Blow-up profile for solutions of a fourth order nonlinear equation

L. D'Ambrosio*[1], J.-P. Lessard†[2], and A. Pugliese‡[1]

[1]Dipartimento di Matematica, Università di Bari, via E. Orabona, 4, I-70125 Bari - Italy
[2]Université Laval, Département de Mathématiques et de Statistique, 1045 avenue de la Médecine, Québec, QC, G1V 0A6, Canada.

December 10, 2014

Dedicated to Enzo Mitidieri
with friendship and esteem

Abstract

It is well known that the nontrivial solutions of the equation

$$u''''(r) + \kappa u''(r) + f(u(r)) = 0$$

blow up in finite time under suitable hypotheses on the initial data, κ and f. These solutions blow up with large oscillations. Knowledge of the blow-up profile of these solutions is of great importance, for instance, in studying the dynamics of suspension bridges. The equation is also commonly referred to as extended Fisher-Kolmogorov equation or Swift-Hohenberg equation.

In this paper we provide details of the blow-up profile. The key idea is to relate this blow-up profile to the existence of periodic solutions for an auxiliary equation.

Keywords: Suspension bridges, Fisher-Kolmogorov, Swift-Hohenberg, blow-up profile, Computer assisted proof

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*email: lorenzo.dambrosio@uniba.it
†email: jean-philippe.lessard@mat.ulaval.ca
‡email: alessandro.pugliese@uniba.it

1 Introduction

This paper is devoted to the study of the blow-up profile of solutions of

$$u''''(r) + \kappa u''(r) + f(u(r)) = 0\,, \tag{1.1}$$

where $\kappa \in \mathbb{R}$ and f is a locally Lipschitz nonlinear function. Equation (1.1) arises in several contexts. Depending on the value of κ and on the form of f, many authors refer to (1.1) as extended Fisher-Kolmogorov equation or Swift-Hohenberg equation. This equation serves as a model for pattern formation in many physical, chemical or biological systems, and is often used to investigate localization and spreading of deformation of a strut confined by an elastic foundation. We refer to the book by Peletier and Troy [18] for more applications of (1.1), and for further references. Equation (1.1) is also connected to the dynamics of suspension bridges through a model proposed by Lazer and McKenna [16]. For further details and references on this subject, we also refer to the works [11, 12, 2, 1] which contain a new point of view on the mathematical explanation of instability (and collapse) of suspension bridges.

Much attention has received the case of the nonlinearity being superlinear, namely

$$\text{there exist } q > 1 \text{ and } c > 0 \text{ such that } f(t)t \geq c\,|t|^{q+1} \text{ for any } t \in \mathbb{R}\,, \tag{1.2}$$

and, in particular, the case

$$f(t) = \mu t + |t|^{q-1} t\,,$$

with $\mu \in \mathbb{R}$. In [7], the authors prove the following result.

Theorem 1.1 ([7]). *Let $q > 1$ and u be a global solution of*

$$uu''''(r) + |u(r)|^{q+1} \geq 0, \;\; r \in \mathbb{R}\,,$$

then $u \equiv 0$.

The previous theorem implies that, if $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ satisfies (1.2) and u solves

$$u''''(r) + f(u(r)) = 0 \tag{1.3}$$

with non trivial initial conditions, then u cannot exist on the whole real line, that is u blows up in finite (forward and/or backward) time.

In [3], the authors prove that:

Theorem 1.2 ([3]). *Let u be a solution of (1.1) which blows up at time $T > 0$. Then*

$$\liminf_{r\to T} u(r) = -\infty, \quad \limsup_{r\to T} u(r) = +\infty.$$

Much effort has been devoted to understanding in detail the behavior of these widely oscillating solutions, for instance the distance between consecutive zeros, the distance between consecutive local extrema as well as the values attained at those extrema, e.g. see [11, 12].

The main purpose of this paper is to draft a global picture of the behavior of solutions of (1.1) by investigating their blow-up profile. We will preliminarily focus our attention on a prototype case, namely

$$u''''(r) + |u(r)|^{q-1}\, u(r) = 0. \tag{$\mathbf{U_q}$}$$

We will give several general results on solutions of ($\mathbf{U_q}$) for all $q > 1$, but will be able to obtain the strongest results by restricting ourselves to some special values of q. We believe that the reason for this disparity is solely technical, and that the general picture is much the same for all values of $q > 1$. This claim will be supported by numerical evidence, see Section 6.

Here below we disclose, in a simplified form, one of the main results of this paper.

Theorem 1.3. *There exist a τ-periodic function $\gamma : \mathbb{R} \to \mathbb{R}$, a set $\Omega \subset \mathbb{R}^4$ unbounded, arc-connected, symmetric with respect to the origin, with non-empty interior, and a constant $a > 0$, such that, for any solution u of*

$$u''''(r) + u^3(r) = 0 \tag{1.4}$$

with initial condition in Ω, we have, up to a phase-shift of γ,

$$\left| u(r) - \frac{1}{(T-r)^2}\,\gamma\left(\ln\left(\frac{T}{T-r}\right)\right)\right| < c(T-r)^a, \quad \textit{for all } r \in [0,T)\,,$$

for some $T, c > 0$ that depend on the initial condition. Moreover, the function γ satisfies:

i) $\gamma(s + \tau/2) = -\gamma(s)$, *for all $s \in \mathbb{R}$;*

ii) γ *vanishes exactly twice on* $[0,\tau)$.

Details on how the set Ω and the periodic function γ are constructed, information on how the blow-up time T depends on the initial conditions, as well as tight estimates on τ and a, will be given in Section 2 and Section 3, whereas an extensive version of Theorem 1.3 will be given in Section 3. Before then, we want to emphasize that Theorem 1.3 gives insight into the nature of the qualitative behavior predicted by Theorem 1.2 for a (fairly large) family of solutions of (1.4), see Figure 1.

Further refinements will allow us to extend the results of Theorem 1.3 to a wider class of equations. For instance, via a continuation argument which is presented in full generality in the Appendix, we are able to extend the validity of Theorem 1.3 to values q in a neighborhood of 3. Moreover, we will also prove the following two results.

Theorem 1.4. *There exist* $\mu_0 > 0$ *and* $\Omega \subset \mathbb{R}^4$ *open set such that, if* u *is a solution of*

$$u''''(r) + \mu u(r) + u^3(r) = 0 \tag{1.5}$$

with initial condition in Ω *and* $|\mu| < \mu_0$, *then the conclusions of Theorem 1.3 hold.*

Theorem 1.5. *There exist a* τ*-periodic function* $\gamma : \mathbb{R} \to \mathbb{R}$, $\kappa_0 > 0$, *an open set* $\Omega \subset \mathbb{R}^4$ *and* $a > 0$ *such that, for any solution* u *of*

$$u''''(r) + \kappa u''(r) + u^3(r) = 0 \tag{1.6}$$

with initial condition in Ω *and* $|\kappa| < \kappa_0$, *we have that* u *blows up at* $T > 0$ *and up to a phase-shift of* γ,

$$\left| (T-r)^2 u(r) - \gamma\left(\ln\left(\frac{T}{T-r}\right)\right)\right| < c(T-r)^a, \quad \textit{for all } r \in [0,T)\,,$$

for some $T, c > 0$ *that depend on the initial condition. Moreover, the function* γ *satisfies i) and ii) of Theorem 1.3.*

The impact of this results is connected to the simplicity of the equations under scrutiny, and more important to their implications for the study of stability of suspension bridges. In fact, solutions u of (1.1) represent the vertical displacement of the bridge. Our results provide a precise blow-up rate for a "measure" of vertical displacement as well as vertical acceleration of the bridge, see Proposition 3.7. Again, we refer the interested reader to the papers [11, 12] for a detailed description of this phenomenon. Furthermore, Theorem 1.5 gives a partial answer to some questions posed in [12] about the behavior of solutions of (1.6) for $\kappa > 0$, see Section 5. In this direction see also the recent papers [10, 19, 8].

