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OPTIMAL COST AND AVAILABILITY REPLACEMENT MODELS FOR MULTI-COMPONENT SYSTEMS

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RÉSUMÉ

Dans ce travail une stratégie de maintenance préventive est proposée pour un système multi composants. Cette stratégie suggère d'effectuer des remplacements préventifs de certains composants du système à tous les *s* unités de temps. Si une défaillance accidentelle survenait, une réparation minimale est aussitôt entreprise pour ramener le système défaillant en état d'opération sans affecter son taux de panne. Après *n* remplacements préventifs, le système est remis à neuf.

La stratégie considérée est alors définie par deux paramètres *n* et *s*. À chaque action de maintenance, on associe une durée et un coût. Deux modèles mathématiques ont été développés pour déterminer le couple optimal (s*, n*). Le premier modèle permet de trouver le couple (s*, n*) qui minimise le coût total moyen sur un horizon infini. Le second modèle permet de déterminer le couple optimal (s*, n*) qui maximise la disponibilité stationnaire du système. Des procédures numériques ont été mises au point pour traiter les deux modèles. Plusieurs résultats numériques ont été obtenus. La stratégie proposée peut s'appliquer à plusieurs systèmes multi composants. Les modèles analytiques développés peuvent servir de base au développement de nouvelles stratégies. Un modèle d'optimisation permettant de déterminer le couple (s*, n*) qui permet de respecter un seuil de disponibilité requis, à coût minimal, est actuellement en phase de développement.

SUMMARY

In this study, a preventive maintenance strategy is proposed for multi-component systems. This strategy suggests performing a preventive replacement of certain components at the end of each time interval s. In the case of failure, a minimal repair is carried out to bring the system back to operating state without affecting its failure rate. According to this strategy, after n component overhaul the system must be completely renewed. The proposed maintenance strategy is defined by the couple of decision variables (s, n).

Time duration and cost, which are supposed to be known, are associated with each maintenance action. Two mathematical models were developed to determine the optimal couple (s*, n*). The first model is used to find the couple (s*, n*), which minimizes the average total cost over an infinite time horizon. The second model is used to determine the optimum couple (s*, n*) which maximizes the availability of the equipment. Numerical methods were developed to determine the optimal couple (s*, n*) for each mathematical model under consideration. Several numerical results have been obtained. The proposed strategy could be applied many industrial systems. The developed analytical models may be used for modeling other maintenance strategy for multi-component systems.

PREFACE

This work would not be possible without the help and collaboration of the persons mentioned below.

I wish to express my sincere thanks to my supervisor, Professor Daoud AIT-KADI for introducing me to the subject area of reliability, availability and maintainability, for his thoughtful supervision, steady support, guidance, and allowing me access to valuable resources throughout the course of this work.

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I take this opportunity to express my deep and heartfelt gratitude to my mother, my father and my little brother, who are at present in Turkey, for their loving support and encouragement. Finally I wish to express my gratitude to my brother Mustafa and his wife for their support, kindness, and encouragement, during my stay in Quebec.

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CHAPTER I

Introduction

Today's industry is mainly characterized by a strong competition depending on the quality and the cost criteria among its sector. Companies strive towards this accomplishment by using equipment with high performance and new technology [1]. The performance of equipment depends on how fast it can resume operation after a system failure or breakdown. Companies can achieve this objective by using suitable and planned maintenance activities.

Maintenance management covers all technical and administrative actions, including supervision. It intends to keep a component in function or restore it to a level where it can perform the required function [2].

Maintenance costs are the major parts of the total operating costs for every manufacturer. They may represent about 15 to 60 percent of the total production cost [4]. To reduce these costs, maintenance strategies must be developed. In this field, several studies have been carried out [e.g. 8, 9, 10, 11, and 12].

Optimization problems have always been of interest in both academic research areas and industrial design. One of the important problems in our area of interest is to minimize the cost in the factory production line of [56]. Generally, this cost, includes the minimal repair costs (expense) and the preventive maintenance costs. Optimal maintenance policies aim at providing optimum system reliability/availability and safety performance at the lowest possible maintenance costs [5].

Optimal preventive maintenance policies are determined for repairable devices by minimizing total cost, maximizing availability, or optimizing some other objectives. There are many optimal models developed for these types of maintenance strategies [e.g. 11, 36 and 53]. Zhang and Jardine`s model [7] is one of the most important examples of these models. Their mathematical model takes into account the optimal cost and determines the optimal overhaul interval s and the number n of overhauls in a renewal cycle. This model considers only the cost criteria for solving the problems.

The present thesis analyses a preventive maintenance strategy for multi-component systems wherein specified components should be replaced at the end of each time period s. The proposed maintenance strategy suggests the performance of three actions, namely minimal repair, periodic overhaul, and complete renewal. An overhaul usually consists of a set of preventive maintenance actions such as oil change, cleaning, greasing, and replacing some worn components of the system, and it is often performed in a workshop [7].

From manufacturers' views, both maintenance cost and availability are crucial factors to be considered. These two performance indicators are strongly correlated. So, there is a need to develop new maintenance strategies which allow one to increase the system availability at a minimal cost.

1.1. Objectives of the present study

The objective of this research is to find the optimal overhaul interval (s) and the number (n) of overhauls in a renewal cycle firstly by maximizing the availability and secondly by minimizing the cost. Based on the model proposed by Zhang and Jardine [7], this study proposes two extensions:

- The first model aims to find the optimal couple (s*,n*) to minimize cost function f(s*, n*) with an availability constraint.
- The second model aims to find the optimal couple (s*,n*) to maximize availability function A(s*, n*) with a budget constraint.

1.2. Methodology

During every maintenance action, we consider *n* time intervals each having time duration *s* for overhaul action in a renewal cycle. At failure time, a minimal repair is carried out to bring back the system to operation state without affecting its failure rate. Once it reaches a certain age or after *n* overhauls, the equipment is completely renewed.

In the present study, we choose two criteria, namely availability and cost criteria. To find the optimal overhaul interval *s* and number *n*, both criteria are used simultaneously; one is used for maximizing the availability and the other for minimizing the cost of the system.

Two mathematical models were developed to determine the optimum couple (s*, n*):

The first model is used to find the couple (s*, n*) which minimizes the average total cost over an infinite time horizon. We added a predetermined level of availability constraint to reach more sensible values.

The second one is used to determine the optimal couple (s*, n*) which maximizes the availability of the equipment. In order to achieve this model, we included a cost constraint in the model.

We developed two models by choosing the most common distribution functions, i.e. Exponential and Weibull to express the failure rate function. We applied minimal repair, periodic overhaul and renewal replacement required in maintenance actions, to the models with *p, q* rule. An item is repaired at failure, with probability *p*; the repair is a perfect repair. With probability *q*, the repair is a minimal repair [42]. The models derived by using integer linear programming. As a comparison, numerical results obtained by our models are compared with those ones presented by [7] in order to confirm the validity of our models. Therefore, it is necessary to select an efficient optimization program that enables us for easy, quick and reliable operation. Lingo programming language is selected to solve the proposed models.

1.3. Structure of the research study

The first chapter (Introduction) is divided into two sections: the first section describes the background and the research problem. The second section presents objectives, limitations, and the structure of the thesis. In the second chapter, a detailed literature review is presented. In the third chapter, a brief review of Zhang and Jardine [7], including their maintenance strategy models and basic definitions are presented. The chapters 4 and 5 describe fundamental issues regarding models, their assumptions and results. The chapters 4 and 5 present the main ideas of extended models with their assumptions and results. Finally, in chapter 6, general conclusions of the present study, discussion and some further research perspectives are presented.

CHAPTER II

Theoretical Frame Work and Basic Definitions

2.1. Introduction

The performance of equipment used in manufacturing industry decline with usage and age. This issue is often reflected in higher production costs and lower product quality. To keep production costs down while insuring good quality, maintenance is often performed on such systems. Here, the fundamental objective is to optimize the system performance by the optimal level of maintenance. The measure of the system performance considered for optimization is the average cost benefit and availability due to maintenance. In this regard, time of replacement and the quality of system performance are important required information [18].

2.2. Literature review

In the past several decades, maintenance, replacement and inspection problems have been discussed in many literatures [e.g. 2, 14, 43 and 16]. The preventive maintenance (PM) policies are adapted to reduce the degradation process of the system in operating conditions and to prolong the system life. A huge number of PM policies have been proposed in the literature [3, 10, 15 and 13]. These policies are defined as the activity undertaken regularly at preselected intervals while system is satisfactorily working [58]. One of these policies is minimal repair. The basic idea of this policy is to repair the system with a minimal effort when the failure occurs. The concept of minimal repair was first introduced by Barlow and Hunter [17] proposes a preventive maintenance strategy with minimal repair at failure. Many extensions of this basic model have been published [e.g. 57, 10, 16, 19, 20 and 21].

In most studies concerned with the reliability and maintenance of a system, the literature indicates that a system, after being subjected to a corrective or preventive maintenance action, will not become "as good as new" but younger. In the recent literature, this kind of maintenance is often called as "imperfect maintenance" [5, 24, 25 and 42]. Usually, it is assumed that imperfect maintenance restores the system operating state to somewhere between as good as new and as bad as old. Clearly, imperfect repair (maintenance) is a general repair which can include two extreme cases: minimal and perfect repair (maintenance) [5]. Imperfect maintenance problems have received more and more attention in literature [25, 26, 27, 28, 29, 30, 31, 32, 33, 35, 36 and 37].

The system improvement due to an imperfect repair is fundamental for establishing an appropriate maintenance model [7]. Three system improvement models have appeared in the literature; the first is Malik's model [40], introducing the concept of virtual age or improvement factor. This model explains that an imperfect repair makes the system "younger" than it was before the action. One of the advantages of Malik's model is that it is relatively easy to analyze. However, there is a concern over this model, which is the assumption that imperfect repair (an overhaul in our case) while only makes the system "younger", it never alters the system failure rate function. The second model belongs to Nakagawa [41], who has assumed that an imperfect repair returns the system to "as bad as old", with a probability ∂ and "as good as new" with a probability $(1-\partial)$. This model seems to be more realistic than the Malik's, because as he said the failure rate function after an imperfect repair will be different from the one before the action. Zhang and Jardine [7]_proposed two optimization models for minimizing the expected unit-time cost or total discounted cost for describing a system improvement due to overhauls. Thus, theirs models provide solutions for situations where the following assumptions are justified], [7].

The well-known treatment method of imperfect repair is the so-called treatment method of (p,q) rule [26, 35, 32, 29, 33, 5, and 37], in which the component is returned to the "as good as new" state (perfect) with probability p and it is returned to the "as bad as old" state (minimal) with probability $q=1-p$ after PM.

Another equally well-known method for modeling imperfect repair is the $\left(p(t),q(t)\right)$ rule, in which the imperfect repair for one-unit system is age dependent. A repair with probability $p(t)$ and with $q(t)=1-p(t)$, is said to be a perfect one. The repair is a minimal repair, where t is the age of the item in use at the failure time (since the last perfect repair) [5, 25, 26, 27, 28, 30, 31 and 36]. Note that when $p=1$ in the (p,q) rule or $p(t)=1$ in the $(p(t),q(t))$ rule, the repair becomes a perfect repair; and when $p=0$ in the (p,q) rule or $p=0$ in the $(p(t),q(t))$ rule, the repair becomes a minimal repair. In other words, both (p,q) and $(p(t),q(t))$ are rules that apply for imperfect repair which include perfect repair and minimal repair as special cases [6].

