &= \frac{\phi_{z+n\delta}}{1-\phi_{z+n\delta}}\left(\frac{n}{\beta}\left\{\sum_{l=0}^{n-2}\frac{(n-1)!}{l!}\frac{1}{\beta^{n-1-l}}\widetilde{m}_{\delta,l}(z)\right\}+\frac{n}{\beta}\widetilde{m}_{\delta,n-1}(z)\right) \\ &= \frac{\phi_{z+n\delta}}{1-\phi_{z+n\delta}}\left(\frac{n}{\beta}\frac{1-\phi_{z+(n-1)\delta}}{\phi_{z+(n-1)\delta}}\widetilde{m}_{\delta,n-1}(z)+\frac{n}{\beta}\widetilde{m}_{\delta,n-1}(z)\right) \\ &= \frac{n}{\beta}\frac{\phi_{z+n\delta}}{1-\phi_{z+n\delta}}\frac{1}{\phi_{z+(n-1)\delta}}\widetilde{m}_{\delta,n-1}(z),\end{aligned}$$

for $n=2,3,\ldots$ It follows that

$$\widetilde{m}_{\delta,n}(z)=\frac{1}{z}\frac{n!}{\beta^n}\frac{\phi_{z+n\delta}}{1-\phi_{z+n\delta}}C_{n-1,\delta}(z), \tag{3.33}$$

where

$$C_{n,\delta}(z) = \prod_{j=1}^{n} \frac{1}{1-\phi_{z+j\delta}}.$$

To invert (3.33) wrt z, we first devote our attention to $C_{n,\delta}(z)$. By definition,

$$C_{n,\delta}(z) = \frac{1}{1-\phi_{z+n\delta}} C_{n-1,\delta}(z),$$

for $n = 2, 3, ...$ where

$$C_{1,\delta}(z) = \frac{1}{1-\phi_{z+\delta}}.$$

From Landriault et al. (2011), ϕ_δ is the unique solution (in x) within the unit circle of

$$x - \widetilde{k}(\delta + c\beta(1-x)) = 0.$$

Using Lagrange's expansion theorem with $f(x) = (1-x)^{-1}$ (see Cohen (1969, pp.624-625)), we obtain

$$(1-\phi_\delta)^{-1} = 1 + \int_0^\infty h_\delta(y)\, dy,$$

where

$$h_\delta(y) = e^{-\delta y} b(y),$$

and

$$\begin{aligned} b(y) &= \sum_{n=1}^{\infty} k^{*n}(y) \sum_{j=1}^{n} \frac{n-j+1}{n} \frac{(c\beta y)^{j-1} e^{-c\beta y}}{(j-1)!} \\ &= \frac{1}{\beta} \sum_{n=1}^{\infty} k^{*n}(y) \sum_{j=1}^{n} \frac{n-j+1}{n} \tau_{\beta,j}(cy) \\ &= \frac{1}{\beta} \sum_{j=1}^{\infty} \left\{ \sum_{n=j}^{\infty} k^{*n}(y) \frac{n-j+1}{n} \right\} \tau_{\beta,j}(cy). \end{aligned} \tag{3.34}$$

Thus,

$$(1-\phi_{z+j\delta})^{-1} = 1 + \int_0^\infty e^{-zy} h_{j\delta}(y)\, dy,$$

Letting

$$C_{n,\delta}(z) = 1 + \int_0^\infty e^{-zy} \{r_{\delta,n}(y)\}\, dy,$$

it follows that

$$r_{\delta,1}(y) = h_\delta(y), \tag{3.35}$$

and

$$r_{\delta,n}(y) = r_{\delta,n-1}(y) + h_{n\delta}(y) + \int_0^y h_{n\delta}(x)\, r_{\delta,n-1}(y-x)\, dx, \tag{3.36}$$

for $n = 2, 3, ...$ One concludes that

$$\begin{aligned}\widetilde{m}_{\delta,n}(z) &= \frac{1}{z}\frac{n!}{\beta^n}\left(\int_0^\infty e^{-zy}h_{n\delta}(y)\,dy\right)\left(1+\int_0^\infty e^{-zy}\left\{r_{\delta,n-1}(y)\right\}dy\right)\\ &= \frac{1}{z}\frac{n!}{\beta^n}\int_0^\infty e^{-zy}\left(h_{n\delta}(y)+\int_0^y h_{n\delta}(x)\,r_{\delta,n-1}(y-x)\right)dy\\ &= \frac{1}{z}\frac{n!}{\beta^n}\int_0^\infty e^{-zy}\left(r_{\delta,n}(y)-r_{\delta,n-1}(y)\right)dy. \qquad (3.37)\end{aligned}$$

Inverting (3.32) and (3.37) wrt z, we get

$$m_{\delta,n}(t) = \frac{n!}{\beta^n}\left(R_{\delta,n}(t) - R_{\delta,n-1}(t)\right),$$

for $n = 1, 2, ...$ where $R_{\delta,0}(t) = 0$ for all $t \geq 0$ and $R_{\delta,n}(t) = \int_0^t r_{\delta,n}(x)\,dx$.

Mixed Erlang interclaim times

In this section, we assume that the interclaim times have a mixed Erlang distribution with LT $\widetilde{k}(s) = C\left(\frac{\lambda}{\lambda+s}\right)$ where $C(z) = \sum_{i=1}^{\infty} c_i z^i$ and $\{c_i\}_{i\geq 1}$ is a pmf. Let c^{*n} be the pmf of the n-fold convolution of the pmf c. To obtain an expression for the moments of $Z_\delta(t)$, we first examine the function $b(y)$ defined in (3.34) :

$$\begin{aligned}b(y) &= \frac{1}{c\beta}\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\left\{\sum_{i=1}^{\infty} c_i^{*n}\tau_{\lambda,i}(y)\right\}\frac{n-j+1}{n}\tau_{c\beta,j}(y)\\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{n=j}^{\infty} c_i^{*n}\frac{n-j+1}{n}\frac{\lambda^i}{(i-1)!}\frac{(c\beta)^{j-1}}{(j-1)!}y^{i+j-2}e^{-(\lambda+c\beta)y}\\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{n=j}^{\infty} c_i^{*n}\frac{n-j+1}{n}\binom{i+j-2}{i-1}\left(\frac{\lambda}{\lambda+c\beta}\right)^i\left(\frac{c\beta}{\lambda+c\beta}\right)^{j-1}\tau_{\lambda+c\beta,i+j-1}(y).\end{aligned} \qquad (3.38)$$

Simple manipulations of (3.38) result in

$$b(y) = \sum_{i=1}^{\infty}\theta_i\tau_{\lambda+c\beta,i}(y),$$

where

$$\theta_i = \sum_{j=0}^{i-1}\binom{i-1}{j}\left(\frac{\lambda}{\lambda+c\beta}\right)^{i-j}\left(\frac{c\beta}{\lambda+c\beta}\right)^j\sum_{n=j+1}^{\infty}c_{i-j}^{*n}\left(1-\frac{j}{n}\right).$$

One easily obtains that h_δ is also of a mixed Erlang form, i.e.

$$h_\delta(y) = \sum_{i=1}^{\infty}\theta_{i,\delta}\tau_{\lambda+c\beta+\delta,i}(y), \qquad (3.39)$$

where $\theta_{i,\delta} = \theta_i\left(\frac{\lambda+c\beta}{\lambda+c\beta+\delta}\right)^i$.

Using (3.36) together with (3.39), we propose to first identity the functional form of $r_{\delta,l}$.

Proposition 3.3.3. *$r_{\delta,l}$ has a mixed Erlang representation of the form*

$$r_{\delta,l}\left(y\right)=\sum_{i=1}^{\infty}\pi_{l,i}\tau_{\lambda+c\beta+l\delta,i}\left(y\right), \tag{3.40}$$

where the weights $\pi_{l,i}$ are obtained recursively via

$$\begin{aligned}\pi_{l,i} &= \sum_{k=1}^{j}\pi_{l-1,k}q_{k,j}\left(\frac{\lambda+c\beta+\left(n-1\right)\delta}{\lambda+c\beta+n\delta}\right)+\theta_{i,l\delta}\\ &\quad+\sum_{j=1}^{i-1}\sum_{k=1}^{j}\pi_{l-1,k}q_{k,j}\left(\frac{\lambda+c\beta+\left(n-1\right)\delta}{\lambda+c\beta+n\delta}\right)\theta_{i-j,l\delta},\end{aligned}$$

for $l=2,3,\ldots$ where $\pi_{1,i}=\theta_{i,\delta}$.

Proof : Using the identity (3.35), (3.39) implies that (3.40) holds at $l=1$ where $\pi_{1,i}=\theta_{i,\delta}$. Henceforth, we assume that (3.40) is valid for $l=1,2,\ldots,n-1$, and show that (3.40) holds for $l=n$.

From the recursion (3.36), we have that the two RHS functions, namely $r_{\delta,n-1}$ and $h_{n\delta}$, are respectively expressed in terms of the Erlang densities $\tau_{\lambda+c\beta+(n-1)\delta,i}$ and $\tau_{\lambda+c\beta+n\delta,i}$. Using Proposition 3.2.1, $r_{\delta,n-1}\left(y\right)$ can be rewritten as

$$\begin{aligned}r_{\delta,n-1}\left(y\right) &= \sum_{i=1}^{\infty}\pi_{n-1,i}\sum_{j=i}^{\infty}q_{i,j}\left(\frac{\lambda+c\beta+\left(n-1\right)\delta}{\lambda+c\beta+n\delta}\right)\tau_{\lambda+c\beta+n\delta,j}\left(y\right)\\ &= \sum_{j=1}^{\infty}\eta_{n-1,j}\tau_{\lambda+c\beta+n\delta,j}\left(y\right),\end{aligned} \tag{3.41}$$

for $n=2,3,\ldots$ where

$$\eta_{n-1,j}=\sum_{i=1}^{j}\pi_{n-1,i}q_{i,j}\left(\frac{\lambda+c\beta+\left(n-1\right)\delta}{\lambda+c\beta+n\delta}\right). \tag{3.42}$$

Finally, substituting (3.41) and (3.39) into (3.36) yields

$$r_{\delta,n}\left(y\right)=\sum_{i=1}^{\infty}\pi_{n,k}\tau_{\lambda+c\beta+n\delta,k}\left(y\right),$$

where

$$\pi_{n,k}=\eta_{n-1,k}+\theta_{k,n\delta}+\sum_{j=1}^{k-1}\eta_{n-1,j}\theta_{k-j,n\delta}. \tag{3.43}$$

Substituting (3.42) into (3.43) completes the proof of this proposition. □

Using (3.40) at $l=n$ together with (3.41), a closed-form expression for the n-th moment of $Z_{\delta}\left(t\right)$ is obtained.

Proposition 3.3.4. *The n-th moment of $Z_\delta(t)$ is given by*

$$m_{\delta,n}(t) = \frac{n!}{\beta^n} \sum_{i=1}^{\infty} (\pi_{n,i} - \eta_{n-1,i}) \left(1 - \sum_{j=0}^{i-1} \frac{((\lambda + c\beta + n\delta)\, t)^j \, e^{-(\lambda + c\beta + n\delta)t}}{j!} \right).$$

for $t \geq 0$ and $n = 1, 2, ...$ where $\eta_{0,i} = 0$ $(i = 1, 2, ...)$.

3.4 Application - Distribution of $Z_\delta(t)$

Capitalizing on the results developed in the earlier sections, we further analyze the distribution of $Z_\delta(t)$ through an approximation technique together with a simulation study. Note that the cdf of $Z_\delta(t)$ is of the form

$$F_{Z_\delta(t)}(x) = 1 - \psi(0,t) + \psi(0,t)\, F_{Y_\delta(t)}(x),$$

where $\psi(0,t)$ is the probability of ruin within $[0,t]$ with an initial surplus of 0, and $F_{Y_\delta(t)}$ is the cdf of a strictly positive rv $Y_\delta(t)$ whose moments are given by

$$E[Y_\delta(t)^n] = \frac{E[Z_\delta(t)^n]}{\psi(0,t)} = \frac{m_{\delta,n}(t)}{\psi(0,t)}, \tag{3.44}$$

for $n = 1, 2, ...$ Our objective is to provide a moment-matching method to approximate the distribution of $Y_\delta(t)$ given that its moments can be obtained from (3.44) and the material presented in the earlier sections. Moment-matching techniques have been extensively studied in the literature. The reader is referred to Johnson and Taffe (1989), Feldmann and Whitt (1998), and references therein.

In this section, we intent to use a technique proposed by Cossette et al. (2011b) to approximate the distribution of $Y_\delta(t)$ which capitalizes on the denseness of the mixed Erlang class (see Tijms (1994)). Indeed, for a given number of moments (say ζ), we consider all mixed Erlang densities of the form

$$E[\tau_{\beta,M}(t)],$$

where M is a rv with ζ atoms on the set of positive integers $\{1, 2, ..., \zeta\}$ (for a previously chosen ζ) with matches the first ζ moments of $Y_\delta(t)$. Note that ζ should be chosen large enough to ensure that a solution exists - see Cossette et al. (2011b) for further details. Among this set of eligible approximations, as a rule of thumb we select the one that minimizes the difference between their $(\zeta + 1)$-th moments. As an illustration, we consider the following numerical example.

Example 3.4.1. *Assume that both interclaim times and claim sizes are exponentially distributed with mean* 1. *We consider a constant premium rate of* 1.2 *and a force of interest of* 4%.

The values of the first 6 *moments of* $Z_\delta(t)$ *are calculated for* $t = 10,\ 50,\ 100$ *and* 200 *:*

n / t	*10*	*50*	*100*	*200*
1	*2.0904*	*2.7911*	*2.8291*	*2.8309*
2	*10.3922*	*16.5036*	*16.8041*	*16.8173*
3	*71.3258*	*132.4999*	*135.4742*	*135.5999*
4	*612.7045*	*1315.6872*	*1350.1879*	*1351.605563*
5	*6254.4329*	*15381.4273*	*15837.2994*	*15855.6367*
6	*73472.8075*	*205368.4629*	*212098.8478*	*212364.9329*

As expected, we observe that the moments of $Z_\delta(t)$ *converge as* t *becomes large, since* $Z_\delta(t)$ *converges to* $Z_\delta(\infty)$*, due to the presence of the strictly positive solvency margin. We also point out that the probability that no ascending ladder heights occurs in* $[0,t]$ $(t = 10,\ 50,\ 100,\ 200)$ *are provided*

t	*10*	*50*	*100*	*200*
$1-\psi(0,t)$	*0.25519*	*0.18129*	*0.16996*	*0.16738*

which immediately leads to the moments of $Y_\delta(t)$*. The values of* $\psi(0,t)$ *obtained with 100000 Monte Carlo simulations of* $Z_\delta(t)$*.*

For the approximation, we consider a time horizon of $t = 200$ *and assume that* M *is a rv with atoms on the integers* $\{1, 2, ..., 10\}$*. For convenience, assume that* M *has pmf* ρ*. Based on the approximation technique described above, the following mixed Erlang models were found to approximate* $Y_\delta(t)$ *when the first* ζ *moments are matched :*

ζ	β	ρ_1	ρ_2	ρ_3	ρ_4	ρ_6	ρ_7
2	0.9662	–	0.7428	–	–	–	0.2572
3	0.7578	0.3829	–	0.5028	–	0.1143	–
4	0.7911	0.3033	0.2899	–	0.3469	–	0.0600
5	0.7452	0.3507	0.1543	0.2816	0.1249	0.0884	–

In Figure 3.2, we compare the cdf of these 4 mixed Erlang distributions with the empirical cdf that resulted from 10 million Monte Carlo simulations of $Z_\delta(t)$*. Clearly, the approximations obtained with 3 and more moments are very good. Also, for these four models, we compare the values of* $VaR_\kappa(Z_\delta(200))$ *and* $TVaR_\kappa(Z_\delta(200))$ *with those obtained through the simulation study :*

$\zeta\backslash\kappa$	0.95	0.99	0.995
2	*9.0707*	*12.3937*	*13.6421*
3	*8.7561*	*12.5794*	*14.0878*
4	*8.7496*	*12.5541*	*14.0702*
5	*8.7614*	*12.5548*	*14.0611*
simulation	*8.7625*	*12.5484*	*14.0566*

Table 3 : Values of $VaR_\kappa(Z_\delta(200))$

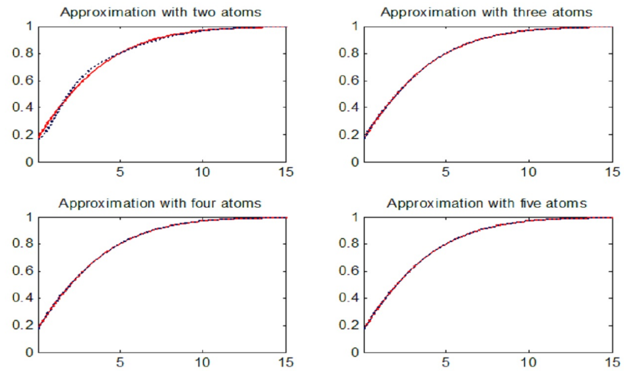


Figure 3.2: Cdf of the 4 mixed Erlang distributions ($n = 2, 3, 4$ and 5 atoms) with the empirical cdf that resulted from 10 millions MC simulations of $Z_\delta(t)$.

$\zeta \backslash \kappa$	0.95	0.99	0.995
2	*11.1205*	*14.1240*	*15.2955*
3	*11.1108*	*14.6846*	*16.1164*
4	*11.0931*	*14.6722*	*16.1138*
5	*11.0997*	*14.6626*	*16.0984*
simulation	*11.0959*	*14.6619*	*16.1005*

Table 4 : Values of $TVaR_\kappa(Z_\delta(200))$

With the first 5 moments matched, one observes that the VaR and TVaR risk measures compare very well with their simulated counterparts.

3.5 Acknowledgements

Support for Helene Cossette, David Landriault and Etienne Marceau from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged. Support from the Chaire d'actuariat de l'Université Laval is also gratefully acknowledged by Hélène Cossette, Etienne Marceau and Khouzeima Moutanabbir.