Before diving into the details, we want to present the general strategy behind the proof of our results. The main idea lies in the following ansatz:

$$u \text{ solves } (\mathbf{U_q}) \text{ and blows up at } T \;\Rightarrow\; u \text{ has the form } u(r) = \frac{c}{(T-r)^\eta} w(\varphi(r)) \tag{1.7}$$

where φ is a suitable change of variable and w is a function defined on $[0+\infty)$. Ideally, the purpose of this transformation is to damp the oscillations in u and scale T to $+\infty$, see Figure 1. Much effort will be devoted to studying the differential equation satisfied

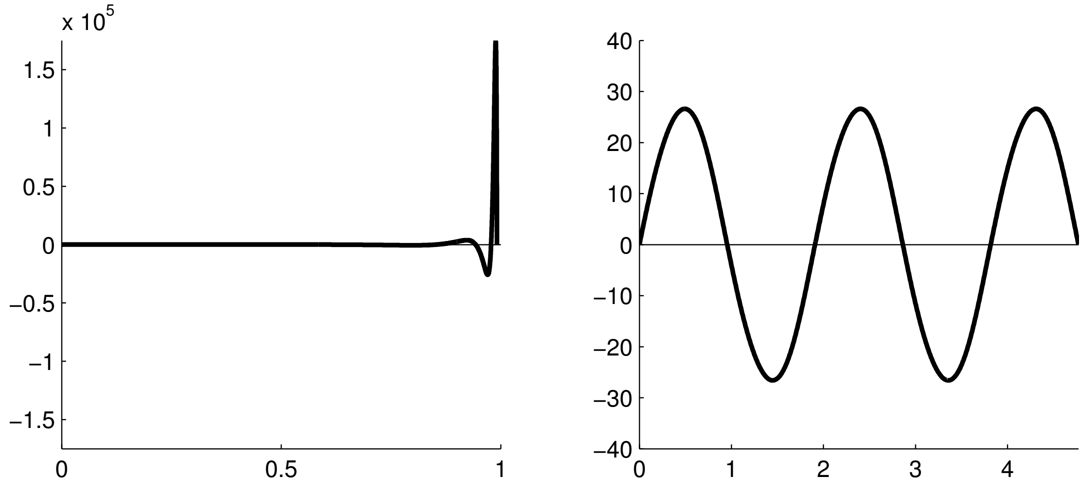

Figure 1: On the left a solution u of $(\mathbf{U_q})$ that blows up at finite time $T = 1$, on the right the function w obtained by appropriately re-scaling/damping u as indicated in (1.7).

by w, that will be referred to as the "auxiliary equation", and labelled as $(\mathbf{W_q})$. Note that knowledge about the existence of non-trivial bounded solutions for $(\mathbf{W_q})$ immediately translates into a picture of the blow-up behavior for solutions u of $(\mathbf{U_q})$. In fact, we will prove that, for q in a neighborhood of 3, $(\mathbf{W_q})$ admits a periodic solution. We also wish to point out that there is actually a strong interplay between equations $(\mathbf{U_q})$ and $(\mathbf{W_q})$ so that we often resorted to one in order to obtain information on the other.

Besides playing a key role in the theoretical analysis of this paper, the transformation (1.7) turns out to be of great help also in the numerical investigation of equation $(\mathbf{U_q})$, as it renders a problem which is generally much easier to treat numerically. We will elaborate more on this matter in Section 6.

The proof of existence of a periodic solution requires several distinct steps. The first step is to setup an equivalent formulation of the form $F(x) = 0$ (where $F : X \to Y$ with X and Y two infinite dimensional Banach spaces) whose solution $x \in X$ corresponds to the targeted periodic solution. Setting up the operator F requires expanding the solution using Fourier series. A point $x \in X$ identifies the period and the Fourier coefficients of the periodic solution. The next step is to consider a finite dimensional Galerkin projection of F, to apply Newton's method on it and to obtain a numerical approximation $\bar{x}$ of $F = 0$. With the help of the computer, we then construct an injective approximate inverse A of $DF(\bar{x})$ so that $AF : X \to X$. We define a Newton-like operator $T : X \to X$ by $T(x) = x - AF(x)$, and we aim at obtaining

(a) the existence of $\tilde{x} \in X$ such that $T(\tilde{x}) = \tilde{x}$, or equivalently (since A is injective) such that $F(\tilde{x}) = 0$;

(b) the existence of an explicit and small $r > 0$ such that $\|\tilde{x} - \bar{x}\|_X \leq r$.

The existence of $\tilde{x} \in X$ and of r is obtained by applying the radii polynomial approach which is Newton-Kantorovich type argument. The radii polynomials provide an efficient mean of determining a closed ball $B_{\bar{x}}(r) \subset X$ of radius r centered at the numerical approximation $\bar{x}$ on which the Newton-like operator $T(x) = x - AF(x)$ is a contraction. Once the assumptions are satisfied, we obtain the proof of existence of the periodic solution.

The radii polynomial approach was introduced in [5] to study equilibria of PDEs. Since then, it was adapted to many different situations, e.g. to the study of higher-dimensional PDEs [9], delay equations [17], Euler-Lagrange equations [4], radially symmetric localized solutions of PDEs [22] and many more. In these previous work, the computer-assisted proofs were all obtained in Banach spaces of solutions with low regularity. In the context of the present work, we use heavily the rigorous numerical method of [15], which adapted the radii polynomial approach to prove existence of analytic solutions of differential equations, and in particular analytic periodic solutions. In this case, the Banach space X is a weighed ℓ^1 space consisting of Fourier coefficients decaying exponentially fast to 0.

We notice that for $q = 3$ our solutions are analytic, and this allows us to use the above mentioned rigorous numerical method. Moreover, since we are looking for a periodic solution possessing some specified symmetries (see *i)* of Theorem 1.3), we seek existence of fixed points of T in a proper subspace inheriting these symmetries. Let us mention that we had to adapt slightly the approach of [15] in order to show that the fixed point has the proper symmetry. See Sections 4 for more details.

Finally, we point out that the reasons why we deal with a single specific value of q relies on the fact that, in order to apply the rigorous numerical method, we have to fix the parameters of the equation under study, and work out ad hoc estimates. We are convinced that the rigorous numerical method employed in this paper can be adapted to get similar results for any $q > 1$ odd (and consequently, by continuation arguments, to open neighborhoods of any such q).

A plan of this paper is as follows. In Section 2 we introduce the transformation hinted at in (1.7), and give several preliminary results about equations ($\mathbf{U_q}$) and ($\mathbf{W_q}$). In Section 3 we present the main results of the paper, chiefly concerned with the blow-up profile of solutions of ($\mathbf{U_q}$). Section 4 is devoted to the computer assisted proof of existence of a periodic solution of ($\mathbf{W_q}$), and to some of its relevant properties such as symmetries, zeros, and stability. In Section 5 we show how some of the main results obtained in Section 3 for ($\mathbf{U_q}$) can be appropriately extended to some instances to the more general equation (1.1). In Section 6 we present some numerical experiments aimed at clarifying how the idea behind the ansatz (1.7) is also of practical help in the numerical investigation of (1.1), and conclude with two conjectures. Finally, in the appendix we present a general result on persistence of periodic solutions under perturbations where one is interested in retaining certain symmetries.

Notations. For convenience, here we list the notations used in this paper in the same order as they appear.

$\mathbf{u}_0$ — The vector of initial conditions for equation

$$u''''(r) + |u(r)|^{q-1}u(r) = 0. \qquad (\mathbf{U_q})$$

$\phi(\cdot, \mathbf{u}_0)$ — The solution of ($\mathbf{U_q}$) with initial condition $\mathbf{u}_0$.

(R_-, R_+) — The lifespan of $\phi(\cdot, \mathbf{u}_0)$.

$\mathbf{\Phi}(\cdot, \mathbf{u}_0)$	The solution of $(\mathbf{U_q})$ with initial condition $\mathbf{u}_0$ in the phase space $\mathbb{R}^4$.
$\mathcal{O}(\mathbf{u}_0)$	The orbit trought $\mathbf{u}_0$, $\mathcal{O}(\mathbf{u}_0) := \{\mathbf{\Phi}(r, \mathbf{u}_0) : r \in (R_-, R_+)\}$.
η	$\frac{4}{q-1}$.
$\varphi(r)$	$\varphi(r) := -\ln(1-r)$, see (2.4).
$\varphi^{-1}(s)$	$\varphi^{-1}(s) = 1 - e^{-s}$, see (2.4).
$N(w)$	$c_0 w + c_1 w' + c_2 w'' + c_3 w'''$, where

$$\begin{aligned} c_0 &:= 8\,\frac{(q+3)\,(3\,q+1)\,(q+1)}{(q-1)^4}, & c_1 &:= 2\,\frac{(3\,q+5)\,\left(q^2+10\,q+5\right)}{(q-1)^3}, \\ c_2 &:= \frac{11\,q^2+50\,q+35}{(q-1)^2}, & c_3 &:= 2\,\frac{3\,q+5}{q-1}. \end{aligned} \tag{2.6}$$

$\mathbf{w}_0$ — The initial condition for the auxiliary equation

$$w'''' + N(w) + |w|^{q-1}w = 0. \tag{$\mathbf{W_q}$}$$

$\psi(\cdot, \mathbf{w}_0)$	The solution of $(\mathbf{W_q})$ with initial condition $\mathbf{w}_0$.
$\mathbf{\Psi}(\cdot, \mathbf{u}_0)$	The solution of the auxiliary equation $(\mathbf{W_q})$ with initial condition $\mathbf{w}_0$ in phase space $\mathbb{R}^4$.
(S_-, S_+)	The lifespan of $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$.
$D(\alpha)$	The matrix $\operatorname{diag}([\alpha^\eta, \alpha^{\eta+1}, \alpha^{\eta+2}, \alpha^{\eta+3}])$, see (2.8).
$u_\alpha(r)$	$\alpha^\eta u(\alpha r)$.
J	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. The matrix defined in (2.12).
L	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\eta & 1 & 0 & 0 \\ \eta^2 & -2\eta-1 & 1 & 0 \\ -\eta^3 & 3\eta^2+3\eta+1 & -3\eta-3 & 1 \end{bmatrix}$. The matrix defined in (2.14).
$\mathcal{D}M$	$\bigcup_{p \in M} \bigcup_{\alpha > 0} D(\alpha)p$.
γ	A non trivial periodic solution of $(\mathbf{W_q})$.
Γ	$\mathcal{O}(\gamma)$, the orbit of the periodic solution γ.
$\mathcal{B}$	$\{\mathbf{w}_0 \in \mathbb{R}^4 : \mathbf{\Psi}(s, \mathbf{w}_0) \to 0 \text{ as } s \to +\infty\}$, that is the basin of attraction of the origin for the problem $(\mathbf{W_q})$.
$\operatorname{gauge}_M(\mathbf{p})$	$\sup\{\alpha > 0 : D(\alpha)\mathbf{p} \in M\}$.

