Sensitivity Analysis of Oscillating Hybrid Systems

by

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Abstract

Many models of physical systems oscillate periodically and exhibit both discrete-state and continuous-state dynamics. These systems are called oscillating hybrid systems and find applications in diverse areas of science and engineering, including robotics, power systems, systems biology, and so on. A useful tool that can provide valuable insights into the influence of parameters on the dynamic behavior of such systems is sensitivity analysis. A theory for sensitivity analysis with respect to the initial conditions and/or parameters of oscillating hybrid systems is developed and discussed. Boundary-value formulations are presented for initial conditions, period, period sensitivity and initial conditions for the sensitivities. A difference equation analysis of general homogeneous equations and parametric sensitivity equations with linear periodic piecewise continuous coefficients is presented. It is noted that the monodromy matrix for these systems is not a fundamental matrix evaluated after one period, but depends on one. A three part decomposition of the sensitivities is presented based on the analysis. These three parts classify the influence of the parameters on the period, amplitude and relative phase of the limit-cycles of hybrid systems, respectively. The theory developed is then applied to the computation of sensitivity information for some examples of oscillating hybrid systems using existing numerical techniques and methods. The relevant information given by the sensitivity trajectory and its parts can be used in algorithms for different applications such as parameter estimation, control system design, stability analysis and dynamic optimization.

Thesis Supervisor: Paul I. Barton
Title: Lammot du Pont Professor of Chemical Engineering

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Chapter 1

Introduction

Sensitivity analysis is a useful tool for analyzing a dynamic system. It can be used to predict the change in the behavior of the system with an infinitesimal perturbation in parameters appearing in the model for the system/and or initial conditions. This information can be utilized in various engineering and scientific applications. For example model reduction, stability analysis (of for example power systems [26]), control system design, parameter estimation, experimental design, process sensitivity studies and numerical optimal control.

There exist higher-order sensitivities in other studies, but the sensitivities which are discussed in this thesis are first-order sensitivities, defined as:

$$\frac{\partial x}{\partial \alpha_j}(i, t, \alpha) = \lim_{\epsilon \to 0} \frac{x(i, t, \alpha + \epsilon \mathbf{e}_j) - x(i, t, \alpha)}{\epsilon},$$

where $t \in [\sigma_i, \tau_i]$, $\mathbf{e}_j$ is the jth unit vector, and $x(i, t, \alpha)$ is a scalar or vector of state variables which changes in time according to the equations of the hybrid system:

$$\frac{dx}{dt}(i, t, \alpha) = f(m_i, x(i, t, \alpha), \alpha),$$

where m_i is the mode of the hybrid system in the ith epoch, which defines the right-hand side of the differential equations, and α is a scalar or vector-valued quantity which can be either a model parameter, or initial conditions of hybrid system or a combination of both as mentioned earlier.

The concept of sensitivity analysis is well understood for continuous dynamic systems [14][45], continuous oscillating systems [46, 47] and hybrid systems [15][6]. This study focuses on sensitivity analysis of limit-cycle oscillators in hybrid systems. In particular, the parameter sensitivities are calculated for the state variables, period of the system, amplitude of the state variables and different phases of the system.

1.1 Oscillating Hybrid Systems

Hybrid systems are those systems that exhibit both discrete-state and continuous-state dynamics. These systems are characterized by interactions between discrete and continuous states which are significant to an extent that they cannot be decoupled and analyzed simultaneously. These are modeled in the past often by partitioning into discrete and continuous parts. There a diverse range of applications where hybrid systems are common. Robotics [35], manufacturing [33], air traffic control [44], power systems [20], safety interlock systems and embedded systems are a few examples. The sensitivity analysis for such systems has been established in [15] by extending the sensitivity analysis for discontinuous systems by [37]. A hybrid system can be described by a collection of different systems of differential equations. In this thesis we limit ourselves to the study of differential-algebraic equations (DAEs) and ordinary differential equations (ODEs).

Many hybrid systems exhibit periodic behavior. An oscillating hybrid system has a mix of discrete and continuous state variables which repeat values as time progresses. There are different classes of oscillating dynamical systems: limit-cycle oscillators (LCOs), non-limit-cycle oscillators (NLCOs) and intermediate type oscillators. A comprehensive guide for the sensitivity analysis of these three different classes of oscillating system is presented in [46, 47]. Oscillating hybrid systems also have these three different classes but this thesis sticks to sensitivity analysis of LCOs in hybrid systems. Such systems can be found in cell cycles [9] and robotic motions which are naturally periodic. Stability analysis of limit cycles in hybrid systems has been presented in [19]. A model for a compass gait biped robot [17, 18] is analyzed in [19].

Limit cycles have a closed and isolated periodic orbit [40]. This orbit is solely determined by the parameters of the system. The limit cycles can be stable (attracting) or unstable (repelling). This work focuses on stable limit cycles in hybrid systems. The shape and position of the limit cycle is independent of the initial conditions as long as the initial conditions lie within the region of attraction. It is shown later how the initial conditions for a LCO in hybrid systems are calculated by solving a boundary-value problem.

1.2 Motivational Example of Oscillating Hybrid Systems: Raibert's Hopper

This is an application of oscillating hybrid systems in robotics described in [4]. It is a control problem involving dynamical behavior and stability of a hopping robot. A simplified model for the machine built by Raibert [36] is described in [24], which is used here. Raibert's hopper consists of two main components: a body which has a control mechanism and a compressible leg as shown in the Figure 1-1. The leg is modeled and constructed as a pneumatic cylinder which has gas whose pressure is subject to feedback control. The robot is dropped from a short distance above the surface and after some transient time, it hops periodically in the vertical direction for some parameter values.

The hopping robot system dynamics are described by four phases: flight, compression, thrust, and decompression. The robot is dropped from a height $x_{1,0}$ and it is in the flight phase until the bottom surface of the cylinder comes in contact with the surface below. During this phase, the gas in the cylinder is at pressure p_{init} and the leg is at its fully extended length l. As soon as the leg makes contact with the surface, the gas inside the cylinder compresses and the compression phase begins. The compression is modeled using a nonlinear spring with a spring constant η and mechanical damping with a coefficient of friction γ. At the point of maximal leg compression, or minimum value of x_1, the thrust phase begins and it lasts for a fixed period of time δ. During the thrust phase, gas at p_{th} is admitted into the cylinder exerting a constant force τ to move the body upwards. The thrust phase starts at t_b and ends at $t_b + \delta$ with a body height of $x_{1,et}$. At the end of thrust phase, the gas in cylinder starts decompressing. The decompression phase is modeled as nonlinear spring with spring constant $\tau x_{1,et}$. The decompression phase ends when the leg reaches its fully extended length l and it lifts from the surface going to the flight phase. Therefore, the robot hops periodically going into four phases in each cycle. The four phases mentioned here are later referred to as *modes* in the text for a general hybrid system. The four phases can be noticed in the periodic orbit in the phase portrait plotted in Figure 1-2(a). Figure 1-2(b) displays the height of the body as a function of time. Each hop is represented by a single cycle in the figure and the maximum height of the hop is constant. The limit cycle in this hybrid system is attracting and is approached asymptotically from any initial conditions within the region of attraction. The system of ODEs for the four phases are given as:

$$\text{Flight}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g \end{cases} \text{when}\,(x_1 > l)\,,$$

$$\text{Compression}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{\eta}{x_1} - \gamma x_2 - g \end{cases} \text{when}\,(x_1 < l) \wedge (x_2 < 0)\,,$$

$$\text{Thrust}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \tau - \gamma x_2 - g \end{cases} \text{when}\,(x_1 < l) \wedge (x_2 > 0) \wedge (t_b < t < t_b + \delta)\,,$$

$$\text{Decompression}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{\tau x_{1,et}}{x_1} - \gamma x_2 - g \end{cases} \text{when}\,(x_1 < l) \wedge (x_2 > 0) \wedge (t > t_b + \delta)\,.$$

This system has six parameters: l, g, τ, δ, η and γ. The sensitivity analysis with respect to these parameters is useful in control design of such systems.

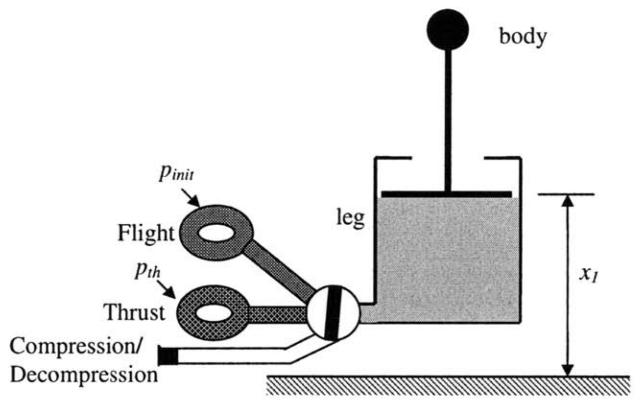
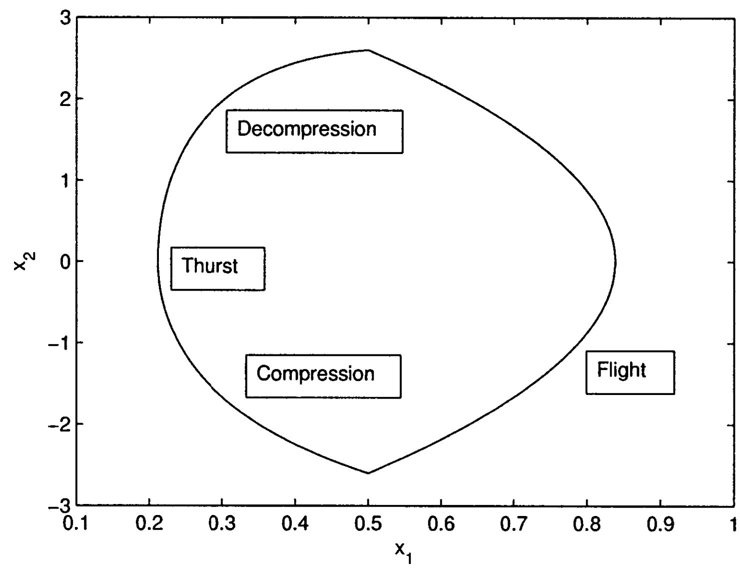


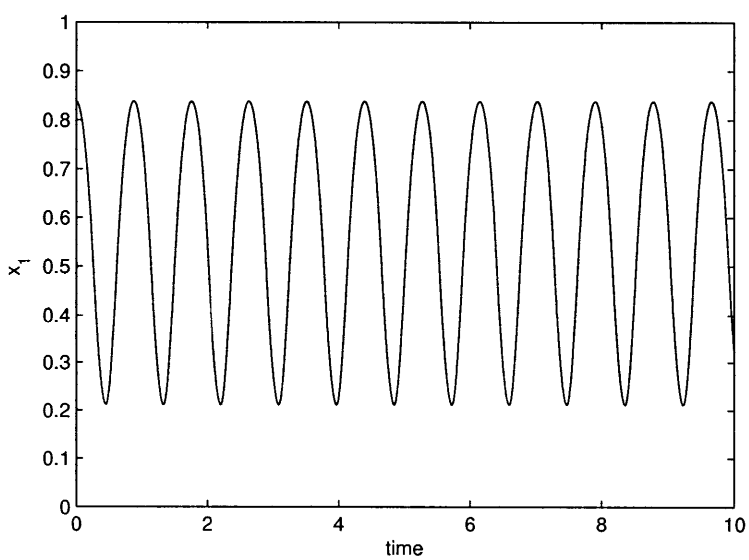
Figure 1-1: Simplified model of Raibert's hopper [24].

1.3 Organization

This thesis is organized as follows. Theoretical background which is used in the thesis to develop the theory of sensitivity analysis of limit cycles of hybrid systems is presented in Chapter 2. A description of ODE systems, hybrid systems and LCOs along with general sensitivity theory of these systems is presented. The theory of sensitivity analysis of limit cycles of hybrid systems is developed in Chapter 3. It is shown how a fundamental matrix evaluated after one period is different from the monodromy matrix for oscillating hybrid systems, in contrast to regular LCOs. Using difference-equation analysis, the properties of the initial-condition sensitivities are proved. A similar analysis is done for the parametric sensitivities to obtain a general solution for sensitivity equations for the limit cycles of hybrid systems. This analysis shows that the sensitivities can be decomposed into an unbounded and a periodic part much like regular LCOs. The periodic part can further be decomposed into periodic effects of shape and phase change in the limit cycles of hybrid systems. Numerical implementation of the developed theory is discussed in Chapter 4. Chapter 5 discusses some of the applications of the analysis to simple oscillating hybrid systems. The work is concluded in Chapter 6 with recommendations for future work.



(a)



(b)

Figure 1-2: Dynamics of Raibert's hopping robot: (a) shows the phase portrait as the robot hops and (b) shows the height of the body x_1 plotted as a function of time.

Chapter 2

Theoretical Background

In this chapter, theoretical background on ODEs, ODE-embedded hybrid systems and LCOs is presented. After describing these systems, the theory of sensitivity analysis (developed earlier) of such systems is also presented briefly.

2.1 ODEs and Linear Systems Theory

Many engineering and scientific problems are described by ODEs and DAEs. Since the focus of this thesis is on ODEs, some theory for systems which are represented by a system of ODEs is presented in this section. Consider such a general system with parameters:

$$\frac{d\mathbf{x}}{dt}(\mathbf{p},t) = \mathbf{F}(\mathbf{x}(\mathbf{p},t),\mathbf{p},t),\ \forall t,\ \mathbf{x}(\mathbf{p},t_0) = \mathbf{x}_0(\mathbf{p}), \tag{2.1}$$

where $\mathbf{x}(\mathbf{p},t) \in X \subset \mathbb{R}^{n_x}$, $\mathbf{p} \in P \subset \mathbb{R}^{n_p}$, t_0 is the initial time and $\mathbf{F} : X \times P \times \mathbb{R} \to \mathbb{R}^{n_x}$ is a vector containing real continuous functions. Here n_x and n_p are the dimension and number of parameters in the ODE system, respectively. Existence and uniqueness of the solutions of the above initial-value problem is discussed in [10]. According to the Picard-Lindelof theorem, if $\mathbf{F}$ is a vector of real continuous functions and satisfies the Lipschitz condition, then there exists a unique solution of Equation (2.1).

2.1.1 Linear Homogeneous Systems

A particularly interesting system is the linear homogeneous system given by equations:

$$\frac{d\mathbf{x}}{dt}(\mathbf{p},t) = \mathbf{A}(\mathbf{p},t)\mathbf{x}(\mathbf{p},t),\ \forall t,\ \mathbf{x}(\mathbf{p},t_0) = \mathbf{x}_0(\mathbf{p}), \tag{2.2}$$

where $\mathbf{x}(\mathbf{p},t) \in X \subset \mathbb{R}^{n_x}$, $\mathbf{p} \in P \subset \mathbb{R}^{n_p}$, and the elements of $\mathbf{A}(\mathbf{p},t)$ are continuous functions of t. For such a linear system where the elements of $\mathbf{A}(\mathbf{p},t)$ are continuous on $[t_0, t_f]$, there is one and only one solution $\mathbf{x}(\mathbf{p},t;\mathbf{x}_0,t_0)$ of the Equation (2.2) passing through state $\mathbf{x}_0$ at time t_0 [10]. Let $\mathbf{A}(\mathbf{p},t)$ be an integrable function of t such that $\|\mathbf{A}(\mathbf{p},t)\| < \alpha(t)$ and $\int_{t_0}^{t_f} \alpha(t)\ dt < +\infty$, then the unique solution satisfies the following equation [49]:

$$\mathbf{x}(\mathbf{p},t;\mathbf{x}_0,t_0) = \mathbf{x}_0(\mathbf{p}) + \int_{t_0}^{t} \mathbf{A}(\mathbf{p},\tau)\mathbf{x}(\mathbf{p},\tau;\mathbf{x}_0,t_0)\ d\tau.$$

Let $\mathbf{\Phi}(\mathbf{p},t,t_0)$ be the $n_x \times n_x$ matrix which is the solution of the equations:

$$\frac{d\mathbf{\Phi}}{dt}(\mathbf{p},t,t_0) = \mathbf{A}(\mathbf{p},t)\mathbf{\Phi}(\mathbf{p},t,t_0),\ \forall t,\ \mathbf{\Phi}(\mathbf{p},t_0,t_0) = \mathbf{I}. \tag{2.3}$$

Then the solution of Equation (2.2) is given by:

$$\mathbf{x}(\mathbf{p},t;\mathbf{x}_0,t_0) = \mathbf{\Phi}(\mathbf{p},t,t_0)\mathbf{x}_0(\mathbf{p}),\ \forall t,\ \forall \mathbf{x}_0(\mathbf{p}).$$

The matrix $\mathbf{\Phi}(\mathbf{p},t,t_0)$ is called the *state transition matrix* or *principal fundamental matrix*. A necessary and sufficient condition that a solution matrix $\mathbf{\Phi}$ of Equation (2.3) be a *fundamental matrix* is that $\det \mathbf{\Phi}(\mathbf{p},t,t_0) \neq 0,\ \forall t$ [10].

2.1.2 Properties of $\mathbf{\Phi}(\mathbf{p},t,t_0)$

The following are some properties of the state transition matrix $\mathbf{\Phi}(\mathbf{p},t,t_0)$:

1. The state transition matrix has the group property:

$$\mathbf{\Phi}(\mathbf{p},t_1,t_2)\mathbf{\Phi}(\mathbf{p},t_2,t_3) = \mathbf{\Phi}(\mathbf{p},t_1,t_3),\ \forall t_1,t_2,t_3.$$

2. An immediate consequence of Property 1 is:

$$\mathbf{\Phi}^{-1}(\mathbf{p}, t_1, t_2) = \mathbf{\Phi}(\mathbf{p}, t_2, t_1).$$

3. If, for all t, $\int_{t_0}^{t_f} \mathbf{A}(\tau)\, d\tau$ and $\mathbf{A}(t)$ commute, then:

$$\mathbf{\Phi}(\mathbf{p}, t, t_0) = \exp\left[\int_{t_0}^{t_f} \mathbf{A}(\mathbf{p}, \tau)\, d\tau\right].$$

4. Let $\mathbf{\Phi}(\mathbf{p}, t, t_0)$ be the state transition matrix, then:

$$\det \mathbf{\Phi}(\mathbf{p}, t, t_0) = \exp\left[\int_{t_0}^{t_f} \mathrm{Tr}\mathbf{A}(\mathbf{p}, \tau)\, d\tau\right].$$

2.1.3 Inhomogeneous Linear Systems

The inhomogeneous linear system is given by the equations:

$$\frac{d\mathbf{x}}{dt}(\mathbf{p}, t) = \mathbf{A}(\mathbf{p}, t)\mathbf{x}(\mathbf{p}, t) + \mathbf{b}(\mathbf{p}, t),\ \forall t,\ \mathbf{x}(\mathbf{p}, t_0) = \mathbf{x}_0(\mathbf{p}), \tag{2.4}$$

where $\mathbf{x}(\mathbf{p}, t) \in X \subset \mathbb{R}^{n_x}$, $\mathbf{p} \in P \subset \mathbb{R}^{n_p}$, and the elements of the $\mathbf{A}(\mathbf{p}, t)$ matrix and $\mathbf{b}(\mathbf{p}, t)$ vector are continuous functions of t. The solution of Equation (2.4) that goes through state $\mathbf{x}_0(\mathbf{p})$ at t_0, is given by following equation [10]:

$$\mathbf{x}(\mathbf{p}, t; \mathbf{x}_0, t_0) = \mathbf{\Phi}(\mathbf{p}, t, t_0)\mathbf{x}_0(\mathbf{p}) + \int_{t_0}^{t} \mathbf{\Phi}(\mathbf{p}, t, \tau)\mathbf{b}(\mathbf{p}, \tau)\, d\tau.$$

It can be noted that $\mathbf{\Phi}(t, \tau) = \exp\left[\mathbf{A}(t - \tau)\right]$ in the case of time-invariant systems.

2.2 ODE-embedded Multi-stage Hybrid systems

Hybrid systems exhibit both the continuous state and discrete state dynamics which cannot be decoupled and must be analyzed simultaneously. These systems are conveniently modeled by partitioning into discrete and continuous states. In general, there is a continuous or discrete time formulation for modeling of hybrid systems. In continuous time formulation, there is a variety of embedded differential equation subsystems including ODEs, DAEs, and PDEs. The focus of this thesis is on ODEs embedded in oscillating hybrid system and hence a formulation is presented for them.

A modification of modeling framework presented in [27] is used in Chapter 3. That framework was based on *hybrid automaton* representation for hybrid systems which is useful for mathematical and numerical analysis.

The evolution of a hybrid system through time consists of starting at an initial time with initial conditions for the discrete and continuous state variables of the hybrid system. The continuous state variables evolve according to differential equations which depend on the discrete state of the system. At some point of time, a change or transition may occur in the system and the continuous state variables then evolve according to different differential equations corresponding to the new discrete state described by a new value for the discrete state variable. After some more time, again a transition occurs and the cycle is repeated indefinitely.

The time axis is called the time horizon, which is further divided into time intervals called *epochs.* The discrete and continuous subsystems interact via discrete changes or *transitions* at points in time called *events.* Each epoch is a closed time interval $[\sigma_i, \tau_i]$, with $\sigma_{i+1} = \tau_i$ and $\tau_i \leqslant \tau_{i+1}$ for all $i \in \mathcal{E}$ where $\mathcal{E}$ is a finite set of epochs, with initial time σ_1. In the ith epoch, the system evolves continuously by allowing time to pass if $\sigma_i < \tau_i$. The evolution of the hybrid system stops at final time $t_f = \tau_{n_e}$, where n_e is the total number of epochs in the time horizon.

The hybrid system can be viewed as a directed graph whose vertices are the continuous state subsystems, called *modes*, and edges are the possible transitions. A hybrid system consists of the following elements:

1. A finite set index $\mathcal{M}$ for the modes, $\mathcal{M} = \{1, 2, 3, ..., n_m\}$ where n_m is the total number of modes in the hybrid system. A sequence of modes corresponding to the time evolution of hybrid system, is called the hybrid mode trajectory $T_\mu = \{m_i\}_{i=1}^{n_e}$, $m_i \in \mathcal{M}$ where m_i is the mode in the ith epoch. For the class of problems discussed here, the transitions which occur are known a priori, and

hence the evolution follows a *fixed mode sequence* for all the parameter values.

2. A set of variables $\{\mathbf{x}, \mathbf{p}, t\}$, where $\mathbf{x}(i, \mathbf{p}, t) \in X \subset \mathbb{R}^{n_x}$ are the state variables in an ODE-embedded hybrid system. The time-invariant parameters $\mathbf{p} \in P \subset \mathbb{R}^{n_p}$, and time $t \in \mathbb{R}$, are the independent variables. Also, for $t \in (\sigma_i, \tau_i]$, $\mathbf{x}(i, \mathbf{p}, t)$ evolves according to the differential equations in mode m_i.

3. A set of equations for each mode $m_i \in \mathcal{M}$. The state of the hybrid system evolves according to the dynamics of the system, which are represented by ODEs given by:
$$\frac{d\mathbf{x}}{dt}(i, \mathbf{p}, t) = \mathbf{F}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p}),$$
where $\mathbf{F} : \mathcal{M} \times X \times P \to \mathbb{R}^{n_x}$. A set of initial conditions is given for the first epoch,
$$\mathbf{x}(1, \mathbf{p}, \sigma_1) = \mathbf{x}_0(\mathbf{p}).$$

4. A set of transitions in a fixed mode sequence from predecessor mode m_i to successor mode m_{i+1}. The transitions are described by:

   (a) Transition conditions $\mathcal{L}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p})$, where $\mathcal{L} : \mathcal{M} \times X \times \mathbb{R}^{n_p} \to \mathbb{R}$, determining the transition times at which switching from mode m_i to m_{i+1} occurs. At the start of the ith epoch, it is assumed that the transition condition satisfies:
   $$\mathcal{L}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p}) > 0.$$
   The earliest time at which the transition condition crosses zero defines the transition time $\tau_i(\mathbf{p})$.

   (b) Each transition has a transition function $\mathcal{T}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p})$ which relates the final condition in mode m_i to the initial condition in the next mode m_{i+1} at the transition time $t = \tau_i$:
   $$\mathbf{x}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) = \mathcal{T}(m_i, \mathbf{x}(i, \mathbf{p}, \tau_i(\mathbf{p})), \mathbf{p}), \forall i = 1, ..., n_e - 1.$$

2.3 Limit-Cycle Oscillators

Oscillating dynamical systems described by ODEs as in Equation (2.1) can be classified into three different classes: LCOs, NLCOs and intermediate-type oscillators. A short discussion on the three is given in [47].

This work is focused on LCOs, in which the periodic orbit is isolated and closed [40]. Unlike the other two classes, the period and location of the limit-cycle are independent of the initial conditions and are solely determined by the parameters of the system. Limit cycles can be further classified into two types: stable and unstable. Stable limit cycles, which are the focus of the present work, are approached from any initial condition as $t \to +\infty$ within the region of attraction. On the other hand, unstable limit cycles are repelling and are approached from any initial condition as $t \to -\infty$. The stability of limit cycles is determined by the characteristic multipliers. An explanation for that, which is known as Floquet theory, is given later in this chapter in the section on Floquet theory. To analyze a limit-cycle trajectory, initial conditions on the limit cycle must be identified. These initial conditions depend on the parameters of the system.

Consider a LCO with parameters $\mathbf{p}$ modeled by nonlinear ODEs. A boundary value problem (BVP) is formulated for $\mathbf{x}_0(\mathbf{p})$ and $T(\mathbf{p})$ subject to:

$$\mathbf{x}\left(\mathbf{p}, T(\mathbf{p}); \mathbf{x}_0(\mathbf{p}), 0\right) - \mathbf{x}_0(\mathbf{p}) = \mathbf{0}, \tag{2.5}$$

$$\dot{x}_i(t_0, \mathbf{p}; \mathbf{x}_0(\mathbf{p}), 0) = 0, \tag{2.6}$$

for some arbitrary $i \in \{1, \ldots, n_x\}$ with $\mathbf{x}\left(\mathbf{p}, t; \mathbf{x}_0(\mathbf{p}), 0\right) \in X \subset \mathbb{R}^{n_x}$ is given by the solution of:

$$\frac{d\mathbf{x}}{dt}(\mathbf{p}, t) = \mathbf{F}(\mathbf{x}(\mathbf{p}, t), \mathbf{p}), \ \forall t, \ \mathbf{x}(\mathbf{p}, 0) = \mathbf{x}_0(\mathbf{p}), \tag{2.7}$$

Solving this BVP yields initial conditions $\mathbf{x}_0(\mathbf{p})$ corresponding to a point on a limit cycle, defined by Eq. (2.6), and the period $T(\mathbf{p})$.

Equation (2.6) fixes zero time for the BVP to a point on the limit cycle and is known as a *phase locking condition* (PLC) [47]. A valid PLC is required to find the solution of the BVP. A PLC is valid if it defines an isolated point on the periodic orbit and it yields a solution that is unique and smooth in a neighborhood of $\mathbf{p}$.

2.4 Sensitivity Analysis: ODE Systems, ODE-embedded Multi-stage Hybrid Systems, LCOs

Sensitivity analysis is the study of the influence of infinitesimal perturbations in parameters and/or initial conditions on the state of a system. It plays an important role in design, modeling, parameter estimation and optimization of systems. This section gives some theoretical background on sensitivity analysis of ODE systems, ODE-embedded multi-stage hybrid systems (with fixed hybrid mode trajectory) and oscillating dynamical systems (limit cycles).

2.4.1 Sensitivity Analysis of ODE Systems

The theory for sensitivity analysis of systems with continuous dynamics is well established [14, 8]. Consider the system defined by Equation (2.1). Sensitivity analysis entails finding the partial derivative of the solution with respect to the parameters $\mathbf{p}$. The sensitivity trajectory given by the matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t) \in \mathbb{R}^{n_x \times n_p}$ is a continuous function of time and satisfies the following inhomogeneous linear system of differential equations:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)(\mathbf{p}, t) = \mathbf{A}(\mathbf{p}, t)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t) + \mathbf{B}(\mathbf{p}, t), \ \forall t, \ \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t_0) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}), \tag{2.8}$$

where $\mathbf{A}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}(\mathbf{p}, t), \mathbf{p}, t)$, and $\mathbf{B}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{p}}(\mathbf{x}(\mathbf{p}, t), \mathbf{p}, t)$ with elements which are continuous functions of t. Equation (2.8) is an inhomogeneous linear system of equations and the solution of such systems was given in the Section 2.1.3, which can be written in terms of the fundamental matrix $\mathbf{\Phi}(\mathbf{p}, t, t_0)$ for the homogeneous system.

In [28], the simultaneous corrector method was proposed to compute the parametric sensitivity which reduced computational cost compared to earlier efforts. In [13], the staggered corrector method was developed and demonstrated for solving stiff ODEs and sensitivities. This algorithm was shown to have a number of advantages over that of the simultaneous corrector algorithm. The sensitivities are calculated by integrating the sensitivity Equation (2.8) using the staggered corrector method given in [13], which solves the sensitivity system after completing the corrector iteration for the state variables.

2.4.2 Sensitivity Analysis of ODE-embedded Multi-stage Hybrid Systems

The theory for sensitivity analysis of hybrid systems has been recently developed [15] which was an extension of work done earlier in [38].

Existence and uniqueness theorems for sensitivity functions of hybrid systems are given in [15] and used later in Chapter 3. Let us consider an ODE-embedded hybrid system with fixed hybrid mode trajectory $T_\mu = \{m_1, m_2, m_3, ..., m_{n_e}\}$ as described in Section 2.2. Suppose that for $t \in (\sigma_i, \tau_i]$, $i \in \mathcal{E}$, the partial derivatives $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p})$ and $\frac{\partial \mathbf{F}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p})$ exist and are continuous in a neighborhood of the solution $\mathbf{x}(i, \mathbf{p}, t)$. In [15], it has been proved that the sensitivity trajectories matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t)$, $\forall i \in \mathcal{E}$ exist and satisfy the differential equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)(i, \mathbf{p}, t) = \mathbf{A}(\mathbf{p}, t)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) + \mathbf{B}(\mathbf{p}, t), \forall t \in (\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \\ \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}), \end{aligned} \tag{2.9}$$

where $\mathbf{A}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p}), \forall t \in (\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E}$ and $\mathbf{B}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p})$, $\forall t \in (\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})]$, $\forall i \in \mathcal{E}$ with elements continuous functions of t.

Now, consider an event between the ith and $i+1$th epochs where state continuity is employed as the transition function:

$$\mathbf{x}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) = \mathbf{x}(i, \mathbf{p}, \tau_i(\mathbf{p})), \forall i = 1, 2, ..., n_e - 1.$$

At $\sigma_{i+1}(\mathbf{p}) = \tau_i(\mathbf{p})$, the relationship between the final values of the sensitivities in epoch i and the initial values of the sensitivities in epoch $i+1$ is determined by differentiation of the transition function:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \tau_i(\mathbf{p})) + (\dot{\mathbf{x}}(i, \mathbf{p}, \tau_i(\mathbf{p})) - \dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p}))) \frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p}),$$

$$\begin{aligned} &\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) = \\ &\quad \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \tau_i(\mathbf{p})) + (\mathbf{F}(m_i, \mathbf{x}(i, \mathbf{p}, \tau_i(\mathbf{p})), \mathbf{p}) - \mathbf{F}(m_{i+1}, \mathbf{x}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p})) \\ &\qquad\qquad \times \frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p}). \end{aligned}$$

The above equation reveals the qualitative behavior of the sensitivities at an event. The equation indicates that the sensitivities will "jump" at an event when two conditions are both satisfied: (a) the vector field is discontinuous and (b) the event time is sensitive to the parameters $\mathbf{p}$. The event-time sensitivity is calculated by using the fact that the transition condition is zero at the event. The transition condition triggering the event between the ith and $i+1$th epochs at the event is:

$$\mathcal{L}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p}) = 0. \tag{2.10}$$

Differentiating Equation (2.10) with respect to the parameters $\mathbf{p}$:

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p}) \left(\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) \frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) \right) \\ &\qquad\qquad + \frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p}) = \mathbf{0}. \end{aligned} \tag{2.11}$$

Equation (2.11) is linear and can be solved for a unique $\frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p})$, $\forall i = 1, 2, ..., n_e - 1$, provided that:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p})\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) \neq 0.$$

The expression obtained for $\frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p})$ is:

$$\frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p}) = -\left(\frac{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p})\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) + \frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p})}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p})\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p}))}\right). \tag{2.12}$$

The above equations can be solved for the evolution of the states and sensitivities in a mode by using the staggered corrector method given in [13]. Correct location of the state events is done using the state event location algorithm given in [32]. The above algorithms have been implemented in DSL48SE.

2.4.3 Sensitivity Analysis of Limit-Cycle Oscillators

The sensitivity analysis for LCOs and NLCOs was formulated separately and discussed in [47]. The matrix of parametric sensitivities $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t) \in \mathbb{R}^{n_x \times n_p}$ is obtained by taking the partial derivatives of a LCO described by the ODE system in the Equations (2.7) and satisfy the following differential equations:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)(\mathbf{p}, t) = \mathbf{A}(\mathbf{p}, t)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t) + \mathbf{B}(\mathbf{p}, t),\ \forall t,\ \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, 0) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}), \tag{2.13}$$

where $\mathbf{A}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}(\mathbf{p}, t), \mathbf{p})$, and $\mathbf{B}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{p}}(\mathbf{x}(\mathbf{p}, t), \mathbf{p})$ have elements which are continuous and periodic in time t with period $T(\mathbf{p})$ because $\mathbf{x}(\mathbf{p}, t)$ is periodic. Since the initial conditions on the limit cycle depend on the parameters of LCOs, the sensitivity initial conditions $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ cannot be set to zero, as is usually done for dynamic systems when the initial conditions are independent of parameters. To determine the correct initial conditions $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$, a BVP is formulated by differentiating Equations (2.5) and (2.6) with respect to the parameters $\mathbf{p}$ [47]:

$$\begin{aligned}\frac{d\mathbf{x}}{dt}(\mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, T(\mathbf{p}))\right)_{\mathbf{x}_0 = const.} + \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, T(\mathbf{p}))\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ - \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, 0)\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) = \mathbf{0},\end{aligned} \tag{2.14}$$

$$\frac{\partial F_i}{\partial \mathbf{x}}(\mathbf{x}_0(\mathbf{p}), \mathbf{p}) \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial F_i}{\partial \mathbf{p}}(\mathbf{x}_0(\mathbf{p}), \mathbf{p}) = \mathbf{0}. \tag{2.15}$$

The BVP in Equations (2.14) and (2.15) can be solved to obtain sensitivity initial conditions $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ and period sensitivities $\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})$.

Before proceeding to the solution of Equation (2.13), it is interesting to look at the solution of the homogeneous linear system which is satisfied by the partial derivatives of the solution with respect to the initial conditions $\mathbf{x}_0$, resulting in the matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t) \in \mathbb{R}^{n_x \times n_x}$.