3.6 References

1. Andersen, E. S. 1957. On the collective theory of risk in case of contagion between the claims. Transactions XVth International Congress of Actuaries, New York, II, 219-229.
2. Asmussen, S., and H. Albrecher. 2010. *Ruin Probabilities.* Advanced Series on Statistical Science and Applied Probability. New Jersey : World Scientific Publishing Co.
3. Bargès, M., Cossette, H., and E. Marceau. 2009. TVaR-based capital allocation with copulas. *Insurance : Mathematics and Economics* 45(3) : 348-361.
4. Borovkov, K.A., and D.C.M. Dickson. 2008. On the Ruin Time Distribution for a Sparre Andersen Risk Process with Exponential Claim Sizes. *Insurance : Mathematics and Economics* 42 : 1104-1108.
5. Cossette, H., Landriault, D., Marceau, E. and K. Moutanabbir. 2011a. Risk measures and maximal aggregate loss. *To be submitted.*
6. Cossette, H., Landriault, D., Marceau, E. and K. Moutanabbir. 2011b. Moment-based approximations with finite mixtures of Erlang of common scale. *To be submitted.*
7. Dickson, D.C.M., and G.E. Willmot. 2005. The density of the time to ruin in the classical Poisson risk model. *Astin Bulletin* 35(1) : 45-60.
8. Feldmann, A., and W. Whitt. 1998. Fitting mixtures of exponentials to long-tail distributions to analyze network performance models. *Performance Evaluation* 31(8) :963-976.
9. Feller, William. 1968. *An Introduction to Probability Theory and Its Applications.* Volume 1. 3rd ed. New York : Wiley.
10. Gerber, H.U. 1979. *An Introduction to Mathematical Risk Theory.* Philadelphia : S.S. Heubner Foundation Monograph Series 8.
11. Gerber, H.U., and E. S.W. Shiu. 1998. On the time value of ruin. *North American Actuarial Journal* 2(1) : 48-72.
12. Johnson, M.A. and M. R. Taffe. 1989. Matching moments to phase distributions : Mixtures of Erlang distribution of common order. Stochastic Models 5 : 711-743.
13. Kulenko, N., and H. Schmidli. 2008. Optimal dividend strategies in a Cramer–Lundberg model with capital injections. *Insurance : Mathematics and Economics* 43(2) : 270–278.
14. Landriault, D., Shi, T., and G.E. Willmot. 2011. Joint density involving the time to ruin in the Sparre Andersen risk model with exponential claims. *Insurance : Mathematics and Economics* 49(3) : 371-379.
15. Landriault, D., and G.E. Willmot. 2009. On the Joint Distributions of the Time to Ruin, the Surplus Prior to Ruin and the Deficit at Ruin in the Classical Risk Model. *North American Actuarial Journal* 13(2) : 252-270.

16. Léveillé, G., and J. Garrido. 2001a. Moments of Compound Renewal Sums with Discounted Claims. *Insurance : Mathematics and Economics* 28(2) : 217–31.

17. Léveillé, G., and J. Garrido. 2001b. Recursive Moments of Compound Renewal Sums with Discounted Claims. *Scandinavian Actuarial Journal* 2001(2) : 98-110.

18. Lin, X.S., and G.E. Willmot. 1999. Analysis of a defective renewal equation arising in ruin theory. *Insurance : Mathematics and Economics* 25(1) : 63-84.

19. Lin, X.S., and G.E. Willmot. 2000. The Moments of the Time of Ruin, the Surplus Before Ruin, and the Deficit at Ruin. *Insurance : Mathematics and Economics* 27(1) : 19-44.

20. Ren, J. 2007. The discounted joint distribution of the surplus prior to ruin and the deficit at ruin in a Sparre Andersen model. *North American Actuarial Journal* 11(3) : 128–136.

21. Ren, J. 2008. On the Laplace transform of the aggregate discounted claims with Markovian arrivals. *North American Actuarial Journal* 12(2) : 198-206.

22. Rolski, T., Schmidli, H., Schmidt, V., and J. Teugels. 1999. *Stochastic processes for insurance and finance*. Wiley Series in Probability and Statistics. Chichester : John Wiley & Sons.

23. Tijms, H.C. 1994. *Stochastic Models : An algorithmic approach*. Chichester : John Wiley.

24. Trufin J., Albrecher, H., and M. Denuit. 2010. Properties of a risk measure derived from ruin theory. The Geneva Risk and Insurance Review. Advance online publication, August 24, 2010 ; doi :10.1057/grir.2010.10.

25. Tsai, C.C.-L., and G.E. Willmot. 2002. On the moments of the surplus process perturbed by diffusion. *Insurance : Mathematics and Economics* 31(3) : 327-350.

26. Willmot, G.E., and X.S. Lin. 2001. *Lundberg Approximations for Compound Distributions with Insurance Applications*. New York : Springer.

27. Willmot, G.E., and X.S. Lin. 2011. Risk modelling with the mixed Erlang distribution. *Applied Stochastic Models in Business and Industry* 27(1) : 8-22.

28. Willmot, G.E., and J.K. Woo. 2007. On the class of Erlang mixtures with risk theoretic applications. *North American Actuarial Journal* 11 (2) : 99-115.

Chapitre 4

A Stochastic International Investment Model and Risk Allocation

Résumé

Les modèles d'investissement servent à décrire le comportement de plusieurs classes d'investissement. Dans ce chapitre, on présente un modèle d'investissement qui est constitué de quarte modules dans le cas de deux économies : l'économie canadienne et l'économie américaine.

Le premier module décrit le comportement du taux d'intérêt et propose une modélisation conjointe des structures par terme dans les deux économies. Ce module sert à évaluer tous les produits à revenus fixes tels que les obligations et les fonds obligataires dans les deux pays.

Le second module décrit le comportement conjoint des indices boursiers canadien et américain : le S&P TSX (indice canadien) et le S&P 500 (américain). Le troisième module propose une modélisation du comportement de l'inflation dans les deux pays. Ce modèle d'investissement décrit aussi la dynamique du taux de change canadien/américain.

À l'aide de ce modèle, on étudie le problème d'allocation et de quantification des risques d'investissement, d'inflation et de taux de change. Cette étude a été élaborée en utilisant des simulations générées par notre modèle et à l'aide de la règle d'allocation TVaR.

Dans le premier module, on va considérer un modèle de taux d'intérêt de la classe affine à 3 facteurs. Ces modèles ont l'avantage d'offrir une description adéquate du comportement de la structure à terme des taux d'intérêt comme il est expliqué dans Dai et Singleton (2000). Ce modèle suppose que les dynamiques des taux d'intérêt dans les deux pays sont déterminées en termes de trois facteurs communs. Ces trois facteurs suivent une dynamique stochastique affine et les rendements à échéance des obligations zéros coupons sont des fonctions affines de

ces facteurs.

Pour le second module, on a recours aux modèles à changement de régime multivariés pour décrire le comportement des rendements sur les indices boursiers. Ces modèles connaissent un intérêt croissant et ils ont été appliqués avec succès par différents auteurs dont Guidolin Timmerman (2007).

Dans le troisième module, le comportement conjoint de l'inflation est déterminé à l'aide d'un modèle à deux facteurs avec un retour à la moyenne pour chaque économie. Une structure de dépendance est introduite entre les différentes composantes du modèle. Le dernier module explique la dynamique du taux de change en terme des autres variables : les taux court terme, les rendements boursiers et l'inflation.

L'estimation des paramètres de ce modèle d'investissement est faite en ayant recours aux données financières couvrant la période de janvier 1987 jusqu'à janvier 2010. La méthode d'estimation proposée est une variante de la méthode d'estimation par le filtre de Kalman. Il s'agit d'une adaptation de ce filtre dans le cas de changement de régime pour des processus non-gaussiens. Cette méthode d'estimation ainsi que tous les résultats de l'estimation sont détaillés dans ce chapitre.

Dans ce chapitre, on présente quelques applications de notre modèle dans le cadre de la quantification et de l'allocation des risques. Pour cette fin, on génère des scénarios en utilisant notre modèle d'investissement puis on détermine une allocation du risque à l'aide de la règle d'allocation TVaR. Cette technique est très flexible ce qui nous permet de donner une quantification à la fois du risque d'investissement, risque d'inflation et risque du taux de change. Dans ce chapitre, on étudie aussi le comportement de l'allocation du risque en présence d'un passif.

4.1 Introduction

Stochastic investment models (SIM) play an important role in actuarial science and quantitative risk management. These models are very useful for long-term actuarial applications that require consistent simulations to develop scenarios for several economic variables. In insurance, SIMs form the basis of dynamic financial analysis (DFA) and risk assessment. These models are also important for regulatory institutions and rating agencies to test insurer's positions and to establish company ratings. Moreover, SIMs provide a useful tool for asset allocation decision making which is an important part of a company's risk management. For pension funds and life insurance companies, SIMs are used in valuation of liabilities and in asset liability management. They also are an important component in determining mismatching reserves and in setting premiums for insurance contracts with complex financial characteristics. Ahlgrim et al. (2004) and Ahlgrim et al.(2008) provide an excellent review on economic scenario generators based on SIMs with applications in actuarial science.

To the best of our knowledge, the report by Maturity Guarantees Working Party (MGWP (1980)) was the first academic contribution in the actuarial literature on SIMs. MGWP (1980) develops a time series model for a variety of financial variables. This SIM was used to calculate quantile reserves for unit-linked life insurance contracts incorporating financial guarantees. This work has led to the publication of the Wilkie SIM described in Wilkie (1986). The SIM proposed by Wilkie (1986) is a multivariate time series model which has four components : retail price inflation, share yield, share dividend and long-term interest rates. Wilkie's model assumes that inflation is an independent component and uses a cascade approach to introduce the other components. This model is updated in Wilkie (1995) by including other components for a wage index, short-term interest rates, real estate and foreign exchange. Sahin et al. (2008) revisited Wilkie's model by discussing some empirical performances of the model and examining the parameter stability. More comments and discussions on Wilkie's model are given in Sahin et al. (2008) and references therein. Other models were proposed in the vein of Wilkie SIM see e.g. Yakoubov et al (1999) and Whitten and Thomas (1999).

Hibbert et al. (2001) describe a SIM that generates values for the term structure of interest rates, both real and nominal, inflation rates, equity returns and dividend payouts. Their proposed real interest rate model is a 2-factor Hull-White model. Inflation is also a two-factor model with a double mean reversion process. The equity model determines the equity return in excess of the nominal interest rate as a Markov regime-switching model. The equity dividend yield model is a one factor first order autoregressive process.

Other models have been developed in continuous time (Battocchio et al. (2004) and Munk et al. (2004)). Battocchio et al. (2004) propose a continuous-time SIM that they apply to analyze the pension fund dynamic and to study the optimal dynamic asset allocation problem for defined contribution plans. Their SIM has three components. The nominal interest rate follows an Ornstein-Uhlenbeck process as in Vasicek (1977) and the dynamic of the stock price contains a risk premium that describes the excess of equity return over the the short interest rate. Battocchio et al (2004) take into account the inflation risk and provide a stochastic model for the consumption price index. Munk et al. (2004) use a continuous-time SIM to analyze rational asset allocation in a dynamic setting. This SIM considers a Vasicek model for interest rates and a model with mean-reverting excess for stock returns. It also incorporates inflation uncertainty by assuming a two-factor model for the inflation dynamic. Munk et al. (2004) calibrate their model using U.S. data. They use their SIM to illustrate the behavior of the optimal asset allocations using power utility and the investment model is solved using a dynamic programming approach.

Over the last years, the investment markets have developed significantly which has lead to more attention given to overseas assets and a great need to set up a multi-economy model. Furthermore, the recent openings of financial markets increase the need for such models. The purpose of this article is to present a SIM in the context where investors are allowed to hold assets in two different economies. In the calibration of the model, we consider the Canadian and American markets but other countries could have been used. The aim of this SIM is to manage Risk/Return tradeoffs by representing uncertainty capturing market and exchange rate risks. Our SIM is a continuous-time model with four components : the joint interest rate model, the joint stock markets model, the joint inflation model and the exchange rate dynamic. We use a joint affine term structure model with three factors for interest rates. For the joint stock market, a two-state continuous time Markov regime switching model is used. The inflation rates in both economies are determined by a joint dynamic based on a two factor model. In our SIM, we explain the exchange rate dynamic regarding what happens in stock markets, interest rates and inflation in both countries.

For investors, it is well known that risk drives returns. This means that investors have to analyze their risk allocation according to their asset allocation strategy. Risk allocation is an approach to analyze one's exposure and to determine the contribution of each risk factor to the total risk of the portfolio. The portfolio's risk is measured with a suitable risk measure and the economic capital of the whole portfolio is determined using this risk measure. Then, this global economic capital is allocated to different risks according to their contributions to the global risk. These risk contributions are calculated using a given allocation rule. In the present paper, we provide a useful risk decomposition that helps to analyze this risk allocation. For this purpose, we define a loss random variable (rv) for the whole portfolio. Then, we write this loss rv as a sum of specific loss rvs related to different risks. Using this risk decomposition and based on simulations from our SIM, we evaluate the contribution of investment risks (bonds and equities), the inflation risk and the exchange risk to the global risk. The economic capital is measured by the Tail-Value-at-Risk (TVaR) and the capital allocation is determined by the TVaR-based allocation rule.

The structure of this paper is as follows : in Section 2, we present our proposed stochastic investment model. In section 3, we explain the statistical method used to estimate and calibrate our investment model and then present and comment the result of the estimation. In Section 4, we apply the stochastic investment model to analyze the impact of the asset allocation on risk allocation. These analyses are based on simulations from our financial model and use Value-at-Risk and Tail-Value-at-Risk risk measures. The risk allocation is determined using the TVaR-based allocation rule.

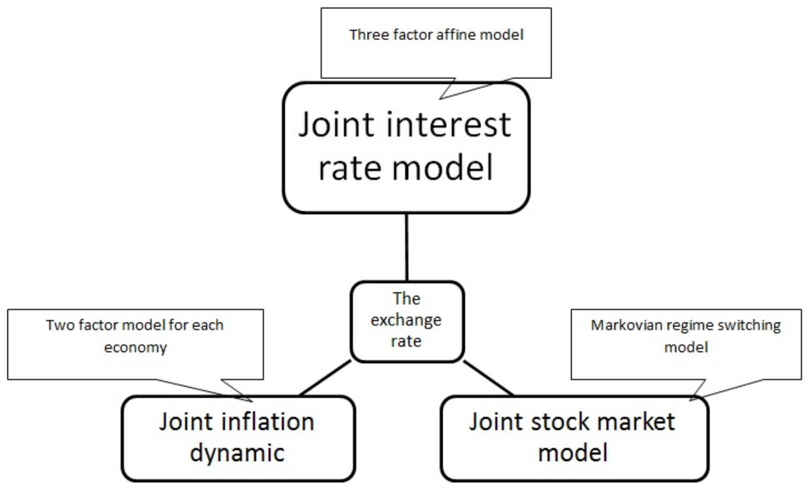


Figure 4.1: Model scheme and structure

4.2 Stochastic investment model

This section presents the SIM assuming that investments are allowed in both domestic and foreign markets. Our proposed SIM includes bank accounts, nominal bonds and one stock index from each country. The main goal of this model is to capture the dependence structure between and within each economy. For this purpose, the following stochastic framework is considered : (i) a joint affine interest rates model which assumes that the same set of factors affect both the domestic and foreign bond indices, (ii) a continuous time Markov switching model for stock markets, (iii) an exchange rate model in order to link the asset models of different economies and (iv) inflation risk measured by modeling the consumer price indexes with a two-factor stochastic dynamic. Figure 4.1 schematizes the structure of our stochastic investment model.

The model structure is subdivided into four major components : the joint interest rates market, the stock markets, the inflation in each economy and the exchange rate between the two economies. The uncertainty of the two economies is described by a complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, where Ω is the set of all possible outcomes and $\mathbf{P}$ represents the physical measure. The set $\mathfrak{F} = \{\mathfrak{F}(t), t \geq 0\}$ is a σ-algebra, where $\mathfrak{F}(t)$ is the available information in both economies up to time t. We also define $\mathbf{Q}^d$ and $\mathbf{Q}^f$ the risk neutral measures for the Canadian economy (d for domestic economy) and the American economy (f for foreign economy), respectively.

4.2.1 The interest rates component

The joint affine term structure model

The literature on bond pricing, starting with Vasicek (1977) and Cox et al. (1985), has focused on the expression of the bond yields as an affine function of some state variables. Many extensions of these models have been proposed (see e.g. James and Webber (2000) or Brigo and Mercurio (2006) for a review). Duffie and Kan (1996) provide a complete characterization of the models known as affine term structure models (ATSM). Dai and Singleton (2000) introduce a classification of these models in terms of the generating processes of these state variables. As noted in Piazzesi (2003), the ATSMs allow a good tractability and are widely used in bond pricing. They are regarded as the cornerstone of modern fixed income theory. Extensions of the single bond market ATSM to international bond markets were considered. Ahn (2004) and Dewachter and Maes (2001) present multi-national three factor affine models and Brennan and Xia (2004) propose a multi-country pure Gaussian term structure model. Mosburger and Schneider (2005) and Egorov et al. (2008) investigate the performance of international ATSMs.

The interest rate component in our SIM is inspired by the work of Dewachter and Maes (2001), Ahn (2004) and Mosburger and Schneider (2005). The basic idea is to suppose that the term structure dynamics in Canada and the U.S. are driven by a vector of n state variables (factors). To determine the number n of factors, we perform a principal component analysis (PCA) using both Canadian and American yield to maturity for τ year, with $\tau = [0.5; 1; 2; 5; 7; 10; 20]$, for the period from January 1987 to January 2010. This PCA examines the number of factors needed to satisfactorily capture the joint dynamics of the Canadian and American term structures. The obtained principal components (PCs) show that 99.44% of the total variations is explained by the first three PCs.

Based on this analysis, we consider a three-factor joint term structure model, i.e. $n = 3$. It means that the joint term structure is driven by three state variables represented by $\mathbf{F} = \{\underline{F}(t) = (F_1(t), F_2(t), F_3(t))', t \geq 0\}$ [1]. These factors may be local, i.e. country-specific factors, or common to both economies. Mosburger and Schneider (2005) study the existence of local factors. Their results provide conclusive evidence against local factors in the joint UK-US term structure. In the present paper, there is no prior specification for local factors.

