Rigorous numerics for analytic solutions of differential equations: the radii polynomial approach

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Abstract

Judicious use of interval arithmetic, combined with careful pen and paper estimates, leads to effective strategies for computer assisted analysis of nonlinear operator equations. The method of radii polynomials is an efficient tool for bounding the smallest and largest neighborhoods on which a Newton-like operator associated with a nonlinear equation is a contraction mapping. The method has been used to study solutions of ordinary, partial, and delay differential equations such as equilibria, periodic orbits, solutions of initial value problems, heteroclinic and homoclinic connecting orbits in the $C^{\mathbf{k}}$ category of functions. In the present work we adapt the method of radii polynomials to the analytic category. For ease of exposition we focus on studying periodic solutions in Cartesian products of infinite sequence spaces. We derive the radii polynomials for some specific application problems, and give a number of computer assisted proofs in the analytic framework.

1 Introduction

Spatiotemporal patterns in applied mathematical problems are often described by special solutions of evolution equations. These special solutions may represent coherent structures a diverse as traveling waves and pulses, spots, fronts, breathers, snakes, isolas, modulated wave trains, spiral wave defects, and shocks to name only a few. Special solutions of evolution equations also describe the classical building block solutions of dynamical systems theory such as equilibria, periodic orbits, heteroclinic and homoclinic connecting orbits of ordinary, delay, and partial differential equations. By appending appropriate phase or symmetry conditions to the evolution equation it is possible to see these special solutions as isolated zeros of nonlinear operator equations on a Banach space. This philosophy connects the study of patterns and structure in applied mathematics to the tools of nonlinear functional analysis. For a much more nuanced discussion of this point we refer to the review article [1].

In practice the obstruction to this program is the fact that the nonlinear functional equation is still difficult to solve. For a given problem it may be impossible, outside the perturbative regime, to obtain useful information about a solution by hand. Numerical methods illuminate the structure of the problem by providing accurate approximate solutions. Since the seminal work of Lanford [2] on the Feigenbaum conjectures in the early

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1980s a great deal of work has gone into developing methods for mathematically rigorous computer assisted analysis of solutions of nonlinear equations. A thorough review of the literature on computer assisted proof lies far outside the scope of this paper but the interested reader might consult [3, 4, 5, 6, 7, 8].

Newton's method has long been known as a powerful tool for nonlinear analysis, and many computer assisted proof strategies are based on studying Newton-like operators. Newton-like operators are discussed formally in Section 2, but the main idea is this: given an approximate solution of the equation and a choice of approximate inverse for the differential, one defines a new map whose fixed points correspond to solutions of the original equation. A constructive computer assisted proof of the existence of a solution of the original equation is obtained as soon as one shows that the Newton-like operator is a contraction mapping on some neighborhood of the approximate solution. This requires a mixture of analytic bounds as well as deliberate management of round-off error. The Banach space in which one decides to work determines the regularity properties of the validated solution, and also influences the estimates which appear in the proof.

Given a particular approximate solution, a particular choice of approximate inverse, and a choice of the Banach space on which to formulate the problem, the method of radii polynomials (first introduced in [9]) is an efficient strategy for obtaining bounds on the smallest and largest neighborhoods of the approximate solution on which the corresponding Newton-like operator is a contraction mapping (see Proposition 2). The size of the smallest of these neighborhoods provides tight bounds on the location of the true solution of the problem. The size of the largest of these neighborhoods provides information about the isolation of the true solution of the problem. The continuity of the radii polynomials can be exploited in order to smoothly connect the results of one computer assisted proof to another, and can also be exploited in the implementation of bisection-type algorithms for optimizing computer assisted proofs.

The method of radii polynomials has been employed in mathematically rigorous computer assisted study of a wide variety of problems in differential equations and dynamical systems. For example equilibria and periodic orbits of ordinary, delay, partial differential equations (PDEs) as well as systems of PDEs are considered in [10, 11, 12, 13, 14, 15, 16, 17]. The same approach is applied in order to validate transverse connecting orbits for ordinary differential equations in [18, 19, 20], to study symmetric pulses, kinks and radially symmetric solutions in reaction diffusion equations [21, 22], to solve initial and boundary value problems for ordinary differential equations in a mathematically rigorous way [19, 23], and to validate series expansions for the Floquet normal form for linear differential equations with periodic coefficients. This leads to methods for validated computations of the linear stable and unstable bundles of periodic orbits in differential equations [24]. Exploiting the isolation bounds as well as continuity of the radii polynomials facilitates the study of problems which depend on parameters via rigorous one- and multi-parameter continuation [25, 26, 27]. We note that the works just mentioned develop the theory of radii polynomials in the context of a $C^{\mathbf{k}}$ function space setup.

In the present work we develop a radii polynomial approach for studying Newton-like operators on spaces of analytic functions. For the sake of simplicity we focus on spaces corresponding to analytic functions which are periodic on some complex strip (more precisely they are analytic on an open strip containing the real axis, and continuous on the closure of the strip). These spaces are isomorphic (through the $\mathbb{S}^1$ Fourier transform) to certain classical sequence spaces. Namely, the spaces utilized in the present work are products of

$$\ell_\nu^1 \stackrel{\text{def}}{=} \left\{ c = \{c_k\}_{k \in \mathbb{Z}} \ : \ \|c\|_\nu < \infty \right\}, \tag{1.1}$$

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equipped with a "weighted ell one norm"

$$\|c\|_\nu \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} |c_k| \nu^{|k|}, \tag{1.2}$$

for some fixed weight $\nu \geq 1$. The little ell moniker distinguishes these sequence spaces from the classical Lebesgue function spaces endowed with integral norms.

In our context, $\|\cdot\|_\nu$ is a weighted sum of the absolute values of Fourier coefficients (see Section 2). The space ℓ^1_ν is a Banach algebra under the discrete convolution product [28, 29]. This fact facilitates nonlinear analysis. We can also exploit the fact that ℓ^1_ν has a well understood dual space in order to study linear functionals and operators which arise naturally in the course of analyzing the Newton-like operators.

It is important to say that the use of analytic function spaces in computer assisted analysis is far from new. Indeed the work of [2] on the Feigenbaum conjectures was formulated using the same sequence spaces used here, and since then many authors have employed this functional analytic framework. The novelty of the present work is the adaptation of the radii polynomial approach to the analytic setting.

Prior to now the radii polynomial approach has been developed in the context of $C^{\mathbf{k}}$ function spaces with $0 \leq \mathbf{k} < \infty$. The case $\mathbf{k} > 0$ is considered in [10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 23, 27] using the space of sequences

$$\Omega^s \stackrel{\text{def}}{=} \left\{ c = \{c_k\}_{k \in \mathbb{Z}} \; : \; \|c\|_s \stackrel{\text{def}}{=} \sup_{k \in \mathbb{Z}} |c_k| |k|^s < \infty \right\},$$

for $s > 1$. The case $\mathbf{k} = 0$ is considered in [18, 21] and the proofs were performed using C^0 splines with the supremum norm.

Before proceeding further, let us mention that the rigorous verification methods based on the $C^{\mathbf{k}}$ and analytic categories are similar in spirit. The main difference is that every estimate has to be done in a different norm. While in this work we do not provide extensive practical comparisons, we discuss general advantages and disadvantages.

The $C^{\mathbf{k}}$ category Ω^s has the major advantage of being applicable to a broader class of differential equations, and the norm $\|\cdot\|_s$ is stable numerically. However, Ω^s is not naturally a Banach algebra under discrete convolutions, and special convolution estimates have to be developed in order to study the nonlinearities. Its dual space is difficult to exploit and estimating linear functionals requires special efforts. Moreover, the optimal choice of the algebraic decay rate $s > 1$ is not known, as sometimes $s \in (1, 2)$ is preferable to $s \geq 2$ while sometimes it is the other way around (e.g. see [15]).

The analytic category ℓ^1_ν has a natural Banach algebra structure under discrete convolutions which provides sharp estimates needed when studying nonlinearities. Moreover, since the dual of ℓ^1_ν is well known (it is a weighted ℓ^∞ space discussed below), this facilitates the analysis of obtaining bounds involving linear functionals over ℓ^1_ν. However, one the greatest disadvantage of using ℓ^1_ν is that its norm $\|\cdot\|_\nu$ can be numerically unstable. Indeed, for large k, the coefficients $|c_k|$ may be very small and the factor $\nu^{|k|}$ may be very large, and therefore the multiplication $|c_k| \nu^{|k|}$ may be difficult to compute accurately. On the other hand, the space admits an a-priori choice for ν that is the most numerically stable: namely $\nu = 1$. While ℓ^1_1 does not provide analyticity of the solutions, if a computer assisted proof is performed at $\nu = 1$ then the continuity (in ν) of the radii polynomials (see Proposition 3) gives that there exists a $\nu > 1$ on which the proof goes through. It follows that there is a complex strip on which the solution is analytic. If explicit bounds on the size of the strip are desired (i.e. explicit bounds on the exponential decay rate of the Fourier coefficients of the solution) then the proof can be repeated with $\nu > 1$. In

fact the continuity of the radii polynomials can be exploited via a bisection algorithm in order to maximize the $\nu > 1$ on which the proof works.

Let us emphasize that this work is by no means a criticism of the above mentioned rigorous methods developed in the $C^{\mathbf{k}}$ category. However, it is important to realize that we cannot get analytic information of the solutions in this category. In fact a motivation for the present work is the development of validated numerics for studying analytic parameterizations of stable/unstable invariant manifolds of periodic orbits of differential equations. The inputs for such a method will have to be analytic representations of the periodic orbit as well as its stable and unstable bundle. This could be done by combining the methods of the present work with the validated methods for computation of the Floquet normal form developed in [24], and will lead to methods for studying heteroclinic and homoclinic orbits connecting to periodic orbits of differential equations in the analytic category. This informs the choice of example problems discussed in this work.

We have chosen to focus on periodic problems in the present work in order to simplify the presentation. However the approach taken here extends naturally to other spectral bases. For example there is much recent interest in using Chebyshev series in conjunction with the radii polynomial approach in order to develop computer assisted proofs for boundary and initial value problems [19, 30]. Moreover, the examples studied here illustrate that the general $C^{\mathbf{k}}$ radii polynomial approach of [10, 12, 13, 15, 16, 19, 31] used to validate periodic solutions of delay equations, periodic solutions of Hamiltonian systems, equilibria of systems of PDEs, periodic orbits of PDEs and equilibria of PDEs defined on domains of dimension greater than one can be extended to the analytic category.

We have also chosen to restrict our discussion to quadratic and cubic nonlinearities, again in order to simplify the exposition. There is no loss of generality in this restriction as long as one is interested in problems with nonlinearities built from "the elementary functions of mathematical physics" (powers, exponential, trig, rational, Bessel, elliptic integrals, etc.) This is because such nonlinearities are themselves solutions of first or second order linear differential equations. These differential equations can then be appended to the original problem of interest in order to obtain a strictly polynomial nonlinearity, albeit in a higher number of variables. This is a standard trick which we learned from [32], and which is sometimes employed in software packages which manipulate formal series expansions. See for example the discussion in [33]. Therefore the ideas as presented here apply with only small modification to many problems of interest.

Finally, we do not intend to give the reader the impression that the functional analytic approach to computer assisted proof provides the only successful methods for studying nonlinear equations. In fact nothing could be further from the truth. Methods based on topological analysis appear in the literature as early as [34] and the use of topological methods for computer assisted proof remains a rapidly growing field. The interested reader might consult [35] for more discussion of this exciting area. We also refer to the review article of [2] for a broad overview of the field of validated numerical methods. It is our view that topological and analytical methods provide complementary tools for computer assisted study of problems in nonlinear analysis.

We conclude this introduction by summarizing the example applications discussed more fully in the remainder of the paper.

Application 1. We use the methods of the present work in order to study periodic orbits in the Lorenz equations

$$u_1' = \sigma(u_2 - u_1)$$
$$u_2' = \rho u_1 - u_2 - u_1 u_3 \tag{1.3}$$
$$u_3' = u_1 u_2 - \beta u_3.$$

For example at $\sigma = 10$, $\beta = 8/3$ and $\rho = 13.92657$ we computed an approximate periodic solution with period roughly 5.8162. This orbit is approximated using $m = 320$ Fourier modes. We prove that the ℓ_ν^1 error between the approximate solution and the true solution is no greater than $r = 2.3267 \times 10^{-7}$ with $\nu = 1.027$. Then the domain of analyticity is a strip in the complex plane about the real axis whose width is not less than 0.024662. The ℓ_ν^1 norm bounds the C^0 norm so that r also provides a bound on the error in phase space between the true periodic orbit and the image of the approximate Fourier series. We refer to Figure 1 to see the profile of the solution and to Section 4 for the details of the approach and for more examples.
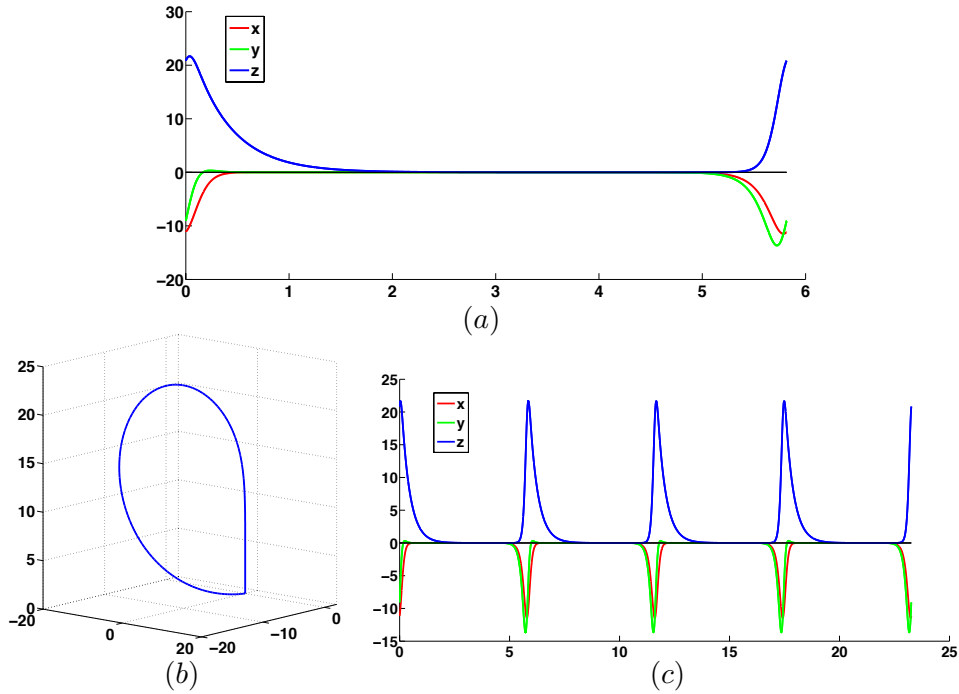

Figure 1: Periodic orbit in the Lorenz equations at $\rho = 13.92657$ with period $T \approx 5.8162$. The validated error bound for the orbit is much smaller than the width of the lines in the figure.