According to treatment methods, work on imperfect maintenance can be classified as in Table 2.1. As clearly seen in the table, the "(p,q) rule" and "$(p(t),q(t))$ rule" are popular in treating imperfect maintenance. This is partly because of these two rules that make imperfect maintenance modeling mathematically tractable [5].

Table 2.1: Reference classification by treatment methods [5]

Modeling method	***References***
(p, q) rule	Chan and Downs, 1978; Helvic, 1980; Nakagawa, 1979a, 1980; Nakagawa and Yasui, 1987; Brown and Proschan, 1983; Fontenot and Proschan, 1984; Lie and Chun, 1986; Yun and Bai, 1987; Bhattacharjee, 1987; Rangan and Grace, 1989; Gael et al., 199la,b; Sheu and Liou, 1992; Srivastava and Wu, 1993; Wang and Pham, 1996a,b,c
$(p(t), q(t))$ rule	Beichelt, 1980, 1981; Block et al., 1985, 1988; Abdel-Hameed, 1987a,b; Whitaker and Samaniego, 1989; Sheu, 1991, 1992; Sheu and Griffith, 1992; Sheu and Kuo, 1994; Sheu et al., 1993, 1995; Makis and Jardine. 1991; lyer, 1992; Hollander et al., 1992; Sheu and Kuo, 1994
Improvement factor	Mahk, 1979; Canfield, 1986; Lie and Chun, 1986; Jayabalan and Chaudhuri, 1992a,b,c, 1995; Chan and Shaw, 1993; Suresh and Chaudhuri, 1994
Virtual age	Uematsu and Nishida, 1987; Kijima et al., 1988; Kijima, 1989; Makis and Jardine, 1993; Liu et al., 1995
Shock	Bhattacharjee, 1987; Kijima and Nakagawa, 1991; Kijima and Nakagawa, 1992; Sheu and Liou, 1992
(α, β) rule	Wang and Pham, 1996a,b,c
Multiple (p, q) rule	Shaked and Shanthikumar, 1986; Sheu and Griffith, 1992
Other	Nakagawa, 1979b, 1980; Nakagawa, 1986, 1988; Subramanian and Natarajan, 1990; Nguyen and Murthy, 198 1 a,b,c; Yak et al., 1984; Yun and Bai, 1988; Dias, 1990; Subramanian and Natarajan, 1990, Zheng and Fard, 1991; Jack, 199 1; Chun, 1992; Dagpunar and Jack, 1994; Zhao, 1994

Another important constraint to optimize the system performance is the criterion of availability for repairable system. In the literature, this system constraint is emphasized by a wide variety of models and methodologies to maximize the availability [39, 44, 45, 46 and 47]. Reliability, which is another system constraint, and availability may or may not be directly related to each other. It is possible to have a piece of equipment that breaks down frequently, but for a short period of time, it also has a reasonable level of availability. Similarly, it is possible to have a piece of equipment that is highly reliable, but in the meantime, has a low level of availability because it is out of service for a long period of time.

Pham H., Wang H. [5] has summarized a periodic preventive maintenance policy in Table 2.2.

Table 2.2: Periodic PM policy [5]

Reference	*PM*	*CM*	*Treatment method*	*Optimality criterion*	*Modeling tool*	*Planning horizon*
Nakagawa 1979a	Imperfect	Minimal	(p, q) rule	Cost rate	Renewal theory	infinite
Nakagawa 1980	Imperfect	Perfect	(p, q) rule	Cost rate	Renewal theory	infinite
		Minimal	X rule			
Beichelt, 1981	perfect	imperfect	(p(t), q(t)) rule	Cost rate	Renewal theory	infinite
Fontenot and Proschan, 1984	perfect	imperfect	(p, q) rule	Cost rate	renewal theory	infinite
Nakagawa, 1986	imperfect	minimal	different failure rates	Cost rate	Renewal theory	infinite
Abdel-Hameed, 1987a	perfect	imperfect	(p(t), q(t)) rule	Cost rate	stochastic process	infinite
Nakagawa and Yasui, 1987	imperfect	perfect	(p, q) rule	availability	Renewal theory	infinite
Kijima et al., 1988	perfect	imperfect	Virtual age	cost rate	Renewal theory	infinite
Kijima and Nakagawa, 1991	imperfect	perfect	Shock model	Cost rate	Renewal theory	infinite
Jack, 1991	perfect	Imperfect	others	total cost	Renewal theory	finite
Chun, 1992	Imperfect	minimal	X rule	Total cost	probability	finite
Sheu, 1992	Perfect	Imperfect	(p(t), q(t)) rule	cost rate	Renewal theory	infinite
Liu et al., 1995	imperfect	Minimal	Virtual age	cost rate	Renewal theory	infinite
Wang and Pham, 1996	Imperfect	imperfect	(p, q) rule	cost rate availability	Renewal theory	infinite
			(α, β) rule	availability		

2.3. Basic definitions

Reliability: According to Leith [23], the reliability of a product is the measure of its ability to perform its function, when required, for a specified time, in a particular environment. Reliability is also defined as the probability that a system (component) will function over some time period t [22]. To express reliability function, a continuous random variable t ($t \geq 0$) is defined to be the time at which the system failure occurs. The reliability R(t) can now be expressed as:

$$R(t) = \Pr\{T > t\}, \qquad t \quad \text{in} \quad [0, \infty) \tag{2.1}$$

where $R(t) \geq 0$, and $R(0) = 1$.

Another function of interest is the *failure rate* or *hazard rate* of failure. The failure rate function enables us to determine the number of failures occurring per unit time [22]. The failure rate (λ) is mathematically given by equation 2.4.

$$R(t) + F(t) = 1 \tag{2.2}$$

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \tag{2.3}$$

$$\lambda(t) = \frac{f(t)}{R(t)} \tag{2.4}$$

Where

F(t) : Failure function

f(t) : Probability density function

***Maintainability*:** A measure of the ease and rapidity with which a system or equipment can be restored to an operational state following a failure or be retained in a specified condition. It is characterized by equipment design and installation, personnel availability in the required skill levels, adequacy of maintenance procedures and test equipment, and the physical environment in which maintenance is performed. An alternative expression of the definition of maintainability would be or is the probability that an item will be retained in or restored to a specified condition, within a given period of time, when the maintenance is performed in accordance with prescribed procedures and resources [55].

A proper maintainability as illustrated in Figure 2.1 must fulfill some requirements. It must be: a) initially planned and included within the overall planning documentation for a given program or project; b) specified in the top-level specification for the applicable system/product; c) designed through the iterative process of functional analysis, requirement allocation, trade-off and optimization, synthesis, and component selection; and d) measured in terms of adequacy through system test and evaluation.

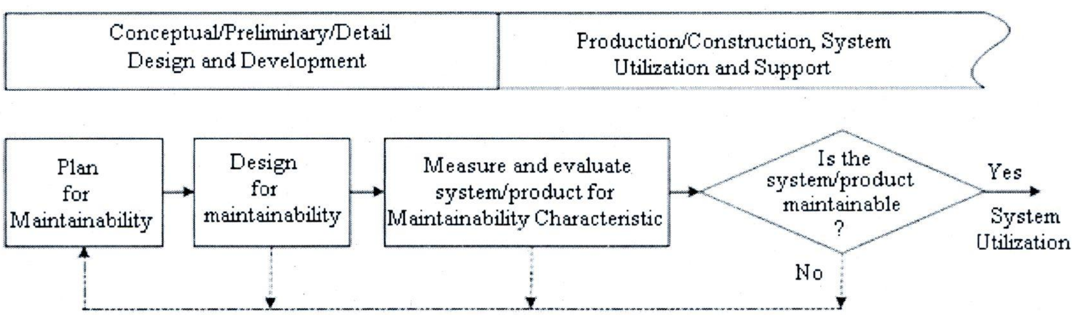

Figure 2.1: Maintainability requirements [38]

The Maintainability that is defined in the broadest sense could be measured in terms of a combination of different maintenance factors. From a system perspective, it is assumed that maintenance can be classified into the following general categories:

1. ***Preventive maintenance* (PM):** actions performed to retain an item in a satisfactory operational condition. These actions consist precisely of undertaking systematic inspection, detection, and prevention of incipient failures [55].

2. **Corrective maintenance (CM):** actions performed to restore an item to a satisfactory condition, by correcting the malfunction that has caused the degradation of the item in question below the specified performance [55].

Maintenance downtime constitutes the total elapsed time required (when the system is not operational) to repair and restore a system to full operating status, or retain a system in that condition. Figure 2.2 illustrates the relationship of the various downtime factors within the context of the overall time domain.

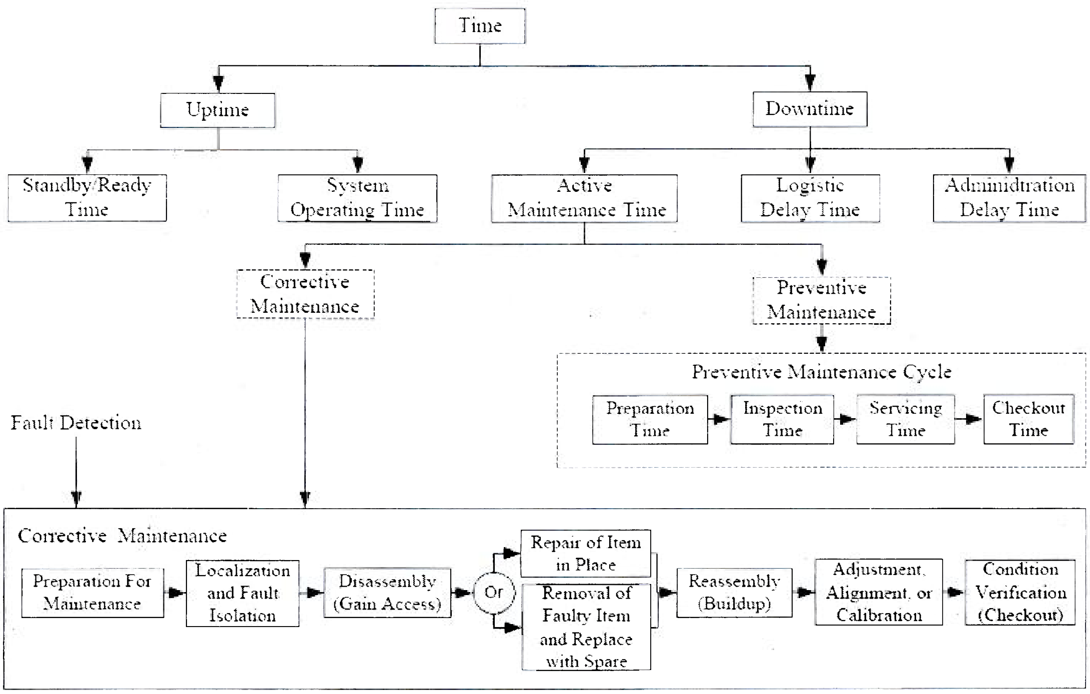

Figure 2.2: Composite view of uptime/downtime factors [34]

In order to increase maintainability, the repair time must be reduced somehow. There are several key concepts that should be followed as part of any design activity that support this reduction. The inner circle in figure 2.3 identifies inherent maintainability design features, and the outer circle lists secondary features that affect the determination of the total system downtime. Secondary factors affecting maintainability focus on the maintenance and supply resources are necessary to support the repair process. Establishing and maintaining the proper levels of these resources are is often considered parts of the logistic process.