1. A' denotes the transpose of the matrix A

We assume that the instantaneous short rate $r^j(t)$, for economy $j\epsilon\{d,f\}$, is an affine function of the three state variables $\underline{F}(t)$, i.e.

$$r^j(t) = \delta_0^j + \underline{\delta}_F^{j'}\underline{F}(t), \qquad\qquad j\epsilon\{d,f\}, \tag{4.1}$$

where δ_0^j is a scalar and $\underline{\delta}_F^j$is 3×1 vector, for $j\epsilon\{d,f\}$.

Furthermore, the latent vector $\underline{F}(t)$ is supposed to follow an affine diffusion process under the physical measure $\mathbf{P}$, with

$$d\underline{F}(t) = \underline{K}(\underline{\theta} - \underline{F}(t))dt + \sqrt{\underline{\Sigma}_{F(t)}}d\underline{W}^r(t), \tag{4.2}$$

where $\underline{W}^r(t) = (W_1^r(t), W_2^r(t), W_3^r(t))'$ is a three-dimensional independent standard Brownian motion under $\mathbf{P}$, $\underline{K}$ and $\underline{\Sigma}_{F(t)}$ are 3×3 matrices and $\underline{\theta} = (\theta_1,\theta_2,\theta_3)'$ is a 3×1 vector. The matrix $\underline{\Sigma}_{F(t)}$ is a diagonal matrix with elements of the main diagonal given as affine functions of $\underline{F}(t)$.

The parameters $\underline{K}$, $\underline{\Sigma}_{F(t)}$ and $\underline{\theta}$ cannot be chosen arbitrarily as discussed in Dai and Singleton (2000). To ensure admissibility, we work with the canonical representation of ATSMs introduced by Dai and Singleton (2000). With three factors, this representation classifies each ATSM into four subfamilies based on the number of factors driving the conditional variance. For $m = 0,1,2,3$, the m^{th} subfamily is noted by $A_m(3)$. The specifications of $\underline{K}$, $\underline{\Sigma}_{F(t)}$ and $\underline{\theta}$ differ for each subfamily $A_m(3)$, $(m = 0,1,2,3)$. These specifications are provided in Dai and Singleton (2000). We have estimated these fours ATSMs and analyzed the fit to Canadian and American data.

For these four models, we compare the fit and we analyse the fitting errors in Section 3. According to this empirical analysis, we choose to use the model $A_2(3)$ which provides a better fit to the observed data. In the rest of this paper, we consider the ATSM $A_2(3)$ with the following forms of the drift matrix K and the diffusion matrix $\Sigma_{F(t)}$

$$\underline{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & 0 \\ \kappa_{21} & \kappa_{22} & 0 \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix}, \tag{4.3}$$

and

$$\underline{\Sigma}_{F(t)} = \begin{bmatrix} F_1(t) & 0 & 0 \\ 0 & F_2(t) & 0 \\ 0 & 0 & 1 + \beta_{31}F_1(t) + \beta_{32}F_2(t) \end{bmatrix}. \tag{4.4}$$

The drift matrix $\underline{K}$ allows mutual correlation through feedback effects and mean reverting to the long run mean $\underline{\theta}$. The specific form of the diagonal diffusion matrix $\underline{\Sigma}_{F(t)}$ suggests that only $F_1(t)$ and $F_2(t)$ affect the volatility. According to the dynamic (4.2) and using the classification of ATSMs presented in Dai and Singleton (2000), this model has two square-root factors $(F_1(t), F_2(t))$ and a gaussian factor $(F_3(t))$.

Bond prices and Yields

The analytic tractability of bond yields accounts for much of the popularity of ATSMs. We denote by $P^j(t,\tau)$ the time-t price of a zero-coupon maturing at time $t+\tau$ denominated in the currency of country j, $j\epsilon\{d,f\}$. The expression for $P^j(t,\tau)$ is provided in the following proposition.

Proposition 4.2.1. *In order to avoid arbitrage opportunities, $P^j(t,\tau)$ is given by the Feynman-Kac representation*

$$P^j(t,\tau) = E_t^{\mathbf{Q}^j}\left[\exp(-\int_t^{t+\tau} r^j(s)ds)\right], \tag{4.5}$$

where $E_t^{\mathbf{Q}^j}$ is the $\mathfrak{F}_t-$ expectation under the risk neutral measure $\mathbf{Q}^j$ for the economy j, $j\epsilon\{d,f\}$.

Démonstration. See e.g. Lemke (2005). □

To compute (4.5), we need to work with the factor dynamic under the equivalent probability measure $\mathbf{Q}^j$. Let $\underline{W}^{r,j} = \left(W_1^{r,j}, W_2^{r,j}, W_3^{r,j}\right)'$ denote a 3×1 vector of $\mathbf{Q}^j$-Brownian motions, which are given by

$$d\underline{W}^{r,j}(t) = d\underline{W}^r(t) + \underline{\Lambda}^j(t)dt,$$

where the 3×1 vector $\underline{\Lambda}^j(t)$, for $j\epsilon\{d,f\}$, defines the market price of risk of each factor for the economy j. In order to preserve the affine form of (4.2), we adopt the same definition introduced by Dai and Singleton (2000) for $\underline{\Lambda}^j(t)$:

$$\underline{\Lambda}^j(t) = \sqrt{\underline{\Sigma}_{F(t)}}\underline{\lambda}^j, \qquad j\epsilon\{d,f\}, \tag{4.6}$$

where the 3×1 vector $\underline{\lambda}^j$ is given by $\underline{\lambda}^j = \left(\lambda_1^j, \lambda_2^j, \lambda_3^j\right)'$.

From the expression of $\underline{\Sigma}_{F(t)}$ given in (4.4), one can write

$$\underline{\Lambda}^j(t) = \begin{bmatrix} \lambda_1^j\sqrt{F_1(t)} \\ \lambda_2^j\sqrt{F_2(t)} \\ \lambda_3^j\sqrt{1+\beta_{31}F_1(t)+\beta_{32}F_2(t)} \end{bmatrix}. \tag{4.7}$$

Assuming (4.6) and using Girsanov's theorem, one can prove that the dynamic of $\underline{F}(t)$ is still affine under the measure $\mathbf{Q}^j$

$$d\underline{F}(t) = \underline{K}^j(\underline{\theta}^j - \underline{F}(t))dt + \sqrt{\underline{\Sigma}_{F(t)}}d\underline{W}^{r,j}(t), \tag{4.8}$$

where $\underline{K}^j = \underline{K} + \underline{\phi}^j$ and $\underline{\theta}^j = (\underline{K}+\underline{\phi}^j)^{-1}(\underline{K\theta} - \underline{\psi}^j)$, with $\underline{\phi}^j$ and $\underline{\psi}^j$ given by

$$\underline{\phi}^j = \begin{bmatrix} \lambda_1^j & 0 & 0 \\ 0 & \lambda_2^j & 0 \\ \lambda_3^j\beta_{31} & \lambda_3^j\beta_{32} & 0 \end{bmatrix}, \tag{4.9}$$

and

$$\underline{\psi}^j = \begin{bmatrix} 0 \\ 0 \\ \lambda_3^j \end{bmatrix}. \tag{4.10}$$

Given the risk-neural dynamics of $\underline{F}(t)$ under both $\mathbf{Q}^d$ and $\mathbf{Q}^f$ in (4.8) together with the short rates in (4.1), Duffie and Kan (1996) show that bond prices $P^j(t,\tau)$ are given by

$$P^j(t,\tau) = \exp(A^j(\tau) - B^j(\tau)'F(t)), \tag{4.11}$$

where loading terms A^j and $\underline{B}^j$ satisfy a system of ordinary differential equations (ODE)

$$\begin{cases} \dfrac{dA^j(\tau)}{d\tau} &= \quad -\underline{\theta}^{j\prime}\,\underline{K}^{j\prime}\,\underline{B}^j(\tau) + \dfrac{1}{2}[B_3^j(\tau)]^2 - \delta_0^j, \\ \dfrac{d\underline{B}^j(\tau)}{d\tau} &= \quad -\underline{K}^{j\prime}\,\underline{B}^j(\tau) - \dfrac{1}{2}\underline{N}^j + \underline{\delta}_F^j, \end{cases}$$

with

$$B^j(\tau) = (B_1^j(\tau), B_2^j(\tau), B_3^j(\tau))', \tag{4.12}$$

and

$$\underline{N}^j = \begin{bmatrix} (B_1^j(\tau))^2 \\ (B_2^j(\tau))^2 \\ \beta_{31}(B_1^j(\tau))^2 + \beta_{32}(B_2^j(\tau))^2 \end{bmatrix}. \tag{4.13}$$

The system of ordinary differential equations (ODE) can be solved numerically using the initial conditions $A(0) = 0$ and $B(0) = 1_{3\times 1}$.

Furthermore, the yields at time t of the zero-coupon bond that matures at time τ are expressed as an affine function of the factors $\underline{F}(t)$,

$$Y^j(t,\tau) = -\frac{1}{\tau}A^j(\tau) + \frac{1}{\tau}\underline{B}^j(\tau)'\underline{F}(t), \tag{4.14}$$

for $j\epsilon\{d, f\}$.

4.2.2 The stock markets component

In the present paper, a continuous-time Markov regime switching approach is adopted to model the joint stock market. It is assumed that two stock indices are traded and let $\underline{S} = \{\underline{S}(t) = (S^d(t), S^f(t)), t \geq 0\}$ denote the joint process of these two stock indices. This observed process can be seen as a diffusion where the drift and the volatility coefficients can take two possible values. The switch between these values is governed by a state process $\Upsilon = \{y(t), t \geq 0\}$ which is a continuous time Markov chain. Markov regime models were used in economic and finance in continuous time (e.g. Buffington and Elliott (2002) ; Guo (2001)). In discrete time, these models were applied in econometrics in Engel and Hamilton (1990) and in actuarial science Hardy (2001) and Hibbert et al. (2001).

The dynamic of the process $\underline{S}$ is given by

$$\begin{bmatrix} \frac{dS^d(t)}{S^d(t)} \\ \frac{dS^f(t)}{S^f(t)} \end{bmatrix} = \begin{bmatrix} \mu^d_{y(t)} \\ \mu^f_{y(t)} \end{bmatrix} dt + \begin{bmatrix} \sigma^d_{y(t)} & 0 \\ \sigma^{d,f}_{y(t)} & \sigma^f_{y(t)} \end{bmatrix} \begin{bmatrix} dW^{S^d}(t) \\ dW^{S^f}(t) \end{bmatrix}, \tag{4.15}$$

where $\underline{W}^S = (W^{S^d}, W^{S^f})$ is a 2-dimensional independent standard Brownian motion under $\mathbf{P}$.

Drift and volatility parameters in (4.15) depend explicitly on the economic regime. This regime switching allows the stock price process to switch between 2 regimes randomly. Hardy (2001) compares two-regimes and three-regimes discrete-time models and has found no significant improvement in the fit for the Canadian data and only marginal improvement for the U.S. data. Guidolin and Timmermann (2007) apply a four-regimes continuous-time model to explore the asset allocation problem. The matter of the number of states is discussed in Otranto and Gallo (2002) and Guidolin and Timmermann (2005). A Bayesian non-parametric approach to choose the number of regimes is described in Otranto and Gallo (2002). In order to make our SIM tractable, a two-state Markov regime switching is considered and Υ takes value in a finite state space $\{1, 2\}$.

Let $\underline{G} = (g_{ik})$ be the transition rate matrix of Υ. The transition probabilities are given by

$$P_{ik}(t) = P(y(t+dt) = k | y(t) = i) = \delta_{ik} + g_{ik} dt + o(dt), \text{ for } \ i, k \epsilon \{1, 2\}, \tag{4.16}$$

where δ_{ik} is Kronecker's delta. For a fixed t, the matrix $P_{ik}(t)$ is a stochastic matrix.

The proposed structure for drifts and volatilities is given by

$$\begin{bmatrix} \mu^d_{y(t)} \\ \mu^f_{y(t)} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mu^d_1 \\ \mu^f_1 \end{bmatrix}, & \text{if } y(t) = 1, \\ \\ \begin{bmatrix} \mu^d_2 \\ \mu^f_2 \end{bmatrix}, & \text{if } y(t) = 2, \end{cases} \tag{4.17}$$

$$\begin{bmatrix} \sigma^d_{y(t)} & 0 \\ \sigma^{d,f}_{y(t)} & \sigma^f_{y(t)} \end{bmatrix} = \begin{cases} \begin{bmatrix} \sigma^d_1 & 0 \\ \sigma^{d,f}_1 & \sigma^f_1 \end{bmatrix}, & \text{if } y(t) = 1, \\ \\ \begin{bmatrix} \sigma^d_2 & 0 \\ \sigma^{d,f}_2 & \sigma^f_2 \end{bmatrix}, & \text{if } y(t) = 2, \end{cases} \tag{4.18}$$

In our model, there are two main sources of randomness, the Brownian motions and the Markov chain Υ. We assume that they are mutually independent since the randomness due to the Markov chain can be seen as an exogeneous factor.

We allow correlation between the interest rate component and the stock market component by assuming that the Brownian motion vectors $\underline{W}^r$ and $\underline{W}^S$ are correlated. We define the following correlations for $j = 1, 2$ and $k = 1, 2, 3$:

$$dW^{S^j}(t)dW_k^r(t) = \rho_{S,r}^{j,k}dt. \tag{4.19}$$

4.2.3 The inflation component

The inflation risk needs to be taken into account since long investment periods are considered. We adapt the inflation component of the SIM proposed by Munk et al (2004) to the context of two economies.

The stochastic differential equation describing the evolution of the joint consumption price index CPI $\underline{\Psi} = (\Psi^d, \Psi^f)$ is supposed to have the following dynamic

$$\begin{bmatrix} \frac{d\Psi^d(t)}{\Psi^d(t)} \\ \frac{d\Psi^f(t)}{\Psi^f(t)} \end{bmatrix} = \begin{bmatrix} \Pi^d(t) \\ \Pi^f(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_\Psi^d & 0 \\ \sigma_\Psi^{d,f} & \sigma_\Psi^f \end{bmatrix} \begin{bmatrix} dW_\Psi^d(t) \\ dW_\Psi^f(t) \end{bmatrix}, \tag{4.20}$$

where $\underline{W}_\Psi = \left(W_\Psi^d, W_\Psi^f\right)$ is a standard Brownian motion vector under the objective measure **P**.

In (4.20), it is assumed that the observed changes in the CPIs at time t are driven by an expected inflation process $\underline{\Pi}(t) = (\Pi^d(t), \Pi^d(t))$ and an unexpected short-run inflation measured by the matrix $\underline{\Sigma}_\Psi$ defined by

$$\underline{\Sigma}_\Psi = \begin{bmatrix} \sigma_\Psi^d & 0 \\ \sigma_\Psi^{d,f} & \sigma_\Psi^f \end{bmatrix}. \tag{4.21}$$

The dependence between the interest rate component and the inflation component is introduced by assuming that standard Brownian motion vectors $\underline{W}^r$ and $\underline{W}_\Psi$ are correlated. We define the corresponding correlations for $j = 1, 2$ and $k = 1, 2, 3$ by

$$dW_\Psi^j(t)dW_k^r(t) = \rho_{\Psi,r}^{j,k}dt. \tag{4.22}$$

We also assume correlation between the stock component and inflation. The correlations between $\underline{W}^S$ and $\underline{W}_\Psi$ are given by

$$dW_\Psi^j(t)dW_k^S(t) = \rho_{\Psi,S}^{j,k}dt, \tag{4.23}$$

for $j = 1, 2$ and $k = 1, 2$.

The dynamic of the joint expected inflation at time t, $\Pi(t) = (\Pi^d(t), \Pi^d(t))$, is modeled by a bivariate Ornstein-Uhlenbeck diffusion process as follows

$$\begin{bmatrix} d\Pi^d(t) \\ d\Pi^f(t) \end{bmatrix} = \begin{bmatrix} \beta_\Pi^d(\mu_\Pi^d - \Pi^d(t)) \\ \beta_\Pi^f(\mu_\Pi^f - \Pi^f(t)) \end{bmatrix} dt + \begin{bmatrix} \sigma_\Pi^d & 0 \\ \sigma_\Pi^{d,f} & \sigma_\Pi^f \end{bmatrix} \begin{bmatrix} dW_d^\Pi(t) \\ dW_f^\Pi(t) \end{bmatrix}, \tag{4.24}$$

where $\underline{W}_\Pi = \left(W_\Pi^d, W_\Pi^f\right)$ is a standard Brownian motion vector under the objective measure **P**.

In (4.24), μ_Π^j describes the long-run mean of the rate of inflation or the long-term inflation target and β_Π^j corresponds to the degree of mean reversion. The volatility of inflation rates is determined by the matrix Σ_Π given by

$$\underline{\Sigma}_\Pi = \begin{bmatrix} \sigma_\Pi^d & 0 \\ \sigma_\Pi^{d,f} & \sigma_\Pi^f \end{bmatrix}. \tag{4.25}$$

The Brownian motion vectors $\underline{W}^r$ and $\underline{W}^\Pi$ are assumed to be correlated. The corresponding correlations for $j = 1, 2$ and $k = 1, 2, 3$ are given by

$$dW_\Pi^j(t) dW_k^r(t) = \rho_{\Pi,r}^{j,k} dt. \tag{4.26}$$

We also allow correlation between W^S and W_Π where the correlations for $j = 1, 2$ and $k = 1, 2$ are given by

$$dW_\Pi^j(t) dW_k^S(t) = \rho_{\Pi,S}^{j,k} dt. \tag{4.27}$$

4.2.4 The exchange rate component

To manage an international portfolio, one needs to consider the currency risk. Biger and Hull (1983) and Garman and Kohlhagen (1983) use a geometric Brownian motion with constant exchange rate volatility, along with constant interest rates to model exchange rate movements. More recent models of the joint dynamics of exchange rates and interest rates are seen in Bakshi and Chen (1997) and Brandt and Santa-Clara (2002) among others. Joint modeling of exchange rates and bond yields across the US and UK is studied in Mosburger (2005) using an international ATSM. They specify a kernel pricing dynamic from the joint ATSM and they try to evaluate the exchange rate dynamic under no arbitrage assumption. This approach is also proposed by Dewachter and Maes (2001) which conclude that bond market accounts for little variability in the exchange rate dynamic.