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Application 2. The methods of the present work can also be used to study solutions of PDEs. The Swift-Hohenberg PDE with even periodic boundary conditions is

$$u_t = (\lambda - 1)u - 2u_{yy} - u_{yyyy} - u^3 , \quad \text{in } \Omega = [0, \frac{2\pi}{L}] \tag{1.4}$$
$$u(y,t) = u(y + 2\pi/L, t) , \ u(y,t) = u(-y,t) , \quad \text{on } \partial\Omega.$$

This model was originally introduced to describe the onset of Rayleigh-Bénard heat convection [36], where L is a fundamental wave number for the system size $\frac{2\pi}{L}$. The parameter λ corresponds to the Rayleigh number and its increase is associated with the appearance of multiple solutions that exhibit complicated patterns. For the computations presented here we fixed $L = 0.65$. At $\lambda = (1 - 4L^2)^2$, there is a pitchfork bifurcation from $u \equiv 0$. The bifurcating solution corresponds to the solution $\cos(2Ly)$. Using a numerical continuation method based on a predictor corrector algorithm we continued to a solution at $\lambda = 3.5 \times 10^8$, and proved that near the numerical approximation there exists an exact solution. The proof used a Fourier approximation to $m = 2103$ modes. The ℓ^1 error between the approximate solution and the exact solution is smaller than $r = 2.6536 \times 10^{-4}$. The proof is discussed in Section 5 and uses the notion of radii polynomials as introduced in Section 3.
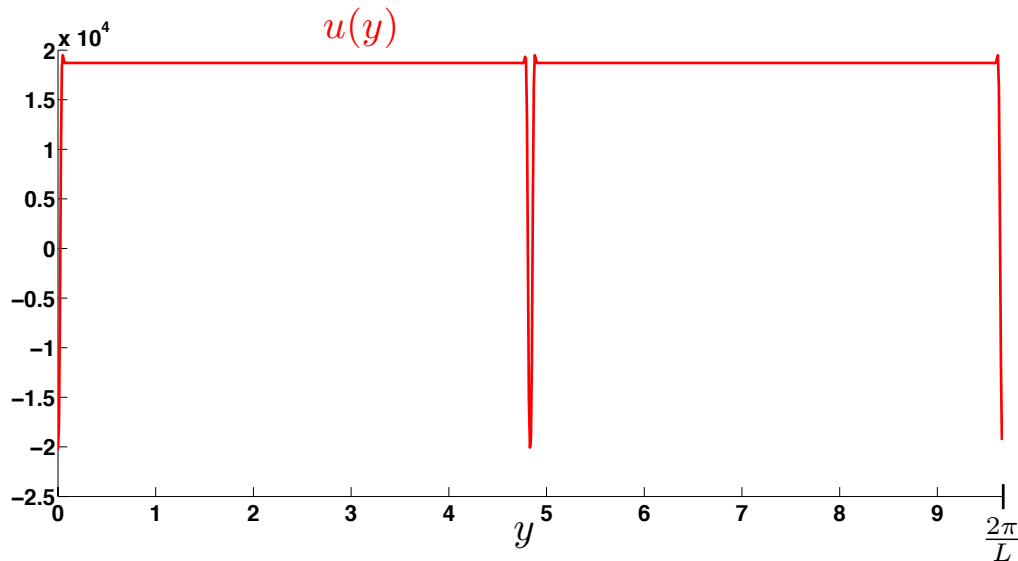

Figure 2: Equilibrium solution of (1.4) at $\lambda = 3.5 \times 10^8$. The computation is rigorously validated. The resulting error is smaller than the width of the curve. So we can say for example that the true solution exhibits the small spiking behavior just before and just after the large spike, as shown in the figure. In this case, this phenomenon is not due to the numerical error associated with the "Gibbs effect".

The remainder of the paper is organized as follows. In Section 2, we introduce the functional analytic background necessary to perform the computer-assisted proofs in the analytic category. In Section 3, we present the new adaptation of the radii polynomial approach to the analytic category setting. In Section 4, we apply the method to prove existence of periodic solutions in the Lorenz equations and finally in Section 5, we apply the method to prove existence of equilibria of the Swift-Hohenberg PDE.

2   Background

2.1   Sequence Spaces

In (1.1) and (1.2) we defined the ν-weighted ell-one space of infinite sequences. We note that ℓ_ν^1 is a Banach space and moreover has the property of being a Banach algebra under discrete convolution defined as

$$a * b = \left\{ \sum_{k_1+k_2=k} a_{k_1} b_{k_2} \right\}_{k \in \mathbb{Z}}, \qquad a, b \in \ell_\nu^1.$$

More explicitly, if $\nu \geq 1$ and $a, b \in \ell_\nu^1$, then $a * b \in \ell_\nu^1$ and $\|a * b\|_\nu \leq \|a\|_\nu \|b\|_\nu$.

Note that with $\nu = 1$ the space ℓ_1^1 is the classical Wiener algebra. We also recall the classical fact that the dual space of ℓ_1^1, which is denoted $(\ell_1^1)^*$, is the space ℓ^∞. Similarly if $\nu > 1$ then the dual of ℓ_ν^1 is a weighted "ell-infinity" space which we define now. For a bi-infinite sequence of complex numbers $c = \{c_k\}_{k \in \mathbb{Z}}$, the ν-weighted supremum norm is defined by

$$\|c\|_\nu^\infty \overset{\text{def}}{=} \sup_{k \in \mathbb{Z}} \frac{|c_k|}{\nu^{|k|}}. \tag{2.1}$$

Let

$$\ell_\nu^\infty = \left\{ c = \{c_k\}_{k \in \mathbb{Z}} \mid c_k \in \mathbb{C} \text{ for all } k \in \mathbb{Z}, \text{ and } \|c\|_\nu^\infty < \infty \right\}. \tag{2.2}$$

The key to the proof that $\ell_\nu^\infty = (\ell_\nu^1)^*$ is the following bound which is itself useful in the sequel.

Lemma 1. *Suppose that $a \in \ell_\nu^1$ and $c \in \ell_\nu^\infty$. Then*

$$\left| \sum_{k \in \mathbb{Z}} c_k a_k \right| \leq \sum_{k \in \mathbb{Z}} |c_k| |a_k| \leq \|c\|_\nu^\infty \|a\|_\nu.$$

The following result states that ℓ_ν^∞ is the dual of ℓ_ν^1, in the sense of isometric isomorphism. It follows that any linear functional on ℓ_ν^1 can be represented as an element of ℓ_ν^∞, and that the operator norm can be computed by taking the weighted "ell-infinity" norm of the corresponding sequence.

Theorem 1. *For any $\nu \geq 1$ we have that $(\ell_\nu^1)^* \cong \ell_\nu^\infty$.*

A related result, which is not usually stated but which is useful in the work to follow, is the following isometric isomorphism theorem for linear maps from $\mathbb{C}$ into ℓ_ν^1.

Lemma 2. *The set $B(\mathbb{C}, \ell_\nu^1)$ of bounded linear maps from $\mathbb{C}$ into ℓ_ν^1 is isometrically isomorphic to ℓ_ν^1. Specifically $l \in B(\mathbb{C}, \ell_\nu^1)$ if and only if there exists $a \in \ell_\nu^1$ so that $l(z) = za$, for all $z \in \mathbb{C}$. Moreover $\|l\|_{B(\mathbb{C}, \ell_\nu^1)} = \|a\|_\nu$.*

The following result is a consequence of Lemma 1, and provides a useful and explicit bound on the norm of an "eventually diagonal" linear operator on ℓ_ν^1. The proof is a direct computation.

Denote by $B(\ell_\nu^1, \ell_\nu^1)$ the set of bounded linear operators from ℓ_ν^1 to ℓ_ν^1, and given $A \in B(\ell_\nu^1, \ell_\nu^1)$, denote its operator norm by $\|A\|_{B(\ell_\nu^1, \ell_\nu^1)}$.

Corollary 1. *Let $A^{(m)}$ be an $(2m-1) \times (2m-1)$ matrix with complex valued entries, $\{\delta_k\}_{|k| \geq m}$ a bi-infinite sequence of complex numbers and $\delta > 0$ a real number such that*

$$|\delta_k| \leq \delta, \quad \text{for all } |k| \geq m.$$

Given $a = (a_k)_{k \in \mathbb{Z}} \in \ell_\nu^1$, denote by $a^{(m)} = (a_{-m+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{m-1}) \in \mathbb{C}^{2m-1}$. Define the map $A : \ell_\nu^1 \to \ell_\nu^1$ by

$$[A(a)]_k = \begin{cases} [A^{(m)}a^{(m)}]_k, & \text{if } |k| < m \\ \delta_k a_k, & \text{if } |k| \geq m. \end{cases}$$

Then $A \in B(\ell_\nu^1, \ell_\nu^1)$ and

$$\|A\|_{B(\ell_\nu^1, \ell_\nu^1)} \leq \max(K, \delta),$$

where

$$K \overset{\text{def}}{=} \max_{|n| < m} \frac{1}{\nu^{|n|}} \sum_{|k| < m} |A_{k,n}| \nu^{|k|}. \tag{2.3}$$

The following elementary results, whose proofs are standard, are going to be useful in the computation of the bounds required to construct the radii polynomials of Definition 1.

Lemma 3. *Let $\nu \geq 1$ and let $a \in \ell_\nu^1$. The function $l_a^k : \ell_\nu^1 \to \mathbb{C}$ defined by*

$$l_a^k(c) = \sum_{k_1 + k_2 = k} a_{k_1} c_{k_2}$$

with $c \in \ell_\nu^1$, is a bounded linear functional.

Corollary 2. *Let $a \in \ell_\nu^1$ and $k \in \mathbb{Z}$. Then*

$$\left\| l_a^k \right\|_\nu^\infty \leq \sup_{i \in \mathbb{Z}} \frac{|a_{k-i}|}{\nu^{|i|}}.$$

Fix a truncation mode to be m. Given $a \in \ell_\nu^1$, we also use the notation $a^{(m)}$ and set $a^{(m)} \overset{\text{def}}{=} (\ldots, 0, 0, a_{-m+1}, \ldots, a_{m-1}, 0, 0, \ldots) \in \ell_\nu^1$ and $a^I \overset{\text{def}}{=} a - a^{(m)} \in \ell_\nu^1$. For $a, b \in \ell_\nu^1$ the truncation to m modes of the k-th convolution coefficient is

$$(a * b)_k^{(m)} \overset{\text{def}}{=} \sum_{\substack{k_1 + k_2 = k \\ |k_1|, |k_2| < m}} a_{k_1} b_{k_2},$$

and the tail of the k-th convolution coefficient is

$$(a * b)_k^I \overset{\text{def}}{=} \sum_{\substack{k_1 + k_2 = k \\ |k_1| \geq m \text{ or } |k_2| \geq m}} a_{k_1} b_{k_2}.$$

We define the operators

$$l_{a,m}^k(c) = (a * c)_k^{(m)},$$

and

$$l_{a,I}^k(c) = (a * c)_k^I.$$

Note that $l_{a,m}^k$, $l_{a,I}^k$ are bounded linear functionals on ℓ_ν^1. The following technical bound will play a small but important role in the truncation error analysis of nonlinearities in Sections 4 and 5.

Corollary 3. *Let $a \in \ell_\nu^1$ be a sequence truncated at the m-th mode, that is suppose that $a = a^{(m)}$. Suppose that $|k| < m$ and define $\hat{l}_a^k \in (\ell_\nu^1)^*$ by*

$$\hat{l}_a^k(c) = (a * c)_k^I = \sum_{\substack{k_1 + k_2 = k \\ |k_2| \geq m}} a_{k_1} c_{k_2}.$$

Then

$$\left\| \hat{l}_a^k(c) \right\|_\nu^\infty \leq \Psi_k(a) \overset{\text{def}}{=} \begin{cases} \displaystyle\max_{k \leq j \leq -1} \frac{|a_{k-j+m-1}|}{\nu^{m-1+|j|}}, & \text{if } k < 0 \\ 0, & \text{if } k = 0 \\ \displaystyle\max_{1 \leq j \leq k} \frac{|a_{k-j-m-1}|}{\nu^{m-1+j}}, & \text{if } k > 0 \end{cases} \tag{2.4}$$

The proof follows from the bound in Corollary 2 by considering the terms which remain when a is a finite sequence and $|k| < m$. We will use the notation $\hat{l}_a^k$ only when it is understood that $a = a^{(m)}$.

In applications we are often interested in differential equations subject to some number of scalar constraint equations. When studying such problems the product space

$$X_\nu^{j_1, j_2} = \mathbb{R}^{j_1} \times \left(\ell_\nu^1\right)^{j_2},$$

is needed. Here j_1 corresponds to the number of scalar constraint equations and j_2 corresponds to the number of unknown scalar functions. When ν, j_1, and j_2 are understood from context we simplify the notation and write $X = X_\nu^{j_1, j_2}$. We denote by $x = (x_1, \ldots, x_{j_1}, a_1, \ldots, a_{j_2})$ an element of X and endow the space with the norm

$$\|x\|_X = \max\left(|x_1|, \ldots, |x_{j_1}|, \|a_1\|_\nu, \ldots, \|a_{j_2}\|_\nu\right). \tag{2.5}$$

Again when there is no cause for confusion we sometimes simply write $\|x\|_X = \|x\|$.

3   The radii polynomial approach on X

Now we are interested in developing the radii polynomial approach to solve nonlinear equations of the form

$$F(x) = 0, \tag{3.1}$$

with $x = (x_1, \ldots, x_{j_1}, a_1, \ldots, a_{j_2}) \in X = \mathbb{R}^{j_1} \times \left(\ell_\nu^1\right)^{j_2}$ and F a nonlinear map on X. At the moment, we stay slightly informal and do not specify the range of F. In fact, the beginning paragraphs of the present section are meant as preparation for the main setting, which is introduced formally from (3.5) and onwards.