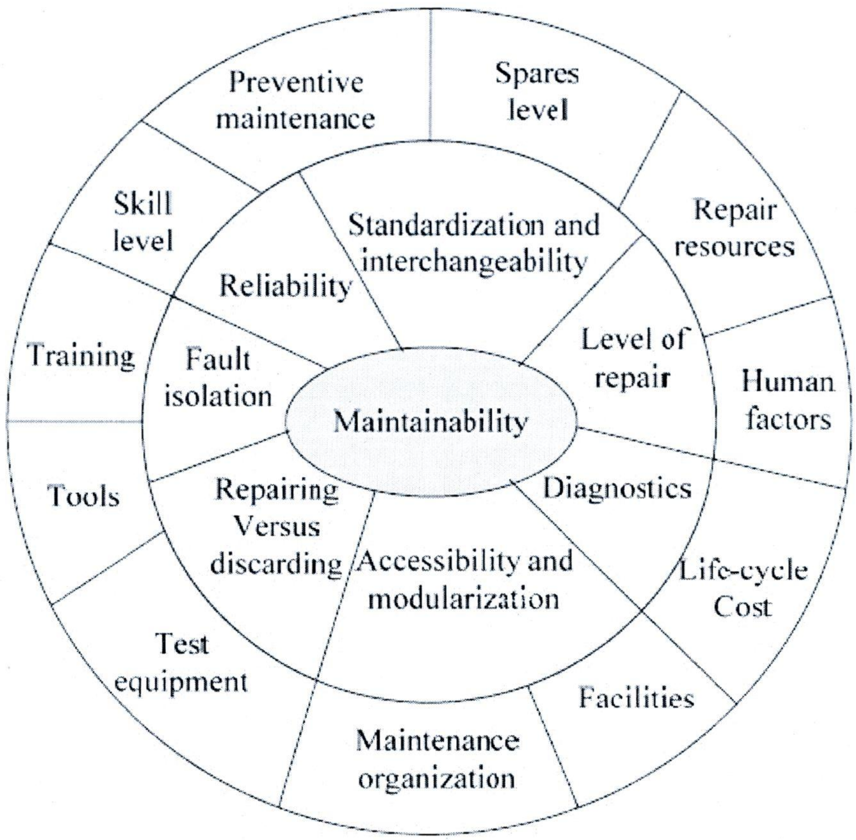

Figure 2.3: Inherent and secondary maintainability design features [22]

***Availability*:** It is a measure of the degree to which an item is in an operable state and can be used at the start of a mission, while the mission is called for an unknown (random) point in time. Availability as measured by the user is a function of how often failures occur and corrective maintenance is required, how often preventative maintenance is performed, how quickly indicated failures can be isolated and repaired, how quickly preventive maintenance tasks can be performed, and how long logistics support delays contribute to downtime [54].

There are several different forms of the steady-state availability that depend on the definitions of uptime and downtime. Some of these definitions are discussed in the following.

***Availability (achieved)*:** Achieved availability is the probability that a system or equipment, when used under stated conditions in an ideal support environment (i.e. available tools, parts, manpower, manuals, etc.), shall operate satisfactorily at a given time. Note that supply downtime and waiting or administrative downtime are excluded [55].

The availability function of a system, denoted by (A_a) is defined as the probability that the system is available at time t in preventive maintenance environment. Achieved availability can be measured by the fallowing equation:

$$A_a = \frac{OT}{OT + TCM + TPM} \tag{2.5}$$

Where

***OT*:** operating time during a given calendar time period,

***TCM*:** total corrective (unscheduled) maintenance downtime during a given calendar time period,

***TPM*:** total preventive (scheduled) maintenance downtime during a given calendar time period.

***Availability (inherent)*:** It is defined as the probability that the system is available at time t. Inherent availability (A_i) can be measured by the fallowing equation:

$$A_i = \frac{OT}{OT + TCM} \tag{2.6}$$

***Availability (operational)*:** A measure of the degree to which an item is either operating or is capable of operating at any random point in time, when used in a typical maintenance and supply environment. Operational availability (A_o) can be measured by the fallowing equation:

$$A_o = \frac{OT + ST}{OT + ST + TCM + TPM + TALDT} \quad (2.7)$$

Where

ST: standby time (not operating, but assumed operable) during a given calendar time period,

TALDT: total administrative and logistic downtime spent waiting for parts, maintenance personnel, or transportation during a given calendar time period,

CHAPTER III

Zhang and Jardine Replacement Model

3.1. Introduction

In this chapter, the study of Zhang and Jardine [7] is summarized. Hereafter we call this paper as Zhang-Jardine model whenever we refer it. The study describes the maintenance strategy of a repairable system, in which three kinds of maintenance actions are considered: minimal repair, overhaul and (complete) renewal. It proposed a new model for describing a system improvement due to overhauls. Based on the improvement model, they established two optimization models, namely discounted cost model and unit time cost model, which are used to minimize the expected time unit cost. The established cost model is proposed for determining the optimal overhaul interval and the numbers of overhauls in a renewal cycle that minimize the expected unit-time cost.

3.2. Problem definition

The system is subjected to three kinds of maintenance action, namely, minimal repair, overhaul and (complete) renewal, with different costs. The system undergoes a minimal repair whenever a failure occurs and it is completely renewed once it reaches a certain age after the last renewal. In the cycle between two consecutive renewals, a fixed number n of overhauls are performed by dividing the cycle into $n+1$ period with an equal length s. An overhaul improves the system, while a minimal repair returns the system to the condition just before that failure.

3.3. The objectives of the article

- The study proposes a new improvement model that can overcome the existing models proposed by Malik [40] and Nakagawa [41].
- Based on the improvement model, Zhang-Jardine established cost models to find optimal couple (n, s) so that the expected unit-time cost or the total discounted cost can be minimized.

3.4. System improvement model

The improvement model assumes that each overhaul brings the system failure rate back to between "bad as old" and "good as previous overhaul period" with a fixed degree. As a result, the model allows the system failure rate function to change from one overhaul period to another overhaul period.

"A system is improved if its failure rate is reduced"

Let $v_{k-1}(t)$ be the system failure rate function just before the overhaul, $v_k(t)$ is the failure rate function right after the overhaul, s is the overhaul interval and $p \in [1,0]$ is a constant. The system is improved by a degree p by the overhaul if, for all t after this overhaul,

$$v_k(t) = p v_{k-1}(t-s) + (1-p) v_{k-1}(t) \quad (3.1)$$

If the improvement degree p equals 0, then $v_k(t) = v_{k-1}(t)$. In this case, the failure rate is not disturbed and the overhaul is equivalent to a minimal repair.

If $p = 1$, then $v_k(t) = v_{k-1}(t-s)$. In this case, each overhaul restores the system to the condition of the previous overhaul period, and thus it is equivalent to a complete renewal. Note that, although the definition is made in terms of overhauls, it can be applied to any regular maintenance action that improves the system.

Optimal maintenance models

The main result of this section, concerning the failure rate of a system after a series of overhauls, is given below.

Suppose that each overhaul improves a system by a degree of p. Let $v(t)$ denote the failure rates of the system without overhaul, $\hat{v}(t)$ denote the system with periodic overhauls with interval s. Then, for all integer k $\geq$1 and $t \in [o, s)$,

$$\hat{\upsilon}(ks+t) = \sum_{i=0}^{k} \binom{k}{i} p^{k-i} q^{i} \upsilon(is+t) \qquad (3.2)$$

3.5. Unit-time cost model

3.5.1. Notations

$\upsilon(t)$: Original failure rate of the system without overhaul;

$\hat{\upsilon}(t)$: Actual failure rate of the system with periodic overhauls;

$H(t) = \int_0^t \upsilon(x)dx$: Originally expected failures in the interval [0,t), which is ;

$\hat{H}(t) = \int_0^t \hat{\upsilon}(x)dx$: Actually expected failures in the interval [0,t), which is;

s : The overhaul interval;

n : One plus the number of overhauls in a renewal cycle;

p,q: p is the improvement degree and $q = 1 - p$;

c_m, c_0, c_r : Costs of minimal repair, of overhaul and of renewal, respectively;

$f(n,s)$: The expected unit-time cost when the system is overhauled $n-1$ times with interval s in a renewal cycle.

3.5.2. Assumptions

(1) An overhaul improves the system with a fixed degree p.

(2) A minimal repair does not change the failure rate.

(3) All renewal cycles have the same number of overhauls dividing each cycle into equal length periods.

(4) All renewal cycles have the same length (ns).

(5) The time spent on repairs and overhauls are ignored.

(6) p, c_m, c_o, c_r, $\upsilon(t)$ and $H(t)$ are known; $c_r > c_o > 0$ and $c_r > c_m > 0$; $p < 1$.

3.5.3. The objective function

The expected cost in a renewal cycle is $c_r + c_0(n-1) + c_m\hat{H}(ns)$ and the length of the cycle is ns. Thus, the expected unit-cost over an infinite time horizon is given by:

$$f(n,s) = \frac{c_r + c_0(n-1) + c_m\hat{H}(ns)}{ns}, \qquad (3.3)$$

Where,

$$\hat{H}(ns) = \sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}H(is), \qquad (3.4)$$

3.5.4. Important properties

Under the condition that the function $\upsilon(t)$ is continuous, non-decreasing and unbounded, the following consequences regarding the minimization of $f(n,s)$ expression can be given:

(1) When n is fixed, an optimal solution to $\min_s f(n,s)$ exists and it is a solution to the equation below

$$\hat{H}'_s(ns)s - \hat{H}(ns) - [c_r + c_0(n-1)]/c_m = 0, \qquad (3.5)$$

(2) $\min_s f(n,s)$ has finite optimal solutions and there exists a bound n_b such that

$$\min_{s,n} f(n,s) = \min\{\min_s f(n,s) : 1 \le n \le n_b\} \qquad (3.6)$$

(3) For every fixed $s > 0$, an optimal solution to $\min_s f(n,s)$ exists and can be obtained as the first integer n such that $f(n+z),s) \ge f(n,s)$

Based on the above properties, minimizing $f(n,s)$ can be achieved according to the following procedure; first, estimate a range in which the optimal number of overhauls is located, then find s_n that minimizes $f(n,s)$ for each fixed n within that range, and finally select n within that range and s_n such that $f(n,s_n)$ is minimized.

3.5.5. For Exponential failure rate

For the frequently used Exponential failure rate; we have the expression $\upsilon = \exp(\alpha_0 + \alpha_1 t)$. $\hat{H}(ns)$ and $f(n,s)$ are simplified, and they illustrate how to use the model through a numerical example.

Suppose $\upsilon(t) = \exp(\alpha_0 + \alpha_1 t)$ with $\alpha_1 > 0$;

$$\hat{H}(ns) = e^{\alpha_0}\left[\left(p + qe^{\alpha_1 s}\right)^n - 1\right]/\left(q\alpha_1\right) \tag{3.7}$$

$$f(n,s) = \frac{c_r + c_0(n-1) + c_m e^{\alpha_0}\left[\left(p + qe^{\alpha_1 s}\right)^n - 1\right]/\left(q\alpha_1\right)}{ns} \tag{3.8}$$

Suppose $v(t) = \exp(\alpha_0 + \alpha_1 t)$. Then, for each fixed integer $n \geq 1$, s^* minimizes $f(n,s)$ if and only if it is a solution of

$$e^{\alpha_1 s}\left(ns - \frac{1}{\alpha_1}\right) - \left[\frac{\left(c_r + c_0(n-1)\right)q}{c_m e^{\alpha_0}} - \frac{1}{\alpha_1}\right]\left(p + qe^{\alpha_1 s}\right)^{1-n} = \frac{p}{\alpha_1} \tag{3.9}$$

Example 1: Suppose that costs of maintaining the system are $c_r = \$200{,}000$, $c_o = \$8{,}000$ and $c_m = \$2{,}000$ respectively. The original failure rate is given as $\upsilon = \exp(\alpha_0 + \alpha_1 t)$, where $\alpha_0 = -15$ and $\alpha_1 = 0.01$.