2 Preliminary results and an auxiliary problem

In this Section we present several preliminary results and introduce a transformation that will be key in proving the main results of this paper.

Let $\mathbf{u}_0 \in \mathbb{R}^4$, with $\phi(\cdot, \mathbf{u}_0)$ we denote the solution of

$$u''''(r) + |u(r)|^{q-1}u(r) = 0 \tag{$\mathbf{U_q}$}$$

with initial condition $[u(0), u'(0), u''(0), u'''(0)] = \mathbf{u}_0$. The maximal interval of existence (or lifespan) of $u(\cdot) = \phi(\cdot, \mathbf{u}_0)$ will be denoted by $(R_-, R_+) = (R_-(\mathbf{u}_0), R_+(\mathbf{u}_0))$.

The scalar equation ($\mathbf{U_q}$) can be canonically recast as a first order vector equation in its phase coordinates $\mathbf{u}(r) = [u_0(r), u_1(r), u_2(r), u_3(r)] := [u(r), u'(r), u''(r), u'''(r)]$, namely

$$\mathbf{u}'(r) = \mathbf{f}(\mathbf{u}(r)) := \begin{bmatrix} u_1(r) \\ u_2(r) \\ u_3(r) \\ -|u_0(r)|^{q-1} u_0(r) \end{bmatrix}. \tag{2.1}$$

The solution of (2.1) with initial condition $\mathbf{u}_0 \in \mathbb{R}^4$ will be denoted by $\mathbf{\Phi}(\cdot, \mathbf{u}_0)$, and the family of maps $\{\mathbf{\Phi}(r, \cdot)\}_r$ will be referred to as its flow. By virtue of the equivalence of ($\mathbf{U_q}$) and (2.1), we will refer to both as ($\mathbf{U_q}$).

For any $\mathbf{u}_0 \in \mathbb{R}^4$, the set

$$\mathcal{O}(\mathbf{u}_0) = \{\mathbf{\Phi}(r, \mathbf{u}_0) : r \in (R_-, R_+)\}$$

will be referred to as the orbit of ($\mathbf{U_q}$) through $\mathbf{u}_0$. If $u = \phi(\cdot, \mathbf{u}_0)$, we may also write $\mathcal{O}(u)$ instead of $\mathcal{O}(\mathbf{u}_0)$. A set $M \subset \mathbb{R}^4$ will be said invariant with respect to ($\mathbf{U_q}$) if

$$\mathbf{u}_0 \in M \implies \mathcal{O}(\mathbf{u}_0) \subset M\,.$$

Throughout the rest of the paper, for any $q > 1$, we set $\eta := \frac{4}{q-1}$. Given a solution u of ($\mathbf{U_q}$), we consider the following transformation:

$$w(s) = e^{-\eta s} u(1 - e^{-s})\,, \tag{2.2}$$

or, equivalently,

$$u(r) = \left(\frac{1}{1-r}\right)^{\eta} w\left(\ln\left(\frac{1}{1-r}\right)\right). \tag{2.3}$$

Setting

$$s = \varphi(r) = -\ln(1-r), \text{ with inverse } r = \varphi^{-1}(s) = 1 - e^{-s}, \tag{2.4}$$

it follows that the transformations (2.2) and (2.3) are valid for $\varphi^{-1}(s) \in (R_-, R_+)$.

Therefore, if u and w are related by (2.2), then u is a solution of ($\mathbf{U_q}$) if and only if w is a solution of the following equation

$$w'''' + N(w) + |w|^{q-1}w = 0, \tag{$\mathbf{W_q}$}$$

where

$$N(w) := \; c_0 w + c_1 w' + c_2 w'' + c_3 w'''\,, \tag{2.5}$$

$$c_0 := 8\,\frac{(q+3)\,(3\,q+1)\,(q+1)}{(q-1)^4}\,, \quad c_1 := 2\,\frac{(3\,q+5)\,\left(q^2+10\,q+5\right)}{(q-1)^3}\,, \\ c_2 := \frac{11\,q^2+50\,q+35}{(q-1)^2}\,, \qquad c_3 := 2\,\frac{3\,q+5}{q-1}\,. \tag{2.6}$$

Letting $\mathbf{w}(s) = [w_0(s), w_1(s), w_2(s), w_3(s)] := [w(s), w'(s), w''(s), w'''(s)]$, equation $(\mathbf{W_q})$ can be rewritten in vector form as

$$\mathbf{w}'(s) = \mathbf{g}(\mathbf{w}(s)) := \begin{bmatrix} w_1(s) \\ w_2(s) \\ w_3(s) \\ -N(\mathbf{w})(s) - |w_0(s)|^{q-1}\, w_0(s) \end{bmatrix}, \tag{2.7}$$

where $N(\mathbf{w}) := c_0 w_0 + c_1 w_1 + c_2 w_2 + c_3 w_3$.

For solutions of $(\mathbf{W_q})$ we will adopt notations similar to those introduced for $(\mathbf{U_q})$, but the solutions will be denoted by $\psi(\cdot, \mathbf{w}_0)$ (or $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$), and their lifespan by $(S_-, S_+) = (S_-(\mathbf{w}_0), S_+(\mathbf{w}_0))$. Analogously, we shall refer indifferently to $(\mathbf{W_q})$ or to (2.7) as $(\mathbf{W_q})$.

We point out that, throughout the whole paper, we will reserve the letters u and w for solutions of, respectively, $(\mathbf{U_q})$ and $(\mathbf{W_q})$, and omit to reference the relevant equation whenever no confusion arises. Moreover, by *periodic solution* or τ-periodic function we always mean a nontrivial periodic function with least period $\tau > 0$.

Clearly, if u and w are related through the transformation (2.2), then their lifespan are related by the function φ defined in (2.4).

Remark 2.1. *Let $u = u(r)$ be a solution of $(\mathbf{U_q})$, and let $w = w(s)$ be the corresponding solution of $(\mathbf{W_q})$ defined as in (2.2). Then we have:*

i) if u blows up at $R_+ < 1$, then w blows up at $S_+ = \ln\left(\frac{1}{1-R^+}\right) < \infty$;

ii) if u blows up at finite $R_+ \geq 1$, or u exists for all $r \geq 0$, then w exists for all $s \geq 0$; in particular, if $R_+ > 1$ (possibly $R_+ = +\infty$), then $w \to 0$ exponentially as $s \to +\infty$.

Viceversa: Let $w = w(s)$ be a solution of $(\mathbf{W_q})$, and let $u = u(r)$ be the corresponding solution of $(\mathbf{U_q})$ defined as in (2.3). Then we have:

iii) if w blows up at $S_+ < \infty$, then u blows up at $R_+ = 1 - e^{-S_+} < 1$;

iv) if w exists for all $s \geq 0$, then $R_+ \geq 1$;

v) if w exists for all $s \geq 0$, and $\limsup_{s \to +\infty} |w(s)| > 0$, then u blows up at $R_+ = 1$;

vi) if w exists for all $s \in \mathbb{R}$, w is bounded on $\mathbb{R}$, and $\limsup_{s \to +\infty} |w(s)| > 0$, then u has lifespan $(-\infty, 1)$, and $u \to 0$ as $r \to -\infty$.