This approach requires first the computation of a numerical approximation that is obtained by computing on a finite dimensional projection. As before, given $c = (c_k)_{k \in \mathbb{Z}} \in \ell_\nu^1$ denote by $c^{(m)} = (c_k)_{|k| < m} \in \mathbb{C}^{2m-1}$ a finite part of c of size $2m - 1$. Consider a finite dimensional projection $F^{(m)}$ of (3.1) given by

$$F^{(m)}(x_1, \ldots, x_{j_1}, a_1^{(m)}, \ldots, a_{j_2}^{(m)}) = \begin{pmatrix} F_1(x_1, \ldots, x_{j_1}, a_1^{(m)}, \ldots, a_{j_2}^{(m)}) \\ \vdots \\ F_{j_1}(x_1, \ldots, x_{j_1}, a_1^{(m)}, \ldots, a_{j_2}^{(m)}) \\ F_{j_1+1}^{(m)}(x_1, \ldots, x_{j_1}, a_1^{(m)}, \ldots, a_{j_2}^{(m)}) \\ \vdots \\ F_{j_1+j_2}^{(m)}(x_1, \ldots, x_{j_1}, a_1^{(m)}, \ldots, a_{j_2}^{(m)}) \end{pmatrix}, \tag{3.2}$$

where $F_j^{(m)}(x_1, \ldots, x_{j_1}, a_1^{(m)}, \ldots, a_{j_2}^{(m)}) \in \mathbb{C}^{2m-1}$ $(j = j_1 + 1, \ldots, j_1 + j_2)$ corresponds to the finite part of F_j of size $2m-1$. We have that $F^{(m)} : \mathbb{R}^{j_1} \times \mathbb{C}^{j_2(2m-1)} \to \mathbb{R}^{j_1} \times \mathbb{C}^{j_2(2m-1)}$, and we seek a numerical solution of the finite dimensional problem $F^{(m)} = 0$ using Newton's method. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{j_1}, \bar{a}_1, \ldots, \bar{a}_{j_2}) \in \mathbb{R}^{j_1} \times \mathbb{C}^{j_2(2m-1)}$ be the approximate solution of $F^{(m)}$ so obtained, with each $\bar{a}_i \in \mathbb{C}^{2m-1}$.

We would now like to employ some kind of Newton-Kantorovich argument in order to establish the existence of a true solution of F near $\bar{x}$. However it is not the case in general that F maps X into itself. This is because a differential operator is in general unbounded on ℓ_ν^1. In order to overcome this problem we look for an injective linear smoothing operator A such that

$$AF : X \to X, \tag{3.3}$$

and also that

$$\|I - A \cdot DF(\bar{x})\|_X \ll 1. \tag{3.4}$$

Equation (3.3) says that A is a smoothing operator, which sends $F(x)$ back into the space ℓ_ν^1. Equation (3.4) says that A is a left approximate inverse for $DF(\bar{x})$. Note that the approximate inverse condition need only hold for the Frechet derivative at $\bar{x}$, while the smoothing condition must apply in a neighborhood of the approximate solution.

The choice of the approximate inverse A is an application dependent problem; we discuss it in the context of specific applications in Sections 4 and 5. For now we take A as given and define the *Newton-like operator* $T : X \to X$ by

$$T(x) = x - AF(x), \tag{3.5}$$

for x in some neighborhood of $\bar{x}$.

More formally, the framework in which we prove existence of solutions is under the assumptions that

- A is an injective linear operator such that $AF : X \to X$;

- $T \in C^1(X)$.

The injectivity of A implies that x is a solution of $F(x) = 0$ if and only if it is a fixed point of T. Moreover since T now maps X back into itself we study (3.5) via the contraction mapping theorem applied on closed balls centered at the numerical approximation $\bar{x}$.

Recall the definition of the norm on X in (2.5), denote by $B(r) = \{x : \|x\|_X \le r\} \subset X$ the closed ball of radius r in X and denote

$$B_{\bar{x}}(r) \stackrel{\text{def}}{=} \bar{x} + B(r).$$

Given $x \in X = \mathbb{R}^{j_1} \times (\ell_\nu^1)^{j_2}$, $x_j \in \mathbb{R}$ for $j = 1, \ldots, j_1$ and $x_j \in \ell_\nu^1$ for $j = j_1 + 1, \ldots, j_1 + j_2$. Given $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{j_1}, \bar{a}_1, \ldots, \bar{a}_{j_2})$, with $\bar{a}_j = ((\bar{a}_j)_{-m+1}, \ldots, (\bar{a}_j)_{m-1})$, define the bounds

$$Y = (Y_1, \ldots, Y_{j_1+j_2}) \in \mathbb{R}^{j_1+j_2}$$

$$Z(r) = (Z_1(r), \ldots, Z_{j_1+j_2}(r)) \in \mathbb{R}^{j_1+j_2}$$

such that

$$\left|(T(\bar{x}) - \bar{x})_j\right| \le Y_j, \quad \sup_{b,c \in B(r)} |DT_j(\bar{x} + b)c| \le Z_j(r), \quad \text{for } j = 1, \ldots, j_1 \tag{3.6}$$

$$\left\|(T(\bar{x}) - \bar{x})_j\right\|_\nu \le Y_j, \quad \sup_{b,c \in B(r)} \|(DT_j(\bar{x} + b)c)\|_\nu \le Z_j(r), \quad \text{for } j = j_1 + 1, \ldots, j_1 + j_2.$$

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Proposition 1. *Consider the bounds $Y, Z(r) \in \mathbb{R}^{j_1+j_2}$ satisfying the component-wise inequalities (3.6). If $\max\limits_{j=1,\ldots,j_1+j_2} \{Z_j(r) + Y_j\} < r$, then $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction. Moreover, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $F(\tilde{x}) = 0$.*

Proof. Letting $x \in B_{\bar{x}}(r)$, we first show that $T(x) \in B_{\bar{x}}(r)$. Note that $y \overset{\text{def}}{=} x - \bar{x} \in B(r)$. For each $j = 1, \ldots, j_1$, there exists $\xi = \xi(j) \in [0,1]$ such that

$$
\begin{aligned}
|(T(x) - \bar{x})_j| &\leq |T_j(x) - T_j(\bar{x})| + |T_j(\bar{x}) - \bar{x}_j| \\
&= |DT_j(\bar{x} + \xi y)y| + |T_j(\bar{x}) - \bar{x}_j| \\
&\leq Z_j(r) + Y_j.
\end{aligned}
$$

Similarly, for each $j = j_1 + 1, \ldots, j_1 + j_2$ and each $k \in \mathbb{Z}$, there exists $\xi = \xi(k,j) \in [0,1]$ such that

$$
\begin{aligned}
|((T(x) - \bar{x})_j)_k| &\leq |(T_j(x) - T_j(\bar{x}))_k| + |((T(\bar{x}) - \bar{x})_j)_k| \\
&= |(DT_j(\bar{x} + \xi y)y)_k| + |((T(x) - \bar{x})_j)_k|,
\end{aligned}
$$

and then

$$
\begin{aligned}
\|(T(x) - \bar{x})_j\|_\nu &= \sum_{k \in \mathbb{Z}} |((T(x) - \bar{x})_j)_k| \nu^{|k|} \\
&\leq \sum_{k \in \mathbb{Z}} |(DT_j(\bar{x} + \xi y)y)_k| \nu^{|k|} + \sum_{k \in \mathbb{Z}} |((T(x) - \bar{x})_j)_k| \nu^{|k|} \\
&\leq Z_j(r) + Y_j,
\end{aligned}
$$

since $y \in B(r)$ and $\bar{x} + \xi y \in B_{\bar{x}}(r)$. Therefore,

$$
\begin{aligned}
\|T(x) - \bar{x}\|_X &= \max \big(|(T(x) - \bar{x})_1|, \ldots, |(T(x) - \bar{x})_{j_1}|, \\
&\qquad \|(T(x) - \bar{x})_{j_1+1}\|_\nu, \ldots, \|(T(x) - \bar{x})_{j_1+j_2}\|_\nu \big) \\
&\leq \max_{j=1,\ldots,j_1+j_2} \{Z_j(r) + Y_j\} < r.
\end{aligned}
$$

Hence $T(x) \in B_{\bar{x}}(r)$ for any $x \in B_{\bar{x}}(r)$. Therefore $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$. Let us now show that T is a contraction. Consider $x, y \in B_{\bar{x}}(r)$ such that $x \neq y$. Then, for each $j = 1, \ldots, j_1$, there exists $\xi = \xi(j) \in [0,1]$ such that

$$
\begin{aligned}
|(T(x) - T(y))_j| &= |DT_j(\xi x + (1-\xi)y)(x-y)| \\
&= \left| DT_j(\xi x + (1-\xi)y)(x-y) \left(\frac{r}{\|x-y\|_X} \right) \right| \frac{\|x-y\|_X}{r} \\
&\leq \frac{Z_j(r)}{r} \|x-y\|_X.
\end{aligned}
$$

Similarly, for each $j = j_1+1, \ldots, j_1+j_2$ and each $k \in \mathbb{Z}$, there exists $\xi = \xi(k,j) \in [0,1]$ such that

$$
\begin{aligned}
|((T(x) - T(y))_j)_k| &= |(DT_j(\xi x + (1-\xi)y)(x-y))_k| \\
&= \left| \left(DT_j(\xi x + (1-\xi)y) \left(\frac{r(x-y)}{\|x-y\|_X} \right) \right)_k \right| \frac{\|x-y\|_X}{r},
\end{aligned}
$$

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and then

$$\|(T(x) - T(y))_j\|_\nu \quad = \quad \frac{\|x - y\|_X}{r} \sum_{k \in \mathbb{Z}} \left| \left(DT_j(\xi x + (1 - \xi)y) \left(\frac{r(x - y)}{\|x - y\|_X} \right) \right)_k \right| \nu^{|k|}$$

$$\leq \quad \frac{Z_j(r)}{r} \|x - y\|_X.$$

Now, since $\max_j \{Z_j(r)\} \leq \max_j \{Z_j(r) + Y_j\} < r$, then

$$\kappa \stackrel{\text{def}}{=} \frac{1}{r} \max_{j=1,\dots,j_1+j_2} \{Z_j(r)\} < 1. \tag{3.7}$$

Therefore,

$$\|T(x) - T(y)\|_X \quad = \quad \max \left(|(T(x) - T(y))_1|, \dots, |(T(x) - T(y))_{j_1}|, \right.$$
$$\left. \|(T(x) - T(y))_{j_1+1}\|_\nu, \dots, \|(T(x) - T(y))_{j_1+j_2}\|_\nu \right)$$
$$\leq \quad \max \left(\frac{Z_1(r)}{r} \|x - y\|_X, \dots, \frac{Z_{j_1+j_2}(r)}{r} \|x - y\|_X \right)$$
$$= \quad \kappa \|x - y\|_X.$$

This implies that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction with contraction constant $\kappa < 1$ defined by (3.7). By the contraction mapping theorem, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $T(\tilde{x}) = \tilde{x} = \tilde{x} - AF(\tilde{x})$. Since A is injective, it follows that there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $F(\tilde{x}) = 0$. $\qquad\square$

Definition 1. Given bounds Y and $Z(r)$ satisfying (3.6), define $p_1(r), \dots, p_{j_1+j_2}(r)$ by

$$p_j(r) \quad \stackrel{\text{def}}{=} \quad Z_j(r) - r + Y_j. \tag{3.8}$$

If for each $j = 1, \dots, j_1 + j_2$, the bound $Z_j(r)$ is polynomial in r, then $p_j(r)$ is polynomial in r. In this case, the polynomials $p_1(r), \dots, p_{j_1+j_2}$ are called the *radii polynomials*.

The definition of the radii polynomials is based under the assumption that each component of the bound $Z(r)$ can be obtained as a polynomial in r. Let us briefly mention why this assumption is achievable for a large class of problems.

Remark 1 (**The Z bound as a polynomial in r**). *The computation of the Z bound requires estimating each component of $DT(\bar{x} + b)c$ for all $b, c \in B_0(r)$. This is equivalent to estimating each component of $DT(\bar{x} + ur)vr$ for all $v, r \in B_0(1)$. If the nonlinearities of the original differential equation are polynomials of order less or equal to n, then F will consists of discrete convolutions with power at most n. Since $T(x) = x - AF(x)$ and $DT(\bar{x} + ur)vr \in X = \mathbb{R}^{j_1} \times \left(\ell_\nu^1\right)^{j_2}$, then each component of $DT(\bar{x} + ur)vr$ can be expanded as a nth order polynomial in r with the coefficients being either in $\mathbb{R}$ or in ℓ_ν^1. Bounding the resulting coefficients independently of r yields polynomial bounds.*

Another important features of the radii polynomials is that their coefficients can assume to be constructed as continuous functions of ν (this feature is useful when studying the domain of analyticity of the solutions as we see in Proposition 3 and Remark 3). The following remark briefly mention why this assumption is achievable.

Remark 2 (**Continuity of the coefficients of the radii polynomials in ν**). The coefficients of the radii polynomials p_j can assume to be constructed as continuous functions of ν. To see this, recall that the dual space of ℓ_ν^1 is ℓ_ν^∞ given in (2.2) with norm (2.1). As we will see in the applications, the coefficients of the p_j depend on two types of quantities:

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- the ℓ^1_ν norm of finite dimensional vectors (these quantities will be analytic in ν);

- the ℓ^∞_ν norm of finite dimensional vectors and matrices (these quantities will only be continuous in ν).

This implies that the coefficients of the p_j can be assumed continuous in ν. Hence, we have that $p_j = p_j(r, \nu)$ with the dependency in the variable radius r being polynomial and the dependency in the exponential decay rate ν being continuous.

The next result shows that the radii polynomials provide an efficient strategy for obtaining sets on which the corresponding Newton-like operator is a contraction mapping.

Proposition 2. *Fix $\nu \geq 1$ an exponential decay rate and construct the radii polynomials $p_j = p_j(r, \nu)$ for $j = 1, \ldots, j_1 + j_2$ of Definition 1. Define*

$$\mathcal{I} = \mathcal{I}(\nu) \overset{\text{def}}{=} \bigcap_{j=1}^{j_1+j_2} \{r > 0 \mid p_j(r, \nu) < 0\}. \tag{3.9}$$

If $\mathcal{I} \neq \emptyset$, then $\mathcal{I}$ is an open interval, and for any $r \in \mathcal{I}$, the ball $B_{\bar{x}}(r)$ contains a unique solution $\tilde{x}$ such that $F(\tilde{x}) = 0$. Note that $\tilde{x}$ is the same solution for all $r \in \mathcal{I}$.

Proof. Assume that the highest degree of the polynomial nonlinearities of the original differential equation is n. Fix $\nu \geq 1$ and $j \in \{1, \ldots, j_1 + j_2\}$. From Remark 1, the coefficients of the radii polynomials will be of the form

$$p_j(r, \nu) = a_n^{(j)} r^n + a_{n-1}^{(j)} r^{n-1} + \cdots + a_1^{(j)} r - r + a_0^{(j)},$$

with $a_i^{(j)} \geq 0$ for all $i = 0, \ldots, n$. Since $\mathcal{I} \neq \emptyset$, then $a_1^{(j)} - 1 < 0$. Otherwise we would not be able to find $r > 0$ such that $p_j(r, \nu) < 0$. By Descartes' rule of signs and since $\mathcal{I} \neq \emptyset$, each radii polynomial p_j has exactly two positive real zeros that we denote by $r_-^{(j)} < r_+^{(j)}$. Defining $\mathcal{I}_j = (r_-^{(j)}, r_+^{(j)})$, we obtain that $\mathcal{I} = \cap_{j=1}^{j_1+j_2} \mathcal{I}_j$. This implies that $\mathcal{I}$ is an open interval.