Linear Homogeneous System with Periodic Coefficients

The initial-condition sensitivity matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t)$ is the solution of the linear system with periodically time-varying coefficients:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)(\mathbf{p}, t) = \mathbf{A}(\mathbf{p}, t)\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t), \forall t, \ \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, 0) = \mathbf{I}. \tag{2.16}$$

If $\mathbf{\Phi}(\mathbf{p}, t, 0)$ is the state transition matrix of Equation (2.16), then its solution will be given [10]:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t; \mathbf{I}, 0) = \mathbf{\Phi}(\mathbf{p}, t, 0)\mathbf{I} = \mathbf{\Phi}(\mathbf{p}, t, 0), \forall t. \tag{2.17}$$

Hence the initial-condition sensitivity matrix is actually the state transition matrix for the homogeneous linear system and follows all the properties of the state transition matrix given in Section 2.1.2. An expression for the $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t)$ matrix has been obtained in [37]:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t) = \mathbf{K}(\mathbf{p}, t) \exp\left(\mathbf{N}(\mathbf{p})t\right),$$

where the matrix $\mathbf{K}(\mathbf{p}, t)$ is nonsingular and satisfies the conditions:

$$\mathbf{K}(\mathbf{p}, t + T(\mathbf{p})) = \mathbf{K}(\mathbf{p}, t), \ \mathbf{K}(\mathbf{p}, 0) = \mathbf{I},$$

and the constant matrix $\mathbf{N}(\mathbf{p})$ is given by:

$$\mathbf{N}(\mathbf{p}) = \frac{1}{T(\mathbf{p})} \ln\left[\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, T(\mathbf{p}))\right].$$

The matrix

$$\mathbf{M}(\mathbf{p}) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, T(\mathbf{p})) = \exp\left(\mathbf{N}(\mathbf{p})T(\mathbf{p})\right)$$

is called the monodromy matrix and has the property:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t + T(\mathbf{p})) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{p}, t)\mathbf{M}(\mathbf{p}).$$

Floquet Theory

The Floquet theorem says that if state transition matrix is diagonalizable, we have [11]:

$$\mathbf{D}(\mathbf{p}, t - s) = \operatorname{diag}\left[\exp\left(\lambda_1(\mathbf{p})(t - s)\right), ..., \exp\left(\lambda_{n_x}(\mathbf{p})(t - s)\right)\right],$$

then the state transition matrix of the linear system in Equation (2.16) can be written in the form:

$$\mathbf{\Phi}(\mathbf{p}, t, s) = \mathbf{U}(\mathbf{p}, t)\mathbf{D}(\mathbf{p}, t - s)\mathbf{V}(\mathbf{p}, s), \tag{2.18}$$

where $\mathbf{U}(\mathbf{p}, t) \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{V}(\mathbf{p}, t) \in \mathbb{R}^{n_x \times n_x}$ are both $T(\mathbf{p})$-periodic and non-singular (for all t) and satisfy:

$$\mathbf{U}(\mathbf{p}, t) = \mathbf{V}^{-1}(\mathbf{p}, t).$$

$\lambda_i(\mathbf{p})$ are called the characteristic (Floquet) exponents of the Equation (2.16), $\rho_i(\mathbf{p}) = \exp\left[\lambda_i(\mathbf{p})T(\mathbf{p})\right]$ are the eigenvalues of $\mathbf{M}(\mathbf{p})$ and are called the (Floquet) characteristic multipliers [37] given by the solutions of the characteristic equation:

$$\det\left[\mathbf{M}(\mathbf{p}) - \rho(\mathbf{p})\mathbf{I}\right] = 0.$$

These characteristic multipliers are used to determine whether a periodic steady-state solution of Equations (2.5) and (2.6), i.e., the limit cycle, is stable. A limit cycle is orbitally stable if one of the multipliers is equal to 1 and all others lie strictly inside the unit circle [2]. The work done in this thesis involves only limit cycles which are stable.

For $s = 0$, it follows that $\mathbf{\Phi}(\mathbf{p}, t, 0)$ can be written as:

$$\mathbf{\Phi}(\mathbf{p}, t, 0) = \mathbf{\Phi}_1(\mathbf{p}, t, 0) + \mathbf{\Phi}_2(\mathbf{p}, t, 0),$$

where $\mathbf{\Phi}_1(\mathbf{p}, t, 0)$ and $\mathbf{\Phi}_2(\mathbf{p}, t, 0)$ are both solutions to Equation (2.16) and given by:

$$\mathbf{\Phi}_1(\mathbf{p}, t, 0) = \left[\begin{array}{cc} \dot{\mathbf{x}}(\mathbf{p}, t) & \mathbf{O}_{n_x, n_x - 1} \end{array} \right] \mathbf{V}(\mathbf{p}, 0),$$

$$\mathbf{\Phi}_2(\mathbf{p}, t, 0) = \left[\begin{array}{cc} \mathbf{O}_{n_x, 1} & \mathbf{G}(\mathbf{p}, t) \end{array} \right] \mathbf{V}(\mathbf{p}, 0),$$

with $\mathbf{O}_{i,k}$ is the zero matrix with i rows and k columns [37, 7]. The matrix $\mathbf{\Phi}_2(\mathbf{p}, t, 0)$ decays for large times t but $\mathbf{\Phi}_1(\mathbf{p}, t, 0)$ is $T(\mathbf{p})$ periodic, so that $\mathbf{\Phi}(\mathbf{p}, t, 0) \to \mathbf{\Phi}_1(\mathbf{p}, t, 0)$ as $t \to +\infty$. So the expression for $\mathbf{\Phi}(\mathbf{p}, t, 0)$ is given by:

$$\mathbf{\Phi}(\mathbf{p}, t, 0) = \left[\begin{array}{cc} \dot{\mathbf{x}}(\mathbf{p}, t) & \mathbf{G}(\mathbf{p}, t) \end{array} \right] \mathbf{V}(\mathbf{p}, 0). \tag{2.19}$$

Since the LCO has an oscillatory mode, one of the Floquet exponents is zero, say $\lambda_1 = 0$. For $s = 0$, Eq. (2.18) will then become:

$$\mathbf{\Phi}(\mathbf{p}, t, 0) = \mathbf{U}(\mathbf{p}, t)\mathbf{D}(\mathbf{p}, t)\mathbf{V}(\mathbf{p}, 0), \tag{2.20}$$

where $\mathbf{D}(\mathbf{p}, t) = \text{diag}\left[1, \exp\left(\lambda_2(\mathbf{p})t\right), \ldots, \exp\left(\lambda_{n_x}(\mathbf{p})t\right)\right]$. Let $\mathbf{u}(\mathbf{p}, t)$ be the first column of $\mathbf{U}(\mathbf{p}, t)$. Eq. (2.20) becomes:

$$\mathbf{\Phi}(\mathbf{p}, t, 0) = \left[\begin{array}{cc} \mathbf{u}(\mathbf{p}, t) & \mathbf{G}(\mathbf{p}, t) \end{array} \right] \mathbf{V}(\mathbf{p}, 0). \tag{2.21}$$

Comparing Eq. (2.21) with Eq. (2.19), it shows that $\dot{\mathbf{x}}(\mathbf{p}, t)$ is the first column of $\mathbf{U}(\mathbf{p}, t)$.

Inhomogeneous Linear Systems with Periodic Coefficients

The sensitivity equations which were described in Equation (2.13) are inhomogeneous linear systems with periodic coefficients as described in Section 2.1.3. The solution of Equation (2.13) can be given in terms of the state transition matrix $\mathbf{\Phi}(\mathbf{p}, t, 0)$:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t) = \mathbf{\Phi}(\mathbf{p}, t, 0)\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \int_0^t \mathbf{\Phi}(\mathbf{p}, t, \tau)\mathbf{B}(\mathbf{p}, \tau)\, d\tau. \tag{2.22}$$

It was shown in [37], that this solution given in Equation (2.22) can be written in the form:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{p}, t) = t\mathbf{R}(\mathbf{p}, t) + \mathbf{Z}(\mathbf{p}, t), \forall t,\ \mathbf{Z}(\mathbf{p}, 0) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}),$$

where $\mathbf{R}(\mathbf{p}, t)$ is $T(\mathbf{p})$-periodic in time and contains the influence of the period on the sensitivity:

$$\mathbf{R}(\mathbf{p}, t) = -\frac{\dot{\mathbf{x}}(\mathbf{p}, t)}{T(\mathbf{p})}\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}),$$

and $\mathbf{Z}(\mathbf{p}, t)$ is also periodic in time with period $T(\mathbf{p})$, and corresponds to the partial derivatives of the state variables with respect to the parameters keeping the period constant:

$$\mathbf{Z} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)_{T(\mathbf{p})=const.}.$$

It was also sometimes referred to as the "cleaned-out" sensitivity and was reported to contain the influence of the parameters on the shape and phase behavior of the limit cycle. This can be further decomposed into two parts [46]:

$$\mathbf{Z}(\mathbf{p}, t) = \mathbf{W}(\mathbf{p}, t) + \dot{\mathbf{x}}(\mathbf{p}, t)\boldsymbol{\delta}(\mathbf{p}, t),$$

where $\mathbf{W}(\mathbf{p}, t)$ is a $T(\mathbf{p})$-periodic matrix, containing information on the shape of the limit cycle, and is constructed from $\mathbf{Z}(\mathbf{p}, t)$ using the projection:

$$\mathbf{W}(\mathbf{p}, t) = \left(\mathbf{I} - \frac{\dot{\mathbf{x}}(\mathbf{p}, t)\dot{\mathbf{x}}(\mathbf{p}, t)^{\mathrm{T}}}{\|\dot{\mathbf{x}}(\mathbf{p}, t)\|^2}\right)\mathbf{Z}(\mathbf{p}, t),$$

and $\boldsymbol{\delta}(\mathbf{p}, t)$ is also a $T(\mathbf{p})$-periodic row vector known as the phase sensitivity, containing information on the phase behavior of the limit cycle, and is constructed from

$\mathbf{Z}(\mathbf{p}, t)$ using the projection onto the direction of $\dot{\mathbf{x}}(\mathbf{p}, t)$:

$$\boldsymbol{\delta}(\mathbf{p}, t) = \frac{1}{\|\dot{\mathbf{x}}(\mathbf{p}, t)\|^2} \left[\dot{\mathbf{x}}(\mathbf{p}, t)^{\mathrm{T}} \mathbf{Z}(\mathbf{p}, t)\right].$$

The quantities which are relevant to calculate for a LCO given in [47] were amplitude sensitivities and peak-to-peak sensitivities. It was shown in [47] that amplitude sensitivities can be obtained from:

$$\frac{\partial \Omega_i}{\partial \mathbf{p}}(\mathbf{p}) = \mathbf{s}_i\left(\mathbf{p}, t_{i,\max}(\mathbf{p})\right) - \mathbf{s}_i\left(\mathbf{p}, t_{i,\min}(\mathbf{p})\right),$$

where $\Omega_i(\mathbf{p})$ is the amplitude for the state variable x_i, $\mathbf{s}_i$ represents the ith row of the full sensitivity matrix, $t_{i,\max}(\mathbf{p})$ and $t_{i,\min}(\mathbf{p})$ are the times at which x_i attains its supremum and infimum value, respectively. The peak-to-peak sensitivities represent the influence of the parameters on a phase ($\beta(\mathbf{p})$), which is defined as the time difference between the peak of one state variable to the peak of another state variable. It can be calculated by solving following equation for $\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p})$:

$$\frac{\partial F_j}{\partial \mathbf{x}}\left(\mathbf{x}\left(\mathbf{p}, \beta(\mathbf{p})\right), \mathbf{p}\right)\left(\dot{\mathbf{x}}\left(\mathbf{p}, \beta(\mathbf{p})\right)\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \mathbf{x}}{\partial \mathbf{p}}\left(\mathbf{p}, \beta(\mathbf{p})\right)\right) + \frac{\partial F_j}{\partial \mathbf{p}}\left(\mathbf{x}\left(\mathbf{p}, \beta(\mathbf{p})\right), \mathbf{p}\right) = \mathbf{0}$$

where F_j is the right-hand side of the differential equation for the state variable x_j, and $\beta(\mathbf{p})$ is the time elapsed at the extremum of x_j relative to the extremum of x_i, for $i \neq j$.

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Chapter 3

Sensitivity Analysis of Oscillating Hybrid Systems

3.1 Hybrid Systems: Definition

The hybrid systems analyzed in this thesis are defined using a modification of the modeling framework in [27] as a basis:

Definition 1. *The hybrid system considered is a 12-tuple* $\mathcal{H} = (\mathcal{M},\ \mathcal{E},\ T_\mu,\ \mathbf{p},\ \mathbf{x},\ \mathbf{F},\ \mathbf{x}_0,\ T,\ \mathcal{L},\ \boldsymbol{\sigma},\ \boldsymbol{\tau},\ \boldsymbol{\mathcal{T}})$, *where*

- $\mathcal{M} = \{1,\ldots,n_m\}$, $1 \le n_m < \infty$,
- $\mathcal{E} = \{1,\ldots,n_e + 1\}$, $1 \le n_e < \infty$,
- $T_\mu = \{m_i\}_{i\in\mathcal{E}}$, $m_i \in \mathcal{M}$, $m_{n_e+1} = m_1$,
- $\mathbf{p} \in P \subset \mathbb{R}^{n_p}$,
- $\mathbf{x} : \mathcal{E} \times P \times \mathbb{R} \to X, X \subset \mathbb{R}^{n_x}$,
- $\mathbf{F} : \mathcal{M} \times X \times P \to \mathbb{R}^{n_x}$,
- $\mathbf{x}_0 : P \to X$,
- $T : P \to \mathbb{R}$,

- $\mathcal{L} : \mathcal{M} \times X \times P \rightarrow \mathbb{R}$,
- $\boldsymbol{\sigma} : P \rightarrow \mathbb{R}^{n_e+1}$,
- $\boldsymbol{\tau} : P \rightarrow \mathbb{R}^{n_e+1}$, and
- $\boldsymbol{\mathcal{T}} : \mathcal{M} \times X \times P \rightarrow X$.

The elements of $\mathcal{M}$ are called the *modes* of $\mathcal{H}$ and $\mathcal{E}$ is the index set for the *epochs* which are illustrated in Figure 3-1 for one period of the cycle. T_μ is called the *hybrid mode trajectory.* $\mathbf{p}$ is the vector of parameters, $\mathbf{x}$ is the vector of continuous state variables, and $\mathbf{F}$ is the vector field for $\mathbf{x}$. $\mathbf{x}_0$ are the initial conditions, T is the period, $\mathcal{L}$ is the *transition condition*, $\boldsymbol{\sigma}$ are the initial times of epochs, $\boldsymbol{\tau}$ are the end times of epochs (also known as *events*) and $\boldsymbol{\mathcal{T}}$ are the *transition functions.*

While Figure 3-1 shows the event times for one period, in general the event times can be defined for each period as:

$$\begin{aligned}\sigma_{i,N}(\mathbf{p}) &\equiv NT(\mathbf{p}) + \sigma_i(\mathbf{p}), \forall i = 2, \ldots, n_e + 1, \forall N \in \{0, 1, \ldots, \infty\}, \qquad (3.1)\\ \tau_{i,N}(\mathbf{p}) &\equiv NT(\mathbf{p}) + \tau_i(\mathbf{p}), \forall i = 1, \ldots, n_e, \forall N \in \{0, 1, \ldots, \infty\}.\end{aligned}$$

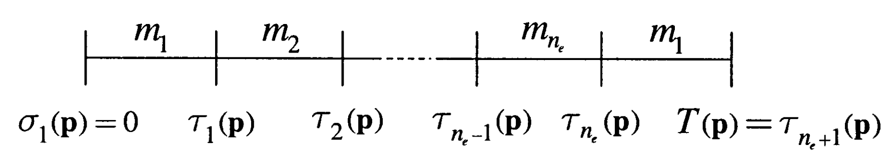


Figure 3-1: Epochs on the hybrid time trajectory.

3.2 Sensitivity Analysis of the Limit-Cycles of Hybrid Systems

In this work, sensitivity analysis of hybrid dynamic systems that are limit-cycle oscillators is investigated. A hybrid dynamic system is one that exhibits both discrete-state and continuous-state dynamics. Limit-cycle oscillators (LCOs) are common in hybrid systems and can be stable or unstable. A limit-cycle oscillator has an isolated and closed periodic orbit. In other words, the periodic orbit is solely determined by the parameters of the system and does not depend on the initial conditions. The focus of the present work is stable hybrid LCOs, in which the periodic orbit is approached asymptotically from any initial condition that lies within the region of attraction. If the initial condition is on the limit cycle, the hybrid system follows its dynamics and returns to this initial condition after time T.

3.2.1 Boundary Value Problem

Given $\mathbf{p} \in P$, a *boundary value problem* (BVP) is formulated to define $\mathbf{x}_0(\mathbf{p})$, $T(\mathbf{p})$, $\boldsymbol{\sigma}(\mathbf{p})$ and $\boldsymbol{\tau}(\mathbf{p})$ implicitly:

$$\mathbf{x}(n_e + 1, \mathbf{p}, (N+1)T(\mathbf{p})) - \mathbf{x}_0(\mathbf{p}) = \mathbf{0}, \tag{3.2}$$

$$\dot{x}_j(1, \mathbf{p}, \sigma_{1,N}(\mathbf{p})) = 0, \tag{3.3}$$

for some arbitrary $j \in \{1, \ldots, n_x\}$ and $N \in \{0, 1, \ldots, \infty\}$ with $\mathbf{x}(i, \mathbf{p}, t)$, $\forall i \in \mathcal{E}$ given by

$$\frac{d\mathbf{x}}{dt}(i, \mathbf{p}, t) = \mathbf{F}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p}), \ \forall t \in (\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})], \tag{3.4}$$

$$\mathbf{x}(1, \mathbf{p}, \sigma_{1,N}(\mathbf{p})) = \mathbf{x}_0(\mathbf{p}),$$

where the event times $\sigma_i(\mathbf{p})$ and $\tau_i(\mathbf{p})$ within first period (for $N = 0$) are determined by the transition condition $\mathcal{L}$ and the definition $\sigma_1(\mathbf{p}) = 0$. At the start of an epoch, it is assumed that the transition condition satisfies:

$$\mathcal{L}(m_i, \mathbf{x}(i, \mathbf{p}, t), \mathbf{p}) > 0. \tag{3.5}$$

Then, the earliest time at which the transition condition crosses zero defines $\tau_i(\mathbf{p})$, as illustrated in Figure 3-2. Furthermore, $\sigma_{i+1}(\mathbf{p}) = \tau_i(\mathbf{p}), \forall i = 1, \ldots, n_e$. Assuming $\mathcal{L}(m, \cdot, \cdot)$ is a continuous function $\forall m \in \mathcal{M}$, this implies that $\tau_i(\mathbf{p}) > \sigma_i(\mathbf{p}), \forall i =$

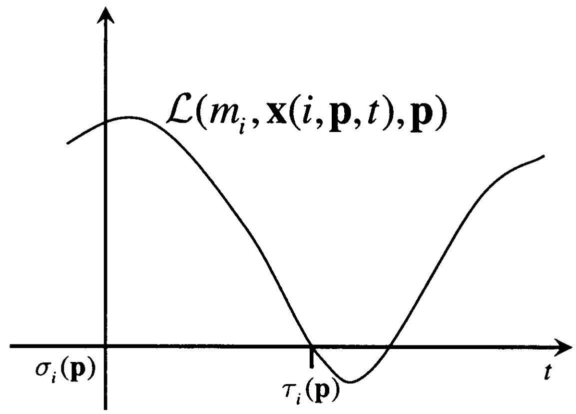


Figure 3-2: Transition time $\tau_i(\mathbf{p})$ for the ith epoch

$1,\ldots,n_e$. Finally, $T(\mathbf{p}) \equiv \tau_{n_e+1}(\mathbf{p})$, and we assume that $T(\mathbf{p}) > \sigma_{n_e+1}(\mathbf{p})$.

The transition functions map the final values of the continuous state variables in the predecessor mode m_i to their initial values in the successor mode m_{i+1} at time $t = \tau_{i,N}(\mathbf{p})$:

$$\mathbf{x}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) = \boldsymbol{\mathcal{T}}(m_i, \mathbf{x}(i,\mathbf{p},\tau_{i,N}(\mathbf{p})),\mathbf{p}),\ \forall i = 1,\ldots,n_e,\ \forall N \in \{0,1,\ldots,\infty\}.$$

By solving this BVP for given values of $\mathbf{p}$ and $N = 0$, initial conditions $\mathbf{x}_0(\mathbf{p})$ for the continuous state variables that lie on a limit cycle are obtained, as well as the period of oscillation, $T(\mathbf{p})$. Eq. (3.2) defines the trajectory to be a limit cycle. Eq. (3.3) is one possible example of a valid *phase locking condition* (PLC), which fixes the zero time to an isolated point on the limit cycle, and determines the event times $\boldsymbol{\sigma}(\mathbf{p})$ and $\boldsymbol{\tau}(\mathbf{p})$ with respect to this reference point. We assume that the BVP as defined has a solution $\mathbf{x}(i,\mathbf{p},\cdot)$, $\forall i \in \mathcal{E}$, $\forall t \geq 0$ where defined, that exists, is unique and satisfies the assumptions imposed. Furthermore, for the sensitivity analysis, we assume the functions:

$$\mathbf{F}(m,\cdot,\cdot), \forall m \in \mathcal{M},$$

$$\mathcal{L}(m,\cdot,\cdot), \forall m \in \mathcal{M},$$

$$\mathcal{T}(m,\cdot,\cdot), \forall m \in \mathcal{M},$$

are continuously differentiable on their domains, assumed to be open sets.

3.2.2 Homogeneous Linear Differential Equations with Piecewise Continuous Periodic Coefficients

Homogeneous Equation and Fundamental Matrix

Given $\mathbf{x}(i, \mathbf{p}, \cdot), \forall i \in \mathcal{E}$ as defined in Section 2.1, define

$$\mathbf{A}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, t - NT(\mathbf{p})), \mathbf{p}), \ \forall t \in (\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})],$$

for $N = 0, 1, \ldots$. Hence, the elements of $\mathbf{A}(\mathbf{p}, t)$ are piecewise continuous and periodic functions of t with period $T(\mathbf{p})$.

Let $\mathbf{\Phi}(\mathbf{p}, t, t_0)$ be the matrix which is the solution of the equations

$$\frac{d\mathbf{\Phi}}{dt}(\mathbf{p}, t, t_0) = \mathbf{A}(\mathbf{p}, t)\mathbf{\Phi}(\mathbf{p}, t, t_0), \ \forall t > 0, \ \mathbf{\Phi}(\mathbf{p}, t_0, t_0) = \mathbf{I}, \ \forall t_0 \geq 0. \tag{3.6}$$

The matrix $\mathbf{\Phi}(\mathbf{p}, t, t_0)$ is called the principal fundamental matrix or the state transition matrix [49]. The solution of Eq. (3.6) will exist and be unique in the sense of Carathéodory [10].

Initial-Condition Sensitivities

While the state variables are $T(\mathbf{p})$ periodic where they are defined, neither the initial-condition sensitivities nor the parametric sensitivities are $T(\mathbf{p})$ periodic. Suppose that *state continuity* is employed as the transition function for each mode so that:

$$\mathbf{x}(i + 1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) = \mathbf{x}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})), \ \forall i = 1, \ldots, n_e, \forall N \in \{0, 1, \ldots, \infty\}. \tag{3.7}$$

Recalling that $\sigma_{i+1,N}(\mathbf{p}) = \tau_{i,N}(\mathbf{p}), \forall N \in \{0, 1, \ldots, \infty\}$ and differentiating Eq. (3.7) with respect to initial conditions $\mathbf{x}_0$ yields:

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) =& \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) \\ &+ (\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) - \dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})))\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p}), \\ &\forall i = 1, \ldots, n_e, \forall N \in \{0, 1, \ldots, \infty\}, \end{aligned} \tag{3.8}$$

where $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p})$ are determined by the transition conditions. Note that $\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))$ and $\dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))$ are the left-hand and right-hand derivatives, respectively.

At $\tau_{i,N}(\mathbf{p}) > \sigma_{i,N}(\mathbf{p})$:

$$\mathcal{L}(m_i, \mathbf{x}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})), \mathbf{p}) = 0. \tag{3.9}$$

Again, recalling that $\sigma_{i+1,N}(\mathbf{p}) = \tau_{i,N}(\mathbf{p})$ and differentiating Eq. (3.9) with respect to $\mathbf{x}_0$ we have,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p})\left(\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p}) + \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))\right) = \mathbf{0}. \tag{3.10}$$

The above linear equations can be solved for unique $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p})$, $\forall i = 1, \ldots, n_e$, $\forall N \in \{0, 1, \ldots, \infty\}$, provided that:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p})\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) \neq 0.$$

The expression obtained for $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p})$ after solving Eq. (3.10) is:

$$\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p}) = -\left(\frac{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p})\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p})\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))}\right). \tag{3.11}$$

Note that $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p})$ can be different for each N because $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))$ is not necessarily periodic. Consider the homogeneous linear system with state jumps, for

$N \in \{0, 1, \ldots, \infty\}$:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)(i, \mathbf{p}, t) = \mathbf{A}(\mathbf{p}, t)\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t), \ \forall t \in (\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})], \ \forall i \in \mathcal{E}, \tag{3.12}$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \mathbf{I},$$

where $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, \sigma_{i,N}(\mathbf{p}))$ is given by Eq. (3.8) for $i = 2, \ldots, n_e + 1$. From continuity of the vector field at $t = NT(\mathbf{p})$:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1, \mathbf{p}, NT(\mathbf{p})) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e + 1, \mathbf{p}, NT(\mathbf{p})), \forall N \in \{0, 1, \ldots, \infty\}.$$

The solution of this system $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, \cdot)$, $\forall i \in \mathcal{E}$, gives the sensitivity with respect to the initial conditions $\mathbf{x}_0(\mathbf{p})$ [15] but in general it is not a fundamental matrix because of the potential state jumps at the epoch boundaries. Then, the solution of Eq. (3.12) is:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1, \mathbf{p}, t) = \mathbf{\Phi}(\mathbf{p}, t, \sigma_1(\mathbf{p})), \ \forall t \in [\sigma_1(\mathbf{p}), \tau_1(\mathbf{p})],$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1, \mathbf{p}, t) = \mathbf{\Phi}(\mathbf{p}, t, \sigma_{1,N}(\mathbf{p}))\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e + 1, \mathbf{p}, NT(\mathbf{p})),$$
$$\forall t \in [\sigma_{1,N}(\mathbf{p}), \tau_{1,N}(\mathbf{p})], \ \forall N \in \{1, 2, \ldots, \infty\},$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t) = \mathbf{\Phi}(\mathbf{p}, t, \sigma_{i,N}(\mathbf{p}))\left[\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i - 1, \mathbf{p}, \tau_{i-1,N}(\mathbf{p})) + \mathbf{\Delta}^0_{i-1,N}(\mathbf{p})\right],$$
$$\forall t \in [\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})], \forall i = 2, \ldots, n_e + 1, \forall N \in \{0, 1, \ldots, \infty\}, \tag{3.13}$$

where the $\mathbf{\Delta}^0_{i,N}(\mathbf{p})$ are given by

$$\begin{aligned}
\mathbf{\Delta}^0_{i,N}(\mathbf{p}) &= \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) - \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i,\mathbf{p},\tau_{i,N}(\mathbf{p})) \\
&= (\dot{\mathbf{x}}(i,\mathbf{p},\tau_{i,N}(\mathbf{p})) - \dot{\mathbf{x}}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})))\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{x}_0}(\mathbf{p}) \\
&= -(\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) - \dot{\mathbf{x}}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))) \\
&\quad \times \left(\frac{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p})\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p})\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))} \right) \\
&= -\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})), \forall i = 1,\ldots,n_e, \;\; \forall N \in \{0,1,\ldots,\infty\}.
\end{aligned}$$

Note that all terms in the premultiplying matrix are $T(\mathbf{p})$ periodic, so $\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))$ can be defined as:

$$\begin{aligned}
\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p})) &= (\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p})) - \dot{\mathbf{x}}(i+1,\mathbf{p},\sigma_{i+1}(\mathbf{p}))) \\
&\quad \times \left(\frac{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p})),\mathbf{p})}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p})),\mathbf{p})\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))} \right), \qquad (3.14) \\
&\quad \forall i = 1,\ldots,n_e.
\end{aligned}$$

Rewriting Eq. (3.13) for epoch $i+1$, $N=0$ and substituting for $\mathbf{\Delta}^0_{i,0}(\mathbf{p})$ we have:

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t) &= \mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\left[\mathbf{I} - \mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))\right]\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p})), \\
&\quad \forall t \in [\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i = 1,\ldots,n_e. \qquad (3.15)
\end{aligned}$$

Since $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1,\mathbf{p},\sigma_1(\mathbf{p})) = \mathbf{I}$, we have $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1,\mathbf{p},\tau_1(\mathbf{p})) = \mathbf{\Phi}(\mathbf{p},\tau_1(\mathbf{p}),\sigma_1(\mathbf{p}))$. Expanding the recursion formula in Eq. (3.15) we have,

$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1, \mathbf{p}, t) = {} & \mathbf{\Phi}(\mathbf{p}, t, \sigma_{i+1}(\mathbf{p}))\,[\mathbf{I} - \mathbf{C}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p}))]\,\mathbf{\Phi}(\mathbf{p}, \tau_i(\mathbf{p}), \sigma_i(\mathbf{p})) \\
& \times [\mathbf{I} - \mathbf{C}(i-1, \mathbf{p}, \sigma_i(\mathbf{p}))] \ldots \mathbf{\Phi}(\mathbf{p}, \tau_2(\mathbf{p}), \sigma_2(\mathbf{p})) \\
& \times [\mathbf{I} - \mathbf{C}(1, \mathbf{p}, \sigma_2(\mathbf{p}))]\,\mathbf{\Phi}(\mathbf{p}, \tau_1(\mathbf{p}), \sigma_1(\mathbf{p})) \\
& \times [\mathbf{I} - \mathbf{C}(0, \mathbf{p}, \sigma_1(\mathbf{p}))]\,\mathbf{\Phi}(\mathbf{p}, \tau_0(\mathbf{p}), \sigma_0(\mathbf{p})) \qquad (3.16) \\
= {} & \mathbf{\Phi}(\mathbf{p}, t, \sigma_{i+1}(\mathbf{p})) \times \\
& \prod_{k=0}^{i} [\mathbf{I} - \mathbf{C}(i-k, \mathbf{p}, \sigma_{i+1-k}(\mathbf{p}))]\,\mathbf{\Phi}(\mathbf{p}, \tau_{i-k}(\mathbf{p}), \sigma_{i-k}(\mathbf{p})), \\
& \forall t \in [\sigma_{i+1}(\mathbf{p}), \tau_{i+1}(\mathbf{p})], \forall i = 0, \ldots, n_e,
\end{aligned}
$$

where $\mathbf{C}(0, \mathbf{p}, \sigma_1(\mathbf{p})) \equiv \mathbf{0}$ and $\mathbf{\Phi}(\mathbf{p}, \tau_0(\mathbf{p}), \sigma_0(\mathbf{p})) \equiv \mathbf{I}$. Let us define:

$$
\begin{aligned}
& \boldsymbol{\mathcal{A}}_i(t) \equiv \mathbf{\Phi}(\mathbf{p}, t, \sigma_i(\mathbf{p})), \forall t \in [\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E}, \\
& \boldsymbol{\mathcal{A}}_i \equiv \mathbf{\Phi}(\mathbf{p}, \tau_i(\mathbf{p}), \sigma_i(\mathbf{p})), \forall i = 1, \ldots, n_e, \\
& \boldsymbol{\mathcal{A}}_0 \equiv \mathbf{I}, \\
& \boldsymbol{\mathcal{C}}_i \equiv \mathbf{C}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \forall i = 1, \ldots, n_e, \\
& \boldsymbol{\mathcal{C}}_0 \equiv \mathbf{0}.
\end{aligned}
$$

Rewriting Eq. (3.16) in terms of $\boldsymbol{\mathcal{A}}_i$s and $\boldsymbol{\mathcal{C}}_i$s we have,

$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1, \mathbf{p}, t) = \boldsymbol{\mathcal{A}}_{i+1}(t) \left(\prod_{k=0}^{i} [\mathbf{I} - \boldsymbol{\mathcal{C}}_{i-k}] \boldsymbol{\mathcal{A}}_{i-k} \right), \\
\forall t \in [\sigma_{i+1}(\mathbf{p}), \tau_{i+1}(\mathbf{p})], \forall i = 0, \ldots, n_e.
\end{aligned}
\qquad (3.17)
$$

Theorem 1. *The right-hand side of equation* (3.17) *can be expressed as,*

$$
\begin{aligned}
\boldsymbol{\mathcal{A}}_{i+1}(t) \left(\prod_{k=0}^{i} [\mathbf{I} - \boldsymbol{\mathcal{C}}_{i-k}] \boldsymbol{\mathcal{A}}_{i-k} \right) = \boldsymbol{\mathcal{A}}_{i+1}(t) \left(\sum_{\mathbf{b} \in B_{i+1}} \left(\prod_{k=0}^{i} (-1)^{b_{k+1}} \boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}} \boldsymbol{\mathcal{A}}_{i-k} \right) \right), \\
\forall t \in [\sigma_{i+1}(\boldsymbol{p}), \tau_{i+1}(\boldsymbol{p})], \forall i = 0, \ldots, n_e, \qquad (3.18)
\end{aligned}
$$

where $B_i = \{0,1\}^i, \forall i \in \mathcal{E}$.