In this paper, a different approach for the exchange rate is considered. We assume that three factors affect the exchange rate : interest rates, stock markets and inflation.

Let $X(t)$ denote the exchange rate, i.e. $X(t)$ is the value of the Canadian currency expressed in American dollar. The dynamic for the exchange rate is explained by differentials in interest rates, stock returns and inflation rates as follows

$$dx(t) = \alpha_r \left(r^d(t) - r^f(t)\right) + \alpha_S \left(\frac{dS^d(t)}{S^d(t)} - \frac{dS^f(t)}{S^f(t)}\right) + \alpha_\Psi \left(\frac{d\Psi^d(t)}{\Psi^d(t)} - \frac{d\Psi^f(t)}{\Psi^f(t)}\right) + \sigma_x dW^x(t), \tag{4.28}$$

where $x(t) = log(X(t))$ measures the continuous change in the Canadian/U.S exchange rate at time t and W^x is a standard Brownian motion under measure **P**.

The interpretation of (4.28) is that the instantaneous variation of the exchange rates is explained by four components. The first component is the interest rate parity (IRP) multiplied by α_r. The second component is the purchasing power parity (PPP) multiplied by α_Ψ. For the third component, the parameter α_S introduces the relationship between exchange rates and stock price indices and the impact of the competitiveness of the financial markets on the exchange rate evolution. For the fourth component, the dynamic (4.28) assumes that there are other exogeneous effects on the exchange rate described by the random effect $\sigma_x dW^x(t)$. We assume independence between the Brownian motion W^x and the other Brownian motion vectors.

4.3 Estimation

In the following subsections, we estimate the SIM presented in Section 2. First, the estimation method used in our empirical calibration is explained. Then, the numerical results of our estimation are presented.

4.3.1 Estimation method

A state-space estimation approach is adopted to estimate the parameters involved in the SIM. A state space model is a representation of the joint dynamic evolution of an observable random vector Z_t and a generally unobservable state vector β_t. The state space model contains a measurement equation and a transition equation. The transition equation governs the evolution of the state vector, the measurement equation specifies how the state interacts with the vector of observations. State space models have a wide range of potential applications in econometrics. Engle and Watson (1981) apply them to modeling the behavior of wage rates. Burmeister and Wall (1982) and Burmeister et al. (1986) use it in estimating expected inflation. For more surveys and applicability of state space models, refer to Hamilton (1994). Lemke (2005), De Jong (1999) apply a state-space framework to estimate ATSM model and Munk et al (2004) use a similar estimation approach to estimate their SIM. See e.g. Hamilton (1994) for an introduction to the state-space estimation method.

The basic tool used to deal with the standard state space model is the Kalman filter. The Kalman filter is a recursive procedure that allows to obtain the relevant log-likelihood function to be maximized. This procedure also computes the estimator of the unobserved state vector at time t, based on available information at time t. Usually, the Kalman filter estimation approach is used when the transition density is multivariate Gaussian and the measurement equation is linear in the state variables. Extended Kalman filter is considered to alleviate the computational problems for the general nonlinear and/or non-Gaussian cases, see e.g. Lemke (2005) for more details.

In the present paper, we use an extended Kalman filter in the case of regime switching.

Proposed by Kim (1994), this method is based on filtering and smoothing algorithms for Markov switching state space model. Kim and Nelson (1998) extend Hamilton's Markov switching model to the state space representation of the general dynamic linear model. They also provide some applications of this filtering method. Kim's algorithm combines Kalman filter, Hamilton's filter and Kim's approximation, see e.g. Kim and Nelson (1998) for more details.

We consider the following state-space representation

$$Z_t = H_{y_t}\beta_t + e_t, \tag{4.29}$$

$$\beta_t = \mu_{y_t} + F_{y_t}\beta_{t-1} + G_{y_t}\nu_t, \tag{4.30}$$

$$\begin{pmatrix} e_t \\ \nu_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} R_{y(t)} & 0 \\ 0 & Q_{y(t)} \end{pmatrix}\right), \tag{4.31}$$

where (4.29) is the measurement equation which determines the observed variables Z_t as a function of the state variables β_t and (4.30) is the transition equation that describes the evolution of the state vector β_t. Some parameters in these models are dependent on an two-state Markov-switching variable $y(t) = 1, 2$ with monthly transition probabilities given by the following matrix

$$Pr = \begin{pmatrix} P_{11} & 1 - P_{22} \\ 1 - P_{11} & P_{22} \end{pmatrix},$$

Our estimation methodology is based on a simple discretization of the SIM using the Euler method. We approximate the non-gaussian process by a gaussian process in order to apply the Kalman filter. Estimation of the SIM is then performed using the quasi maximum likelihood (QML) method. The state-vector Z_t consists of the nine state variables $F_1(t)$, $F_2(t)$, $F_3(t)$, $S^d(t)$,$S^f(t)$, $\Psi^d(t)$, $\Psi^f(t)$, $\Pi^d(t)$ and $\Pi^f(t)$. The observed variables consist in part of seven yields to maturity zero coupon bonds in economy $j = d, f$ with times to maturity τ_k, $k = 1, ..., 7$, which are related to the state variables as follows

$$Y^j(t, \tau_k) = -\frac{1}{\tau}A^j(\tau_k) + \frac{1}{\tau_k}\underline{B}^j(\tau_k)'\underline{F}(t) + \varepsilon_t^j.$$

It is assumed that yields are observed with errors and the rv ε_t^j is the measurement noise. For each economy $j = d, f$, the measurement noise terms on the zero-coupon yields are assumed to be uncorrelated and normally distributed with mean zero and standard deviation h^j. In addition, the measurement equation consists of observing $S^d(t)$, $S^f(t)$, $\Psi^d(t)$, $\Psi^f(t)$ without measurement noise and $x(t)$ using (4.28). We present the detailed filtering method in the appendix and Figure 4.2 schematizes the estimation procedure.

4.3.2 Constraints on admissible ATSM and optimization method

We should constraint the domain of the parameters with respect to the existence and boundary conditions as it is shown in Dai and Singleton (2000). We have also some stationarity conditions that add constraints on parameters(see e.g. Ait-Sahalia et al. (2009) who highlight these facts). Working with a constraint optimization makes the numerical resolution of this problem more complex. To avoid these numerical difficulties, we transform the optimization problem with constraints to sequential non-constraint optimizations using the interior point algorithm with a logarithm barrier function. See e.g. Forsgren et al. (2002) for more details about this optimization method.

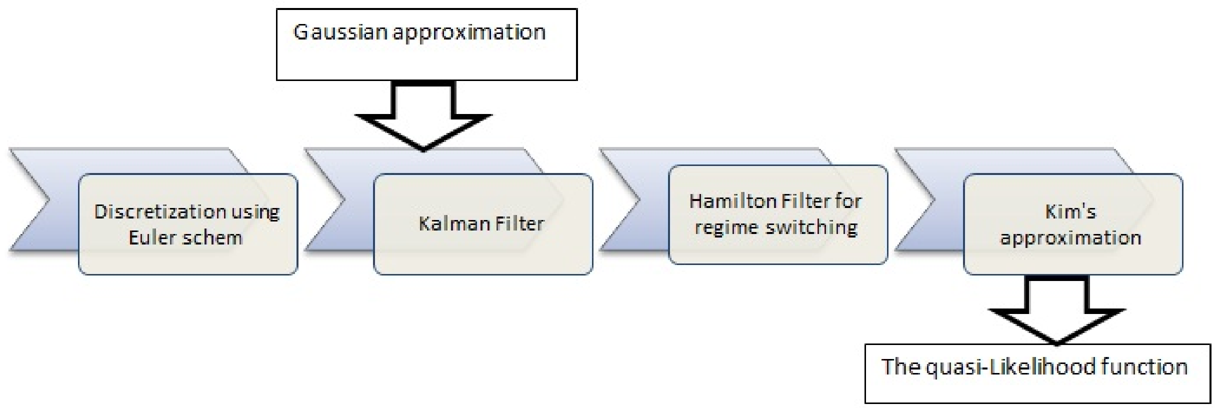


Figure 4.2: Estimation procedure with Kim-Hamilton-Kalman filter

4.3.3 Data and estimation

In this paper, the parameters of the SIM are estimated using Canadian and American data. Our data sample consists of monthly Canadian and American data, with a period of calibration from January 1987 to January 2010. We use a sample of seven yields for each interest rates market with maturities $\tau = [0.5; 1; 2; 5; 7; 10; 20]$ and two stock indices are considered : the American index S&P500 and the Canadian index TSX. We use the zero-coupon data published by the bank of Canada via the institution web site which also provides us the Canadian CPI's data. For the American zero-coupon yields, the constant maturities yields from the federal reserve data-base (http ://research.stlouisfed.org/fred2/) are considered. This source also provides a database of the American CPI. The stock markets data are available in the Data-stream database.

4.3.4 Result

This subsection presents the empirical results of our SIM estimation. We analyze each module's output separately.

ATSM component

The corresponding estimates of the ATSM component are displayed in Table 4.1. We mention that the matrix K has higher values on the diagonal which explains the mean reverting of the factors $F(t)$ in Figure 4.3 that displays the filtered factor. Note that only the third factor takes negative values (gaussian process) with negative long-run mean $\theta_3 = -0.2845$.

Parameter	Estimate	Parameter	Estimate
κ_{11}	0.5735 (0.0342)	δ_2^d	3.68 e-05 (0.0002)
κ_{21}	-0.0016 (0.0064)	δ_3^d	7.33 e-04 (0.007)
κ_{31}	-0.1379 (0.084)	δ_0^f	3.84 e-04 (0.01)
κ_{12}	-0.0041 (2.4 e-03)	δ_1^f	9.87 e-05 (7.1 e-05)
κ_{22}	0.2981 (0.1356)	δ_2^f	2.41 e-04 (3.5 e-04)
κ_{32}	-1.138 e-05 (2.7 e-04)	δ_3^f	6.09e-04 (2.01 e-03)
κ_{33}	0.0344 (0.054)	λ_1^d	-2.31 e-05 (9.4 e-05)
θ_1	11.8979 (1.6349)	λ_2^d	-0.0011 (2.1 e-04)
θ_2	10.2247 (0.9481).	λ_3^d	-0.2037 (0.067)
θ_3	-0.2845 (0.1782)	λ_1^f	9.607 e-04 (1.1 e-04)
β_{31}	0.0062 (0.0024)	λ_2^f	-0.0049 (4.5 e-03)
β_{32}	0.0181 (0.0283)	λ_3^f	-0.1637 (0.023)
δ_0^d	0.0026 (0.001)	h^d	3.1655 e-06 (8.4 e-05)
δ_1^d	1.610 e-04 (6.3 e-04)	h^f	4.72 e-08 (3.1 e-05)

Table 4.1: Estimates of the affine term structure parameters

In risk management, it is very important to extract the correlation between the yields in each market and across markets. Table 4.2 displays the fitted correlation and the observed correlation. We present in Figure 4.4 the filtered yield and we compare it to the observed yield. It shows that the model provides a more accurate fit to the American yield. For both markets, the ATSM is able to provide the joint behavior of the bond markets in Canada and the U.S.

Stock component

Table 4.3 presents the estimates of the drift's parameters in (4.15) and the transition rates matrix. From the drift of each index under different regimes, the first regime is identified as the stressed regime with negative expected return on stock indices and the second regime as the growth regime when positive returns are expected. This fact is confirmed by the diffusion matrix in Table 4.4. Indeed, during the stressed regime months, there is more volatility than in the 'ordinary' months. The diffusion parameters introduce the correlation between the American index S&P500 and the Canadian index TSX.

The estimates of this parameter in each state justify the switch from high correlation during

–	Can1Y	Can2Y	Can5Y	Can10Y	Can15Y	US1Y	US2Y	US5Y	US10Y	US15Y
Can1Y	1	0.99	0.99	0.98	0.97	0.91	0.92	0.96	0.96	0.96
Can2Y	0.99	1	0.99	0.98	0.97	0.89	0.91	0.95	0.96	0.96
Can5Y	0.96	0.98	1	0.99	0.99	0.86	0.88	0.94	0.97	0.97
Can10Y	0.91	0.95	0.98	1	0.99	0.82	0.85	0.92	0.96	0.98
Can15Y	0.84	0.89	0.95	0.98	1	0.80	0.83	0.91	0.96	0.98
US1Y	0.86	0.85	0.80	0.73	0.66	1	0.99	0.97	0.91	0.86
US2Y	0.88	0.89	0.86	0.80	0.73	0.99	1	0.98	0.94	0.89
US5Y	0.92	0.94	0.94	0.91	0.86	0.93	0.97	1	0.98	0.96
US10Y	0.92	0.95	0.97	0.95	0.92	0.86	0.92	0.98	1	0.99
US15Y	0.90	0.94	0.97	0.97	0.95	0.80	0.87	0.96	0.99	1

Table 4.2: Observed values of linear correlation (lower triangular) Vs the filtered values of linear correlation (upper triangular) of the bonds yields in both markets

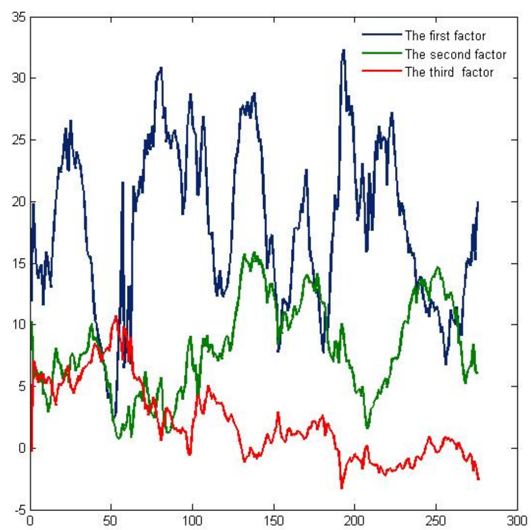


Figure 4.3: Filtred factor for the affine term structure

Parameter	Estimate
μ_1^d	-0.6868 (0.1741)
μ_1^f	-0.4812 (0.0936)
μ_2^d	0.2374 (0.0824)
μ_2^f	0.2123 (0.1065)
g_{12}	6.6077 (0.8571)
g_{21}	0.6415 (0.3592)

Table 4.3: Regime-switching parameters estimates

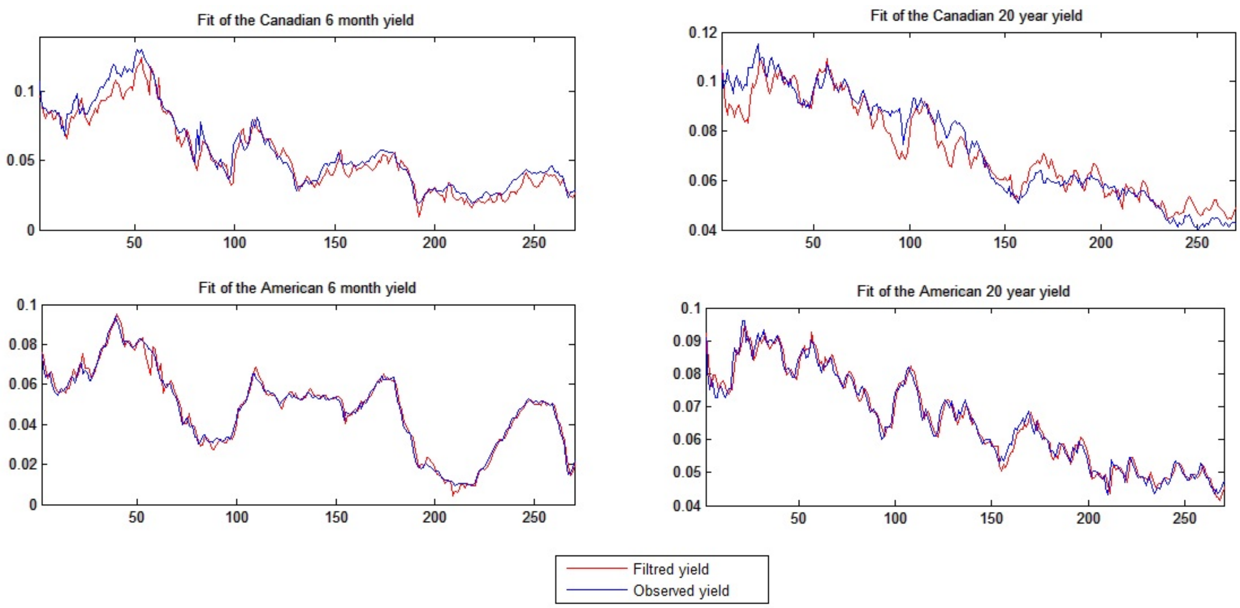


Figure 4.4: Filtered yields and the observed yield

the first regime($\sigma_2^{d,f} = 0.1409$) and the less correlated stock indices during the growth regime ($\sigma_1^{d,f} = 0.0686$)

Regime	1='Stressed'	2='Growth'
σ_R^d	0.2443 (0.0953)	0.1346 (0.1032)
$\sigma_R^{d,f}$	0.1314 (0.1075)	0.0688 (0.070)
σ_R^f	0.1131 (0.0957)	0.1128 (0.0823)

Table 4.4: Regime-switching diffusion matrix estimates

The monthly transition probability matrix is given by

$$Pr = \begin{pmatrix} 0.6219 & 0.0411 \\ 0.3781 & 0.9589 \end{pmatrix} \tag{4.32}$$

According to this transition matrix, the economy spends in average about one month in the stressed state and more than ten months in the growth state. The stationary mean return for the Canadian index is 14.68% and 14.43% for the American index.

One advantage of a filtering based method is the ability to extract some information from the sample about which regime occurred in the past. The Kalman-Hamilton-Kim filter produces the 'smoothed' probabilities of being in regime 1 or 2 conditionally to the whole information. Figure 4.5 shows the smoothed the probabilities that economy lies in the stressed state : 'probabilities of crashes' or the probability that the stock market lies in the stressed state.