Consider now $r \in \mathcal{I}$. Hence $p_j(r, \nu) < 0$ for all $j = 1, \ldots, j_1 + j_2$, and therefore

$$\max_{j=1,\ldots,j_1+j_2} \{Y_j + Z_j(r)\} = \max_{j=1,\ldots,j_1+j_2} (p_j(r, \nu) + r) < r.$$

The result follows from Proposition 1. $\qquad\square$

If $\mathcal{I} \neq \emptyset$, then let $r_- < r_+$ such that $\mathcal{I} = (r_-, r_+)$. Proposition 2 demonstrates that the radii polynomial approach provides a strategy for obtaining bounds on the smallest ball (given by $B_{\bar{x}}(r_-)$) and largest ball (given by $B_{\bar{x}}(r_+)$) about the approximate solution on which the corresponding Newton-like operator is a contraction mapping. The radius r_- provides tight bounds on the location of the true solution of the problem.

We now show that if a proof is performed at $\nu = 1$, we get analyticity of the solution.

Proposition 3. *Assume that the coefficients of the radii polynomials (3.8) are continuous in $\nu \geq 1$. Set $\nu = 1$ and assume that $\mathcal{I}(1) \neq \emptyset$. From Proposition 2, there exists a unique $\tilde{x} \in \{x : \|x - \bar{x}\|_1 \leq r\}$ such that $F(\tilde{x}) = 0$, for any $r \in \mathcal{I}$. Then there exists $\nu > 1$ such that the solution $\tilde{x}$ actually lies in the smoother space $\{x : \|x - \bar{x}\|_\nu \leq r\}$.*

Proof. For any $r \in \mathcal{I}(1)$ and for any $j = 1, \ldots, j_1 + j_2$, $p_j(r, 1) < 0$. By continuity of the radii polynomials in ν, there exists $\nu > 1$ such that $p_j(r, \nu) < 0$ for all $r \in \mathcal{I}(1)$ and $j = 1, \ldots, j_1 + j_2$. The conclusion follows from Proposition 2. $\qquad\square$

Remark 3 (**Maximizing bounds on the domain of analyticity**). Proposition 3 provides a starting point for an algorithm to maximize the validated domain of analyticity of the solutions we are studying. Assume that the radii polynomials are all negative at $\nu = 1$ on the interval of $\mathcal{I}$, that is $\mathcal{I}(1) \neq \emptyset$, and that $\mathcal{I}(\nu_1) = \emptyset$ for some $\nu_1 > 1$ fixed. There is a natural bisection algorithm that allows maximizing the domain of analyticity as follows. Initialize $\nu_{\text{left}} = 1$ and $\nu_{\text{right}} = \nu_1$. Set $\nu_{\text{mid}} = \frac{\nu_{\text{left}} + \nu_{\text{right}}}{2}$ and construct $\mathcal{I}(\nu_{\text{mid}})$. If $\mathcal{I}(\nu_{\text{mid}}) = \emptyset$, then set $\nu_{\text{right}} \hookleftarrow \nu_{\text{mid}}$ and start over. If $\mathcal{I}(\nu_{\text{mid}}) \neq \emptyset$, then set $\nu_{\text{left}} \hookleftarrow \nu_{\text{mid}}$ and start over. By fixing at the beginning a number of maximal bisection steps and a tolerance on $|\nu_{\text{right}} - \nu_{\text{left}}|$, the algorithm terminates and we set $\nu_{\max} = \nu_{\text{mid}}$.

The previous remark provides a general strategy to study periodic solutions in the analytic category. We choose a finite dimensional projection and compute an approximate solution. Based on that, we construct the radii polynomials at $\nu = 1$ and construct $\mathcal{I}(1)$. If $\mathcal{I}(1) \neq \emptyset$, we apply the strategy of Remark 3 to maximize the domain of analyticity of the periodic solutions. If $\mathcal{I}(1) = \emptyset$, then we can increase the size of the finite dimensional projection, and start over.

Remark 4 (**Real Solutions**). Given $z \in \mathbb{C}$, denote by $\text{conj}(z)$ the complex conjugate of z. A solution of $F(x) = 0$ with F given in (3.1) corresponds to a real valued function if and only if $(a_i)_{-k} = (\text{conj}(a_i))_k$ for all $k \in \mathbb{Z}$. If our goal is to show that the problem has a real solution then we must take this symmetry into account. One possibility is to impose the condition in the sequence space ℓ^1_ν defined in (1.1). In other words we take X to be the collection of all sequences in ℓ^1_ν which also satisfy the symmetry, that is we consider the *symmetric* sequence space

$$\ell^1_\nu \overset{\text{def}}{=} \left\{ a = \{a_k\}_{k \in \mathbb{Z}} \,|\, a_k \in \mathbb{C}, \ a_{-k} = \text{conj}(a_k) \ \text{ for all } k \in \mathbb{Z}, \ \text{ and } \ \|a\|_\nu < \infty \right\}.$$

Now we must check that the map F, as well as the linear operator A, are maps which take the symmetric sequence space into itself. This is usually clear for F. However we must take care in the definition of A that the symmetry is preserved. We illustrate this procedure in Section 4. Another possibility is to decompose the map F into real and imaginary parts, that is to work with sine and cosine as basis functions instead of the complex exponential. In this case we end up working in one sided sequence spaces, however the analysis is often a little more tedious. The choice is just a matter of convenience, and is best considered on a problem by problem basis. We illustrate the one sided sequence approach in Section 5.

We are now ready to present some applications.

4 Periodic orbits in the Lorenz equations

Recall the Lorenz equations given by (1.3) and denote by $\gamma(t) = (u_1, u_2, u_3)(t)$ an a priori unknown $\frac{2\pi}{\omega}$-periodic solution to this system. Denote its Fourier expansion as

$$u_1(t) = \sum_{k \in \mathbb{Z}} (a_1)_k e^{i\omega k t}, \qquad u_2(t) = \sum_{k \in \mathbb{Z}} (a_2)_k e^{i\omega k t}, \qquad u_3(t) = \sum_{k \in \mathbb{Z}} (a_3)_k e^{i\omega k t}.$$

The unknowns for this problem are the frequency ω and the three sequences of Fourier coefficients a_1, a_2 and a_3 of the components $u_1(t)$, $u_2(t)$ and $u_3(t)$ respectively. Therefore, the infinite dimensional vector of unknowns is given by $x \overset{\text{def}}{=} (\omega, a_1, a_2, a_3)$. The function space in which the unknown x lives is then $X = X_\nu^{1,3} = \mathbb{R} \times \ell^1_\nu \times \ell^1_\nu \times \ell^1_\nu$.

In order to isolate the periodic solution in the function space X, we set a *phase condition* for any potential orbit γ to be $\dot{\bar{\gamma}}_0 \cdot (\bar{\gamma}_0 - \gamma(0)) = 0$, for fixed $\bar{\gamma}_0 = (\bar{\gamma}_{0,1}, \bar{\gamma}_{0,2}, \bar{\gamma}_{0,3}) \in \mathbb{R}^3$ and $\dot{\bar{\gamma}}_0 = (\dot{\bar{\gamma}}_{0,1}, \dot{\bar{\gamma}}_{0,2}, \dot{\bar{\gamma}}_{0,3}) \in \mathbb{R}^3$. In terms of the Fourier coefficients of $\gamma(t)$, this is equivalent to

$$\bar{\gamma}_{0,1} \dot{\bar{\gamma}}_{0,1} + \bar{\gamma}_{0,2} \dot{\bar{\gamma}}_{0,2} + \bar{\gamma}_{0,3} \dot{\bar{\gamma}}_{0,3} - \sum_{k \in \mathbb{Z}} (\dot{\bar{\gamma}}_{0,1}(a_1)_k + \dot{\bar{\gamma}}_{0,2}(a_2)_k + \dot{\bar{\gamma}}_{0,3}(a_3)_k) = 0.$$

However, this condition can be relaxed, by fixing $k_0 \in \mathbb{N}$, to the following

$$F_0(x) \stackrel{\text{def}}{=} \bar{\gamma}_{0,1} \dot{\bar{\gamma}}_{0,1} + \bar{\gamma}_{0,2} \dot{\bar{\gamma}}_{0,2} + \bar{\gamma}_{0,3} \dot{\bar{\gamma}}_{0,3} - \sum_{|k| \le k_0} (\dot{\bar{\gamma}}_{0,1}(a_1)_k + \dot{\bar{\gamma}}_{0,2}(a_2)_k + \dot{\bar{\gamma}}_{0,3}(a_3)_k) = 0. \quad (4.1)$$

Thus, if we have a periodic orbit with the phase condition (4.1), then the coefficients in its Fourier expansion satisfy

$$(F_1(x))_k \stackrel{\text{def}}{=} -\mathbf{i}\omega k(a_1)_k + \sigma\left((a_2)_k - (a_1)_k\right) = 0$$

$$(F_2(x))_k \stackrel{\text{def}}{=} -\mathbf{i}\omega k(a_2)_k + \rho(a_1)_k - (a_2)_k - \sum_{k_1+k_2=k} (a_1)_{k_1}(a_3)_{k_2} = 0 \quad (4.2)$$

$$(F_3(x))_k \stackrel{\text{def}}{=} -\mathbf{i}\omega k(a_3)_k - \beta(a_3)_k + \sum_{k_1+k_2=k} (a_1)_{k_1}(a_2)_{k_2} = 0.$$

Since we are interested in real periodic solutions of (1.3), we impose the complex conjugacy condition $a_{-k} = \text{conj}(a_k)$ directly in the space of two-tailed complex sequences, i.e.

$$\ell^1_\nu = \left\{ a = (a_k)_{k \in \mathbb{Z}} : a_k \in \mathbb{C}, \ a_{-k} = \text{conj}(a_k), \ ||a||_\nu = \sum_{k \in \mathbb{Z}} |a_k| \nu^{|k|} < \infty \right\}.$$

Note that even if we impose the complex conjugacy condition in the space ℓ^1_ν, it still remains a Banach algebra under discrete convolutions. Indeed, given $a, b \in \ell^1_\nu$

$$(a * b)_{-k} = \sum_{k_1+k_2=-k} a_{k_1} b_{k_2} = \sum_{-k_1-k_2=k} a_{k_1} b_{k_2}$$

$$= \sum_{k_1+k_2=k} a_{-k_1} b_{-k_2} = \sum_{k_1+k_2=k} \text{conj}(a_{k_1})\text{conj}(b_{k_2}) = \text{conj}((a*b)_k).$$

Recalling (4.2), this implies that for $j = 1, 2, 3$ and for any $x \in X^{1,3}_\nu$, we have that $(F_j(x))_{-k} = \text{conj}\left((F_j(x))_k\right)$. Now, given any $x \in X^{1,3}_\nu$, set

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} F_0(x) \\ F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix}.$$

Note that for F does not map $X^{1,3}_\nu$ into itself, because the map $\mathcal{L}(a) \stackrel{\text{def}}{=} (ka_k)_{k \in \mathbb{Z}}$ $(a \in \ell^1_\nu)$ is unbounded on ℓ^1_ν. However, for any $\nu' < \nu$, $F : X^{1,3}_\nu \to X^{1,3}_{\nu'}$ since $\mathcal{L} : \ell^1_\nu \to \ell^1_{\nu'}$ is bounded. To see this let $\delta \stackrel{\text{def}}{=} \nu/\nu' > 1$. Then there exists $\bar{k} \in \mathbb{N}$ such that $\frac{|k|}{\delta^{|k|}} \le 1$ for

any $|k| > \bar{k}$. Therefore, for any $a \in \ell_\nu^1$

$$\|\mathcal{L}(a)\|_{\nu'} = \sum_{k \in \mathbb{Z}} |ka_k|(\nu')^{|k|} = \sum_{k \in \mathbb{Z}} \frac{|k|}{\delta^{|k|}}|a_k|\nu^{|k|} = \sum_{|k| \leq \bar{k}} \frac{|k|}{\delta^{|k|}}|a_k|\nu^{|k|} + \sum_{|k| \geq \bar{k}} \frac{|k|}{\delta^{|k|}}|a_k|\nu^{|k|}$$

$$\leq \sum_{|k| \leq \bar{k}} \frac{|k|}{\delta^{|k|}}|a_k|\nu^{|k|} + \sum_{|k| \geq \bar{k}} |a_k|\nu^{|k|} < \infty.$$

On the other hand one can find sequences $a \in \ell_\nu^1$ so that $\|\mathcal{L}(a)\|_\nu$ is a divergent series.

Since we may only deal with a finite number of Fourier modes numerically we define projections of the infinite dimensional space $X_\nu^{1,3}$ onto a finite-dimensional space. Set $a_j^{(m)} = ((a_j)_k)_{|k|<m} \in \mathbb{C}^{2m-1}$ for $j = 1, 2, 3$, and $x^{(m)} = (\omega, a_1^{(m)}, a_2^{(m)}, a_3^{(m)}) \in \mathbb{R} \times \mathbb{C}^{3(2m-1)}$. We also truncate each F_j at the mth and $-m^{th}$ mode (with $m > k_0$). Recall (3.2) and define $F^{(m)} : \mathbb{R} \times \mathbb{C}^{3(2m-1)} \to \mathbb{R} \times \mathbb{C}^{3(2m-1)}$ component-wise by

$$F_0^{(m)}(x^{(m)}) = \bar{\gamma}_{0,1}\dot{\gamma}_{0,1} + \bar{\gamma}_{0,2}\dot{\gamma}_{0,2} + \bar{\gamma}_{0,3}\dot{\gamma}_{0,3} - \sum_{k=-k_0}^{k_0} (\dot{\gamma}_{0,1}(a_1)_k + \dot{\gamma}_{0,2}(a_2)_k + \dot{\gamma}_{0,3}(a_3)_k)$$

$$F_1^{(m)}(x^{(m)}) = \left[-\mathbf{i}k\omega(a_1)_k + \sigma(a_2)_k - \sigma(a_1)_k \right]_{k=-m+1}^{m-1}$$

$$F_2^{(m)}(x^{(m)}) = \left[-\mathbf{i}\omega k(a_2)_k + \rho(a_1)_k - (a_2)_k - \sum_{\substack{k_1+k_2=k \\ |k_1|,|k_2|<m}} (a_1)_{k_1}(a_3)_{k_2} \right]_{k=-m+1}^{m-1}$$

$$F_3^{(m)}(x^{(m)}) = \left[-\mathbf{i}\omega k(a_3)_k - \beta(a_3)_k + \sum_{\substack{k_1+k_2=k \\ |k_1|,|k_2|<m}} (a_1)_{k_1}(a_2)_{k_2} \right]_{k=-m+1}^{m-1}.$$

Assume that using Newton's method, we numerically found $\bar{x} \in \mathbb{R} \times \mathbb{C}^{3(2m-1)}$ such that $F^{(m)}(\bar{x}) \approx 0$. We want to use the radii polynomial approach of Section 3 to show that near $\bar{x}$, there is an exact solution $\tilde{x}$ of $F(\tilde{x}) = 0$. The next step is to define an approximate inverse A for $DF(\bar{x})$.