Now, for each $\alpha > 0$, we define:

$$D(\alpha) := \begin{bmatrix} \alpha^\eta & 0 & 0 & 0 \\ 0 & \alpha^{\eta+1} & 0 & 0 \\ 0 & 0 & \alpha^{\eta+2} & 0 \\ 0 & 0 & 0 & \alpha^{\eta+3} \end{bmatrix}. \tag{2.8}$$

The following remark clarifies the role played by the matrix D when we rescale solutions of $(\mathbf{U_q})$ by

$$r \longmapsto \alpha r\,.$$

Remark 2.2. *Let u be a solution of $(\mathbf{U_q})$. Then, for any $\alpha > 0$, the function u_α defined as*

$$u_\alpha(r) := \alpha^\eta u(\alpha r)$$

is a solution of $(\mathbf{U_q})$. More precisely, if $u(r) = \phi(r, \mathbf{u}_0)$, then $u_\alpha(r) = \phi(r, D(\alpha)\mathbf{u}_0)$. In vector form, we have

$$D(\alpha)\mathbf{\Phi}(\alpha r, \mathbf{u}_0) = \mathbf{\Phi}(r, D(\alpha)\mathbf{u}_0). \tag{2.9}$$

Finally, if the lifespan of u is (R_-, R_+), then the lifespan of u_α is $\left(\frac{R_-}{\alpha}, \frac{R_+}{\alpha}\right)$.

Remark 2.3. *Let u be a solution of $(\mathbf{U_q})$ that blows up at $T > 0$. As observed in the previous remark, u_T blows up at 1. Therefore, using (2.2), we have that*

$$w(s) = T^\eta e^{-\eta s} u(T(1 - e^{-s})) \tag{2.10}$$

solves $(\mathbf{W_q})$ for all $s \geq 0$. Viceversa, if w solves $(\mathbf{W_q})$ for all $s \geq 0$ and $\limsup_{s\to+\infty} |w(s)| > 0$, then

$$u(r) = \left(\frac{1}{T - r}\right)^\eta w\left(\ln\left(\frac{T}{T - r}\right)\right) \tag{2.11}$$

solves $(\mathbf{U_q})$ and blows up at $T > 0$.

We conclude this subsection with a remark on the effect of the change of variables

$$r \longmapsto -r$$

on solutions of $(\mathbf{U_q})$. Let

$$J := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.12}$$

Remark 2.4. *For any $\mathbf{u}_0 \in \mathbb{R}^4$, we have $R_-(J\mathbf{u}_0) = -R_+(\mathbf{u}_0)$, $R_+(J\mathbf{u}_0) = -R_-(\mathbf{u}_0)$, and*

$$\mathbf{\Phi}(r, J\mathbf{u}_0) = \mathbf{\Phi}(-r, \mathbf{u}_0),\;\; r \in (R_-(J\mathbf{u}_0), R_+(J\mathbf{u}_0)).$$

2.1 Some explicit computations about the transformation

In this subsection we state some explicit computations that will be useful throught the rest of the paper. Let $u = \phi(\cdot, \mathbf{u}_0)$ and $w = \psi(\cdot, \mathbf{w}_0)$ be solutions of, respectively, $(\mathbf{U_q})$ and $(\mathbf{W_q})$ related by (2.2). Then, by repeated differentiation of (2.2), we obtain:

$$\begin{aligned}
w(s) &= e^{-\eta s} u(\varphi^{-1}(s))\,, \\
w'(s) &= -\,\eta e^{-\eta s} u(\varphi^{-1}(s)) + e^{-(\eta+1)s} u'(\varphi^{-1}(s))\,, \\
w''(s) &= \eta^2 e^{-\eta s} u(\varphi^{-1}(s)) - (2\eta + 1) e^{-(\eta+1)s} u'(\varphi^{-1}(s)) + e^{-(\eta+2)s} u''(\varphi^{-1}(s))\,, \\
w'''(s) &= -\,\eta^3 e^{-\eta s} u(\varphi^{-1}(s)) + (3\eta^2 + 3\eta + 1) e^{-(\eta+1)s} u'(\varphi^{-1}(s)) + \\
&\quad -\,(3\eta + 3) e^{-(\eta+2)s} u''(\varphi^{-1}(s)) + e^{-(\eta+3)s} u'''(\varphi^{-1}(s))\,,
\end{aligned}$$

where we recall that $\eta = \dfrac{4}{q-1}$, and that φ (and consequently φ^{-1}) has been defined in (2.4).

We can write the relations above as

$$\begin{bmatrix} w(s) \\ w'(s) \\ w''(s) \\ w'''(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\eta & 1 & 0 & 0 \\ \eta^2 & -2\eta-1 & 1 & 0 \\ -\eta^3 & 3\eta^2+3\eta+1 & -3\eta-3 & 1 \end{bmatrix} \begin{bmatrix} e^{-\eta s}u(\varphi^{-1}(s)) \\ e^{-(\eta+1)s}u'(\varphi^{-1}(s)) \\ e^{-(\eta+2)s}u''(\varphi^{-1}(s)) \\ e^{-(\eta+3)s}u'''(\varphi^{-1}(s)) \end{bmatrix}. \tag{2.13}$$

Finally, letting

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\eta & 1 & 0 & 0 \\ \eta^2 & -2\eta-1 & 1 & 0 \\ -\eta^3 & 3\eta^2+3\eta+1 & -3\eta-3 & 1 \end{bmatrix}, \tag{2.14}$$

and making use of (2.8), we can rewrite (2.13) in compact form as

$$\mathbf{w}(s) = LD(e^{-s})\mathbf{u}(\varphi^{-1}(s)) = LD(1-\varphi^{-1}(s))\mathbf{u}(\varphi^{-1}(s)). \tag{2.15}$$

In particular, setting $r = s = 0$, we get the following relation for the vectors of initial conditions $\mathbf{u}_0$ and $\mathbf{w}_0$:

$$\mathbf{w}_0 = L\mathbf{u}_0. \tag{2.16}$$

By inverting the relation (2.15) we obtain

$$\mathbf{u}(r) = D\left(\frac{1}{1-r}\right) L^{-1}\mathbf{w}(\varphi(r)). \tag{2.17}$$

From (2.15) and (2.17) we easily deduce the following.

Proposition 2.5. *Let $\mathbf{w}_0 \in \mathbb{R}^4$. Then we have*

$$\mathbf{\Psi}(s, \mathbf{w}_0) = LD(e^{-s})\mathbf{\Phi}(\varphi^{-1}(s), L^{-1}\mathbf{w}_0), \text{ for all } s \in (S_-, S_+). \tag{2.18}$$

Let $\mathbf{u}_0 \in \mathbb{R}^4$. Then we have

$$\mathbf{\Phi}(r, \mathbf{u}_0) = D\left(\frac{1}{1-r}\right) L^{-1}\mathbf{\Psi}(\varphi(r), L\mathbf{u}_0), \text{ for all } r \in (R_-, \min\{1, R_+\}). \tag{2.19}$$

2.2 Invariant sets

Let $M \subset \mathbb{R}^4$, by $\mathcal{D}M$ we denote the collection of the supports of all the curves $\alpha > 0 \mapsto D(\alpha)\mathbf{p}$, $\mathbf{p} \in M$, that is

$$\mathcal{D}M := \bigcup_{\mathbf{p}\in M} \bigcup_{\alpha>0} D(\alpha)\mathbf{p}\,. \tag{2.20}$$

For $\mathbf{u}_0 \in \mathbb{R}^4$, the set $\mathcal{DO}(\mathbf{u}_0)$ can be parametrized as

$$(\alpha, r) \in (0, +\infty) \times (R_-(\mathbf{u}_0), R_+(\mathbf{u}_0)) \longmapsto \mathcal{D}(\alpha)\mathbf{\Phi}(r, \mathbf{u}_0). \tag{2.21}$$

If $\mathbf{u}_0 \neq 0$, we can say more.

Proposition 2.6. *For any* $\mathbf{u}_0 \neq 0$, $\mathcal{DO}(\mathbf{u}_0)$ *is a regular 2-dimensional manifold which is invariant with respect to* $(\mathbf{U_q})$.

Proof. The parametrization (2.21) is regular and it defines a 2-dimensional manifold provided that, for any given $(\alpha, r) \in (0, +\infty) \times (R_-(\mathbf{u}_0), R_+(\mathbf{u}_0))$, there exists a 2-dimensional tangent space, i.e. the vectors

$$\frac{\partial}{\partial \alpha} D(\alpha)\mathbf{\Phi}(r, \mathbf{u}_0), \quad \frac{\partial}{\partial r} D(\alpha)\mathbf{\Phi}(r, \mathbf{u}_0)$$

are linearly independent. Clearly, none of the two vectors can be zero since $\mathbf{u}_0 \neq 0$ and $\alpha \neq 0$. Arguing by contradiction, assume that at a point $(\alpha_0, r_0) \in (0, +\infty) \times (R_-(\mathbf{u}_0), R_+(\mathbf{u}_0))$ those vectors are parallel, i.e. there exists $\nu \in \mathbb{R}$, $\nu \neq 0$, such that

$$\frac{\partial}{\partial \alpha} D(\alpha)\mathbf{\Phi}(r, \mathbf{u}_0) = \nu \frac{\partial}{\partial r} D(\alpha)\mathbf{\Phi}(r, \mathbf{u}_0) \,.$$

Then, r_0 is a solution of the following system of equations

$$\begin{cases} \eta u(r) = \nu \alpha_0 u'(r), \\ (\eta + 1)u'(r) = \nu \alpha_0 u''(r), \\ (\eta + 2)u''(r) = \nu \alpha_0 u'''(r), \\ (\eta + 3)u'''(r) = \nu \alpha_0 u''''(r), \end{cases}$$

where $u(r) = \phi(r, \mathbf{u}_0)$. Using all the equations above, and the fact that u solves $(\mathbf{U_q})$, we have

$$u(r_0) = \frac{(\nu\alpha_0)^4}{\eta(\eta+1)(\eta+2)(\eta+3)} u^{(4)}(r_0) = -\frac{(\nu\alpha_0)^4}{\eta(\eta+1)(\eta+2)(\eta+3)} |u(r_0)|^{q-1} u(r_0).$$

This is possible if and only if $u(r_0) = 0$, which forces all derivatives of u to be zero at $r = r_0$ as well, and this contradicts $\mathbf{u}_0 = 0$.