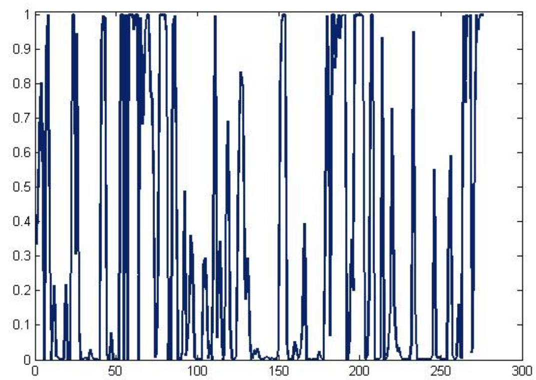


Figure 4.5: Smoothed probabilities that economy lies in the stressed state

Note that the stressed periods include months when TSX and S&P500 returns are negative with high volatility level. We also give in Figure 4.6 and Figure 4.7 the probability density function generated by the regime switching model and we compare it to the histogram built from the observed data for each stock index.

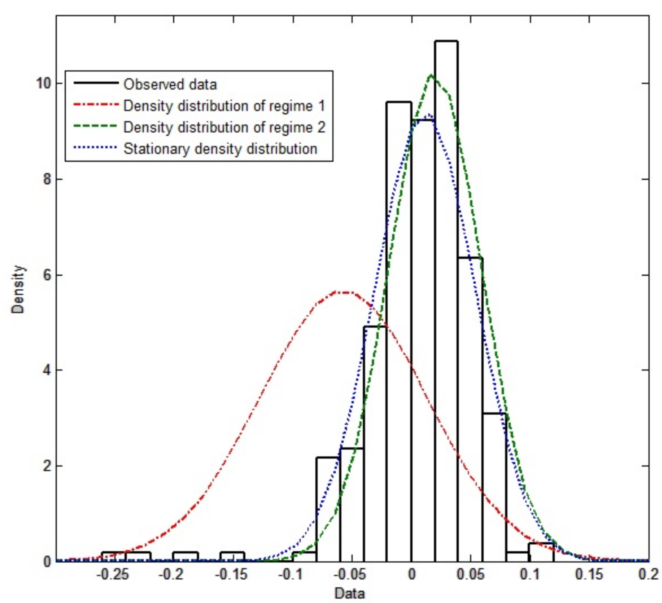


Figure 4.6: Comparison of the Canadian stock market return and the distribution of each regime

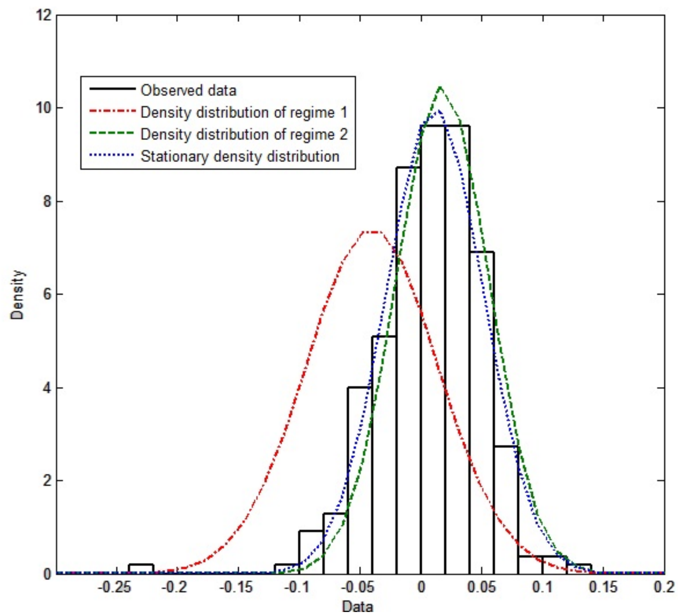


Figure 4.7: Comparison of the American stock market return and the distribution of each regime

Inflation component

For the inflation module, we obtain estimates that are in harmony with the monetary policy in both country and the previous studies. For example, the level 2.36% for the equilibrium Canadian inflation rate looks credible with respect to the interval of The Bank of Canada inflation target [1% ;3%]. In Figure 4.8, the filtered (unobserved) inflation expected over the calibration period is displayed. This Figure shows the monetary policy shift adopted by the Central banks in the early 1990's. The values of the mean-reverting parameters $\beta^{d}_{\Pi} = 0.1291$ and $\beta^{f}_{\Pi} = 0.8684$ explain the behavior of the filtered inflation in Figure 4.8. The corresponding volatility parameters are shown in Table 4.6.

–	Canada	U.S
Mean-reverting	0.1291(0.0762)	0.8684 (0.054)
Mean inflation	2.36% (1.0452)	2.75% (0.9037)

Table 4.5: Inflation parameters estimates

Exchange rate component

From Table 4.7, we conclude that the exchange dynamic is better explained by the interest rate parity and the competitiveness of the financial markets than the purchasing power parity.

Parameter	Estimate
σ_{ψ}^{d}	0,0121 (0.0074)
$\sigma_{\psi}^{d,f}$	0,0042 (0.0942)
σ_{ψ}^{f}	0.0121 (0.0428)
σ_{Π}^{d}	0.0149 (0.0126)
$\sigma_{\Pi}^{d,f}$	-0.0002 (0.023)
σ_{Π}^{f}	0.0214 (0.0076)

Table 4.6: Inflation volatility parameters estimates

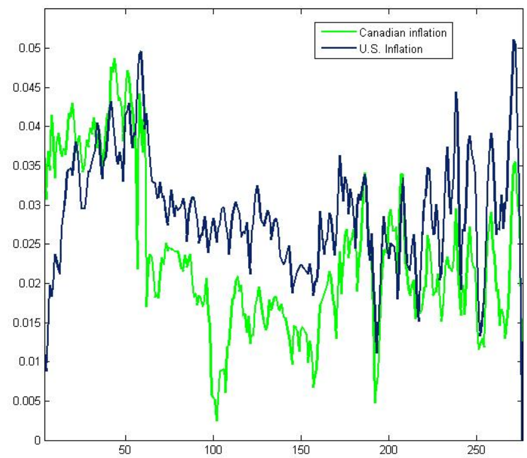


Figure 4.8: Filtered inflation

This estimated model explains more than 45% of the changes of the exchange rate. Also, we have fitted the models derived from the approach based on the ATSM used in Dewachter and Maes (2001) and Mosburger (2005) to the data. We find that our proposed estimated model provides a better fit than their model.

Parameter	Estimation
α_r	0.2203 (0.0847)
α_S	0.2035 (0.1093)
α_{ψ}	0.1299 (0.1406)
σ_X	0.0002 (0.0005)

Table 4.7: Exchange rate parameter estimates

4.3.5 Correlation parameters

We display in Table 4.8 the different values of the correlation parameters that introduce dependence between different components. The estimates of correlations imply that stock returns are negatively correlated with the nominal interest rates. One also remarks that for the correlation between assets and inflation is slightly negative for some assets and slightly negative for others.

—	Parameter	Estimate	Parameter	Estimate
Interest rate vs stock	$\rho^{1,1}_{S,r}$	0.0017 (0.0731)	$\rho^{2,1}_{S,r}$	0.0002 (0.0076)
	$\rho^{1,2}_{S,r}$	-0.0742 (0.0693)	$\rho^{2,2}_{S,r}$	0.0006 (0.0026)
	$\rho^{1,3}_{S,r}$	-0.3085 (0.1581)	$\rho^{2,3}_{S,r}$	0.0002 (0.0015)
Interest rate vs CPIs	$\rho^{1,1}_{\Psi,r}$	0.0004 (0.0051)	$\rho^{2,1}_{\Psi,r}$	-0.0008 (0.0291)
	$\rho^{1,2}_{\Psi,r}$	0.0068 (0.0579)	$\rho^{2,2}_{\Psi,r}$	0.0002 (0.0063)
	$\rho^{1,3}_{\Psi,r}$	0.0004 (0.0194)	$\rho^{2,3}_{\Psi,r}$	-0.0019 (0.0031)
Stock vs CPIs	$\rho^{1,2}_{\Psi,S}$	0.0678 (0.0841)	$\rho^{2,1}_{\Psi,S}$	0.0047 (0.0752)
	$\rho^{1,2}_{\Psi,S}$	0.0002 (0.0047)	$\rho^{2,2}_{\Psi,S}$	-0.0007 (0.0031)
Interest rate vs Inflation	$\rho^{1,1}_{\Pi,r}$	0.0002 (0.0137)	$\rho^{2,1}_{\Pi,r}$	-0.0002 (0.0078)
	$\rho^{1,2}_{\Pi,r}$	-0.0001 (0.0181)	$\rho^{2,2}_{\Psi,r}$	-0.0005 (0.0067)
	$\rho^{1,3}_{\Pi,r}$	0.0003 (0.0011)	$\rho^{2,3}_{\Pi,r}$	0.0003 (0.0019)
Stock vs Inflation	$\rho^{1,1}_{\Pi,S}$	0.0009 (0.0501)	$\rho^{2,1}_{\Pi,S}$	0.0010 (0.0094)
	$\rho^{1,2}_{\Pi,S}$	-0.0004 (0.0014)	$\rho^{2,2}_{\Pi,S}$	0.0001 (0.0009)

Table 4.8: Estimates of the correlation parameters

4.4 Risk allocation using the SIM

4.4.1 Context

We consider the situation in which a Canadian investor holds a portfolio over a time period $[0, t]$ and makes an initial investment $V(0)$ Can \$ at time 0. The asset composition of the portfolio may include assets in the Canadian and American markets and l asset classes are considered. Once the portfolio is selected, its composition is not adjusted and static asset-allocation strategies are considered, i.e. 'buy and hold' strategies.

The portfolio may contain l asset classes with ξ_j the percentage associated to asset j, $j = 1, 2, \ldots, l$. Let $\underline{\xi} = [\xi_1, \ldots, \xi_l]$ be a $l \times 1$ vector describing the percentage of holdings in each asset. We denote by $V(t)$ the accumulated value of the investment at time t expressed in Canadian currency. This rv is given by

$$V(t) = \sum_{i=1}^{l} V_i(t), \tag{4.33}$$

where $V_i(t)$, for $i = 1, 2, ..., l$, denotes the accumulated value corresponding to the class i defined as follows

$$V_i(t) = V_i(0)e^{R_i(0,t)}, \tag{4.34}$$

where the rv $R_i(0,t)$ is the return rate of the asset i over $[0,t]$. These return rates are determined using our SIM introduced in Section 2.

According to the adopted investment strategy, the following constraints hold

$$V_i(0) = \xi_i V(0), \tag{4.35}$$

for $i = 1, 2, ..., l$.

The main goal of this section is to give a general overview of the impact of each component on the risk allocation. First, we define some risk measures and the risk allocation rule that we use in our analysis. Then, we provide the risk allocation to the financial risks during the investment accumulation period and also in the case of the presence of a liability. We conclude this section by investigating the impact of the currency and inflation risks. Ten asset classes are considered, i.e. $l = 10$: (1) domestic and foreign equities (two equity classes), (2) American and Canadian T-bills and (3) six government fixed income funds, three in each economy, with the following maturities : short-term (2 years), mid-term (7 years) and long-term (20 years).

4.4.2 Risk measures and risk allocation

In order to have an appropriate idea of the risk exposure of his portfolio, an investor may often use risk measures. A risk capital must be held by the investor to insure a safety financial level and this capital is commonly determined using an adequate risk measure. Artzner et al. (1999) give an axiomatic definition of a risk measure and introduce the concept of coherent measures of risk. In our analysis, portfolio risk is measured by the Value-at-Risk (VaR) risk measure and the Tail-Value-at-Risk (TVaR) risk measure. For an arbitrary rv X, we define the VaR of X at a given confidence level $0 < \kappa < 1$ by

$$VaR_\kappa (X) = \inf \{x \epsilon \mathbb{R} : F_X(x) \geq \kappa\}.$$

Given that the VaR risk measure does not meet the subadditivity criteria, this risk measure is not coherent. Artzner (1999) and Wang (2002) suggest using the TVaR as an alternative risk measure to replace the VaR, since it is a coherent risk measure. At a given confidence level $0 < \kappa < 1$, we define the TVaR of X by

$$TVaR_\kappa (X) = \frac{E\left[X \times 1_{\{X > VaR_\kappa(X)\}}\right] + VaR_\kappa (X) \left(F_X \left(VaR_\kappa (X)\right) - \kappa\right)}{1 - \kappa}. \tag{4.36}$$

When X is a continuous rv, (4.36) becomes

$$TVaR_{\kappa}(X) = \frac{E\left[X 1_{\{X>VaR_{\kappa}(X)\}}\right]}{1-\kappa}.$$

Consider the case where the rv X can be expressed as a sum of n rvs X_i, $i = 1, 2, ..., l$, i.e.

$$X = \sum_{i=1}^{l} X_l.$$

In this case, the investor analyzes the contribution of each asset class to the global risk. This risk allocation analysis aims to compute the contribution of each component X_i, $i = 1, 2, ..., l$, to the global risk measured by $TVaR_{\kappa}(X)$. Using the TVaR-based allocation rule, we decompose the TVaR risk measure into the sum of contributions as follows

$$TVaR_{\kappa}(X) = TVaR_{\kappa}(X_1; X) + TVaR_{\kappa}(X_2; X) + \ldots + TVaR_{\kappa}(X_l; X).$$

The contribution of the i^{th} risk, $TVaR_{\kappa}(X_i; X)$, which represents the part of the capital that is allocated to risk i, for $i = 1, 2, ..., l$, can be expressed as

$$TVaR_{\kappa}(X_i, X) = \frac{E\left[X_i 1_{\{X>VaR_{\kappa}(X)\}}\right] + \beta_X E\left[X_i 1_{\{X=VaR_{\kappa}(X)\}}\right]}{1-\kappa}, \tag{4.37}$$

with

$$\beta_X = \begin{cases} \frac{Pr(X \le VaR_{\kappa}(X)) - \kappa}{Pr(X = VaR_{\kappa}(X))}, & \text{if } Pr(X = VaR_{\kappa}(X)) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For continuous rvs, the expression of the allocation based on the TVaR-rule (4.37) becomes

$$TVaR_{\kappa}(X_i, X) = \frac{E\left[X_l 1_{\{X>VaR_{\kappa}(X)\}}\right]}{1-\kappa}.$$

See McNeil et al. (2005) for more details on the TVaR-based allocation rule.

4.4.3 The accumulation phase analysis

In this subsection, we show how to use the TVaR-based allocation rule and our SIM's outputs to analyze the risk allocation during the investment period T. We are interested in the accumulated fund value which depends on the investment policy $\underline{\xi}$ adopted at time $t = 0$. For each portfolio $\underline{\xi}$, let us define the following loss rvs at time T

$$L(T) = V(0) - V(T), \tag{4.38}$$

and

$$L^*(T) = E[V(T)] - V(T). \tag{4.39}$$

Losses in (4.38) and (4.39)are described as a deviation from a target value. We consider a financial goal equal to the initial value $V(0)$ and the loss rv L is the nominal loss. The rv L^* introduces the expected value $E[V(T)]$ as a financial goal for each investment strategy.

Given the portfolio composition, we also have the following decompositions

$$L(T) = \sum_{i=1}^{l} L_{V_i}(T), \tag{4.40}$$

where

$$L_{V_i}(T) = V_i(0) - V_i(T),$$

and

$$L^*(T) = \sum_{i=1}^{l} L^*_{V_i}(T), \tag{4.41}$$

where

$$\begin{aligned} L^*_{V_i}(T) &= E[V_i(t)] - V_i(T) \\ &= V_i(0)E\left[e^{R_i(0,T)}\right] - V_i(T). \end{aligned}$$

The rv $V_i(T)$ and the constant $V_i(0)$, for $i = 1, 2, ..., l$, are given in (4.34) and (4.35) respectively.

These decompositions of the loss rvs provide the risk contribution measured by the TVaR-based rule of each asset class on the global risk. The following example illustrates the impact of asset allocation on this risk allocation.

Example 4.4.1. *We consider a time period of ten years, i.e.* $t = 10$*. Using our SIM, 10000 scenarios of the return rates* $R_i(0,t)$*, for* $i = 1, 2, ..., 10$*, are generated. Three strategies are compared (see Table 4.9) : (i) a Canadian portfolio with 20% in each Canadian asset class, (ii) an American portfolio with 20% in each American asset class and (iii) an international portfolio with 10% in each asset class. For these three strategies, we have* $V(0) = 1000$*.*

Based on these simulations, the risk measures $VaR_\kappa(L(10))$*,* $TVaR_\kappa(L(10))$*,* $VaR_\kappa(L^*(10))$ *and* $TVaR_\kappa(L^*(10))$ *are computed and the results are displayed in Table 4.11. We give in Table 4.10 the mean of the accumulated real value for each portfolio. It is clear that the Canadian portfolio provides a higher return level and higher risk than both the American and the international portfolio.*

According to the TVaR-based allocation rule and using the decomposition in (4.40) and (4.41), we compute the TVaR-based capital allocation. Figure 4.9 shows the impact of asset allocation on risk allocation for the international portfolio. One can observe that equities account for

Asset Strategy	Canadian portfolio	American portfolio	International portfolio
Canadian T-bills	20%	0%	10%
American T-bills	0%	20%	10%
Canadian equity	20%	0%	10%
American equity	0%	20%	10%
Canadian 2 year bond fund	20%	0%	10%
American 2 year bond fund	0%	20%	10%
Canadian 7 year bond fund	20%	0%	10%
American 7 year bond fund	0%	20%	10%
Canadian 20 year bond fund	20%	0%	10%
American 20 year bond fund	0%	20%	10%

Table 4.9: Portfolio weights for each strategy

20% of the portfolio, yet are responsible for more than 85% of the total risk. Asset allocation remains an important factor to consider when constructing a portfolio but it cannot be considered in isolation of other events. If the investor thinks about investments only in terms of what asset classes he owns (e.g., bonds, equities) without taking into consideration their exposures to different risks, he has less information on what is going to drive that asset in extreme times. Furthermore, a risk allocation analysis provides a good view of the portfolio's risk profile.

It is important to mention that the international portfolio provides a diversification opportunity. By holding Canadian and American assets, we subdivide the risk contributed to equities into two parts : 49% from the Canadian equity market and 37% from the American equity market. This diversification reduces the expected value of the accumulated fund, as shown in Table 4.10, which leads to a Risk/Return tradeoff.