Given $x = (\omega, a_1, a_2, a_3) \in X_\nu^{1,3}$ note that the Fréchet derivative $DF(\bar{x})$ can be visualized as

$$DF(\bar{x}) = \begin{bmatrix} \partial_\omega F_0(\bar{x}) & D_{a_1}F_0(\bar{x}) & D_{a_2}F_0(\bar{x}) & D_{a_3}F_0(\bar{x}) \\ \partial_\omega F_1(\bar{x}) & D_{a_1}F_1(\bar{x}) & D_{a_2}F_1(\bar{x}) & D_{a_3}F_1(\bar{x}) \\ \partial_\omega F_2(\bar{x}) & D_{a_1}F_2(\bar{x}) & D_{a_2}F_2(\bar{x}) & D_{a_3}F_2(\bar{x}) \\ \partial_\omega F_3(\bar{x}) & D_{a_1}F_3(\bar{x}) & D_{a_2}F_3(\bar{x}) & D_{a_3}F_3(\bar{x}) \end{bmatrix},$$

where

$$\begin{cases} \partial_\omega F_0(\bar{x}) : \mathbb{R} \to \mathbb{R} \\ \partial_\omega F_j(\bar{x}) : \mathbb{R} \to \ell_\nu^1 \text{ for } j = 1, 2, 3, \\ D_{a_i}F_0(\bar{x}) : \ell_\nu^1 \to \mathbb{R} \text{ are linear functionals } (i = 1, 2, 3) \\ D_{a_i}F_j(\bar{x}) : \ell_\nu^1 \to \ell_{\nu'}^1 \text{ are linear operators for } i, j = 1, 2, 3 \text{ with } \nu' < \nu. \end{cases}$$

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Given the numerical solution $\bar{x} = (\bar{\omega}, \bar{a}_1^{(m)}, \bar{a}_2^{(m)}, \bar{a}_3^{(m)}) = (\bar{\omega}, \bar{a}_1, \bar{a}_2, \bar{a}_3) \in \mathbb{R} \times \mathbb{C}^{3(2m-1)}$, we first approximate $DF(\bar{x})$ with the operator

$$A^\dagger \stackrel{\text{def}}{=} \begin{bmatrix} A^\dagger_{\omega,0} & A^\dagger_{a_1,0} & A^\dagger_{a_2,0} & A^\dagger_{a_3,0} \\ A^\dagger_{\omega,1} & A^\dagger_{a_1,1} & A^\dagger_{a_2,1} & A^\dagger_{a_3,1} \\ A^\dagger_{\omega,2} & A^\dagger_{a_1,2} & A^\dagger_{a_2,2} & A^\dagger_{a_3,2} \\ A^\dagger_{\omega,3} & A^\dagger_{a_1,3} & A^\dagger_{a_2,3} & A^\dagger_{a_3,3} \end{bmatrix},$$

which acts on $b = (b_0, b_1, b_2, b_3) \in X_\nu^{1,3}$ component-wise as

$$(A^\dagger b)_0 = A^\dagger_{\omega,0} b_0 + \sum_{i=1}^3 A^\dagger_{a_i,0} b_i \stackrel{\text{def}}{=} \partial_\omega F_0^{(m)}(\bar{x}) b_0 + \sum_{i=1}^3 D_{a_i^{(m)}} F_0^{(m)}(\bar{x}) b_i^{(m)}$$

$$(A^\dagger b)_j = A^\dagger_{\omega,j} b_0 + \sum_{i=1}^3 A^\dagger_{a_i,j} b_i \in \ell_{\nu'}^1, \qquad (j = 1, 2, 3),$$

where $A^\dagger_{\omega,j} = \partial_\omega F_j^{(m)}(\bar{x})$ and $A^\dagger_{a_i,j} b_i \in \ell_{\nu'}^1$ is defined component-wise by

$$\left(A^\dagger_{a_i,j} b_i \right)_k = \begin{cases} \left(D_{a_i} F_j^{(m)}(\bar{x}) b_i^{(m)} \right)_k, & |k| < m \\ (-\mathbf{i}\bar{\omega}\delta_{i,j}) k (b_i)_k, & |k| \ge m. \end{cases}$$

Let $A^{(m)}$ a finite dimensional approximate inverse of $DF^{(m)}(\bar{x})$ (which will usually be obtained numerically). Define the decomposition

$$A^{(m)} = \begin{bmatrix} A^{(m)}_{\omega,0} & A^{(m)}_{a_1,0} & A^{(m)}_{a_2,0} & A^{(m)}_{a_3,0} \\ A^{(m)}_{\omega,1} & A^{(m)}_{a_1,1} & A^{(m)}_{a_2,1} & A^{(m)}_{a_3,1} \\ A^{(m)}_{\omega,2} & A^{(m)}_{a_1,2} & A^{(m)}_{a_2,2} & A^{(m)}_{a_3,2} \\ A^{(m)}_{\omega,3} & A^{(m)}_{a_1,3} & A^{(m)}_{a_2,3} & A^{(m)}_{a_3,3} \end{bmatrix} \in \mathbb{C}^{(6m-2)\times(6m-2)},$$

where $A^{(m)}_{\omega,0} \in \mathbb{R}$, $A^{(m)}_{a_i,0} \in \mathbb{C}^{1\times(2m-1)}$, $A^{(m)}_{\omega,j} \in \mathbb{C}^{(2m-1)\times 1}$ and $A^{(m)}_{a_i,j} \in \mathbb{C}^{(2m-1)\times(2m-1)}$. Assume moreover that $A^{(m)}$ satisfies the following *symmetry* assumptions:

1. $(A^{(m)}_{a_i,0})_{-j} = \text{conj}\left((A^{(m)}_{a_i,0})_j \right)$, $\quad j = -m+1, \ldots, m-1$,

2. $(A^{(m)}_{\omega,i})_{-k} = \text{conj}\left((A^{(m)}_{\omega,i})_k \right)$, $\quad k = -m+1, \ldots, m-1$, $\qquad$ (4.3)

3. $(A^{(m)}_{a_i,\ell})_{-k,-j} = \text{conj}\left((A^{(m)}_{a_i,\ell})_{k,j} \right)$, $\quad k, j = -m+1, \ldots, m-1$.

One consequence of assumption 1. of (4.3) is that $(A^{(m)}_{a_i,0})_0 \in \mathbb{R}$ for $i = 1, 2, 3$ while one consequence of assumption 2. is that $(A^{(m)}_{\omega,i})_0 \in \mathbb{R}$ for $i = 1, 2, 3$. Let us now verify that (3.3) holds.

Lemma 4. *Let $x \in X$. Then $AF(x) \in X$.*

Proof. Consider $x = (\omega, a_1, a_2, a_3) \in X$ and let $F(x) = (F_0(x), F_1(x), F_2(x), F_3(x))$, with $F_0(x)$ given in (4.1) and $F_i(x)$ given in (4.2) for $i = 1, 2, 3$. For sake of simplicity of the presentation, we denote $F_i = F_i(x)$. To show that $AF(x) \in X$, we need to show that $(AF(x))_0 \in \mathbb{R}$ and for $i = 1, 2, 3$ that $((AF(x))_i)_{-k} = \text{conj}\left(((AF(x))_i)_k \right)$

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and that $\|(AF(x))_i\|_\nu < \infty$. It is a simple exercise to show that $F_0 \in \mathbb{R}$ and that $(F_i)_{-k} = \text{conj}\,((F_i)_{-k})$ for $i = 1, 2, 3$. Now, by assumption 1. of (4.3),

$$
\begin{aligned}
(AF(x))_0 &= A_{\omega,0}^{(m)} F_0 + \sum_{i=1}^{3} \left(\sum_{j=1}^{m-1} (A_{a_i,0}^{(m)})_{-j}(F_i)_{-j} + (A_{a_i,0}^{(m)})_0(F_i)_0 + \sum_{j=1}^{m-1} (A_{a_i,0}^{(m)})_j(F_i)_j \right) \\
&= A_{\omega,0}^{(m)} F_0 + \sum_{i=1}^{3} (A_{a_i,0}^{(m)})_0(F_i)_0 \\
&\quad + \sum_{i=1}^{3} \left(\sum_{j=1}^{m-1} \text{conj}\left((A_{a_i,0}^{(m)})_j(F_i)_j \right) + (A_{a_i,0}^{(m)})_j(F_i)_j \right) \in \mathbb{R}.
\end{aligned}
$$

By assumptions 2 and 3 of (4.3), for $|k| < m$, we have

$$
\begin{aligned}
((AF(x))_i)_{-k} &= \left(A_{\omega,1}^{(m)} \right)_{-k} F_0 + \sum_{\ell=1}^{3} \sum_{j=-m+1}^{m-1} \left(A_{a_i,\ell}^{(m)} \right)_{-k,-j} \left(F_\ell^{(m)} \right)_{-j} \\
&= \text{conj}\,(((AF(x))_i)_k),
\end{aligned}
$$

and for $|k| \geq m$, we easily see that $((AF(x))_i)_{-k} = \text{conj}\,(((AF(x))_i)_k)$. The final step is to show that $\|(AF(x))_i\|_\nu < \infty$ for $i = 1, 2, 3$, and is left to the reader. $\qquad\square$

By *approximate inverse* we mean that for some ϵ with $0 < \epsilon \ll 1$,

$$
\left\| I_{\mathbb{R} \times (\mathbb{C}^{2m-1})^3} - A^{(m)} DF^{(m)}(\bar{x}) \right\| \leq \epsilon. \tag{4.4}
$$

We define the approximate inverse A of the infinite dimensional operator $DF(\bar{x})$ by

$$
A \stackrel{\text{def}}{=} \begin{bmatrix} A_{\omega,0} & A_{a_1,0} & A_{a_2,0} & A_{a_3,0} \\ A_{\omega,1} & A_{a_1,1} & A_{a_2,1} & A_{a_3,1} \\ A_{\omega,2} & A_{a_1,2} & A_{a_2,2} & A_{a_3,2} \\ A_{\omega,3} & A_{a_1,3} & A_{a_2,3} & A_{a_3,3} \end{bmatrix}.
$$

A acts on $b = (b_0, b_1, b_2, b_3) \in X_\nu^{1,3}$ component-wise as

$$
\begin{aligned}
(Ab)_0 &= A_{\omega,0}^{(m)} b_0 + \sum_{i=1}^{3} A_{a_i,0}^{(m)} b_i^{(m)} \\
(Ab)_j &= A_{\omega,j}^{(m)} b_0 + \sum_{i=1}^{3} A_{a_i,j} b_i \in \ell_\nu^1, \qquad (j = 1, 2, 3),
\end{aligned}
$$

where $A_{\omega,j}^{(m)} \in \mathbb{C}^{(2m-1)\times 1}$ is understood to be an element of ℓ_ν^1 by *padding* the tail with zeros, and $A_{a_i,j} b_i \in \ell_\nu^1$ is defined component-wise by

$$
(A_{a_i,j} b_i)_k = \begin{cases} \left(A_{a_i,j}^{(m)} b_i^{(m)} \right)_k, & |k| < m \\ \dfrac{\delta_{i,j}}{(-\mathbf{i}\bar{\omega})k}(b_i)_k, & |k| \geq m. \end{cases}
$$

Having defined A piece by piece, we can now define the Newton-like operator by

$$
T(x) = x - AF(x).
$$

Recalling (2.5), the norm on $X = X_\nu^{1,3}$ is given by

$$\|x\|_X = \max\left(|\omega|, \|a_1\|_\nu, \|a_2\|_\nu, \|a_3\|_\nu\right).$$

Proposition 4. $T : X_\nu^{1,3} \to X_\nu^{1,3}$.

The proof is a direct consequence of Lemma 4.

Remark 5 (Obtaining the approximate solution $\bar{x}$). Throughout the previous section we assume that an approximate solution $\bar{x} = (\bar{\omega}, \bar{a}_1^{(m)}, \bar{a}_2^{(m)}, \bar{a}_3^{(m)})$ is given. In practice we compute a candidate typically in one of the following three ways. The first possibility is to numerically continue from a Hopf bifurcation until we reach a parameter value where we want to validate the solution. This continuation could be done using a general purpose numerical package such as AUTO or MatCont. Once we reach the desired parameter we take $\bar{\omega}$ to be the numerical frequency of the orbit and compute the approximate Fourier coefficients $\bar{a}_1^{(m)}, \bar{a}_2^{(m)}$, and $\bar{a}_3^{(m)}$ numerically via the fast Fourier transform (FFT). A second possibility for locating approximate orbits is to numerically integrate a point on the Lorenz attractor (using any convenient numerical package) until the point returns to near where it started. This integration time is taken as the approximate period and again the Fourier coefficients are computed numerically using the FFT. In both of the first two cases, the initial guess can be refined by applying the Newton iteration in sequence space in order to obtain a solution with numerical defect as small as possible. A third possibility is to use a predictor-corrector continuation algorithm directly in the space of Fourier coefficients. This is how the approximations are obtained in the present work. Of course all the above possibilites are standard techniques and the remark is included only for the sake of completeness.