Resolution:

$$f(n,s) = \frac{200000 + 8000(n-1) + 2000e^{-15}\left[\left(p + qe^{0.01s}\right)^n - 1 \Big/ (0.01q)\right]}{ns}$$

Table 3.1: Optimal solutions for different p

P	n^*	s^*	$f(n^*,s^*)$
0.5	6	260.3	165.1
0.6	8	223.3	153.9
0.7	11	195.6	138.7
0.8	15	186.2	118.5

Minimizing $f(n,s)$ for $p = 0.7$, we obtain $n^* = 11$, $s^* = 195.6$ and $f(n^*,s^*) = 138.7$.

The result may be interpreted as fallows: the optimal overhaul interval is 195.6 days; the system should be completely renewed after 2,156 days or about 6 years. The maintenance cost per day is $138.7. Table 3.1 shows the optimal solutions for different p.

3.5.6. For a Weibull failure rate

The Weibull function: $v(t) = \left(\beta/\eta\right)\left(t/\eta\right)^{\beta-1}$

Suppose $v(t) = \left(\beta/\eta\right)\left(t/\eta\right)^{\beta-1}$ with $\beta > 1$.

$$\hat{H}(ns) = \left(\beta/\eta\right)^{\beta} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \tag{3.10}$$

And

$$f(n,s) = \left(c_r + c_0(n-1) + c_m(s/\eta)^{\beta} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \right) \Big/ ns \tag{3.11}$$

The expression of $f(n,s)$ is obtained by bringing in the expression of $\hat{H}(ns)$

Suppose $v(t) = (\beta/\eta)(t/\eta)^{\beta-1}$. Then, (n^*, s^*) minimizes $f(n,s)$ if and only if n^* minimizes, equation 3.12 below

$$\min \left\{ \left((c_r/c_o - 1 + n)^{\beta-1} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \right) \Big/ n^{\beta} \quad \text{for all } n \geq 1 \right\}, \tag{3.12}$$

And

$$s^* = \sqrt[\beta]{\frac{\left(c_r + (n^* - 1)c_o\right)}{(\beta - 1)c_m \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta}}}\,\eta \quad . \tag{3.13}$$

Proof: For each n, let $F(n)$ denote $\min_s f(n,s)$. We notice that (n^*, s^*) minimizes $f(n,s)$ if and only if n^* minimizes $F(n)$ and s^* minimizes $f(n^*, s)$.

For each fixed n, according to equation 3.1, s minimizes $f(n,s)$ if it satisfies (3.5), which becomes;

$$c_r + c_0(n-1) + c_m (s/\eta)^{\beta} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta}$$

$$= c_m (s/\eta)^{\beta} \beta \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta}$$

So,

$$(s/\eta)^{\beta} = \left(c_r + c_0(n-1)\right) \Big/ \left(c_m (\beta - 1) \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \right),$$

And

$$s = \sqrt[\beta]{\left(c_r + c_0(n-1)\right) \Big/ \left(c_m (\beta - 1) \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \right)}\,\eta \, .$$

By replacing s in $f(n,s)$ with the above expression, we obtain;

$$F(n)=\sqrt[\beta^{\beta}]{[c_r+c_0(n-1)]^{\beta-1}c_m(\beta-1)\sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}i^{\beta}}\Big/\left(\eta n\sqrt[\beta]{(\beta-1)^{\beta-1}}\right)$$

$$=\left(\left(\sqrt[\beta]{[c_r+c_0(n-1)]^{\beta-1}c_m(\beta-1)\sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}i^{\beta}}\right)\Big/n\right)\times\frac{\beta\sqrt[\beta]{c_m c_o^{\beta-1}}}{\eta\sqrt[\beta]{(\beta-1)^{\beta-1}}}$$

Clearly, n^* minimizes $F(n)$ if and only if it minimizes

$$\left((c_r/c_o-1+n)^{\beta-1}\sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}i^{\beta}\right)\Big/n^{\beta}$$

If β happens to be an integer greater than 1, it is possible to find a short expression of

$$\sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}i^{\beta}$$

For example, when $n\geq\beta$,

$$\sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}i^{\beta}$$

$$=\begin{cases} n & for\ \beta=1 \\ n^2q+np & for\ \beta=2 \\ n(n-1)(n-2)q^2+3n(n-1)q+n & for\ \beta=3 \end{cases}$$

Example 2: Suppose that $c_r=\$200{,}000$, $c_o=\$8{,}000$, $c_m=\$2{,}000$, $p=0.7$ and $v(t)=(\beta/\eta)(t/\eta)^{\beta-1}$ with $\beta=2$ and $\eta=100$. Since

$$\sum_{i=0}^{n}\binom{n}{i}p^{n-i}q^{i-1}i^{\beta}=n^2q+np,$$

The objective function of equation 3.1 becomes

$$(c_r/c_o - 1 + n)(n^2 q + np)/n^2 = q(c_r/c_o - 1) + p + nq + p(c_r/c_o - 1)/n,$$

And it is minimized if $nq = p(c_r/c_o - 1)/n$. Thus, n^* is either

$$\left\lfloor \sqrt{p(c_r/c_o - 1)/q} \right\rfloor = 7 \quad \text{or} \quad \left\lfloor \sqrt{p(c_r/c_o - 1)/q} \right\rfloor + 1 = 8.$$

Since $q(c_r/c_o - 1) + p + nq + p(c_r/c_o - 1)/n = 12.4$ for $n = 7$ and 8, both $n^* = 7$ and $n^* = 8$ minimize $F(n)$. For $n^* = 7$

$$s = \sqrt{(c_r + c_0(n-1))/(c_m n^*(n^* q + p))\eta} = 269$$

3.6. Conclusion

The Zhang-Jardine model which is an optimization model is able to minimize expected unit-time cost. We split their model into two new models in order to firstly minimize cost and secondly maximize availability while they are under availability and cost constraints respectively.

CHAPTER IV

Minimal Cost / Fixed Availability

4.1. Introduction

The costs of maintainability and availability are important factors for developing the optimal maintenance model which considers minimal repair, overhaul and complete renewal. The basic objective of the maintenance models is to minimize the total costs of maintainability and to increase availability. The cost model presented by Zhang-Jardine can be efficiently solved using the LINGO programming language. This chapter presents additional aspects to extend the model proposed by Zhang-Jardine which minimizes the cost while availability is kept fixed.

Zhang-Jardine model is proposed to determine the optimal overhaul interval and the number of overhauls in a renewal cycle. The model is able to minimize the expected unit-time cost, however; the availability criterion was not considered in the model. Our extended model is derived from Zhang-Jardine model with same assumptions by considering an availability criterion. As a result, the optimal values of n^* and s^* could be acceptable not only for the estimation of the unit cost, but also for the level of the availability.

The use of computer programming is also important in the modeling operations and/or optimization studies. It is necessary to select an efficient modeling program for an easy, quick and reliable operation. The LINGO programming language was selected for determining the optimal maintenance conditions, because it is a comprehensive tool designed to help build and solve linear optimization models quickly, easily, and efficiently [48]. Many researchers have also used the LINGO for this kind of studies [51, 52 and 53]. The LINGO programming language allows the user to obtain the following important values:

- Minimum cost (per unit),
- Optimal s^*,
- And optimal n^*.

Finally, the objective of this chapter is to determine the values of n^* and s^* that minimize the total cost per unit of time while satisfying a predetermined availability level D. The mathematical expression is expressed as:

$$\text{Min } f(n,s)$$

Subjected to:

$$A(n,s) \geq D$$

where A is the availability of the system expressed by %.

The section 2 of this chapter presents the notations that will be served in this study. The sections 3 and 4 present the general assumptions and the main models chosen for the study. Furthermore, Exponential and Weibull distributions will be used to determine the failure rate. The sections 5 and 6 of the chapter present the Lingo language programming and its implementation in detail. Finally, at the end of this chapter, the results obtained from the models will be discussed.

4.2. Notations

The notations used in following chapters are as follows:

T_m, T_0, T_r : Downtime of minimal repair, overhaul and renewal, respectively. These durations are constant and known with: $T_r > T_o > T_m$;

$f(n,s)$: The expected unit-time cost when the system is overhauled $n-1$ times with interval s in a renewal cycle;

A : Availability predetermined level of the system;

$A(n,s)$: Steady-state availability of the system;

C : Represent $f(n,s)$ in the model.

4.3. Numerical data

The following numerical data are used for our mathematical model:

- C_r = \$200,000, C_o = \$8,000, C_m = \$2,000 (taken from Zhang-Jardine model)
- T_r = 150, T_o = 100, T_m = 50 of hours
- $p = 0.7$ (taken from Zhang-Jardine model)

4.4. Main model

In the present study, a new extension of Zhang-Jardine model is proposed as:

$$f(n,s) = C = \frac{c_r + c_0(n-1) + c_m\hat{H}(ns)}{ns} \quad (4.1)$$

We included availability criteria into the Zhang-Jardine model in order to make it more effective to find optimal values of n and s, so the availability of the system is obtained by following formula:

$$A(n,s) = A = 1 - \frac{Downtime}{Totaltime} \tag{4.2}$$

$$Total\ time = Up\ time + Down\ time$$

$$Downtime\ of\ the\ system = T_r + T_o(n-1) + T_c \hat{H}(ns) = T(n,s)$$

$$Up\ Time\ of\ the\ system = ns$$

Thus, the new model can be given as;

$$A = 1 - \frac{T_r + T_o(n-1) + T_c \hat{H}(ns)}{ns + T_r + T_o(n-1) + T_c \hat{H}(ns)} \tag{4.3}$$

$$A = \frac{ns}{ns + T_r + T_o(n-1) + T_c \hat{H}(ns)} \tag{4.4}$$

$$A = \frac{ns}{T(n,s)} \qquad \begin{cases} \dfrac{\partial A}{\partial s} = 0 \\ \dfrac{\partial A}{\partial n} = 0 \end{cases}$$

Prior to solve the model, we need to obtain the expression of $\hat{H}(ns)$. We will consider two specific failure rate functions, namely, Exponential failure rate distribution and Weibull failure rate distribution.

4.5. Exponential failure rate distribution

The Exponential distribution is commonly used in the development of reliability practices, standards and methods. Mathematically, it is a fairly simple distribution, which is frequently used in inappropriate situations. It is, in fact, a special case of the Weibull distribution, the case where $\beta = 1$. The Exponential distribution is used to model the behavior of units that have a constant failure rate (or units that do not degrade with time or wear out) [49]. In this case this distribution function is used to model the failure rate.

The model obtained for an Exponential failure rate, its assumptions and results are detailed in the following subsections.

4.5.1. Assumptions

The assumptions of the Exponential failure rate distribution are given by the following equations.

$$\upsilon(t) = \exp(\alpha_0 + \alpha_1 t) \quad \text{with} \quad \alpha_1 > 0\ ; \tag{4.5}$$

$$\hat{H}(ns) = e^{\alpha_0}\left[\left(p + qe^{\alpha_1 s}\right)^n - 1\right]/(q\alpha_1) \tag{4.6}$$

In addition, the Exponential distribution characteristics "α_0" and "α_1 " were selected as -15 and 0.01, respectively. These values are the same as the ones used in Zhang-Jardine model. It provides opportunities to compare the results from Zhang-Jardine models.