Portfolio	$E[V(10)]$
Canadian portfolio	2113.89
American portfolio	1982.65
International portfolio	2048.27

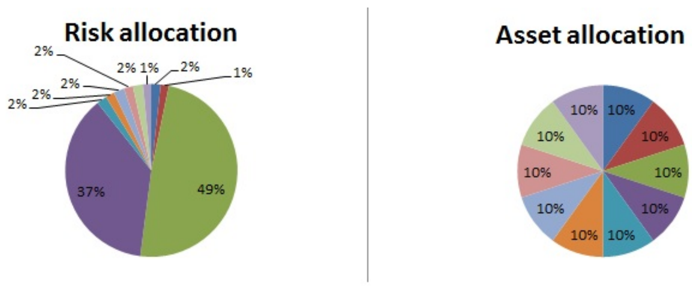
Table 4.10: Mean value for each strategy

4.4.4 Risk allocation and liabilities

In this subsection, we are interested in the risk allocation when the investor has to accumulate enough funds to finance a liability at the end of the investment period. In this case, we define the following loss rv

$$L(T) \quad = \quad P(T) - V(T), \tag{4.42}$$

which quantifies the ability of the investor to meet his liability by the time horizon T. Note that the initial investment amount $V(0)$ is determined such that $E[P(t)] = E[V(t)]$, i.e. the



(a) International portfolio

(b) Canadian portfolio

(c) American portfolio

Figure 4.9: Risk allocation and asset allocation for the three portfolios

Risk measure	κ	0.9	0.95	0.99	0.995
$VaR_\kappa(L(10))$	International portfolio	-412.44	-324.95	-191.90	-161.49
	Canadian portfolio	-339.99	-253.80	-126.07	-85.91
	American portfolio	-442.54	-362.11	-226.05	-194.11
$TVaR_\kappa(L(10))$	International portfolio	-308.47	-245.72	-149.98	-122.83
	Canadian portfolio	-235.39	-174.32	-80.47	-51.40
	American portfolio	-342.94	-280.99	-186.52	-160.52
$VaR_\kappa(L^*(10))$	International portfolio	635.82	723.32	856.36	886.78
	Canadian portfolio	773.90	860.09	987.819	1027.98
	American portfolio	540.11	620.53	756.59	788.54
$TVaR_\kappa(L^*(10))$	International portfolio	739.80	802.55	898.28	925.43
	Canadian portfolio	786.32	854.29	970.81	1009.14
	American portfolio	639.71	701.66	796.12	822.13

Table 4.11: Risk measures comparison

expected loss is 0.

The quantity $L(T)$ can be expressed as follows

$$L(T) = L_P(T) - L_V(T), \tag{4.43}$$

where

$$L_P(T) = P(T) - E[P(T)],$$

$$L_V(T) = E[V(T)] - V(T).$$

The rv $L_P(T)$ measures the liability side contribution to the global loss while $L_V(T)$ is the part of the loss that depends on investments (see the previous subsection).

According to (4.40), we provide another decomposition of (4.42) as follows

$$L(T) = L_P(T) - \sum_{i=1}^{10} L_{V_i}(T), \tag{4.44}$$

where $L_{V_i}(t)$ is given by

$$L_{V_i}(T) = E[V_i(T)] - V_i(T). \tag{4.45}$$

This decomposition allows us to determine the contribution of each asset class to the global risk.

This context is similar to many situations in asset liability management in life insurance and pension fund management. For example, we assume that the investor is x years old at time $t = 0$ and that he must purchase a level life annuity in T years. It is assumed that the accumulated fund $V(T)$ is used immediately to purchase a level life annuity at a price $\ddot{a}_{x,T}$ per 1 Can\$. This annuity pays annually the amount of g Can\$. The price $\ddot{a}_{x,T}$ depends on probabilities of survival above the age $x+T$ and on prices of the Canadian zero-coupon bonds in t years as follows

$$\ddot{a}_{x,T} = \sum_{k=0}^{\infty} P^d(T, T+k) Pr(\text{survive from age x+T to age x+T+k}),$$

where $P^d(T, T+k)$ is obtained using (4.5).

Then, the rv $P(T)$ corresponds to the value of the annuity at time T defined as

$$P(T) = g\ddot{a}_{x,T}.$$

The liability value $P(T)$ depends on the life annuity rate g and the present value of a life annuity at age $x+T$, $\ddot{a}_{x,T}$. The value of $\ddot{a}_{x,T}$ is a function of the survival probabilities over the age of $x+T$ and the zero coupon bond prices at time T. We use the survival probabilities from the mortality table provided from the Institute of Canadian Actuaries (I.C.A.). The zero coupon bond and the inflation rates are generated from our SIM presented in Section 2.

Example 4.4.2. *In our analysis, two strategies, S_1 and S_2, are considered. In Table 4.13, we give the corresponding weight for each strategy. Furthermore, we consider a 55 years old Canadian investor meaning $x = 55$ with $T = 10$ and $g = 10000$. Using 10000 scenarios simulated from our SIM, the values of the risk measures $VaR_\kappa(L(10))$ and $TVaR_\kappa(L(10))$ for each portfolio and at different κ levels are computed and provided in Table 4.15. We observe that strategy S_2 is riskier and requires more capital because the part of equities in the portfolio rises from 20% in S_1 to 30% in S_2. Table 4.12 shows the values of V_0 for each strategy and it is clear that the initial investment for S_2 is less than the initial capital for S_1.*

The TVaR-based capital allocation rule is used to compare the capital allocation between liability and asset and also between liability and each asset class. Figure 4.10 and Figure 4.11 display the impact of the asset's weight on the risk allocation and that the risk allocation is not proportional to the asset allocation. Moreover, the contribution of the liability to the global TVaR risk measure $TVaR_\kappa(L(10))$ decreases as the percentage of equities in the portfolio rises. By increasing the proportion invested in equities, the risk is transferred from the liability side to the asset side. We could conclude that a pension fund manager with appetite for higher returns reduces the cost of the initial funding of this pension fund (as shown in Table 4.12) but holds a riskier investment position.

Strategy	S_1	S_2
$V(0)$	49445.13	43249.48

Table 4.12: Initial investment value

Asset Strategy	S_1	S_2
Canadian T-bills	10%	15%
American T-bills	10%	10%
Canadian equity	10%	15%
American equity	10%	15%
Canadian 2 year bond fund	10%	10%
American 2 year bond fund	10%	5%
Canadian 7 year bond fund	10%	10%
American 7 year bond fund	10%	5%
Canadian 20 year bond fund	10%	10%
American 20 year bond fund	10%	5%

Table 4.13: Portfolio weight for each strategy

κ	0.9	0.95	0.99	0.995
$VaR_\kappa (L(10))$ for portfolio S_1	43281.21	53018.51	66323.80	71138.86
$VaR_\kappa (L(10))$ for portfolio S_1	47534.88	57821.43	72212.86	77234.97
$TVaR_\kappa (L(10))$ for portfolio S_2	54784.21	61635.94	72297.01	75821.12
$TVaR_\kappa (L(10))$ for portfolio S_2	59662.62	67006.59	78631.01	82631.79

Table 4.14: Risk measures for each portfolio

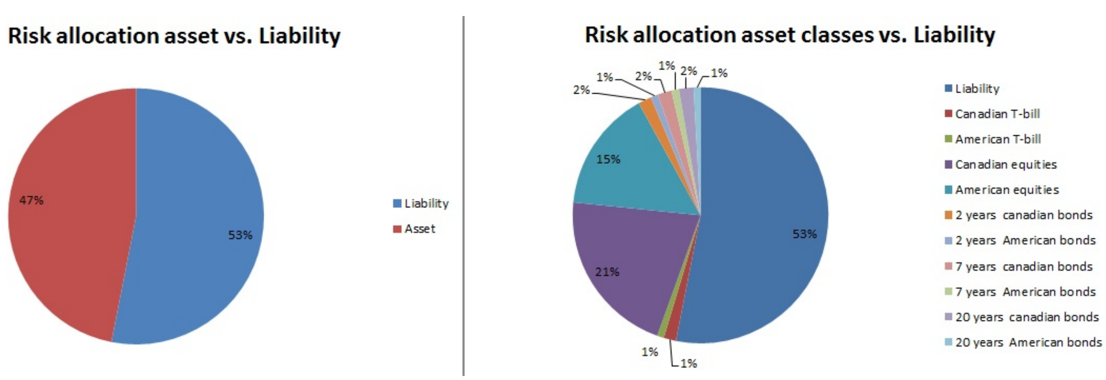


Figure 4.10: Risk allocation vs asset allocation for strategy S_1.

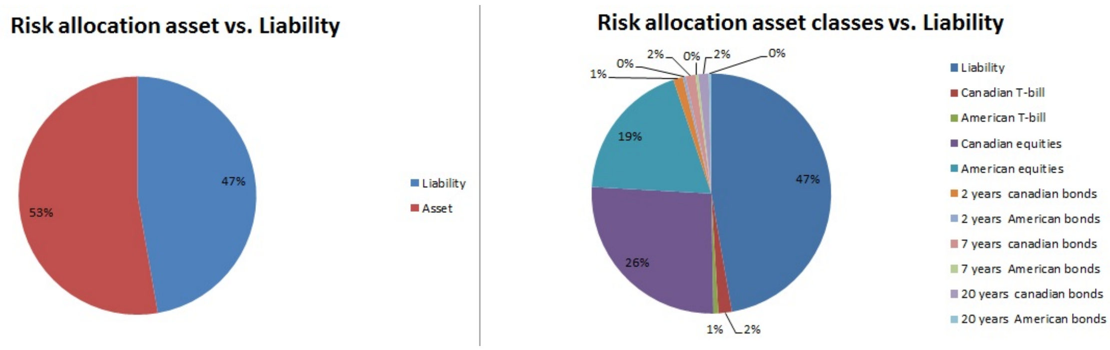


Figure 4.11: Risk allocation between asset and liability S_2.

4.4.5 The exchange rate risk

By holding foreign assets, the Canadian investor must take into account the exchange rate risk. In this subsection, we assume that the investor only holds American assets to analyze the risk allocation between investment and exchange rate risks. Let $X(T)$ denote the exchange rate, i.e. $X(T)$ is the value of the Canadian currency expressed in American dollar.

The loss rv defined in (4.38) can be written as follows

$$\begin{aligned} L(T) &= V(0) - V(T) \\ &= V(0) - U(T)V(0) + V(0)U(T) - V(T) \\ &= V(0)\left[1 - U(T)\right] + V(0)U(T)\left[1 - \frac{X(0)}{X(T)}\right], \end{aligned}$$

where $U(T)$ is the accumulated value of 1 U.S.\$ invested in the American market. We also define the following loss rvs

$$L_I(T) = V(0)\left[1 - U(T)\right], \tag{4.46}$$

and

$$L_X(T) = V(0)U(t)\left[1 - \frac{X(0)}{X(T)}\right]. \tag{4.47}$$

The loss rv L_I measures the loss from investing in the American market and the loss rv L_X quantifies the contribution of the exchange rate risk to the global loss.

Example 4.4.3. *For an investment horizon of ten years (i.e. T=10) and* $V(0) = 1000$*, we use 10000 simulations from our SIM to compute the risk measures and risk allocation based on the TVaR rule. Results, displayed in Table 4.15, show that the exchange rate contributes significantly to the global loss. According to the calibration obtained for our SIM, the Canadian dollar is expected to become stronger than the American dollar in 10 years which would lead to a reduction of the gain from investing in the American market.*

κ	$TVaR_\kappa(L_I(10), L(10))$	$TVaR_\kappa(L_X(10), L(10))$	$TVaR_\kappa(L(10))$	$VaR_\kappa(L(10))$
0.9	-398.73	55.79	-342.94	-442.54
0.95	-350.68	69.69	-280.99	-362.11
0.99	-273.99	87.47	-186.52	-226.05
0.995	-252.98	92.46	-160.52	-194.11

Table 4.15: Risk measures and risk allocation to the investment risk and the exchange rate risk

4.4.6 The inflation risk

This subsection focusses on the inflation risk in the Canadian economy measured at time T by the Canadian CPI $\Psi^d(T)$. We define the following loss rv at time T

$$L(T) = V(0)\frac{\Psi^d(T)}{\Psi^d(0)} - V(T). \tag{4.48}$$

Hence, the rv $L(T)$ determines the real loss or the loss adjusted to inflation. This loss rv can be decomposed as follows

$$L(T) = L_{inf}(T) + L_{nom}(T),$$

where $L_{inf}(T)$ is the loss component due to changes in the price level and $L_{nom}(T)$ is the nominal loss. These two rvs are given by

$$L_{inf}(T) = V(0)\frac{\Psi^d(T)}{\Psi^d(0)} - V(0),$$

and

$$L_{nom}(T) = V(0) - V(T).$$

Example 4.4.4. *We consider the same three strategies of Subsection 4.3 and we use 10000 scenarios simulated from our SIM over ten years. We compute $TVaR_\kappa(L(10))$, $TVaR_\kappa(L_{inf}(10), L(10))$ and $TVaR_\kappa(L_{nom}(10), L(10))$ for each strategy. Results are given in Table 4.16. From these results, one could conclude that investing in the Canadian market reduces the inflation risk. The American portfolio has less real losses than the Canadian portfolio and the international portfolio. Investing in the American market gives a better diversification between the investment risks and the inflation risk than investing solely in the Canadian market.*

4.5 Conclusion

In this paper, we have introduced an international stochastic investment model. The model is estimated using Canadian and American data and based on an extended Kalman filter. We apply this SIM to analyze the impact of asset selection on risk allocation. We also provide a

Risk measure	κ	0.9	0.95	0.99	0.995
$TVaR_{\kappa}(L(10))$	International portfolio	78.03	188.79	403.57	472.85
	Canadian portfolio	147.27	260.23	480.28	550.98
	American portfolio	51.03	154.76	358.86	427.37
$TVaR_{\kappa}(L_{inf}(10), L(10))$	International portfolio	480.45	549.11	699.58	756.40
	Canadian portfolio	466.03	544.97	694.81	755.23
	American portfolio	491.96	564.59	713.23	758.771
$TVaR_{\kappa}(L_{nom}(10), L(10))$	International portfolio	-402.42	-360.31	-296.01	-283.55
	Canadian portfolio	-318.76	-284.74	-214.52	-204.24
	American portfolio	-440.92	-409.83	-354.37	-331.40

Table 4.16: Risk measures and risk allocations for the inflation risk

capital allocation based on the TVaR capital rule. Using simulations from our SIM and the TVaR-based rule, we determine risk contribution of asset, inflation and the exchange rate risks.

4.6 Acknowledgements

Support for Helene Cossette, Patrice Gaillardetz and Etienne Marceau from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged. Support from the Chaire d'actuariat de l'Université Laval is also gratefully acknowledged by Hélène Cossette, Etienne Marceau, and Khouzeima Moutanabbir.

The authors would like to acknowledge the valuable comments made by Jean-Philippe Lemay. Special thanks also go to Ivor Krol a senior associate at Aon Hewitt for his advice and help.

4.7 References

1. Ahlgrim, K.C., D'Arcy, S.P., Gorvett, R.W., 2004. Modeling of Economic Series Coordinated with Interest Rate Scenarios. Society of Actuaries.
2. Ahlgrim, K.C., D'Arcy, S.P., Gorvett, R.W., 2008. A Comparison of Actuarial Fi nancial Scenario Generators. Society of Actuaries.
3. Ahn, D., 2004. Common Factors and Local Factors : Implications for Term Structures and Exchange Rates, Journal of Financial and Quantitative Analysis, 39, 69-101.
4. Ait-Sahalia, Y. and Kimmel, R., 2009. Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions. Fisher College of Business Working Paper.
5. Artzner, P., 1999. Application of Coherent Risk Measures to Capital Requirements in Insurance. North American Actuarial Journal 3 (2).

6. Artzner, P., Delbaen, F., Eber, J-M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9 (3), 203–228.
7. Bakshi, G. S. and Chen, Z., 1997. Equilibrium Valuation of Foreign Exchange Claims. Journal of Finance, 52, 799-826.
8. Battocchio, P., Menoncin,F., 2004. Optimal Pension Management in a Stochastic Framework. Insurnance : Mathematics and Economics, 34, 79-95.
9. Biger, N., and Hull, J., 1983. The Valuation of Currency Options. Financial Management,12,24-28.
10. Brandt, M.W. and Santa-Clara, P., 2002. Simulated Likelihood Estimation of Diffusions with an Application to Exchange Rate Dynamics in Incomplete Markets. Journal of Financial Economics, 63, 161-210.
11. Brennan, M. and Xia Y.,2000. Dynamic Asset Allocation Under Inflation. AFA 2002 Atlanta Meetings.
12. Brigo, D., and Mercurio, F., 2006. Interest rate models-theory and practice (2nd edition). Springer-Verlag, Berlin.
13. Buffington, J., Elliott, R.J., 2002. American Options with Regime Switching. International Journal of Theoretical and Applied Finance.5(5).
14. Burmeister, E., Wall, K.D., 1982. Kalman filtering estimation of unobserved rational expectations with an application to the German hyperinflation. Journal of Econometrics, 20, 255-284.
15. Burmeister, E., Wall, K.D., Hamilton, J.D., 1986. Estimation of Unobserved Expected Monthly Inflation Using Kalman Filtering. Journal of Business and Economic Statistics, American Statistical Association. 4(2), 147-60.
16. Cox, J.C., Ingersoll, J. and Ross S., 1985 A Theory of the Term Structure of Interest Rates. Econometrica 53.
17. Dai, Q. and Singleton, K., 2000. Specification Analysis of Affine Term Structure Models. New York University, Leonard N. Stern School Finance Department Working Paper Seires.
18. De Jong, F., 2000. Time Series and Cross-Section Information in Affine Term-Structure Models. Journal of Business and Economic Statistics, 18, issue 3, p. 300-314.
19. Dewachter, H. and Maes, K., 2001. An Affine Model for International Bond Markets. EFA 2001 Barcelona Meetings.
20. Duffie, D. and Kan, R., 1996. A Yield-factor Model of Interest Rates. Mathematical Finance, 6, 379-406.
21. Egorov, A., Li, H. and Ng, D.,2008. A Tale of Two Yield Curves : Modeling the Joint Term Structure of Dollar and Euro Interest Rates. Journal of Econometrics 2008.