Proof. The result can be proved using mathematical induction. Let $P(i)$ be the statement in the equation (3.18) for any non-negative integer i. $P(0)$ asserts $\boldsymbol{\mathcal{A}}_1(t)\boldsymbol{\mathcal{A}}_0 = \boldsymbol{\mathcal{A}}_1(t)\boldsymbol{\mathcal{A}}_0$, which is true. $P(1)$ asserts $\boldsymbol{\mathcal{A}}_2(t)[\mathbf{I} - \boldsymbol{\mathcal{C}}_1]\boldsymbol{\mathcal{A}}_1 = \boldsymbol{\mathcal{A}}_2(t)\boldsymbol{\mathcal{A}}_1 - \boldsymbol{\mathcal{A}}_2(t)\boldsymbol{\mathcal{C}}_1\boldsymbol{\mathcal{A}}_1, \forall t \in [\sigma_2(\mathbf{p}), \tau_2(\mathbf{p})]$, which is also clearly true. We assume there is a i for which $P(i)$ is true, then we must prove for this same i, $P(i+1)$ is true, i.e.,

$$\boldsymbol{\mathcal{A}}_{i+2}(t)\left(\prod_{k=0}^{i+1}[\mathbf{I} - \boldsymbol{\mathcal{C}}_{i+1-k}]\boldsymbol{\mathcal{A}}_{i+1-k}\right) = \boldsymbol{\mathcal{A}}_{i+2}(t)\left(\sum_{\mathbf{b}\in B_{i+2}}\left(\prod_{k=0}^{i+1}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i+1-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i+1-k}\right)\right),$$
$$\forall t \in [\sigma_{i+2}(\mathbf{p}), \tau_{i+2}(\mathbf{p})],$$

where $B_{i+2} = \{0,1\}^{i+2}$. Simplifying the LHS and using $P(i)$,

$$\begin{aligned}
\boldsymbol{\mathcal{A}}_{i+2}(t)\left(\prod_{k=0}^{i+1}[\mathbf{I} - \boldsymbol{\mathcal{C}}_{i+1-k}]\boldsymbol{\mathcal{A}}_{i+1-k}\right) &= \boldsymbol{\mathcal{A}}_{i+2}(t)[\mathbf{I} - \boldsymbol{\mathcal{C}}_{i+1}]\boldsymbol{\mathcal{A}}_{i+1}\left(\prod_{k=0}^{i}[\mathbf{I} - \boldsymbol{\mathcal{C}}_{i-k}]\boldsymbol{\mathcal{A}}_{i-k}\right) \\
&= \boldsymbol{\mathcal{A}}_{i+2}(t)[\mathbf{I} - \boldsymbol{\mathcal{C}}_{i+1}]\boldsymbol{\mathcal{A}}_{i+1} \\
&\quad \times \left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right) \\
&= \boldsymbol{\mathcal{A}}_{i+2}(t)\left[\boldsymbol{\mathcal{C}}_{i+1}^{0}\boldsymbol{\mathcal{A}}_{i+1} - \boldsymbol{\mathcal{C}}_{i+1}^{1}\boldsymbol{\mathcal{A}}_{i+1}\right] \\
&\quad \times \left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right) \\
&= \boldsymbol{\mathcal{A}}_{i+2}(t)\left(\sum_{\mathbf{b}\in B_{i+2}}\left(\prod_{k=0}^{i+1}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i+1-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i+1-k}\right)\right), \\
&\quad \forall t \in [\sigma_{i+2}(\mathbf{p}), \tau_{i+2}(\mathbf{p})].
\end{aligned}$$

Hence $P(i+1)$ is true. $\square$

Equation (3.17) can be rewritten as:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t) = \boldsymbol{\mathcal{A}}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right), \tag{3.19}$$
$$\forall t \in [\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i = 0,\ldots,n_e.$$

The state transition matrix has the group property [49]:

$$\boldsymbol{\Phi}(\mathbf{p},t_1,t_2)\boldsymbol{\Phi}(\mathbf{p},t_2,t_3) = \boldsymbol{\Phi}(\mathbf{p},t_1,t_3) \quad \forall t_1,t_2,t_3 \geq 0. \tag{3.20}$$

Simplifying equation (3.19) and rewriting as:

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t) = {} & \boldsymbol{\mathcal{A}}_{i+1}(t)\prod_{k=0}^{i}\boldsymbol{\mathcal{A}}_{i-k}\\ & + \boldsymbol{\mathcal{A}}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}\setminus\{0\}^{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right),\\ & \forall t \in [\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i = 0,\ldots,n_e,\end{aligned}$$

where $\boldsymbol{\mathcal{A}}_{i+1}(t)\prod_{k=0}^{i}\boldsymbol{\mathcal{A}}_{i-k}$ is simplified further by using the property in Equation (3.20):

$$\begin{aligned}\boldsymbol{\mathcal{A}}_{i+1}(t)\prod_{k=0}^{i}\boldsymbol{\mathcal{A}}_{i-k} &= \boldsymbol{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\boldsymbol{\Phi}(\mathbf{p},\tau_i(\mathbf{p}),\sigma_i(\mathbf{p}))\ldots\boldsymbol{\Phi}(\mathbf{p},\tau_2(\mathbf{p}),\sigma_2(\mathbf{p}))\\ &\quad \times \boldsymbol{\Phi}(\mathbf{p},\tau_1(\mathbf{p}),\sigma_1(\mathbf{p}))\\ &= \boldsymbol{\Phi}(\mathbf{p},t,\sigma_1(\mathbf{p})), \forall i = 0,\ldots,n_e.\end{aligned}$$

Then Equation (3.17) becomes:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t) = \boldsymbol{\Phi}(\mathbf{p},t,\sigma_1(\mathbf{p})) + \boldsymbol{\mathcal{A}}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}\setminus\{0\}^{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right),$$
$$\forall t \in [\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i = 0,\ldots,n_e.$$

At $t = T(\mathbf{p})$ and $i = n_e$ we have:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1, \mathbf{p}, T(\mathbf{p})) = \boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p})) \left(\sum_{\mathbf{b} \in B_{n_e+1}} \left(\prod_{k=0}^{n_e} (-1)^{b_{k+1}} \boldsymbol{\mathcal{C}}_{n_e-k}^{b_{k+1}} \boldsymbol{\mathcal{A}}_{n_e-k} \right) \right). \quad (3.21)$$

For $t = NT(\mathbf{p}), i = n_e$ and $N = 0, 1, \dots,$

$$\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e + 1, \mathbf{p}, NT(\mathbf{p})) = \\
&\quad \mathbf{\Phi}(\mathbf{p}, NT(\mathbf{p}), (N-1)T(\mathbf{p}) + \sigma_{n_e+1}(\mathbf{p})) \left[\mathbf{I} - \mathbf{C}(n_e, \mathbf{p}, \sigma_{n_e+1}(\mathbf{p}))\right] \dots \\
&\quad \times \left[\mathbf{I} - \mathbf{C}(1, \mathbf{p}, \sigma_2(\mathbf{p}))\right] \mathbf{\Phi}(\mathbf{p}, (N-1)T(\mathbf{p}) + \tau_1(\mathbf{p}), (N-1)T(\mathbf{p}) + \sigma_1(\mathbf{p})) \\
&\quad \times \mathbf{\Phi}(\mathbf{p}, (N-1)T(\mathbf{p}), (N-2)T(\mathbf{p}) + \sigma_{n_e+1}(\mathbf{p})) \\
&\quad \times \left[\mathbf{I} - \mathbf{C}(n_e, \mathbf{p}, \sigma_{n_e+1}(\mathbf{p}))\right] \dots \left[\mathbf{I} - \mathbf{C}(1, \mathbf{p}, \sigma_2(\mathbf{p}))\right] \\
&\quad \times \mathbf{\Phi}(\mathbf{p}, (N-2)T(\mathbf{p}) + \tau_1(\mathbf{p}), (N-2)T(\mathbf{p}) + \sigma_1(\mathbf{p})) \dots \\
&\quad \times \mathbf{\Phi}(\mathbf{p}, T(\mathbf{p}), \sigma_{n_e+1}(\mathbf{p})) \left[\mathbf{I} - \mathbf{C}(n_e, \mathbf{p}, \sigma_{n_e+1}(\mathbf{p}))\right] \dots \left[\mathbf{I} - \mathbf{C}(1, \mathbf{p}, \sigma_2(\mathbf{p}))\right] \\
&\quad \times \mathbf{\Phi}(\mathbf{p}, \tau_1(\mathbf{p}), \sigma_1(\mathbf{p})) \left[\mathbf{I} - \mathbf{C}(0, \mathbf{p}, \sigma_1(\mathbf{p}))\right] \mathbf{\Phi}(\mathbf{p}, \tau_0(\mathbf{p}), \sigma_0(\mathbf{p})). \qquad (3.22)
\end{aligned}$$

The state transition matrix also has the property [49]:

$$\mathbf{\Phi}^{-1}(\mathbf{p}, t, t_0) = \mathbf{\Phi}(\mathbf{p}, t_0, t), \forall t, t_0 \geq 0. \quad (3.23)$$

Proposition 1. *If* $\mathbf{\Phi}(\mathbf{p}, t, t_0)$ *is the state transition matrix of a system of equations with piecewise continuous periodic coefficients* $\mathbf{A}(\mathbf{p}, t)$ *with period* $T(\mathbf{p})$ *as defined in Eq.* (3.6)*, then:*

$$\mathbf{\Phi}(\mathbf{p}, NT(\mathbf{p}) + t, NT(\mathbf{p})) = \mathbf{\Phi}(\mathbf{p}, t, 0), \forall t \geq 0, \forall N \in \{0, 1, \dots, \infty\}. \quad (3.24)$$

Proof. From Eq. (3.6):

$$\frac{d\mathbf{\Phi}}{dt}(\mathbf{p}, NT(\mathbf{p}) + t, NT(\mathbf{p})) = \mathbf{A}(\mathbf{p}, NT(\mathbf{p}) + t)\mathbf{\Phi}(\mathbf{p}, NT(\mathbf{p}) + t, NT(\mathbf{p})),$$
$$\forall t > -NT(\mathbf{p}), \ \mathbf{\Phi}(\mathbf{p}, NT(\mathbf{p}), NT(\mathbf{p})) = \mathbf{I}.$$

Because $\mathbf{A}(\mathbf{p},t)$ is periodic with period $T(\mathbf{p})$, this is equivalent to:

$$\frac{d\mathbf{\Phi}}{dt}(\mathbf{p},NT(\mathbf{p})+t,NT(\mathbf{p})) = \mathbf{A}(\mathbf{p},t)\mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+t,NT(\mathbf{p})),\ \forall t>0,$$
$$\mathbf{\Phi}(\mathbf{p},NT(\mathbf{p}),NT(\mathbf{p})) = \mathbf{I}.$$

Now, let us define:

$$\frac{d\mathbf{\Psi}}{dt}(\mathbf{p},t_0+t,t_0) = \mathbf{A}(\mathbf{p},t)\mathbf{\Psi}(\mathbf{p},t_0+t,t_0),\ \forall t>0,\quad \mathbf{\Psi}(\mathbf{p},t_0,t_0) = \mathbf{I}.$$

By inspection:

$$\mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+t,NT(\mathbf{p})) = \mathbf{\Psi}(\mathbf{p},t_0+t,t_0),$$

for any t_0. Also, by comparison with Eq. (3.6), if $t = t + t_0$, then:

$$\mathbf{\Psi}(\mathbf{p},t+t_0,t_0) = \mathbf{\Phi}(\mathbf{p},t,t_0),$$

which implies $t_0 = 0$ and

$$\mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+t,NT(\mathbf{p})) = \mathbf{\Phi}(\mathbf{p},t,0),\ \forall t\geq 0.$$

□

Using Proposition 1, the properties of the state transition matrix in Eq. (3.20) and Eq. (3.23), for $N \in \{0,1,\ldots,\infty\}$ and $i = 1,\ldots,n_e+1$:

$$\begin{aligned}
\mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+&\tau_i(\mathbf{p}),NT(\mathbf{p})+\sigma_i(\mathbf{p}))\\
&= \mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+\tau_i(\mathbf{p}),NT(\mathbf{p}))\mathbf{\Phi}(\mathbf{p},NT(\mathbf{p}),NT(\mathbf{p})+\sigma_i(\mathbf{p})),\\
&= \mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+\tau_i(\mathbf{p}),NT(\mathbf{p}))\mathbf{\Phi}^{-1}(\mathbf{p},NT(\mathbf{p})+\sigma_i(\mathbf{p}),NT(\mathbf{p})),\\
&= \mathbf{\Phi}(\mathbf{p},\tau_i(\mathbf{p}),0)\mathbf{\Phi}^{-1}(\mathbf{p},\sigma_i(\mathbf{p}),0),\\
&= \mathbf{\Phi}(\mathbf{p},\tau_i(\mathbf{p}),0)\mathbf{\Phi}(\mathbf{p},0,\sigma_i(\mathbf{p})),\\
&= \mathbf{\Phi}(\mathbf{p},\tau_i(\mathbf{p}),\sigma_i(\mathbf{p})).
\end{aligned} \tag{3.25}$$

Using Equation (3.25) in Equation (3.22) we have,

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1,\mathbf{p},NT(\mathbf{p})) &= [\mathbf{\Phi}(\mathbf{p},T(\mathbf{p}),\sigma_{n_e+1}(\mathbf{p}))\,[\mathbf{I}-\mathbf{C}(n_e,\mathbf{p},\sigma_{n_e+1}(\mathbf{p}))]\ldots \\ &\quad \times [\mathbf{I}-\mathbf{C}(1,\mathbf{p},\sigma_2(\mathbf{p}))]\,\mathbf{\Phi}(\mathbf{p},\tau_1(\mathbf{p}),\sigma_1(\mathbf{p}))]^N \\ &= \left[\mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\,\mathcal{C}^{b_{k+1}}_{n_e-k}\mathcal{A}_{n_e-k}\right)\right)\right]^N .\end{aligned}$$

Using Equation (3.21) and rewriting the above equation:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1,\mathbf{p},NT(\mathbf{p})) = \left[\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1,\mathbf{p},T(\mathbf{p}))\right]^N, \forall N\in\{0,1,\ldots,\infty\}.$$

Extending this expression for the initial condition sensitivities for all times yields:

$$\begin{aligned}&\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t) \\ &\quad = \mathbf{\Phi}(\mathbf{p},t,NT(\mathbf{p})+\sigma_{i+1}(\mathbf{p}))\,[\mathbf{I}-\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))]\ldots[\mathbf{I}-\mathbf{C}(1,\mathbf{p},\sigma_2(\mathbf{p}))] \\ &\qquad \times \mathbf{\Phi}(\mathbf{p},NT(\mathbf{p})+\tau_1(\mathbf{p}),NT(\mathbf{p})+\sigma_1(\mathbf{p}))\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1,\mathbf{p},NT(\mathbf{p})) \\ &\quad = \mathbf{\Phi}(\mathbf{p},t-NT(\mathbf{p}),\sigma_{i+1}(\mathbf{p}))\,[\mathbf{I}-\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))]\ldots[\mathbf{I}-\mathbf{C}(1,\mathbf{p},\sigma_2(\mathbf{p}))] \\ &\qquad \times \mathbf{\Phi}(\mathbf{p},\tau_1(\mathbf{p}),\sigma_1(\mathbf{p}))\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1,\mathbf{p},NT(\mathbf{p})) \\ &\quad = \mathcal{A}_{i+1}(t-NT(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\,\mathcal{C}^{b_{k+1}}_{i-k}\mathcal{A}_{i-k}\right)\right) \\ &\qquad \times \left[\mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\,\mathcal{C}^{b_{k+1}}_{n_e-k}\mathcal{A}_{n_e-k}\right)\right)\right]^N, \\ &\qquad \forall t\in[\sigma_{i+1,N}(\mathbf{p}),\tau_{i+1,N}(\mathbf{p})], \forall i=0,\ldots,n_e, \\ &\qquad \forall N=0,1,\ldots. \end{aligned} \tag{3.26}$$

3.2.3 The Monodromy Matrix

The monodromy matrix $\mathbf{M}$ of the system described in Section 2.1 can be defined as $\mathbf{M} \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1, \mathbf{p}, T(\mathbf{p}))$ and it follows from Eq. (3.26) that:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t+T(\mathbf{p})) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t)\mathbf{M}, \forall t \in [\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E}. \tag{3.27}$$

The eigenvalues ρ_i of $\mathbf{M}$ are called the multipliers [37] of Equation (3.12). The characteristic exponents of Eq. (3.12) are then $\lambda_i = \frac{1}{T}\ln \rho_i$. A solution of Eqs. (3.2) and (3.4) is orbitally stable if one multiplier is equal to 1 and all others lie strictly inside the unit circle. Throughout this study it is assumed that the solution of Eq. (3.4) is orbitally stable.

3.2.4 Properties of the Matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t)$

Matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t)$ obeys Eq. (3.27) which is exactly the same as in the case of regular limit-cycle oscillators. Hence, it can be written as:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t) = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t) + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_2 (i, \mathbf{p}, t)$$

where $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t)$ and $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_2 (i, \mathbf{p}, t)$ are given by:

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t) = \begin{bmatrix} \mathbf{u}(t) & \mathbf{O}_{n_x, n_x-1} \end{bmatrix} \mathbf{V}(0)$$

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_2 (i, \mathbf{p}, t) = \begin{bmatrix} \mathbf{O}_{n_x, 1} & \mathbf{G}(t) \end{bmatrix} \mathbf{V}(0)$$

where $\mathbf{O}_{i,k}$ is a zero matrix with i rows and k columns [37]. The matrix $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_2 (i, \mathbf{p}, t)$ decays for large times and $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t)$ is $T(\mathbf{p})$-periodic. The next section confirms and derives this expression for steady-state periodic solution for $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t)$ using difference equation analysis.

This decomposition implies that the event-time sensitivities (as given by Eq. (3.11)) will experience an initial transient, but will settle down to constant values as the steady-state periodic solution is reached.

3.2.5 Difference Equation Formulations

From Eq. (3.26), we have:

$$\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t+NT(\mathbf{p})) \\
&\quad = \mathcal{A}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\,\mathcal{C}_{i-k}^{b_{k+1}}\mathcal{A}_{i-k}\right)\right) \\
&\quad \times\left[\mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\,\mathcal{C}_{n_e-k}^{b_{k+1}}\mathcal{A}_{n_e-k}\right)\right)\right]^{N}, \\
&\quad \forall t\in[\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i=0,\ldots,n_e.
\end{aligned} \tag{3.28}$$

Also from Eq. (3.26), we have:

$$\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t+(N+1)T(\mathbf{p})) \\
&\quad = \mathcal{A}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\,\mathcal{C}_{i-k}^{b_{k+1}}\mathcal{A}_{i-k}\right)\right) \\
&\quad \times\left[\mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\,\mathcal{C}_{n_e-k}^{b_{k+1}}\mathcal{A}_{n_e-k}\right)\right)\right]^{N+1}, \\
&\quad \forall t\in[\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i=0,\ldots,n_e.
\end{aligned} \tag{3.29}$$

Using Eq. (3.28) in Eq. (3.29) we have,

$$\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t+(N+1)T(\mathbf{p})) \\
&\quad = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t+NT(\mathbf{p})) \\
&\quad \times\left[\mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\,\mathcal{C}_{n_e-k}^{b_{k+1}}\mathcal{A}_{n_e-k}\right)\right)\right] \\
&\quad = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i+1,\mathbf{p},t+NT(\mathbf{p}))\mathbf{M}, \\
&\quad \forall t\in[\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i=0,\ldots,n_e, \forall N=0,1,\ldots.
\end{aligned} \tag{3.30}$$

Equation (3.30) is the difference equation for initial condition sensitivities.

Steady-State Solution of Difference Equations

If the initial condition sensitivities have a periodic steady-state solution it follows from Eq. (3.30) that it will satisfy:

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_{ss} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_{ss} \mathbf{M}$$

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_{ss} (\mathbf{M} - \mathbf{I}) = \mathbf{0}. \tag{3.31}$$

Uniqueness of the solution of Eq. (3.31) depends on the eigenvalues ρ_i of the matrix $\mathbf{M}$. Since for stable limit cycles, $\mathbf{M}$ has one eigenvalue equal to 1 and the rest lie strictly inside the unit circle, Eq. (3.31) has infinite solutions. Hence there exists a periodic steady-state solution for the initial-condition sensitivities and this orbit is attractive.

Denote the periodic steady-state solution for all times by $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t)$, $\forall t \in [\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})]$, $\forall i \in \mathcal{E}, \forall N = 0, 1, \ldots$. Then, it will satisfy Eq. (3.30):

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t) (\mathbf{M} - \mathbf{I}) = \mathbf{0}. \tag{3.32}$$

Since it is a LCO, then by definition $\mathbf{M}$ has only one eigenvalue equal to 1. Therefore:

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t) = \mathbf{u}(t)\mathbf{v}_1^{\mathrm{T}}, \tag{3.33}$$

where $\mathbf{v}_1$ is a left eigenvector of $\mathbf{M}$ corresponding to the eigenvalue equal to 1. Note that this is a periodic time-varying rank one matrix. Eq. (3.33) is equivalent to the statement:

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, t) = \left[\begin{array}{cc} \mathbf{u}(t) & \mathbf{O}_{n_x, n_x - 1} \end{array}\right] \mathbf{V}(0), \tag{3.34}$$

with the first row of $\mathbf{V}(0)$ as $\mathbf{v}_1^{\mathrm{T}}$ and the remaining rows of $\mathbf{V}(0)$ can be anything.

Also for $t = T(\mathbf{p})$, Eq. (3.28) and Eq. (3.29) imply the additional relation:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, (N+1)T(\mathbf{p})) = \mathbf{M}\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, NT(\mathbf{p})), \forall N = 0, 1, \ldots. \quad (3.35)$$

Therefore, the periodic steady-state solution at $t = T(\mathbf{p})$ also satisfies Eq. (3.35):

$$(\mathbf{M} - \mathbf{I})\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)_1 (i, \mathbf{p}, T(\mathbf{p})) = \mathbf{0},$$

$$\Rightarrow (\mathbf{M} - \mathbf{I})\, \mathbf{u}(T(\mathbf{p}))\mathbf{v}_1^{\mathrm{T}} = \mathbf{0},$$

$$\Rightarrow (\mathbf{M} - \mathbf{I})\, \mathbf{u}(T(\mathbf{p}))\, \|\mathbf{v}_1\|_2^2 = \mathbf{0},$$

and it follows that $\mathbf{u}(T(\mathbf{p}))$ is a right eigenvector of $\mathbf{M}$ corresponding to the eigenvalue equal to 1.

Let $t \geq s$, then the *forward* state transition function for the hybrid system from the initial condition:

$$\mathbf{x}(j, \mathbf{p}, s) = \mathbf{a},$$

is defined as:

$$\begin{aligned} \mathbf{x}(i, \mathbf{p}, t) = \boldsymbol{\mathcal{F}}(t - s, i, j, \mathbf{a}), \forall t \in [\sigma_{i,N_1}(\mathbf{p}), \tau_{i,N_1}(\mathbf{p})], \\ \forall s \in [\sigma_{j,N_2}(\mathbf{p}), \tau_{j,N_2}(\mathbf{p})], \end{aligned} \quad (3.36)$$

for some N_1 and N_2, $N_1 \geq N_2$. From Eq. (3.12):

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t) = \frac{\partial \boldsymbol{\mathcal{F}}}{\partial \mathbf{a}}(t - 0, i, 1, \mathbf{x}(1, \mathbf{p}, 0)).$$

From the above equation and continuity at $t = T(\mathbf{p})$, the expression for the monodromy matrix $\mathbf{M}$ is:

$$\mathbf{M} \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e + 1, \mathbf{p}, T(\mathbf{p})) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(1, \mathbf{p}, T(\mathbf{p})) = \frac{\partial \boldsymbol{\mathcal{F}}}{\partial \mathbf{a}}(T(\mathbf{p}), 1, 1, \mathbf{x}(1, \mathbf{p}, 0)). \quad (3.37)$$

From Eq. (3.36):

$$\mathbf{x}(i, \mathbf{p}, t + T(\mathbf{p})) = \boldsymbol{\mathcal{F}}(T(\mathbf{p}), i, i, \mathbf{x}(i, \mathbf{p}, t)), \forall t \in [\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \ \forall i \in \mathcal{E}. \tag{3.38}$$

Now differentiate Eq. (3.38) with respect to t; at right and left epoch boundaries, left-hand and right-hand derivatives are taken, respectively. This yields:

$$\dot{\mathbf{x}}(i, \mathbf{p}, t + T(\mathbf{p})) = \frac{\partial \boldsymbol{\mathcal{F}}}{\partial \mathbf{a}}(T(\mathbf{p}), i, i, \mathbf{x}(i, \mathbf{p}, t))\dot{\mathbf{x}}(i, \mathbf{p}, t). \tag{3.39}$$

From Eq. (3.39) at $t = 0$, for a LCO:

$$\dot{\mathbf{x}}(1, \mathbf{p}, 0) = \dot{\mathbf{x}}(1, \mathbf{p}, T(\mathbf{p})) = \frac{\partial \boldsymbol{\mathcal{F}}}{\partial \mathbf{a}}(T(\mathbf{p}), 1, 1, \mathbf{x}(1, \mathbf{p}, 0))\dot{\mathbf{x}}(1, \mathbf{p}, 0). \tag{3.40}$$

Using Eq. (3.37) in Eq. (3.40) to obtain:

$$\dot{\mathbf{x}}(1, \mathbf{p}, 0) = \mathbf{M}\dot{\mathbf{x}}(1, \mathbf{p}, 0). \tag{3.41}$$

Eq. (3.41) illustrates an important property of the monodromy matrix, i.e., it has eigenvalue 1 corresponding to a right eigenvector which is $\dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p})) = \dot{\mathbf{x}}(1, \mathbf{p}, 0)$. Hence we can choose $\mathbf{u}(T(\mathbf{p})) = \mathbf{u}(0) = \dot{\mathbf{x}}(1, \mathbf{p}, 0)$.

3.2.6 Parametric Sensitivity Analysis

Given $\mathbf{x}(i, \mathbf{p}, \cdot)$, $\forall i \in \mathcal{E}$ as defined in Section 2.1, define

$$\mathbf{B}(\mathbf{p}, t) = \frac{\partial \mathbf{F}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, t - NT(\mathbf{p})), \mathbf{p}), \ \forall t \in (\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})],$$

for $N \in \{0, 1, \ldots, \infty\}$. Hence, the elements of $\mathbf{B}(\mathbf{p}, t)$ are piecewise continuous and periodic functions of t with period $T(\mathbf{p})$.

Differentiating Eq. (3.7) with respect to parameters $\mathbf{p}$ yields:

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) =& \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) \\ &+ (\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) - \dot{\mathbf{x}}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))) \\ &\times \frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p}),\ \forall i=1,\ldots,n_e, \forall N \in \{0,1,\ldots,\infty\}, \quad (3.42)\end{aligned}$$

where $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p})$ are determined by the transition conditions. Differentiating equation (3.1) with respect to the parameters $\mathbf{p}$ yields:

$$\frac{\partial \sigma_{i,N}}{\partial \mathbf{p}}(\mathbf{p}) = N\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \sigma_i}{\partial \mathbf{p}}(\mathbf{p}), \forall i = 2,\ldots,n_e+1, \forall N \in \{0,1,\ldots,\infty\}. \quad (3.43)$$

Equation (3.43) shows that (on the limit cycle) as one goes further out in time the period stretch has larger and larger influence on the parametric event-time sensitivities and hence on the jumps in the parametric sensitivities. Note that this was not true for the initial-condition sensitivities because initial condition perturbations do not necessarily stay on the limit cycle.

Similar to the initial condition case we have:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p}) \left(\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))\right) \\ + \frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p}) = \mathbf{0}.\end{aligned} \quad (3.44)$$

The above linear equations can be solved for unique $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p})$, $\forall i = 1,\ldots,n_e$, $\forall N \in \{0,1,\ldots,\infty\}$, provided that

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i,\mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p})\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) \neq 0.$$

The expression obtained for $\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p})$, $\forall i = 1,\ldots,n_e, \forall N \in \{0,1,\ldots,\infty\}$ after solving Eq. (3.44) is:

$$\frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p}) = -\frac{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p})\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})) + \frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p})}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p})),\mathbf{p})\dot{\mathbf{x}}(i,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))}. \tag{3.45}$$

The matrix of sensitivities with respect to the parameters $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i,\mathbf{p},t) \in \mathbb{R}^{n_x \times n_p}$ satisfies the following inhomogeneous linear system of differential equations with state jumps, $\forall N \in \{0,1,\ldots,\infty\}$:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)(i,\mathbf{p},t) = \mathbf{A}(\mathbf{p},t)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i,\mathbf{p},t) + \mathbf{B}(\mathbf{p},t),\ \forall t \in (\sigma_{i,N}(\mathbf{p}),\tau_{i,N}(\mathbf{p})], \forall i \in \mathcal{E} \tag{3.46}$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1,\mathbf{p},\sigma_1(\mathbf{p})) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$$

where $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ are the initial conditions for the sensitivities. $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1,\mathbf{p},\sigma_{i+1,N}(\mathbf{p}))$ is given by Eq. (3.42) for $i = 2,\ldots,n_e+1$. From continuity of the vector field at $t = NT(\mathbf{p})$:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1,\mathbf{p},NT(\mathbf{p})) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1,\mathbf{p},NT(\mathbf{p})), \forall N \in \{0,1,\ldots,\infty\}.$$

Note that $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ are not zero because of the dependence of the limit cycle on the parameters.

The solution of Eq. (3.46) is given by:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1,\mathbf{p},t) = \mathbf{\Phi}(\mathbf{p},t,\sigma_1(\mathbf{p}))\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \int_{\sigma_1(\mathbf{p})}^{t} \mathbf{\Phi}(\mathbf{p},t,s)\mathbf{B}(\mathbf{p},s)ds,\ \forall t \in [\sigma_1(\mathbf{p}),\tau_1(\mathbf{p})],$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1,\mathbf{p},t) = \mathbf{\Phi}\left(\mathbf{p},t,\sigma_{1,N}(\mathbf{p})\right)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\left(n_e+1,\mathbf{p},NT(\mathbf{p})\right) + \int_{\sigma_{1,N}(\mathbf{p})}^{t} \mathbf{\Phi}(\mathbf{p},t,s)\mathbf{B}(\mathbf{p},s)ds,$$
$$\forall t \in [\sigma_{1,N}(\mathbf{p}),\tau_{1,N}(\mathbf{p})],\ \forall N \in \{1,2,\ldots,\infty\},$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) = \mathbf{\Phi}(\mathbf{p}, t, \sigma_{i,N}(\mathbf{p})) \left[\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i-1, \mathbf{p}, \tau_{i-1,N}(\mathbf{p})) + \mathbf{\Delta}^{\mathbf{p}}_{i-1,N}(\mathbf{p}) \right]$$
$$+ \int_{\sigma_{i,N}(\mathbf{p})}^{t} \mathbf{\Phi}(\mathbf{p}, t, s)\mathbf{B}(\mathbf{p}, s) ds,$$
$$\forall t \in [\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})],\ \forall i = 2, \ldots, n_e + 1,\ \forall N \in \{0, 1, \ldots, \infty\}, \quad (3.47)$$

where the $\mathbf{\Delta}^{\mathbf{p}}_{i,N}(\mathbf{p})$ are given by

$$\mathbf{\Delta}^{\mathbf{p}}_{i,N}(\mathbf{p}) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) - \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p}))$$
$$= (\dot{\mathbf{x}}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})) - \dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))) \frac{\partial \sigma_{i+1,N}}{\partial \mathbf{p}}(\mathbf{p})$$
$$= -(\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) - \dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})))$$
$$\times \left(\frac{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p}) \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p}) \dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))} + \frac{\frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p})}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})), \mathbf{p}) \dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))} \right)$$
$$= -\mathbf{C}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) - \mathbf{D}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})),$$
$$\forall i = 1, \ldots, n_e, \forall N \in \{0, 1, \ldots, \infty\}.$$

$\mathbf{C}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p}))$ is defined by Eq. (3.14) and since $\mathbf{D}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p}))$ is $T(\mathbf{p})$-periodic, it can be given by:

$$\mathbf{D}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) = (\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})) - \dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})))$$
$$\times \frac{\frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p})}{\frac{\partial \mathcal{L}}{\partial \mathbf{x}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \mathbf{p}) \dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p}))}, \quad (3.48)$$
$$\forall i = 1, \ldots, n_e.$$

Rewriting Eq. (3.47) for epoch $i+1, N=0$ and substituting for $\mathbf{\Delta}^{\mathbf{p}}_{i,N}(\mathbf{p})$ we have,

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1,\mathbf{p},t) &= \mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\left[\mathbf{I}-\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))\right]\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))\\ &\quad -\mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\mathbf{D}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))+\int_{\sigma_{i+1}(\mathbf{p})}^{t}\mathbf{\Phi}(\mathbf{p},t,s)\mathbf{B}(\mathbf{p},s)ds,\\ &\quad \forall t\in[\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})],\ \forall i=1,\ldots,n_e. \end{aligned} \tag{3.49}$$

Expanding the recursion formula in Eq. (3.49) we have,

$$\begin{aligned}&\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1,\mathbf{p},t)\\ &= \mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\left(\prod_{k=0}^{i}[\mathbf{I}-\mathbf{C}(i-k,\mathbf{p},\sigma_{i+1-k}(\mathbf{p}))]\mathbf{\Phi}(\mathbf{p},\tau_{i-k}(\mathbf{p}),\sigma_{i-k}(\mathbf{p}))\right)\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})\\ &\quad+\mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\left[\mathbf{I}-\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))\right]\\ &\quad\times\left(\sum_{j=1}^{i-1}\left(\prod_{k=0}^{i-j-1}\mathbf{\Phi}(\mathbf{p},\tau_{i-k}(\mathbf{p}),\sigma_{i-k}(\mathbf{p}))\left[\mathbf{I}-\mathbf{C}(i-1-k,\mathbf{p},\sigma_{i-k}(\mathbf{p}))\right]\right)\right.\\ &\qquad\qquad\left.\times\left(\int_{\sigma_j(\mathbf{p})}^{\tau_j(\mathbf{p})}\mathbf{\Phi}(\mathbf{p},\tau_j(\mathbf{p}),s)\mathbf{B}(\mathbf{p},s)ds\right)\right)\\ &\quad+\mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\left[\mathbf{I}-\mathbf{C}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p}))\right]\left(\int_{\sigma_i(\mathbf{p})}^{\tau_i(\mathbf{p})}\mathbf{\Phi}(\mathbf{p},\tau_i(\mathbf{p}),s)\mathbf{B}(\mathbf{p},s)ds\right)\\ &\quad-\mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\left(\sum_{j=1}^{i-1}\left(\prod_{k=0}^{i-j-1}[\mathbf{I}-\mathbf{C}(i-k,\mathbf{p},\sigma_{i+1-k}(\mathbf{p}))]\,\mathbf{\Phi}(\mathbf{p},\tau_{i-k}(\mathbf{p}),\sigma_{i-k}(\mathbf{p}))\right)\right.\\ &\qquad\qquad\left.\times\,\mathbf{D}(j,\mathbf{p},\sigma_{j+1}(\mathbf{p}))\right)\\ &\quad+\int_{\sigma_{i+1}(\mathbf{p})}^{t}\mathbf{\Phi}(\mathbf{p},t,s)\mathbf{B}(\mathbf{p},s)ds-\mathbf{\Phi}(\mathbf{p},t,\sigma_{i+1}(\mathbf{p}))\mathbf{D}(i,\mathbf{p},\sigma_{i+1}(\mathbf{p})),\\ &\quad\forall t\in[\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})],\forall i=0,\ldots,n_e, \end{aligned} \tag{3.50}$$

where $\mathbf{D}(0, \mathbf{p}, \sigma_1(\mathbf{p})) \equiv \mathbf{0}$. Let us define:

$$\boldsymbol{\mathcal{I}}_i(t) \equiv \int\limits_{\sigma_i(\mathbf{p})}^{t} \mathbf{\Phi}(\mathbf{p}, t, s)\mathbf{B}(\mathbf{p}, s)ds, \forall t \in [\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E},$$

$$\boldsymbol{\mathcal{I}}_i \equiv \int\limits_{\sigma_i(\mathbf{p})}^{\tau_i(\mathbf{p})} \mathbf{\Phi}(\mathbf{p}, \tau_i(\mathbf{p}), s)\mathbf{B}(\mathbf{p}, s)ds, \forall i = 1, \ldots, n_e,$$

$$\boldsymbol{\mathcal{I}}_0 \equiv \mathbf{0},$$

$$\boldsymbol{\mathcal{D}}_i \equiv \mathbf{D}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p})), \forall i = 1, \ldots, n_e,$$

$$\boldsymbol{\mathcal{D}}_0 \equiv \mathbf{0}.$$

Rewriting Eq. (3.50) in terms of $\boldsymbol{\mathcal{A}}_i$s, $\boldsymbol{\mathcal{C}}_i$s, $\boldsymbol{\mathcal{D}}_i$s and $\boldsymbol{\mathcal{I}}_i$s we have,

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1, \mathbf{p}, t) = {} & \boldsymbol{\mathcal{A}}_{i+1}(t) \left(\prod_{k=0}^{i} [\mathbf{I} - \boldsymbol{\mathcal{C}}_{i-k}]\boldsymbol{\mathcal{A}}_{i-k} \right) \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\
& + \boldsymbol{\mathcal{A}}_{i+1}(t)[\mathbf{I} - \boldsymbol{\mathcal{C}}_i] \left(\sum_{j=1}^{i-1} \left(\prod_{k=0}^{i-j-1} \boldsymbol{\mathcal{A}}_{i-k}[\mathbf{I} - \boldsymbol{\mathcal{C}}_{i-1-k}] \right) \boldsymbol{\mathcal{I}}_j \right) \\
& - \boldsymbol{\mathcal{A}}_{i+1}(t) \left(\sum_{j=1}^{i-1} \left(\prod_{k=0}^{i-j-1} [\mathbf{I} - \boldsymbol{\mathcal{C}}_{i-k}]\boldsymbol{\mathcal{A}}_{i-k} \right) \boldsymbol{\mathcal{D}}_j \right) \\
& + \boldsymbol{\mathcal{A}}_{i+1}(t)[\mathbf{I} - \boldsymbol{\mathcal{C}}_i]\boldsymbol{\mathcal{I}}_i + \boldsymbol{\mathcal{I}}_{i+1}(t) - \boldsymbol{\mathcal{A}}_{i+1}(t)\boldsymbol{\mathcal{D}}_i, \\
& \forall t \in [\sigma_{i+1}(\mathbf{p}), \tau_{i+1}(\mathbf{p})], \forall i = 0, \ldots, n_e.
\end{aligned} \tag{3.51}$$