4.5.2. Optimization modeling by Lingo

The Exponential distribution model written by Lingo language is shown in Figure 4.1. Minimum unit cost, with optimal n^* and s^* values, can be obtained by the use of this model. As shown in figure 4.1, the model includes two constraints:

- Minimum C;
- Availability > 0.7 (70%)

Effective results can be obtained in a very short period of time.

```
Model:

MIN = C ;

C = Z / (n * s);

A = (n * s) / ((n * s) + F);


Z = 200000 + 8000 * (n - 1) + W;
W = (2000 * @EXP(-15 )*(T - 1)) / ( 0.01 * q ) ;


F = 150 +100*(n-1) + L;
L = (50 * @EXP(-15 )*(T - 1)) / ( 0.01 * q ) ;
T = (p + q * @EXP( 0.01 * s )) ^ n;

A = 0.7;

n > 0;
s > 0;
p = 0.7;
q = 1 - p;
@GIN(n);


end
```

Figure 4.1: The Exponential distribution model written by Lingo language for a fixed availability

4.5.3. The results of Exponential distribution model

The results shown in Figure 4.2 are obtained by solving the model by Lingo. At 70% availability, the values of C, n and s are obtained as 152.0140 $, 5 and 311.4083 days, respectively.

```
Local optimal solution found.
Objective value:                                    152.0140
Extended solver steps:                                     5
Total solver iterations:                                 141


                       Variable           Value        Reduced Cost
                              C        152.0140            0.000000
                              Z        236692.1            0.000000
                              n        5.000000            3.202421
                              s        311.4083            0.000000
                              A       0.7000000            0.000000
                              F        667.3035            0.000000
                              W        4692.140            0.000000
                              T        23009.03            0.000000
                              Q       0.3000000            0.000000
                              L        117.3035            0.000000
                              P       0.7000000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        152.0140           -1.000000
                              2        0.000000           -1.000000
                              3        0.000000            352.2721
                              4        0.000000      -0.6422436E-03
                              5        0.000000      -0.6422436E-03
                              6        0.000000          -0.1108598
                              7        0.000000          -0.1108598
                              8  -0.5916809E-07      -0.6961805E-03
                              9        0.000000           -352.2721
                             10        5.000000            0.000000
                             11        311.4083            0.000000
                             12        0.000000            178.1642
                             13        0.000000           -188.9100
```

Figure 4.2: The solution of the model by Lingo program

Table 4.1 presents the values, *n*, *s* and the corresponding cost rate while the increasing availability. It shows that as the availability increases, the number of overhauls becomes smaller.

Table 4.1: n, s and C results in terms of availability by Exponential distribution model

A	C($)	n	s (days)
0.1	445.0659	20	158.7005
0.2	249.1315	20	151.4616
0.3	185.8880	19	151.2619
0.4	156.4192	18	150.9611
0.5	141.9753	15	163.4382
0.6	139.3545	9	222.8378
0.7	152.0140	5	311.4083
0.8	331.9782	1	602.5252

For further discussion about the effect of availability on the parameters, additional graphical analyses are performed.

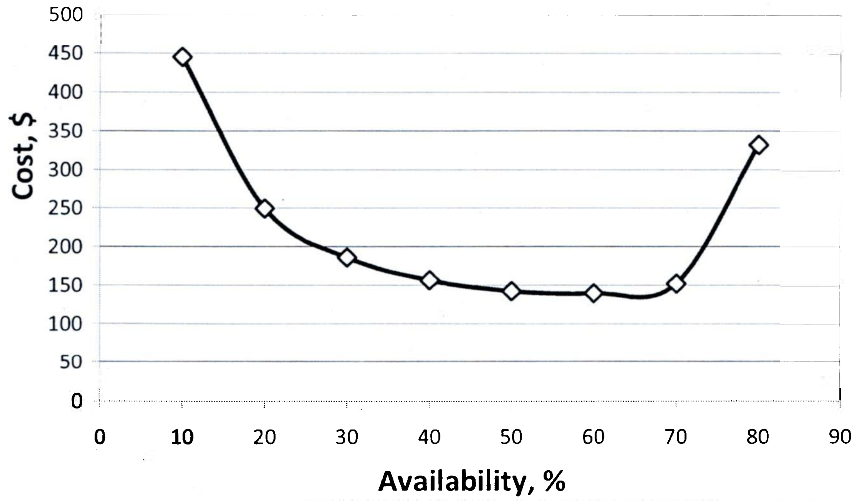

Figure 4.3: Unit-time cost versus availability

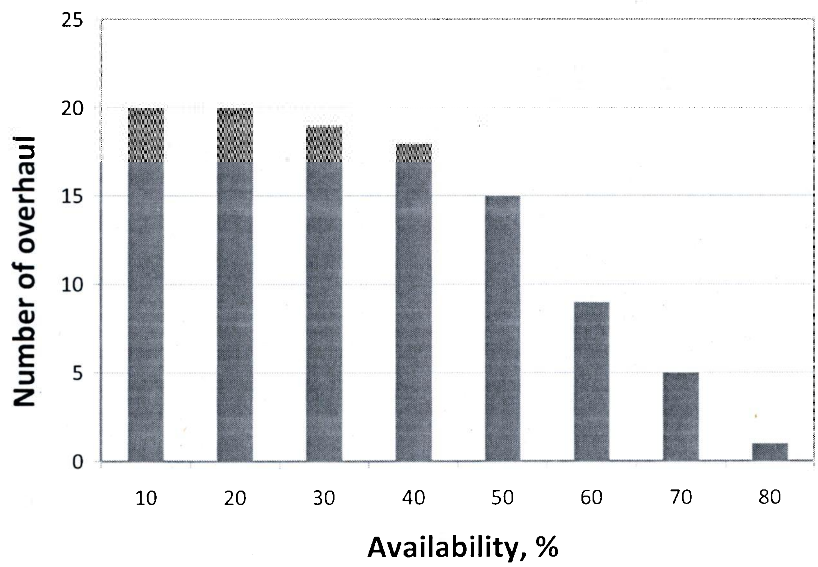

Figure 4.4: Number of overhaul versus availability

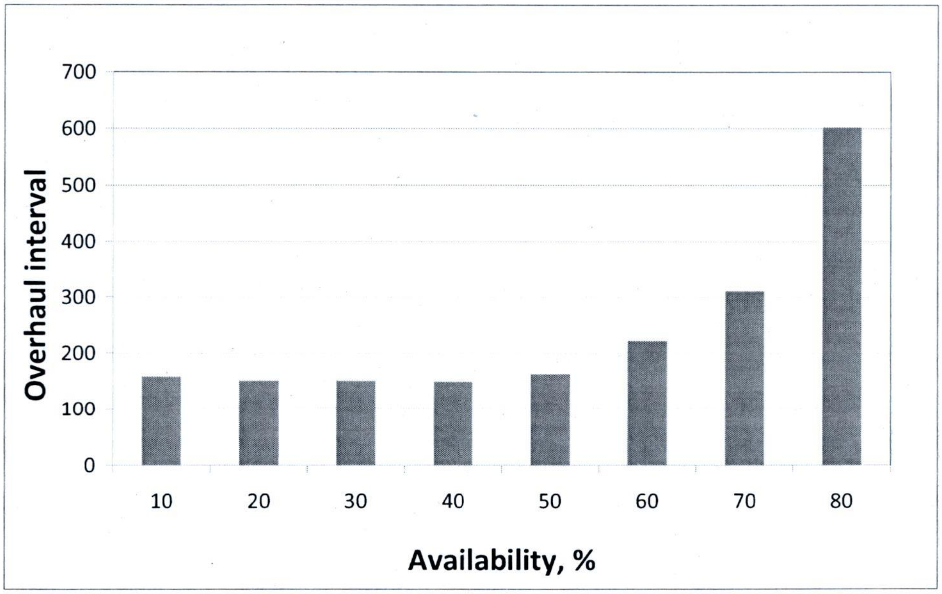

Figure 4.5: Overhaul interval versus availability

Fig. 4.3 shows availability diagram versus unit-time cost. As availability increases from 0.1 to 0.6, the value of C tends to decrease from approximately 445 to 139. From this figure, the curve reaches a point in which the cost has its minimum value, so the availability at this point is optimized. Furthermore, if the availability increases from 0.6 to 0.8, the value of C increases up to 331. Fig. 4.4 shows that as the number of overhauls becomes smaller, the availability becomes higher. Finally, the overhaul interval slightly decreases with the increasing availability up to a point of 40% availability, then it significantly increases from 150.96 to 602.53 with an increase in the availability value up to 0.8 (Figure 4.5).

Parameter estimation is usually a difficult task. However, according to the results obtained by solving the model by Lingo, it could be expressed that the optimal solution of all these local results is located at 60 % of availability. This level of availability is the maximum when the minimum cost is taken into account. In this case, the C is 139.3545 $ for the values of $n^* = 9$ and $s^* = 222.8378$ days.

4.6. Weibull failure rate distribution

The Weibull distribution is one of the most widely used lifetime distributions in reliability engineering. It is a versatile distribution that can take on the characteristics of other types of distributions, based on the value of the shape parameter, β [50].

This section describes Weibull distribution model, its assumption and its results.

4.6.1. Assumptions

The assumptions of this model are similar to the Exponential distribution model and can be expressed as:

For a Weibull failure rate;

$$v(t) = (\beta/\eta)(t/\eta)^{\beta-1} \tag{4.7}$$

$$\hat{H}(ns) = \left(\frac{s}{\eta}\right)^{\beta} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \tag{4.8}$$

$$\sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} = n^2 q + np \qquad \text{for} \quad \beta = 2 \tag{4.9}$$

In addition, Weibull distribution characteristics β and η are selected as 2 and 100, respectively.

4.6.2. Optimization modeling by Lingo

The Weibull distribution model is solved by a program written by Lingo language as given in Fig. 4.6 Minimum unit cost with optimal n^* and s^* values can be obtained by the use of this model. As shown in Fig. 4.6, the model includes two requirements, which are:

- Minimum C and
- Availability > 0.7 (70 %).

```
min = c;

C = (200000 + 8000 *(n-1) + (K * B))/(n*s);

A = (n * s) / ((n * s) + F);

F = 150 +100*(n-1) + L;
L = (50 * B);

B = ((n ^ 2) * q) + (n * p);

K = 2000 * (( s / 100) ^ 2);


A = 0.7;
@GIN(n);
n > 1;
s > 0;
P = 0.7;
q = 1 - p;


end
```

Figure 4.6: The Weibull distribution model written by Lingo language for a fixed availability

The program shown in Fig. 4.6 is only an illustrative example to solve the model when the availability is taken as 70 %.