4.1 The radii polynomials for Lorenz.

Recall that we have a numerical approximation $\bar{x} = (\bar{\omega}, \bar{a}_1, \bar{a}_2, \bar{a}_3) \in X_\nu^{1,3}$. We begin the construction of the radii polynomials (3.8) by constructing the bounds Y_j such that $|(T(\bar{x}) - \bar{x})_0| \le Y_0$ and $\|(T(\bar{x}) - \bar{x})_j\|_\nu \le Y_j$ for $j = 1, 2, 3$. We only show explicitly the calculation of Y_2, as the other cases are similar. We have that

$$
\begin{aligned}
\|(T(\bar{x}) - \bar{x})_2\|_\nu &= \|(AF(\bar{x}))_2\|_\nu \\
&= \|A_{\omega,2}F_0(\bar{x}) + A_{a_1,2}F_1(\bar{x}) + A_{a_2,2}F_2(\bar{x}) + A_{a_3,2}F_3(\bar{x})\|_\nu \\
&= \sum_{k \in \mathbb{Z}} |(A_{\omega,2}F_0(\bar{x}))_k + (A_{a_1,2}F_1(\bar{x}))_k + (A_{a_2,2}F_2(\bar{x}))_k + (A_{a_3,2}F_3(\bar{x}))_k| \, \nu^{|k|} \\
&= \sum_{|k|<m} \left|(A_{\omega,2}^{(m)}F_0^{(m)}(\bar{x}))_k + (A_{a_1,2}^{(m)}F_1^{(m)}(\bar{x}))_k + (A_{a_2,2}^{(m)}F_2^{(m)}(\bar{x}))_k + (A_{a_3,2}^{(m)}F_3^{(m)}(\bar{x}))_k\right| \nu^{|k|} \\
&\qquad + \sum_{|k|\ge m} |(A_{a_2,2}F_2(\bar{x}))_k| \, \nu^{|k|},
\end{aligned}
$$

where the first summand is finite and the second summand satisfies

$$\sum_{|k| \geq m} |(A_{a_2,2} F_2(\bar{x}))_k| \, \nu^{|k|}$$

$$= \sum_{|k| \geq m} \left| \frac{1}{-\mathbf{i}\bar{\omega}k} \left(-\mathbf{i}\bar{\omega}k(\bar{a}_2)_k + \rho(\bar{a}_1)_k - (\bar{a}_2)_k - \sum_{k_1+k_2=k} (\bar{a}_1)_{k_1}(\bar{a}_3)_{k_2} \right) \right| \nu^{|k|}$$

$$= \sum_{m \leq |k| < 2m-1} \left| (\bar{a}_2)_k + \frac{\rho(\bar{a}_1)_k}{-\mathbf{i}\bar{\omega}k} - \frac{(\bar{a}_2)_k}{-\mathbf{i}\bar{\omega}k} - \frac{1}{-\mathbf{i}\bar{\omega}k} \sum_{k_1+k_2=k} (\bar{a}_1)_{k_1}(\bar{a}_3)_{k_2} \right| \nu^{|k|}$$

$$= \sum_{m \leq |k| < 2m-1} \frac{1}{\bar{\omega}|k|} \left| \sum_{k_1+k_2=k} (\bar{a}_1)_{k_1}(\bar{a}_3)_{k_2} \right| \nu^{|k|}, \qquad \text{since } (\bar{a}_1)_k, (\bar{a}_2)_k = 0 \text{ for } |k| \geq m.$$

Since we may do the same with Y_3, and the bound for Y_1 has no convolution terms, we are done by setting

$$Y_0 \overset{\text{def}}{=} \left| A_{\omega,0}^{(m)} F_0^{(m)}(\bar{x}) + \sum_{i=1}^{3} A_{a_i,0}^{(m)} F_i^{(m)}(\bar{x}) \right| \tag{4.5}$$

$$Y_1 \overset{\text{def}}{=} \sum_{|k|<m} \left| (A_{\omega,1}^{(m)} F_0^{(m)}(\bar{x}))_k + \sum_{i=1}^{3} (A_{a_i,1}^{(m)} F_i^{(m)}(\bar{x}))_k \right| \nu^{|k|}, \tag{4.6}$$

$$Y_2 \overset{\text{def}}{=} \sum_{|k|<m} \left| (A_{\omega,2}^{(m)} F_0^{(m)}(\bar{x}))_k + \sum_{i=1}^{3} (A_{a_i,2}^{(m)} F_i^{(m)}(\bar{x}))_k \right| \nu^{|k|} \tag{4.7}$$

$$+ \sum_{m \leq |k| < 2m-1} \frac{1}{\bar{\omega}|k|} \left| \sum_{k_1+k_2=k} (\bar{a}_1)_{k_1}(\bar{a}_3)_{k_2} \right| \nu^{|k|},$$

$$Y_3 \overset{\text{def}}{=} \sum_{|k|<m} \left| (A_{\omega,3}^{(m)} F_0^{(m)}(\bar{x}))_k + \sum_{i=1}^{3} (A_{a_i,3}^{(m)} F_i^{(m)}(\bar{x}))_k \right| \nu^{|k|} \tag{4.8}$$

$$+ \sum_{m \leq |k| < 2m-1} \frac{1}{\bar{\omega}|k|} \left| \sum_{k_1+k_2=k} (\bar{a}_1)_{k_1}(\bar{a}_2)_{k_2} \right| \nu^{|k|}.$$

The next step in the construction of the radii polynomials (3.8) is to construct the bounds $Z_0(r), \ldots, Z_3(r)$. Let $b, c \in B(r) \subset X_\nu^{1,3}$. Then

$$DT(\bar{x} + b)c = [I - ADF(\bar{x} + b)]c = [I - AA^\dagger]c - A[DF(\bar{x} + b) - A^\dagger]c. \tag{4.9}$$

We first bound the quantities involved in the first term of (4.9). Let $B \overset{\text{def}}{=} I - AA^\dagger$, which we express as

$$B = \begin{bmatrix} B_{\omega,0} & B_{a_1,0} & B_{a_2,0} & B_{a_3,0} \\ B_{\omega,1} & B_{a_1,1} & B_{a_2,1} & B_{a_3,1} \\ B_{\omega,2} & B_{a_1,2} & B_{a_2,2} & B_{a_3,2} \\ B_{\omega,3} & B_{a_1,3} & B_{a_2,3} & B_{a_3,3} \end{bmatrix}.$$

Due to the structure of B, we have that $((Bc)_j)_k = 0$ for $|k| \geq m$, $j = 1, 2, 3$ and

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$c \in B(r) \subset \mathbb{R} \times (\ell_\nu^1)^3$. Define

$$Z_0^{(0)} \overset{\text{def}}{=} |B_{\omega,0}| + \sum_{i=1}^3 \left(\max_{|k|<m} \frac{|(B_{a_i,0})_k|}{\nu^{|k|}} \right) \tag{4.10}$$

$$Z_j^{(0)} \overset{\text{def}}{=} \sum_{|k|<m} |(B_{\omega,j})_k| \nu^{|k|} + \sum_{i=1}^3 \left(\max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |(B_{a_i,j})_{k,n}| \nu^{|k|} \right), \tag{4.11}$$

for $j = 1, 2, 3$. Now, recalling (2.1) and Lemma 1, we have that

$$|(Bc)_0| = \left| B_{\omega,0} c_0 + \sum_{i=1}^3 \sum_{k \in \mathbb{Z}} (B_{a_i,0})_k (c_i)_k \right| \le \left(|B_{\omega,0}| + \sum_{i=1}^3 \|B_{a_i,0}\|_\nu^\infty \right) r = Z_0^{(0)} r.$$

Thus, for $j = 1, 2, 3$, recalling Lemma 2, Corollary 1 and (2.3), we get that

$$\|(Bc)_j\|_\nu = \left\| B_{\omega,j} c_0 + \sum_{i=1}^3 B_{a_i,j} c_i \right\|_\nu \le \left(\|B_{\omega,j}\|_\nu + \sum_{i=1}^3 \|B_{a_i,j}\|_{B(\ell_\nu^1, \ell_\nu^1)} \right) r \le Z_j^{(0)} r,$$

which bounds the first term of (4.9).

Next, we bound the quantities involved in the second term: Denote $b = (b_0, b_1, b_2, b_3) \in B(r) \subset \mathbb{R} \times (\ell_\nu^1)^3$. For $j = 0, 1, 2, 3$, let $z_j \overset{\text{def}}{=} \left([DF(\bar{x} + b) - A^\dagger]c \right)_j$ and set $z \overset{\text{def}}{=} (z_0, z_1, z_2, z_3)$. Now, since $m > k_0$, $z_0 = 0$. For $j = 1, 2, 3$,

$$\begin{aligned}
z_1 &= \{-\mathbf{i}(b_0(c_1)_k + c_0(b_1)_k)k\}_{k \in \mathbb{Z}} + \{\sigma((c_2)_k - (c_1)_k)\}_{|k| \ge m} \\
z_2 &= \{-\mathbf{i}(b_0(c_2)_k + c_0(b_2)_k)k - (b_1 * c_3)_k - (c_1 * b_3)_k\}_{k \in \mathbb{Z}} \\
&\quad + \{\rho(c_1)_k - (c_2)_k - (\bar{a}_1 * c_3)_k - (c_1 * \bar{a}_3)_k\}_{|k| \ge m} \\
&\quad + \left\{-(\bar{a}_1 * c_3^I)_k - (c_1^I * \bar{a}_3)_k\right\}_{|k| < m} \\
z_3 &= \{-\mathbf{i}(b_0(c_3)_k + c_0(b_3)_k)k + (b_1 * c_2)_k + (c_1 * b_2)_k\}_{k \in \mathbb{Z}} \\
&\quad + \{-\beta(c_3)_k + (\bar{a}_1 * c_2)_k + (c_1 * \bar{a}_2)_k\}_{|k| \ge m} \\
&\quad + \left\{(\bar{a}_1 * c_2^I)_k + (c_1^I * \bar{a}_2)_k\right\}_{|k| < m}.
\end{aligned}$$

The second term of (4.9) is $A[DF(\bar{x} + b) - A^\dagger]c = Az$ given component-wise by

$$\left(A[DF(\bar{x} + b) - A^\dagger]c \right)_j = (Az)_j = \sum_{i=1}^3 A_{a_i,j} z_i.$$

Consider $\tilde{b} = (\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$, $\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \in B(1)$ such that $b = \tilde{b}r$ and $c = \tilde{c}r$ for $r > 0$. For $j = 0$ and each $i = 1, 2, 3$, define the vector $\tilde{A}_{a_i,j} = \{k(A_{a_i,j})_k\}_{k \in \mathbb{Z}}$. We now construct an upper bound for $|(Az)_j|$ for the cases $j = 0, 1, 2, 3$.

Case 1: a bound on $|(Az)_0|$.

$$
\begin{aligned}
(Az)_0 &= \sum_{i=1}^{3} A_{a_i,0} z_i \\
&= \tilde{A}_{a_1,0} \left\{ -\mathbf{i}(\tilde{b}_0(\tilde{c}_1)_k + \tilde{c}_0(\tilde{b}_1)_k) \right\}_{k\in\mathbb{Z}} r^2 \\
&\quad + \tilde{A}_{a_2,0} \left\{ -\mathbf{i}(\tilde{b}_0(\tilde{c}_2)_k + \tilde{c}_0(\tilde{b}_2)_k) \right\}_{k\in\mathbb{Z}} r^2 - A_{a_2,0} \left\{ (\tilde{b}_1 * \tilde{c}_3)_k + (\tilde{c}_1 * \tilde{b}_3)_k \right\}_{k\in\mathbb{Z}} r^2 \\
&\quad + A_{a_2,0} \left\{ -(\bar{a}_1 * \tilde{c}_3^I)_k - (\tilde{c}_1^I * \bar{a}_3)_k \right\}_{|k|<m} r \\
&\quad + \tilde{A}_{a_3,0} \left\{ -\mathbf{i}(\tilde{b}_0(\tilde{c}_3)_k + \tilde{c}_0(\tilde{b}_3)_k) \right\}_{k\in\mathbb{Z}} r^2 + A_{a_3,0} \left\{ (\tilde{b}_1 * \tilde{c}_2)_k + (\tilde{c}_1 * \tilde{b}_2)_k \right\}_{k\in\mathbb{Z}} r^2 \\
&\quad + A_{a_3,0} \left\{ (\bar{a}_1 * \tilde{c}_2^I)_k + (\tilde{c}_1^I * \bar{a}_2)_k \right\}_{|k|<m} r.
\end{aligned}
$$

Note that $|\tilde{b}_0|, \|\tilde{b}_1\|_\nu, \|\tilde{b}_2\|_\nu, \|\tilde{b}_3\|_\nu, |\tilde{c}_0|, \|\tilde{c}_1\|_\nu, \|\tilde{c}_2\|_\nu, \|\tilde{c}_3\|_\nu \le 1$. Using Lemma 1, Corollary 3 and (2.4), we get that

$$
\begin{aligned}
\left| \sum_{i=1}^{3} A_{a_i,0} z_i \right| &\le 2 \left(\|\tilde{A}_{a_1,0}\|_\nu^\infty + \|\tilde{A}_{a_2,0}\|_\nu^\infty + \|A_{a_2,0}\|_\nu^\infty + \|\tilde{A}_{a_3,0}\|_\nu^\infty + \|A_{a_3,0}\|_\nu^\infty \right) r^2 \\
&\quad + \left(\sum_{|k|<m} |(A_{a_2,0})_k| \left(|\hat{l}_{\bar{a}_1}^k(\tilde{c}_3)| + |\hat{l}_{\bar{a}_3}^k(\tilde{c}_1)| \right) + \sum_{|k|<m} |(A_{a_3,0})_k| \left(|\hat{l}_{\bar{a}_1}^k(\tilde{c}_2)| + |\hat{l}_{\bar{a}_2}^k(\tilde{c}_1)| \right) \right) r \\
&\le Z_0^{(2)} r^2 + Z_0^{(1)} r \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.12) \\
&\stackrel{\text{def}}{=} 2 \left(\|\tilde{A}_{a_1,0}\|_\nu^\infty + \|\tilde{A}_{a_2,0}\|_\nu^\infty + \|A_{a_2,0}\|_\nu^\infty + \|\tilde{A}_{a_3,0}\|_\nu^\infty + \|A_{a_3,0}\|_\nu^\infty \right) r^2 \\
&\quad + \left(\sum_{|k|<m} |(A_{a_2,0})_k| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_3) \right) + |(A_{a_3,0})_k| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_2) \right) \right) r
\end{aligned}
$$

where we recall that Ψ_k is defined in (2.4).

Case 2: a bound on $\|(Az)_j\|_\nu$, $j = 1, 2, 3$.

