Polynomial Policies in Supply Chain Networks

by

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Abstract

This thesis aims to solve the periodic-reviewed inventory control problem in supply chain networks with uncertain demand so as to minimize the overall cost of the system over a fixed planning time horizon. In such problems, one seeks to optimally determine ordering quantities at different stages in time. We investigate the class of polynomial policies, where the control policy is directly parametrized polynomially in the observed uncertainties of previous stages. We use sum-of-square relaxations to reformulate the problem into a single semidefinite optimization problem for a specific polynomial degree. We consider both robust and stochastic approaches in order to address the uncertainties in demand.

In extensive numerical studies, we find that polynomial policies exhibit better performance over basestock policies across a variety of networks and demand distributions under the mean and standard deviation criteria. However, when the uncertainty set turns out to be larger than planned, basestock policies start outperforming polynomial policies. Comparing the policies obtained under the robust and stochastic frameworks, we find that they are comparable in the average performance criterion, but the robust approach leads to better tail behavior and lower standard deviation in general.

Thesis Supervisor: Dimitris Bertsimas
Title: Boeing Professor of Operations Research

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Chapter 1

Introduction

In this thesis, we study the inventory management problem of a multi-echelon supply chain network with stochastic demands under a fixed planning time horizon. It is one of the core problems in supply chain management offering significant opportunities for increasing the operational and financial efficiency of a company.

The general setting of inventory management in a supply chain network is as follows. In a multi-echelon assembly network, the storage hubs receive their supplies from outside sources and send items through the network; each processing node assembles parts from upstream nodes and ships the intermediate products to downstream nodes until the final destinations, the sinks of the network, where the final products are assembled to satisfy unknown sequences of demands from customers. If the underlying system is a distribution network, the distribution centers receive their supplies from outside manufacturing plants with infinite capacity and send items through the network by intermediate warehouses to bring them closer to the stores, the sinks of the network, where unknown sequences of demands occur. In both types of networks, external demand can also occur at intermediate nodes of the networks.

In either networks, at each time period, each node orders from its upstream suppliers at a per unit variable cost. For simplicity, we have assumed that there is no lead time. The order size is constrained to be nonnegative. In each time period, each node also incurs per unit inventory holding cost for all the amount of materials stored there, or per unit backordering penalty for any quantity or demand that is not satisfied in the current period. All unsatisfied demands are fully backlogged in the system until they are satisfied in subsequent periods. The objective is to find an inventory control policy on the ordering quantity in each time period that minimizes the overall cost of the system over a fixed planning time horizon, which could be expected average cost or worst case cost depending on the company's interest or risk aversion.

1.1 Literature Review

It has been a fundamental yet challenging problem to design a computationally efficient and practically implementable inventory control policy for supply chain networks. As a result, it has attracted a large interest from diverse research communities. The most established methodology for dealing with such problems is dynamic programming. It has been very successful in theoretically characterizing the structure of optimal policies in simple structures. Basestock type policies were proven to be optimal for multiple variants of the problem. Clark and Scarf [11] first proved the optimality of basestock policies for a serial network with independent and identical distributed demand in the absence of capacity constraints. Subsequently researchers generalized the proof to other models and more general assumptions on the demand distributions [13, 17]. In basestock type policies, there is a lower bound inventory level defined as echelon basestock level. When the echelon inventory is higher than the basestock level, no order will be placed. However, if the echelon inventory falls below the basestock level, it will make order to bring the inventory to an upper bound level defined as the echelon order-up-to level.

The main drawback of dynamic programming approach is *the curse of dimensionality*, in that the complexity of the underlying recursive equations grows exponentially with the size of the state-space, making the computation of actual policy sometimes infeasible. Therefore, one has to resort to numerical or approximate methods in practice [1, 12].

Researchers in the stochastic programming community proposed an alternative approach to consider control policies parametrized directly in the sequence of observed uncertainties in past periods [10, 14], which is typically referred to as *recourse decision rules.* Gartska and Wets [14] showed that piece-wise affine decision rules are optimal for the case of linear constraints on the controls, with random uncertainties having bounded support and known distributions, and minimizing piece-wise quadratic, convex cost as the objective.

In recent years, *disturbance feedback parameterizations* have been adopted in robust control and optimization. Some of initial steps in analyzing the properties of disturbance-affine policies were taken by Kerrigan and Maciejowski [16] and Goulart and Kerrigan [15], where they showed that the resulting parametrization has desirable system theoretic properties of stability and robust invariance under certain suitable conditions, and that the class of affine disturbance feedback policies incorporate the open-loop and pre-stabilizing control policies, as it is equivalent to the class of affine state feedback policies with memory of prior states. Subsequently, in the papers [2, 5, 6, 19, 22], it was demonstrated how reformulations can be done to allow the computation of the policy parameters by solving convex optimization problems which vary from linear and quadratic to second-order conic and semidefinite programs. Bertsimas et al. [7] also showed that, in the case of one-dimensional systems with independent state and control constraints, linear control costs and any affine state costs, disturbance-affine policies are optimal and can be found effectively.

Robust optimization addresses the issue of data uncertainty. Bertsimas and Thiele [8] first applied robust optimization to inventory theory with encouraging numerical results. Ben-Tal et al. [3] advanced this approach by computing affine order policies for a two-echelon supply chain. Bertsimas et al. [6] implemented polynomial policies to single echelon and serial supply chain networks and reported that the optimality gap can be essentially closed with increasing the degree of the polynomials.

1.2 Main Contributions

In this thesis, we address the problem of computing polynomial policies for multi-echelon supply chain networks with general topologies by robust and stochastic optimization, respectively. The demand distribution is not specified, but assumed to lie in some polyhedral uncertainty set. Then we perform numerical simulations to analyze the behavior of these two approaches. Our main contributions are summarized below:

- We implement control policies that depend polynomially on the observed disturbances, and reformulate the constraints and objective function using Sum-of-Square (SOS) relaxations on the uncertainty set. We solve the resulting convex reformulation efficiently as a single semidefinite optimization problem in polynomial time for a fixed precision. We consider both the robust min-max objective and the stochastic expectation objective for each polynomial policy.

- In our numerical study, we compare the performance of our framework with basestock policies and observe a steady improvement in reducing cost. We also notice the better performance of affine policies compared to constant policies; we find, however, quadratic policies do not really improve upon affine policies. We further find that the robust approach outperforms the stochastic approach in most demand realizations in mean value, standard deviation and 5%-tail. Across various realizations of demand, we observe that the polynomial policies work best if the demand sequence follows a discrete distribution; multi-modal continuous distributions lead to lower cost than unimodal continuous distributions; and negatively correlated demand realization gives better performance of polynomial policies especially in standard deviation and 5% tail than positively correlated demand. We also investigate the scaling of this framework and establish the size of problems (size of network and time horizon) that our approach is able to solve.

1.3 Thesis Structure

The thesis is organized as follows. Chapter 2 presents the general mathematical formulation of the problem for a fixed polynomial degree and discussed the related techniques. Chapter 3 reports the numerical study of the framework on supply chain networks with varying complexity, compares the polynomial policies with basestock policies, and evaluates performances of robust and stochastic approaches. Chapter 4 concludes the thesis and proposes directions of future research.

1.4 Notation and Definitions

Throughout the rest of the thesis, we use lowercase, non-bold face symbols (e.g. $x \in \mathbb{R}$) to denote scalar quantities; lowercase, bold face symbols (e.g. $\mathbf{x} \in \mathbb{R}^n, n > 1$) to vector quantities; and uppercase symbols (e.g. $\mathbf{X} \in \mathbb{R}^{n \cdot n}, n > 1$) to matrices. Also we use operator to denote vertical vector concatenation, e.g. with $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$, then $(\mathbf{x}, \mathbf{y}) := (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$. We denote quantities specific to time period t by using a subscript, e.g. x_t, and we refer to the k-th component of a vector at time t as $x_k(t)$.

With $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote by $\mathbb{R}[\mathbf{x}]$ the ring of polynomials in variables $x_1, \ldots, x_n$, and by $\mathcal{P}_d[\mathbf{x}]$ the $\mathbb{R}$-vector space of polynomials in $x_1, \ldots, x_n$, with degree at most d. We denote the canonical basis of $\mathcal{P}_d[\mathbf{x}]$,

$$\mathcal{B}_d(\mathbf{x}) := \left(1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots, x_n^d\right)$$

and its dimension $s(d) = \binom{n+d}{d}$. We write polynomial $f \in \mathcal{P}_d[\mathbf{x}]$ as a finite linear combination of monomials,

$$p(\mathbf{x}) = p(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha = \mathbf{p}^T \mathcal{B}_d(\mathbf{x}),$$

where $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and the sum is taken over all n-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ satisfying $\sum_{i=1}^n \alpha_i \leq d$. In the expression above, $\mathbf{p} = (p_\alpha) \in \mathbb{R}^{s(d)}$ is the vector of coefficients of $p(\mathbf{x})$ in the basis $\mathcal{B}_d(\mathbf{x})$.

A polynomial $p(\mathbf{x})$ is called a sum of squares polynomial if $p(\mathbf{x})$ can be written as $p(\mathbf{x}) = \sum_{i=1}^s q_i^2(\mathbf{x})$ for some polynomials $q_1(\mathbf{x}), \ldots, q_s(\mathbf{x})$.

Chapter 2

Problem Formulation

2.1 Problem Description

In the inventory control problem in the multi-echelon system we introduced in Chapter 1, the state of the system, i.e., the vector of inventory levels at each node, is represented by $\mathbf{x}_t$, and the control at each time period, i.e., the vector of orders made by each node, is represented by $\mathbf{u}_t$. With matrices of appropriate dimensions $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{C}_t$ describing the evolution of the system, the dynamics of the system can be expressed as:

$$\mathbf{x}_{t+1} = \mathbf{A}_t\mathbf{x}_t + \mathbf{B}_t\mathbf{u}_t + \mathbf{C}_t\mathbf{w}_t, \tag{2.1}$$

over a finite planning horizon $t = 0, 1, \ldots, T-1$. We assume that the initial state is known. The system is affected by unknown external disturbances, $\mathbf{w}_t$, that lie in a given compact semi-algebraic set

$$\mathcal{W}_t := \left\{\mathbf{w}_t \in \mathbb{R}^{n_w} : g_j(\mathbf{w}_t) \geq 0, j = 1, \ldots, m_t\right\}, \tag{2.2}$$

where g_j are multivariate polynomials depending on the vector of uncertainties at time t, $\mathbf{w}_t$.

The states and controls satisfy linear inequalities,

$$\mathbf{E}_t\mathbf{x}_t + \mathbf{F}_t\mathbf{u}_t \leq \mathbf{f}_t, t = 0, 1, \ldots, T-1, \tag{2.3}$$

$$\mathbf{E}_T\mathbf{x}_T \leq \mathbf{f}_T. \tag{2.4}$$

Also, the cost incurred by the system can be expressed as piece-wise affine and convex function in the states and controls:

$$q_t(\mathbf{x}_t, \mathbf{u}_t) = \max_{i=1,\ldots,r_t} \left[c_t^0(i) + \mathbf{c}_t^u(i)^T\mathbf{u}_t + \mathbf{c}_t^x(i)^T\mathbf{x}_t\right].$$

Given that the demand $\mathbf{w}_t$ is uncertain, we choose two modeling frameworks to model uncertainties: a) robust optimization and b) stochastic optimization.