4.6.3. Weibull distribution model results

The Results shown in Figure 4.7 are obtained by solving the Weibull distribution model by Lingo. As seen in the Table, the values of C, n and s are found to be 299.6460 $, 4 and 484.1667 days, respectively, for a fixed availability of 0.7 (70%).

```
Local optimal solution found.
Objective value:                              299.6460
Extended solver steps:                               3
Total solver iterations:                            95

                 Variable           Value        Reduced Cost
                        C        299.6460            0.000000
                        n        4.000000            8.175045
                        K        46883.47            0.000000
                        B        7.600000            0.000000
                        s        484.1667            0.000000
                        A       0.7000000            0.000000
                        F        830.0000            0.000000
                        L        380.0000            0.000000
                        Q       0.3000000            0.000000
                        P       0.7000000            0.000000

                      Row    Slack or Surplus      Dual Price
                        1        299.6460           -1.000000
                        2        0.000000           -1.000000
                        3        0.000000            325.3362
                        4        0.000000       -0.8231402E-01
                        5        0.000000       -0.8231402E-01
                        6        0.000000           -28.32404
                        7   -0.1818000E-05      -0.3924268E-02
                        8        0.000000           -325.3363
                        9        3.000000            0.000000
                       10        484.1667            0.000000
                       11        0.000000            339.8885
                       12        0.000000           -453.1846
```

Figure 4.7: The solution of the model by Lingo program

According to the results, the local objective value (minimum cost per unit) is 299.6460 $ for an availability of 70% and the optimal n^*, s^*.

Table 4.2 shows estimated (calculated) values of C, n and s in different levels of availability. It is concluded that the optimal solution among these local results is located between

50% and 60% while the cost is minimal. In this case, the $n^* = 5 - 7$ and $s^* = 247 - 330$ days values are acceptable when C is between 282 and 286 $ (Table 4.2).

Table 4.2: n, s and C results in terms of availability by Weibull distribution model

A	C($)	n	s (days)
0.1	345.4831	32	68.50694
0.2	306.5995	19	105.6579
0.3	290.7706	13	143.077
0.4	283.7563	10	193.3333
0.5	281.7526	7	247.1429
0.6	285.8061	5	330
0.7	299.646	4	484.1667
0.8	334.421	2	760
0.9	471.1108	1	1799.998

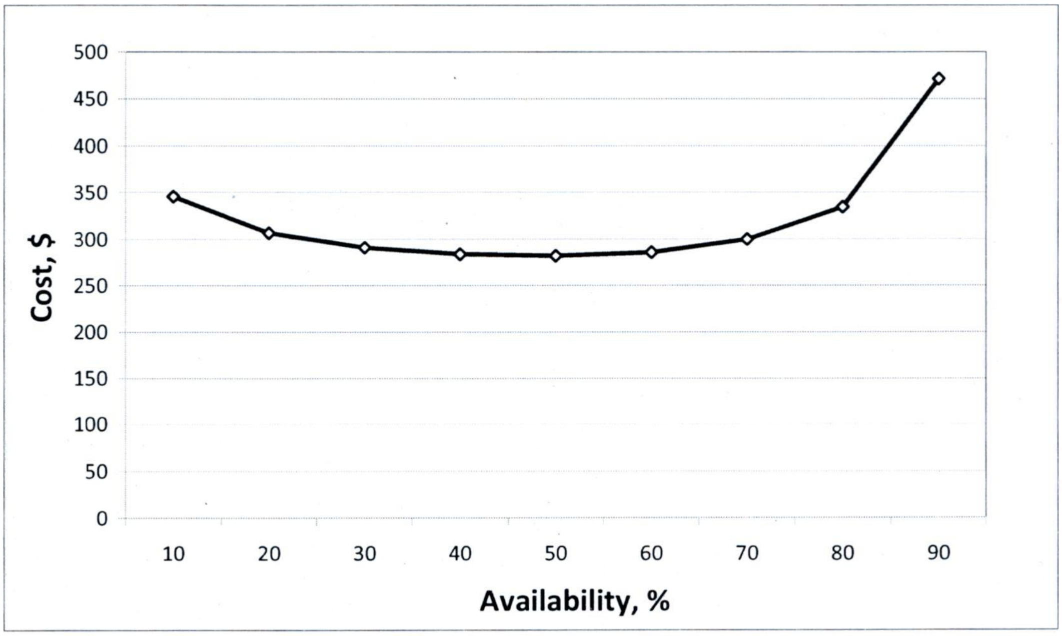

Figure 4.8: Unit-time cost versus availability

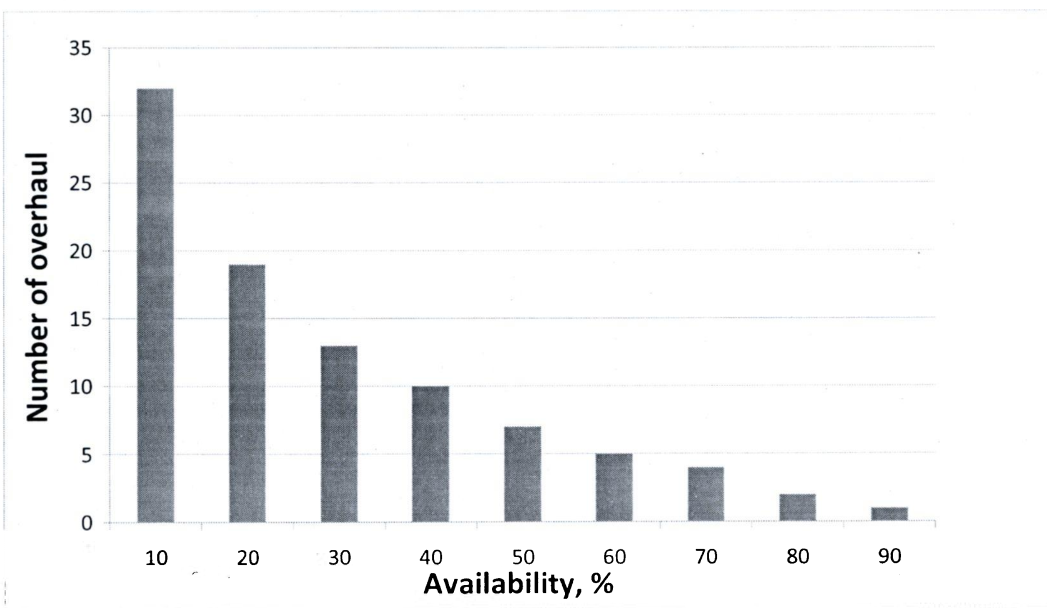

Figure 4.9: Number of overhaul versus availability

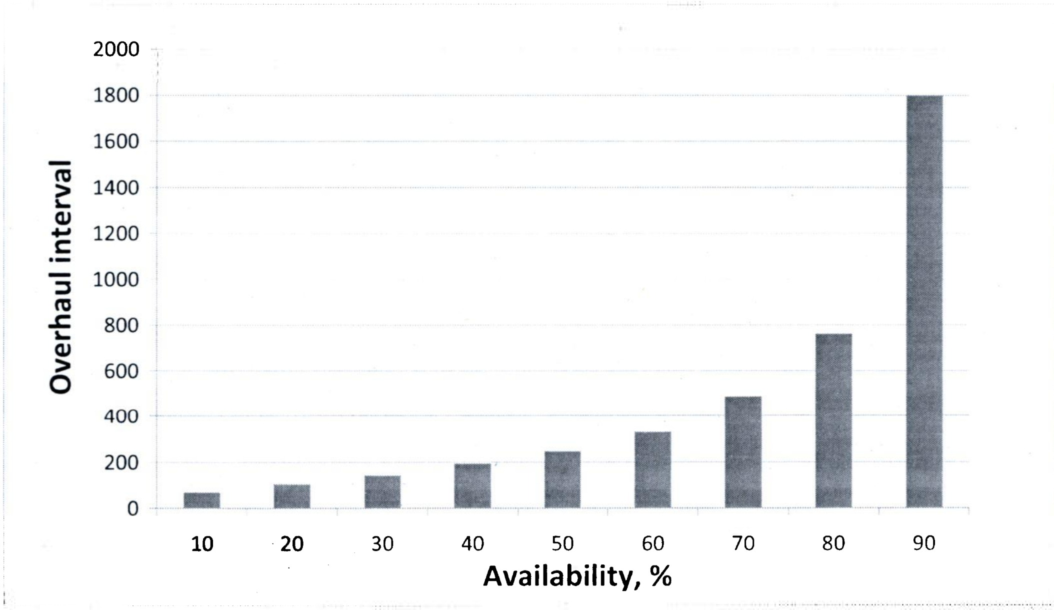

Figure 4.10: Overhaul interval versus availability

Fig. 4.8 shows the effect of availability on the unit-time cost. The value of C declines from 346 to 281 while A increases from 0.1 to 0.5. On the contrary when the A is increased from 0.5 to 0.9, C value starts to increase from 281 to 471 (Fig. 4.8). This result indicates that the minimum unit-time cost can be achieved with an availability of 5 %. However, the availability increases as the number of overhaul become smaller (Fig. 4.9). The overhaul interval increases proportionally to the availability (Fig. 4.10).

4.7. Results and comparison

The following table (or Table 4.3) compares our results with the results obtained by Zhang-Jardine model.

Table 4.3: The comparison of the results obtained by both models according to Exponential failure rate

Exponential failure rate	*Zhang and Jardine*	*This work*
p	0.7	0.7
n*	11	9
s* (day)	195.6	222.8378
Complete Renewal (days)	2,151.6	2,005.5402
$f(n^*,s^*)$ ($)	138.7	139.3545
Availability (%)	Not considered	60

Table 4.4: The comparison of the results obtained by both models according to Weibull failure rate

Weibull failure rate	*Zhang and Jardine*	*This work*
p	0.7	0.7
n*	7	5 - 7
s* (day)	269	247 – 330
Complete Renewal (days)	1,883	1,235 – 2,310
$f(n^*,s^*)$ ($)	282.3447	282 – 286
Availability (%)	Not considered	50 - 60

According to above tables (4.3, 4.4); both results are similar. We can clearly see the advantage of using the availability criterion. If we look at the results of the Table 4.3; for $p = 0.7$, we find the Exponential failure rate as 9 with an overhaul interval of 222.8378 days, and a cost per day of 139.3545 $. Although, the cost per day is slightly expensive comparing with the result of Zhang-Jardine model, the difference may be attributed to the effect of the availability criterion. The Figure 4.3 clearly shows that how the cost changes with the availability. The optimal solution is taken when the point shows the minimal cost with maximum availability. Finally, we can say that the model working well with two criteria.

CHAPTER V

Maximal Availability / Fixed Cost

5.1. Introduction

Availability, as mentioned before, is a very important criterion for the development of optimal maintenance strategies. For that reason finding the exact availability value will be beneficial for the any system. This chapter deals with the finding of the acceptable value of availability by using the fixed unit-time cost constraint. Our extended model is based on Zhang-Jardine model with same assumptions by considering a cost constraint.

In order to maximize the availability, the unit-time cost is kept fixed. A general mathematical expression of this objective can be denoted as:

$$\text{Max } A(n,s)$$

Subject is:

$$C(n,s) \leq B,$$

where B is the constant.

LINGO program can be used for solving this kind of equation and displays three main results, which are availability, optimal n^* and s^* values. These values depend on the B value chosen for unit-time cost. The optimal n^* and s^* values were obtained from LINGO with maximizing the availability.

Second and third sections of this chapter indicate general assumptions and main models, respectively. The fourth and fifth sections of the chapter also include detailed information (Model by Lingo language, calculation etc) about the distributions.

5.2. Exponential failure rate distribution

In this section, assumptions, the model written by Lingo language for Exponential distribution and its results is explained in detail.

5.2.1. Analytical expression

For the Exponential failure rate;

$$\upsilon(t) = \exp(\alpha_0 + \alpha_1 t) \text{ with } \alpha_1 > 0 \; ; \tag{5.1}$$

$$\hat{H}(ns) = e^{\alpha_0}\left[\left(p + qe^{\alpha_1 s}\right)^n - 1\right] / (q\alpha_1) \tag{5.2}$$

In addition, the Exponential distribution characteristics "α_0" and "α_1" were selected as -15 and 0.01 respectively.