Next, in order to prove the invariance of the manifold with respect to $(\mathbf{U_q})$, consider $\mathbf{p} \in \mathcal{DO}(\mathbf{u}_0)$, i.e. $\mathbf{p} = D(\alpha_0)\mathbf{\Phi}(r_0, \mathbf{u}_0)$ for some (α_0, r_0). In light of Remark 2.2, we have

$$\mathbf{\Phi}(r, \mathbf{p}) = \mathbf{\Phi}(r, D(\alpha_0)\mathbf{\Phi}(r_0, \mathbf{u}_0)) = D(\alpha_0)\mathbf{\Phi}(\alpha_0 r, \mathbf{\Phi}(r_0, \mathbf{u}_0)) = D(\alpha_0)\mathbf{\Phi}(\alpha_0 r + r_0, \mathbf{u}_0)\,,$$

which, by construction, belongs to $\mathcal{DO}(\mathbf{u}_0)$. □

The previous proposition generalizes to the following result.

Proposition 2.7. *Let* $M \subset \mathbb{R}^4$ *be invariant with respect to* $(\mathbf{U_q})$, *then also* $\mathcal{D}M$ *is invariant with respect to* $(\mathbf{U_q})$.

Proof. Let $\mathbf{p} \in \mathcal{D}M$, i.e. $\mathbf{p} = D(\alpha_0)\mathbf{p}_0$ for some $\alpha_0 > 0$ and $\mathbf{p}_0 \in M$. Since M is invariant, $\mathcal{O}(\mathbf{p}_0) \subset M$, and hence $\mathcal{DO}(\mathbf{p}_0) \subset \mathcal{D}M$. But $\mathbf{p} \in \mathcal{DO}(\mathbf{p}_0)$, so the claim follows from Proposition 2.6. □

With additional information on the blow-up time of a given solution of $(\mathbf{U_q})$ we can construct more invariant sets for $(\mathbf{U_q})$ and $(\mathbf{W_q})$.

Proposition 2.8. *Let* $\mathbf{w}_0 \neq 0$ *and assume* $R_+(L^{-1}\mathbf{w}_0) \leq 1$. *Then, the set* $L\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$ *is a 2-dimensional manifold invariant with respect to* $(\mathbf{W_q})$.

Proposition 2.9. *Let $\mathcal{M} \subset \mathbb{R}^4$ be invariant with respect to* ($\mathbf{W_q}$) *and such that, for any* $\mathbf{w}_0 \in \mathcal{M}$*, we have* $R_+(L^{-1}\mathbf{w}_0) \leq 1$.

Then, $\mathcal{D}L^{-1}\mathcal{M}$ *is invariant with respect to* ($\mathbf{U_q}$).

The proofs of the two propositions above rely on the following lemma, and we postpone their proof to that of the lemma.

Lemma 2.10. *Let* $\mathbf{w}_0 \in \mathbb{R}^4$, $\mathbf{w}_0 \neq 0$. *Then:*

i) $\mathcal{O}(\mathbf{w}_0) \subset L\mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0)$,

ii) $L\mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0)$ *is invariant with respect to* ($\mathbf{W_q}$),

iii) $\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0) \subset \mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0)$,

iv) if $R_+(L^{-1}\mathbf{w}_0) \leq 1$*, then*

$$\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0) = \mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0). \tag{2.22}$$

Proof. Let $\mathbf{w}(s) = \mathbf{\Psi}(s, \mathbf{w}_0)$ be defined for $s \in (S_-, S_+)$, and set $\mathbf{u}_0 = L^{-1}\mathbf{w}_0$ and $\mathcal{M} = \mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0)$. We begin recalling that $\mathcal{M}$ is a 2-dimensional manifold invariant with respect to ($\mathbf{U_q}$) (see Proposition 2.6).

i) Let $\mathbf{p} \in \mathcal{O}(\mathbf{w}_0)$, i.e. $\mathbf{p} = \mathbf{\Psi}(s_0, \mathbf{w}_0)$ for some $s_0 \in (S_-, S_+)$. Let $r_0 = \varphi^{-1}(s_0)$ and $\alpha_0 = 1 - r_0$. Note that, by the definition of φ in (2.4), we have $r_0 < 1$, hence $\alpha_0 > 0$.

From (2.18), we have

$$\begin{aligned} \mathbf{p} = \mathbf{\Psi}(s_0, \mathbf{w}_0) &= LD(e^{-s_0})\mathbf{\Phi}(\varphi^{-1}(s_0), L^{-1}\mathbf{w}_0) = LD(1 - r_0)\mathbf{\Phi}(r_0, \mathbf{u}_0) = \\ &= LD(\alpha_0)\mathbf{\Phi}(r_0, \mathbf{u}_0) \in L\mathcal{M}, \end{aligned}$$

and hence the claim follows.

ii) Let $\mathbf{p} \in L\mathcal{M}$. We have to show that $\mathbf{\Psi}(s, \mathbf{p}) \in L\mathcal{M}$, for any $s \in (S_-, S_+)$. Let $\mathbf{p}_0 = L^{-1}\mathbf{p} \in \mathcal{M}$, $s \in (S_-, S_+)$, $r = \varphi^{-1}(s)$, and recall that $r < 1$. From (2.18), we have

$$\mathbf{\Psi}(s, \mathbf{p}) = LD(e^{-s})\mathbf{\Phi}(\varphi^{-1}(s), L^{-1}\mathbf{p}) = LD(1 - r)\mathbf{\Phi}(r, \mathbf{p}_0).$$

The claim will follow once we show that $D(1 - r)\mathbf{\Phi}(r, \mathbf{p}_0) \in \mathcal{M}$. To this end, observe that, since $\mathbf{p}_0 \in \mathcal{M}$ and $\mathcal{M}$ is invariant with respect to ($\mathbf{U_q}$), we have that $\mathbf{\Phi}(r, \mathbf{p}_0) = D(\alpha_1)\mathbf{\Phi}(r_1, \mathbf{u}_0)$, for some suitable α_1 and r_1, and hence

$$D(1 - r)\mathbf{\Phi}(r, \mathbf{p}_0) = D(1 - r)D(\alpha_1)\mathbf{\Phi}(r_1, \mathbf{u}_0) = D(\alpha_2)\mathbf{\Phi}(r_1, \mathbf{u}_0) \in \mathcal{M},$$

with $\alpha_2 = (1 - r)\alpha_1 > 0$.

iii) It follows applying first L^{-1} to both sides of inclusion in i), and then $\mathcal{D}$.

iv) We only need to show that $\mathcal{M} \subset \mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$. Let $\mathbf{p} \in \mathcal{M}$, that is $\mathbf{p} = D(\alpha_0)\mathbf{\Phi}(r_0, \mathbf{u}_0)$, for some α_0 and r_0. Since $r_0 < R_+(\mathbf{u}_0) \leq 1$, we can write $s_0 = \varphi(r_0)$. Setting $\alpha_1 = \alpha_0 e^{s_0}$, we have

$$D(\alpha_0)\mathbf{\Phi}(r_0, \mathbf{u}_0) = D(\alpha_1)D(e^{-s_0})\mathbf{\Phi}(\varphi^{-1}(s_0), L^{-1}\mathbf{w}_0),$$

which belongs to $\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$ since the set $\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$ can be written as

$$\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0) = \bigcup_{\alpha>0} \bigcup_{s\in(S_-,S_+)} D(\alpha)L^{-1}\mathbf{\Psi}(s,\mathbf{w}_0) = \bigcup_{\alpha,s} D(\alpha)D(e^{-s})\mathbf{\Phi}(\varphi^{-1}(s), L^{-1}\mathbf{w}_0).$$

This concludes the proof. □

Proof of Proposition 2.8. The fact that $L\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$ is invariant with respect to $(\mathbf{W_q})$ is a consequence of *ii)* and *iv)* in Lemma 2.10. As for $L\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$ being a 2-dimensional manifold, it follows from the fact that it is the image of a regular 2-dimensional manifold (see Proposition 2.6) under the (linear) invertible map L. □