Under robust optimization, we assumed that the uncertainty sequence is in the set $\mathcal{W}_0 \times \mathcal{W}_1 \times \ldots \times \mathcal{W}_{T-1}$. The objective is to find non-anticipatory control policies $\mathbf{u}_t, t = 0, \ldots, T-1$ under the constraints that minimize the cost incurred by the system in the worst-case realization of the uncertainties. Specifically,

$$\begin{aligned}\min\nolimits_{\mathbf{u}_0} \Big\{ q_0(\mathbf{x}_0, \mathbf{u}_0) + & \max\nolimits_{\mathbf{w}_0} \min\nolimits_{\mathbf{u}_1} \Big\{ q_1(\mathbf{x}_1, \mathbf{u}_1) + \ldots \\ & + \max\nolimits_{\mathbf{w}_{T-2}} \min\nolimits_{\mathbf{u}_{T-1}} \left\{ q_{T-1}(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) + \max\nolimits_{\mathbf{w}_{T-1}} q_T(\mathbf{x}_T) \right\} \ldots \Big\} \Big\}.\end{aligned}$$

$$\begin{aligned}\text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{A}_t\mathbf{x}_t + \mathbf{B}_t\mathbf{u}_t + \mathbf{C}_t\mathbf{w}_t \quad && \forall t = 0, \ldots, T-1, \\ & \mathbf{E}_t\mathbf{x}_t + \mathbf{F}_t\mathbf{u}_t \leq \mathbf{f}_t && \forall t = 0, \ldots, T-1, \\ & \mathbf{E}_T\mathbf{x}_T \leq \mathbf{f}_T. \end{aligned}$$

Under stochastic optimization, we assume that the uncertainties are random variables with bounded support given by $\mathcal{W}_0 \times \mathcal{W}_1 \times \ldots \times \mathcal{W}_{T-1}$. The objective is to find non-anticipatory control policies $\mathbf{u}_t$, $t = 0, \ldots, T-1$ obeying the constraints almost surely, and minimizing the expected cost. Specifically,

$$
\begin{aligned}
\min_{\mathbf{u}_0} \Bigg\{ q_0(\mathbf{x}_0, \mathbf{u}_0) \quad & + \mathbb{E}_{\mathbf{w}_0} \min_{\mathbf{u}_1} \Big\{ q_1(\mathbf{x}_1, \mathbf{u}_1) + \dots \\
& + \mathbb{E}_{\mathbf{w}_{T-2}} \min_{\mathbf{u}_{T-1}} \left\{ q_{T-1}(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) + \mathbb{E}_{\mathbf{w}_{T-1}} q_T(\mathbf{x}_T) \right\} \dots \Big\} \Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{u}_t + \mathbf{C}_t \mathbf{w}_t \quad && \forall t = 0, \dots, T-1, \\
& \mathbf{E}_t \mathbf{x}_t + \mathbf{F}_t \mathbf{u}_t \le \mathbf{f}_t && \forall t = 0, \dots, T-1, \\
& \mathbf{E}_T \mathbf{x}_T \le \mathbf{f}_T.
\end{aligned}
$$

2.2 Polynomial Policies and SDP Formulation

In this thesis, we explore the performance of polynomial policies, namely setting the control as a polynomial function in past observed uncertainties. We assume that every node in the network is able to "see" all external demands, so the policies at each node will depend on all past observed external uncertainties incurred to the entire system. At the initial time period, since there is no uncertainty observed yet, the first policy will be only constant. For a specific polynomial degree d, we have

$$
\mathbf{u}_t = \mathbf{L}_t \mathcal{B}_d(\boldsymbol{\xi}_t),
$$

where $\boldsymbol{\xi}_t$ denotes the vector of all past disturbances,

$$
\boldsymbol{\xi}_t = [\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{t-1}].
$$

and lies in a compact basic semi-algebraic set

$$
\Xi \equiv \left\{ \boldsymbol{\xi}_t \in \mathbb{R}^{tn_w} : g_j(\boldsymbol{\xi}_t) \ge 0, j = 1, \dots, m \right\},
$$

where g_j are all polynomial functions describing the set $\Xi \equiv \mathcal{W}_0 \times \cdots \times \mathcal{W}_{t-1}$. $\mathcal{B}_d(\boldsymbol{\xi}_t)$ is the canonical basis of $\mathcal{P}_d[\boldsymbol{\xi}_t]$, the vector space of polynomials in variables $\boldsymbol{\xi}_t$ with degrees at most d. $\mathbf{L}_t$ is the matrix of coefficients of the polynomials in the basis $\mathcal{B}_d(\boldsymbol{\xi}_t)$. The new decision variables become the coefficients in $\mathbf{L}_t$ instead of the order quantity itself. Consequently from the dynamics, we can also transform all the states into polynomials in basis $\mathcal{B}_d(\boldsymbol{\xi}_t)$.

With the polynomial parametrization of controls and states, all the constraints can now be written as:

$$p(\boldsymbol{\xi}_t) \geq 0, \ \forall \boldsymbol{\xi}_t \in \mathcal{W}_0 \times \cdots \times \mathcal{W}_{t-1},$$

where $p(\boldsymbol{\xi}_t)$ is a polynomial in variables $\boldsymbol{\xi}_t$. A sufficient condition for a polynomial to be nonnegative on a basic semi-algebraic set like Ξ is that it can be written in the following form:

$$p = \tau_0 + \sum_{j=1}^{m} \tau_j g_j, \tag{2.5}$$

where τ_j are sums of squares (SOS) polynomials in variables $\boldsymbol{\xi}_t$, and g_j are the polynomial functions describing the uncertainty set. Utilizing this condition, testing the non-negativity of p on the set Ξ can be transformed into a system of linear equality constraints on the coefficients of p and τ_j, and testing whether τ_j are SOS, which is equivalent to solving a semidefinite optimization problem (SDP). For a fixed precision, solving a SDP problem can be done in polynomial time by interior point methods [18, 21].

The stage cost at node k of time t, $q_k(t)$, is a convex piecewise affine function of the state $x_k(t)$ and the control $u_k(t)$. Since the control and inventory both are parametrized as polynomials of the past uncertainties, $q_k(t)$ becomes a piecewise polynomial function in the past uncertainties, i.e., a maximum of several polynomials.

$$q_k(t) = \max_{i=1,\ldots,r_t} p_k^i(t)(\boldsymbol{\xi}_t).$$

Therefore it is natural to introduce an upper bound polynomial $\pi_k(t) \in \mathcal{P}_d\left[\boldsymbol{\xi}_t\right]$ for each stage cost so that it is greater than every piece of the stage cost function under every possibility of uncertainties.

$$\pi_k(t)(\boldsymbol{\xi}_t) \geq p_k^i(t)(\boldsymbol{\xi}_t), \ \forall \boldsymbol{\xi}_t \in \mathcal{W}_0 \times \ldots \times \mathcal{W}_{t-1}, \forall i$$

We impose this condition by setting that $\pi_k(t)(\boldsymbol{\xi}_t) - p_k^i(t)(\boldsymbol{\xi}_t)$ is SOS.

In robust optimization, we are looking for a polynomial policy, denoted by ROB, that minimizes the overall cost in the worst case realization, so we introduce an upper bound value on the overall modified cost such that

$$J \geq \sum_{k=1}^{N} \sum_{t=0}^{T} \pi_k(t)(\boldsymbol{\xi}_t), \forall \boldsymbol{\xi}_T \in \mathcal{W}_0 \times \ldots \times \mathcal{W}_{T-1}$$

This inequality can be transformed into SOS condition as well. In the end, the objective is

$$\min J.$$

In contrast to the robust approach, we are also interested in the polynomial policy that minimizes the expectation of the overall cost, denoted by STO, by solving

$$\min \mathbb{E}_{\boldsymbol{\xi}_\mathbf{T}} \left[\sum_{k=1}^{N} \sum_{t=0}^{T} \pi_k(t)(\boldsymbol{\xi}_t) \right].$$

The sum of modified costs is a polynomial in past uncertainties in the form of $\mathbf{p}^T \mathcal{B}_d(\boldsymbol{\xi}_T)$, where the expectation can be evaluated by using moments of $\boldsymbol{\xi}_\mathbf{T}$, so the objective is an affine function in the coefficients.

Algorithm for computing optimal polynomial policies of degree d:

1. Consider polynomial control policies in the uncertainties, $\mathbf{u}_t(\boldsymbol{\xi}_t) = \mathbf{L}_t \mathcal{B}_d(\boldsymbol{\xi}_t)$.

2. Substitute a typical stage cost $q_k(t) = \max_{i=1,\ldots,r_t} p_k^i(t)(\boldsymbol{\xi}_t)$ with a modified stage cost $\pi_k(t) \in \mathcal{P}_d[\boldsymbol{\xi}_t]$, constrained to satisfy $\pi_k(t)(\boldsymbol{\xi}_t) \geq p_k^i(t)(\boldsymbol{\xi}_t), \forall \boldsymbol{\xi}_t \in \mathcal{W}_0 \times \ldots \times \mathcal{W}_{t-1}, \forall i$.

3. Substitute the overall cost with the sum of the modified stage costs $\sum_{k=1}^{N}\sum_{t=0}^{T}\pi_k(t)(\boldsymbol{\xi}_t)$.

4. Substitute a typical constraint $p(\boldsymbol{\xi}_t) \geq 0, \forall \boldsymbol{\xi}_t \in \{\boldsymbol{\xi} : g_j(\boldsymbol{\xi}) \geq 0, j = 1, \ldots, m\}$ (for either state-control constraints or modified costs constraints) with the condition:

   (a) Linear equality constraints on coefficients: $p = \tau_0 + \sum_{j=1}^{m}\tau_j g_j$;

   (b) SDP constraints: τ_j SOS, $deg(\tau_0) \leq d, deg(\tau_j g_j) \leq d, j = 1, \ldots, m.$

5. Formulate the objective function to find the coefficients of the respective policies.

   (a) ROB policy: introduce an upper bound J on the overall cost such that

   $$J \geq \sum_{k=1}^{N}\sum_{t=0}^{T}\pi_{k,t}(\boldsymbol{\xi}_t), \forall \boldsymbol{\xi}_T \in \mathcal{W}_0 \times \ldots \times \mathcal{W}_{T-1}$$

   and repeat Step 4 for this constraint, and solve the resulting SDP.

   (b) STO policy: find the expectation of the overall cost, $\mathbb{E}_{\boldsymbol{\xi}_\mathbf{T}}\left[\sum_{k=1}^{N}\sum_{t=0}^{T}\pi_k(t)(\boldsymbol{\xi}_t)\right]$, by moments of $\boldsymbol{\xi}_\mathbf{T}$, and solve the resulting SDP.

Example We take an example of a single echelon network with two planning time periods to illustrate the above framework. Let x, u, and w represent the inventory, order and demand, respectively. We assume the demand has mean μ and standard deviation σ, and we design the uncertainty set as $w_i \in [\mu - \sigma, \mu + \sigma]$. Since the initial inventory is zero, the dynamics are as follows:

$$x_1 = u_0 - w_0,$$

$$x_2 = x_1 + u_1 - w_1.$$

With c, h and p represents the ordering, holding and backlogging cost, respectively, the costs are:

$$q_0 = cu_0,$$

$$q_1 = cu_1 + \max(hx_1, -px_1),$$

$$q_2 = \max(hx_2, -px_2).$$

We have constraints:

$$u_0 \geq 0, u_1 \geq 0.$$

Applying our framework,

1. For polynomial policy of degree d, we have

$$u_0 = l_0 \geq 0, \tag{2.6}$$

$$u_1 = \mathbf{l}_1{}^T \mathcal{B}_d(\boldsymbol{\xi}_1) \geq 0, \tag{2.7}$$

where $\boldsymbol{\xi}_1 = [w_0]$.