Using Eq. (3.18) and simplifying Eq. (3.51), we have

$$
\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1,\mathbf{p},t) = \\
&\quad \mathcal{A}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\,\mathcal{C}_{i-k}^{b_{k+1}}\mathcal{A}_{i-k}\right)\right)\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\
&\quad + \mathcal{A}_{i+1}(t)[\mathbf{I}-\mathcal{C}_i]\left(\sum_{j=1}^{i-1}\left(\sum_{\mathbf{b}\in B_{i-j}}\left(\prod_{k=0}^{i-j-1}(-1)^{b_{k+1}}\,\mathcal{A}_{i-k}\mathcal{C}_{i-1-k}^{b_{k+1}}\right)\right)\mathcal{I}_j\right) \\
&\quad - \mathcal{A}_{i+1}(t)\left(\sum_{j=1}^{i-1}\left(\sum_{\mathbf{b}\in B_{i-j}}\left(\prod_{k=0}^{i-j-1}(-1)^{b_{k+1}}\,\mathcal{C}_{i-k}^{b_{k+1}}\mathcal{A}_{i-k}\right)\right)\mathcal{D}_j\right) \\
&\quad + \mathcal{A}_{i+1}(t)[\mathbf{I}-\mathcal{C}_i]\mathcal{I}_i + \mathcal{I}_{i+1}(t) - \mathcal{A}_{i+1}(t)\mathcal{D}_i, \\
&\quad \forall t\in[\sigma_{i+1}(\mathbf{p}),\tau_{i+1}(\mathbf{p})], \forall i = 0,\ldots,n_e.
\end{aligned}
\tag{3.52}
$$

After one time period $t = T(\mathbf{p})$ and $i = n_e$ we have,

$$
\begin{aligned}
&\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1,\mathbf{p},T(\mathbf{p})) = \\
&\quad \mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\,\mathcal{C}_{n_e-k}^{b_{k+1}}\mathcal{A}_{n_e-k}\right)\right)\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\
&\quad + \mathcal{A}_{n_e+1}(T(\mathbf{p}))[\mathbf{I}-\mathcal{C}_{n_e}]\left(\sum_{j=1}^{n_e-1}\left(\sum_{\mathbf{b}\in B_{n_e-j}}\left(\prod_{k=0}^{n_e-j-1}(-1)^{b_{k+1}}\,\mathcal{A}_{n_e-k}\mathcal{C}_{n_e-1-k}^{b_{k+1}}\right)\right)\mathcal{I}_j\right) \\
&\quad - \mathcal{A}_{n_e+1}(T(\mathbf{p}))\left(\sum_{j=1}^{n_e-1}\left(\sum_{\mathbf{b}\in B_{n_e-j}}\left(\prod_{k=0}^{n_e-j-1}(-1)^{b_{k+1}}\,\mathcal{C}_{n_e-k}^{b_{k+1}}\mathcal{A}_{n_e-k}\right)\right)\mathcal{D}_j\right) \\
&\quad + \mathcal{A}_{n_e+1}(T(\mathbf{p}))[\mathbf{I}-\mathcal{C}_{n_e}]\mathcal{I}_{n_e} + \mathcal{I}_{n_e+1}(T(\mathbf{p})) - \mathcal{A}_{n_e+1}(T(\mathbf{p}))\mathcal{D}_{n_e}.
\end{aligned}
\tag{3.53}
$$

By definition the $\mathcal{C}_i$s, and $\mathcal{D}_i$s are constant. From Equation (3.53) for $t = NT(\mathbf{p})$

and $i = n_e$ it follows:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, NT(\mathbf{p})) =$$

$$\begin{aligned}
&\left(\boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{n_e-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{n_e-k}\right)\right)\right)^N \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\
&+\left(\sum_{q=0}^{N-1}\left(\boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p}))\left(\sum_{\mathbf{b}\in B_{n_e+1}}\left(\prod_{k=0}^{n_e}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{n_e-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{n_e-k}\right)\right)\right)^q\right) \\
&\quad\times\Bigg[\boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p}))[\mathbf{I}-\boldsymbol{\mathcal{C}}_{n_e}]\left(\sum_{j=1}^{n_e-1}\left(\sum_{\mathbf{b}\in B_{n_e-j}}\left(\prod_{k=0}^{n_e-j-1}(-1)^{b_{k+1}}\boldsymbol{\mathcal{A}}_{n_e-k}\boldsymbol{\mathcal{C}}_{n_e-1-k}^{b_{k+1}}\right)\right)\boldsymbol{\mathcal{I}}_j\right) \\
&\quad-\boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p}))\left(\sum_{j=1}^{n_e-1}\left(\sum_{\mathbf{b}\in B_{n_e-j}}\left(\prod_{k=0}^{n_e-j-1}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{n_e-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{n_e-k}\right)\right)\boldsymbol{\mathcal{D}}_j\right) \\
&\quad+\boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p}))[\mathbf{I}-\boldsymbol{\mathcal{C}}_{n_e}]\boldsymbol{\mathcal{I}}_{n_e}+\boldsymbol{\mathcal{I}}_{n_e+1}(T(\mathbf{p}))-\boldsymbol{\mathcal{A}}_{n_e+1}(T(\mathbf{p}))\boldsymbol{\mathcal{D}}_{n_e}\Bigg], \\
&\quad\forall N\in\{0,1,\ldots,\infty\}.
\end{aligned} \tag{3.54}$$

Extending the expression for the parametric sensitivities for all times yields:

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i+1,\mathbf{p},t) &= \boldsymbol{\mathcal{A}}_{i+1}(t)\left(\sum_{\mathbf{b}\in B_{i+1}}\left(\prod_{k=0}^{i}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1,\mathbf{p},NT(\mathbf{p})) \\
&+\boldsymbol{\mathcal{A}}_{i+1}(t)[\mathbf{I}-\boldsymbol{\mathcal{C}}_i]\left(\sum_{j=1}^{i-1}\left(\sum_{\mathbf{b}\in B_{i-j}}\left(\prod_{k=0}^{i-j-1}(-1)^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\boldsymbol{\mathcal{C}}_{i-1-k}^{b_{k+1}}\right)\right)\boldsymbol{\mathcal{I}}_j\right) \\
&-\boldsymbol{\mathcal{A}}_{i+1}(t)\left(\sum_{j=1}^{i-1}\left(\sum_{\mathbf{b}\in B_{i-j}}\left(\prod_{k=0}^{i-j-1}(-1)^{b_{k+1}}\boldsymbol{\mathcal{C}}_{i-k}^{b_{k+1}}\boldsymbol{\mathcal{A}}_{i-k}\right)\right)\boldsymbol{\mathcal{D}}_j\right) \\
&+\boldsymbol{\mathcal{A}}_{i+1}(t)[\mathbf{I}-\boldsymbol{\mathcal{C}}_i]\boldsymbol{\mathcal{I}}_i+\boldsymbol{\mathcal{I}}_{i+1}(t)-\boldsymbol{\mathcal{A}}_{i+1}(t)\boldsymbol{\mathcal{D}}_i, \\
&\forall t\in[\sigma_{i+1,N}(\mathbf{p}),\tau_{i+1,N}(\mathbf{p})], \forall i=0,\ldots,n_e,
\end{aligned} \tag{3.55}$$

for $\forall N \in \{0,1,\ldots,\infty\}$ and where $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1,\mathbf{p},NT(\mathbf{p}))$ is given by Eq. (3.54).

Difference Equation for Parametric Sensitivity Analysis

Let us define $\mathbf{P}$ as:

$$\begin{aligned}\mathbf{P} = {} & \mathcal{A}_{n_e+1}(T(\mathbf{p}))[\mathbf{I} - \mathcal{C}_{n_e}] \left(\sum_{j=1}^{n_e-1} \left(\sum_{\mathbf{b} \in B_{n_e-j}} \left(\prod_{k=0}^{n_e-j-1} (-1)^{b_{k+1}} \mathcal{A}_{n_e-k} \mathcal{C}_{n_e-1-k}^{b_{k+1}} \right) \right) \mathcal{I}_j \right) \\ & - \mathcal{A}_{n_e+1}(T(\mathbf{p})) \left(\sum_{j=1}^{n_e-1} \left(\sum_{\mathbf{b} \in B_{n_e-j}} \left(\prod_{k=0}^{n_e-j-1} (-1)^{b_{k+1}} \mathcal{C}_{n_e-k}^{b_{k+1}} \mathcal{A}_{n_e-k} \right) \right) \mathcal{D}_j \right) \\ & + \mathcal{A}_{n_e+1}(T(\mathbf{p}))[\mathbf{I} - \mathcal{C}_{n_e}]\mathcal{I}_{n_e} + \mathcal{I}_{n_e+1}(T(\mathbf{p})) - \mathcal{A}_{n_e+1}(T(\mathbf{p}))\mathcal{D}_{n_e}.\end{aligned} \tag{3.56}$$

Then Eq. (3.54) can be written as,

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, NT(\mathbf{p})) = \mathbf{M}^N \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \sum_{q=0}^{N-1} \mathbf{M}^q \mathbf{P} \tag{3.57}$$

for $N \in \{0, 1, \ldots, \infty\}$. Rewriting Eq. (3.57) for $N+1$:

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, (N+1)T(\mathbf{p})) &= \mathbf{M}^{N+1} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \sum_{q=0}^{N} \mathbf{M}^q \mathbf{P}, \\ &= \mathbf{M}\left(\mathbf{M}^N \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \sum_{q=0}^{N-1} \mathbf{M}^q \mathbf{P} \right) + \mathbf{P}, \\ \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, (N+1)T(\mathbf{p})) &= \mathbf{M} \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, NT(\mathbf{p})) + \mathbf{P}.\end{aligned} \tag{3.58}$$

Equation (3.58) is the difference equation for parametric sensitivities. But, it will not be of this form for $t \neq NT(\mathbf{p})$.

General Solution of the Sensitivity Equations

The difference equation in Eq. (3.58) can be reduced to a general solution of the sensitivity equations as a consequence of the following theorem.

Theorem 2. *Suppose the matrices* $\mathbf{Z}, \mathbf{R} \in \mathbb{R}^{n_x \times n_p}$ *satisfy the following equations:*

$$\mathbf{MR} = \mathbf{R}, \tag{3.59}$$

$$(\mathbf{M} - \mathbf{I})\mathbf{Z} - \mathbf{R} + \mathbf{P} = \mathbf{0}, \tag{3.60}$$

and $\mathbf{S}_N \in \mathbb{R}^{n_x \times n_p}$ *is given by*

$$\mathbf{S}_{N+1} = \mathbf{MS}_N + \mathbf{P}, \forall N = 0, 1, \ldots, \tag{3.61}$$

where $\mathbf{S}_0 \in \mathbb{R}^{n_x \times n_p}$ *is any arbitrary matrix. Then:*

$$\mathbf{S}_N = N\mathbf{R} + \mathbf{Z} + \mathbf{M}^N(\mathbf{S}_0 - \mathbf{Z}), \tag{3.62}$$

for $N \in \{0, 1, \ldots, \infty\}$.

Proof. This result can be proved by using mathematical induction. Let $P(N)$ be the statement in Eq. (3.62) for any positive integer N. $P(1)$ asserts

$$\mathbf{S}_1 = \mathbf{R} + \mathbf{Z} + \mathbf{M}(\mathbf{S}_0 - \mathbf{Z}).$$

Manipulating Eq. (3.61) for $N = 0$, using equation (3.60) we have,

$$\begin{aligned} \mathbf{S}_1 &= \mathbf{MS}_0 + \mathbf{P} \\ &= \mathbf{MS}_0 + \mathbf{R} - (\mathbf{M} - \mathbf{I})\mathbf{Z} \\ &= \mathbf{R} \; + \mathbf{Z} + \mathbf{M}(\mathbf{S}_0 - \mathbf{Z}). \end{aligned}$$

Hence $P(1)$ is true. Now we assume there is a N for which $P(N)$ is true then we must prove for the same N, $P(N+1)$ is true, i.e.,

$$\mathbf{S}_{N+1} = (N+1)\mathbf{R} + \mathbf{Z} + \mathbf{M}^{N+1}(\mathbf{S}_0 - \mathbf{Z}).$$

Manipulating Eq. (3.61) using Eq. (3.62), Eq. (3.60) and Eq. (3.59) we have,

$$
\begin{aligned}
\mathbf{S}_{N+1} &= \mathbf{M}\mathbf{S}_N + \mathbf{P}, \\
&= \mathbf{M}\big[N\mathbf{R} + \mathbf{Z} + \mathbf{M}^N(\mathbf{S}_0 - \mathbf{Z})\big] + \mathbf{P}, \\
&= N\mathbf{M}\mathbf{R} + \mathbf{M}\mathbf{Z} + \mathbf{M}^{N+1}(\mathbf{S}_0 - \mathbf{Z}) + \mathbf{P}, \\
&= N\mathbf{M}\mathbf{R} + \mathbf{M}\mathbf{Z} + \mathbf{M}^{N+1}(\mathbf{S}_0 - \mathbf{Z}) + \mathbf{R} - (\mathbf{M} - \mathbf{I})\mathbf{Z}, \\
&= (N+1)\mathbf{R} + \mathbf{Z} + \mathbf{M}^{N+1}(\mathbf{S}_0 - \mathbf{Z}).
\end{aligned}
$$

Hence $P(N+1)$ is true. □

Proposition 2. *If* $\mathbf{R}$ *is expressed as:*

$$
\mathbf{R} = -\dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}),
$$

where the left-hand derivative is taken at $t = T(\mathbf{p})$ *then*

$$
\mathbf{M}\mathbf{R} = \mathbf{R}.
$$

Proof. From Eq. (3.41) and continuity of the vector field at $T(\mathbf{p})$:

$$
\mathbf{M}\dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p})) = \dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p})).
$$

Hence,

$$
\begin{aligned}
\mathbf{M}\mathbf{R} &= \mathbf{M}\left(-\dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})\right) \\
&= -\dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \\
&= \mathbf{R}.
\end{aligned}
$$

□

Under the hypothesis of Theorem 2, the general solution of the difference Equations (3.61) is given by Eq. (3.62). The third term in Eq. (3.62) shows the influence of initial conditions for the sensitivities, $\mathbf{S}_0$. Since the initial condition for the system depend on the parameters, the matrix $\mathbf{S}_0$ cannot be set to zero and needs to be determined. Similar to as shown in [37] for regular oscillating systems, the solution of these difference equations takes the form

$$\mathbf{S}_N = N\mathbf{R} + \mathbf{Z}, \tag{3.63}$$

when $\mathbf{Z} = \mathbf{S}_0$.

In [15], a detailed theory of sufficient conditions for existence and uniqueness of sensitivity functions of hybrid systems with ODEs is developed and it is proved that under the assumptions already made Eq. (3.2) and Eq. (3.3) involve continuously differentiable mappings wrt $\mathbf{p}$. Differentiating Eq. (3.2) for $N = 0$ with respect to the parameters $\mathbf{p}$:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1, \mathbf{p}, T(\mathbf{p}))\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) - \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \dot{\mathbf{x}}(n_e+1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, T(\mathbf{p}))\right)_{\mathbf{x}_0=const.} = \mathbf{0},$$

$$\Rightarrow (\mathbf{M} - \mathbf{I})\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \dot{\mathbf{x}}(n_e+1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, T(\mathbf{p}))\right)_{\mathbf{x}_0=const.} = \mathbf{0}. \tag{3.64}$$

Hence $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ must satisfy Eq. (3.64).

Let $\mathbf{S}_0 = \mathbf{Z} = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$. Then Eq. (3.53) and the expression defining $\mathbf{P}$ in Eq. (3.56) indicate that $\mathbf{P}$ corresponds to the partial derivatives of the state variables with respect to parameters while keeping the initial conditions constant. Choosing $\mathbf{R} = -\dot{\mathbf{x}}(n_e+1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})$ and $\mathbf{Z} = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ in Eq. (3.60) shows that it is satisfied by Eq. (3.64). Hence from Eq. (3.58) and Eq. (3.63), for $N \in \{0, 1, \ldots, \infty\}$:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, NT(\mathbf{p})) = -N\dot{\mathbf{x}}(n_e+1, \mathbf{p}, NT(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}). \tag{3.65}$$

Expressions for $\mathbf{R}(i, \mathbf{p}, t)$, $\mathbf{P}(i, \mathbf{p}, t)$ and $\mathbf{Z}(i, \mathbf{p}, t)$

Theorem 3. *Suppose* $\mathbf{R}(i, \mathbf{p}, t)$ *is a solution of:*

$$\frac{d\mathbf{R}}{dt}(i, \mathbf{p}, t) = \mathbf{A}(t)\mathbf{R}(i, \mathbf{p}, t), \ \forall t \in (\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E} \tag{3.66}$$

with

$$\mathbf{R}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = -\dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}), \tag{3.67}$$

and

$$\mathbf{R}(i+1, \mathbf{p}, \sigma_{i+1}) = \mathbf{R}(i, \mathbf{p}, \sigma_{i+1}) + (\dot{\mathbf{x}}(i, \mathbf{p}, \sigma_{i+1}) - \dot{\mathbf{x}}(i+1, \mathbf{p}, \sigma_{i+1}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}),$$
$$\forall i = 1, \ldots, n_e.$$

Then, $\mathbf{R}(i, \mathbf{p}, t)$ *is given by:*

$$\mathbf{R}(i, \mathbf{p}, t) = -\dot{\mathbf{x}}(i, \mathbf{p}, t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}), \forall t \in [\sigma_i(p), \tau_i(p)], \forall i \in \mathcal{E}. \tag{3.68}$$

Proof. The above result can be proved by simply substituting it into the equations (3.66) and (3.67). Since $\dot{\mathbf{x}}(i, \mathbf{p}, \cdot)$ is a continuous $T(\mathbf{p})$ periodic function where it is defined, the expression for $\mathbf{R}(i, \mathbf{p}, t)$ satisfies Equation (3.67). Differentiating Equation (3.68) yields:

$$\frac{d\mathbf{R}}{dt}(i, \mathbf{p}, t) = -\ddot{\mathbf{x}}(i, \mathbf{p}, t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) = -\mathbf{A}(t)\dot{\mathbf{x}}(i, \mathbf{p}, t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) = \mathbf{A}(t)\mathbf{R}(i, \mathbf{p}, t),$$
$$\forall t \in (\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E}.$$

At the right boundaries, left-hand derivatives are taken. Hence Eq. (3.68) satisfies Eq. (3.66). The expression for $\mathbf{R}(i, \mathbf{p}, t)$ also satisfies the jumps at the boundaries. □

Let $\mathbf{P}(i, \mathbf{p}, t)$ represent the sensitivities with respect to parameters keeping the initial conditions constant, hence it is the solution of following system of differential equations for $N \in \{0, 1, \ldots, \infty\}$:

$$\frac{d\mathbf{P}}{dt}(i, \mathbf{p}, t) = \mathbf{A}(t)\mathbf{P}(i, \mathbf{p}, t) + \mathbf{B}(t), \forall t \in (\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})], \forall i \in \mathcal{E},$$

with

$$\mathbf{P}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \mathbf{0},$$

and

$$\begin{aligned}\mathbf{P}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) = (\mathbf{I} - \mathbf{C}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})))\,\mathbf{P}(i, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p}))\\ - \mathbf{D}(i+1, \mathbf{p}, \sigma_{i+1}(\mathbf{p})),\ \forall i = 1, \ldots, n_e,\ \forall N \in \{0, 1, \ldots, \infty\},\end{aligned}$$

and from continuity of the vector field

$$\mathbf{P}(1, \mathbf{p}, NT(\mathbf{p})) = \mathbf{P}(n_e + 1, \mathbf{p}, NT(\mathbf{p})), \forall N \in \{0, 1, \ldots, \infty\}.$$

Note that $\mathbf{P} = \mathbf{P}(n_e + 1, \mathbf{p}, T(\mathbf{p}))$.

In general, similar to the case for regular oscillating systems [37], for $N = 0, 1, \ldots$ and for any $t \in [\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})]$, we will show that the parametric sensitivities can be written as

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) = \frac{t}{T(\mathbf{p})}\mathbf{R}(i, \mathbf{p}, t) + \mathbf{Z}(i, \mathbf{p}, t), \forall i \in \mathcal{E}. \tag{3.69}$$

Eq. (3.69) satisfies the expression for parametric sensitivities for $t = NT(\mathbf{p})$ and $i = n_e + 1$ (Eq. (3.65)), if it is shown that the matrix $\mathbf{Z}(i, \mathbf{p}, \cdot)$ given by

$$\mathbf{Z}(i, \mathbf{p}, t) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) - \frac{t}{T(\mathbf{p})}\mathbf{R}(i, \mathbf{p}, t) \tag{3.70}$$

is $T(\mathbf{p})$ periodic. Note also that:

$$\mathbf{Z}(1, \mathbf{p}, 0) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1, \mathbf{p}, 0) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}.$$

Since $\mathbf{x}(i, \mathbf{p}, \cdot)$ is $T(\mathbf{p})$ periodic where it is defined, it follows for any fixed $t \in$

$[\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})]$ that:

$$\mathbf{x}(i, \mathbf{p}, t + T(\mathbf{p})) = \mathbf{x}(i, \mathbf{p}, t). \tag{3.71}$$

Eq. (3.71) involves continuously differentiable mappings with respect to $\mathbf{p}$ [15] and hence differentiating with respect to the parameters $\mathbf{p}$ yields:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t + T(\mathbf{p})) + \dot{\mathbf{x}}(i, \mathbf{p}, t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t). \tag{3.72}$$

Rewriting Eq. (3.70) for $t + T(\mathbf{p})$ and simplifying using Eq. (3.68) and Eq. (3.72)

$$\begin{aligned}
\mathbf{Z}(i, \mathbf{p}, t + T(\mathbf{p})) &= \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t + T(\mathbf{p})) - \frac{(t + T(\mathbf{p}))}{T(\mathbf{p})}\mathbf{R}(i, \mathbf{p}, t + T(\mathbf{p})), \\
&= \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) - \dot{\mathbf{x}}(i, \mathbf{p}, t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) - \frac{t}{T(\mathbf{p})}\mathbf{R}(i, \mathbf{p}, t) + \dot{\mathbf{x}}(i, \mathbf{p}, t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}), \\
&= \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) - \frac{t}{T(\mathbf{p})}\mathbf{R}(i, \mathbf{p}, t), \\
&= \mathbf{Z}(i, \mathbf{p}, t).
\end{aligned}$$

So, matrix $\mathbf{Z}(i, \mathbf{p}, \cdot)$ is continuous $T(\mathbf{p})$ periodic where it is defined and hence the general expression for the sensitivities in Eq. (3.69) satisfies Eq. (3.65). As mentioned in [45] for regular oscillating systems, $\mathbf{Z}(i, \mathbf{p}, t)$ are the cleaned-out sensitivities, as illustrated below.

Introduce the "cyclic time" $\tau = \frac{t}{T(\mathbf{p})}$, then $\hat{\mathbf{x}}(i, \mathbf{p}, \tau(t, T(\mathbf{p})))$ can be defined such that $\hat{\mathbf{x}}(i, \mathbf{p}, \tau(t, T(\mathbf{p}))) \equiv \mathbf{x}(i, \mathbf{p}, t)$ and the parametric sensitivity will be given by

$$\begin{aligned}
&\frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{p}}(i, \mathbf{p}, \tau(t, T(\mathbf{p}))) \\
&\quad = \left(\frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{p}}(i, \mathbf{p}, \tau(t, T(\mathbf{p})))\right)_{\tau(t,T(\mathbf{p}))=const.} \\
&\quad + \frac{d\hat{\mathbf{x}}}{d\tau}(i, \mathbf{p}, \tau(t, T(\mathbf{p})))\left(\frac{\partial \tau}{\partial t}(t, T(\mathbf{p}))\frac{\partial t}{\partial \mathbf{p}} + \frac{\partial \tau}{\partial T}(t, T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})\right), \\
&\quad = \left(\frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{p}}(i, \mathbf{p}, \tau(t, T(\mathbf{p})))\right)_{\tau(t,T(\mathbf{p}))=const.} - \frac{d\hat{\mathbf{x}}}{d\tau}(i, \mathbf{p}, \tau(t, T(\mathbf{p})))\frac{t}{(T(\mathbf{p}))^2}\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}), \\
&\quad \forall \tau \in \left[\frac{\sigma_i(\mathbf{p})}{T(\mathbf{p})}, \frac{\tau_i(\mathbf{p})}{T(\mathbf{p})}\right], \forall i \in \mathcal{E}.
\end{aligned} \tag{3.73}$$

Since

$$\frac{d\mathbf{x}}{d\tau}(i,\mathbf{p},t(\tau)) = \frac{d\mathbf{x}}{dt}(i,\mathbf{p},t(\tau))\frac{dt}{d\tau} = T(\mathbf{p})\frac{d\mathbf{x}}{dt}(i,\mathbf{p},t(\tau)),$$

and by definition:

$$\frac{d\hat{\mathbf{x}}}{d\tau}(i,\mathbf{p},\tau(t,T(\mathbf{p}))) = \frac{d\mathbf{x}}{d\tau}(i,\mathbf{p},t),$$

so that:

$$\frac{d\hat{\mathbf{x}}}{d\tau}(i,\mathbf{p},\tau(t,T(\mathbf{p}))) = T(\mathbf{p})\frac{d\mathbf{x}}{dt}(i,\mathbf{p},t),$$

and using the equivalent terms, Eq. (3.73) can be written as:

$$\frac{\partial\mathbf{x}}{\partial\mathbf{p}}(i,\mathbf{p},t)) = \left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}(i,\mathbf{p},t)\right)_{T(\mathbf{p})=const.} - \frac{t}{T(\mathbf{p})}\dot{\mathbf{x}}(i,\mathbf{p},t)\frac{\partial T}{\partial\mathbf{p}}(\mathbf{p}), \qquad (3.74)$$
$$\forall t \in [\sigma_i(\mathbf{p}),\tau_i(\mathbf{p})], \forall i \in \mathcal{E}.$$

Comparing Eq. (3.69) for $N = 0$ with Eq. (3.74):

$$\mathbf{Z}(i,\mathbf{p},t) = \left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}(i,\mathbf{p},t)\right)_{T(\mathbf{p})=const.}, \forall t \in [\sigma_i(\mathbf{p}),\tau_i(\mathbf{p})], \forall i \in \mathcal{E}. \qquad (3.75)$$

Eq. (3.75) shows that $\mathbf{Z}(i,\mathbf{p},t)$ can be interpreted as the part of the sensitivity keeping the period constant.

3.2.7 Boundary Value Formulation for the Period Sensitivities

As discussed earlier, the initial conditions for the sensitivities $\frac{\partial\mathbf{x}}{\partial\mathbf{p}}$ cannot be set to zero and need to be determined. In [15], a detailed theory of sufficient conditions for existence and uniqueness of sensitivity functions of hybrid systems with ODEs is developed and it is proved that under the assumptions already made Eq. (3.2) and Eq. (3.3) involve continuously differentiable mappings wrt $\mathbf{p}$. Hence, differentiating Eq. (3.2) and Eq. (3.3) with respect to the parameters $\mathbf{p}$ for $N = 0$, the following equations are obtained:

$$\begin{aligned}\frac{d\mathbf{x}}{dt}(n_e+1,\mathbf{p},T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1,\mathbf{p},T(\mathbf{p}))\right)_{\mathbf{x}_0=const.} \\ + \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(n_e+1,\mathbf{p},T(\mathbf{p}))\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) - \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) = \mathbf{0},\end{aligned} \tag{3.76}$$

$$\frac{\partial F_j}{\partial \mathbf{x}}(m_1,\mathbf{x}_0(\mathbf{p}),\mathbf{p})\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial F_j}{\partial \mathbf{p}}(m_1,\mathbf{x}_0(\mathbf{p}),\mathbf{p}) = \mathbf{0}.$$

In matrix form,

$$\begin{bmatrix} (\mathbf{M}(\mathbf{p})-\mathbf{I}) & \dot{\mathbf{x}}(n_e+1,\mathbf{p},T(\mathbf{p})) \\ \frac{\partial F_j}{\partial \mathbf{x}}(m_1,\mathbf{x}_0(\mathbf{p}),\mathbf{p}) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\mathbf{P}(n_e+1,\mathbf{p},T(\mathbf{p})) \\ -\frac{\partial F_j}{\partial \mathbf{p}}(m_1,\mathbf{x}_0(\mathbf{p}),\mathbf{p}) \end{bmatrix}.$$

It has been shown in [46], that the solution of this equation exists and is unique. The following matrix of unknowns are determined by this equation,

$$\begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix},$$

obtaining a set of initial conditions for the sensitivities $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$ and the period sensitivities.

3.2.8 Decomposition of the $\mathbf{Z}(i,\mathbf{p},t)$ matrix

Similar to regular oscillating systems [25], we will argue that $\mathbf{Z}(i,\mathbf{p},t)$ represents the information on how the parameters effect the shape of the limit cycle and contain information about phase behavior of the limit cycle. Hence the $\mathbf{Z}(i,\mathbf{p},t)$ matrix can be decomposed into two parts corresponding to these two effects of the parameters on the limit cycle by taking an orthogonal projection of $\mathbf{Z}(i,\mathbf{p},t)$ onto $\dot{\mathbf{x}}(i,\mathbf{p},t)$. Then, the expression for $\mathbf{Z}(i,\mathbf{p},t)$ will be:

$$\begin{aligned}\mathbf{Z}(i,\mathbf{p},t) = \mathbf{W}(i,\mathbf{p},t) + \dot{\mathbf{x}}(i,\mathbf{p},t)\boldsymbol{\delta}(i,\mathbf{p},t), \\ \forall t \in [\sigma_i(\mathbf{p}),\tau_i(\mathbf{p})], \forall i \in \mathcal{E},\end{aligned}$$

where left-hand and right-hand derivatives are taken at the boundaries. The matrix $\mathbf{W}(i, \mathbf{p}, t)$ and $\boldsymbol{\delta}(i, \mathbf{p}, t)$ can be obtained from $\mathbf{Z}(i, \mathbf{p}, t)$ by projection

$$\mathbf{W}(i, \mathbf{p}, t) = \left(\mathbf{I} - \frac{\dot{\mathbf{x}}(i, \mathbf{p}, t)\dot{\mathbf{x}}(i, \mathbf{p}, t)^{\mathrm{T}}}{\|\dot{\mathbf{x}}(i, \mathbf{p}, t)\|^2}\right)\mathbf{Z}(i, \mathbf{p}, t), \quad \forall t \in [\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E}, \tag{3.77}$$

$$\boldsymbol{\delta}(i, \mathbf{p}, t) = \frac{\left[\dot{\mathbf{x}}(i, \mathbf{p}, t)^{\mathrm{T}}\mathbf{Z}(i, \mathbf{p}, t)\right]}{\|\dot{\mathbf{x}}(i, \mathbf{p}, t)\|^2}, \quad \forall t \in [\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})], \forall i \in \mathcal{E}.$$

Since $\mathbf{Z}(i, \mathbf{p}, \cdot)$ and $\dot{\mathbf{x}}(i, \mathbf{p}, \cdot)$ are both continuous $T(\mathbf{p})$ periodic where they are defined, so are $\mathbf{W}(i, \mathbf{p}, \cdot)$ and $\boldsymbol{\delta}(i, \mathbf{p}, \cdot)$.

Overall sensitivities can be expressed in terms of the decomposition of the $\mathbf{Z}(i, \mathbf{p}, t)$ matrix as:

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(i, \mathbf{p}, t) = &-\frac{t}{T(\mathbf{p})}\dot{\mathbf{x}}(i, \mathbf{p}, t - NT(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \mathbf{W}(i, \mathbf{p}, t - NT(\mathbf{p}))\\ &+ \dot{\mathbf{x}}(i, \mathbf{p}, t - NT(\mathbf{p}))\boldsymbol{\delta}(i, \mathbf{p}, t),\\ &\forall t \in [\sigma_{i,N}(\mathbf{p}), \tau_{i,N}(\mathbf{p})], \forall i \in \mathcal{E}, \forall N = 0, 1, \ldots.\end{aligned}$$

3.2.9 Amplitude Sensitivities

The amplitude of state variable x_j for some $i_{j,max} \in \mathcal{E}$ and $i_{j,min} \in \mathcal{E}$ can be defined as

$$\begin{aligned}\Omega_j(\mathbf{p}) \equiv\ & x_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p})) - x_j(i_{j,\min}, \mathbf{p}, t_{j,\min}(\mathbf{p})),\\ & t_{j,\max}(\mathbf{p}) \in [\sigma_{i_{j,\max}}(\mathbf{p}), \tau_{i_{j,\max}}(\mathbf{p})],\\ & t_{j,\min}(\mathbf{p}) \in [\sigma_{i_{j,\min}}(\mathbf{p}), \tau_{i_{j,\min}}(\mathbf{p})],\end{aligned} \tag{3.78}$$

where $t_{j,\max}(\mathbf{p})$ and $t_{j,\min}(\mathbf{p})$ are times at which x_j attains its infimum and supremum value with respect to time, respectively.

Differentiating Eq. (3.78) with respect to the parameters $\mathbf{p}$ yields:

$$\begin{aligned}\frac{\partial \Omega_j}{\partial \mathbf{p}}(\mathbf{p}) = \mathbf{s}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p})) + \dot{x}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p}))\frac{\partial t_{j,\max}}{\partial \mathbf{p}}(\mathbf{p}) \\ - \mathbf{s}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}(\mathbf{p})) - \dot{x}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}(\mathbf{p}))\frac{\partial t_{j,\min}}{\partial \mathbf{p}}(\mathbf{p})\end{aligned} \tag{3.79}$$

where $\mathbf{s}_j$ represents the jth row of the sensitivity matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}$ and as usual, left-hand and right-hand time derivatives are taken at epoch boundaries. The second and fourth terms in Eq. (3.79) are nonzero only if both (a) the vector fields of state variable x_j at $t_{j,\max}(\mathbf{p})$ and $t_{j,\min}(\mathbf{p})$ and (b) the minimum and maximum time sensitivities $\frac{\partial t_{j,\min}}{\partial \mathbf{p}}(\mathbf{p})$ and $\frac{\partial t_{j,\max}}{\partial \mathbf{p}}(\mathbf{p})$ are nonzero. Both of these conditions can be possibly true when the infimum and/or supremum is attained at event times, i.e., at the epoch boundaries. In such a case, the minimum- and maximum-time sensitivities $\frac{\partial t_{j,\min}}{\partial \mathbf{p}}(\mathbf{p})$ and $\frac{\partial t_{j,\max}}{\partial \mathbf{p}}(\mathbf{p})$ are obtained from Equation (3.45). If the extremum occurs at an epoch boundary and both the left- and right-time derivative limits are nonzero, the amplitude sensitivity exists [15]. The case where one or both limits are zero is discussed next.

Consider the situation when a maximum occurs at an epoch boundary $\tau_i(\mathbf{p})$ and one of time-derivative limits is zero, the three cases that can occur are:

(a) The time derivative of the variable is continuous at the epoch boundary:

$$\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) = \dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})).$$

This case is shown in the Figure 3-3(a). There is no problem in calculating the amplitude sensitivity because the extremum is stationary point and not caused by the event.

(b) The time derivative jumps from zero to a negative value at the epoch boundary:

$$\begin{aligned}&\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) = 0, \\ &\dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})) < 0.\end{aligned}$$

This case is shown in Figure 3-3(b).