Proof of Proposition 2.9. Using the fact that $\mathcal{M}$ is invariant with respect to $(\mathbf{W_q})$, i.e. $\mathbf{w}_0 \in \mathcal{M}$ if and only if $\mathcal{O}(\mathbf{w}_0) \subset \mathcal{M}$, we can write $\mathcal{D}L^{-1}\mathcal{M}$ as

$$\mathcal{D}L^{-1}\mathcal{M} = \bigcup_{\mathbf{w}_0\in\mathcal{M}} \mathcal{D}L^{-1}\mathbf{w}_0 = \bigcup_{\mathbf{w}_0\in\mathcal{M}} \mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0).$$

By (2.22) we obtain

$$\mathcal{D}L^{-1}\mathcal{M} = \bigcup_{\mathbf{w}_0\in\mathcal{M}} \mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0).$$

Proposition 2.6 assures that the manifolds $\mathcal{D}\mathcal{O}(L^{-1}\mathbf{w}_0)$ are invariant with respect to $(\mathbf{U_q})$, therefore, since we have written $\mathcal{D}L^{-1}\mathcal{M}$ as the union of sets which are invariant with respect to $(\mathbf{U_q})$, the proof is concluded. □

Corollary 2.11. *Assume that* $(\mathbf{W_q})$ *has a nontrivial periodic solution* γ*, and let* $\Gamma = \mathcal{O}(\gamma)$ *be its orbit. Then,* $L\mathcal{D}L^{-1}\Gamma$ *is a 2-dimensional manifold invariant with respect to* $(\mathbf{W_q})$*.*

The manifold $L\mathcal{D}L^{-1}\Gamma$ plays an important role in the study of $(\mathbf{W_q})$, most notably it provides information about the unstable manifold associated to a periodic solution γ. See the next subsection, and in particular Corollary 2.14.

Proof. The proof follows from Proposition 2.8 once we note that, if $\mathbf{w}_0 \in \Gamma$, then $u = \phi(\cdot, L^{-1}\mathbf{w}_0)$ is a solution of $(\mathbf{U_q})$ that blows up at 1 (see Remark 2.1). □

Proposition 2.12. *Assume that* $(\mathbf{W_q})$ *has a nontrivial periodic solution* γ*, and let* $\Gamma = \mathcal{O}(\gamma)$ *be its orbit. Then,* $\mathcal{D}L^{-1}\Gamma$ *and* $J\mathcal{D}L^{-1}\Gamma$ *are invariant with respect to* $(\mathbf{U_q})$*.*

The manifolds $\mathcal{D}L^{-1}\Gamma$ and $J\mathcal{D}L^{-1}\Gamma$ will be helpful in studying solutions of $(\mathbf{U_q})$ which vanish as $|r| \to +\infty$. See Theorem 3.4.

Proof. We prove that $\mathcal{D}L^{-1}\Gamma$ is invariant with respect to $(\mathbf{U_q})$. The rest of the claim follows by inverting the direction of time, see Remark 2.4.

Let $\mathbf{u}_0 \in \mathcal{D}L^{-1}\Gamma$, i.e. $\mathbf{u}_0 = D(\alpha)L^{-1}\boldsymbol{\gamma}(s_0)$ for some $\alpha > 0$ and $s_0 \in \mathbb{R}$. By virtue of Remark 2.2 and relation 2.18, we have

$$\begin{aligned}\mathbf{\Phi}(r,\mathbf{u}_0) &= \mathbf{\Phi}(r, D(\alpha)L^{-1}\boldsymbol{\gamma}(s_0)) = D(\alpha)\mathbf{\Phi}(\alpha r, L^{-1}\boldsymbol{\gamma}(s_0)) = \\ &= D(\alpha)D\left((1-\alpha r)^{-1}\right)\mathbf{\Psi}(\varphi(\alpha r), \boldsymbol{\gamma}(s_0)),\end{aligned} \tag{2.23}$$

for $\alpha r < \min\{1, R_+(L^{-1}\boldsymbol{\gamma}(s_0))\}$. Now recall that $R_+(L^{-1}\boldsymbol{\gamma}(0)) = 1$ (see v) of Remark 2.1), and therefore $R_+(\mathbf{u}_0) = 1/\alpha$. Hence, we have that relation (2.23) holds for $r < 1/\alpha$ and, since $\mathbf{\Psi}(\cdot, \boldsymbol{\gamma}(s_0))$ is periodic, the claim follows. □

2.3 On the unstable manifold for the auxiliary equation

Here we present a theorem that concerns the stability of bounded solutions of ($\mathbf{W_q}$) on $(-\infty, 0]$. Ultimately, this result will provide a characterization of the unstable manifold for a periodic solution of ($\mathbf{W_q}$), see Section 2.5.

Theorem 2.13. *Let $\mathbf{w}_0 \in \mathbb{R}^4$, $\mathbf{w}_0 \neq 0$ and consider $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$. Assume that $S^-(\mathbf{w}_0) = -\infty$, and that $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ is bounded on $(-\infty, 0]$.*

Then, for any $\mathbf{p} \in L\mathcal{D}L^{-1}\mathcal{O}(\mathbf{w}_0)$, $\mathbf{\Psi}(\cdot, \mathbf{p})$ approaches $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ with asymptotic phase as $s \to -\infty$. More precisely, there exists $c > 0$ such that

$$\|\mathbf{\Psi}(s, \mathbf{p}) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0)\| \leq ce^s, \text{ for all } s \in (-\infty, -|s_0|], \tag{2.24}$$

where s_0 is implicitly given by $\mathbf{p} = LD(\alpha)L^{-1}\mathbf{\Psi}(\ln(\alpha) - s_0, \mathbf{w}_0)$.

Moreover, if $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ is τ-periodic, and $\mathbf{p} \notin \mathcal{O}(\mathbf{w}_0)$, then there exist $\bar{s} = \bar{s}(\mathbf{w}_0) \in \mathbb{R}$ and a vector $\mathbf{v} = \mathbf{v}(\mathbf{w}_0) \in \mathbb{R}^4$, $\mathbf{v} \neq 0$, such that

$$\frac{\mathbf{\Psi}(\bar{s} - n\tau, \mathbf{p}) - \mathbf{\Psi}(\bar{s} - n\tau - s_0, \mathbf{w}_0)}{e^{-n\tau}} \to \frac{(1-\alpha)e^{\bar{s}}}{\alpha}\mathbf{v}, \text{ for } n \to +\infty. \tag{2.25}$$

Proof. Notice that the equation $\mathbf{p} = LD(\alpha)L^{-1}\mathbf{\Psi}(\ln(\alpha) - s_0, \mathbf{w}_0)$ admits a solution. Indeed, let $\mathbf{p}_0 \in \mathcal{O}(\mathbf{w}_0)$ be such that $\mathbf{p} = L\mathcal{D}(\alpha)L^{-1}\mathbf{p}_0$, then clearly we can find an $s_0 \in \mathbb{R}$ such that $\mathbf{p}_0 = \mathbf{\Psi}(\ln(\alpha) - s_0, \mathbf{w}_0)$.

For any $s \leq 0$, let $r = \varphi^{-1}(s) := 1 - e^{-s}$, and note that $r \leq 0$. Using (2.18) and (2.19), we obtain

$$\begin{aligned}\mathbf{\Psi}(s, \mathbf{p}) &= LD(e^{-s})\mathbf{\Phi}(\varphi^{-1}(s), L^{-1}\mathbf{p}) = LD(e^{-s})\mathbf{\Phi}(\varphi^{-1}(s), D(\alpha)L^{-1}\mathbf{p}_0) = \\ &= LD(e^{-s})D(\alpha)\mathbf{\Phi}(\alpha\varphi^{-1}(s), L^{-1}\mathbf{p}_0) = LD(e^{-s})D(\alpha)\mathbf{\Phi}(\alpha r, L^{-1}\mathbf{p}_0) = \\ &= LD(e^{-s})D(\alpha)D\left(\frac{1}{1-\alpha r}\right)L^{-1}\mathbf{\Psi}(\varphi(\alpha r), \mathbf{p}_0).\end{aligned}$$