2. We introduce polynomials π_0, π_1 and π_2 such that

$$\pi_0 = cl_0, \tag{2.8}$$

$$\pi_1 \geq cu_1 + hx_1, \tag{2.9}$$

$$\pi_1 \geq cu_1 - px_1, \tag{2.10}$$

$$\pi_2 \geq hx_2, \tag{2.11}$$

$$\pi_2 \geq px_2. \tag{2.12}$$

π_0 is a scalar, $\pi_1 = \bar{\mathbf{l}}_1{}^T \mathcal{B}_d(\boldsymbol{\xi}_1)$ is a polynomial of degree d in w_0, and $\pi_2 = \bar{\mathbf{l}}_2{}^T \mathcal{B}_d(\boldsymbol{\xi}_2)$ is a polynomial of degree d in w_0 and w_1, where $\boldsymbol{\xi}_2 = [w_0, w_1]^T$.

3. The overall cost of the system is $\pi_0 + \pi_1 + \pi_2$, which is a polynomial in $\boldsymbol{\xi}_2$.

4. Perform SOS relaxations to constraints (2.7),(2.9),(2.10),(2.11),and (2.12), we have

$$u_1 = \tau_0^u + \tau_1^u(w_0 - \mu + \sigma) + \tau_2^u(\mu + \sigma - w_0),$$

where $\tau_0^u = \mathbf{l}_0^{uT}\mathcal{B}_d(\boldsymbol{\xi}_1)$, $\tau_1^u = \mathbf{l}_1^{uT}\mathcal{B}_{d-1}(\boldsymbol{\xi}_1)$ and $\tau_2^u = \mathbf{l}_2^{uT}\mathcal{B}_{d-1}(\boldsymbol{\xi}_1)$ are SOS.

$$\pi_1 - cu_1 - hx_1 = \tau_0^{h_1} + \tau_1^{h_1}(w_0 - \mu + \sigma) + \tau_2^{h_1}(\mu + \sigma - w_0),$$

$$\pi_1 - cu_1 + px_1 = \tau_0^{p_1} + \tau_1^{p_1}(w_0 - \mu + \sigma) + \tau_2^{p_1}(\mu + \sigma - w_0),$$

where $\tau_0^{h_1} = \mathbf{l}_0^{h_1T}\mathcal{B}_d(\boldsymbol{\xi}_1)$, $\tau_0^{p_1} = \mathbf{l}_0^{p_1T}\mathcal{B}_d(\boldsymbol{\xi}_1)$, $\tau_1^{h_1} = \mathbf{l}_1^{h_1T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_1)$, $\tau_2^{h_1} = \mathbf{l}_2^{h_1T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_1)$, $\tau_1^{p_1} = \mathbf{l}_1^{p_1T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_1)$ and $\tau_2^{p_1} = \mathbf{l}_2^{p_1T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_1)$ are SOS.

$$\pi_2 - hx_2 = \tau_0^{h_2} + \tau_1^{h_2}(w_0 - \mu + \sigma) + \tau_2^{h_2}(\mu + \sigma - w_0) + \tau_3^{h_2}(w_1 - \mu + \sigma) + \tau_4^{h_2}(\mu + \sigma - w_1),$$

$$\pi_2 + px_2 = \tau_0^{p_2} + \tau_1^{p_2}(w_0 - \mu + \sigma) + \tau_2^{p_2}(\mu + \sigma - w_0) + \tau_3^{p_2}(w_1 - \mu + \sigma) + \tau_4^{p_2}(\mu + \sigma - w_1),$$

where $\tau_0^{h_2} = \mathbf{l}_0^{h_2T}\mathcal{B}_d(\boldsymbol{\xi}_2)$, $\tau_0^{p_2} = \mathbf{l}_0^{p_2T}\mathcal{B}_d(\boldsymbol{\xi}_2)$, $\tau_1^{h_2} = \mathbf{l}_1^{h_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_2^{h_2} = \mathbf{l}_2^{h_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_3^{h_2} = \mathbf{l}_3^{h_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_4^{h_2} = \mathbf{l}_4^{h_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_1^{p_2} = \mathbf{l}_1^{p_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_2^{p_2} = \mathbf{l}_2^{p_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_3^{p_2} = \mathbf{l}_3^{p_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$ and $\tau_4^{p_2} = \mathbf{l}_4^{p_2T}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$ are SOS.

5. (a) Robust approach: We introduce an upper bound value J on the overall cost such that

$$J \geq \pi_0 + \pi_1 + \pi_2.$$

Doing SOS relaxation,

$$\begin{aligned} J - \pi_0 - \pi_1 - \pi_2 = \tau_0^J + \tau_1^J(w_0 - \mu + \sigma) + \tau_2^J(\mu + \sigma - w_0) \\ + \tau_3^J(w_1 - \mu + \sigma) + \tau_4^J(\mu + \sigma - w_1), \end{aligned}$$

where $\tau_0^J = \mathbf{l}_0^{JT}\mathcal{B}_d(\boldsymbol{\xi}_2)$, $\tau_1^J = \mathbf{l}_1^{JT}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_2^J = \mathbf{l}_2^{JT}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, $\tau_3^J = \mathbf{l}_3^{JT}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$, and $\tau_4^J = \mathbf{l}_4^{JT}\mathcal{B}_{d-1}(\boldsymbol{\xi}_2)$ are SOS. As a result, the robust formulation is :

$$
\begin{aligned}
\min \quad & J \\
s.t. \quad J - \pi_0 - \pi_1 - \pi_2 = \; & \tau_0^J + \tau_1^J(w_0 - \mu + \sigma) + \tau_2^J(\mu + \sigma - w_0) \\
& + \tau_3^J(w_1 - \mu + \sigma) + \tau_4^J(\mu + \sigma - w_1) \\
\pi_1 - cu_1 - hx_1 = \; & \tau_0^{h_1} + \tau_1^{h_1}(w_0 - \mu + \sigma) + \tau_2^{h_1}(\mu + \sigma - w_0) \\
\pi_1 - cu_1 + px_1 = \; & \tau_0^{p_1} + \tau_1^{p_1}(w_0 - \mu + \sigma) + \tau_2^{p_1}(\mu + \sigma - w_0) \\
\pi_2 - hx_2 = \; & \tau_0^{h_2} + \tau_1^{h_2}(w_0 - \mu + \sigma) + \tau_2^{h_2}(\mu + \sigma - w_0) \\
& + \tau_3^{h_2}(w_1 - \mu + \sigma) + \tau_4^{h_2}(\mu + \sigma - w_1) \\
\pi_2 + px_2 = \; & \tau_0^{p_2} + \tau_1^{p_2}(w_0 - \mu + \sigma) + \tau_2^{p_2}(\mu + \sigma - w_0) \\
& + \tau_3^{p_2}(w_1 - \mu + \sigma) + \tau_4^{p_2}(\mu + \sigma - w_1) \\
u_0 = \; & l_0 \\
u_1 = \; & \tau_0^u + \tau_1^u(w_0 - \mu + \sigma) + \tau_2^u(\mu + \sigma - w_0) \\
l_0 \geq 0, \; & \text{all } \tau \text{ are SOS}
\end{aligned}
$$

(b) Stochastic approach: We evaluate the expectation of $\pi_0 + \pi_1 + \pi_2$ using moments up to degree d of w_0 and w_1, so

$$
\mathbb{E}_{w_0,w_1}\left[\pi_0 + \pi_1 + \pi_2\right] = f(l_0, \bar{\mathbf{l}}_1, \bar{\mathbf{l}}_2),
$$

where f is an affine function in $(l_0, \bar{\mathbf{l}}_1, \bar{\mathbf{l}}_2)$. As a result, the stochastic formulation is:

$$
\begin{aligned}
\min \quad & f(l_0, \bar{\mathbf{l}}_1, \bar{\mathbf{l}}_2) \\
s.t. \quad \pi_1 - cu_1 - hx_1 = \; & \tau_0^{h_1} + \tau_1^{h_1}(w_0 - \mu + \sigma) + \tau_2^{h_1}(\mu + \sigma - w_0) \\
\pi_1 - cu_1 + px_1 = \; & \tau_0^{p_1} + \tau_1^{p_1}(w_0 - \mu + \sigma) + \tau_2^{p_1}(\mu + \sigma - w_0) \\
\pi_2 - hx_2 = \; & \tau_0^{h_2} + \tau_1^{h_2}(w_0 - \mu + \sigma) + \tau_2^{h_2}(\mu + \sigma - w_0) \\
& + \tau_3^{h_2}(w_1 - \mu + \sigma) + \tau_4^{h_2}(\mu + \sigma - w_1) \\
\pi_2 + px_2 = \; & \tau_0^{p_2} + \tau_1^{p_2}(w_0 - \mu + \sigma) + \tau_2^{p_2}(\mu + \sigma - w_0) \\
& + \tau_3^{p_2}(w_1 - \mu + \sigma) + \tau_4^{p_2}(\mu + \sigma - w_1) \\
u_0 = \; & l_0 \\
u_1 = \; & \tau_0^u + \tau_1^u(w_0 - \mu + \sigma) + \tau_2^u(\mu + \sigma - w_0) \\
l_0 \geq 0, \; & \text{all } \tau \text{ are SOS}
\end{aligned}
$$

2.3 Special Cases of SDP

In this section, we show that under affine policies and polyhedral uncertainty sets, the resulting problem reduces into a linear optimization problem instead of a semidefinite one.

Firstly, in order for the resulting problem of the framework to be affine, the degree of the polynomial policies can only be either 0 or 1. In degree 0 policies, $\mathbf{u}_t(\boldsymbol{\xi}_t) = \boldsymbol{\theta}_t^0$, which are actually constants and independent of the uncertainties. In degree 1 policies, $\mathbf{u}_t(\boldsymbol{\xi}_t) = \boldsymbol{\Theta}_t\boldsymbol{\xi}_t + \boldsymbol{\theta}_t$. By the linear dynamics of the system (2.1), the states $\mathbf{x}_t$ are affine in the uncertainties, so the constraints (2.3) and (2.4) are either independent of or affine in the uncertainties. Each piece of the stage cost $p_k^i(t)(\boldsymbol{\xi}_t)$ becomes affine in the past uncertainties, so as the modified stage cost $\pi_k(t)(\boldsymbol{\xi}_t)$.

Secondly, the uncertainty sets have to be polytopic, i.e., all the functions g_j describing the uncertainty set $\mathcal{W}_k(t)$ in (2.2) are affine so that the SDP constraints in Step 4 reduces into positivity constraints. For a polyhedral uncertainty set, Bempored et al. [1] proved that piecewise affine policies are optimal and the min-max values is achieved at some extreme points of the polyhedron.

If the constraint $p(\boldsymbol{\xi}_t)$ is of degree 0, it will be substituted as:

$$p(\boldsymbol{\xi}_t) = \tau_0, \text{ where } \tau_0 \text{ is scalar and } \tau_0 \geq 0.$$

If the constraint $p(\boldsymbol{\xi}_t)$ is of degree 1, we will have:

$$p = \tau_0 + \textstyle\sum_{j=1}^m \tau_j g_j, \text{ where } \tau_j \text{ is scalar and } \tau_j \geq 0, \forall j = 0, 1, \dots, m.$$

We can see that in the resulted formulation, there are no SOS constraints, and all the constraints and objective function are linear, so it is indeed a linear optimization problem.