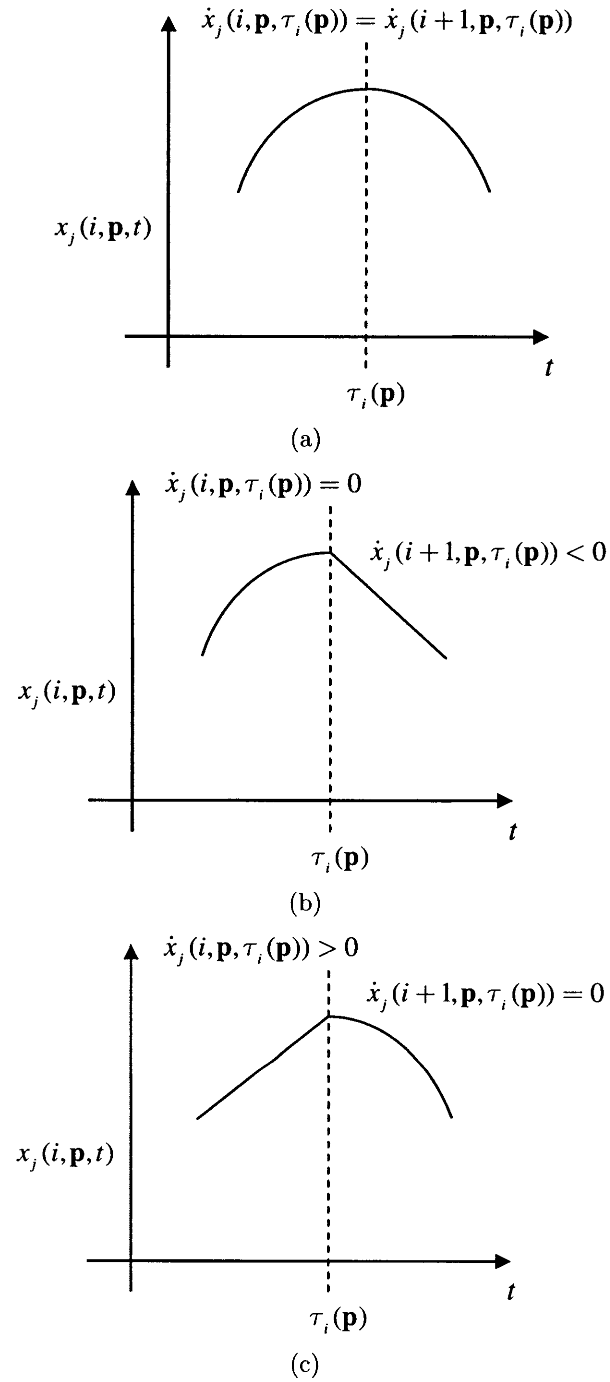


Figure 3-3: Maximum at an epoch boundary when one of time-derivative limits is zero: (a) Continuous vector field $\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) = \dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p}))$, (b) $\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) = 0$ and $\dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})) < 0$ and (c) $\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) > 0$ and $\dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})) = 0$.

(c) The time derivative jumps from a positive value to zero at the epoch boundary:

$$\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) > 0,$$
$$\dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})) = 0.$$

This case is shown in Figure 3-3(c).

In the second and third cases above, the amplitude is unlikely to be a smooth function of the parameters and hence the amplitude sensitivity will not exist.

For a minimum at an epoch boundary and one of time derivative limits is zero, again there are three cases:

(a) The time derivative of the variable is continuous at the epoch boundary:

$$\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) = \dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})).$$

(b) The time derivative jumps from zero to a positive value at the epoch boundary:

$$\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) = 0,$$
$$\dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})) > 0.$$

(c) The time derivative jumps from a negative value to zero at the epoch boundary:

$$\dot{x}_j(i, \mathbf{p}, \tau_i(\mathbf{p})) < 0,$$
$$\dot{x}_j(i+1, \mathbf{p}, \tau_i(\mathbf{p})) = 0.$$

Again, the amplitude is unlikely to be a smooth function of the parameters for the second and third cases.

The expression for the amplitude sensitivity in Eq. (3.79) can be further simplified in two different cases:

(a) If $t_{j,\max}(\mathbf{p})$ and $t_{j,\min}(\mathbf{p})$ are interior to epochs, i.e.:

$$\sigma_{i_{j,\max}}(\mathbf{p}) < t_{j,\max}(\mathbf{p}) < \tau_{i_{j,\max}}(\mathbf{p}),$$

and

$$\sigma_{i_{j,\min}}(\mathbf{p}) < t_{j,\min}(\mathbf{p}) < \tau_{i_{j,\min}}(\mathbf{p}),$$

then $\dot{x}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}) = 0$ and $\dot{x}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}) = 0$, respectively. Equation (3.79) reduces to:

$$\frac{\partial \Omega_j}{\partial \mathbf{p}}(\mathbf{p}) = \mathbf{s}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p})) - \mathbf{s}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}(\mathbf{p})). \tag{3.80}$$

In this case, the amplitude sensitivity can be calculated using $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}$, $\mathbf{Z}$ or $\mathbf{W}$ as:

$$\mathbf{s}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p})) = \mathbf{z}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p})) = \mathbf{w}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}(\mathbf{p})).$$

Also from this, it can be concluded that the amplitude sensitivity does not vary from period to period in this case because $\mathbf{z}_j$ and $\mathbf{w}_j$ are $T(\mathbf{p})$-periodic.

(b) If one of $t_{j,\max}(\mathbf{p})$ or $t_{j,\min}(\mathbf{p})$ are at one of the epoch boundaries, for e.g. $t_{j,\max}(\mathbf{p}) = \sigma_{i_{j,\max}}(\mathbf{p})$ and

$$\sigma_{i_{j,\min}}(\mathbf{p}) < t_{j,\min}(\mathbf{p}) < \tau_{i_{j,\min}}(\mathbf{p}),$$

then $\dot{x}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}) = 0$ but $\dot{x}_j(i_{j,\max}, \mathbf{p}, t_{j,\max}) \neq 0$. Equation (3.79) reduces to:

$$\begin{aligned} \frac{\partial \Omega_j}{\partial \mathbf{p}}(\mathbf{p}) = {} & \mathbf{s}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p})) + \dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p})) \frac{\partial \sigma_{i_{j,\max}}}{\partial \mathbf{p}}(\mathbf{p}) \\ & - \mathbf{s}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}(\mathbf{p})) \end{aligned} \tag{3.81}$$

Note that the infimum and supremum repeat at same time relative to start of the period in each period of the limit cycle. To check whether the amplitude sensitivities change from period to period in this case, let us consider the above equation for amplitude sensitivity for each period:

$$\begin{aligned}\frac{\partial \Omega_{j,N}}{\partial \mathbf{p}}(\mathbf{p}) = \; & \mathbf{s}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max},N}(\mathbf{p})) + \dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max},N}(\mathbf{p}))\frac{\partial \sigma_{i_{j,\max},N}}{\partial \mathbf{p}}(\mathbf{p}) \\ & - \mathbf{s}_j(i_{j,\min}, \mathbf{p}, t_{j,\min,N}(\mathbf{p})), \\ & \forall N \in \{0, 1, \ldots, \infty\} \end{aligned} \tag{3.82}$$

Putting the expression for $\mathbf{s}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max},N}(\mathbf{p}))$ and $\mathbf{s}_j(i_{j,\min}, \mathbf{p}, t_{j,\min,N}(\mathbf{p}))$ in Eq. (3.82) yields:

$$\begin{aligned}\frac{\partial \Omega_{j,N}}{\partial \mathbf{p}}(\mathbf{p}) = & -\frac{\sigma_{i_{j,\max},N}(\mathbf{p})}{T(\mathbf{p})}\dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max},N}(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \\ & + \mathbf{z}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max},N}(\mathbf{p})) \\ & + \dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{j,\max,N}(\mathbf{p}))\frac{\partial \sigma_{j,\max,N}}{\partial \mathbf{p}}(\mathbf{p}) \\ & + \frac{t_{j,\min,N}(\mathbf{p})}{T(\mathbf{p})}\dot{x}_j(i_{j,\min}, \mathbf{p}, t_{j,\min,N}(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) - \mathbf{z}_j(i_{j,\max}, \mathbf{p}, t_{j,\min,N}(\mathbf{p})) \\ & \forall N \in \{0, 1, \ldots, \infty\} \end{aligned} \tag{3.83}$$

Again, recalling that $\mathbf{z}_j$ is T-periodic, $\dot{x}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}) = 0$ and using Eq. (3.43) to reduce Eq. (3.83) to:

$$\begin{aligned}\frac{\partial \Omega_{j,N}}{\partial \mathbf{p}}(\mathbf{p}) = & -\frac{NT(\mathbf{p}) + \sigma_{i_{j,\max}}(\mathbf{p})}{T(\mathbf{p})}\dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \\ & + \mathbf{z}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p})) \\ & + \dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{j,\max}(\mathbf{p}))\left(N\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \sigma_{i_{j,\max}}}{\partial \mathbf{p}}(\mathbf{p})\right) \\ & - \mathbf{z}_j(i_{j,\max}, \mathbf{p}, t_{j,\min}(\mathbf{p})) \\ = & -\frac{\sigma_{i_{j,\max}}(\mathbf{p})}{T(\mathbf{p})}\dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \mathbf{z}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p})) \\ & + \dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{j,\max}(\mathbf{p}))\frac{\partial \sigma_{i_{j,\max}}}{\partial \mathbf{p}}(\mathbf{p}) - \mathbf{z}_j(i_{j,\max}, \mathbf{p}, t_{j,\min}(\mathbf{p})) \\ = & \; \mathbf{s}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p})) + \dot{x}_j(i_{j,\max}, \mathbf{p}, \sigma_{i_{j,\max}}(\mathbf{p}))\frac{\partial \sigma_{i_{j,\max}}}{\partial \mathbf{p}}(\mathbf{p}) \\ & - \mathbf{s}_j(i_{j,\min}, \mathbf{p}, t_{j,\min}(\mathbf{p})) \end{aligned}$$

Hence in this case also, above equation shows that the amplitude sensitivity do not change from one period to another.

As in the case of regular LCOs, it will be shown that the continuous matrix $\mathbf{W}(i, \mathbf{p}, t)$ is uniquely defined by $\mathbf{x}(i, \mathbf{p}, t)$ on each point of the limit cycle away from the events.

Take two sensitivity $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)_1$ and $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)_2$ solutions with initial conditions $\left(\frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}\right)_1(\mathbf{p})$ and $\left(\frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}\right)_2(\mathbf{p})$, obtained from PLC_1 and PLC_2, respectively. At some point of time on the limit cycle $\mathbf{x}_1(i, \mathbf{p}, t = \alpha) = \mathbf{x}_2(j, \mathbf{p}, s = \beta)$ due to two PLCs. Assume that $\alpha \in (\sigma_i(\mathbf{p}), \tau_i(\mathbf{p})), \beta \in (\sigma_j(\mathbf{p}), \tau_j(\mathbf{p}))$ are not equal to event times. Differentiate this with respect to parameters to obtain:

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)_1 (i, \mathbf{p}, t = \alpha) + \dot{\mathbf{x}}_1(i, \mathbf{p}, t = \alpha)\frac{\partial \alpha}{\partial \mathbf{p}}(\mathbf{p}) = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)_2 (j, \mathbf{p}, s = \beta) + \dot{\mathbf{x}}_2(j, \mathbf{p}, s = \beta)\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}). \tag{3.84}$$

Eq. (3.84) shows that the two sensitivity solutions satisfying the BVP in Eq. (3.76) differ only in parts in the direction of $\dot{\mathbf{x}}_1(i, \mathbf{p}, t = \alpha) = \dot{\mathbf{x}}_2(j, \mathbf{p}, s = \beta)$. This difference is eliminated from $\mathbf{W}(i, \mathbf{p}, t)$ by taking the projection of $\mathbf{Z}(i, \mathbf{p}, t)$ in a direction perpendicular to $\dot{\mathbf{x}}(i, \mathbf{p}, t)$ as illustrated in the Eq. (3.77). Hence $\mathbf{W}(i, \mathbf{p}, t)$ is uniquely defined by the value $\mathbf{x}(i, \mathbf{p}, t)$ for each point on the limit cycle away from the events.

Note that the possibility of x_j attaining a supremum or an infimum at a time that its time derivative is not equal to zero does not occur for a regular LCO.

3.2.10 Phase Sensitivities

Relative Phase Sensitivities $\boldsymbol{\delta}(i, \mathbf{p}, t)$

It was suggested in the previous study of oscillating dynamical systems [46], that $\boldsymbol{\delta}(i, \mathbf{p}, t)$ is a relative phase sensitivity of the limit cycle, where relative phase is defined as the time difference between two points described by two different PLCs. Consider $\mathbf{x}^*(l, \mathbf{p}, t)$ to be the solution of the BVP with initial conditions $\mathbf{x}_0^*(\mathbf{p})$ using

PLC_1. Then the sensitivity solution with respect to this PLC will be given by:

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{p}}(l,\mathbf{p},t) = -\frac{t}{T(\mathbf{p})}\dot{\mathbf{x}}^*(l,\mathbf{p},t)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \mathbf{W}^*(l,\mathbf{p},t) + \dot{\mathbf{x}}^*(l,\mathbf{p},t)\boldsymbol{\delta}^*(l,\mathbf{p},t),$$
$$\forall t \in [\sigma_{l,N}(\mathbf{p}), \tau_{l,N}(\mathbf{p})],\ \forall l \in \mathcal{E},\ \forall N \in \{0,1,\ldots,\infty\}. \tag{3.85}$$

Let $\mathbf{x}^{**}(o,\mathbf{p},s)$ to be the solution of the BVP with initial conditions $\mathbf{x}_0^{**}(\mathbf{p})$ using PLC_2. Then the sensitivity solution with respect to this PLC will be given by

$$\frac{\partial \mathbf{x}^{**}}{\partial \mathbf{p}}(o,\mathbf{p},s) = -\frac{s}{T(\mathbf{p})}\dot{\mathbf{x}}^{**}(o,\mathbf{p},s)\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \mathbf{W}^{**}(o,\mathbf{p},s) + \dot{\mathbf{x}}^{**}(o,\mathbf{p},s)\boldsymbol{\delta}^{**}(o,\mathbf{p},s),$$
$$\forall s \in [\sigma_{o,N}(\mathbf{p}), \tau_{o,N}(\mathbf{p})],\ \forall o \in \mathcal{E},\ \forall N \in \{0,1,\ldots,\infty\}. \tag{3.86}$$

Defining a pair $(\alpha(\mathbf{p}),\beta(\mathbf{p}))$ for some l and o by

$$\mathbf{x}^*(l,\mathbf{p},t=\beta(\mathbf{p})) = \mathbf{x}^{**}(o,\mathbf{p},s=\alpha(\mathbf{p})), \tag{3.87}$$

and thus it follows:

$$\dot{\mathbf{x}}^*(l,\mathbf{p},\beta(\mathbf{p})) = \dot{\mathbf{x}}^{**}(o,\mathbf{p},\alpha(\mathbf{p})).$$

Differentiating Eq. (3.87) with respect parameters $\mathbf{p}$ to obtain:

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{p}}(l,\mathbf{p},\beta(\mathbf{p})) + \dot{\mathbf{x}}^*(l,\mathbf{p},\beta(\mathbf{p}))\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}) = \frac{\partial \mathbf{x}^{**}}{\partial \mathbf{p}}(o,\mathbf{p},\alpha(\mathbf{p})) + \dot{\mathbf{x}}^{**}(o,\mathbf{p},\alpha(\mathbf{p}))\frac{\partial \alpha}{\partial \mathbf{p}}(\mathbf{p}).$$

Using Eqs. (3.85) and (3.86) to cancel identical terms:

$$\dot{\mathbf{x}}^*(l,\mathbf{p},\beta(\mathbf{p}))\Bigg(-\frac{\beta(\mathbf{p})}{T(\mathbf{p})} + \frac{\alpha(\mathbf{p})}{T(\mathbf{p})}\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}) - \frac{\partial \alpha}{\partial \mathbf{p}}(\mathbf{p}) + \boldsymbol{\delta}^*(l,\mathbf{p},\beta(\mathbf{p}))$$
$$- \boldsymbol{\delta}^{**}(o,\mathbf{p},\alpha(\mathbf{p}))\Bigg) + \mathbf{W}^*(l,\mathbf{p},\beta(\mathbf{p})) - \mathbf{W}^{**}(o,\mathbf{p},\alpha(\mathbf{p})) = \mathbf{0}.$$

As shown in the last section, $\mathbf{W}^*(l,\mathbf{p},\beta(\mathbf{p})) = \mathbf{W}^{**}(o,\mathbf{p},\alpha(\mathbf{p}))$ and the system is not stationary, so that

$$\frac{\partial(\alpha(\mathbf{p})-\beta(\mathbf{p}))}{\partial\mathbf{p}} = \frac{(\alpha(\mathbf{p})-\beta(\mathbf{p}))}{T(\mathbf{p})}\frac{\partial T}{\partial\mathbf{p}}(\mathbf{p}) + \boldsymbol{\delta}^*(l,\mathbf{p},\beta(\mathbf{p})) - \boldsymbol{\delta}^{**}(o,\mathbf{p},\alpha(\mathbf{p})). \quad (3.88)$$

The first term on the right-hand side of the above result is the contribution of period sensitivity to the phase sensitivity while the second term is the phase sensitivity while keeping the period constant.

Peak-to-Peak Phase Sensitivities

As defined previously for regular LCOs [46], peak-to-peak sensitivities are phase sensitivities where the relative phase is the time difference between the extrema in different state variables. First, it is assumed that the extrema occur interior to epochs.

Consider the time scale defined by the PLC with initial conditions $\mathbf{x}_0(\mathbf{p})$:

$$\dot{x}_1(1,\mathbf{p},\sigma_1(\mathbf{p})) = 0. \quad (3.89)$$

Then for some $l \in \mathcal{E}$, define $\beta(\mathbf{p}) \in (\sigma_l(\mathbf{p}), \tau_l(\mathbf{p}))$ as the time of the extremum of x_j relative to extremum of x_1 using the equation:

$$\dot{x}_j(l,\mathbf{p},\beta(\mathbf{p})) = 0, \quad (3.90)$$

which can be written as:

$$F_j(m_l,\mathbf{x}(l,\mathbf{p},\beta(\mathbf{p})),\mathbf{p}) = 0. \quad (3.91)$$

Differentiate Eq. (3.91) with respect to the parameters $\mathbf{p}$ to obtain:

$$\begin{aligned} \frac{\partial F_j}{\partial\mathbf{x}}(m_l,\mathbf{x}(l,\mathbf{p},\beta(\mathbf{p})),\mathbf{p})\left(\dot{\mathbf{x}}(l,\mathbf{p},\beta(\mathbf{p}))\frac{\partial\beta}{\partial\mathbf{p}}(\mathbf{p}) + \frac{\partial\mathbf{x}}{\partial\mathbf{p}}(l,\mathbf{p},\beta(\mathbf{p}))\right)& \\ + \frac{\partial F_j}{\partial\mathbf{p}}(m_l,\mathbf{x}(l,\mathbf{p},\beta(\mathbf{p})),\mathbf{p}) = \mathbf{0}.& \end{aligned} \quad (3.92)$$

Eq. (3.92) can solved directly for $\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p})$ provided that:

$$\frac{\partial F_j}{\partial \mathbf{x}}(m_l, \mathbf{x}(l, \mathbf{p}, \beta(\mathbf{p})), \mathbf{p})\dot{\mathbf{x}}(l, \mathbf{p}, \beta(\mathbf{p})) \neq 0.$$

The PLC chosen to define the time scale should be valid as in the case of regular LCOs.

To extend the peak-to-peak sensitivities for all the periods, let us define $\beta_N(\mathbf{p}) \in (\sigma_{l,N}(\mathbf{p}), \tau_{l,N}(\mathbf{p})), \forall N \in \{0, 1, \ldots, \infty\}$ as the time extremum of x_j in the $N + 1$th period relative to extremum of x_1. Since the vector field is $T(\mathbf{p})$-periodic, $\beta_N(\mathbf{p})$ can be written in terms of $\beta(\mathbf{p})$ as:

$$\beta_N(\mathbf{p}) = NT(\mathbf{p}) + \beta(\mathbf{p}), \forall N \in \{0, 1, \ldots, \infty\}. \tag{3.93}$$

Differentiate Eq. (3.93) with respect to the parameters $\mathbf{p}$ to obtain:

$$\frac{\partial \beta_N}{\partial \mathbf{p}}(\mathbf{p}) = N\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}), \forall N \in \{0, 1, \ldots, \infty\}. \tag{3.94}$$

Eq. (3.94) shows that the peak-to-peak sensitivity remains constant within a period and increases by a constant amount equal to period sensitivity $\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})$ at the start of the succeeding period. This is intuitively expected as the time $\beta_N(\mathbf{p})$ of extremum of x_j relative to x_1 increases by $T(\mathbf{p})$ when moving to a succeeding period and period stretch due to perturbations in parameters should affect this time. Also, Eq. (3.94) can be used to calculate the peak-to-peak sensitivity after solving for $\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p})$ using Eq. (3.92).

Consider $\mathbf{x}(l, \mathbf{p}, t)$ to be the solution using Eq. (3.89) as PLC_1 and $\mathbf{x}^{**}(o, \mathbf{p}, s)$ to be the solution using Eq. (3.90) as PLC_2 and $\alpha = 0$, so that $\mathbf{x}(l, \mathbf{p}, \beta(\mathbf{p})) = \mathbf{x}^{**}(1, \mathbf{p}, 0)$.

Then Eq. (3.92) can be written as:

$$\frac{\partial F_j}{\partial \mathbf{x}}(m_l, \mathbf{x}(l,\mathbf{p},\beta(\mathbf{p})),\mathbf{p})\Bigg(\dot{\mathbf{x}}(l,\mathbf{p},\beta(\mathbf{p}))\left(\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}) - \frac{\beta(\mathbf{p})}{T(\mathbf{p})}\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \boldsymbol{\delta}(l,\mathbf{p},\beta(\mathbf{p}))\right) + \mathbf{W}(l,\mathbf{p},\beta(\mathbf{p}))\Bigg) + \frac{\partial F_j}{\partial \mathbf{p}}(m_l,\mathbf{x}(l,\mathbf{p},\beta(\mathbf{p})),\mathbf{p}) = \mathbf{0}. \tag{3.95}$$

From PLC_2 it can obtained:

$$\frac{\partial F_j}{\partial \mathbf{x}}(m_l, \mathbf{x}^{**}(1,\mathbf{p},0),\mathbf{p})\left(\dot{\mathbf{x}}^{**}(1,\mathbf{p},0)\boldsymbol{\delta}^{**}(1,\mathbf{p},0) + \mathbf{W}^{**}(1,\mathbf{p},0)\right) + \frac{\partial F_j}{\partial \mathbf{p}}(m_l,\mathbf{x}^{**}(1,\mathbf{p},0),\mathbf{p}) = \mathbf{0}. \tag{3.96}$$

As was proved earlier, $\mathbf{W}^{**}(1,\mathbf{p},0) = \mathbf{W}(l,\mathbf{p},\beta(\mathbf{p}))$ and thus using Eq. (3.96) to simplify Eq. (3.95) to obtain:

$$\frac{\partial F_j}{\partial \mathbf{x}}(m_l, \mathbf{x}(l,\mathbf{p},\beta(\mathbf{p})),\mathbf{p})\dot{\mathbf{x}}(l,\mathbf{p},\beta(\mathbf{p}))\left(\frac{\partial \beta}{\partial \mathbf{p}}(\mathbf{p}) - \frac{\beta}{T(\mathbf{p})}\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \boldsymbol{\delta}(l,\mathbf{p},\beta) - \boldsymbol{\delta}^{**}(1,\mathbf{p},0)\right) = \mathbf{0}, \tag{3.97}$$

which can also be obtained from Eq. (3.88).

Considering a special case where the phase is the time difference between the peak of one state variable and the next peak in the same state variable, which is after time period $T(\mathbf{p})$. Then PLC_2 can be written as:

$$F_1(m_{n_e+1}, \mathbf{x}(n_e+1,\mathbf{p},T(\mathbf{p})),\mathbf{p}) = 0.$$

Differentiating the above equation with respect to $\mathbf{p}$ to obtain:

$$\frac{\partial F_1}{\partial \mathbf{x}}(m_{n_e+1}, \mathbf{x}(n_e+1,\mathbf{p},T(\mathbf{p})),\mathbf{p})\left(\dot{\mathbf{x}}(n_e+1,\mathbf{p},T(\mathbf{p}))\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1,\mathbf{p},T(\mathbf{p}))\right) + \frac{\partial F_1}{\partial \mathbf{p}}(m_{n_e+1},\mathbf{x}(n_e+1,\mathbf{p},T(\mathbf{p})),\mathbf{p}) = \mathbf{0} \tag{3.98}$$

Using the fact that F_1 is $T(\mathbf{p})$ periodic in time, it can be shown that the Eq. (3.98) can be reduced to the BVP for period sensitivities described in Eq. (3.76). Hence, as shown for regular LCOs, the period sensitivity is a special case of peak-to-peak sensitivities for oscillating hybrid systems as well.

Also, consider the case when the extremum of the variable x_j relative to variable x_1 happens at an epoch boundary. For the present work, a sufficient condition for this extremum to be an isolated extremum is that the time derivative of variable x_j should change sign at the epoch boundary. This condition is illustrated in Figure 3-4. Figure 3-4(a) shows that a maximum of the variable x_j occurs at the epoch boundary when the time derivative of x_j changes sign from positive to negative and vice versa for the minimum, which is also shown in Figure 3-4(b). Similar to the amplitude sensitivities, if an extremum occurs at an epoch boundary it is required to have to have both left- and right-hand time derivatives nonzero or both equal to zero otherwise the relative phase between peaks of x_j and x_1 will unlikely to be a smooth function of the parameters. The discussion in Section 3.2.9 on an extremum at an epoch boundary with one of the time-derivative limits as zero, also applies here.

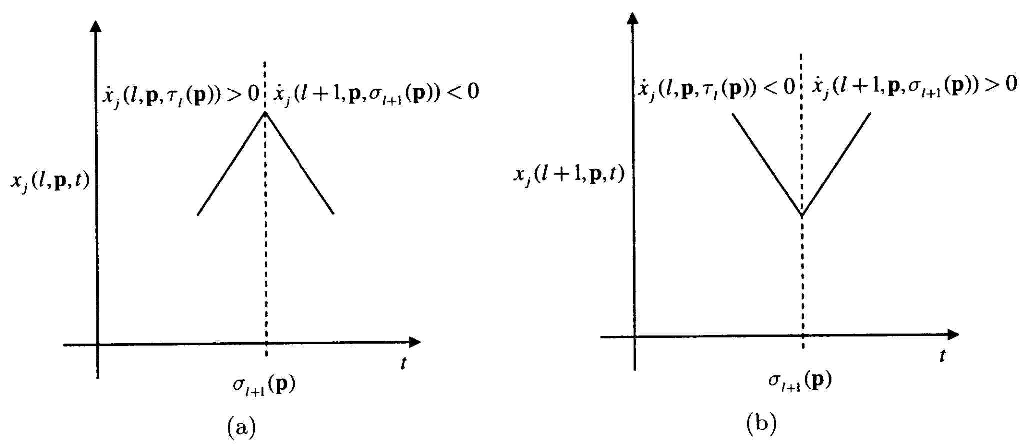


Figure 3-4: Extremum of variable x_j at epoch boundaries: (a) maximum and (b) minimum.

The phase here will be given by the event time, i.e., $\beta(\mathbf{p}) = \sigma_{l+1}(\mathbf{p})$. Then the peak-to-peak sensitivity will be given by the event sensitivity, with some conditions that the event time remains an extrema for small parameter variations. In this case, peak-to-peak sensitivity can be calculated by Eq. (3.45) for $N = 0$, given that $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}$ is fixed by Eq. (3.89). Peak-to-peak sensitivity for all periods will be given by:

$$\frac{\partial \beta_N}{\partial \mathbf{p}}(\mathbf{p}) = \frac{\partial \sigma_{l+1,N}}{\partial \mathbf{p}}(\mathbf{p}) = N\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) + \frac{\partial \sigma_{l+1}}{\partial \mathbf{p}}(\mathbf{p}), \forall N \in \{0, 1, \ldots, \infty\}. \tag{3.99}$$

The above equations shows that peak-to-peak sensitivity remains constant within a period and increase by a constant amount equal to the period sensitivity $\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})$ at the start of the next period.

Chapter 4

Numerical Methods

The importance of simulation of hybrid systems in different applications has motivated the development of many software packages (some of them are mentioned in [31]). Simulation of pure continuous and pure discrete systems is well-understood. These systems are commonly modeled with ODEs and DAEs. A number of robust codes exist for solving purely continuous systems, e.g., DSL48S [5], CVODES [39], DASSL [34]. In general, these simulation packages use implicit linear multi-step method, e.g., Gear's BDF method [16] or implicit Runga Kutta methods [3]. This thesis involves simulation of hybrid systems, which exhibit both continuous state and discrete state behavior, to solve the BVP for the limit cycle. DSL48SE [42] is used in the current work for solving the stiff ODEs and to compute the sensitivities according to the staggered-corrector method given in [13].

A general description of working with DSL48SE:

1. The model is compiled and validated.

2. The current mode is set to initial mode.

3. A consistent initialization is performed to determine a set of initial condition values for the continuous state and sensitivity variables.

4. The original ODE augmented with the sensitivity system is numerically integrated efficiently according to the staggered-corrector algorithm. After the

integration step, DSL48SE performs state event location using the algorithm given in [32]. One of the following occurs upon the completion of the step:

(a) integration is advanced a single step containing no state events.

(b) an event occurs and the integration is advanced to the earliest state event time and consistent state, time derivatives, and sensitivities at this point are returned.

5. If the integration step was successful and no events occurred, the calculation is advanced by calling DSL48SE again.

6. If the event was identified, the model is locked into a the new mode (the state condition causing the event is used to determine which mode is active), a consistent re-initialization is performed for the new model to compute values for the states and time derivatives at the start of the new mode, and the integration is advanced by calling DSL48SE again. During the first call to DSL48SE after an event is located, the jump in the sensitivities is computed automatically.

Hence, DSL48SE is called repeatedly to perform a series of integration steps to compute state and sensitivity trajectories similar to DSL48S (in fact, DSL48S is called within DSL48SE to perform the actual step). As in DSL48E [41], DSL48SE is called with the model *locked* into the current mode by performing *discontinuity locking*.

The state event location algorithm employed by DSL48SE consists of three main phases: state event detection/location, state event polishing, and computation of the jumps in sensitivities.

Event detection is performed during each integration step and it determines whether or not one or more events occur over the integration step just taken. If an event occurs, the event time is determined using the algorithm described in [32] and the earliest state event time is guaranteed to be found.

Event polishing involves adjusting the state event time to prevent discontinuity sticking and determining consistent values for state variables, derivatives, and sensitivities at this time.

The computation of the jumps in the sensitivities is done by using the equations provided in [15].

Some of the features of DSL48SE code:

1. The large unstructured sparse linear algebra package MA48 [12] is embedded in DSL48SE and DSL48S for the solution of the corrector equation. The MA48 package is especially suited for the types of problems that arise in chemical engineering, as well as other applications.

2. It offers to identify sensitivity variables with respect to a subset of the parameters as identically zero which helps in improving the efficiency of the control parametrization method.

3. There is an option to include or exclude the sensitivity variables from the truncation error test. This improves the efficiency of some applications that do not require guaranteed accuracy for sensitivity variables.

4. The staggered-corrector method has been employed in DSL48SE. It offers the options for solving the system alone or with sensitivities.

5. DSL48SE provides the capability of explicitly computing the sensitivity jumps and thus, time events and state events can be both handled by DSL48SE.

The additional information required to perform all the steps while using DSL48SE is generated by using DAEPACK [43]. Only a subroutine returning the residuals of the original hybrid system is provided to generate this information. The residual files for the three examples discussed in this thesis have been provided in the Appendix.

A technique used earlier to solve the BVP, which is computationally the most demanding part of the simulation, is followed here.

4.1 Transformation of the BVP

Using the cyclic time $\tau = \frac{t}{T(\mathbf{p})}$, the BVP described in Eqs. (3.2) and (3.3) was transformed to:

$$\hat{\mathbf{x}}(n_e + 1, \mathbf{p}, N + 1) - \hat{\mathbf{x}}_0(\mathbf{p}) = \mathbf{0}, \tag{4.1}$$

$$f_j\left(m_1, \hat{\mathbf{x}}\left(1, \mathbf{p}, \frac{\sigma_{1,N}(\mathbf{p})}{T(\mathbf{p})}\right), \mathbf{p}\right) = 0, \tag{4.2}$$

where $j \in \{1, \ldots, n_x\}$ and $N = 0, 1, \ldots$ with $\hat{\mathbf{x}}(i, \mathbf{p}, \tau(t, T(\mathbf{p})))$, $\forall i \in \mathcal{E}$ given by

$$\frac{d\hat{\mathbf{x}}}{d\tau}(i, \mathbf{p}, \tau(t, T(\mathbf{p}))) = T(\mathbf{p}) \cdot \mathbf{F}(m_i, \hat{\mathbf{x}}(i, \mathbf{p}, \tau(t, T(\mathbf{p}))), \mathbf{p}), \ \forall \tau \in \left(\frac{\sigma_{i,N}(\mathbf{p})}{T(\mathbf{p})}, \frac{\tau_{i,N}(\mathbf{p})}{T(\mathbf{p})}\right],$$

$$\mathbf{x}\left(i + 1, \mathbf{p}, \frac{\sigma_{i+1,N}(\mathbf{p})}{T(\mathbf{p})}\right) = \mathbf{x}\left(i, \mathbf{p}, \frac{\tau_{i,N}(\mathbf{p})}{T(\mathbf{p})}\right), \ \forall i = 1, \ldots, n_e, \ \forall N \in \{0, 1, \ldots, \infty\},$$

$$\hat{\mathbf{x}}\left(1, \mathbf{p}, \frac{\sigma_{1,N}(\mathbf{p})}{T(\mathbf{p})}\right) = \hat{\mathbf{x}}_0(\mathbf{p}), \ \forall N \in \{0, 1, \ldots, \infty\}.$$

The above transformation allows us to integrate to time 1 for solving the BVP for $N = 0$ and thus reducing the required computational time.

4.2 Solution of the BVP

The BVP which is given in the Eq. (4.1) and Eq. (4.2) for the initial conditions $\hat{\mathbf{x}}_0(\mathbf{p})$ and time period $T(\mathbf{p})$, was solved using the shooting method by converting it into an Initial Value Problem (IVP).

An initial guess was provided for the initial conditions $\hat{\mathbf{x}}_0(\mathbf{p})$ and the period $T(\mathbf{p})$. This guess needs to be in the region of attraction for the limit cycle so that Newton's method can converge to a point on the cycle. The transformed BVP was then integrated from 0 to 1. The values of the state variables after integration should be equal to the initial conditions. This forms a system of nonlinear equations in $\hat{\mathbf{x}}_0(\mathbf{p})$ and $T(\mathbf{p})$ which are solved using Newton's method. The set of equations are given by the BVP in Eq. (4.1) and Eq. (4.2).

The Newton step requires a calculation of the Jacobian matrix for deciding the direction to move towards the solution. The Jacobian matrix $\mathbf{J}$ for the Newton step is given by:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \hat{\mathbf{x}}}{\partial \hat{\mathbf{x}}_0}(n_e+1,\mathbf{p},1) - \mathbf{I} & \frac{\partial \hat{\mathbf{x}}}{\partial T}(n_e+1,\mathbf{p},1) \\ T(\mathbf{p})\frac{\partial F_j}{\partial \hat{\mathbf{x}}_0}\left(m_1, \hat{\mathbf{x}}\left(1,\mathbf{p},\frac{\sigma_1(\mathbf{p})}{T(\mathbf{p})}\right),\mathbf{p}\right) & F_j\left(m_1, \hat{\mathbf{x}}\left(1,\mathbf{p},\frac{\sigma_1(\mathbf{p})}{T(\mathbf{p})}\right),\mathbf{p}\right) \end{bmatrix} \tag{4.3}$$

Calculation of the Jacobian given by the Equation (4.3), needs the evaluation of the initial-condition sensitivities and sensitivities with respect to the period $T(\mathbf{p})$. The last row of the Jacobian is just given by the derivatives of the PLC in Eq. (4.2) with respect to $\hat{\mathbf{x}}_0$ and T.