Observing that

$$D(e^{-s})D(\alpha)D\left(\frac{1}{1-\alpha r}\right) = D\left(\frac{\alpha e^{-s}}{1-\alpha(1-e^{-s})}\right) = D\left(\left(1 - \frac{\alpha-1}{\alpha}e^s\right)^{-1}\right),$$

$$\varphi(\alpha r) = -\ln(1-\alpha r) = -\ln(1-\alpha+\alpha e^{-s}) = s - \ln\alpha - \ln\left(1 - \frac{\alpha-1}{\alpha}e^s\right),$$

and

$$\mathbf{\Psi}(s, \mathbf{p}_0) = \mathbf{\Psi}(s, \mathbf{\Psi}(\ln(\alpha) - s_0, \mathbf{w}_0)) = \mathbf{\Psi}(s + \ln(\alpha) - s_0, \mathbf{w}_0),$$

we have

$$\begin{aligned}\mathbf{\Psi}(s, \mathbf{p}) &= LD\left(\left(1 - \frac{\alpha-1}{\alpha}e^s\right)^{-1}\right)L^{-1}\mathbf{\Psi}\left(s - \ln\alpha - \ln\left(1 - \frac{\alpha-1}{\alpha}e^s\right), \mathbf{p}_0\right) \\ &= LD\left(\left(1 - \frac{\alpha-1}{\alpha}e^s\right)^{-1}\right)L^{-1}\mathbf{\Psi}\left(s - s_0 - \ln\left(1 - \frac{\alpha-1}{\alpha}e^s\right), \mathbf{w}_0\right).\end{aligned}$$

Now, for the sake of brevity, we set $x = x(s) := \frac{\alpha-1}{\alpha}e^s$, and observe that $x \to 0$ as $s \to -\infty$. Adding and subtracting $\mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0)$, and writing $\mathbf{\Psi}(\cdot, \mathbf{w}_0) = LL^{-1}\mathbf{\Psi}(\cdot, \mathbf{w}_0)$, we have

$$\begin{aligned}
&\mathbf{\Psi}(s, \mathbf{p}) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0) = \\
&\quad = L\left[D\left((1-x)^{-1}\right)L^{-1}\mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0) - L^{-1}\mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0)\right] \\
&\qquad + \mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0) \\
&\quad = L\left[D\left((1-x)^{-1}\right) - I_4\right]L^{-1}\mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0) \\
&\qquad + \mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0),
\end{aligned} \tag{2.26}$$

where I_4 is the 4×4 identity matrix. Passing to the norm, we have

$$\begin{aligned}
&\|\mathbf{\Psi}(s, \mathbf{p}) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0)\| \\
&\quad \le \|L\| \left\|D\left((1-x)^{-1}\right) - I_4\right\| \left\|L^{-1}\right\| \|\mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0)\| \\
&\qquad + \|\mathbf{\Psi}(s - s_0 - \ln(1-x), \mathbf{w}_0) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0)\| \\
&\quad \le c\,\|\mathbf{\Psi}(\cdot, \mathbf{w}_0)\|_\infty \left\|D\left((1-x)^{-1}\right) - I_4\right\| + \left\|\mathbf{\Psi}'(\cdot, \mathbf{w}_0)\right\|_\infty |\ln(1-x)|,
\end{aligned}$$

where we have used the Mean Value Theorem and the fact that also $\mathbf{\Psi}'(\cdot, \mathbf{w}_0)$ is bounded in $(-\infty, -|s_0|]$.

Since

$$\lim_{x \to 0} \frac{1}{x}\left(D\left((1-x)^{-1}\right) - I_4\right) = \begin{bmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta+1 & 0 & 0 \\ 0 & 0 & \eta+2 & 0 \\ 0 & 0 & 0 & \eta+3 \end{bmatrix} =: \Lambda,$$

we conclude that

$$\|\mathbf{\Psi}(s, \mathbf{p}) - \mathbf{\Psi}(s - s_0, \mathbf{w}_0)\| \le c\,|x| \le ce^s$$

for $-s$ large, and hence, possibly with a suitable replacement of the constant c, for any $s \le -|s_0|$.

To conclude the proof, let $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ be τ-periodic, and $\mathbf{p} \notin \mathcal{O}(\mathbf{w}_0)$. Clearly we must have $\alpha \neq 1$, and hence $x(s) \neq 0$ for all s. Let $\bar{s} \in \mathbb{R}$ be such that

$$\mathbf{v} := L\Lambda L^{-1}\mathbf{\Psi}(\bar{s} - s_0, \mathbf{w}_0) + \mathbf{\Psi}'(\bar{s} - s_0, \mathbf{w}_0) \neq 0. \tag{2.27}$$

Note that a such a value $\bar{s}$ must exist, otherwise the vector $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ would solve the linear differential equation (2.27), and hence its components would be exponentials, contradicting the hypothesis that $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ is periodic.

Now, set $s_n := \bar{s} - n\tau$, and consequently $x_n := x(s_n) = \frac{\alpha-1}{\alpha}e^{s_n}$. Plugging $s = s_n$ in (2.26), we obtain

$$\begin{aligned}
&\mathbf{\Psi}(s_n, \mathbf{p}) - \mathbf{\Psi}(s_n - s_0, \mathbf{w}_0) \\
&\quad = L\left[D\left((1-x_n)^{-1}\right) - I_4\right]L^{-1}\mathbf{\Psi}(s_n - s_0 - \ln(1-x_n)), \mathbf{w}_0) \\
&\qquad + \mathbf{\Psi}(s_n - s_0 - \ln(1-x_n), \mathbf{w}_0) - \mathbf{\Psi}(s_n - s_0, \mathbf{w}_0) \\
&\quad = L\left[D\left((1-x_n)^{-1}\right) - I_4\right]L^{-1}\mathbf{\Psi}(\bar{s} - s_0 - \ln(1-x_n)), \mathbf{w}_0) \\
&\qquad + \mathbf{\Psi}(\bar{s} - s_0 - \ln(1-x_n), \mathbf{w}_0) - \mathbf{\Psi}(\bar{s} - s_0, \mathbf{w}_0).
\end{aligned}$$

The last chain of equalities implies that

$$\frac{\mathbf{\Psi}(s_n, \mathbf{p}) - \mathbf{\Psi}(s_n - s_0, \mathbf{w}_0)}{x_n} \to L\Lambda L^{-1}\mathbf{\Psi}(\bar{s} - s_0, \mathbf{w}_0) + \mathbf{\Psi}'(\bar{s} - s_0, \mathbf{w}_0) = \mathbf{v} \neq 0\,,$$

as $n \to +\infty$, which is the claim. □

Corollary 2.14. *Assume that* ($\mathbf{W_q}$) *admits a non trivial periodic solution* γ. *Then,* γ *has an unstable manifold that contains the 2-dimensional manifold* $L\mathcal{D}L^{-1}\mathcal{O}(\gamma)$.

2.4 On the basin of attraction of the origin for the auxiliary equation

Let $A := \mathbf{g}'(0)$, where $\mathbf{g}$ is the right hand side of equation ($\mathbf{W_q}$). We have that

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 \end{bmatrix}, \tag{2.28}$$

where the c_j's are defined in (2.6). Direct computation shows that the eigenvalues of A are

$$\lambda_1 = -\eta,\ \lambda_2 = -\eta - 1,\ \lambda_3 = -\eta - 2,\ \lambda_4 = -\eta - 3\,.$$

Since all the eigenvalues of A strictly negative, the origin is an asymptotically stable equilibrium for ($\mathbf{W_q}$), and its basin of attraction

$$\mathcal{B} := \{\mathbf{w}_0 \in \mathbb{R}^4 : \mathbf{\Psi}(s, \mathbf{w}_0) \to 0 \text{ as } s \to +\infty\}$$

is an open set. We also recall that both $\mathcal{B}$ and its boundary $\partial\mathcal{B}$ are invariant with respect to ($\mathbf{W_q}$). See for instance [21].

It turns out that investigating the properties of $\mathcal{B}$ is of exceptional help in gaining knowledge of the blow-up time of solutions of ($\mathbf{U_q}$). In this section we collect several results about $\mathcal{B}$, particularly about $\partial\mathcal{B}$ and its relation with the lifespan of solutions of ($\mathbf{U_q}$).

First, we introduce the notion of *gauge* of a point with respect to a set. For a set $M \subset \mathbb{R}^4$, $M \neq \emptyset$, and for any given $\mathbf{p} \in M$ we define

$$\text{gauge}_M(\mathbf{p}) := \sup\{\alpha > 0 : D(\alpha)\mathbf{p} \in M\}, \tag{2.29}$$

with the agreement that $\text{gauge}_M(\mathbf{p}) = -\infty$ when the sup is taken over the empty set.

The following proposition reveals the key role played by the function gauge in the study of the blow-up time of solutions of ($\mathbf{U_q}$).