Chapter 3

Computational Results

In this section, we test the polynomial policy framework on multi-echelon networks of varying size and topology so as to obtain insights on the effectiveness of the policies, and the framework (robust vs. stochastic) models . There specific questions that we aim to investigate are:

- how performance depends on the polynomial policy degree;
- comparing the performance of polynomial and basestock policies;
- the relative performance evaluation of the robust and the stochastic framework;
- how the performance of polynomial policies depends on different realization distribution of the demand sequences;
- examining the scaling of the polynomial policy framework with the problem size, namely the complexity of the network and time horizon; and
- investigating the performance of the polynomial policies across different types of networks.

For these purposes, we consider three assembly networks containing three, five and eight installations with polyhedral type uncertain demands. The details are described below.

Networks:

Figure 3-1, 3-2, and 3-3 depict system setups of three-echelon, five-echelon and eight-echelon networks respectively, and the inventory holding cost h, backlogging cost b, variable ordering cost c are summarized in Table 3.1. We obtained the networks from Bertsimas et al. [9], where they considered basestock policies.

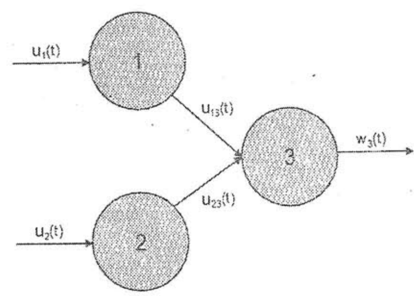

Figure 3-1: 3-Installation System Setup.

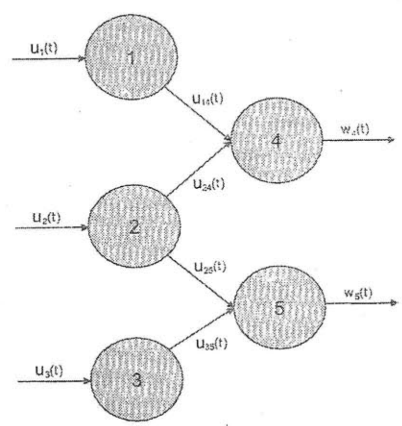

Figure 3-2: 5-Installation System Setup.

Firstly, we would like to consider polynomial policies in assembly networks. We assume that at each node, all the parts ordered from each suppliers are equal in quantities, since any extra amount of parts will incur higher holding cost than in upstream nodes, e.g. $u_{13}(t) = u_{23}(t)$ in the three-echelon network and both will be denoted by $u_3(t)$. In these three systems, there is no constraint on the inventory levels, but the

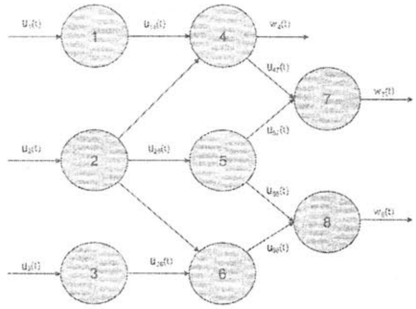

Figure 3-3: 8-Installation System Setup.

Network size	Installation	h	b	c
3-installation	1	6	10	1
	2	8	15	3
	3	10	28	4
5-installation	1	6	10	1
	2	8	15	2
	3	8	15	2
	4	14	30	10
	5	15	35	10
8-installation	1	6	10	1
	2	8	15	2
	3	8	15	2
	4	10	20	6
	5	16	28	6
	6	16	28	2
	7	25	45	8
	8	30	55	8

Table 3.1: Cost parameters for the networks.

order quantity has to be nonnegative, i.e., the flow of items is only one-way. The planning time horizons considered for these three networks are $T = 15$, $T = 10$ and $T = 10$ respectively.

With $x_k(t)$ denoting the inventory of node k at time t, $u_k(t)$ denoting the order of node k placed to its suppliers at time t, and $w_k(t)$ denoting the external demand incurred at node k of time t, we have the following dynamics:

$$x_k(t+1) = x_k(t) + u_k(t) - \sum_{i \in D(k)} u_i(t) - w_k(t)\mathbb{1}_{[k \in S]}, \tag{3.1}$$

where $D(k)$ represents the set of downstream nodes of node k, i.e., the nodes making orders from node k, and S denotes the set of sink nodes where external demands occur. For simplicity, we assume the initial states at all the nodes are zero, i.e. there is no initial inventory in the system prior to the beginning of planning periods. The cost incurred at node k time period t is

$$q_k(t) = c_k(t)u_k(t) + \max(h_k(t)x_k(t+1), -b_k(t)x_k(t+1)), \tag{3.2}$$

and the system is constrained by

$$u_k(t) \geq 0, \forall k, t.$$

Uncertainty set:

The external demands at all the sinks are assumed to be i.i.d and have mean $\mu = 100$ and standard deviation $\sigma = 20$. To find the robust and stochastic polynomial policies for the three networks, we used the same uncertainty set as in [9], which is specified by a combination of individual demand uncertainty and cumulative demand uncertainty.

$$w_s(t) \in [\mu - \sigma, \mu + \sigma]\, \forall s \in S, t = 0, \ldots, T-1, \tag{3.3}$$

$$\frac{\left|\sum_{i=1}^{t}(w_s(i) - \mu)\right|}{\sqrt{t}} \leq 3\sigma \forall s \in S, t = 0, \ldots, T-1. \tag{3.4}$$

The first constraint of the polyhedral uncertainty set limits each individual demand within one standard deviation from the average. The second constraint, by central limit theorem, covers 99% of the possible occurrences of demand sequences and eliminates the unlikely scenario where all the realized demands are dominated at one end of the interval.

3.1 Performance Analysis as a Function of Polynomial Policy Degree

Applying the framework in Chapter 2 under uncertainty set given in (3.3)-(3.4), we found both the robust and stochastic optimization solutions denoted by ROB and STO respectively for the three networks under consideration. With the policies, firstly we would like to measure their effectiveness with respect to the worst case demand realization of varying sizes of polyhedral uncertainty. The results are presented in Table 3.2 and Table 3.3 for degree 0 and degree 1 policies respectively.

network	policy	$[\mu - \sigma, \mu + \sigma]$	$[\mu - 2\sigma, \mu + 2\sigma]$	$[\mu - 3\sigma, \mu + 3\sigma]$
3-installation	STO	60232	**88180**	**155380**
	ROB	**48134**	115334	182534
5-installation	STO	90130	**122030**	**180610**
	ROB	**67948**	139176	210676
8-installation	STO	188530	**260030**	**349330**
	ROB	**141933**	273266	405266

Table 3.2: Max cost of degree 0 policies for assembly networks assuming demand lies in a polyhedron of varying size.

From these two tables, we can see that the cost by degree 0 policy is very high. At degree 0, the ordering policy is independent of the past realized uncertainties, it is solved as a closed-loop problem with all the policies become static ordering quantities regardless the actual realization of demand sequences. However, in the polynomial policy of degree $d \geq 1$, the exact ordering quantities are not fixed, and the open-loop

network	policy	$[\mu-\sigma,\mu+\sigma]$	$[\mu-2\sigma,\mu+2\sigma]$	$[\mu-3\sigma,\mu+3\sigma]$
3-installation	STO	16595	**25755**	**36355**
	ROB	**16111**	26751	37391
5-installation	STO	32290	**46430**	**63290**
	ROB	**31016**	47876	64736
8-installation	STO	74850	**105690**	**139770**
	ROB	**72052**	106132	140212

Table 3.3: Max cost of degree 1 policies for assembly networks assuming demand lies in a polyhedron of varying size.

policy adapts to the available information of realized demand sequence at hand to compute the next ordering quantities. In order to reduce the cost of degree 0 policy, we also would like to make use of some of the available information at each time period to decide the order quantities. Therefore, when implementing the degree 0 policy, we resolved the problem every two time periods, corresponding to the commonly used bi-weekly review policy in industry, based on the instant inventory level.

Next we evaluate the performance of the policies by generating demand sequences under the following distributions with the same mean $\mu = 100$ and standard deviation $\sigma = 20$.

- Unimodal continuous distribution: This class of distribution is commonly used to model the demand distribution. We draw demands from three such distributions - Normal(μ, σ), Lognormal(μ, σ), and Gamma(μ, σ).

- Multi-modal continuous distribution: We used uniform and mixture of two normal distributions to simulate demands with the original μ and σ.

- Discrete distribution: Each demand is generated to be either $\mu + \sigma$ or $\mu - \sigma$ with 50% probability for each. This is a very special demand realization since the generated demand sequences are at the vertices of the original assumed uncertainty set in obtaining the ROB and STO.

- Correlated distribution: In finding the policies, the demand sequences are assumed to be independent. We would like to test the effect of correlation on the time horizon of demand sequence, which exists in some commodities. We used both positive correlation and negative correlation when $\rho = 0.5$ and $\rho = -0.5$ respectively. We note that the correlation here only exits on the time horizon, so demands at multiple sinks are still uncorrelated.

For each of the above distribution, we generated 20000 instances for degree 1 policies and 1000 instances for degree 0 policies as it is very time consuming for the resolving in implementing degree 0 policies. We calculated the total cost for each instance, and, to compare the performance, we reported the mean cost, standard deviation of costs, and average of 5% highest costs. Tables 3.4 to 3.9 present all the results.

demand distribution	policy	mean	std	5% tail
normal	STO	18103	**705**	**19941**
	ROB	**17841**	866	20155
lognormal	STO	18340	**865**	**20311**
	ROB	**18059**	970	20360
gamma	STO	18213	**902**	20596
	ROB	**17897**	941	**20329**
mixture of two normals	STO	17867	**517**	**19002**
	ROB	**17747**	657	19338
uniform	STO	17934	**582**	**19011**
	ROB	**17838**	776	19656
discrete	STO	**17288**	709	**18488**
	ROB	17384	**554**	18522
50% correlation	STO	18636	**1358**	**22339**
	ROB	**18517**	1507	22639
-50% correlation	STO	17767	**513**	18818
	ROB	**17462**	575	**18715**

Table 3.4: Performance of degree 0 policies for 3-installation assembly network.