22. Engle, R. and Watson, M. W., 1981. A One Factor Multivariate Time Series Model of Metropolitan Wage Rates. Journal of the American Statistical Association, 76 (376), 774-781.

23. Engel, C., Hamilton, J. Long Swings in the Dollar : Are They in the Data and Do Markets Know It ?. American Economic Review.

24. Forsgren, A., Gill, P.E., Wright, M.H., 2002. Interior Methods for Nonlinear Optimization. Society for Industrial and Applied Mathematics Review. 44 (4), 525-597.

25. Garman, M., and Kohlhagen, S., 1983. Foreign Currency Options Values. Journal of International Money and Finance., 2,231-237.

26. Guo, X., 2001. An Explicit Solution to an Optimal Stopping Problem with Regime Switching. Journal of Applied Probability. 38(2) :464-481, 2001.

27. Guidolin, M., Timmermann, A., 2005. Strategic asset allocation and consumption decisions under multivariate regime switching. Federal Reserve Bank of St. Louis Working Paper No. 2005-002A.

28. Guidolin, M., and Timmermann, A., 2007. Asset Allocation under Multivariate Regime Switching. Journal of Economic Dynamics and Control, 31 , 3503-3544.

29. Hamilton, J.D., 1994. Time Series Analysis. Princeton Univ. Press.

30. Hardy M.R. 2001. A regime switching model of long term stock returns . North American Actuarial Journal 5.2 41-53.

31. Hibbert, E-J., Mowbray, M-J.,Turnbull, J., 2001. A Stochastic Asset Model and Calibration for the Long-Term Financial Planning Purposes. Technical Report, Barrie and Hibbert Limited.

32. James, J., and Webber, N., 2000. Interest rate modelling. Wiley.

33. Kim, C-J., 1994. Dynamic linear models with Markov-switching. Journal of Econometrics, 60(1-2), 1-22.

34. Kim, C-J. and Nelson,C-R., 1998. State-space models with regime switching : classical and Gibbs-sampling approaches with applications. Cambridge, MA.

35. Lemke 2006 Lemke, W., 2006. Term Structure Modeling and Estimation in a State Space Framework. Lecture Notes in Economics and Mathematical Systems, Vol. 565, Springer.

36. McNeil, A.J., Frey, R., Embrechts, P., 2005. Quantitative Risk Management. Princeton University Press, Princeton.

37. MGWP, 1980. Report of the Maturity Guarantees Working Party. Journal of the Institute of Actuaries, 107.

38. Mosburger, G. and Schneider, P., 2005, Modelling International Bond Markets with Affine Term Structure Models. Working paper.

39. Munk, C., Sorensen, C., Nygaard Vinther, T., 2004. Dynamic Asset Allocation under Mean-Reverting Returns, Stochastic Interest Rates, and Inflation Uncertainty : Are Popular Recommendations Consistent with Rational Behavior ?. International Review of Economics and Finance, 13, 2.

40. Otranto, E. and Gallo, G., 2002. A Nonparametric Bayesian Approach To Detect The Number Of Regimes In Markov Switching Models. Econometric Reviews, Taylor and Francis Journals. 21(4), 477-496.

41. Piazzesi, M., 2003. Affine Term Structure Models. The Handbook of Financial Econometrics, Elsevier.

42. Sahin, S., Cairns, A., Kleinow, T., Wilkie, A.D., 2008. Revisiting the Wilkie Investment Model. 18th International AFIR Colloquium.

43. Vasicek, O.A., 1977. An Equilibrium Characterization of the Term Structure. Journal of Financial Economics, 5.

44. Wang, S.S., 2002. A Set of New Methods and Tools for Enterprise Risk Capital Management and Portfolio Optimization. Working paper.

45. Whitten, S.P. and Thomas, R.G., 1999. A Non-Linear Stochastic Model for Actuarial Use. British Actuarial Journal, 5.

46. Wilkie, A.D., 1986, A Stochastic Investment Model for Actuarial Use. Transactions of the Faculty of Actuaries, 39.

47. Wilkie, A.D., 1995, More on a Stochastic Asset Model for Actuarial Use. British Actuarial Journal,1.

48. Yakoubov, Y., Teeger, M. and Duval, D., 1999. A Stochastic Investment Model for Asset and Liability Management. Proceedings of the 30th International ASTIN Colloquium and 9th International AFIR Colloquium, Volume J.

4.8 Appendix

In this appendix, we show the QML derivation from the regime switching Kalman-Filter adapted to the estimation of the model in Section 3.

Specification of the model We consider the following state-space representation

$$Y_t = H_{y_t}\beta_t + A_{y(t)}z_t + e_t, \tag{4.49}$$

$$\beta_t = \mu_{y(t)} + F_{y(t)}\beta_{t-1} + G_{y(t)}\nu_t, \tag{4.50}$$

$$\begin{pmatrix} e_t \\ \nu_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} R_{y(t)} & 0 \\ 0 & Q_{y(t)} \end{pmatrix}\right), \tag{4.51}$$

where (4.49) is the measurement equation which determines the observed variables Y_t as a function of the state variables β_t and (4.50) is the transition equation that describes the evolution of the state vector β_t. Some parameters in these model are dependent on an two-state Markov-switching variable $y(t) = 1, 2$ with transition probabilities given by the following matrix

$$Pr = \begin{pmatrix} P_{11} & 1 - P_{22} \\ 1 - P_{11} & P_{22} \end{pmatrix},$$

The following equations show how to calculate the log-likelihood function $l(\underline{Y})$ defined by

$$l(\underline{Y}) = \sum_{t=1}^{T} \ln(f(Y_t/\phi_{t-1})),$$

where ϕ_t denotes the available information up to time t and T is the length of the sample.

Conditional on $y(t-1) = i$ and $y(t) = j$, we have

$$f(Y_t/\phi_{t-1}) = \sum_{j=1}^{2}\sum_{i=1}^{2} f(Y_t/y_t = j, y_{t-1} = i, \phi_{t-1})Pr(y_t = j, y_{t-1} = i/\phi_{t-1}),$$

In our framework presented is section 2, the transition equation of the state-vector consisting of nine state variables $\beta_t = \left(F_1(t), F_2(t), F_3(t), \frac{dS^d(t)}{S^d(t)}, \frac{dS^f(t)}{S^f(t)}, \frac{d\Psi^d(t)}{\Psi^d(t)}, \frac{d\Psi^f(t)}{\Psi^f(t)}, \Pi^d(t), \Pi^d(t)\right)$

Using a combination of the Kalman filter and Hamilton filter with appropriate approximation, we compute $f(Y_t/\phi_{t-1})$ for $t = 1, 2, ..., T$.

The Kalman filter Conditional on $y(t-1) = i$ and $y(t) = j$, the Kalman filter algorithm is as follows :

$$\beta_{t|t-1}^{(i,j)} = \mu_j + F_j\beta_{t-1|t-1}^{i},$$

$$P_{t|t-1}^{(i,j)} = F_j P_{t-1|t-1}^{i} F_j^{'} + G_j Q_j G_j^{'},$$

$$\eta_{t|t-1}^{(i,j)} = Y_t - H_j\beta_{t|t-1}^{(i,j)} - A_j z_t,$$

$$f_{t|t-1}^{(i,j)} = H_j P_{t|t-1}^{(i,j)} H_j^{'} + R_j,$$

$$\beta_{t|t}^{(i,j)} = \beta_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} H_j^{'} \left(f_{t|t-1}^{(i,j)}\right)^{-1} \eta_{t|t-1}^{(i,j)},$$

$$P_{t|t}^{(i,j)} = \left(I - P_{t|t-1}^{(i,j)} H_j^{'} \left(f_{t|t-1}^{(i,j)}\right)^{-1} H_j\right) P_{t|t-1}^{(i,j)},$$

The Hamilton filter :

According to Hamilton filter, the filtered probabilities $Pr(y_t/\phi_t)$ is given by

$$Pr(y_t, y_{t-1}/\phi_{t-1}) = Pr(y_t/y_{t-1})Pr(y_{t-1}/\phi_{t-1}),$$

$$f((Y_t/\phi_{t-1}) = \sum_{y_t}\sum_{y_{t-1}} f(Y_t/y_t, y_{t-1}, \phi_{t-1})Pr(y_t, y_{t-1}/\phi_t),$$

$$Pr(y_t, y_{t-1}/\phi_t) = \frac{f(Y_t/y_t, y_{t-1}, \phi_{t-1})Pr(y_t, y_{t-1}/\phi_t)}{f(Y_t/\phi_{t-1})},$$

$$Pr(y_t/\phi_t) = \sum_{y_{t-1}} Pr(y_t, y_{t-1}/\phi_t),$$

Kim's approximation :

$$\beta^j_{t|t} = \frac{\sum_{i=1}^2 Pr(y_t = j, y_{t-1} = i/\phi_t)\beta^{(i,j)}_{t|t}}{Pr(y_t = j/\phi_t)},$$

$$P^j_{t|t} = \frac{\sum_{i=1}^2 Pr(y_t = j, y_{t-1} = i/\phi_t)(P^{(i,j)}_{t|t} + (\beta^j_{t|t} - \beta^{(i,j)}_{t|t})(\beta^j_{t|t} - \beta^{(i,j)}_{t|t})')}{Pr(y_t = j/\phi_t)}$$

We recall that the conditional density $f(Y_t/y_t = j, y_{t-1} = i, \phi_{t-1})$ is obtained under the gaussian assumption as follows :

$$f(Y_t/y_t = j, y_{t-1} = i, \phi_{t-1}) = (2\pi)^{-\frac{N}{2}}|f^{(i,j)}_{t|t-1}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\eta^{(i,j)}_{t|t-1}{}'(f^{(i,j)}_{t|t-1})^{-1}\eta^{(i,j)}_{t|t-1}\right)$$

for $i, j = 1, 2$.

Conclusion

Dans cette thèse, on a présenté différents aspects de l'évaluation et l'allocation du risque en utilisant des modèles avancés en actuariat.

Dans un premier temps, on a étudié le problème d'agrégation des risques et d'allocation du capital pour des risques dépendants. On a considéré des risques qui suivent une distribution mélange d'Erlang définie avec la copule FGM. En se basant sur certaines propriétés des distributions mélange d'Erlang et de la famille de copules FGM, on a prouvé que le montant agrégé a une distribution mélange d'Erlang dont on a déterminé les caractéristiques. On a aussi trouvé des expressions exactes de la covariance entre les risques, la mesure de risque TVaR, la prime stop-loss, des allocations de capital selon la règle TVaR et la règle Covariance.

Dans le cadre de la théorie de la ruine, on a étudié la distribution de la somme escomptée des *ladder heights* sur un horizon de temps fini ou infini. En particulier, on a trouvé une expression ferme de ces moments dans le cas du modèle classique Poisson-composé et le modèle de Sparre Andersen avec sinistres de loi exponentielle. Pour cette fin, on a appliqué une forme spécifique de la fonction de Gerber-Shiu et en se basant sur certaines caractéristiques de la loi mélange d'Erlang. L'élaboration d'une expression exacte de ces moments nous a permit d'approximer la distribution de la somme escomptée des *ladder heights* par une distribution mélange d'Erlang. Pour établir cette approximation, on a utilisé une méthode basée sur les moments. À l'aide de cette approximation, on a calculé les mesures de risque VaR et TVaR associées à la somme escomptée des *ladder heights*.

Enfin, on a présenté un modèle d'investissement qui est constitué de quatre modules dans le cas de deux économies : l'économie canadienne et l'économie américaine. Le premier module décrit le comportement du taux d'intérêt et propose une modélisation conjointe des structures par terme dans les deux économies. Ce module sert à évaluer tous les produits à revenus fixes tels que les obligations et les fonds obligataires dans les deux pays. Le second module décrit le comportement conjoint des indices boursiers canadien et américain : le S&P TSX (indice canadien) et le S&P 500 (américain). Le troisième module propose une modélisation du comportement de l'inflation dans les deux pays. Ce modèle d'investissement décrit aussi la dynamique du taux de change canadien/américain. À l'aide de ce modèle, on a étudié le problème d'allocation et la quantification des risques d'investissement, d'inflation et de taux de change. Cette étude a été élaborée en utilisant des simulations générées par notre modèle et à l'aide de la règle d'allocation TVaR.

En élaborant ces trois projets, on a essayé de répondre à certaines questions liées aux problèmes d'évaluation et d'allocation du risque. Il serait pertinent de regarder des généralisations des résultats obtenus et aussi des généralisation des applications de ces résultats. À ce propos, plusieurs travaux peuvent être envisagés à base de cette thèse.

Pour généraliser les résultats du deuxième chapitre deux pistes sont possibles. D'abord, on peut travailler avec une famille de copule autre que la famille FGM. Actuellement, on est entrain de développer des travaux similaires en utilisant des familles de copules qui ont la même tractabilité que la famille FGM. De plus, on peut généraliser le modèle multivarié considéré dans ce chapitre en utilisant d'autres lois marginales. Dans ce cas, on peut élaborer des approximations de toutes les lois mutlivariés définie à l'aide de la copule FGM en utilisant l'approximation mélange d'Erlang basées sur les moments qu'on a décrit dans le troisième chapitre. L'étude de la distribution de la somme escomptée des *ladder heights* qu'on a développée dans le troisième chapitre peut être généralisée en tenant compte de l'impact de la politique de distribution des dividendes. On peut aussi s'intéresser au même problème dans le cas d'un processus du risque bivarié. Le modèle d'investissement élaboré dans le quatrième chapitre peut être la base de plusieurs travaux dans le future. À ce sujet, on envisage l'ajout des données de plusieurs économies mais il serait aussi intéressant d'appliquer ce modèle dans d'autre contexte. À titre d'exemple, on peut étudier des problèmes de gestion actif-passif ainsi que des problème liés à l'évaluation des equity-indexed-annuities.

Ces idées développées à travers cette thèse sont l'objet des travaux encours et d'autres qu'on envisage étudier dans le future.

Bibliographie

1. Acerbi, C., Nordio, C. et Sirtori, C. (2001). Expected Shortfall as a Tool for Financial Risk Management. *Working Paper.*
2. Acerbi, C. et Tasche, D., 2002. On the Coherence of Expected Shortfall. *Journal of Banking and Finance.* 26 (7) :1487-1503.
3. Ahlgrim, K.C., D'Arcy, S.P. et Gorvett, R.W. (2004). Modeling of Economic Series Coordinated with Interest Rate Scenarios. *Casualty Actuarial Society.*
4. Ahlgrim, K.C., D'Arcy, S.P. et Gorvett, R.W. (2008). A Comparison of Actuarial Financial Scenario Generators. *Variance* 2(1) :111-134.
5. Ahn, D. (2004). Common Factors and Local Factors : Implications for Term Structures and Exchange Rates, *Journal of Financial and Quantitative Analysis.* 39 : 69-101.
6. Ait-Sahalia, Y. et Kimmel, R. (2010). Estimating Affine Multifactor Term Structure Models Using Closed-Form Likelihood Expansions. *Journal of Financial Economics.* 98 : 113–144.
7. Albrecht, P. (2004). Risk Based Capital Allocation. *Sonderforschungsbereich 504 Publications 03-02, University of Mannheim.*
8. Albrecher H. et Hipp C. (2007). Lundberg's Risk Process with Tax. Blatter der DGVFM. 28(1) : 13-28.
9. Albrecher, H. et Kainhofer, R. (2002). Risk Theory with a Nonlinear Dividend Barrier. *Computing.* 68(4) : 289-311.
10. Albrecher, H. et Teugels, J. L. (2006). Exponential Behavior in the Presence of Dependence in Risk Theory. *Journal of Applied Probababilty.* 43(1) : 257-273.
11. Andersen, S. E. (1957). On the Collective Theory of Risk in Case of Contagion between Claims. *Bull. Inst. Math. Appl.* 12 :275 :279.
12. Ang, A., et Bekaert, G. (2002). Regime Switches in Interest Rates. *Journal of Business and Economic Statistics.* 20(2) : 163-82.
13. Armstrong, M. et Galli, A. (2002). Sequential Nongaussian Simulations Using the FGM Copula. *Working Paper.*

14. Artzner, P. (1999). Application of Coherent Risk Measures to Capital Requirements in Insurance. *North American Actuarial Journal.* 3(2) : 83-109.

15. Artzner, P., Delbaen, F., Eber, J-M. et Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance.* 9(3) : 203-228.

16. Asmussen, S. et H. Albrecher. (2010). Ruin Probabilities (Second Edition). *Advanced Series on Statistical Science and Applied Probability. New Jersey : World Scientific Publishing Co.*

17. Bakshi, G. S. et Chen, Z. (1997). Equilibrium Valuation of Foreign Exchange Claims. *Journal of Finance.* 52(2) : 799-826.

18. Bargès, M., Cossette, H. et Marceau, E. (2009). TVaR-based Capital Allocation with Copulas. *Insurance : Mathematics and Economics.* 45(3) : 348-361.

19. Battocchio, P. et Menoncin,F. (2004). Optimal Pension Management in a Stochastic Framework. *Insurance : Mathematics and Economics.* 34, 79-95.

20. Biger, N., et Hull, J. (1983). The Valuation of Currency Options. *Financial Management*,12,24-28.

21. Borovkov, K.A. et Dickson, D.C.M. (2008). On the Ruin Time Distribution for a Sparre Andersen Risk Process with Exponential Claim Sizes. *Insurance : Mathematics and Economics* 42 : 1104-1108.

22. Brandt, M.W. et Santa-Clara, P. (2002). Simulated Likelihood Estimation of Diffusions with an Application to Exchange Rate Dynamics in Incomplete Markets. *Journal of Financial Economics*, 63, 161-210.

23. Brennan, M. et Xia Y. (2000). Dynamic Asset Allocation Under Inflation. *AFA 2002 Atlanta Meetings.*

24. Brigo, D. et Mercurio, F. (2006). Interest rate models-theory and practice (2nd edition). *Springer-Verlag, Berlin.*

25. Buch, A. et Dorfleitner, G. (2008). Coherent Risk Measures, Coherent Capital Allocations and the Gradient Allocation Principle. *Insurance : Mathematics and Economics.* 42 (1), 235-242.