The sensitivities with respect to the variables of the BVP were integrated along with the original ODE system using DSL48SE. DSL48SE is a stiff initial-value solver with sensitivity capabilities described more in the previous section. The algorithm used for the calculation of the sensitivities in DSL48SE was the staggered-corrector method [13] which has a number of advantages over the simultaneous-corrector algorithm described in [28].

The Newton iteration involves solving the following system of linear equations for the step and then using the step to determine the new approximation for zeros of the nonlinear equations:

$$\mathbf{J}\left(\begin{bmatrix} \hat{\mathbf{x}}_0^k \\ T^k \end{bmatrix}\right)\left(\begin{bmatrix} \hat{\mathbf{x}}_0^{k+1} \\ T^{k+1} \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{x}}_0^k \\ T^k \end{bmatrix}\right) = -\begin{bmatrix} \hat{\mathbf{x}}(n_e+1,\mathbf{p},1) - \hat{\mathbf{x}}_0^k \\ T^k.F_j\left(m_1, \hat{\mathbf{x}}\left(1,\mathbf{p},\frac{\sigma_1(\mathbf{p})}{T(\mathbf{p})}\right),\mathbf{p}\right) \end{bmatrix}. \tag{4.4}$$

Equation (4.4) was solved using Linear Algebra PACKage (LAPACK) [1]. LAPACK is written in Fortran90 and provides subroutines for solving systems of simultaneous systems of equations. Subroutine **dgsev** was used for the calculations here.

Integration was performed using DSL48SE with a relative tolerance of 10^{-8} and absolute tolerance of 10^{-10}. The BVP was solved to an absolute and relative tolerance of 10^{-8} and 10^{-6}, respectively.

It should also be mentioned that for a small system of LCOs that have a short transient time, which means that the approach to the periodic orbit is rapid from any initial condition, one can solve the BVP effectively by integrating over a sufficiently large time span. An event detection function can be used to determine the period of oscillation. However, for more accuracy it is recommended to solve the BVP which gives the value of the initial conditions and time period with more significant figures.

4.3 Solution of the Sensitivity Equations

After solving the BVP using the shooting method for initial conditions $\mathbf{x}_0(\mathbf{p})$ and the time period $T(\mathbf{p})$ of the limit cycle, value of $\mathbf{M}$ is calculated by finding sensitivities with respect to the initial conditions by setting the initial value of the initial condition sensitivities as an identity matrix and integrating Eq. (3.12) over one time period using DSL48SE.

The value of $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(n_e+1, \mathbf{p}, T(\mathbf{p}))\right)_{\mathbf{x}(0)=const.}$ is calculated by integrating the parametric sensitivity equation (3.46) over one period by setting the sensitivity initial conditions as zero. The absolute and relative tolerances were set to 10^{-10} and 10^{-8}, respectively.

The matrix operations to solve the system of linear equations given in Eq. (3.76) were performed in MATLAB. The initial conditions for the sensitivity matrix, $\frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$, and the period sensitivities, $\frac{\partial T}{\partial \mathbf{p}}(\mathbf{p})$, are obtained as a result of these operations.

Using these initial conditions for the sensitivity matrix, the parametric sensitivity equation (3.46) are again integrated to the desired number of periods using DSL48SE to obtain the raw sensitivities. Raw sensitivities are then decomposed to obtain $\mathbf{R}(i, \mathbf{p}, t)$ and $\mathbf{Z}(i, \mathbf{p}, t)$ by solving Eq. (3.68) and Eq. (3.70), respectively using MATLAB. The matrix $\mathbf{W}(i, \mathbf{p}, t)$ and vector $\boldsymbol{\delta}(i, \mathbf{p}, t)$ are constructed by performing matrix operations on $\mathbf{Z}(i, \mathbf{p}, t)$ using MATLAB to project it perpendicular to and onto the direction of $\mathbf{x}(i, \mathbf{p}, t)$, respectively. Finally, matrix operations are performed in MATLAB to calculate the amplitude sensitivities and peak-to-peak phase sensitivities using Equation (3.79) and Equation (3.92), respectively.

Chapter 5

Applications and Results

5.1 Pressure Relief Valve Hybrid System

A pressure relief valve is a type of valve usually used to control or limit the pressure in a vessel which can build up due to a process upset or equipment failure. The pressure is relieved by allowing the pressurized fluid to flow from an auxiliary passage out of the system. The model for the system is shown in Figure 5-1. A fluid enters the vessel with a flow rate, F_{in}. The pressure relief value opens at a predetermined set pressure, P_s and a portion of the fluid is diverted out through the auxiliary route until the pressure in the vessel reaches a predetermined reseat pressure, P_r at which time the valve closes. The system switches from Mode 1 to 2 when $P \geq P_s$ and from Mode 2 to 1 when $P \leq P_r$. The value of n_e is 2 and hybrid mode trajectory is given by $T_\mu = \{1, 2, 1\}$. The value of pressure in the vessel $P(t)$ oscillates in a limit cycle between P_r and P_s. This is an example of an 1-dimensional limit cycle where the system has only one continuous state variable, pressure P. This system has 8 parameters $\mathbf{p} = (R, T_f, V, k, P_a, P_s, P_r, F_{in})$ with $P_s > P_r$. R is the ideal gas constant, T_f is the temperature of the fluid entering the vessel, V is the volume of the vessel, k is the valve constant and P_a is atmospheric pressure. The values of the parameters are given in Table 5.1.

Table 5.1: Values of the parameters in the pressure relief valve hybrid system.

Parameters	Value
R	$8.314472 \times 10^{-5}\ \frac{\text{m}^3\ \text{bar}}{\text{K mol}}$
T_f	300 K
V	1.0221 m^3
k	$20\ \frac{\text{mol}}{\text{s bar}^{0.5}}$
P_a	1.01325 bar
P_s	10 bar
P_r	9 bar
F_{in}	$40\ \frac{\text{mol}}{\text{s}}$

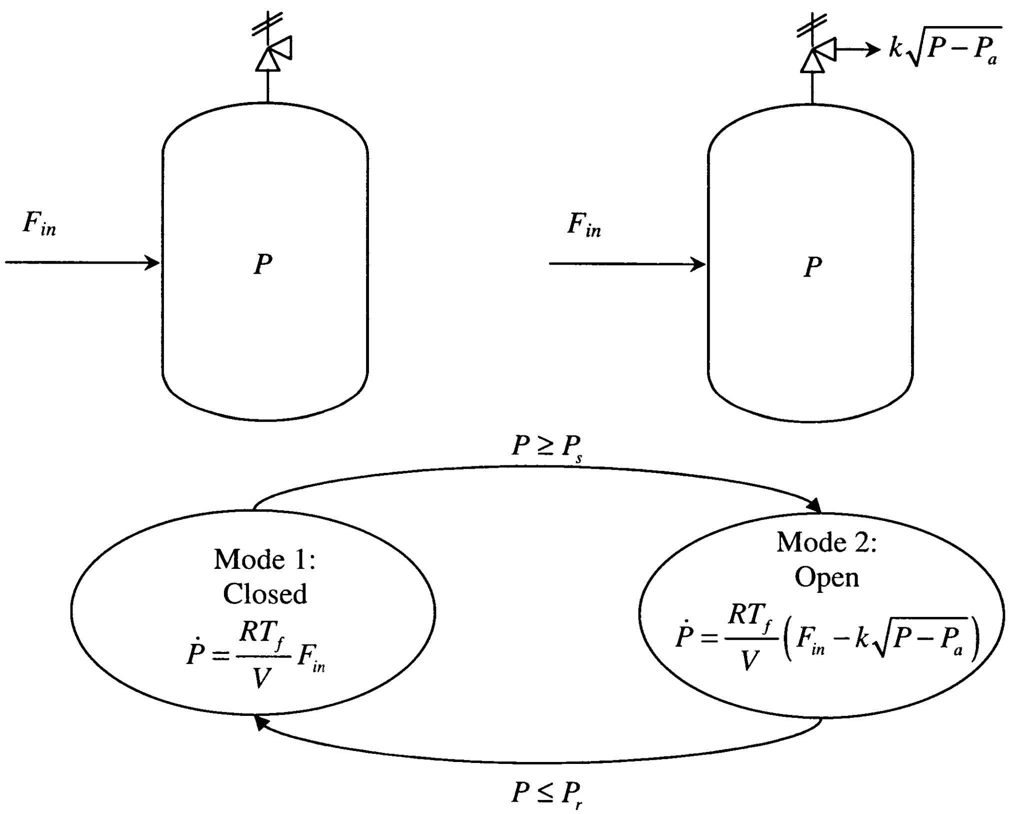


Figure 5-1: Hybrid dynamic model of the pressure relief valve hybrid system.

This system is described by the following sets of ODEs:

$$\text{Mode 1: } \dot{P} = \frac{RT_f}{V} F_{in},$$

$$\text{Mode 2} : \dot{P} = \frac{RT_f}{V} \left(F_{in} - k\sqrt{P - P_a} \right).$$

State continuity is employed at the transitions:

$$P(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) = P(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})), \ \forall i \in \{1, 2\}, \forall N \in \{0, 1, \ldots, \infty\}.$$

Figure 5-2 shows the state trajectory $P(t)$ for the pressure relief valve hybrid system over time. The BVP for the initial conditions and the period given in Eq. (3.2) and Eq. (3.3) was solved using the PLC $P(t = 0) = 9.5$ yielding results given in Table 5.2. The monodromy matrix contains only one value because there is only

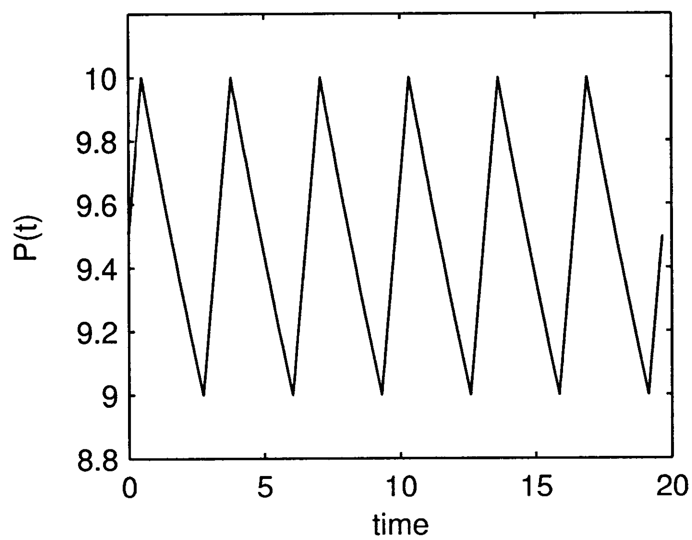


Figure 5-2: State trajectory $P(t)$ for the pressure relief valve hybrid system.

one continuous state variable. The value of the monodromy matrix is:

$$\mathbf{M} = \frac{\partial P}{\partial P_0} = 1.$$

Table 5.2 gives the results for the sensitivity initial conditions as well as period sen-

Table 5.2: Results for the sensitivity analysis of the pressure relief valve hybrid system. The resulting initial conditions were $P(0) = 9.5$ and period $T(\mathbf{p}) = 3.2757$.

Parameters	$\frac{\partial T}{\partial \mathbf{p}}$	$\frac{\partial P_0}{\partial \mathbf{p}}$
R	-39397.86	0
T_f	-0.0109	0
V	3.2049	0
k	-0.3607	0
P_a	0.4268	0
P_s	3.0778	0
P_r	-3.5046	0
F_{in}	0.0984	0

sitivities obtained by solving the BVP given by following system of linear equations:

$$\begin{bmatrix} \frac{\partial P}{\partial P_0} - 1 & \dot{P}(n_e+1, \mathbf{p}, T(\mathbf{p})) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial P_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\mathbf{P}(n_e+1, \mathbf{p}, T(\mathbf{p})) \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0.9762 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial P_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} =$$

$$\begin{bmatrix} -38458.75 & -0.0107 & 3.1285 & -0.3521 & 0.4166 & 3.0044 & -3.4211 & 0.0961 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system of parametric sensitivity ODEs for this example are given by:

$$\text{Mode 1}: \frac{d}{dt}\left(\frac{\partial P}{\partial \mathbf{p}}\right) = 0\frac{\partial P}{\partial \mathbf{p}} + \begin{bmatrix} \frac{T_f}{V}F_{in} & \frac{R}{V}F_{in} & -\frac{RT_f}{V^2}F_{in} & 0 & 0 & 0 & 0 & \frac{RT_f}{V} \end{bmatrix},$$

$$\text{Mode 2}: \frac{d}{dt}\left(\frac{\partial P}{\partial \mathbf{p}}\right) = -\frac{RT_f}{2V}\frac{k}{\sqrt{P-P_a}}\frac{\partial P}{\partial \mathbf{p}} +$$
$$\left[\begin{array}{cccccccc} \frac{T_f\left(F_{in}-k\sqrt{P-P_a}\right)}{V} & \frac{R\left(F_{in}-k\sqrt{P-P_a}\right)}{V} & -\frac{RT_f\left(F_{in}-k\sqrt{P-P_a}\right)}{V^2} & -\frac{RT_f\sqrt{P-P_a}}{V} & \frac{RT_f k}{2V\sqrt{P-P_a}} & 0 & 0 & \frac{RT_f}{V} \end{array}\right].$$

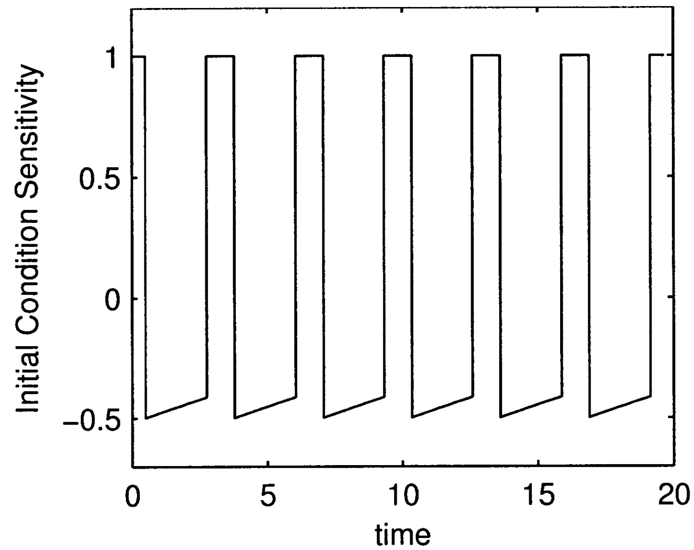


Figure 5-3: Initial condition sensitivity trajectory of P w.r.t. P_0 for the pressure relief valve hybrid system.

Initial-Condition Sensitivity Trajectory: The trajectory for the initial-condition sensitivity of the pressure relief valve hybrid system are shown in Figure 5-3. The initial-condition sensitivities have only a periodic part and a decaying part, mentioned in Section 3.2.4, is zero for all the times.

Sensitivity Trajectories: The sensitivity trajectories for the state variable P with respect to the parameter k are shown in Figure 5-4, along with trajectories for the relevant element of $\mathbf{Z}(t)$, $\mathbf{W}(t)$, and relative phase sensitivity with respect to k, $\delta_k(t)$. The sensitivity in Figure 5-4(a) grows as time evolves because of the unbounded part $\mathbf{R}(t)$ while the other part of the sensitivity $\mathbf{Z}(t)$ (shown in Figure 5-4(b)) is periodic in time. Both the unbounded part as well as the periodic part have jumps in them because the vector field is discontinuous at the transition and the event-time is sensitive to the parameter k (with the exception of the first event). Further decomposition of the periodic part into $\mathbf{W}(t)$ and $\delta_k(t)$ is shown in Figures 5-4(c)

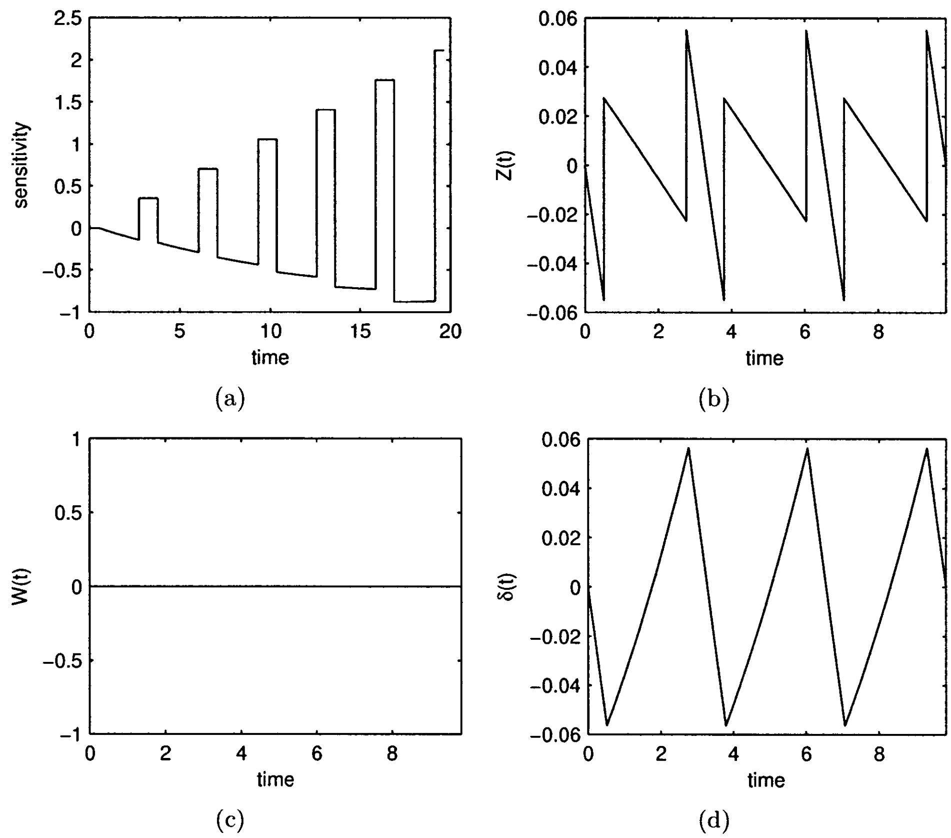


Figure 5-4: Sensitivity trajectories for the pressure relief valve hybrid system, all with respect to the parameter k: (a) full sensitivities of P, when $\frac{\partial P}{\partial \mathbf{p}}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \frac{\partial P_0}{\partial \mathbf{p}}(\mathbf{p})$, (b) period-independent periodic part $\mathbf{Z}(i, \mathbf{p}, t)$, (c) period and phase-independent part $\mathbf{W}(i, \mathbf{p}, t)$ and (d) relative phase sensitivity with respect to k, $\delta_k(i, \mathbf{p}, t)$.

Table 5.3: Results of the amplitude sensitivities for the state variable P in the pressure relief valve hybrid system.

Parameters	$\frac{\partial \Omega}{\partial \mathbf{p}}(\mathbf{p})$
R	0
T_f	0
V	0
k	0
P_a	0
P_s	1
P_r	-1
F_{in}	0

and 5-4(d). Since there is only one state variable for this system, $\mathbf{W}(t)$, which is the projection of $\mathbf{Z}(t)$ perpendicular to $\dot{P}(i, \mathbf{p}, t)$ and is calculated by Eq. (3.77), is zero over time.

Amplitude Sensitivity: The amplitude sensitivities for the state variable P are given in Table 5.3. These sensitivities are calculated by using the formula given in the Eq. (3.79). The amplitude for P is given by the difference in the values of P_s and P_r. Hence the amplitude sensitivity is nonzero only with respect to the parameters P_s and P_r.

5.2 Simple Switching Hybrid System

A simple switching hybrid system has been constructed in earlier work to show Raibert-type bifurcations [22]. It consists of 2 continuous state variables and 3 parameters. The value of n_e is 4 and hybrid mode trajectory is given by $T_\mu = \{1, 2, 3, 4, 1\}$. This system is given by the sets of ODEs which are also shown in Figure 5-5:

$$\text{Mode 1}: \begin{cases} \dot{x} = y \\ \dot{y} = -cx - by, \end{cases}$$

$$\text{Mode 2}: \begin{cases} \dot{x} = y \\ \dot{y} = -cx - by, \end{cases}$$

$$\text{Mode 3}: \begin{cases} \dot{x} = 0 \\ \dot{y} = 1, \end{cases}$$

$$\text{Mode 4}: \begin{cases} \dot{x} = y \\ \dot{y} = -cx - by. \end{cases}$$

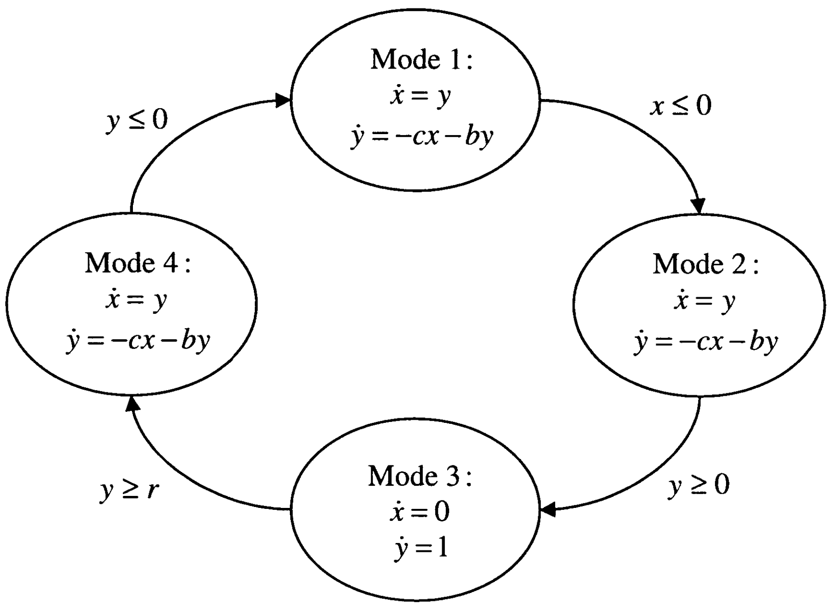


Figure 5-5: Hybrid dynamic model of simple switching hybrid system.

Note that the original formulation of this model [22] only had two modes and included "AND" operators in the transition conditions; these are avoided in the current formulation via the introduction of additional modes. The system switches from Mode 1 to 2 when $x \leqslant 0$, from Mode 2 to 3 when $y \geqslant 0$, from Mode 3 to 4 when $y \geqslant r$ and from Mode 4 to 1 when $y \leqslant 0$. The state variable vector is $\mathbf{x} = (x, y)$. State continuity is employed at the transitions:

$$\mathbf{x}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) = \mathbf{x}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})), \ \forall i \in \{1,2,3,4\}, \forall N \in \{0, 1, \ldots, \infty\}.$$

Figure 5-6 shows the limit cycle on the phase portrait for the simple switching hybrid system and the state trajectories x and y over the time. The parameters $\mathbf{p} = (r, b, c)$ are constrained as $b > 0$, $c > 0$, $(b^2 - 4c) < 0$ and $r > 0$. The parameter values used for this example are $b = 0.1$, $c = 1.5$ and $r = 0.710$. The BVP for the initial conditions and the time period given in Eq. (3.2) and Eq. (3.3) was solved using the PLC $\dot{x}(t = 0) = 0$ yielding the results given in Table 5.4. Table 5.4 gives the results for the sensitivity initial conditions as well as period sensitivities obtained by solving the BVP given by following system of linear equations:

$$\begin{bmatrix} \mathbf{M} - \mathbf{I} & \dot{\mathbf{x}}(n_e + 1, \mathbf{p}, T(\mathbf{p})) \\ 0 \;\; 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\mathbf{P}(n_e + 1, \mathbf{p}, T(\mathbf{p})) \\ 0 \;\; 0 \;\; 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{M} - \mathbf{I} & \begin{matrix} 0 \\ -1.2314 \end{matrix} \\ 0 \;\; 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -0.4718 & 1.6122 & 0.0573 \\ -0.5117 & 0.7237 & 1.6805 \\ 0 & 0 & 0 \end{bmatrix}.$$

The system of parametric sensitivity ODEs for this example are given by:

$$\text{Mode 1}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -y & -x \end{bmatrix},$$

$$\text{Mode 2}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -y & -x \end{bmatrix},$$

$$\text{Mode 3}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Mode 4}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -y & -x \end{bmatrix}.$$

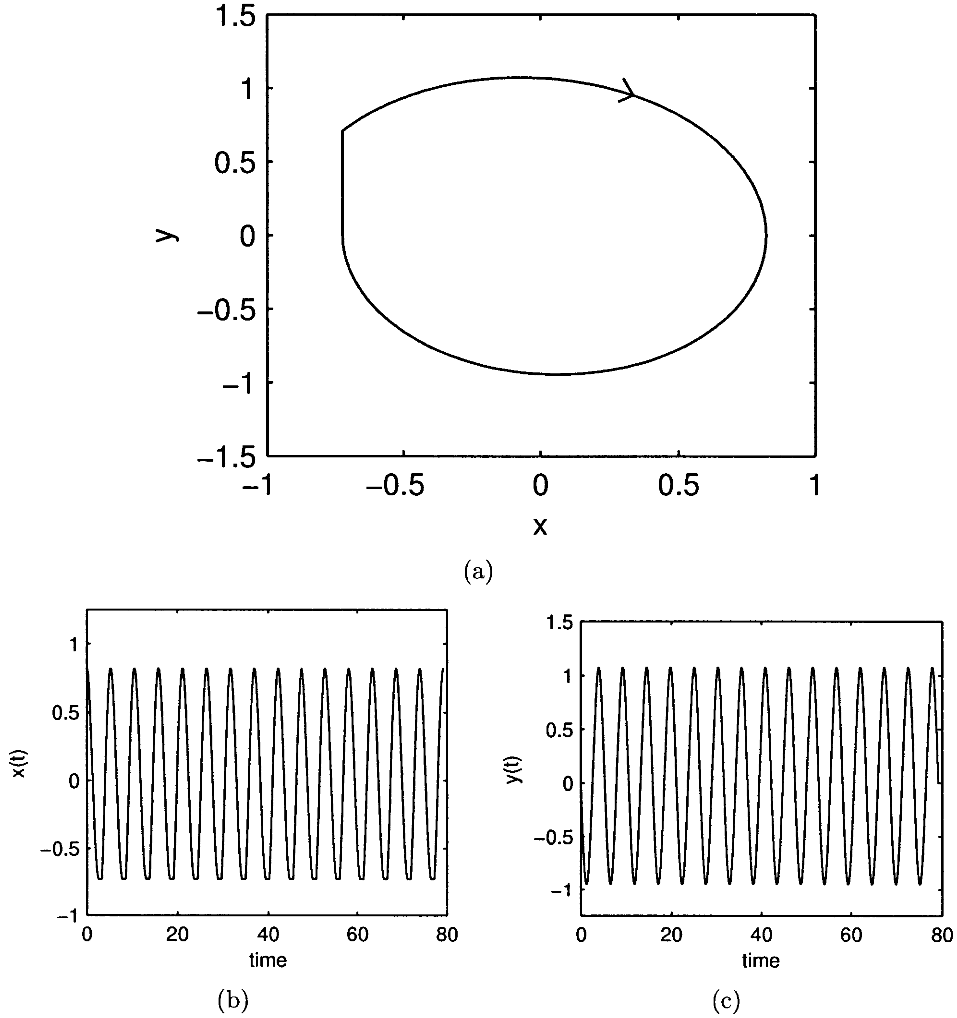


Figure 5-6: Dynamics of simple switching hybrid system: (a) limit cycle, (b) state trajectory $x(t)$ and (c) state trajectory $y(t)$.

Initial-Condition Sensitivity Trajectories: The trajectories for the initial-condition sensitivities of the simple switching hybrid system are shown in Figure 5-7. This figure shows the property of the matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(i, \mathbf{p}, t)$ discussed in Section 3.2.4. The initial-condition sensitivities can be decomposed into a periodic part and a decaying part which vanishes over long times. It can be noticed in the figure that the initial-

Table 5.4: Results for the sensitivity analysis of the simple switching hybrid system. The resulting initial conditions were $x(0) = 0.8209$, $y(0) = 0$ and period $T(\mathbf{p}) = 5.2787$.

Parameters	r	b	c
$\frac{\partial T}{\partial \mathbf{p}}$	1	-2.5849	-1.4357
$\frac{\partial x_0}{\partial \mathbf{p}}$	1.1559	-3.9496	-0.1404
$\frac{\partial y_0}{\partial \mathbf{p}}$	0	0	0

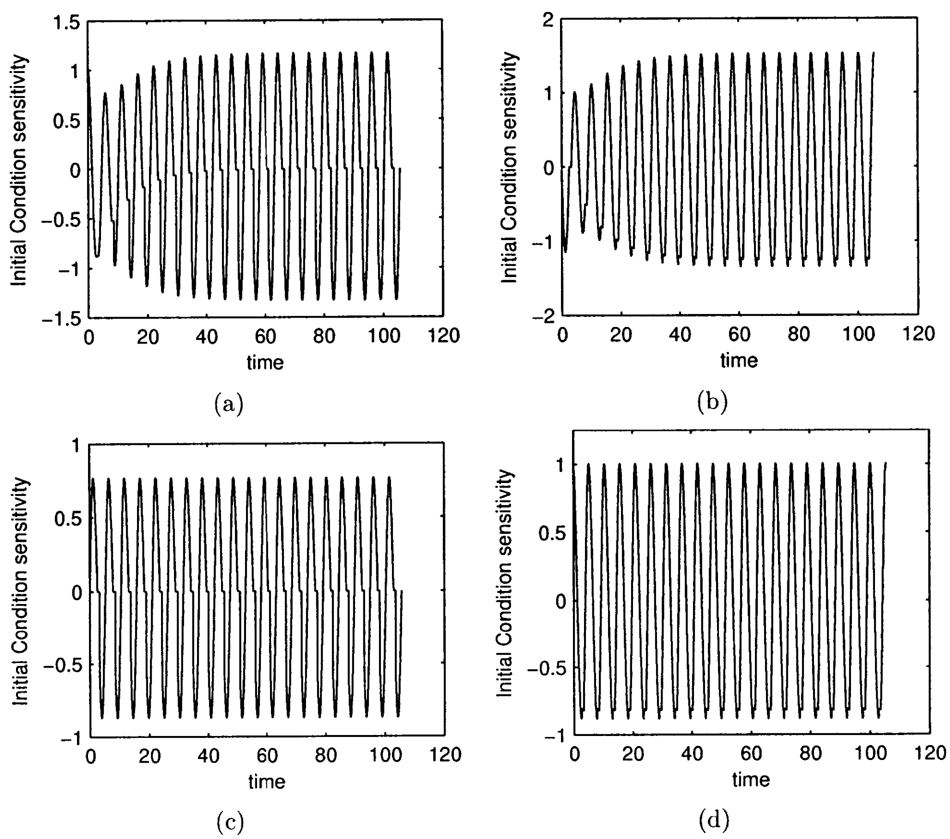
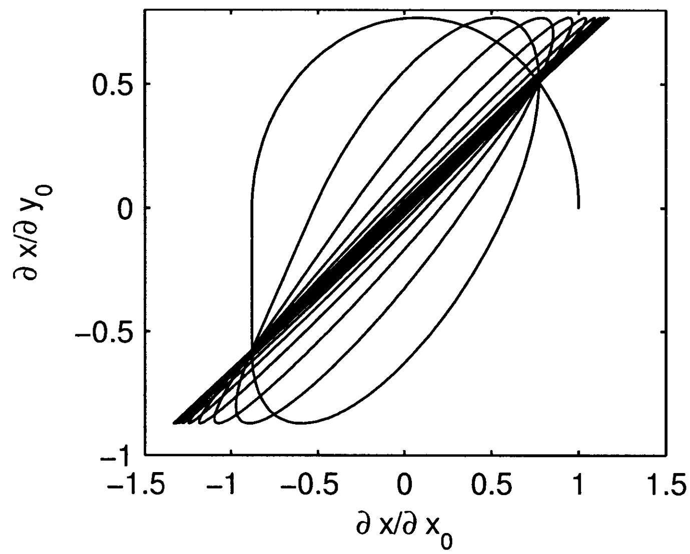
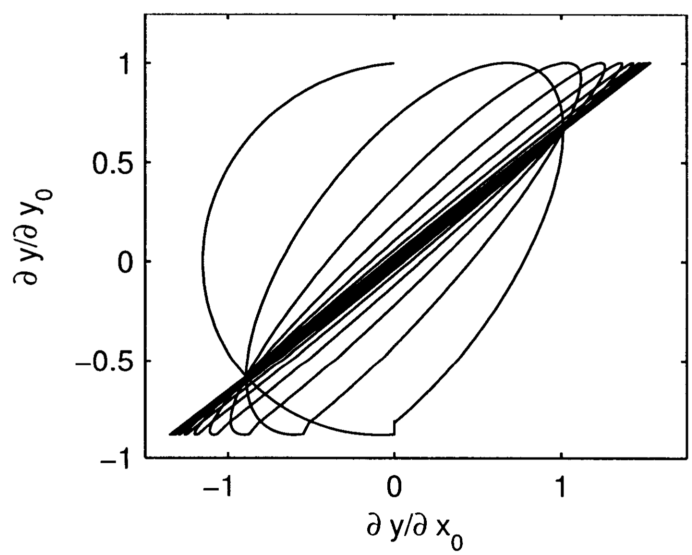


Figure 5-7: Initial condition sensitivity trajectories for the simple switching hybrid system: (a) sensitivities of x w.r.t. x_0, (b) sensitivities of y w.r.t. x_0, (c) sensitivities of x w.r.t. y_0 and (d) sensitivities of y w.r.t. y_0.



(a)



(b)

Figure 5-8: Phase portrait plot of the initial condition sensitivity for simple switching hybrid system: (a) $\frac{\partial x}{\partial y_0}$vs.$\frac{\partial x}{\partial x_0}$ and (b) $\frac{\partial y}{\partial y_0}$vs.$\frac{\partial y}{\partial x_0}$.

condition sensitivities of each state becomes periodic after some amount of time.

The value of these initial-condition sensitivities after one period is the monodromy matrix. The value of the monodromy matrix for this example is:

$$\mathbf{M} = \begin{bmatrix} 0.5918 & 0.0000 \\ 0.6227 & 1.0000 \end{bmatrix}.$$

The eigenvalues of the monodromy matrix are 0.5918 and 1.

Figure 5-8 shows phase portrait plots of the initial-condition sensitivities of the simple switching hybrid system for long times. A plot of initial-condition sensitivity $\frac{\partial x}{\partial y_0}$ versus $\frac{\partial x}{\partial x_0}$ and $\frac{\partial y}{\partial y_0}$ versus $\frac{\partial y}{\partial x_0}$ are shown in Figures 5-8(a) and 5-8(b), respectively. This figure shows that the initial condition sensitivity decays to a periodic solution as $t \to +\infty$, which lies along a line after long time has passed. This is also the conclusion of the Eq. (3.31) given in Section 3.2.5. The periodic solution is given by Eq. (3.33) and the line along which it lies is spanned by $\mathbf{v}_1$, a left eigenvector of $\mathbf{M}$ corresponding to the eigenvalue 1. In this case the left eigenvector is calculated to be $\mathbf{v}_1 = (1.5255, 1)$. Figures 5-8(a) and 5-8(b) confirm that the steady solution lies along the direction of $\mathbf{v}_1$.