5.2.2. Optimization modeling by Lingo

The Exponential distribution's model written by Lingo language is given in Fig. 5.1 maximum availability with optimal n^* and s^* values can be obtained by the use of this model. As shown in Fig. 5.1, the model should contain two conditions, which are:

- Maximum $A(n,s)$; A and
- Unit-time cost $\leq$ B (150\$).

According to these qualifications the model can be written as a Flowsheet shown in Fig. 5.1.

```
Model:

MAX = A ;

C = Z / (n * s);

A = (n * s) / ((n * s) + F);


Z = 200000 + 8000 * (n - 1) + W;
W = (2000 * @EXP(-15 )*(T - 1)) / ( 0.01 * q ) ;


F = 150 +100*(n-1) + L;
L = (50 * @EXP(-15 )*(T - 1)) / ( 0.01 * q ) ;
T = (p + q * @EXP( 0.01 * s )) ^ n;

C < 150;

n > 0;
s > 0;
p = 0.7;
q = 1 - p;
@GIN(N);


end
```

Figure 5.1: The Exponential distribution model written by Lingo language for a fixed unit-time cost

The model shown in Fig. 5.1 is only an example of the program when the unit-time cost is taken as 150$. The results from the several run will be explained in the next section.

5.2.3. Results of Exponential distribution model

Results obtained from the model are shown in Figure 5.2, which shows the objective value, and local optimal s and n values. A typical LINGO result is given in Figure 5.2. It includes several values. Among them, the A, n and s are the most important values.

```
Local optimal solution found.
Objective value:                               0.6923656
Extended solver steps:                                 2
Total solver iterations:                             115


                    Variable           Value        Reduced Cost
                           A       0.6923656            0.000000
                           C        150.0000            0.000000
                           Z        238226.6            0.000000
                           n        5.000000        0.1698522E-03
                           s        317.6354            0.000000
                           F        705.6646            0.000000
                           W        6226.583            0.000000
                           T        30533.21            0.000000
                           Q       0.3000000            0.000000
                           L        155.6646            0.000000
                           P       0.7000000            0.000000

                         Row    Slack or Surplus      Dual Price
                           1       0.6923656            1.000000
                           2     -0.8573949E-08     -0.5007766E-02
                           3        0.000000            1.000000
                           4        0.000000        -0.3153153E-05
                           5        0.000000        -0.3153153E-05
                           6        0.000000        -0.3018367E-03
                           7        0.000000        -0.3018367E-03
                           8     -0.3674900E-03     -0.2181914E-05
                           9        0.000000         0.5007766E-02
                          10        5.000000            0.000000
                          11        317.6354            0.000000
                          12        0.000000          -0.8336759
                          13        0.000000          -0.7914363
```

Figure 5.2: The solution of the model by Lingo program

The *A*, *n* and *s* values are obtained as 0.69 (69%), 5 and 317.64 days, respectively. In this example, the objective value (max. availability) is 69 % for a fixed unit-time cost of 150$. The results shown in Figure 5.2 are obtained by solving the model by Lingo.

Results of the model for the Exponential distribution by using different unit-time costs are summarized in Table 5.1.

Table 5.1: *n*, *s* and *A* results in terms of cost by Exponential distribution model

C($)	*A*	*n*	*s (days)*
140	0.6142611	9	218.5835
145	0.6619363	6	286.3155
150	0.6923656	5	317.6354
155	0.715443	4	370.3245
160	0.7314558	3	463.3627
170	0.7678136	2	626.9631

From the Table 5.1, following results can be drawn;

- Availability increases from 0.61 to 0.77 with the increase in the unit-time cost from 140 to 170 (Fig. 5.3).
- The value of the unit-time cost increases when decreasing *n* value (Fig. 5.4).
- Overhaul interval value increase from 219 to about 627 as the unit-time cost is increased from 140 to 170 (Figure 5.5).

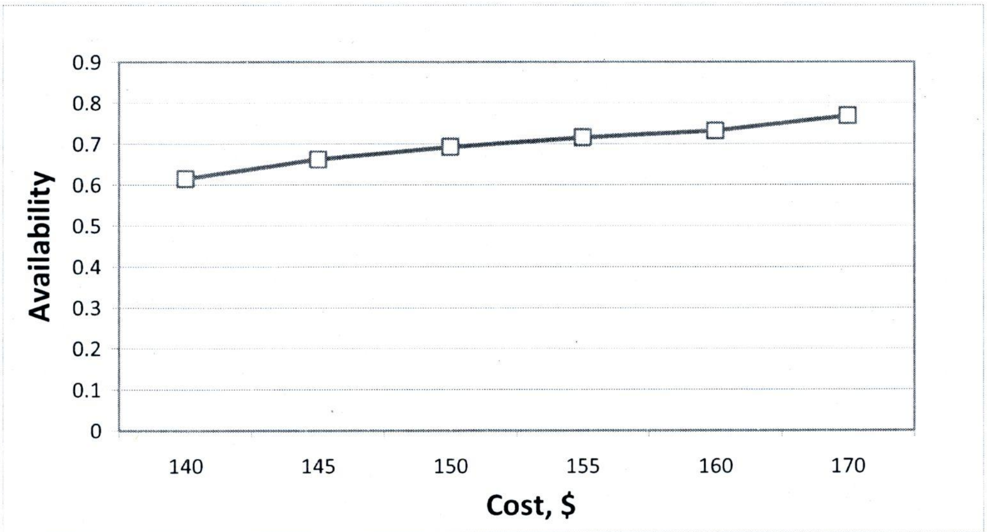

Figure 5.3: Availability versus cost

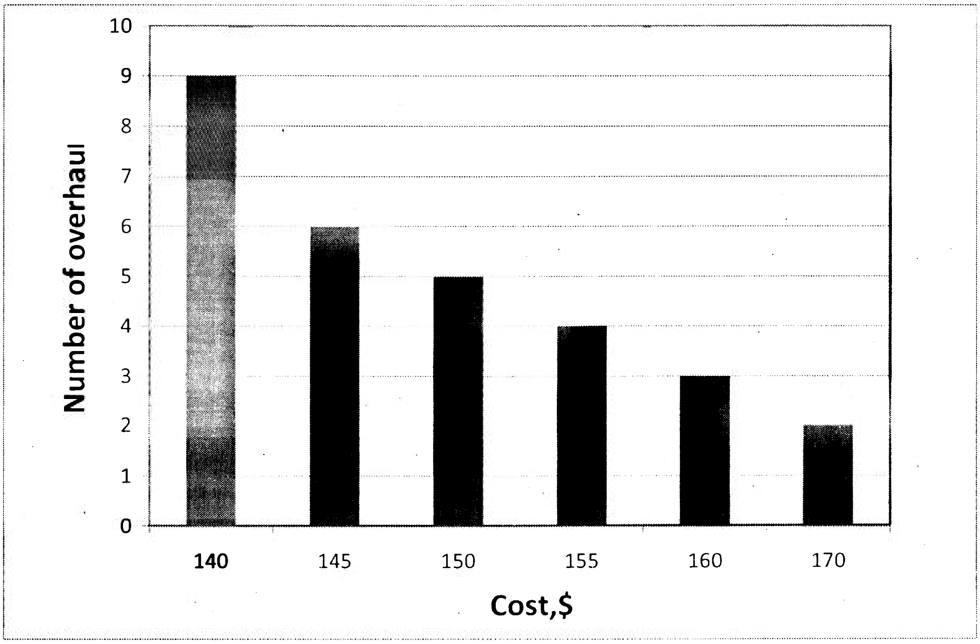

Figure 5.4: Number of overhaul versus cost

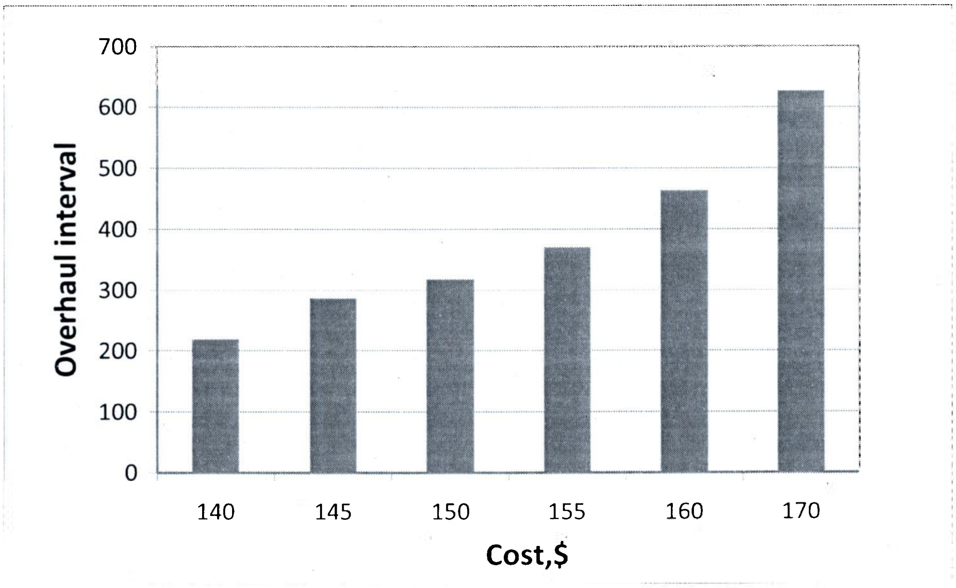

Figure 5.5: Overhaul interval versus cost

These results indicate that in every run the availability increases with an increase in the cost. Minimizing the cost, we obtained an optimal solution $n^* = 9$ and $s^* = 219$ days, with an availability of 61 %.

5.3. Weibull failure rate distribution

In this section, we use Weibull Failure Rate Distribution to maximize availability with optimal n^* and s^* values. The assumptions, the model for Weibull distribution writen by lingo language and its results are discussed in detail.

5.3.1. Assumptions

Suppose ;

$$v(t) = (\beta/\eta)(t/\eta)^{\beta-1} \quad \text{with} \quad \beta=2 \quad \text{and} \quad \eta = 100 .$$

For a Weibull failure rate;

$$\hat{H}(ns) = \left(s/\eta \right)^{\beta} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} \qquad (5.3)$$

$$\sum_{i=0}^{n} \binom{n}{i} p^{n-i} q^{i-1} i^{\beta} = n^2 q + np \qquad \text{for} \quad \beta = 2 \qquad (5.4)$$

According to these assumptions, the model can be written by Lingo language as shown in Fig. 5.5. The details of this model will be explained in the next section.