For a fixed $j = 1, 2, 3$, one has that the expansion of $(Az)_j$ is given by

$$
\begin{aligned}
(Az)_j &= \sum_{i=1}^{3} A_{a_i,j} z_i \\
&= A_{a_1,j} \left\{ -\mathbf{i}(\tilde{b}_0(\tilde{c}_1)_k + \tilde{c}_0(\tilde{b}_1)_k)k \right\}_{k\in\mathbb{Z}} r^2 + A_{a_1,j} \left\{ \sigma((\tilde{c}_2)_k - (\tilde{c}_1)_k) \right\}_{|k|\ge m} r \\
&\quad + A_{a_2,j} \left\{ -\mathbf{i}(\tilde{b}_0(\tilde{c}_2)_k + \tilde{c}_0(\tilde{b}_2)_k)k \right\}_{k\in\mathbb{Z}} r^2 - A_{a_2,j} \left\{ (\tilde{b}_1 * \tilde{c}_3)_k + (\tilde{c}_1 * \tilde{b}_3)_k \right\}_{k\in\mathbb{Z}} r^2 \\
&\quad + A_{a_2,j} \left\{ \rho(\tilde{c}_1)_k - (\tilde{c}_2)_k - (\bar{a}_1 * \tilde{c}_3)_k - (\tilde{c}_1 * \bar{a}_3)_k \right\}_{|k|\ge m} r \\
&\quad + A_{a_2,j} \left\{ -(\bar{a}_1 * \tilde{c}_3^I)_k - (\tilde{c}_1^I * \bar{a}_3)_k \right\}_{|k|<m} r \\
&\quad + A_{a_3,j} \left\{ -\mathbf{i}(\tilde{b}_0(\tilde{c}_3)_k + \tilde{c}_0(\tilde{b}_3)_k)k \right\}_{k\in\mathbb{Z}} r^2 + A_{a_3,j} \left\{ (\tilde{b}_1 * \tilde{c}_2)_k + (\tilde{c}_1 * \tilde{b}_2)_k \right\}_{k\in\mathbb{Z}} r^2 \\
&\quad + A_{a_3,j} \left\{ -\beta(\tilde{c}_3)_k + (\bar{a}_1 * \tilde{c}_2)_k + (\tilde{c}_1 * \bar{a}_2)_k \right\}_{|k|\ge m} r \\
&\quad + A_{a_3,j} \left\{ (\bar{a}_1 * \tilde{c}_2^I)_k + (\tilde{c}_1^I * \bar{a}_2)_k \right\}_{|k|<m} r.
\end{aligned}
$$

Before bounding $|(Az)_j|$ for $j = 1, 2, 3$, we need the following useful result whose proof is a slight modification of Corollary 1.

Corollary 4. *For $i, j = 1, 2, 3$, let*

$$K_{a_{i,j}} \overset{\text{def}}{=} \max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |(A_{a_{i,j}})_{k,n}| \, \nu^{|k|}.$$

$$\tilde{K}_{a_{i,j}} \overset{\text{def}}{=} \max_{|n|<m} \frac{|n|}{\nu^{|n|}} \sum_{|k|<m} |(A_{a_{i,j}})_{k,n}| \, \nu^{|k|}.$$

Let $a = \{a_k\}_{k \in \mathbb{Z}} \in \ell_\nu^1$ and $\tilde{a} \overset{\text{def}}{=} \{k a_k\}_{k \in \mathbb{Z}} \in \ell_\nu^1$. Then

$$\|A_{a_{i,j}} a\|_\nu \leq \max\left\{ K_{a_{i,j}}, \frac{1}{\bar\omega m} \delta_{i,j} \right\} \quad \text{and} \quad \|A_{a_{i,j}} \tilde{a}\|_\nu \leq \max\left\{ \tilde{K}_{a_{i,j}}, \frac{1}{\bar\omega} \delta_{i,j} \right\}.$$

Using the previous result, we obtain that

$$\left\| \sum_{i=1}^{3} A_{a_i,1} z_i \right\|_\nu \leq Z_1^{(2)} r^2 + Z_1^{(1)} r \tag{4.13}$$

$$\overset{\text{def}}{=} 2 \left(\max\left\{ \tilde{K}_{a_1,1}, \frac{1}{\bar\omega} \right\} + \tilde{K}_{a_2,1} + \tilde{K}_{a_3,1} + K_{a_2,1} + K_{a_3,1} \right) r^2$$

$$+ \left[\frac{2|\sigma|}{\bar\omega m} + \sum_{|n|<m} \sum_{|k|<m} |(A_{a_2,1})_{n,k}| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_3) \right) \right.$$

$$\left. + \sum_{|n|<m} \sum_{|k|<m} |(A_{a_3,1})_{n,k}| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_2) \right) \right] r,$$

$$\left\| \sum_{i=1}^{3} A_{a_i,2} z_i \right\|_\nu \leq Z_2^{(2)} r^2 + Z_2^{(1)} r \tag{4.14}$$

$$\overset{\text{def}}{=} 2 \left(\tilde{K}_{a_1,2} + \max\left\{ \tilde{K}_{a_2,2}, \frac{1}{\bar\omega} \right\} + \tilde{K}_{a_3,2} + \max\left\{ K_{a_2,2}, \frac{1}{\bar\omega m} \right\} + K_{a_3,2} \right) r^2$$

$$+ \left[\frac{|\rho| + 1 + \|\bar{a}_3\|_\nu + \|\bar{a}_1\|_\nu}{\bar\omega m} + \sum_{|n|<m} \sum_{|k|<m} |(A_{a_2,2})_{n,k}| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_3) \right) \right.$$

$$\left. + \sum_{|n|<m} \sum_{|k|<m} |(A_{a_3,2})_{n,k}| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_2) \right) \right] r,$$

and

$$\left\| \sum_{i=1}^{3} A_{a_i,3} z_i \right\|_\nu \leq Z_3^{(2)} r^2 + Z_3^{(1)} r \tag{4.15}$$

$$\overset{\text{def}}{=} 2 \left(\tilde{K}_{a_1,3} + \tilde{K}_{a_2,3} + \max\left\{ \tilde{K}_{a_3,3}, \frac{1}{\bar\omega} \right\} + K_{a_2,3} + \max\left\{ K_{a_3,3}, \frac{1}{\bar\omega m} \right\} \right) r^2$$

$$+ \left[\sum_{|n|<m} \sum_{|k|<m} |(A_{a_2,3})_{n,k}| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_3) \right) + \frac{|\beta| + \|\bar{a}_1\|_\nu + \|\bar{a}_2\|_\nu}{\omega m} \right.$$

$$\left. + \sum_{|n|<m} \sum_{|k|<m} |(A_{a_3,3})_{n,k}| \left(\Psi_k(\bar{a}_1) + \Psi_k(\bar{a}_2) \right) \right] r.$$

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Recall (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15), let

$$Z_j(r) \quad \stackrel{\text{def}}{=} \quad (Z_j^{(0)} + Z_j^{(1)})r + Z_j^{(2)}r^2, \quad \text{for } j = 0, 1, 2, 3. \tag{4.16}$$

Using (4.5), (4.6), (4.7), (4.8) and (4.16), we define the radii polynomials by

$$p_j(r) \stackrel{\text{def}}{=} Z_j(r) - r + Y_j, \quad \text{for } j = 0, 1, 2, 3. \tag{4.17}$$

4.2   Validated numerics in Lorenz

Using a numerical continuation method based on a predictor corrector algorithm, we computed a branch of periodic orbits for the Lorenz equations (1.3). We single out the approximate periodic orbits at $\rho = 24.6815$, $\rho = 18.0815$ and $\rho = 13.92657$ for validation. For all computations, we fixed $k_0 = 10$ in the phase condition (4.1). Hence, the minimum number of Fourier modes for the proofs is $m = 11$. We used a computer program in MATLAB with the interval arithmetic library INTLAB [37] to compute the coefficients $p(r, 1)$ as defined in (4.17). We constructed $\mathcal{I}(1) \neq \emptyset$ as defined in (3.9). Then following the idea of Remark 3, we used a bisection algorithm to find the maximal $\nu = \nu_{\max}$ for which $\mathcal{I}(\nu_{\max}) \neq \emptyset$. We could therefore maximize the lower bound on the domain of analyticity of the periodic solutions. At $\rho = 24.6815$ with $m = 60$, we obtained $\nu_{\max} = 12.59$ and that the width of the domain of analyticity is at least 0.26394. At $\rho = 18.0815$ with $m = 60$, we obtained $\nu_{\max} = 2.36$ and that the width of the domain of analyticity is at least 0.14047. At $\rho = 13.92657$ with $m = 320$, we obtained $\nu_{\max} = 1.027$ and that the width of the domain of analyticity is at least 0.024662.

ρ	ν	m	$\mathcal{I}(\nu)$		running time (in secs)
24.6815	1	11	$[1.4637 \times 10^{-11}$	$6.2662 \times 10^{-4}]$	0.462523
24.6815	1	60	$[7.8346 \times 10^{-12}$	$1.1684 \times 10^{-3}]$	2.3219878
24.6815	5	60	$[5.4680 \times 10^{-9}$	$9.4860 \times 10^{-4}]$	2.180022
24.6815	10.21	11	$[8.0895 \times 10^{-5}$	$9.9004 \times 10^{-5}]$	0.462796
24.6815	12.19	30	$[2.1602 \times 10^{-4}$	$2.4747 \times 10^{-4}]$	1.036563
24.6815	12.59	60	$[2.5544 \times 10^{-4}$	$2.9000 \times 10^{-4}]$	2.229176
18.0815	1	15	$[1.3606 \times 10^{-5}$	$9.1574 \times 10^{-3}]$	0.564342
18.0815	1	60	$[1.3184 \times 10^{-12}$	$7.5540 \times 10^{-2}]$	2.177668
18.0815	2.26	33	$[1.2138 \times 10^{-3}$	$1.9007 \times 10^{-3}]$	1.216458
18.0815	2.36	60	$[2.3669 \times 10^{-3}$	$3.7767 \times 10^{-3}]$	2.361393
13.92657	1	180	$[2.1441 \times 10^{-7}$	$2.5715 \times 10^{-7}]$	12.755006
13.92657	1.019	240	$[2.4350 \times 10^{-7}$	$2.6216 \times 10^{-7}]$	24.489942
13.92657	1.027	320	$[2.3267 \times 10^{-7}$	$2.9420 \times 10^{-7}]$	46.63911

Figure 3: Different data for the proofs of three periodic orbits in Lorenz. The orbit associated with the data in the last line of the table is illustrated in Figure 1. Also note that in all cases recorded here we obtain the best error bounds when $\nu = 1$, but increasing ν gives bounds on the decay rate of the Fourier coefficients/width of the domain. The last column shows the running times for each proof. All computations were run on a 2011 MacBook Air with processor 1.7 GHz Intel Core i5 with 4GB of memory.

5 Equilibria of the Swift-Hohenberg PDE

Recalling the Swift-Hohenberg PDE (1.4), we note that the solutions can be expressed via the Fourier expansion

$$u(y,t) = \sum_{k=-\infty}^{\infty} a_k(t)e^{ikLy} = a_0 + 2\sum_{k=1}^{\infty} a_k(t)\cos(kLy), \tag{5.1}$$

where $a_k \in \mathbb{R}$ and $a_{-k} = a_k$. Plugging (5.1) in (1.4) results in the infinite set of ODEs given by

$$\dot{a}_k = F_k(a) \stackrel{\text{def}}{=} \mu_k a_k - \sum_{k_1+k_2+k_3=k} a_{k_1}a_{k_2}a_{k_3}, \tag{5.2}$$

where

$$\mu_k \stackrel{\text{def}}{=} \lambda - \left(1 - k^2 L^2\right)^2 \tag{5.3}$$

is the eigenvalue of the linear part of (1.4). Since $a_{-k} = a_k$, then $F_{-k} = F_k$. This implies that one can consider only the variables a_k for $k \geq 0$ and the functions F_k for $k \geq 0$. This is the reason why in this case we are going to use the one-sided sequences ℓ_ν^1. Note that looking for equilibria of the Swift-Hohenberg PDE (1.4) is equivalent to compute solutions of $F(a) = 0$ in ℓ_ν^1, for some $\nu > 1$ small enough.

We fix $L = 0.65$ and leave λ as a continuation parameter. Equilibria $u = u(y)$ of (1.4) correspond to solutions of $F(a) = 0$, where $a = (a_k)_{k\geq0}$ is the infinite sequence of Fourier coefficients and $F = (F_k)_{k\geq0}$ is given component-wise by (5.2). In this case, the expansion (5.1) reads as

$$u(y) = \sum_{k\in\mathbb{Z}} a_k \cos(kLy) = a_0 + 2\sum_{k\geq1} a_k \cos(kLy). \tag{5.4}$$

Since there is a natural symmetry $a_{-k} = a_k$ in the Fourier coefficients of u, we slightly adjust the definition of the space as follows: given an infinite sequence $a = (a_k)_{k\geq0}$, define

$$\|a\|_\nu \stackrel{\text{def}}{=} \sum_{k\geq0} |a_k|\nu^k$$

and the function space consisting of one-sided sequences

$$\ell_\nu^1 = \left\{a = (a_k)_{k\geq0} \ : \ \|a\|_\nu < \infty\right\}.$$

Endow the one-sided ℓ_ν^1 with the following *extended* discrete convolution product: given $a = (a_k)_{k\geq0}, b = (b_k)_{k\geq0} \in \ell_\nu^1$, extend them with the symmetry $a_{-k} = a_k$ and $b_{-k} = b_k$, and define $a * b$ component-wise by

$$(a * b)_k \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} a_{k_1}b_{k_2}, \quad k \geq 0.$$

Then

$$\|a * b\|_\nu \leq 4\|a\|_\nu\|b\|_\nu, \tag{5.5}$$

as can be checked by direct computation.

Given an infinite dimensional vector $v = (v_k)_{k\geq0}$, denote $v_F = (v_0, v_1, \ldots, v_{m-1}) \in \mathbb{R}^m$ its finite dimensional projection. A Galerkin projection $F^{(m)} : \mathbb{R}^m \to \mathbb{R}^m$ is defined

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by $F^{(m)}(a_F) = (F(a_F, 0))_F$. Assume that using Newton's method, we computed a solution $\bar{a} = (\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{m-1})$ such that $F^{(m)}(\bar{a}) \approx 0$. Consider A_m computed so that $A_m \approx \left(DF^{(m)}(\bar{a})\right)^{-1}$ and assume that A_m is invertible. Recalling (5.3), set

$$
A \stackrel{\text{def}}{=}
\begin{bmatrix}
A_m & & 0 & \\
& \mu_m^{-1} & & \\
0 & & \mu_{m+1}^{-1} & \\
& & & \ddots
\end{bmatrix}.
$$

Take the Galerkin projection dimension m large enough so that $|\mu_k| \geq |\mu_m|$ for all $k \geq m$. Recalling (2.3), set

$$
K \stackrel{\text{def}}{=} \max_{0 \leq n \leq m-1} \frac{1}{\nu^n} \sum_{\ell=0}^{m-1} |(A_m)_{\ell,n}| \nu^\ell,
$$

and define

$$
\alpha_\nu = \max(K, \frac{1}{\mu_m}). \tag{5.6}
$$

Then, by Corollary 1, we have that

$$
\|A\|_{B(\ell_\nu^1, \ell_\nu^1)} \leq \alpha_\nu. \tag{5.7}
$$

Proposition 5. *Define the Newton-like operator by*

$$
T(a) = a - AF(a). \tag{5.8}
$$

Then $T : \ell_\nu^1 \to \ell_\nu^1$.

We omit the elementary proof.