Sensitivity Trajectories: The sensitivity trajectories for the state variable x with respect to the parameter r are shown in Figure 5-9, along with trajectories for the relevant element of $\mathbf{Z}(t)$, $\mathbf{W}(t)$, and the relative phase sensitivity with respect to r, $\delta_r(t)$. These sensitivities are dependent on the initial conditions and hence the choice of the PLC, therefore it is difficult to compare trajectories starting from different initial conditions or reference point. Thus, it is important to have a time reference, i.e., PLC, along with the sensitivities while reporting them. The trajectories have jumps in the sensitivities at the change of the mode in the hybrid trajectory. These are because of the discontinuities in the vector fields and non-zero sensitivity of the event time with respect to the parameter at the epoch boundaries and are given by Eq. (3.42). The sensitivity in Figure 5-9(a) grows as the time evolves because of the unbounded part $\mathbf{R}(t)$ while the other part of the sensitivity $\mathbf{Z}(t)$ (shown in Figure

5-9(b)) is periodic in time. Both the unbounded part as well as the periodic part have jumps in them. Further decomposition of the periodic part into $\mathbf{W}(t)$ and $\boldsymbol{\delta}(t)$ is shown in Figures 5-9(c) and 5-9(d).

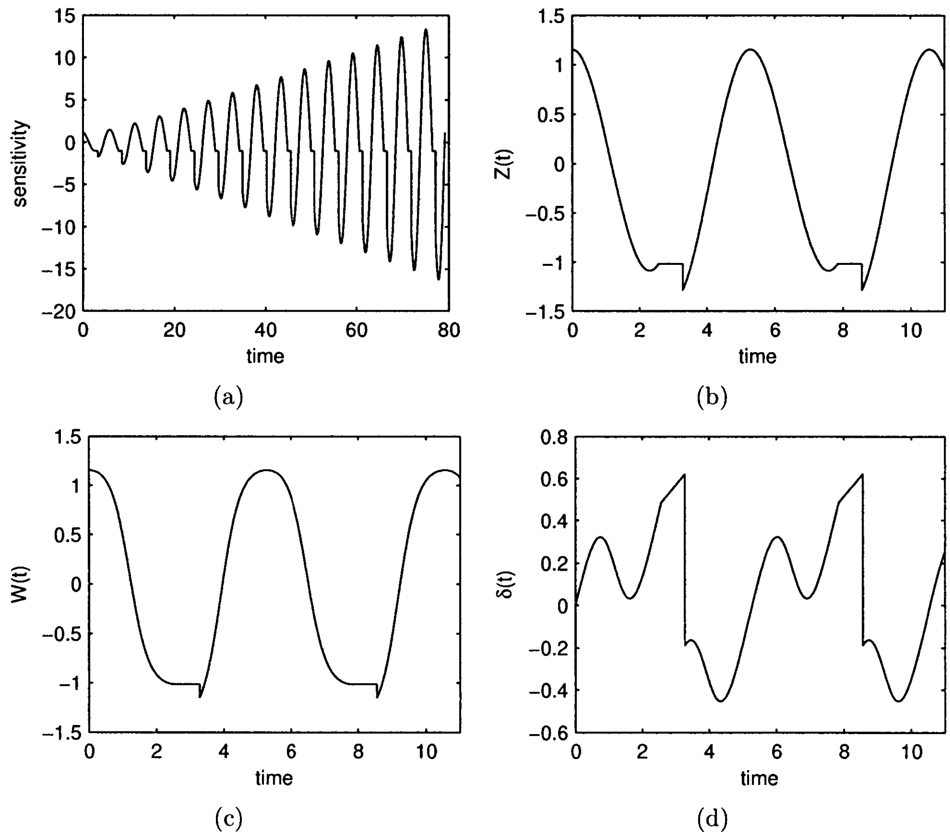


Figure 5-9: Sensitivity trajectories for the simple switching hybrid system, all with respect to the parameter r: (a) full sensitivities of x, when $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$, (b) period-independent periodic part $\mathbf{Z}(i, \mathbf{p}, t)$, (c) period and phase-independent part $\mathbf{W}(i, \mathbf{p}, t)$ and (d) relative phase sensitivity with respect to r, $\delta_r(i, \mathbf{p}, t)$.

It is also important to note here that although the differential equations for the sensitivities with respect to the parameter r are the same as the initial condition sensitivity equations, the two trajectories are different because r appears in the transition conditions for the system. This gives a non-zero value for $\frac{\partial \mathcal{L}}{\partial \mathbf{p}}(m_i, \mathbf{x}(i, \mathbf{p}, \sigma_{i+1}(\mathbf{p}), \mathbf{p})$ in the formula for $\frac{\partial \sigma_{i+1}}{\partial \mathbf{p}}(\mathbf{p})$ in Eq. (3.44). This value is zero in the expression for $\frac{\partial \sigma_{i+1}}{\partial \mathbf{x}_0}(\mathbf{p})$ which is given in Eq. (3.11). Hence, the jumps in the parametric sensitivity and initial-condition sensitivities are different.

The trajectory for the state variable x as well as the relevant element of $\mathbf{Z}(t)$, $\mathbf{W}(t)$ and the relative phase sensitivity, $\delta_c(t)$, with respect to parameter c, which appears in the right-hand side of the ODEs for the simple switching hybrid system, are shown in Figure 5-10. The sensitivity equations in this case are different from initial-condition sensitivity equations because the parameter appears in the right-hand side of the ODEs.

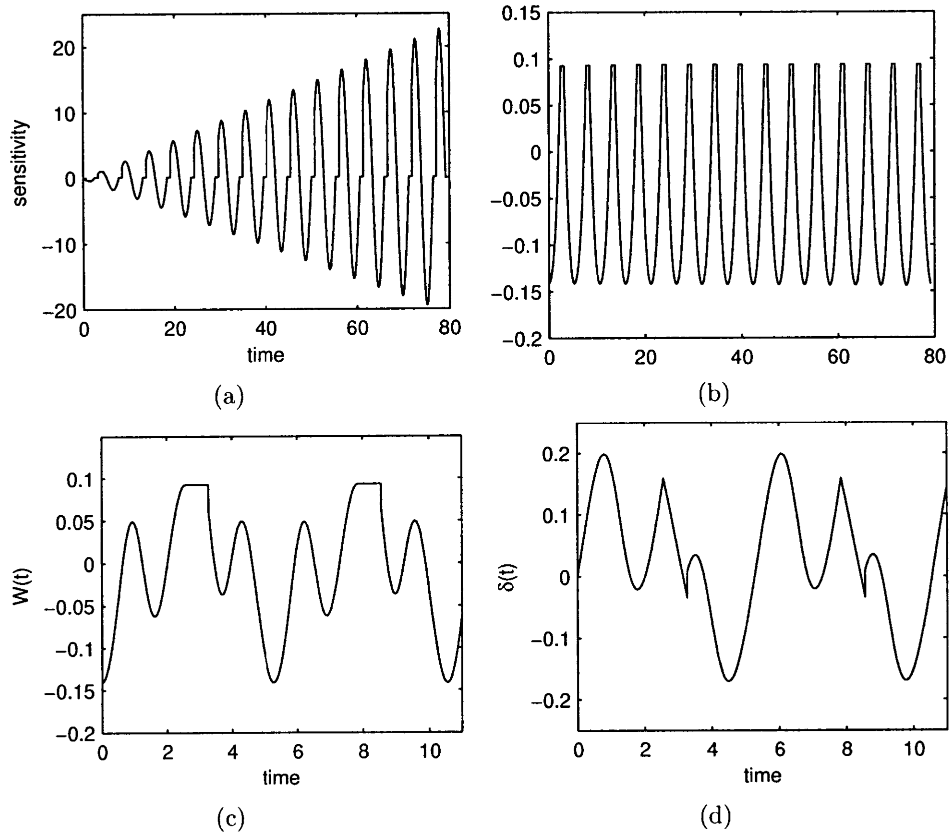


Figure 5-10: Sensitivity trajectories for the simple switching hybrid system, all with respect to the parameter c: (a) full sensitivities of x, when $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}(\mathbf{p})$, (b) period-independent periodic part $\mathbf{Z}(i, \mathbf{p}, t)$, (c) period and phase-independent part $\mathbf{W}(i, \mathbf{p}, t)$ and (d) relative phase sensitivity with respect to c, $\delta_c(i, \mathbf{p}, t)$.

Amplitude Sensitivity: The amplitude sensitivities for the state variables x and y are given in Table 5.5. Figure 5-6 shows that there are non-unique minima for x, i.e., x is at its infimum for the entire time when the system is in mode 3. The value of $\dot{x}$ is zero during mode 3 as well as both limits at the event time where a transition occurs

Table 5.5: Results of the amplitude sensitivities for the state variables in the simple switching hybrid system.

Parameters	r	b	c
For x, $\frac{\partial \Omega_1}{\partial \mathbf{p}}(\mathbf{p})$	2.1710	-8.3469	-0.2329
For y, $\frac{\partial \Omega_2}{\partial \mathbf{p}}(\mathbf{p})$	2.8354	-9.5418	0.3292

from mode 2 to 3. This is just a coincidence because the condition $\dot{x} = 0$ during mode 3 and the transition condition ($y = 0$) at the event are equivalent. This is the case where a minimum occurs at an epoch boundary and both time-derivative limits are zero, discussed in Section 3.2.9. Hence Eq. (3.80) can be used to calculate the amplitude sensitivity for x because the second and fourth term in Eq. (3.79) drop out as $\dot{x}(i_{j,extremum}, \mathbf{p}, t_{j,extremum}) = 0$. The sensitivity differential equation for $\frac{\partial x}{\partial \mathbf{p}}$ in mode 3 has a right-hand side value of zero in mode 3 and the sensitivity $\frac{\partial x}{\partial \mathbf{p}}$ remains at a constant value. This is the reason that the amplitude sensitivity for x given in Table 5.5 is unique even when the minima of x are non-unique.

The extrema for variable y occur away from the event times and the value of $\dot{y}(i_{j,extremum}, \mathbf{p}, t_{j,extremum})$ is zero. Hence the second and fourth terms in Eq. (3.79) drop out to yield Eq. (3.80) for amplitude sensitivity for y. This implies that $\mathbf{z}_j\left(i_{j,extremum}, \mathbf{p}, t_{j,extremum}\right)$ or $\mathbf{w}_j\left(i_{j,extremum}, \mathbf{p}, t_{j,extremum}\right)$ can also be used to calculate the values for amplitude sensitivities.

Peak-to-peak sensitivity: The peak-to-peak sensitivities for the simple switching hybrid system, where the relative phase is the time difference ($\beta(\mathbf{p})$) between the peak of the variable y and the peak of state variable x, are shown in Table 5.6. The results agree well with the finite-difference approximation with a finite difference of $\epsilon = 0.01$, with a maximum deviation of 1%.

5.3 Planar Hybrid System

This system was earlier used in [29] to show how a classical shooting algorithm can be used to compute periodic solutions of piecewise continuous systems. The system

Table 5.6: Results of the peak-to-peak sensitivities for the simple switching hybrid system. $\frac{\partial \beta}{\partial \mathbf{p}}$=peak-to-peak sensitivity, FD = finite-difference approximation of $\frac{\partial \beta}{\partial \mathbf{p}}$ (with a finite-difference of $\epsilon = 0.01$)

Parameters	r	b	c
$\frac{\partial \beta}{\partial \mathbf{p}}$	-0.0026	0.3230	0.4962
FD	-0.0026	0.3230	0.5012

has 2 continuous states, the value of n_e is 4 and hybrid mode trajectory is given by $T_\mu = \{1, 2, 3, 4, 1\}$. Again, additional modes are introduced to avoid the use of "AND" operators. Two parameters $\mathbf{p} = (p_1, p_2)$ having values $p_1 = 0.4$ and $p_2 = 0.75$ can be introduced in this system. The stability of the limit cycle depends upon the values of these parameters and for the values presented, the cycle is stable. The system is given by the following sets of ODEs in the four modes:

$$\text{Mode 1}: \begin{cases} \dot{x} = x\left(1 - \sqrt{x^2+y^2}\right) - y\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right) \\ \dot{y} = y\left(1 - \sqrt{x^2+y^2}\right) + x\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right), \end{cases}$$

$$\text{Mode 2}: \begin{cases} \dot{x} = -p_1 x\left(2 - \sqrt{x^2+y^2}\right) - y\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right) \\ \dot{y} = -p_1 y\left(2 - \sqrt{x^2+y^2}\right) + x\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right), \end{cases}$$

$$\text{Mode 3}: \begin{cases} \dot{x} = -p_1 x\left(2 - \sqrt{x^2+y^2}\right) - y\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right) \\ \dot{y} = -p_1 y\left(2 - \sqrt{x^2+y^2}\right) + x\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right), \end{cases}$$

$$\text{Mode 4}: \begin{cases} \dot{x} = x\left(1 - \sqrt{x^2+y^2}\right) - y\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right) \\ \dot{y} = y\left(1 - \sqrt{x^2+y^2}\right) + x\left(2 - \sqrt{x^2+y^2} - p_2 x/\sqrt{x^2+y^2}\right). \end{cases}$$

The hybrid dynamic model for the planar hybrid system is shown in Figure 5-11. The system switches from Mode 1 to 2 when $y \leqslant 0$, from Mode 2 to 3 when $x \geqslant 0$, from Mode 3 to 4 when $y \geqslant 0$ and from Mode 4 to 1 when $x \leqslant 0$. The state variable

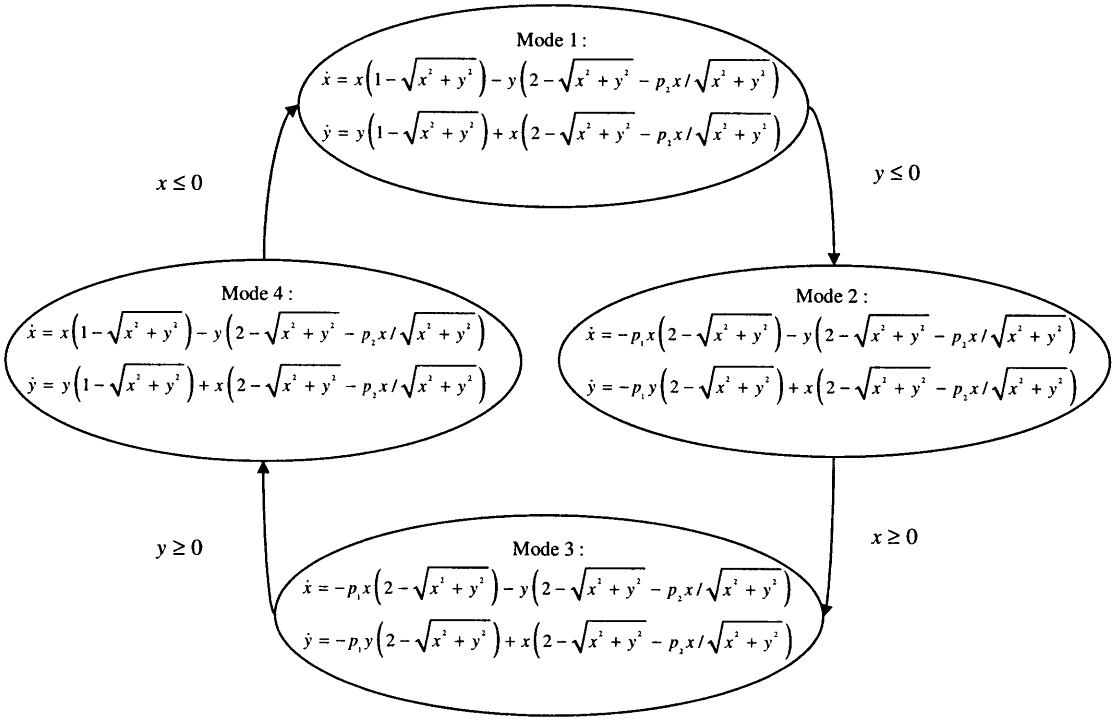


Figure 5-11: Hybrid dynamic model of the planar hybrid system.

Table 5.7: Results for the sensitivity analysis for the planar hybrid system. The resulting initial conditions were $x(0) = 0$, $y(0) = 0.3745$ and period $T(\mathbf{p}) = 4.0835$.

Parameters	p_1	p_2
$\frac{\partial T}{\partial \mathbf{p}}$	-7.0399	1.1166
$\frac{\partial x_0}{\partial \mathbf{p}}$	0	0
$\frac{\partial y_0}{\partial \mathbf{p}}$	-3.2214	0.1095

Table 5.8: Results of the amplitude sensitivities for planar hybrid system.

Parameters	p_1	p_2
For x, $\frac{\partial \Omega_1}{\partial \mathbf{p}}(\mathbf{p})$	-6.5612	80.7430
For y, $\frac{\partial \Omega_2}{\partial \mathbf{p}}(\mathbf{p})$	-6.0026	0.1814

vector is $\mathbf{x} = (x, y)$. State continuity is employed at the transitions:

$$\mathbf{x}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) = \mathbf{x}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})), \ \forall i \in \{1,2,3,4\}, \ \forall N \in \{0,1,\ldots,\infty\},$$

Figure 5-12 shows the limit cycle on the phase portrait for the planar system and the state trajectories x and y over time. The BVP for the initial conditions and the period given in Eq. (3.2) and Eq. (3.3) was solved using the PLC $x(t=0) = 0$, yielding the results given in Table 5.7. The value of the monodromy matrix is:

$$\mathbf{M} = \begin{bmatrix} 1.4112 & 1.0687 \\ -0.2888 & 0.2495 \end{bmatrix}.$$

The eigenvalues of the monodromy matrix are 0.6607 and 1. Table 5.7 gives the results for the sensitivity initial conditions as well as period sensitivities obtained by solving the BVP given by the following system of linear equations:

$$\begin{bmatrix} \mathbf{M}-\mathbf{I} & \dot{\mathbf{x}}(n_e+1, \mathbf{p}, T(\mathbf{p})) \\ 1 \ \ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\mathbf{P}(n_e+1, \mathbf{p}, T(\mathbf{p})) \\ 0 \ \ 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{M}-\mathbf{I} & \begin{matrix} -0.6088 \\ 0.2343 \end{matrix} \\ 1 \ \ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -0.8431 & 0.5627 \\ -0.7682 & -0.1794 \\ 0 & 0 \end{bmatrix}.$$

The system of parametric sensitivity ODEs for this example are given by:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} \frac{\partial F_1}{\partial p_1} & \frac{\partial F_1}{\partial p_2} \\ \frac{\partial F_2}{\partial p_1} & \frac{\partial F_2}{\partial p_2} \end{bmatrix},$$

$$\text{Mode 1}: \begin{cases} \dfrac{\partial F_1}{\partial x} = 1 - \sqrt{x^2+y^2} - \dfrac{x^2}{\sqrt{x^2+y^2}} + \dfrac{y\left(x\left(x^2+y^2\right)+p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial y} = \sqrt{x^2+y^2} - 2 + \dfrac{x(p_2-y)}{\sqrt{x^2+y^2}} + \dfrac{y^2(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial x} = 2 - \sqrt{x^2+y^2} - \dfrac{x(p_2+y)}{\sqrt{x^2+y^2}} - \dfrac{x\left(x\left(x^2+y^2\right)+p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial y} = 1 - \sqrt{x^2+y^2} - \dfrac{y^2}{\sqrt{x^2+y^2}} - \dfrac{xy(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial p_1} = 0, \dfrac{\partial F_1}{\partial p_2} = 0, \dfrac{\partial F_2}{\partial p_1} = \dfrac{xy}{\sqrt{x^2+y^2}}, \dfrac{\partial F_2}{\partial p_2} = -\dfrac{x^2}{\sqrt{x^2+y^2}}, \end{cases}$$

$$\text{Mode 2}: \begin{cases} \dfrac{\partial F_1}{\partial x} = -p_1\left(2 - \sqrt{x^2+y^2} - \dfrac{x^2}{\sqrt{x^2+y^2}}\right) + \dfrac{y\left(x\left(x^2+y^2\right)-p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial y} = \sqrt{x^2+y^2} - 2 + \dfrac{x(p_1y+p_2)}{\sqrt{x^2+y^2}} - \dfrac{y^2(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial x} = 2 - \sqrt{x^2+y^2} - \dfrac{x(p_1y+p_2)}{\sqrt{x^2+y^2}} - \dfrac{x\left(x\left(x^2+y^2\right)+p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial y} = -p_1\left(2 - \sqrt{x^2+y^2} - \dfrac{y^2}{\sqrt{x^2+y^2}}\right) - \dfrac{xy(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial p_1} = -x(2-\sqrt{x^2+y^2}), \dfrac{\partial F_1}{\partial p_2} = -y(2-\sqrt{x^2+y^2}) \\ \dfrac{\partial F_2}{\partial p_1} = \dfrac{xy}{\sqrt{x^2+y^2}}, \dfrac{\partial F_2}{\partial p_2} = -\dfrac{x^2}{\sqrt{x^2+y^2}}, \end{cases}$$

$$\text{Mode 3}: \begin{cases} \dfrac{\partial F_1}{\partial x} = -p_1\left(2 - \sqrt{x^2+y^2} - \dfrac{x^2}{\sqrt{x^2+y^2}}\right) + \dfrac{y\left(x\left(x^2+y^2\right)-p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial y} = \sqrt{x^2+y^2} - 2 + \dfrac{x(p_1y+p_2)}{\sqrt{x^2+y^2}} - \dfrac{y^2(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial x} = 2 - \sqrt{x^2+y^2} - \dfrac{x(p_1y+p_2)}{\sqrt{x^2+y^2}} - \dfrac{x\left(x\left(x^2+y^2\right)+p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial y} = -p_1\left(2 - \sqrt{x^2+y^2} - \dfrac{y^2}{\sqrt{x^2+y^2}}\right) - \dfrac{xy(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial p_1} = -x(2-\sqrt{x^2+y^2}), \dfrac{\partial F_1}{\partial p_2} = -y(2-\sqrt{x^2+y^2}) \\ \dfrac{\partial F_2}{\partial p_1} = \dfrac{xy}{\sqrt{x^2+y^2}}, \dfrac{\partial F_2}{\partial p_2} = -\dfrac{x^2}{\sqrt{x^2+y^2}}, \end{cases}$$

$$\text{Mode 4}: \begin{cases} \dfrac{\partial F_1}{\partial x} = 1 - \sqrt{x^2+y^2} - \dfrac{x^2}{\sqrt{x^2+y^2}} + \dfrac{y\left(x\left(x^2+y^2\right)+p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial y} = \sqrt{x^2+y^2} - 2 + \dfrac{x(p_2-y)}{\sqrt{x^2+y^2}} + \dfrac{y^2(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial x} = 2 - \sqrt{x^2+y^2} - \dfrac{x(p_2+y)}{\sqrt{x^2+y^2}} - \dfrac{x\left(x\left(x^2+y^2\right)+p_2y^2\right)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_2}{\partial y} = 1 - \sqrt{x^2+y^2} - \dfrac{y^2}{\sqrt{x^2+y^2}} - \dfrac{xy(x^2+y^2-p_2x)}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{\partial F_1}{\partial p_1} = 0, \dfrac{\partial F_1}{\partial p_2} = 0, \dfrac{\partial F_2}{\partial p_1} = \dfrac{xy}{\sqrt{x^2+y^2}}, \dfrac{\partial F_2}{\partial p_2} = -\dfrac{x^2}{\sqrt{x^2+y^2}}. \end{cases}$$

The trajectory for the sensitivity of the state variable y with respect to p_1 is shown in Figure 5-13 along with the relevant element of $\mathbf{Z}(t)$, $\mathbf{W}(t)$, and the relative phase sensitivity with respect to p_1, $\delta_{p_1}(t)$. The results for the amplitude sensitivities for the planar hybrid system are given in Table 5.8. The amplitude sensitivity for the variable y was calculated using the Eq. (3.80) as $\dot{y}(i_{y,extremum}, \mathbf{p}, t_{y,extremum})$ are zero. But the value of $\dot{x}(i_{x,min}, \mathbf{p}, t_{x,min})$ is not zero and hence the calculation is done using the equation similar to the Eq. (3.81) obtained by setting $\dot{x}_j(i_{j,max}, \mathbf{p}, t_{j,max})$ equal to zero in Eq. (3.79).

Peak-to-peak sensitivity: To calculate the peak-to-peak sensitivities for the planar hybrid system, $\dot{y}(0) = 0$ is taken to be PLC to define the time scale because y is a smooth function of time and attains its peak away from the events. The BVP for the initial conditions and the period given in Eq. (3.2) and Eq. (3.3) is solved to yield the results given in Table 5.9. The monodromy matrix is given by:

$$\mathbf{M} = \begin{bmatrix} 1 & 1.4567 \\ 0 & 0.6608 \end{bmatrix}.$$

Table 5.9 also gives the results for the sensitivity initial conditions and period sen-

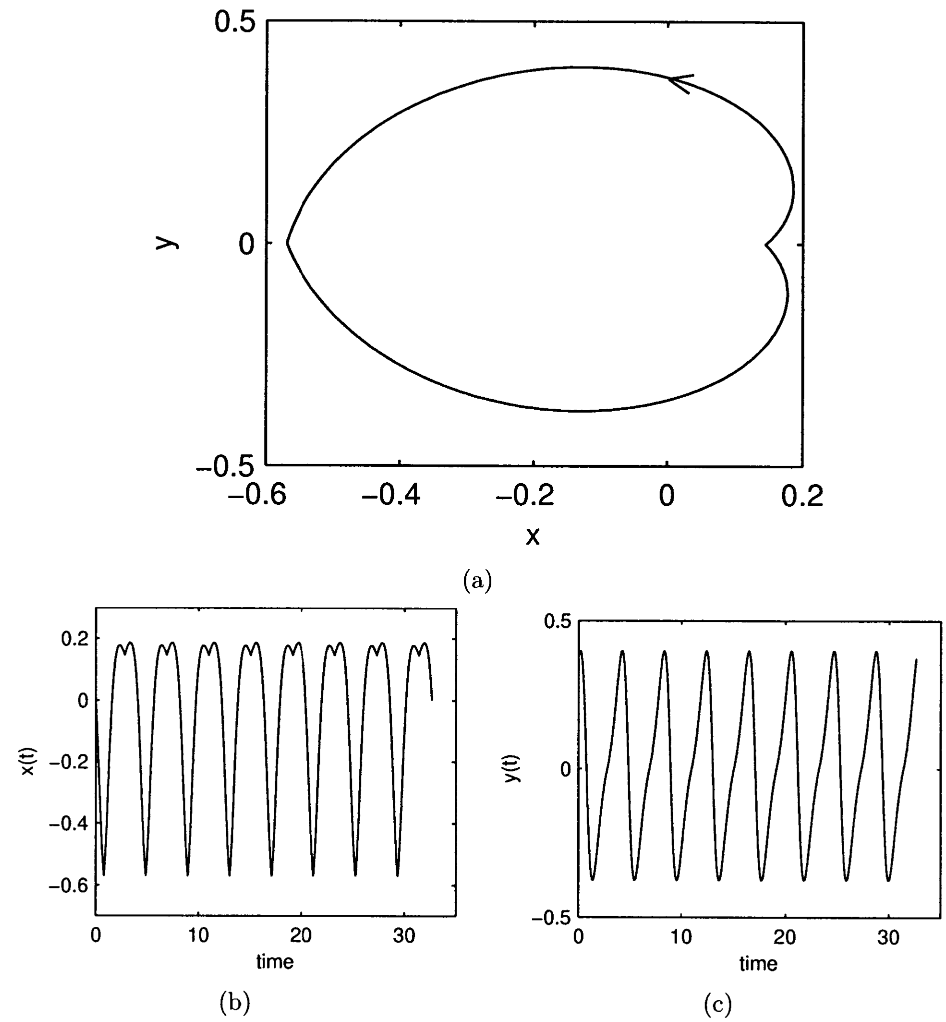


Figure 5-12: Dynamics of simple planar hybrid system: (a) limit cycle, (b) state trajectory $x(t)$ and (c) state trajectory $y(t)$.

Table 5.9: Results for the sensitivity analysis for the planar hybrid system with PLC as $\dot{y}(0) = 0$. The resulting initial conditions were $x(0) = -0.1278$, $y(0) = 0.3978$ and period $T(\mathbf{p}) = 4.0835$.

Parameters	p_1	p_2
$\frac{\partial T}{\partial \mathbf{p}}$	-7.0399	1.1166
$\frac{\partial x_0}{\partial \mathbf{p}}$	0.6090	-0.0014
$\frac{\partial y_0}{\partial \mathbf{p}}$	-3.2657	0.1074

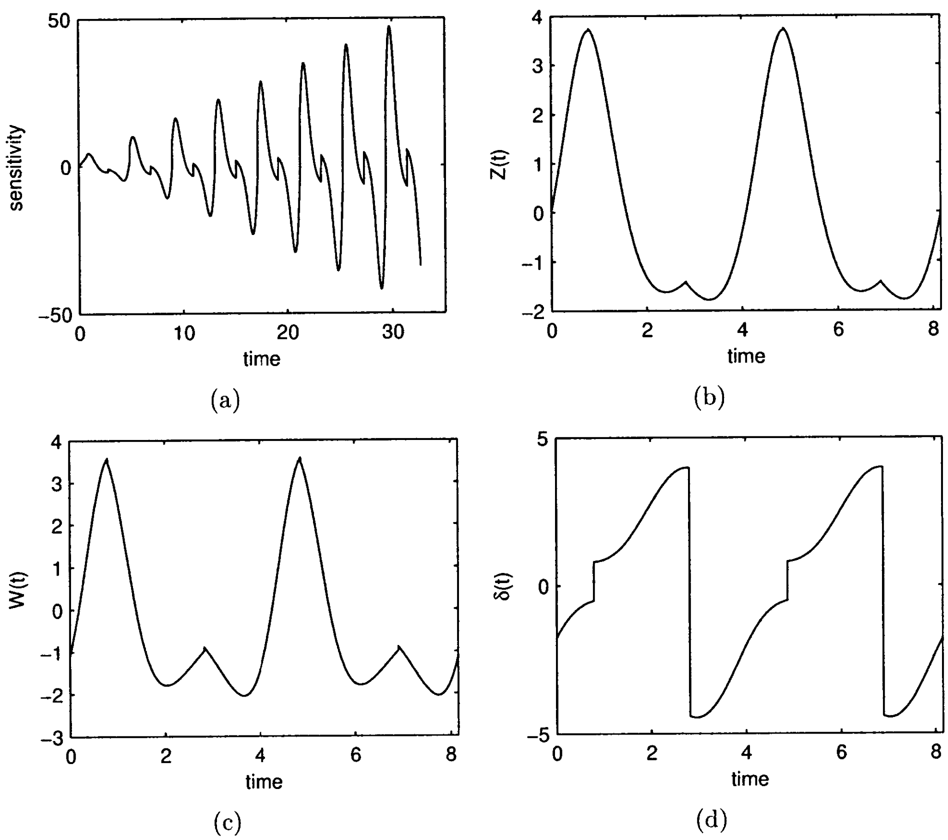


Figure 5-13: Sensitivity trajectories for the planar hybrid system, all with respect to the parameter p_1: (a) full sensitivities of x, when $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p})$, (b) period-independent periodic part $\mathbf{Z}(i, \mathbf{p}, t)$, (c) period and phase-independent part $\mathbf{W}(i, \mathbf{p}, t)$, and (d) relative phase sensitivity with respect to p_1, $\delta_{p_1}(i, \mathbf{p}, t)$.

Table 5.10: Results of the peak-to-peak sensitivities for the planar hybrid system for 4 periods. $\frac{\partial \beta_N}{\partial \mathbf{p}}$=peak-to-peak sensitivity in $N+1$ period, FD = finite-difference approximation of $\frac{\partial \beta_N}{\partial \mathbf{p}}$ (with a finite-difference of $\epsilon = 0.0001$)

Parameters	p_1	p_2
$\frac{\partial \beta_0}{\partial \mathbf{p}}$	-1.6306	-0.1465
FD of $\frac{\partial \beta_0}{\partial \mathbf{p}}$	-1.6254	-0.1468
$\frac{\partial \beta_1}{\partial \mathbf{p}}$	-8.6705	0.9700
FD of $\frac{\partial \beta_1}{\partial \mathbf{p}}$	-8.6554	0.9530
$\frac{\partial \beta_2}{\partial \mathbf{p}}$	-15.7103	2.0866
FD of $\frac{\partial \beta_2}{\partial \mathbf{p}}$	-15.6889	2.0721
$\frac{\partial \beta_3}{\partial \mathbf{p}}$	-22.7499	3.2031
FD of $\frac{\partial \beta_3}{\partial \mathbf{p}}$	-22.7215	3.1904

sitivities obtained by solving the BVP given by following system of linear equations:

$$\begin{bmatrix} \mathbf{M}-\mathbf{I} & \dot{\mathbf{x}}(n_e+1,\mathbf{p},T(\mathbf{p})) \\ \frac{\partial F_2}{\partial x} \quad \frac{\partial F_2}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\mathbf{P}(n_e+1,\mathbf{p},T(\mathbf{p})) \\ \frac{\partial F_2}{\partial p_1} \quad \frac{\partial F_2}{\partial p_2} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{M}-\mathbf{I} & & \begin{matrix} -0.7951 \\ 0 \end{matrix} \\ 2.1022 & 0.3920 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} 0.8398 & -0.7313 \\ 1.1077 & -0.0364 \\ 0 & 0.0391 \end{bmatrix}.$$

The infimum of state variable x is attained at the transition from Mode 1 to 2, where $y = 0$. Hence the peak-to-peak sensitivity is given by the event time sensitivity in Eq. (3.45). The peak-to-peak sensitivity where the relative phase is the time difference between the minimum of x relative to the maximum of x, is shown in Table 5.10 for 4 periods. It can be noted that the results satisfy Eq. (3.99).