Firstly, we notice that the costs of degree 0 policies have been largely reduced by adopting the idea of resolving in every two periods. Then comparing the performance of degree 0 and degree 1 policies for each network, we can see steady improvement in performance, lower mean cost, lower 5% tail, and generally lower standard deviation in all the distribution realizations, when moving from degree 0 policies to degree 1

demand distribution	policy	mean	std	5% tail
normal	STO	16790	**692**	18589
	ROB	**16594**	777	**18579**
lognormal	STO	17023	**907**	19466
	ROB	**16812**	987	**19420**
gamma	STO	16860	**843**	19209
	ROB	**16756**	930	**19196**
mixture of two normals	STO	16597	**456**	**17713**
	ROB	**16499**	529	17787
uniform	STO	16723	**456**	**17772**
	ROB	**16608**	554	17883
discrete	STO	**15835**	248	16295
	ROB	16111	**6.01e-6**	**16111**
50% correlation	STO	16773	**873**	**19398**
	ROB	**16571**	1013	19566
-50% correlation	STO	16789	**741**	18736
	ROB	**16590**	820	**18698**

Table 3.5: Performance of degree 1 policies for 3-installation assembly network.

demand distribution	policy	mean	std	5% tail
normal	STO	37724	1410	40425
	ROB	**37269**	**1214**	**40067**
lognormal	STO	38310	1886	42952
	ROB	**37642**	**1532**	**41333**
gamma	STO	38146	1649	42163
	ROB	**37671**	**1570**	**41910**
mixture of two normals	STO	37957	1280	39846
	ROB	**37262**	**1006**	**39337**
uniform	STO	37883	1336	40328
	ROB	**37328**	**1035**	**39238**
discrete	STO	36428	1579	39143
	ROB	**36185**	**1154**	**38276**
50% correlation	STO	38410	2249	44321
	ROB	**37856**	**2001**	**43339**
-50% correlation	STO	37648	1245	40701
	ROB	**36670**	**1029**	**39257**

Table 3.6: Performance of degree 0 policies for 5-installation assembly network.

demand distribution	policy	mean	std	5% tail
normal	STO	33202	1238	**34147**
	ROB	**33049**	**1111**	35856
lognormal	STO	33492	1558	37347
	ROB	**33435**	**1446**	**37136**
gamma	STO	33419	1454	36975
	ROB	**33322**	**1312**	**36630**
mixture of two normals	STO	32906	1044	35174
	ROB	**32881**	**758**	**34678**
uniform	STO	33096	1037	35328
	ROB	**33035**	**779**	**34797**
discrete	STO	**30629**	928	32047
	ROB	31016	**2.79e-6**	**31016**
50% correlation	STO	34203	1628	38597
	ROB	**34054**	**1487**	**38267**
-50% correlation	STO	34197	1255	37222
	ROB	**34038**	**1141**	**36903**

Table 3.7: Performance of degree 1 policies for 5-installation assembly network.

demand distribution	policy	mean	std	5% tail
normal	STO	79629	2304	85322
	ROB	**78173**	**1981**	**82408**
lognormal	STO	80737	2983	86881
	ROB	**79531**	**2839**	**86209**
gamma	STO	80900	2947	86999
	ROB	**79577**	**2760**	**85232**
mixture of two normals	STO	79559	2235	83985
	ROB	**78087**	**1946**	**81811**
uniform	STO	79968	2714	84915
	ROB	**78652**	**2012**	**82482**
discrete	STO	77472	2282	82397
	ROB	**76422**	**1976**	**80244**
50% correlation	STO	80603	3438	87656
	ROB	**79446**	**3248**	**86959**
-50% correlation	STO	79202	2005	83132
	ROB	**76958**	**1773**	**80211**

Table 3.8: Performance of degree 0 policies for 8-installation assembly network.

demand distribution	policy	mean	std	5% tail
normal	STO	73907	2030	78653
	ROB	**73485**	**1912**	**78127**
lognormal	STO	74502	2534	80691
	ROB	**74168**	**2439**	**80314**
gamma	STO	74358	2347	80007
	ROB	**74023**	**2252**	**79612**
mixture of two normals	STO	73315	1687	76948
	ROB	**73310**	**1341**	**76438**
uniform	STO	73698	1642	77204
	ROB	**73680**	**1378**	**76664**
discrete	STO	**70846**	1557	73708
	ROB	72052	**1.96e-4**	**72052**
50% correlation	STO	73942	2575	80800
	ROB	**73511**	**2509**	**80427**
-50% correlation	STO	73904	2085	78942
	ROB	**73471**	**2049**	**78564**

Table 3.9: Performance of degree 1 policies for 8-installation assembly network.

policies for all the three networks.

However, similar improvement in performance was not observed when moving from affine policies to quadratic policies. Simulation results of quadratic policies for 3-installation system are shown in Table 3.10. Comparing with affine policies, we can see that except for the equal performance in discrete distribution, the costs of quadratic ROB policies appear slightly higher than the affine ROB policies, and costs of quadratic STO policies appear much higher than the affine STO policies. To investigate the underlying reason, we look at the computed policies, which are shown partially in Table 3.11, 3.12, 3.13, and 3.14 for illustration purpose.

In the policy of affine ROB, we can see that they only depend on the constant term and the last demand. In the quadratic ROB policy, all the first order monomials have essentially the same coefficients as in affine ROB policy. The coefficients of second order monomials are almost zero with a few terms having some residuals, which cancel out with the residual in constant term coefficient compared to the constant coefficient

		mean	std	5% tail
normal	STO	462927	214297	960902
	ROB	161604	764	18565
lognormal	STO	481873	216293	978560
	ROB	16835	1001	19499
gamma	STO	469246	216017	965626
	ROB	16778	929	19211
mixture of two normals	STO	296295	144868	670740
	ROB	16508	537	17802
uniform	STO	330245	169181	770180
	ROB	16608	553	17907
50% correlation	STO	512458	233650	1034069
	ROB	16613	1045	19694
-50% correlation	STO	515900	237465	1049546
	ROB	16615	831	18780
discrete	STO	15834	249	16293
	ROB	16111	2.4E-7	16111

Table 3.10: Performance of degree 2 policies for 3-installation assembly network.

monomial	1			
$u_1(1)$	118			
$u_2(1)$	118			
$u_3(1)$	118			
monomial	1	w_1		
$u_1(2)$	100	20		
$u_2(2)$	100	20		
$u_3(2)$	100	20		
monomial	1	w_1	w_2	
$u_1(3)$	100	5e-10	20	
$u_2(3)$	100	5e-10	20	
$u_3(3)$	100	3e-10	20	
monomial	1	w_1	w_2	w_3
$u_1(4)$	100	3e-10	5e-10	20
$u_2(4)$	100	3e-10	5e-10	20
$u_3(4)$	100	2e-10	3e-10	20

Table 3.11: Partial affine ROB policies for 3-installation assembly network.

monomial	1									
$u_1(1)$	118									
$u_2(1)$	118									
$u_3(1)$	118									
monomial	1	w_1	w_1^2							
$u_1(2)$	100	20	-3e-2							
$u_2(2)$	100	20	-3e-2							
$u_3(2)$	100	20	-2e-2							
monomial	1	w_1	w_2	w_1^2	$w_1 w_2$	w_2^2				
$u_1(3)$	100	-6e-11	20	-1e-3	-4e-13	-3e-2				
$u_2(3)$	100	2e-11	20	-1e-3	-4e-13	-3e-2				
$u_3(3)$	100	-1e-11	20	-1e-3	-2e-13	-2e-2				
monomial	1	w_1	w_2	w_3	w_1^2	$w_1 w_2$	$w_1 w_3$	w_2^2	$w_2 w_3$	w_3^2
$u_1(3)$	100	6e-11	-8e-12	20	1e-3	4e-13	-1e-3	-6e-14	-3e-13	-4e-2
$u_2(3)$	100	1e-11	-2e-11	20	2e-3	3e-13	-1e-3	2e-13	-4e-13	-3e-2
$u_3(3)$	100	7e-11	-4e-12	20	1e-3	3e-13	-1e-3	8e-14	-2e-13	-2e-2

Table 3.12: Partial quadratic ROB policies for 3-installation assembly network.

monomial	1			
$u_1(1)$	120			
$u_2(1)$	120			
$u_3(1)$	120			
monomial	1	w_1		
$u_1(2)$	100	20		
$u_2(2)$	100	20		
$u_3(2)$	100	20		
monomial	1	w_1	w_2	
$u_1(3)$	100	-7e-11	20	
$u_2(3)$	100	-2e-11	20	
$u_3(3)$	100	1e-11	20	
monomial	1	w_1	w_2	w_3
$u_1(4)$	100	1e-10	8e-12	20
$u_2(4)$	100	9e-11	8e-11	20
$u_3(4)$	100	6e-11	6e-11	20

Table 3.13: Partial affine STO policies in 3-installation assembly network.

monomial	1									
$u_1(1)$	120									
$u_2(1)$	120									
$u_3(1)$	120									
monomial	1	w_1	w_1^2							
$u_1(2)$	465	20	-365							
$u_2(2)$	415	20	-315							
$u_3(2)$	260	20	-160							
monomial	1	w_1	w_2	w_1^2	$w_1 w_2$	w_2^2				
$u_1(3)$	820	4e-13	20	-352	-2e-12	-368				
$u_2(3)$	700	7e-13	20	-288	-2e-12	-312				
$u_3(3)$	417	6e-13	20	-156	-1e-12	-160				
monomial	1	w_1	w_2	w_3	w_1^2	$w_1 w_2$	$w_1 w_3$	w_2^2	$w_2 w_3$	w_3^2
$u_1(3)$	1223	1e-12	-2e-12	20	-367	1e-13	-371	-7e-13	-2e-12	-386
$u_2(3)$	1032	1e-12	2e-13	20	-300	1e-13	-306	-7e-13	-1e-12	-326
$u_3(3)$	603	1e-12	-6e-14	20	-169	8e-13	-166	-3e-13	-8e-13	-169

Table 3.14: Partial quadratic STO policies in 3-installation assembly network.

in affine policy. It indicates that under our assumptions and conditions the optimal policy is an affine function in the very last observed demand, instead of depending on the entire past history. Also we can conclude that the solution becomes unstable at degree 2, and this instability leads to slightly poorer performance comparing to degree 1. Comparing the STO affine and quadratic policies, we notice much severe instability. The first order monomials similarly have essentially the same coefficients in both affine and quadratic policies as well. Though the instability of second order monomial coefficients will cancel out with the instability of the constant coefficients in expectation, it renders the quadratic STO policy very bad performance in practice. The cost can be 100 times higher as shown in Table 3.10.

By these experiments, we find that affine policies exhibit the best performance in all the three networks for both robust and stochastic frameworks. We notice an interesting point: in all three networks, affine robust policies consistently give almost constant cost in discrete distribution, where all the demand sequences are at the vertices of the polyhedral uncertain set. The constant cost of discrete distribution is the same as the worst case cost of the original uncertainty set in Table 3.3, i.e., 16111, 31016 and 72052 for 3-, 5- and 8-installation network respectively. This observation indicates that almost all the vertices are worst case realizations for the robust affine policy, so as to minimize the corresponding worst case cost, it has to be at some point with equal distance to all of them.

3.2 Robust Policies vs. Stochastic Policies

The purpose of this section is to measure the effectiveness of robust polynomial policy with respect to stochastic polynomial policy in the following aspects: worst case demand realization of polyhedron uncertainty, realization of various demand distributions, and uncertainty in standard deviation of demand distribution. Table 3.15 to 3.19, expressed as $\frac{\text{STO-ROB}}{\text{STO}}(\%)$, present the performances comparison for affine policies.

It is obvious that ROB gives lower worst case cost than STO in the original polyhedron, since ROB minimizes the worst case cost. However, if the uncertainty set turns out to be different as planned, the STO actually performs better in the worst case scenario as shown in Table 3.15.

network	$[\mu-\sigma,\mu+\sigma]$	$[\mu-2\sigma,\mu+2\sigma]$	$[\mu-3\sigma,\mu+3\sigma]$
3-installation	2.92%	-3.87%	-2.85%
5-installation	3.95%	-3.11%	-2.28%
8-installation	3.74%	-0.42%	-0.32%

Table 3.15: Max cost comparison of affine ROB vs. STO $\frac{\text{STO-ROB}}{\text{STO}}$ assuming demand lies in a polyhedron of varying size.