26. Buffington, J. et Elliott, R.J. (2002). American Options with Regime Switching. *International Journal of Theoretical and Applied Finance.* 5(5).

27. Burmeister, E. et Wall, K.D. (1982). Kalman filtering estimation of unobserved rational expectations with an application to the German hyperinflation. *Journal of Econometrics*, 20, 255-284.

28. Burmeister, E., Wall, K.D. et Hamilton, J.D. (1986). Estimation of Unobserved Expected Monthly Inflation Using Kalman Filtering. *Journal of Business and Economic Statistics.* 4(2), 147-60.

29. Chadjiconstantinidis, S. et Vrontos, S. (2012). On a Renewal Risk Process with Dependence under a Farlie-Gumbel-Morgenstern Copula. *Scandinavian Actuarial Journal.* 1-34.

30. Cherubini, U., Mulinacci, S. et Romagnoli, S. (2011). A Copula-based Model of Speculative Price Dynamics in Discrete Time. *Journal of Multivariate Analysis.* 102 (6), 1047-1063.

31. Cossette, H., Landriault, D., Marceau, E. et Moutanabbir, K. (2012). Analysis of the Discounted Sum of Ascending Ladder Heights. *Insurance : Mathematics and Economics.* To appear.

32. Cossette, H., Landriault, D., Marceau, E. et Moutanabbir, K. (2012a). Risk measures and maximal aggregate loss. *To be submitted.*

33. Cossette, H., Landriault, D., Marceau, E. and K. Moutanabbir. 2012b. Moment-based approximations with finite mixtures of Erlang of common scale. *To be submitted.*

34. Cossette, H., Mailhot, M. et Marceau, E. (2012). TVaR-Based Capital Allocation for Multivariate Compound Distributions with Positive Continuous Claim Amounts. *Insurance : Mathematics and Economics.* 50 (2), 247–256.

35. Cossette, H., Marceau, E. et Marri, F. (2008). On the Compound Poisson Risk Model with Dependence based on a Generalized Farlie Gumbel Morgenstern Copula. *Insurance : Mathematics and Economics.* 43, 444-455.

36. Cox, J.C., Ingersoll, J. et Ross S. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53.

37. Cramer, H. (1930). On the Mathematical Theory of Risk. *Skandia Jubilee Volume, Stockholm.*

38. Dai, Q. et Singleton, K. (2000). Specification Analysis of Affine Term Structure Models. *New York University, Leonard N. Stern School Finance Department Working Paper Series.*

39. Daykin, C.D., Pentikäinen, T. et Pesonen, M. (1994). Practical Risk Theory for Actuaries. *Chapman and Hall, London*

40. De Jong, F. (2000). Time Series and Cross-Section Information in Affine Term-Structure Models. *Journal of Business and Economic Statistics*, 18, 3, p. 300-314.

41. De Grauwe, P., Dewachter, H. et Embrechts, M. (1993). Exchange Rate Theories. Chaotic Models of the Foreign Exchange Markets. *Oxford, Black-well.*

42. Denault, M. (2001). Coherent Allocation of Risk Capital . *The journal of Risk.* 4 (1), 24-40.

43. Denuit, M., Dhaene, J., Goovaerts, M.J. et Kaas, R. (2005). Actuarial Theory for Dependent Risks : Measures, Orders and Models. *Wiley, New York.*

44. DeVylder, F. E. et Goovaerts, M. J. (1999). Explicit finite-time and infinite-time ruin probabilities in the continuous case. *Insurance Mathematics and Economics*. 24(3) :155–172.

45. Dewachter, H. et Maes, K. (2001). An Affine Model for International Bond Markets. *EFA 2001 Barcelona Meetings*.

46. Dhaene, J., Henrard, L., Landsman, Z., Vandendorpe, A. et Vanduffel, S. (2008). Some results on the CTE-based capital allocation rule. *Insurance : Mathematics and Economics*. 42 (2), 855-863.

47. Dhaene, J., Tsanakas, A., Valdez, E. et Steven, V. (2009). Optimal capital allocation principles. *MPRA Paper 13574, University Library of Munich, Germany.*

48. Dickson, D. C. M. et Waters, H. R. (1996). Reinsurance and ruin. *Insurance : Mathematics and Economics*. 19(1) :61 – 80.

49. Dickson, D.C.M. et Willmot, G.E. (2005). The density of the time to ruin in the classical Poisson risk model. *Astin Bulletin* 35(1) : 45-60.

50. Duffie, D. et Kan, R. (1996). A Yield-factor Model of Interest Rates.*Mathematical Finance*. 6, 379-406.

51. Egorov, A., Li, H. et Ng, D. (2008). A Tale of Two Yield Curves : Modeling the Joint Term Structure of Dollar and Euro Interest Rates. *Journal of Econometrics*.

52. Engle, R. et Watson, M.W. (1981). A One Factor Multivariate Time Series Model of Metropolitan Wage Rates. *Journal of the American Statistical Association*. 76 (376), 774-781.

53. Engel, C. et Hamilton, J. (1990). Long Swings in the Dollar : Are They in the Data and Do Markets Know It ?. *American Economic Review*.

54. Feldmann, A. et Whitt, W. (1998). Fitting mixtures of exponentials to long-tail distributions to analyze network performance models. *Performance Evaluation*. 31(8) :963-976.

55. Feller, W. (1968). An Introduction to Probability Theory and Its Applications. *Volume 1. 3rd ed. New York : Wiley.*

56. Forsgren, A., Gill, P.E. et Wright, M.H. (2002). Interior Methods for Nonlinear Optimization. *Society for Industrial and Applied Mathematics Review*. 44 (4), 525-597.

57. Gaier, J., Grandits, P. et Schachermayer, W. (2003). Asymptotic ruin probabilities and optimal investment. *Annal of Applied Probability*. 13(3) :1054–1076.

58. Garman, M. et Kohlhagen, S. (1983). Foreign Currency Options Values. *Journal of International Money and Finance*. 2,231-237.

59. Gatfaoui, H. (2005). How does systematic risk impact U.S. credit spreads ? A copula study. *Bankers Markets and Investors*. 77 : 5-12.

60. Gatfaoui, H. (2007). Credit default swap spreads and US financial market : Investigating some dependence. *20th Australasian Finance and Banking Conference*.

61. Gerber, H.U. (1979). An Introduction to Mathematical Risk Theory. *Philadelphia : S.S. Heubner Foundation Monograph Series 8.*

62. Gerber, H.U. (1981). On the probability of ruin in the presence of a linear dividend barrier. *Scandinavian Actuarial Journal.* 2 : 105–115.

63. Gerber, H.U. et Shiu, E. (1998). On the time value of ruin. *North American Actuarial Journal.* 2(1) : 48-72.

64. Goovaerts, M., Van den Borre, E. et Laeven, R. (2005). Managing economic and virtual economic capital within financial conglomerates. *North American Actuarial Journal.* 9(3) :77–89.

65. Gray, S.F. (1996). Modeling the Conditional Distribution of Interest Rates as a Regime-Switching Process. *Journal of Financial Economics.* 42, 27-62.

66. Guidolin, M. et Timmermann, A. (2005). Strategic asset allocation and consumption decisions under multivariate regime switching. *Federal Reserve Bank of St. Louis Working Paper No. 2005-002A.*

67. Guidolin, M. et Timmermann, A. (2007). Asset Allocation under Multivariate Regime Switching. *Journal of Economic Dynamics and Control.* 31 , 3503-3544.

68. Guo, X. (2001). An Explicit Solution to an Optimal Stopping Problem with Regime Switching. *Journal of Applied Probability.* 38(2) :464-481.

69. Hamilton, J.D. (1989). A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica.* 57, 357-384.

70. Hamilton, J.D. (1994). Time Series Analysis. *Princeton Univ. Press.*

71. Hardy M.R. 2001. A regime switching model of long term stock returns. *North American Actuarial Journal.* 5-2 : 41-53.

72. Hesselager, O. et Andersson, U. (2002). Risk Sharing and Capital Allocation. *Tryg Insurance DP.*

73. Hibbert, E-J., Mowbray, M-J. et Turnbull, J. (2001). A Stochastic Asset Model and Calibration for the Long-Term Financial Planning Purposes. *Technical Report, Barrie and Hibbert Limited.*

74. Hipp, C. et Plum, M. (2000). Optimal investment for insurers. *Insurance Mathematics and Economics.* 27(2) :215–228.

75. James, J. et Webber, N. (2000). Interest rate modelling. *Wiley.*

76. Jang, J. et Fu, G. (2011). Measuring tail dependence for aggregate collateral losses using bivariate compound Cox process with shot noise intensity. *Working paper.*

77. Jasiulewicz, H. (2001). Probability of ruin with variable premium rate in a Markovian environment. *Insurance Mathematics and Economics.* 29(2) : 291–296.

78. Joe, H. (1997). Multivariate models and dependence concepts. *Volume 73 de Monographs on Statistics and Applied Probability. Chapman & Hall, London.*

79. Johnson, M.A. et Taffe, M.R. (1989). Matching moments to phase distributions : Mixtures of Erlang distribution of common order. *Stochastic Models*. 5 : 711-743.

80. Kaas, R., Goovaerts, M., Dhaene, J. et Denuit, M. (2001). Modern Actuarial Risk Theory. *Kluwer Academic Publishers, Dordrecht.*

81. Kim, C-J. (1994). Dynamic Linear Models with Markov-Switching. *Journal of Econometrics*. 60(1-2), 1-22.

82. Kim, J et Hardy, M. (2009). A Capital Allocation based on a Solvency Exchange Option. *Insurance : Mathematics and Economics* 44(3) :357–366.

83. Kim, C-J. and Nelson,C-R., 1998. State-Space Models with Regime Switching : Classical and Gibbs-Sampling Approaches with Applications. *Cambridge, MA.*

84. Kulenko, N. et Schmidli, H. (2008). Optimal Dividend Strategies in a Cramer–Lundberg Model with Capital Injections. *Insurance : Mathematics and Economics* . 43(2) : 270-278.

85. Landriault, D., Shi, T. et Willmot, G.E. (2011). Joint Density Involving the Time to Ruin in the Sparre Andersen Risk Model with Exponential Claims. *Insurance : Mathematics and Economics* . 49(3) : 371-379.

86. Landriault, D. et Willmot, G.E. (2009). On the Joint Distributions of the Time to Ruin, the Surplus Prior to Ruin and the Deficit at Ruin in the Classical Risk Model. *North American Actuarial Journal*. 13(2) : 252-270.

87. Lee, S., Lin, X.S. (2010). Modeling and Evaluating Insurance Losses via Mixtures of Erlang Distributions. *North American Actuarial Journal*. 14, 107-130.

88. Lemke, W. (2006). Term Structure Modeling and Estimation in a State Space Framework. *Lecture Notes in Economics and Mathematical Systems, Vol. 565, Springer.*

89. Léveillé, G., et Garrido, J. (2001a). Moments of Compound Renewal Sums with Discounted Claims. *Insurance : Mathematics and Economics*. 28(2) : 217–31.

90. Léveillé, G., and Garrido, J. (2001b). Recursive Moments of Compound Renewal Sums with Discounted Claims. *Scandinavian Actuarial Journal*. 2 : 98-110.

91. Lin, X.S. et Willmot, G.E. (1999). Analysis of a Defective Renewal Equation Arising in Ruin Theory. *Insurance : Mathematics and Economics*. 25(1) : 63-84.

92. Lin, X.S. et Willmot, G.E. (2000). The Moments of the Time of Ruin, the Surplus Before Ruin, and the Deficit at Ruin. *Insurance : Mathematics and Economics*. 27(1) : 19-44.

93. Lindgren, G. (1978). Markov Regime Models for Mixed Distributions and Switching Regressions. *Scandinavian Journal of Statistics*. 5, 81-91.

94. Lundberg, F. (1926). Försäkringsteknisk Riskutjämning. *F. Englunds Boktryckeri AB, Stockholm.*

95. Marceau, E. (2012). Modélisation et Évaluation des Risques en Actuariat. *Statistique et probabilités appliquées, Springer.*

96. McNeil, A.J., Frey, R. etEmbrechts, P. (2005). Quantitative Risk Management. *Princeton University Press, Princeton.*

97. MGWP, (1980). Report of the Maturity Guarantees Working Party. *Journal of the Institute of Actuaries.* 107.

98. Michaud, F. (1996). Estimating the Probability of Ruin for Variable Premiums by Simulation. *Astin Bulletin.* 26(1) :93–105.

99. Mosburger, G. et Schneider, P. (2005). Modelling International Bond Markets with Affine Term Structure Models. *Working paper.*

100. Munk, C., Sorensen, C. et Nygaard Vinther, T. (2004). Dynamic Asset Allocation under Mean-Reverting Returns, Stochastic Interest Rates and Inflation Uncertainty : Are Popular Recommendations Consistent with Rational Behavior ?. *International Review of Economics and Finance.* 13, 2.

101. Myers, S. et Read Jr, J. (2001). Capital allocation for insurance companies. *The Journal of Risk and Insurance.* 68(4) : 545–580.

102. Nelsen, R.B. (2006). An Introduction to Copulas, second edition. *Springer Series in Statistics. Springer, New York.*

103. Otranto, E. and Gallo, G., 2002. A Nonparametric Bayesian Approach To Detect The Number Of Regimes In Markov Switching Models. Econometric Reviews, Taylor and Francis Journals. 21(4), 477-496.

104. Panjer, H.H., (2002). Measurement of Risk, Solvency Requirements and Allocation of Capital within Financial Conglomerates. *Research Report 01 15, Institute of Insurance and Pension Research, University of Waterloo.*

105. Piazzesi, M., (2003). Affine Term Structure Models. *The Handbook of Financial Econometrics, Elsevier.*

106. Prieger J., (2002). A Flexible Parametric Selection Model for Non-Normal Data with an Application to Health Care Usage. *Journal of Applied Econometrics.* 17, 367-392.

107. Ren, J. (2007). The Discounted Joint Distribution of the Surplus Prior to Ruin and the Deficit at Ruin in a Sparre Andersen Model. *North American Actuarial Journal.* 11(3) : 128–136.

108. Ren, J. (2008). On the Laplace Transform of the Aggregate discounted Claims with Markovian Arrivals. *North American Actuarial Journal.* 12(2) : 198-206.

109. Rolski, T., Schmidli, H., Schmidt, V. et Teugels, J. (1999). Stochastic Processes for Insurance and Finance. *Wiley Series in Probability and Statistics. Chichester : John Wiley & Sons.*

110. Sahin, S., Cairns, A., Kleinow, T. et Wilkie, A.D. (2008). Revisiting the Wilkie Investment Model. *18th International AFIR Colloquium.*

111. Schmidli, H. (2002). On Minimizing the Ruin Probability by Investment and Reinsurance. *Annal of Applied Probability.* 12(3) :890–907.

112. Schaller, H. et Van Norden, S. (1997). Regime Switching in Stock Market Returns. *Applied Financial Economics, Taylor and Francis Journals.* 7(2) : 177-191.

113. Sherris, M. et Australia, S. (2006). Solvency, Capital Allocation and Fair Rate of Return in Insurance. *Journal of Risk and Insurance.* 73(1) :71–96.

114. Sklar, A. (1959). Fonctions de Répartition à n Dimensions et leurs Marges. *Publications de l'Institut de Statistique de l'Université de Paris.* 8, 229-231.

115. Tasche, D. (2000). Risk Contributions and Performance Measurement. *Research paper, Zentrum Mathematik (SCA).*

116. Taylor, G. C. (1980). Probability of ruin with variable premium rate. *Scandinavian Actuarial Journal.*(2) :57–76.

117. Tijms, H.C. (1994). Stochastic Models : An algorithmic approach. *Wiley.*

118. Trufin J., Albrecher, H. et Denuit, M. (2010). Properties of a Risk Measure Derived from Ruin Theory. *The Geneva Risk and Insurance Review.*

119. Tsai, C.C. et Willmot, G.E (2002). On the Moments of the Surplus Process Perturbed by Diffusion. *Insurance : Mathematics and Economics.* 31(3) : 327-350.

120. Valdez, E. A. et Chernih, A. (2003). Wang's Capital Allocation Formula for Elliptically Contoured Distributions. *Insurance : Mathematics and Economics.* 33(3) :517–532.

121. Vasicek, O.A., 1977. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics.* 5.

122. Wang, S. (1998). Aggregation of Correlated Risk Portfolios : Models and Algorithms. *Working paper.*

123. Wang, S., (2002). A Set of New Methods and Tools for Enterprise Risk Capital Management and Portfolio Optimization. *Working paper.*

124. Whitten, S.P. et Thomas, R.G. (1999). A Non-Linear Stochastic Model for Actuarial Use. *British Actuarial Journal.* 5.

125. Wilkie, A.D. (1986). A Stochastic Investment Model for Actuarial Use. *Transactions of the Faculty of Actuaries.* 39.

126. Wilkie, A.D., (1995). More on a Stochastic Asset Model for Actuarial Use. *British Actuarial Journal.*1.

127. Willmot, G.E. et Lin, X.S. (2001). Lundberg Approximations for Compound Distributions with Insurance Applications. *New York : Springer.*

128. Willmot, G.E. et Lin, X.S. (2011). Risk Modelling with the Mixed Erlang Distribution. *Applied Stochastic Models in Business and Industry* . 27(1) : 8-22.

129. Willmot, G.E. et Woo, J.K, (2007). On the Class of Erlang Mixtures with Risk Theoretic Applications. *North American Actuarial Journal.* 11 (2) : 99-115.

130. Yakoubov, Y., Teeger, M. et Duval, D. (1999). A Stochastic Investment Model for Asset and Liability Management. *Proceedings of the 30th International ASTIN Colloquium and 9th International AFIR Colloquium, Volume J.*

131. Yeo, K.L. et Valdez, E.A. (2006). Claim Dependence with Common Effects in Credibility Models. *Insurance : Mathematics and Economics.* 38 (3), 609-629.

132. Zhang, Z. et Yang, H., (2011). Gerber-Shiu Analysis in a Perturbed Risk model with Dependence between Claim Sizes and Interclaim Times. *Journal of Computational and Applied Mathematics.* 235, 1189-1204.