5.4 Neural Oscillator

This example is a biological application of oscillating hybrid systems also known as a neural oscillator [23]. Various autonomous oscillators existing in living things are produced by rhythmic activities of the corresponding neural systems for, e.g., locomotion, heart beat, etc. Adaptation of the neurons plays a very important role in the generation and sustenance of the oscillations. A mathematical discussion of the oscillations generated due to mutual inhibition of neurons is presented in [23]. The model suggested in this paper consists of two neurons mutually inhibiting to produce limit-cycle oscillations. This oscillator is used in the area of robotics in feedback with oscillatory systems such as legged locomotion [21][30] and juggling [48]. The neural oscillator consists of 4 continuous states and 3 parameters. The value of n_e is 6 and hybrid mode trajectory is given by $T_\mu = \{1, 2, 3, 4, 5, 6, 1\}$. The systems of ODEs for the neural oscillator are given by:

$$\text{Mode 1}: \begin{cases} \dot{x}_1 = -10(x_1 + bx_2 - 1) \\ \dot{x}_2 = 10\,(x_1 - x_2)\,/\tau \\ \dot{x}_3 = -10(ax_1 + x_3 + bx_4 - 1) \\ \dot{x}_4 = -10x_4/\tau, \end{cases}$$

$$\text{Mode 2}: \begin{cases} \dot{x}_1 = -10(x_1 + ax_3 + bx_2 - 1) \\ \dot{x}_2 = 10\,(x_1 - x_2)\,/\tau \\ \dot{x}_3 = -10(ax_1 + x_3 + bx_4 - 1) \\ \dot{x}_4 = 10\,(x_3 - x_4)\,/\tau, \end{cases}$$

$$\text{Mode 3}: \begin{cases} \dot{x}_1 = -10(x_1 + ax_3 + bx_2 - 1) \\ \dot{x}_2 = -10x_2/\tau \\ \dot{x}_3 = -10(x_3 + bx_4 - 1) \\ \dot{x}_4 = 10\,(x_3 - x_4)\,/\tau, \end{cases}$$

$$\text{Mode 4}: \begin{cases} \dot{x}_1 = -10(x_1 + ax_3 + bx_2 - 1) \\ \dot{x}_2 = -10x_2/\tau \\ \dot{x}_3 = -10(x_3 + bx_4 - 1) \\ \dot{x}_4 = 10\,(x_3 - x_4)\,/\tau, \end{cases}$$

$$\text{Mode 5}: \begin{cases} \dot{x}_1 = -10(x_1 + ax_3 + bx_2 - 1) \\ \dot{x}_2 = 10\,(x_1 - x_2)\,/\tau \\ \dot{x}_3 = -10(ax_1 + x_3 + bx_4 - 1) \\ \dot{x}_4 = 10\,(x_3 - x_4)\,/\tau, \end{cases}$$

$$\text{Mode 6}: \begin{cases} \dot{x}_1 = -10(x_1 + bx_2 - 1) \\ \dot{x}_2 = 10\,(x_1 - x_2)\,/\tau \\ \dot{x}_3 = -10(ax_1 + x_3 + bx_4 - 1) \\ \dot{x}_4 = -10x_4/\tau. \end{cases}$$

The hybrid dynamic model for the neural oscillator is shown in Figure 5-14. Additional modes are introduced in this formulation to avoid the use of "AND" operators. The system switches from Mode 1 to 2 when $x_3 \geqslant 0$, from Mode 2 to 3 when $x_1 \leqslant 0$, from Mode 3 to 4 when $x_3 \leqslant 0.36$, from Mode 4 to 5 when $x_1 \geqslant 0$, from Mode 5 to 6 when $x_3 \leqslant 0$ and from Mode 6 to 1 when $x_1 \geqslant 0.45$. The values of the three parameters $\mathbf{p} = (a, b, \tau)$ are: $a = 2$, $b = 2$ and $\tau = 2$. The state variable vector is $\mathbf{x} = (x_1, x_2, x_3, x_4)$. State continuity is employed at the transitions:

$$\mathbf{x}(i+1, \mathbf{p}, \sigma_{i+1,N}(\mathbf{p})) = \mathbf{x}(i, \mathbf{p}, \tau_{i,N}(\mathbf{p})),\ \forall i \in \{1,2,3,4,5,6\},\ \forall N \in \{0, 1, \ldots, \infty\}.$$

Figure 5-15 shows the limit cycle of the neural oscillator projected onto (a) $x_1 - x_2$ plane, (b) $x_1 - x_3$ plane, and (c) $x_1 - x_4$ plane. The state trajectories for the neural oscillator are shown in Figure 5-16. The BVP for the initial conditions and the period given in Eq. (3.2) and Eq. (3.3) was solved using the PLC $\dot{x}_1(t = 0) = 0$, yielding

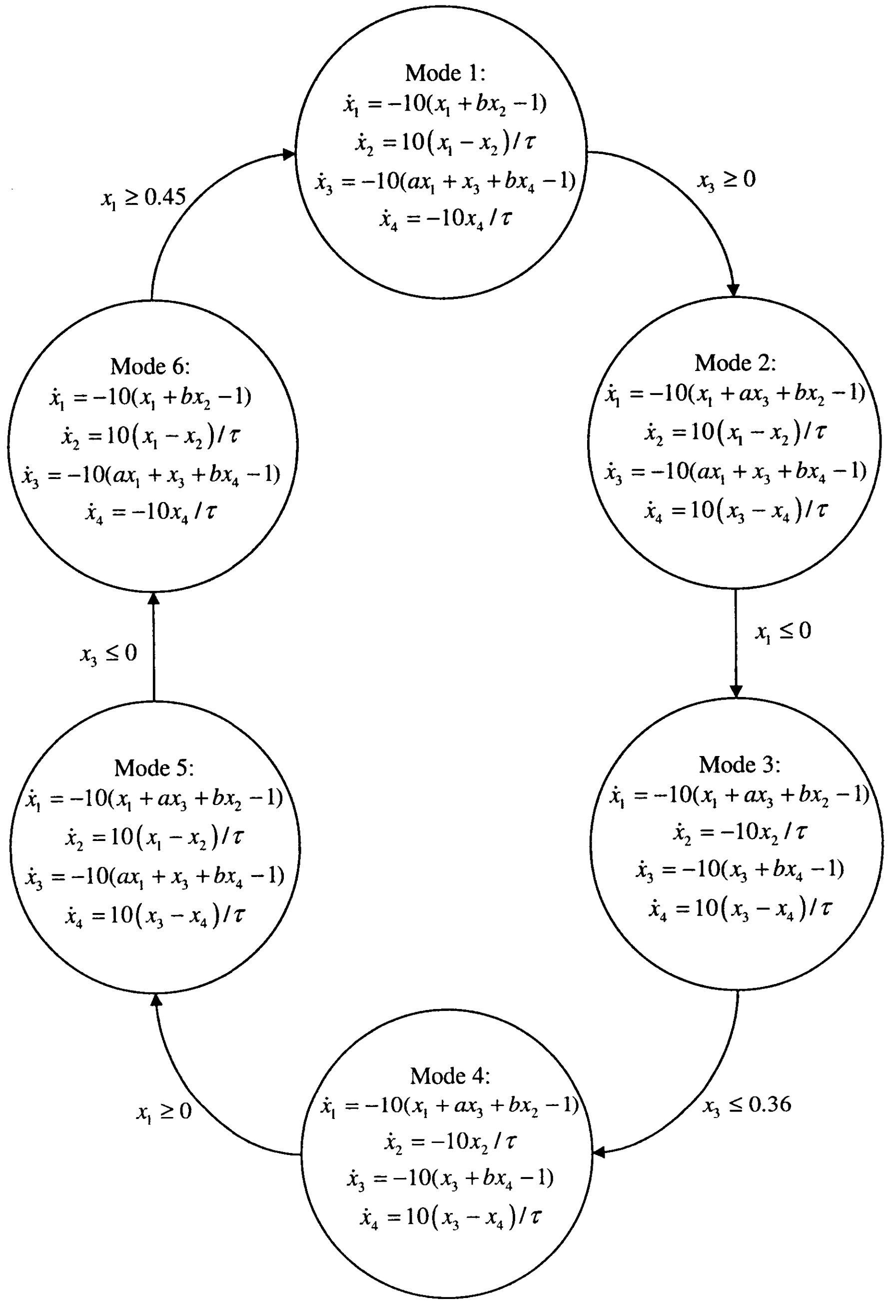


Figure 5-14: Hybrid dynamic model of the neural oscillator.

the results given in Table 5.11. The value of the monodromy matrix is:

$$\mathbf{M} = \begin{bmatrix} -0.0005 & -0.0024 & -0.0005 & -0.0020 \\ 0.0551 & 0.6579 & 0.0570 & -0.6024 \\ -0.0682 & -0.8195 & -0.0702 & 0.7617 \\ -0.0379 & -0.4523 & -0.0392 & 0.4126 \end{bmatrix}$$

The eigenvalues of the monodromy matrix are −0.0002, −0.0008, 0.0008 and 1. Table 5.11 gives the results for the sensitivity initial conditions and period sensitivities obtained by solving the BVP given by following system of linear equations:

$$\begin{bmatrix} \mathbf{M}-\mathbf{I} & \dot{\mathbf{x}}(n_e+1,\mathbf{p},T(\mathbf{p})) \\ -10 \quad -10b \quad 0 \quad 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\mathbf{P}(n_e+1,\mathbf{p},T(\mathbf{p})) \\ 0 \quad -10x_{2,0} \quad 0 \end{bmatrix},$$

$$\begin{bmatrix} & 0 \\ \mathbf{M}-\mathbf{I} & 1.2859 \\ & -1.6134 \\ & -0.8826 \\ -10 \quad -10b \quad 0 \quad 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}}(\mathbf{p}) \\ \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -0.0505 & 0.0973 & -0.0533 \\ 0.4782 & -0.4418 & 0.2822 \\ -0.2388 & 0.5997 & -0.1908 \\ -0.2963 & 0.3699 & -0.2139 \\ 0 & -2.47604 & 0 \end{bmatrix}.$$

The system of parametric sensitivity ODEs for this example is given by:

$$\text{Mode 1}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} -10 & -10b & 0 & 0 \\ \frac{10}{\tau} & -\frac{10}{\tau} & 0 & 0 \\ -10a & 0 & -10 & -10b \\ 0 & 0 & 0 & -\frac{10}{\tau} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} 0 & -10x_2 & 0 \\ 0 & 0 & \frac{-10(x_1-x_2)}{\tau^2} \\ -10x_1 & -10x_4 & 0 \\ 0 & 0 & \frac{10x_4}{\tau^2} \end{bmatrix},$$

$$\text{Mode 2}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} -10 & -10b & -10a & 0 \\ \frac{10}{\tau} & -\frac{10}{\tau} & 0 & 0 \\ -10a & 0 & -10 & -10b \\ 0 & 0 & \frac{10}{\tau} & -\frac{10}{\tau} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} -10x_3 & -10x_2 & 0 \\ 0 & 0 & \frac{-10(x_1 - x_2)}{\tau^2} \\ -10x_1 & -10x_4 & 0 \\ 0 & 0 & \frac{-10(x_3 - x_4)}{\tau^2} \end{bmatrix},$$

$$\text{Mode 3}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} -10 & -10b & -10a & 0 \\ 0 & -\frac{10}{\tau} & 0 & 0 \\ 0 & 0 & -10 & -10b \\ 0 & 0 & \frac{10}{\tau} & -\frac{10}{\tau} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} -10x_3 & -10x_2 & 0 \\ 0 & 0 & \frac{-10x_2}{\tau^2} \\ 0 & -10x_4 & 0 \\ 0 & 0 & \frac{-10(x_3 - x_4)}{\tau^2} \end{bmatrix},$$

$$\text{Mode 4}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} -10 & -10b & -10a & 0 \\ 0 & -\frac{10}{\tau} & 0 & 0 \\ 0 & 0 & -10 & -10b \\ 0 & 0 & \frac{10}{\tau} & -\frac{10}{\tau} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} -10x_3 & -10x_2 & 0 \\ 0 & 0 & \frac{-10x_2}{\tau^2} \\ 0 & -10x_4 & 0 \\ 0 & 0 & \frac{-10(x_3 - x_4)}{\tau^2} \end{bmatrix},$$

$$\text{Mode 5}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} -10 & -10b & -10a & 0 \\ \frac{10}{\tau} & -\frac{10}{\tau} & 0 & 0 \\ -10a & 0 & -10 & -10b \\ 0 & 0 & \frac{10}{\tau} & -\frac{10}{\tau} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} -10x_3 & -10x_2 & 0 \\ 0 & 0 & \frac{-10(x_1 - x_2)}{\tau^2} \\ -10x_1 & -10x_4 & 0 \\ 0 & 0 & \frac{-10(x_3 - x_4)}{\tau^2} \end{bmatrix},$$

$$\text{Mode 6}: \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \begin{bmatrix} -10 & -10b & 0 & 0 \\ \frac{10}{\tau} & -\frac{10}{\tau} & 0 & 0 \\ -10a & 0 & -10 & -10b \\ 0 & 0 & 0 & -\frac{10}{\tau} \end{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \begin{bmatrix} 0 & -10x_2 & 0 \\ 0 & 0 & \frac{-10(x_1 - x_2)}{\tau^2} \\ -10x_1 & -10x_4 & 0 \\ 0 & 0 & \frac{10x_4}{\tau^2} \end{bmatrix}.$$

The time derivatives for the 4 state variables are shown in Figure 5-17. Since the time derivatives shown here are continuous, there are no jumps in the sensitivities. It can be noticed by putting the terms corresponding to the jumps in the present analysis equal to zero that the results will reduce to those for oscillating dynamical

Table 5.11: Results for the sensitivity analysis of the neural oscillator. The resulting initial conditions were $x_1(0) = 0.5048$, $x_2(0) = 0.2476$, $x_3(0) = -0.2013$, $x_4(0) = 0.1765$ and period $T(\mathbf{p}) = 0.8973$.

Parameters	a	b	τ
$\frac{\partial T}{\partial \mathbf{p}}$	0.3708	-0.3686	0.2339
$\frac{\partial x_{1,0}}{\partial \mathbf{p}}$	0.0507	-0.0971	0.0533
$\frac{\partial x_{2,0}}{\partial \mathbf{p}}$	-0.0254	-0.0753	-0.0267
$\frac{\partial x_{3,0}}{\partial \mathbf{p}}$	-0.3300	0.0485	-0.1299
$\frac{\partial x_{4,0}}{\partial \mathbf{p}}$	-0.0145	-0.0149	0.0385

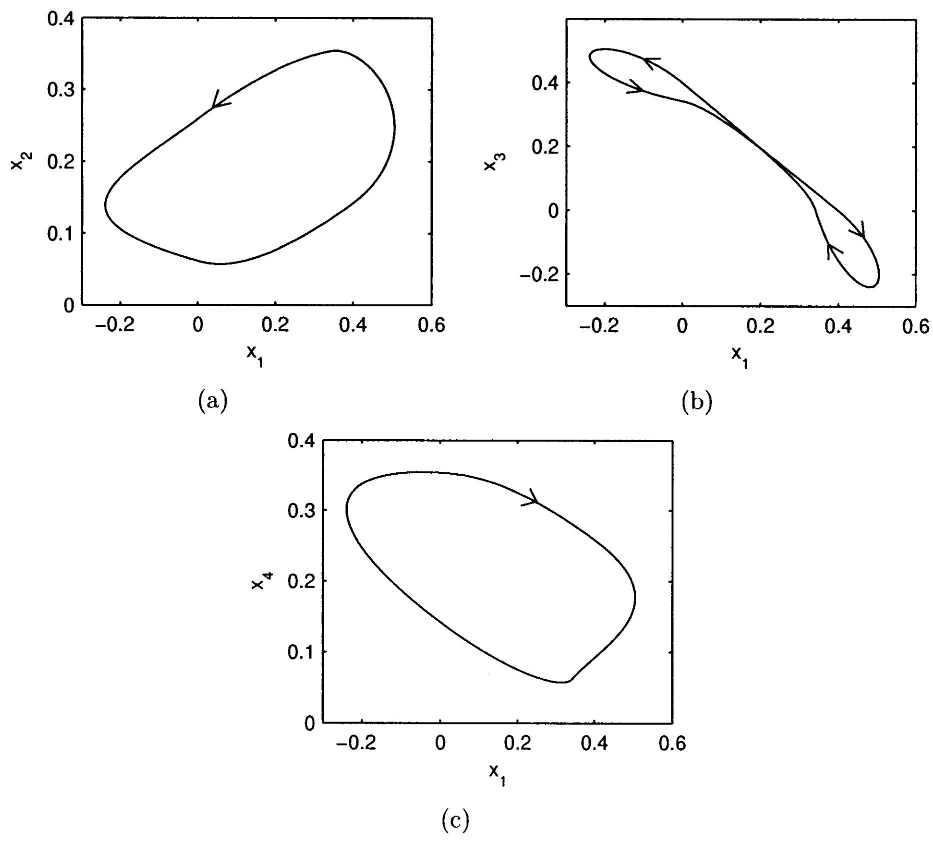


Figure 5-15: Limit cycle of the neural oscillator projected onto: (a) $x_1 - x_2$ plane, (b) $x_1 - x_3$ plane, and (c) $x_1 - x_4$ plane.

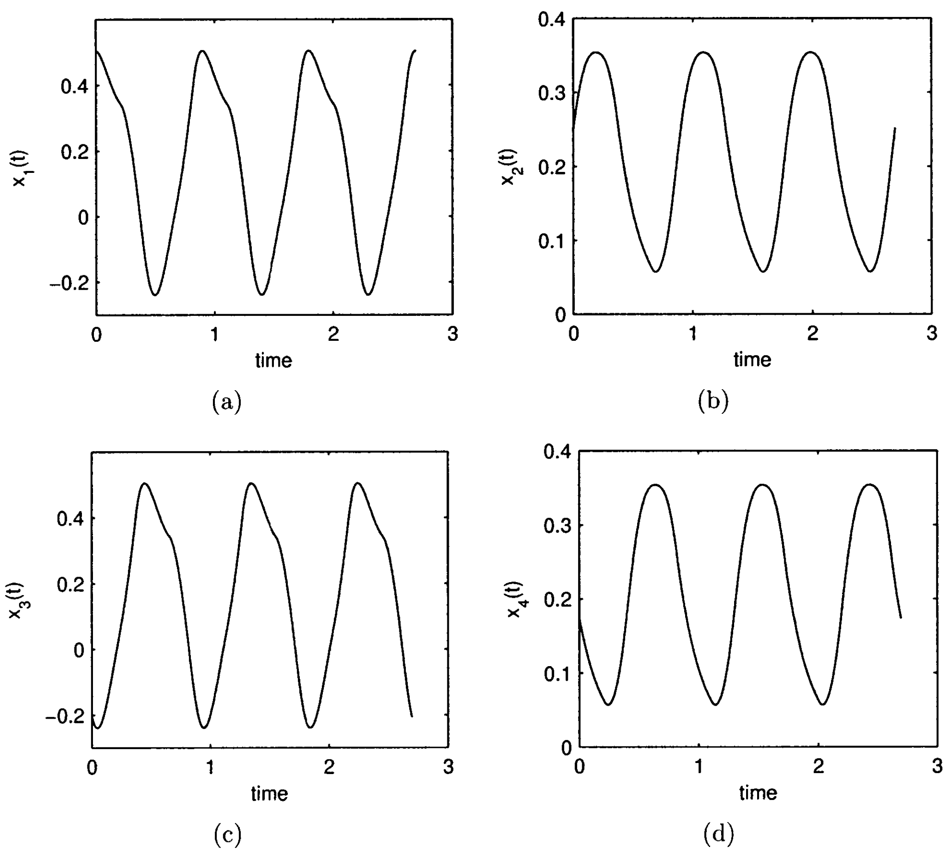


Figure 5-16: State trajectories of the neural oscillator: (a) $x_1(t)$, (b) $x_2(t)$, (c) $x_3(t)$ and (d) $x_4(t)$.

systems [46], in particular LCOs. Hence, all the theory which is applicable for regular LCOs can be used here. The trajectory for the sensitivity of the state variable x_3 with respect to the parameter a, along with the relevant element of $\mathbf{Z}(t)$, $\mathbf{W}(t)$, and the relative phase sensitivity with respect to a, $\delta_a(t)$ are shown in Figure 5-18. The results of the amplitude sensitivities for the 4 state variables for the neural oscillator are shown in Table 5.12. The calculation of amplitude sensitivity is done using Eq. (3.80).

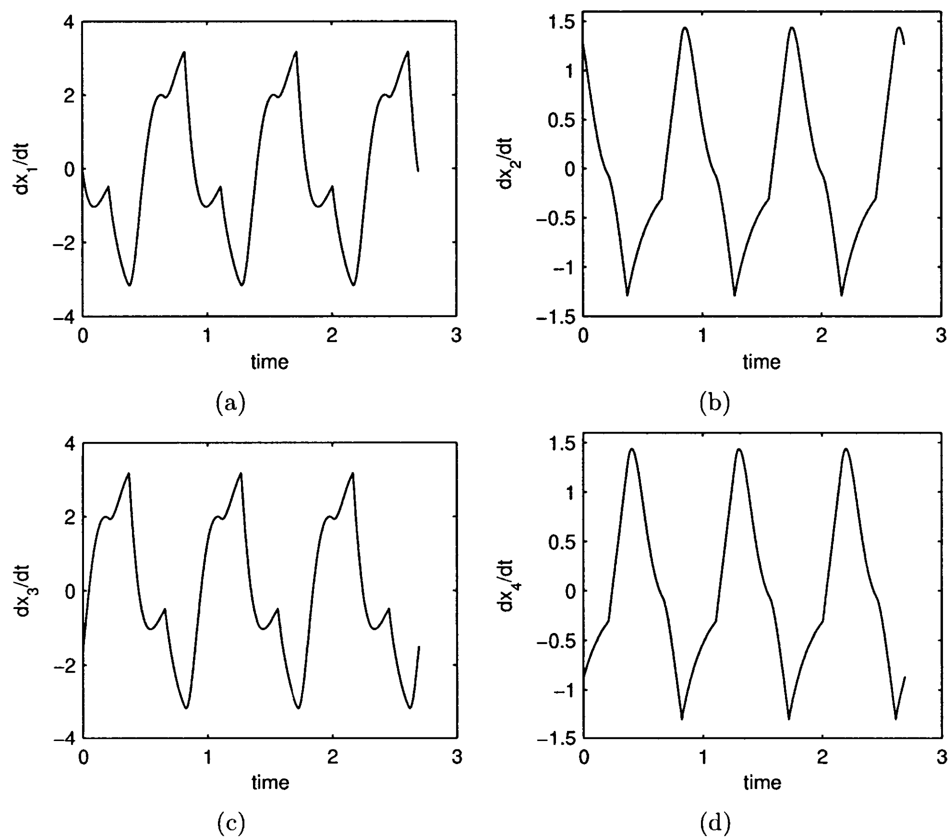


Figure 5-17: Time derivatives for the states (a) x_1, (b) x_2, (c) x_3 and (d) x_4 in the neural oscillator.

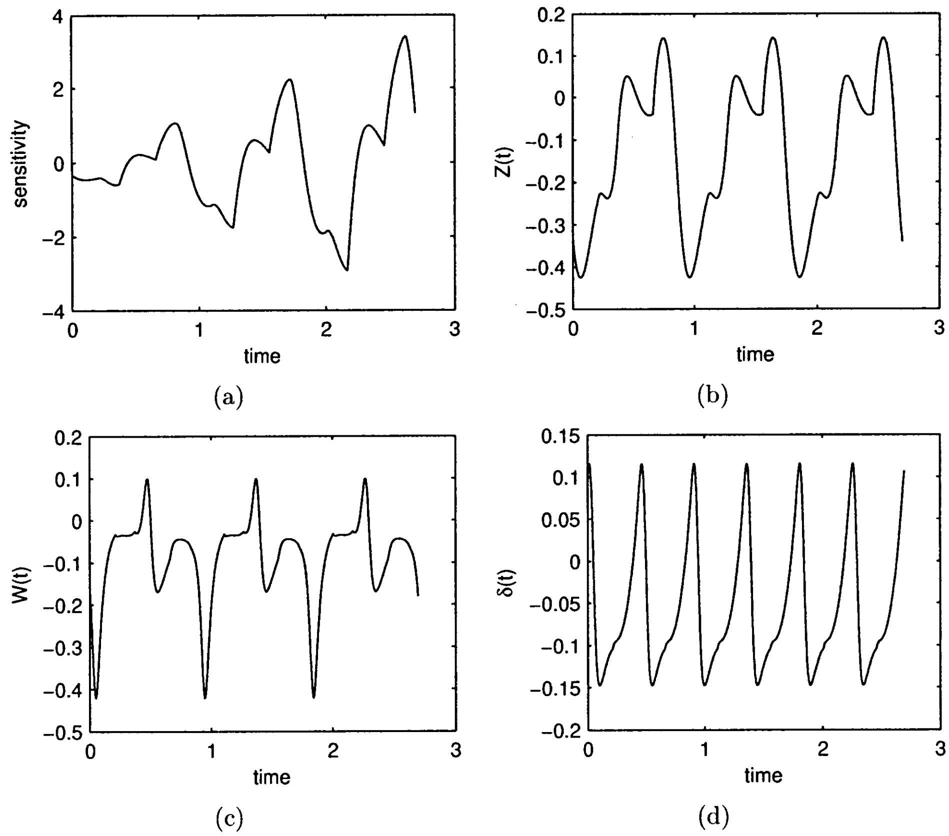


Figure 5-18: Sensitivity trajectories for the neural oscillator, all with respect to the parameter a: (a) full sensitivities of x_3, when $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(1, \mathbf{p}, \sigma_1(\mathbf{p})) = \frac{\partial \mathbf{x_0}}{\partial \mathbf{p}}(\mathbf{p})$, (b) period independent periodic part $\mathbf{Z}(i, \mathbf{p}, t)$, (c) period and phase independent part $\mathbf{W}(i, \mathbf{p}, t)$ and (d) relative phase sensitivity with respect to a, $\delta_a(i, \mathbf{p}, t)$.

Table 5.12: Results of the amplitude sensitivities for the neural oscillator.

Parameters	a	b	τ
For x_1, $\frac{\partial \Omega_1}{\partial \mathbf{p}}(\mathbf{p})$	0.4719	-0.1554	0.2092
For x_2, $\frac{\partial \Omega_2}{\partial \mathbf{p}}(\mathbf{p})$	0.0653	-0.1256	-0.0222
For x_3, $\frac{\partial \Omega_3}{\partial \mathbf{p}}(\mathbf{p})$	0.4783	-0.1617	0.2132
For x_4, $\frac{\partial \Omega_4}{\partial \mathbf{p}}(\mathbf{p})$	0.0673	-0.1274	-0.0210

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Chapter 6

Conclusions and Future Work

In this work, the theory for sensitivity analysis of oscillating hybrid systems (in particular, stable LCOs) is developed and discussed. A BVP is formulated for initial conditions, period, period sensitivities and initial conditions for sensitivities. This BVP is solved for a point on the limit cycle by using a PLC. The PLC defines the time reference and starting point on the cycle with initial conditions for states and sensitivities. A mathematical analysis of the initial-condition sensitivities and parametric sensitivities is presented. Analysis of the solution of general homogeneous linear equations with linear piecewise periodic coefficients is used to obtain an expression for the initial condition sensitivities in terms of fundamental matrices, vector fields and event-time sensitivities for different transition times. This analysis concludes that the monodromy matrix is different from a fundamental matrix evaluated after one period for limit cycles of hybrid systems. Also, a decomposition of the initial condition sensitivity matrix is done into periodic and decaying parts based on this analysis. An expression for the general parametric sensitivity equations for limit cycles of hybrid systems is also obtained. This expression and difference equation analysis suggests a decomposition of the parametric sensitivities into an unbounded part (period dependent) and a periodic part (period independent). For simple LCOs in hybrid systems this periodic part can further be decomposed into two parts which affect shape (independent of the PLC) and phase of the limit-cycle, respectively. This provides a useful framework for calculating relevant quantities such as peak-to-peak sensitivities similar to regular oscillating systems. The trajectory sensitivity and its three part decomposition provides valuable insights into the influence of parameters on the dynamic behavior of oscillating hybrid systems. This is illustrated by applying the analysis to some simple examples.

However, although this analysis covers LCOs in hybrid systems very well, there is still a need for a separate analysis for other types of oscillators, such as NLCOs and intermediate-type oscillators. This work has focused on ODE embedded oscillating hybrid systems which have continuous state variables, but a large number of hybrid systems have discontinuities (jumps) in the states in practical applications. There is a need for an extension of the theory of sensitivity analysis to such systems (represented by DAEs) and applications. The relevant information given by the trajectory sensitivity and its parts can be used in algorithms for different applications such as parameter estimation, control system design, stability analysis and dynamic optimization. This work forms a basis of the extension to these and other applications of oscillating hybrid systems.

Appendix A

Residual Subroutines provided to DSL48SE

A.1 Residual Subroutine for Pressure Relief Valve Hybrid System

```
C### This model implements Pressure Relief Valve Hybrid System

      subroutine hybrid0(neq,t,y,ydot,delta,ires,ichvar,rpar,ipar)
      implicit none
      integer neq,ires,ichvar,ipar(1)
      double precision t,y(neq),ydot(neq),delta(neq),rpar(14)
      double precision x,xdot
      double precision R,Tf,V,k,Pa,Ps,Pr,Fin
      integer ndsc, A,B
      parameter (A=1,B=2)
      common /DISCRETESTATES/ mode
      integer mode
```

```
c###  The discrete state is kept in common block and is written
c###  to by the hybriddriver.f file

      x    =  y(1)

      xdot =  ydot(1)

      R = rpar(1)
      Tf = rpar(2)
      V = rpar(3)
      k = rpar(4)
      Pa = rpar(5)
      Ps = rpar(6)
      Pr = rpar(7)
      Fin = rpar(8)

      if (mode.eq.A.and.x.ge.Ps) then
            mode = B
      end if
      if (mode.eq.B.and.x.le.Pr) then
            mode = A
      end if

c###  Calculating Residuals
      if (mode.eq.A) then
            delta(1) = -xdot+R*Tf*Fin/V
      else if (mode.eq.B) then
            delta(1) = -xdot+R*Tf*(Fin-k*sqrt(x-Pa))/V
      end  if
```

```
c### Debug information
      ires = 0
      return
      end
```

A.2 Residual Subroutine for Simple Switching Hybrid System

```
C### This model implements a Simple Switching Hybrid Systems

      subroutine hybrid0(neq,t,y,ydot,delta,ires,ichvar,rpar,ipar)
      implicit none
      integer neq,ires,ichvar,ipar(1)
      double precision t,y(neq),ydot(neq),delta(neq),rpar(14)
      double precision x1,x2,x1dot,x2dot
      double precision r,b,c
      integer ndsc, A,BB,CC,D
      parameter (A=1,BB=2,CC=3,D=4)
      common /DISCRETESTATES/ mode
      integer mode

c###  The discrete state is kept in common block and is written
c###  to by the hybriddriver.f file

      x1    =  y(1)
      x2    =  y(2)

      x1dot =  ydot(1)
      x2dot =  ydot(2)
```

```
      r = rpar(1)
      b = rpar(2)
      c = rpar(3)

      if (mode.eq.A.and.x1.le.0.0) then
            mode = BB
      end if
      if (mode.eq.BB.and.x2.ge.0.0) then
            mode = CC
      end if
      if (mode.eq.CC.and.x2.ge.r) then
            mode = D
      end if
      if (mode.eq.D.and.x2.le.0.0) then
            mode = A
      end if

c###  Calculating Residuals
      if (mode.eq.A) then
            delta(1) = -x1dot+x2
            delta(2) = -x2dot-c*x1-b*x2
      else if (mode.eq.BB) then
            delta(1) = -x1dot+x2
            delta(2) = -x2dot-c*x1-b*x2
      else if (mode.eq.CC) then
            delta(1) = -x1dot+0.0
            delta(2) = -x2dot+1.0
      else if (mode.eq.D) then
            delta(1) = -x1dot+x2
```

```
          delta(2) = -x2dot-c*x1-b*x2
      end  if

c### Debug information
      ires = 0
      return
      end
```

A.3 Residual Subroutine for Planar Hybrid System

```
C### This model implements Planar Hybrid System

      subroutine hybrid0(neq,t,y,ydot,delta,ires,ichvar,rpar,ipar)
      implicit none
      integer neq,ires,ichvar,ipar(1)
      double precision t,y(neq),ydot(neq),delta(neq),rpar(14)
      double precision x1,x2,x1dot,x2dot
      double precision p1,p2
      integer ndsc, A,B,C,D
      parameter (A=1,B=2,C=3,D=4)
      common /DISCRETESTATES/ mode
      integer mode

c###  The discrete state is kept in common block and is written
c###  to by the hybriddriver.f file

      x1    =  y(1)
      x2    =  y(2)
```

```
      x1dot =  ydot(1)
      x2dot =  ydot(2)

      p1 = rpar(1)
      p2 = rpar(2)

      if (mode.eq.A.and.x2.le.0) then
            mode = B
      end if
      if (mode.eq.B.and.x1.ge.0) then
            mode = C
      end if
      if (mode.eq.CC.and.x2.ge.0) then
            mode = D
      end if
      if (mode.eq.D.and.x1.le.0) then
            mode = A
      end if

c###  Calculating Residuals
      if (mode.eq.A) then
            delta(1) = -x1dot+x1*(1-sqrt(x1**2+x2**2))-x2*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
            delta(2) = -x2dot+x2*(1-sqrt(x1**2+x2**2))+x1*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
      else if (mode.eq.B) then
            delta(1) = -x1dot-p1*x1*(2-sqrt(x1**2+x2**2))-x2*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
            delta(2) = -x2dot-p1*x2*(2-sqrt(x1**2+x2**2))+x1*(2-
```

```
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
      else if (mode.eq.C) then
            delta(1) = -x1dot-p1*x1*(2-sqrt(x1**2+x2**2))-x2*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
            delta(2) = -x2dot-p1*x2*(2-sqrt(x1**2+x2**2))+x1*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
      else if (mode.eq.D) then
            delta(1) = -x1dot+x1*(1-sqrt(x1**2+x2**2))-x2*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
            delta(2) = -x2dot+x2*(1-sqrt(x1**2+x2**2))+x1*(2-
     +                  sqrt(x1**2+x2**2)-p2*x1/(sqrt(x1**2+x2**2)))
      end  if

c### Debug information
      ires = 0
      return
      end
```

A.4 Residual Subroutine for Neural Oscillator

```
C### This model implements Neural Oscillator

      subroutine hybrid0(neq,t,y,ydot,delta,ires,ichvar,rpar,ipar)
      implicit none
      integer neq,ires,ichvar,ipar(1)
      double precision t,y(neq),ydot(neq),delta(neq),rpar(14)
      double precision x,xdot,x1,x2,x3,x1dot,x2dot,x3dot
      double precision aa,TT,bb,x4,x4dot
      integer ndsc, A,B,C,D,E,F
      parameter (A=1,B=2,C=3,D=4,E=5,F=6)
```

```
      common /DISCRETESTATES/ mode
      integer mode

c###  The discrete state is kept in common block and is written
c###  to by the hybriddriver.f file

      x1    =  y(1)
      x2    =  y(2)
      x3    =  y(3)
      x4    =  y(4)

      x1dot =  ydot(1)
      x2dot =  ydot(2)
      x3dot =  ydot(3)
      x4dot =  ydot(4)

      aa =  rpar(1)
      bb =  rpar(2)
      TT =  rpar(3)

      if (mode.eq.A.and.x3.ge.0.0) then
            mode = B
      end if
      if (mode.eq.B.and.x1.le.0.0) then
            mode = C
      end if
      if (mode.eq.C.and.x3.le.0.36) then
            mode = D
      end if
      if (mode.eq.D.and.x1.ge.0.0) then
```

```
            mode = E
      end if
      if (mode.eq.E.and.x3.le.0.0) then
            mode = F
      end if
      if (mode.eq.F.and.x1.ge.0.45) then
            mode = A
      end if


c###  Calculating Residuals
      if (mode.eq.A) then
            delta(1) = -x1dot-10*(x1+bb*x2-1)
            delta(2) = -x2dot+(10*(x1-x2))/TT
            delta(3) = -x3dot-10*(aa*x1+x3+bb*x4-1)
            delta(4) = -x4dot+(10*(-x4))/TT
      else if (mode.eq.B) then
            delta(1) = -x1dot-10*(x1+aa*x3+bb*x2-1)
            delta(2) = -x2dot+(10*(x1-x2))/TT
            delta(3) = -x3dot-10*(aa*x1+x3+bb*x4-1)
            delta(4) = -x4dot+(10*(x3-x4))/TT
      else if (mode.eq.C) then
            delta(1) = -x1dot-10*(x1+aa*x3+bb*x2-1)
            delta(2) = -x2dot+(10*(-x2))/TT
            delta(3) = -x3dot-10*(x3+bb*x4-1)
            delta(4) = -x4dot+(10*(x3-x4))/TT
      else if (mode.eq.D) then
            delta(1) = -x1dot-10*(x1+aa*x3+bb*x2-1)
            delta(2) = -x2dot+(10*(-x2))/TT
            delta(3) = -x3dot-10*(x3+bb*x4-1)
            delta(4) = -x4dot+(10*(x3-x4))/TT
```

```
      else if (mode.eq.E) then
            delta(1) = -x1dot-10*(x1+aa*x3+bb*x2-1)
            delta(2) = -x2dot+(10*(x1-x2))/TT
            delta(3) = -x3dot-10*(aa*x1+x3+bb*x4-1)
            delta(4) = -x4dot+(10*(x3-x4))/TT
      else if (mode.eq.F) then
            delta(1) = -x1dot-10*(x1+bb*x2-1)
            delta(2) = -x2dot+(10*(x1-x2))/TT
            delta(3) = -x3dot-10*(aa*x1+x3+bb*x4-1)
            delta(4) = -x4dot+(10*(-x4))/TT
      end if

c### Debug information
      ires = 0
      return
      end
```

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