From Table 3.16, 3.17 and 3.18, it is obvious that robust policies outperform stochastic policy in almost all the demand realizations except the discrete case in mean, standard deviation and 5% tail of the costs. Comparing the superiority of ROB over STO in the three aspects, we found the superiority in mean in the least

demand distribution	mean	std	5% tail
normal	1.2%	-12.3%	0.1%
lognormal	1.2%	-8.9%	0.2%
gamma	1.2%	-10.3%	0.1%
mixture of two normals	0.6%	-16.2%	-0.4%
uniform	0.7%	-21.6%	-0.6%
discrete	-1.7%	100.0%	1.1%
50% correlation	1.2%	-16.0 %	-0.9%
-50% correlation	1.2%	-10.7%	0.2%

Table 3.16: Performance comparison of affine ROB vs. STO $\frac{\text{STO-ROB}}{\text{STO}}$ for 3-installation assembly network.

demand distribution	mean	std	5% tail
normal	0.5%	10.2%	0.8%
lognormal	0.2%	7.2%	0.6%
gamma	0.3%	9.8%	0.9%
mixture of two normals	0.1%	27.4%	1.4%
uniform	0.2%	24.9%	1.5%
discrete	-1.3%	100.0%	3.2%
50% correlation	0.4%	8.7%	0.9%
-50% correlation	0.5%	9.1%	0.9%

Table 3.17: Performance comparison of affine ROB vs. STO $\frac{\text{STO-ROB}}{\text{STO}}$ for 5-installation assembly network.

demand distribution	mean	std	5% tail
normal	0.6%	5.8%	0.7%
lognormal	0.5%	3.8%	0.5%
gamma	0.5%	4.1%	0.5%
mixture of two normals	1e-2%	20.6%	0.7%
uniform	0.2%	16.1%	0.7%
discrete	-1.7%	100.0%	2.3%
50% correlation	0.6%	2.6%	0.5%
-50% correlation	0.6%	1.7%	0.5%

Table 3.18: Performance comparison of affine ROB vs. STO $\frac{\text{STO-ROB}}{\text{STO}}$ for 8-installation assembly network.

demand distribution	mean	std	5% tail
gamma($\mu, 0.5\sigma$)	1.28%	44.69%	2.27%
gamma($\mu, 0.75\sigma$)	0.73%	19.32%	1.49%
gamma(μ, σ)	0.29%	9.78%	0.93%
gamma($\mu, 1.25\sigma$)	0.13%	7.16%	0.82%
gamma($\mu, 1.5\sigma$)	-0.17%	3.99%	0.32%
gamma($\mu, 1.75\sigma$)	-0.34%	2.72%	0.18%
gamma($\mu, 2\sigma$)	-0.57%	2.12%	0.10%

Table 3.19: Relative performance of affine ROB vs. STO $\frac{\text{STO-ROB}}{\text{STO}}$ as a function of realized $\tilde{\sigma}$ for 5-installation assembly network.

which is usually less than 0.5%. Superiority in 5% tail is the second with around 1% lower, since robust optimization minimizes the worst scenario cost. With the reduction in the high tail, ROB leads to a large reduction in standard deviation of performance, which is usually around 10% and may go up to 20% for multimodal continuous distributions. We indeed have situations that STO gives smaller standard deviation than ROB, when ROB is not able to reduce the tail significantly. These situations occur in the unimodal continuous distributions of 3-installation network. Lastly, in the special discrete demand realization, as explained in Section 3.1, robust affine policy is invariant to the actual demand sequences with constant cost, which is higher than the stochastic affine policy in average but lower in 5% tail.

In all the previous simulations, we assume the standard deviation to be known. However, we would like to measure the effect of uncertainty in σ. Therefore, we experiment on the 5-installation network under gamma distribution of demand sequence, where demands are drawn from gamma distribution with mean μ and standard deviation $\tilde{\sigma} \in \{0.5\sigma, 0.75\sigma, \ldots, 2\sigma\}$. Table 3.19 presents the relative performance of affine ROB vs. STO in this situation. We observed that ROB outperforms STO significantly when σ is small in all of mean standard deviation and 5% tail. However, with increase of σ, the gap between ROS and STO decreases. At 1.5σ, STO starts beating ROB in mean value, however ROB still keeps advantage in minimizing the 5% tail, and lower the standard deviation.

By comparing the affine policies from robust and stochastic frameworks, we find STO is more robust than ROB in larger uncertainty set; ROB generally performs than STO for distribution with presumed standard deviation; however with increasing standard deviation, the superiority of ROB over STO decreases.

3.3 Sensitivity Analysis with Respect to Demand Distributions

In this section, we compare the performance of polynomial policies across different demand realization distributions with reference to Figure 3-4, 3-5 and 3-6.

Firstly, we notice that the policies lead to lowest cost in both average and 5% tail with the smallest standard deviation when realized demand is discretely distributed. This is because the demand sequences all fall in the original planned uncertainty set in discrete distribution. However, substantial amount of the demand sequences of other distribution will be outside of the set $[\mu - \sigma, \mu + \sigma]$.

Secondly, the policies yield better performance in multimodal than unimodal continuous distributions, as the three figures all indicate lower average cost, standard deviation and 5% tail. The two multimodal continuous distributions, mixture of two normals and uniform, lead to very similar performance in all three criteria for both frameworks. In the three unimodal distributions, lognormal is the highest in all three aspects, gamma in the middle and normal is the lowest.

Lastly, comparing the effects of correlation, we noticed the cost of positively correlated demands is higher than the negatively correlated ones. In the positive correlation case, one high demand supports the occurrence of another high demand or one low demand boosts the occurrence of another low demand, so the demand sequence over