5.3.2. Optimization modeling by Lingo

Maximum availability with optimal n^* and s^* values can be obtained by the use of the model for Weibull distribution (Fig. 5.6). As shown in Fig. 5.5, the model should include two conditions. These are:

- Maximum $A(n,s)$; A and
- Unit-time cost $\leq$ B (290$).

```
max = A;

C = (200000 + 8000 *(n-1) + (K * B))/(n*s);

A = (n * s) / ((n * s) + F);

F = 150 +100*(n-1) + L;
L = (50 * B);

B = ((n ^ 2) * q) + (n * p);

K = 2000 * (( s / 100) ^ 2);


C < 290;
@GIN(N);
n > 1;
s > 0;
p = 0.7;
q = 1 - p;


end
```

Figure 5.6: The Weibull distribution model written by Lingo language for a fixed unit-time cost

5.3.3. Results of Weibull distribution model

Figure 5.7 indicates the model results for the objective value, optimal *s* and *n* values of Weibull distribution. The *A*, *C*, *n* and *s* values are obtained as 0.64 (64 %), 290 $, 5 and 385.63 days, respectively.

```
Local optimal solution found.
Objective value:                          0.6367429
Extended solver steps:                            3
Total solver iterations:                        166


                  Variable          Value        Reduced Cost
                         A      0.6367429            0.000000
                         C       290.0000            0.000000
                         n       5.000000        0.2876982E-01
                         K       29742.35            0.000000
                         B       11.00000            0.000000
                         s       385.6316            0.000000
                         F       1100.000            0.000000
                         L       550.0000            0.000000
                         Q      0.3000000            0.000000
                         P      0.7000000            0.000000

                       Row    Slack or Surplus      Dual Price
                         1      0.6367429            1.000000
                         2     -0.6145768E-07      -0.4686405E-02
                         3       0.000000            1.000000
                         4       0.000000          -0.2102740E-03
                         5       0.000000          -0.2102740E-03
                         6       0.000000          -0.8280276E-01
                         7     -0.2066591E-04      -0.2673559E-04
                         8       0.000000           0.4686405E-02
                         9       4.000000            0.000000
                        10       385.6316            0.000000
                        11       0.000000           -2.484083
                        12       0.000000           -2.070069
```

Figure 5.7: The solution of the model by Lingo program

According to results, the objective value (max. availability) is 0.64 for a unit time cost of 290$ and the optimal n^* and s^* values.

Table 5.2 represents the results of the model for Weibull distribution with different *C* values. From the Table, following results are drawn;

- Availability increases from 0.64 to 0.91 with an increase in the unit-time cost from 290 to 500 (Fig. 5. 8).

- n value indicates a decrease from 5 to 1 as the unit-time cost is increased from 290 to 500 (Fig. 5.9).

- s value indicates an increase from 386 to about 2,000 as the unit-time cost is increased from 290 to 500 (Fig. 5.10).

Table 5.2: n, s and A results in terms of cost by Weibull distribution model

C($)	***A***	***n***	***s (days)***
290	0.6367429	5	385.6316
300	0.7010750	4	486.6539
320	0.7699215	3	658.1139
350	0.8262245	2	903.3649
380	0.8523540	2	1,096.862
410	0.8640025	2	1,259.282
440	0.8811216	2	1,408.272
470	0.8907766	2	1,549.553
500	0.9090909	1	2,000

In order to have a better understanding of the optimal maintenance strategies, the results from the model are also given in Figs. 5.8-5.10. From the Figures, the optimal solution (max, availability and min. cost) can be chosen as 0.64 (64%) of availability and the unit-time cost as 290 $. In this case, the model gives the parameters as n*=5, and s*= 386 days.

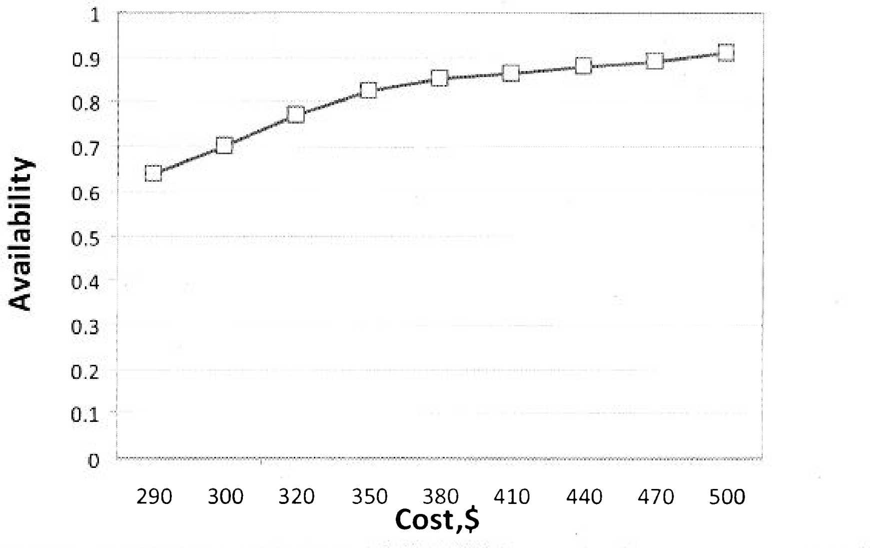

Figure 5.8: Availability versus cost

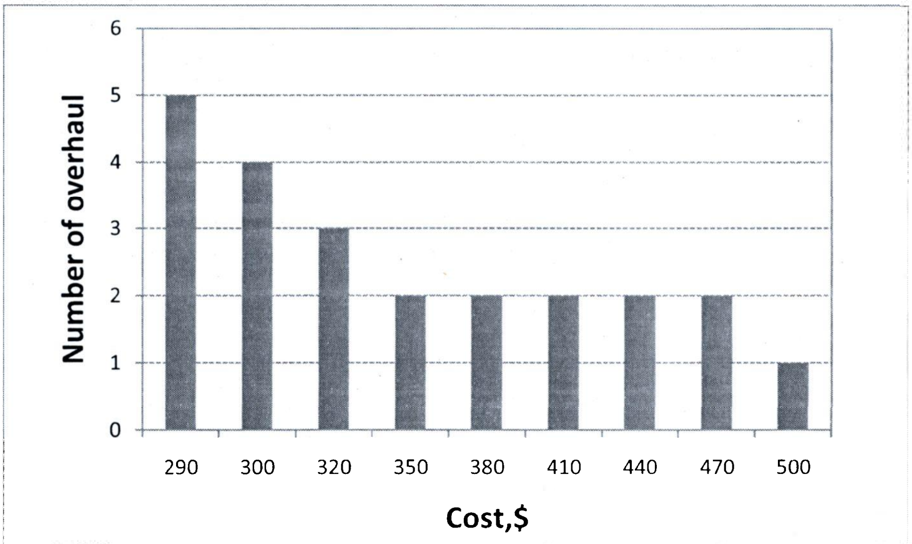

Figure 5.9: Number of overhaul versus cost

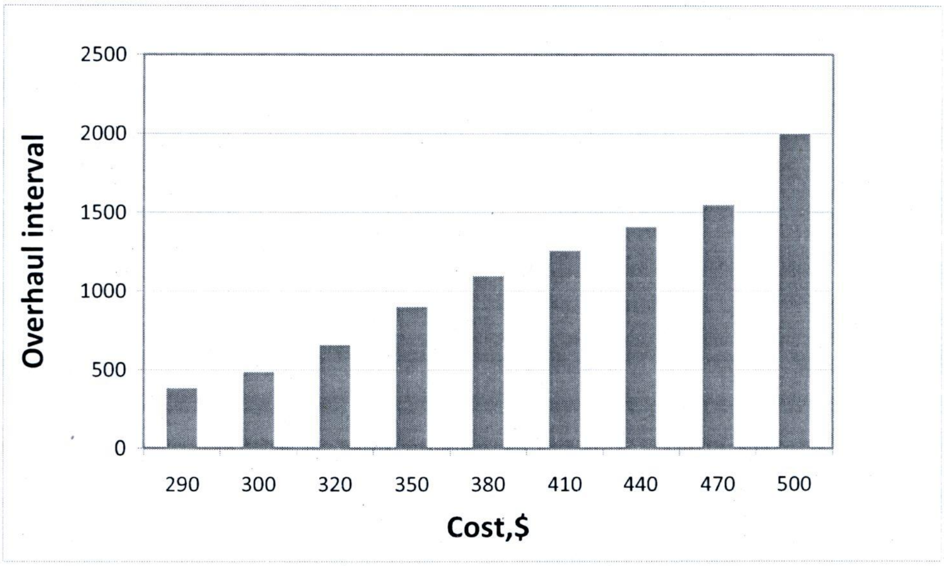

Figure 5.10: Overhaul interval versus cost

5.4. Results and comparison

In this chapter, we have presented a model which determines the acceptable value of availability by using the fixed unit-time cost constraint. A comparison of our findings to the results of Zhang-Jardine model is given in the following Tables.

Table 5.3: The comparison of two works for Exponential failure rate

Exponential failure rate	*Zhang and Jardine*	*This work*
P	0.7	0.7
n*	11	9
s* (day)	195.6	219
$f(n^*,s^*)$ ($)	138.7	140
Availability (%)	Not considering	61

Table 5.4: The comparison of two works for Weibull failure rate

Weibull failure rate	***Zhang and Jardine***	***This work***
p	0.7	0.7
n*	7	5
s* (day)	269	386
$f(n^*,s^*)$ ($)	282.3447	290
Availability (%)	Not considering	64

According to these Tables (5.3 and 5.4) we can conclude;

- Both results are similar but our modified model has an advantage of using the criterion of availability.
- From the Table 5.3; for a unit-cost of 140$, we obtained 5 times of overhaul in which each time of interval is 219 days with 61% of availability.
- From the Table 5.4; when the unit-cost is chosen as 290$, we obtained 5 times of overhaul in which each time of interval is 386 days with 64% of availability.
- In comparison with Zhang-Jardine model, our model has less overhaul with little bit more expensive unit-time cost, which result from the use of two criterions in same model.
- The optimal solution is taken when the point shows the minimal cost with maximum availability.
- We considered the values of 60% or up as maximum availability.

According to the results of two models, the Exponential and Weibull distribution's models; the values of availability (A) increase with the augmentation of the values of unit-time cost (Figs. 5.3 and 5.8). Although, the results obtained do not model a real-world problem they are theoretically suitable and acceptable. We think that our optimal maintenance models developed by lingo language offer a solution more or less integral to problem of maintenance action. The lingo language is also very useful to build up the optimal maintenance model.

CHAPTER VI

Conclusion and Discussion

Maintenance is always a considerable activity in industrial practice. It is very important to develop effective maintenance strategies for reducing maintenance costs and maximizing the availability in all manufacturing and product plants. The most suitable replacement maintenance can be used for maintenance strategies. It is essential to determine when and how this replacement maintenance will be applied. This can be achieved by appropriate models. Several models have been developed by literature. There are also detailed research projects related to this domain that are in progress.

In this study, the cost rate criterion model, which has been proposed by Zhang-Jardine, was improved by integrating the availability criterion. Thus, the model becomes more interesting than the cost rate criterion model by adding the second criterion. The developed model is a simple maintenance model for optimal replacement. The results obtained were compared to Zhang-Jardine model.

The model was used to estimate the optimal $n*$ (number of overhaul) and $s*$ (overhaul interval) for a minimal cost according to the constraint of a fixed availability or a maximum availability with given cost.

The following conclusions can be drawn from the extended model:

- For the minimum cost, the optimal $n*$ and $s*$ values for a fixed availability were obtained with an acceptable correlation with theory.

- For the maximum availability, the optimal n^* and s^* values for a fixed cost were also obtained with an acceptable correlation with theory.
- The values of n and s from our models are in consistent with the results of Zhang-Jardine model.

The advantages of the developed model:

- It contains two criteria,
- More effective comparing a model with one criterion,
- Results comply with each other,
- Results are similar to Zhang-Jardine model but with us model we known the criteria of availability ,
- Easy, reliable and offers quick operation by LINGO.

Future work and discussion;

- It is only limited to two distributions, the model can be developing with other distributions.
- The model was not applied on a real system, only compared with Zhang-Jardine model.

The present model can be calibrated and further evaluated by comparing with the real results. For this modeling study, some modifications may be applied on these parts. Studies along with the modification of the model may offer to model a real-world problem.

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