As explained in Section 3, the goal is to demonstrate that nearby the approximate solution $\bar{a}$, there exists an exact solution of $F(a) = 0$, with F given component-wise by (5.2). In this case, F is defined on the function space $X = X_\nu^{0,1} = \ell_\nu^1$, hence there will be only one radii polynomial. This radii polynomial is denoted here $p(r)$ and it corresponds to (3.8) for the case $j = j_1 + 1 = 1$ since here $j_1 = 0$. Once $p(r)$ is constructed, we use Proposition 2 and attempt to construct an interval $\mathcal{I} \neq \emptyset$ to conclude that for any $r \in \mathcal{I}$, there exists a unique $\tilde{a} \in B_{\bar{a}}(r)$ such that $F(\tilde{a}) = 0$. We now present the explicit construction of the radii polynomial.

5.1 Computation of the radii polynomial

Let us now compute the bounds Y and Z in the context of the equilibria of the Swift-Hohenberg PDE (1.4). Recall that by considering a Galerkin projection $F^{(m)} : \mathbb{R}^m \to \mathbb{R}^m$, one computed a solution $\bar{a} = (\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{m-1})$ such that $F^{(m)}(\bar{a}) \approx 0$. Hence, consider y_k such that

$$
|(T(\bar{a}) - \bar{a})_k| = |(-AF(\bar{a}))_k| \leq y_k. \tag{5.9}
$$

Compute $y_F = (y_0, y_1, \ldots, y_{m-1})^T$ with the formula

$$
y_F = |A_m F^{(m)}(\bar{a})|. \tag{5.10}
$$

Since $\bar{a}_k = 0$ for all $k \geq m$, then $F_k(\bar{a}) = \mu_k \bar{a}_k - (\bar{a} * \bar{a} * \bar{a})_k = 0$ for all $k \geq 3m - 2$. For $k = m, \ldots, 3m - 3$, set

$$
y_k = |-\frac{1}{\mu_k} F_k(\bar{a})| = \frac{1}{|\mu_k|} |(\bar{a} * \bar{a} * \bar{a})_k|. \tag{5.11}
$$

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Using (5.10) and (5.11), one may obtain

$$Y = \|y\|_\nu = \sum_{k=0}^{m-1} \left| [A_m F^{(m)}(\bar{a})]_k \right| \nu^k + \sum_{k=m}^{3m-3} \frac{1}{|\mu_k|} \left| (\bar{a} * \bar{a} * \bar{a})_k \right| \nu^k. \qquad (5.12)$$

To simplify the computation of the bounds $Z_k(r)$ define the operator

$$A^\dagger \overset{\text{def}}{=} \begin{bmatrix} DF^{(m)}(\bar{a}) & & & 0 \\ & \mu_m & & \\ 0 & & \mu_{m+1} & \\ & & & \ddots \end{bmatrix}.$$

Considering $b, c \in B(r)$ and recalling the definition of the Newton-like operator (5.8), notice that

$$DT(\bar{a} + b)c = [I - ADF(\bar{a} + b)]c = [I - AA^\dagger]c - A[DF(\bar{a} + b)c - A^\dagger c]. \qquad (5.13)$$

We now bound the ν−norm of each of the terms in the right hand side of (5.13). Consider $u, v \in B(1)$ such that $b = ur$ and $c = vr$. Let

$$Z^{(0)} \overset{\text{def}}{=} \max_{0 \le n \le m-1} \frac{1}{\nu^n} \sum_{\ell=0}^{m-1} \left| \left(I_m - A_m DF^{(m)}(\bar{a}) \right)_{\ell, n} \right| \nu^\ell. \qquad (5.14)$$

By definition of the diagonal tails of A and $A^\dagger$, the diagonal tail of $I - AA^\dagger$ is zero. Hence, by Corollary 1

$$\|[I - AA^\dagger]c\|_\nu \le Z^{(0)} r.$$

The next step is to bound the term $\| - A[DF(\bar{x} + b)c - A^\dagger c]\|_\nu$. For this, notice that for $k = 0, \ldots, m - 1$,

$$
\begin{aligned}
[DF(\bar{a} + b)c - A^\dagger c]_k &= \mu_k c_k - 3[(\bar{a} + b)^2 c]_k - [DF^{(m)}(\bar{a})c_F]_k \\
&= \mu_k c_k - 3[(\bar{a} + b)^2 c]_k - (\mu_k c_k - 3[\bar{a}^2 c_F]_k) \\
&= -3[\bar{a}^2 c_I]_k - 6[\bar{a}bc]_k - 3[b^2 c]_k \\
&= \left(-3[\bar{a}^2 v_I]_k\right) r + \left(-6[\bar{a}uv]_k\right) r^2 + \left(-3[u^2 v]_k\right) r^3,
\end{aligned}
$$

where

$$
\begin{aligned}
v_I &\overset{\text{def}}{=} (0, 0, \ldots, 0, v_m, v_{m+1}, \ldots) \\
[\bar{a}^2 v_I]_k &= \sum_{\substack{k_1 + k_2 + k_3 = k \\ |k_3| \ge m}} \bar{a}_{k_1} \bar{a}_{k_2} v_{k_3} \\
[\bar{a}uv]_k &= \sum_{k_1 + k_2 + k_3 = k} \bar{a}_{k_1} u_{k_2} v_{k_3} \\
[u^2 v]_k &= \sum_{k_1 + k_2 + k_3 = k} u_{k_1} u_{k_2} v_{k_3}.
\end{aligned}
$$

Similarly, for $k \ge m$,

$$
\begin{aligned}
[DF(\bar{a} + b)c - A^\dagger c]_k &= \mu_k c_k - 3[(\bar{a} + b)^2 c]_k - \mu_k c_k \\
&= \left(-3[\bar{a}^2 v]_k\right) r + \left(-6[\bar{a}uv]_k\right) r^2 + \left(-3[u^2 v]_k\right) r^3.
\end{aligned}
$$

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Using (5.5) and (5.7),

$$
\begin{aligned}
\|A[DF(\bar{a}+b)c - A^\dagger c]\|_\nu &= \sum_{k=0}^{m-1} \left|\left(A_m[(-3[\bar{a}^2 v_I]_F)\, r + (-6[\bar{a}uv]_F)\, r^2 + (-3[u^2 v]_F)\, r^3])_k\right| \nu^k \right. \\
&\quad + \sum_{k \geq m} \left|\frac{1}{\mu_k}\left(-3[\bar{a}^2 v]_k r - 6[\bar{a}uv]_k r^2 - 3[u^2 v]_k r^3\right)\right| \nu^k \\
&\leq \left(3\sum_{k=0}^{m-1}\left|\left(|A_m|[|\bar{a}|^2|v_I|]_F\right)_k\right|\nu^k + \frac{3}{|\mu_m|}\sum_{k\geq m}\left|[\bar{a}^2 v]_k\right|\nu^k\right) r \\
&\quad + 6\|A\|_{B(\ell^1_\nu, \ell^1_\nu)}\|\bar{a}*u*v\|_\nu r^2 + 3\|A\|_{B(\ell^1_\nu, \ell^1_\nu)}\|u^2 * v\|_\nu r^3 \\
&\leq Z^{(1)} r + 6\cdot 16\alpha_\nu\|\bar{a}\|_\nu r^2 + 3\cdot 16\alpha_\nu r^3,
\end{aligned}
$$

where the bound $Z^{(1)}$ can be obtained using Lemma 1. However, such bounds can be computationally expensive. One way to circumvent this issue is to use the following coarser (yet faster to compute!) bound.

$$
\sup_{\|v\|_\nu \leq 1}\left(3\sum_{k=0}^{m-1}\left|\left(|A_m|[|\bar{a}|^2|v_I|]_F\right)_k\right|\nu^k + \frac{3}{|\mu_m|}\sum_{k\geq m}\left|[\bar{a}^2 v]_k\right|\nu^k\right)
$$

$$
\leq 3\sum_{k=0}^{m-1}\left|\left(|A_m|[|\bar{a}|^2\tilde{\omega}]_F\right)_k\right|\nu^k + \frac{48\|\bar{a}\|_\nu^2}{|\mu_m|},
$$

where
$$
\tilde{\omega} \overset{\text{def}}{=} (0, 0, \ldots, 0, \nu^{-m}, \nu^{-(m+1)}, \ldots, \nu^{-(3m-3)}).
$$

The above bound is the worst case scenario, in the sense that each component $k \geq m$ of v_I is replaced by $\frac{1}{\nu^k}$. But this bound is much faster to compute as it requires the evaluation of only one convolution term.

Recall the definition of α_ν in (5.6) and set

$$
Z^{(1)} \overset{\text{def}}{=} 3\sum_{k=0}^{m-1}\left|\left(|A_m|[|\bar{a}|^2\tilde{\omega}]_F\right)_k\right|\nu^k + \frac{48\|\bar{a}\|_\nu^2}{|\mu_m|}, \tag{5.15}
$$

$$
Z^{(2)} \overset{\text{def}}{=} 96\alpha_\nu\|\bar{a}\|_\nu, \tag{5.16}
$$

$$
Z^{(3)} \overset{\text{def}}{=} 48\alpha_\nu. \tag{5.17}
$$

Combining (5.14), (5.15), (5.16) and (5.17), we set

$$
Z(r) \overset{\text{def}}{=} Z^{(3)}r^3 + Z^{(2)}r^2 + (Z^{(1)} + Z^{(0)})r. \tag{5.18}
$$

Finally combining (5.12) and (5.18), one can define the radii polynomial by

$$
p(r, \nu) = Z(r) - r + Y. \tag{5.19}
$$

Next, we show some results about rigorous computations of equilibria of (1.4) using the radii polynomial (5.19) and Proposition 2.

5.2 Validated numerics for equilibria of Swift-Hohenberg: existence, isolation, and domain of analyticity

Recalling the Swift-Hohenberg PDE (1.4), we fix the fundamental wave number $L = 0.65$ for the system size $\frac{2\pi}{L}$. As mentioned in the introduction there is a pitchfork bifurcation from $u \equiv 0$ at $\lambda = \left(1 - 4L^2\right)^2$ which corresponds to the solution $\cos(2Ly)$. Using a numerical continuation method based on a predictor corrector algorithm we computed numerical approximations for a long branch of equilibria and single out the parameter values of $\lambda = 1$, $\lambda = 10$ and $\lambda = 3.5 \times 10^8$ for rigorous validation. Then, we used a computer program in MATLAB to compute with interval arithmetics (again using INT-LAB) the coefficients $p(r, 1)$ as defined in (5.19). We constructed $\mathcal{I}(1) \neq \emptyset$ as defined in (3.9). Then following the idea of Remark 3, we used a bisection algorithm to find the maximal $\nu = \nu_{\max}$ for which $\mathcal{I}(\nu_{\max}) \neq \emptyset$. We could therefore maximize the lower bound on the domain of analyticity of the spatially periodic solutions. At $\lambda = 1$, we obtained $\nu_{\max} = 2.249$ so that the function is analytic on a strip of width at least 1.2469. At $\lambda = 10$, we obtained $\nu_{\max} = 1.584$ and the width of the strip is at least 0.70762. At $\lambda = 3.5 \times 10^8$, we obtained $\nu_{\max} = 1.003$ and the width of the strip is at least 0.0046085.

λ	ν	m	$\mathcal{I}(\nu)$	running time (in secs)
1	1	18	$[2.9972 \times 10^{-13} \quad 0.0039689]$	15.038808
1	2.249	18	$[0.00032037 \quad 0.00040009]$	15.680303
10	1	31	$[5.4594 \times 10^{-12} \quad 6.5776 \times 10^{-3}]$	28.244303
10	1.584	31	$[0.00053541 \quad 0.00079332]$	27.836866
3.5×10^8	1	2103	$[0.00026536 \quad 1.1047]$	1959.475159
3.5×10^8	1.003	2103	$[0.0019089 \quad 0.67048]$	1948.385623

Figure 4: Different data for the proofs. The last column shows the running times for each proof. All computations were run on a 2011 MacBook Air with processor 1.7 GHz Intel Core i5 with 4GB of memory.

6 Conclusion

We have presented a method for studying analytic solutions of differential equations by computer assisted means. The present work focuses on periodic problems. Our implementation exploits the radii polynomials so that rigorous bounds on truncation errors and information about the isolation of the solutions are obtained. The continuity of the radii polynomials facilitates

- Implementation of bisection algorithms for determining lower bounds on the domain of analyticity of the solution, and

- The use of the numerically most numerically stable norm (i.e. the classical space of absolutely summable Fourier series; $\nu = 1$). The continuity of the radii polynomials then implies that there exists a complex strip onto which the function can be continued analytically.

The continuity of the radii polynomials could also be exploited in order to compute branches (even multi-parameter branches) of analytic solutions. Other interesting future projects could be to apply the techniques of the present work to delay equations, equilibria solutions of higher dimensional partial differential equations, and to the study of

periodic solutions of partial differential equations. Problems which have all been studied successfully using the $C^{\mathbf{k}}$ approach.

Another interesting line of future research will be to apply the functional analytic approach in conjunction with the method of radii polynomials in order to study problems which are fundamentally $C^{\mathbf{k}}$. So while it is intellectually interesting to compare the $C^{\mathbf{k}}$ approach to the analytic approach in problems where both apply, a much more interesting problem is to consider the performance of the $C^{\mathbf{k}}$ approach in problems where the analytic approach must fail. In order to make the discussion more concrete consider the following spatially inhomogeneous version of Fisher's Equation

$$u_t = u_{xx} + \mu u(1 - gu), \tag{6.1}$$

subject to Neumann boundary conditions. Here $g(x)$ is a function of the spatial variable. In a cosine basis the problem becomes

$$a'_n = (\mu - n^2)a_n - (c * a * a)_n, \qquad n \geq 0,$$

where $a = \{a_n\}_{n=0}^{\infty}$ are the cosine series coefficients of u and $c = \{c_n\}_{n=0}^{\infty}$ are the cosine series coefficients of g. If g is analytic and satisfies the boundary conditions then the methods of the present work apply directly to compute equilibria of (6.1).

On the other hand suppose that the function g is $C^{\mathbf{k}}$ but not analytic, for example g might be piecewise polynomial (a spline). Then the ℓ_ν^1 norm of $\{c_n\}_{n=0}^{\infty}$ is infinite for any choice $\nu > 1$ and the analytic tools discussed above are not appropriate. On the other hand existing $C^{\mathbf{k}}$ tools, which have thus far always been applied for a-priori analytic problems, are available to study such a $C^{\mathbf{k}}$ problem. The change in category would effect the convergence of the existing methods, and one expects that the numerical portion of the proofs will be more difficult (require more modes, etc). Nevertheless it would be interesting to compare the performance of the $C^{\mathbf{k}}$ methods for problems with varying degrees of regularity. This work is currently under progress and will be the subject of a forthcoming manuscript.

7 Acknowlegments

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