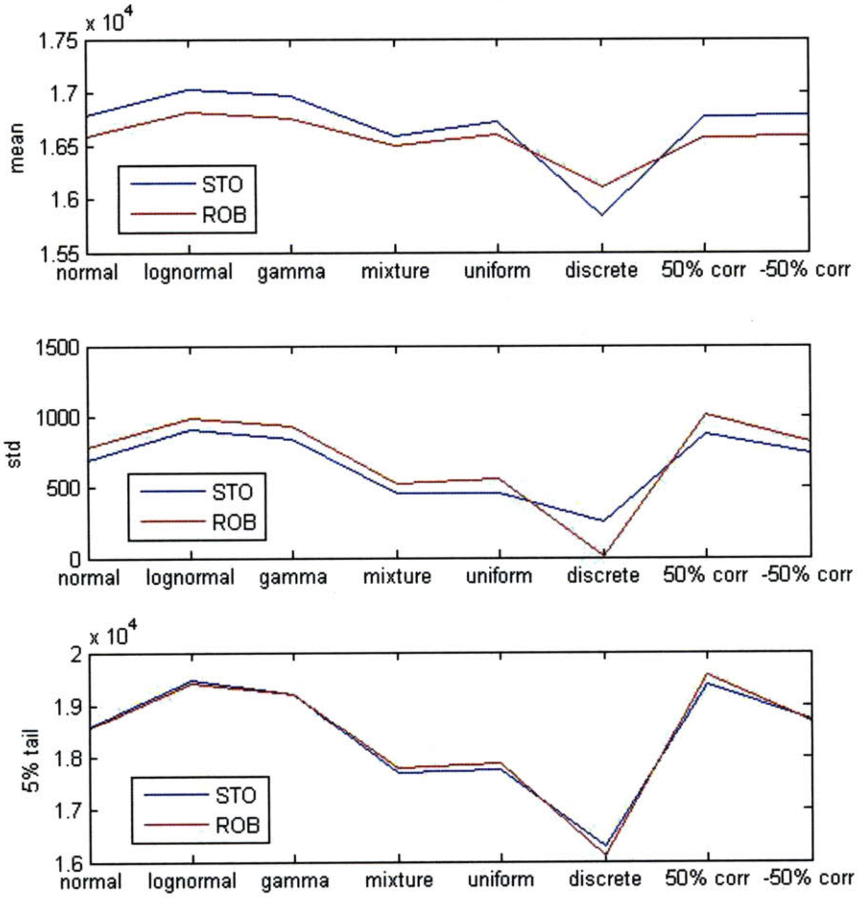

Figure 3-4: Affine STO vs. ROB for 3-Installation Assembly Network

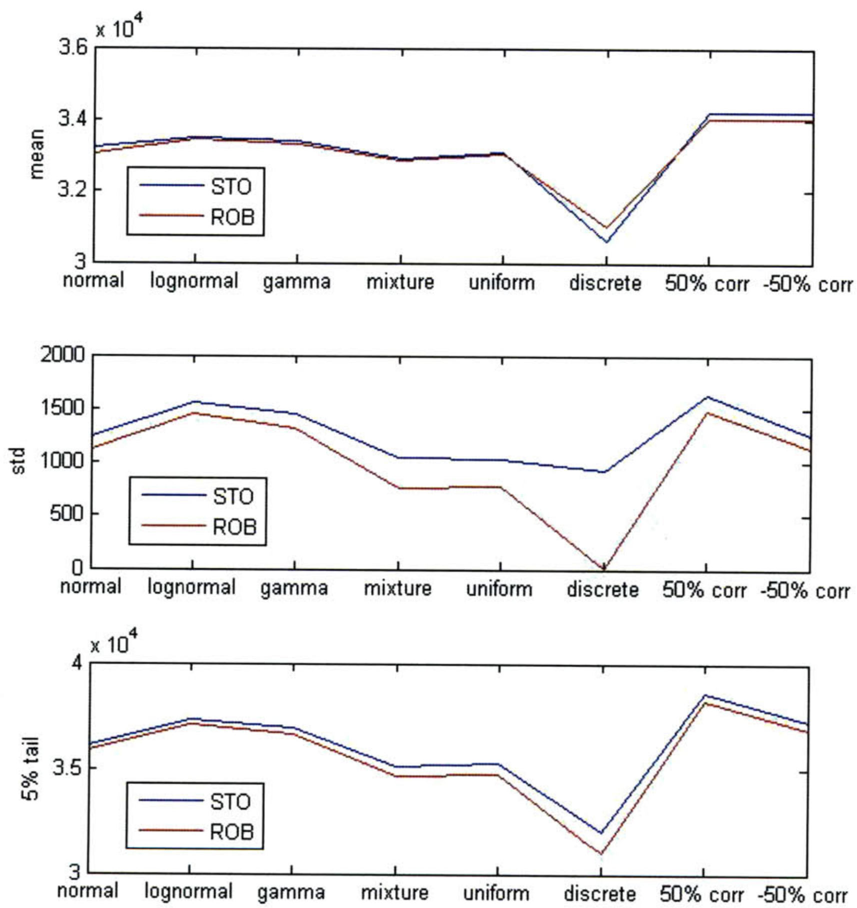

Figure 3-5: Affine STO vs. ROB for 5-Installation Assembly Network

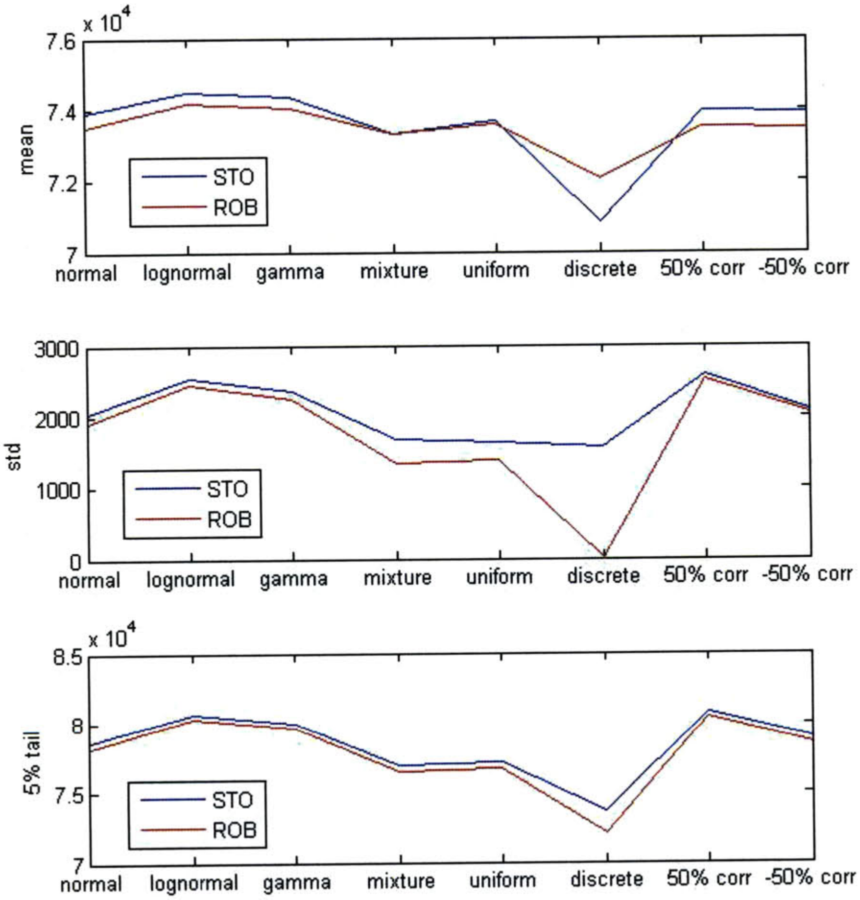

Figure 3-6: Affine STO vs. ROB for 8-Installation Assembly Network

time has a wider distribution and it is more likely to take extreme values. On the other hand, by negative correlation, the total demand is more centered around the average value by the likely coupling across high and low demands. As a result, the positive correlation gives rise to higher 5% tail and standard deviation than negative correlation, but on average they do not differ much.

3.4 Performance Comparison with Basestock Policies

In this section, we compare the performance of polynomial policies, in both robust and stochastic frameworks, with basestock policies presented in Bertsimas et al. [9]. They computed the basestock robust policy as the best basestock strategy for the worst realization of uncertainty under the same assumption (3.3)-(3.4), but they assumed normal distribution $N(\mu, \sigma)$ in the basestock stochastic policy and minimized the sample average objective of $M = 1000$ simulations. They solved both policies by simulated annealing which is well known for finding global optimal solution in practice. After finding the basestock ROB and STO policies, they also evaluated the max cost of two policies under varying size of polyhedron uncertainties, and performed demand realization simulation according to various distributions as in Section 3.1. Table 3.20 to Table 3.23 present the comparison between affine policies and basestock policies. All the results are presented as percentage of $\frac{\text{Basestock-Affine}}{\text{Basestock}}$ for both robust and stochastic policies respectively.

Table 3.20 shows that if the uncertainty set is realized as planned, both polynomial ROB and STO policies achieve lower maximum cost than their basestock counterparts. If the uncertainty set is moderately different from planned, STO polynomial policy still gives better results than basestock policy, but, in robust optimization, both policies give similar results with one policy outperforming the other depending

Network	policy	$[\mu - \sigma, \mu + \sigma]$	$[\mu - 2\sigma, \mu + 2\sigma]$	$[\mu - 3\sigma, \mu + 3\sigma]$
3-installation	STO	20.7%	6.1%	-5.3%
	ROB	11.3%	-1.7%	-12.8%
5-installation	STO	23.0%	12.0%	9.0%
	ROB	10.8%	-1.1%	-11.1%
8-installation	STO	16.9%	7.1%	5.6%
	ROB	8.2%	1.0%	-7.9%

Table 3.20: Max cost comparison of affine policies vs. basestock policies $\frac{\text{Basestock-Affine}}{\text{Basestock}}$ assuming demand lies in polyhedron of varying size.

on the network. However, when the uncertainty set turns out to differ largely from the planned, basestock ROB policy results in around 10% lower cost than polynomial ROB policy for all the networks. In stochastic optimization, basestock policy appears to perform better for small networks and the polynomial policy has the edge in larger networks.

From the comparison of results under various demand distribution on all the three networks, polynomial policies beat basestock in almost all aspects. In discrete demand distribution, polynomial policies outperform basestock policy the most, with more than 10% improvement in average cost, 50% improvement in standard deviation, and around 10% improvement in 5% tail. For unimodal continuous distributions, we still observed around 5% lower average cost, 20% lower standard deviation, and 10% lower in 5% tail. In the correlated distributions, we noted that the improvement in mean cost is the same for both positive and negative correlations, but the advantage of the polynomial policy appears to be stronger in terms of standard deviation and 5% tail in the case of positively rather than negatively correlated distribution. We observed in robust optimization of the 5-installation network that the basestock policy gives smaller standard deviation than the polynomial policy in unimodal continuous distributions and correlated distributions. Across the network size, we noticed the advantage of the polynomial policy is more significant for smaller size networks.

By comparing the affine and basestock policies, we find that affine policies out-

demand distribution	policy	mean	std	5% tail
Normal (μ, σ)	STO	4.6 %	47.8%	10.4%
	ROB	6.3 %	17.5%	7.2%
Lognormal (μ, σ)	STO	3.4 %	39.6%	8.3%
	ROB	6.1 %	12.2%	6.5%
Gamma (μ, σ)	STO	3.7 %	41.0%	8.7%
	ROB	6.1 %	11.5%	6.1%
Mixture of two normals (μ, σ)	STO	8.9 %	58.3%	14.2%
	ROB	7.5 %	22.7%	8.3%
Uniform (μ, σ)	STO	7.1 %	60.5%	13.4%
	ROB	7.1 %	22.2%	8.0%
Discrete(μ, σ)	STO	15.7 %	68.2%	20.0%
	ROB	10.4 %	100.0%	11.7%
50% correlation(μ, σ)	STO	4.6 %	53.3%	14.2%
	ROB	6.3 %	15.1%	6.9%
-50% correlation(μ, σ)	STO	4.6 %	44.7%	9.7%
	ROB	6.3 %	24.4%	8.4%

Table 3.21: Performance comparison of affine policies vs. basestock policies $\frac{\text{Basestock-Affine}}{\text{Basestock}}$ in 3-installation network.

demand distribution	policy	mean	std	5% tail
Normal (μ, σ)	STO	5.7 %	42.0%	9.8%
	ROB	7.1%	-2.0%	6.4%
Lognormal (μ, σ)	STO	4.5 %	34.5%	8.2%
	ROB	7.1 %	-2.2%	6.2%
Gamma (μ, σ)	STO	4.9 %	36.7%	8.7%
	ROB	7.1 %	-1.1%	6.5%
Mixture of two normals (μ, σ)	STO	8.4 %	45.0%	12.2%
	ROB	6.9%	2.2%	6.6%
Uniform (μ, σ)	STO	7.0%	46.7%	11.3%
	ROB	7.0 %	5.1e-3%	6.7%
Discrete(μ, σ)	STO	16.6 %	43.3%	20.0%
	ROB	9.5 %	100.0%	10.9%
50% correlation(μ, σ)	STO	2.5 %	49.0%	10.5%
	ROB	4.3 %	-3.0%	3.7%
-50% correlation(μ, σ)	STO	2.9 %	39.5%	7.4%
	ROB	4.3 %	3.3e-2%	4.0%

Table 3.22: Performance comparison of affine policies vs. basestock policies $\frac{\text{Basestock-Affine}}{\text{Basestock}}$ in 5-installation network.

demand distribution	policy	mean	std	5% tail
Normal (μ, σ)	STO	1.7 %	50.6%	7.1%
	ROB	2.6%	24.9%	4.2%
Lognormal (μ, σ)	STO	0.7 %	45.1%	6.1%
	ROB	2.5 %	20.7%	4.1%
Gamma (μ, σ)	STO	1.0 %	47.2%	6.4%
	ROB	2.5%	22.0%	4.1%
Mixture of two normals (μ, σ)	STO	5.0%	54.7%	10.0%
	ROB	3.9%	33.3%	5.5%
Uniform (μ, σ)	STO	3.6%	56.3%	8.9%
	ROB	3.4 %	32.7%	5.1%
Discrete(μ, σ)	STO	10.1 %	51.9%	14.0%
	ROB	6.6 %	100.0%	9.1%
50% correlation(μ, σ)	STO	1.7 %	58.7%	11.2%
	ROB	2.6 %	27.7%	5.1%
-50% correlation(μ, σ)	STO	1.7 %	45.5%	6.2%
	ROB	2.7 %	26.5%	4.6%

Table 3.23: Performance comparison of affine policies vs. basestock policies $\frac{\text{Basestock-Affine}}{\text{Basestock}}$ in 8-installation network.

perform basestock policies in most scenarios. The advantage of affine over basestock in the stochastic framework is more significant than in the robust framework especially for the worst case cost, standard deviation and 5% tail. However, ROB affine appears to be less robust than ROB basestock in the worst case for large polyhedral uncertainty set.

3.5 Scaling of the Framework

In this section, we are interested in finding out how large problems can be solved for polynomial policies by the presented framework. As discussed previously, the decision variables for polynomial policies of degree d are the matrices of coefficients $\mathbf{L}_t \in \mathbb{R}^{n_u \cdot s}$, $t = 0, \ldots, T-1$, where $s = \binom{t \cdot n_w + d}{d}$ is the dimension of the monomials, n_u is the number of orders in the network which is equal to the number of nodes in assembly network, and n_w is the number of sink nodes. Therefore, with a fixed degree d, the problem size is polynomially bounded by the problem input T, n_u, n_w. From the

NO. of installations	NO. of sinks	Largest time horizon
10	4	11
15	6	7
20	9	5
25	12	3
30	15	3
35	17	2
40	20	2
45	23	1

Table 3.24: Largest solvable problem size as a trade-off between the number of installations and the time horizon.

analysis in Section 3.1, we have found that the performance does not improve when moving from degree 1 to degree 2. Therefore, we will only investigate the largest problem size as a trade-off between the number of nodes and the time horizon that can be solved for affine policy (on a computation machine of 2.26GHz Intel Dual Core 7550 Processor with 2GB of RAM memory, running Windows) as shown in Table 3.24.

We can see as the network size becomes bigger, the planning time horizon that can be solved is reduced. At network of 45 installations and 23 sinks, we will only be able to solve for one planning time period, which is essentially not an open-loop policy and does not depend on the demand information. We found the bottleneck is at the SOS relaxations by using the package YALMIP [20], which has a high requirement on memory.

3.6 Distribution Network Evaluation

In this section, the three networks are treated as distribution networks. In the distribution network, it is assumed there is only one single commodity flowing in the system from a factory of unlimited capacity through distribution centers and warehouses to the stores. In this setting, each node is not required to make orders from all its suppliers since they all supply the same product. Therefore, the cost of the distribution network is expected to be lower than its assembly network counterpart.

With $x_k(t)$ denoting the inventory of node k at time t, and $u_{i,k}(t)$ denoting the order of node k placed from its supplier i at time t, we have the following dynamics:

$$x_k(t+1) = x_k(t) + \sum_{i \in U(k)} u_{i,k}(t) - \sum_{j \in D(k)} u_{k,j}(t) - w_k(t)\mathbb{1}_{[k \in S]}, \tag{3.5}$$

where $U(k)$ and $D(k)$ represent the set of upstream and downstream nodes of node k respectively, and S denotes the set of sink nodes where external demands occur. The cost incurred at node k time period t is

$$q_k(t) = c_k(t) \sum_{i \in U(k)} u_{i,k}(t) + \max\left(h_k(t)x_k(t+1), -b_k(t)x_k(t+1)\right), \tag{3.6}$$

and the system is constrained by

$$u_{i,k}(t) \geq 0, \forall i \in U(k), k, t.$$

We applied the same framework to find the robust and stochastic polynomial policies and carried out the same set of simulations to analyze the performance of polynomial policies in distribution network and compared with assembly network.

Firstly, Table 3.25 and 3.26 present the performance of degree 0 and degree 1 policies in the worst case demand realization of varying sizes of polyhedral uncertainty. Resembling the behavior in assembly network, when the uncertainty is the same as planned, ROB gives better result in worst case; however STO performs better when the uncertainty set turns out to be larger than expected.

Next, we also measured the performance of polynomial policies in distribution networks under various distribution realizations as in the assembly networks case. The results are shown in Table 3.27 to 3.32. It is clear that costs are consistently

network	policy	$[\mu - \sigma, \mu + \sigma]$	$[\mu - 2\sigma, \mu + 2\sigma]$	$[\mu - 3\sigma, \mu + 3\sigma]$
3-installation	STO	54360	**82308**	**149508**
	ROB	**44578**	111778	178978
5-installation	STO	85160	**117060**	**175640**
	ROB	**63874**	135102	206602
8-installation	STO	176529	**248029**	**323129**
	ROB	**121576**	252908	384908

Table 3.25: Max cost of degree 0 policies for distribution networks assuming demand lies in polyhedron of varying size.

network	policy	$[\mu - \sigma, \mu + \sigma]$	$[\mu - 2\sigma, \mu + 2\sigma]$	$[\mu - 3\sigma, \mu + 3\sigma]$
3-installation	STO	12875	**19515**	**29275**
	ROB	**11980**	21780	31580
5-installation	STO	27090	**37070**	**51210**
	ROB	**26260**	38400	51940
8-installation	STO	54880	**73340**	**102960**
	ROB	**49236**	79356	109476

Table 3.26: Max cost of degree 1 policies for distribution networks assuming demand lies in polyhedron of varying size.

reduced from degree 0 to degree 1 polynomial policies. However, policies of degree 2 degenerate into degree 1 and become unstable as well. Very similar observations to the assembly network are found for the distribution network: the ROB policies generally perform better than the STO counterparts, but we did notice STO policy gives lower standard deviation and 5% tail for 3-installation distribution network; lower cost in multi-modal continuous distribution than unimodal; positive correlation gives slightly higher cost than negative correlation, as in assembly networks, but the gap is smaller; and discrete distribution gives the lowest cost among all demand realization distributions and affine ROB is again leads to invariant cost to all sequences.

Lastly, we look at the effects of varying standard deviation of demand realization distribution on the distribution network as shown in Table 3.34. Once again, ROB shows strong superiority over STO when σ is smaller, but with increasing value of σ, the superiority of ROB decreases though at a slower speed compared to assembly network, and STO starts to outperform ROB in cost standard deviation at 1.75σ.

demand distribution	policy	mean	std	5% tail
normal	STO	13554	**731**	15304
	ROB	**13323**	911	**13662**
lognormal	STO	13772	**862**	**15519**
	ROB	**13493**	954	15734
gamma	STO	13704	**846**	15733
	ROB	**13359**	920	**15533**
mixture of two normals	STO	13305	**598**	**14373**
	ROB	**13218**	758	14884
uniform	STO	13345	**667**	**14525**
	ROB	**13307**	856	15375
discrete	STO	**12747**	867	**14275**
	ROB	12906	**680**	14366
50% correlation	STO	14086	**1416**	**17885**
	ROB	**14011**	1558	17986
-50% correlation	STO	13182	**557**	14368
	ROB	**12876**	642	**14311**

Table 3.27: Performance of degree 0 policies for 3-installation distribution network.

demand distribution	policy	mean	std	5% tail
normal	STO	12221	**624**	**13790**
	ROB	**11969**	788	13916
lognormal	STO	12470	**807**	**14615**
	ROB	**12183**	983	14726
gamma	STO	12385	**727**	**14264**
	ROB	**12127**	915	14446
mixture of two normals	STO	12037	**447**	**13040**
	ROB	**12010**	560	13300
uniform	STO	12158	**417**	**13057**
	ROB	**12077**	585	13372
discrete	STO	**11277**	423	12130
	ROB	11980	**1.16e-5**	**11980**
50% correlation	STO	12228	**754**	**14317**
	ROB	**11964**	1010	14833
-50% correlation	STO	12225	**715**	**14094**
	ROB	**11964**	899	14245

Table 3.28: Performance of degree 1 policies for 3-installation distribution network.

demand distribution	policy	mean	std	5% tail
normal	STO	30903	1446	33627
	ROB	**30282**	**1234**	**33062**
lognormal	STO	31477	1865	36005
	ROB	**30644**	**1499**	**34217**
gamma	STO	31288	1616	35024
	ROB	**30646**	**1530**	**34760**
mixture of two normals	STO	31135	1338	33157
	ROB	**30269**	**1067**	**32597**
uniform	STO	31015	1330	33525
	ROB	**30293**	**1054**	**32292**
discrete	STO	29561	1656	32437
	ROB	**29145**	**1250**	**31494**
50% correlation	STO	31566	2242	37238
	ROB	**30842**	**1995**	**36202**
-50% correlation	STO	307768	1277	33921
	ROB	**29629**	**1052**	**32309**

Table 3.29: Performance of degree 0 policies for 5-installation distribution network.

demand distribution	policy	mean	std	5% tail
normal	STO	28280	1193	31081
	ROB	**27828**	**1109**	**30583**
lognormal	STO	28588	1461	32230
	ROB	**28207**	**1411**	**31801**
gamma	STO	28479	1366	31785
	ROB	**28116**	**1315**	**31430**
mixture of two normals	STO	27970	1017	30171
	ROB	**27749**	**780**	**29603**
uniform	STO	28165	980	30276
	ROB	**27893**	**799**	**29712**
discrete	STO	**26707**	949	28360
	ROB	27260	**8.66e-6**	**27260**
50% correlation	STO	28254	**1444**	32135
	ROB	**27826**	1445	**31888**
-50% correlation	STO	28283	1252	31301
	ROB	**27831**	**1205**	**30865**

Table 3.30: Performance of degree 1 policies for 5-installation distribution network.

demand distribution	policy	mean	std	5% tail
normal	STO	58560	**2196**	62081
	ROB	**56157**	2340	**61037**
lognormal	STO	59308	**2773**	65198
	ROB	**57412**	2800	**63358**
gamma	STO	59521	**2623**	64350
	ROB	**57603**	2777	**62661**
mixture of two normals	STO	58609	2588	63716
	ROB	**56265**	**2468**	**61231**
uniform	STO	58891	2652	63494
	ROB	**56853**	**2413**	**61161**
discrete	STO	56008	3242	62410
	ROB	**54437**	**2527**	**59143**
50% correlation	STO	59800	**3396**	67054
	ROB	**57455**	3634	**65831**
-50% correlation	STO	58085	**1797**	62408
	ROB	**55096**	1928	**58787**

Table 3.31: Performance of degree 0 policies for 8-installation distribution network.

demand distribution	policy	mean	std	5% tail
normal	STO	52328	**1733**	56256
	ROB	**50313**	1972	**54918**
lognormal	STO	53036	**2044**	57960
	ROB	**50897**	2380	**56714**
gamma	STO	52827	**1923**	57354
	ROB	**50714**	2237	**56073**
mixture of two normals	STO	51718	1512	54889
	ROB	**50763**	**1443**	**53931**
uniform	STO	52101	**1414**	55048
	ROB	**50791**	1528	**54119**
discrete	STO	**49250**	1696	52686
	ROB	51086	**7.69e-5**	**51086**
50% correlation	STO	52339	**2065**	57333
	ROB	**50288**	2459	**56678**
-50% correlation	STO	52323	**1932**	57009
	ROB	**50262**	2273	**55690**

Table 3.32: Performance of degree 1 policies for 8-installation distribution network.

demand distribution	policy	mean	std	5% tail
normal	STO	642489	528785	692489
	ROB	12017	787	14003
lognormal	STO	692489	691051	2033865
	ROB	12214	970	14777
gamma	STO	663793	594856	2650087
	ROB	12168	920	14573
mixture of two normals	STO	404656	292484	1321264
	ROB	12038	566	13381
uniform	STO	421083	271862	1249664
	ROB	12110	592	13453
discrete	STO	11280	423	12140
	ROB	11980	8.58e-5	11980
50% correlation	STO	12228	754	14317
	ROB	11964	1010	14833
-50% correlation	STO	12225	715	14094
	ROB	11964	899	14245

Table 3.33: Performance of degree 2 policies for 3-installation distribution network.

demand distribution	mean	std	5% tail
gamma($\mu, 0.5\sigma$)	3.41%	40.08%	4.37%
gamma($\mu, 0.75\sigma$)	2.31%	14.08%	2.54%
gamma(μ, σ)	1.27%	3.74%	1.12%
gamma($\mu, 1.25\sigma$)	0.79%	1.58%	0.64%
gamma($\mu, 1.5\sigma$)	0.45%	2.45%	0.69%
gamma($\mu, 1.75\sigma$)	0.04%	-1.47%	-0.38%
gamma($\mu, 2\sigma$)	-0.39%	-1.42%	-0.75%

Table 3.34: Relative performance of affine ROB vs. STO as a function of realized $\tilde{\sigma}$ for 5-installation distribution network.

From these experiments, we find that overall polynomial policies perform consistently to both types of supply chain networks. This is supported in cases involving worst case of varying polyhedral uncertainty, various demand realization distributions, as well as varying standard deviation of demand distribution.

Chapter 4

Conclusions

We solved the inventory control problem of multi-echelon supply chain networks with uncertain demands. The objective is to minimize the cost incurred to the entire system over a fixed planning time horizon, comprising variable ordering cost, inventory holding cost and backlogging cost. We implemented polynomial policies parametrized directly in the sequence of the observed uncertainties, and then carried out SOS relaxations to reformulate the problem as an SDP, applicable to both robust and stochastic optimization frameworks.

By extensive numerical studies on three supply chain assembly networks of varying complexity, we find that affine policies is the best and gives lower cost than constant, quadratic and basestock policies. Comparing the two frameworks, the robust affine policies is overall better if the demand sequences come from the one we planned though the performance of robust affine policies is also satisfactory. However, in larger uncertainty set than planned, affine robust policies is less robust in the worst case compared to affine stochastic and robust basestock policies. Also, with the increasing of standard deviation in demand distribution, stochastic affine policies catch up robust affine policies. Similar findings are also obtained when we apply the same policies and frameworks to distribution networks.

We did not include fixed ordering cost in the cost function when we formulated the optimization problems as it does not fit into the current convex frameworks. An insightful direction is to find ways of solving polynomial policies in mixed integer problem when fixed ordering costs are incorporated. The constraints in our network are relatively simple: only the order quantity has to be nonnegative. Therefore an interesting future direction will be looking at networks with more general constraints on the states and controls. In those cases, the affine policies may not be optimal. Therefore we do need more powerful and stable software packages for interior point methods and dedicated algorithms for SOS problems, since the problem size grows very quickly at higher degrees.

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