Proposition 2.15. *For any* $\mathbf{u}_0 \in \mathbb{R}^4$, *we have:*

$$\text{gauge}_{L^{-1}\mathcal{B}}(\mathbf{u}_0) = R_+(\mathbf{u}_0).$$

Proof. Let $\mathbf{u}_0 \in \mathbb{R}^4$, and $\alpha > 0$ be such that $D(\alpha)\mathbf{u}_0 \in L^{-1}\mathcal{B}$, then from iv) of Remark 2.1 we have $R_+(D(\alpha)\mathbf{u}_0) \geq 1$, and therefore

$$R_+(\mathbf{u}_0) = \alpha R_+(D(\alpha)\mathbf{u}_0) \geq \alpha.$$

Taking the supremum, we obtain

$$\text{gauge}_{L^{-1}\mathcal{B}}(\mathbf{u}_0) \leq R_+(\mathbf{u}_0).$$

To establish the converse inequality, we begin by considering the case $R_+(\mathbf{u}_0) < +\infty$. To simplify the notation, let $T := R_+(\mathbf{u}_0)$, so that $R_+(D(T)\mathbf{u}_0) = 1$. For any $\beta > 1$, we have

$$R_+(D(T/\beta)\mathbf{u}_0) = \beta > 1,$$

which, in light of ii) of Remark 2.1, implies

$$D(T/\beta)\mathbf{u}_0 \in L^{-1}\mathcal{B}.$$

Hence, we have

$$\frac{T}{\beta} \leq \text{gauge}_{L^{-1}\mathcal{B}}(\mathbf{u}_0), \text{ for any } \beta > 1.$$

Letting $\beta \searrow 1$, we obtain

$$T \leq \text{gauge}_{L^{-1}\mathcal{B}}(\mathbf{u}_0),$$

and the claim follows. The case $R_+(\mathbf{u}_0) = +\infty$ follows easily by observing that, in such case, we would as well have $\text{gauge}_{L^{-1}\mathcal{B}}(\mathbf{u}_0) = +\infty$. □

We conjecture (see Section 6) that $\partial\mathcal{B}$ completely characterizes the set of initial conditions that lead to blow-up at time $R_+ = 1$ in $(\mathbf{U_q})$. We are unable to prove this fact, but, instead, present two results (Propositions 2.16 and 2.17) which provide partial steps in that direction.

Proposition 2.16. *Let $\mathbf{u}_0 \in \mathbb{R}^4$. Then,*

i) if $R_+(\mathbf{u}_0) = 1$, then $L\mathbf{u}_0 \in \partial\mathcal{B}$;

ii) if $L\mathbf{u}_0 \in \partial\mathcal{B}$, then $R_+(\mathbf{u}_0) \leq 1$;

iii) if $L\mathbf{u}_0 \in \partial\mathcal{B}$ and $LD(\alpha)\mathbf{u}_0 \in \mathcal{B}$ for any $\alpha \in [0, 1)$, then $R_+(\mathbf{u}_0) = 1$.

Proof. *i)* Let $\mathbf{u}_0 \in \mathbb{R}^4$ be such that $R_+(\mathbf{u}_0) = 1$. Recall that, for any $\alpha > 0$, we have

$$R_+(D(\alpha)\mathbf{u}_0) = \frac{R_+(\mathbf{u}_0)}{\alpha} = \frac{1}{\alpha}.$$

If $\alpha > 1$, we have $R_+(D(\alpha)\mathbf{u}_0) < 1$, which implies that $LD(\alpha)\mathbf{u}_0 \notin \mathcal{B}$. On the other hand, if $\alpha < 1$, we have $R_+(D(\alpha)\mathbf{u}_0) > 1$, and hence $LD(\alpha)\mathbf{u}_0 \in \mathcal{B}$. Since $LD(\alpha)\mathbf{u}_0 \to L\mathbf{u}_0$ as $\alpha \to 1$, we have $L\mathbf{u}_0 \in \partial\mathcal{B}$.

ii) Let $\mathbf{u}_0 \in L^{-1}\partial\mathcal{B}$. This means that there exist points $\mathbf{u}_1 \in L^{-1}\mathcal{B}$ arbitrary close to $\mathbf{u}_0$. For these points, from Proposition 2.15, we have $R_+(\mathbf{u}_1) \geq 1$. It follows from the lower semicontinuity of $\mathbf{u} \mapsto R_+(\mathbf{u})$ that $R_+(\mathbf{u}_0) \leq 1$.

iii) Let $\mathbf{u}_0 \in L^{-1}\partial\mathcal{B}$ and assume $LD(\alpha)\mathbf{u}_0 \in \mathcal{B}$ for any $\alpha \in [0, 1)$. From the last assumption, it follows that $R_+(\mathbf{u}_0) = \text{gauge}_{L^{-1}\mathcal{B}}\,\mathbf{u}_0 \geq 1$. This, together with *ii)*, proves the claim. □

Proposition 2.17. *Let* $\mathbf{w}_0 \in \mathbb{R}^4$.

i) If $\mathbf{w}_0 \in \mathcal{B}$*, then* $LD(\alpha)L^{-1}\mathbf{w}_0 \in \mathcal{B}$ *for all* $\alpha \in [0,1]$*;*

ii) if $\mathbf{w}_0 \in \partial\mathcal{B}$*, then one of the following two alternatives holds:*

a) $LD(\alpha)L^{-1}\mathbf{w}_0 \in \mathcal{B}$ *for all* $\alpha \in [0,1)$*;*

b) there exists $0 < \alpha_0 < 1$ *such that*

$$LD(\alpha)L^{-1}\mathbf{w}_0 \in \partial\mathcal{B} \text{ for all } \alpha \in [\alpha_0, 1],$$
$$LD(\alpha)L^{-1}\mathbf{w}_0 \in \mathcal{B} \text{ for all } \alpha \in [0, \alpha_0).$$

Proof. *i)* Recalling Remarks 2.1 and 2.2, the claim follows by observing that

$$R_+(D(\alpha)L^{-1}\mathbf{w}_0) = \frac{R_+(L^{-1}\mathbf{w}_0)}{\alpha} > 1,$$

for all $\alpha \in (0,1)$.

ii) Assume that $\mathbf{w}_0 \in \partial\mathcal{B}$, and that *a)* does not hold. Let

$$\alpha_0 = \sup\{\alpha > 0 : LD(\alpha)L^{-1}\mathbf{w}_0 \in \mathcal{B}\} = \mathrm{gauge}_{L^{-1}\mathcal{B}}(L^{-1}\mathbf{w}_0).$$

Using the hypotheses, and recalling *i)* and the fact that $\mathcal{B}$ is open, we easily argue that $0 < \alpha_0 < 1$, and $LD(\alpha)L^{-1}\mathbf{w}_0 \in \mathcal{B}$ for all $\alpha \in [0, \alpha_0)$. Since $\mathbf{w}_0 \in \partial\mathcal{B}$, there exists a sequence $(\mathbf{w}_n)_n \subset \mathcal{B}$ such that $\mathbf{w}_n \to \mathbf{w}_0$ as $n \to +\infty$. Now, let $\alpha \in [\alpha_0, 1]$. Again, from *i)* and the fact that $\mathcal{B}$ is open, one can argue that

$$LD(\alpha)L^{-1}\mathbf{w}_0 \notin \mathcal{B},$$
$$LD(\alpha)L^{-1}\mathbf{w}_n \in \mathcal{B} \text{ for any } n,$$
$$LD(\alpha)L^{-1}\mathbf{w}_n \to LD(\alpha)L^{-1}\mathbf{w}_0 \text{ as } n \to +\infty.$$

It follows that $LD(\alpha)L^{-1}\mathbf{w}_0 \in \partial\mathcal{B}$ for all $\alpha \in [\alpha_0, 1]$. This completes the proof. □

Remark 2.18. *Part i) of Proposition 2.16 can be rephrased as*

$$\{\mathbf{u}_0 \in \mathbb{R}^4 : R_+(\mathbf{u}_0) = 1\} \subset L^{-1}\partial\mathcal{B}.$$

Alternative ii), b) of Proposition 2.17 is the only obstruction towards proving that

$$\{\mathbf{u}_0 \in \mathbb{R}^4 : R_+(\mathbf{u}_0) = 1\} = L^{-1}\partial\mathcal{B}. \tag{2.30}$$

We believe that ii), b) does not occur (see Conjecture 2), in which case, i) and iii) of Proposition 2.16 would imply (2.30).

In the remaining part of this subsection, we present some results which hold true under the additional hypothesis that $(\mathbf{W_q})$ admits a non-trivial periodic solution γ. This hypothesis is motivated by the results in the next section. First, we observe that the (possible) stable manifold of γ must be contained in $\partial\mathcal{B}$. This fact if a straightforward consequence of Remarks 2.18 and 2.1.

Remark 2.19. *Let γ be a nontrivial periodic solution of* $(\mathbf{W_q})$*, and let $\mathcal{S}$ be its stable manifold. Then*

$$\mathcal{S} \subset \{\mathbf{w}_0 \in \mathbb{R}^4 : R_+(L^{-1}\mathbf{w}_0) = 1\} \subset \partial\mathcal{B}.$$

Last, we conclude with considerations on the size of $\mathcal{B}$. Under the hypothesis that $(\mathbf{W_q})$ admits a non-trivial periodic solution, the basin of attraction of the origin is unbounded, and contains an unbounded 2-dimensional manifold.

Proposition 2.20. *Let γ be a nontrivial periodic solution of* $(\mathbf{W_q})$*, and let $\Gamma = \mathcal{O}(\gamma)$. Then,*

$$L\mathcal{D}JL^{-1}\Gamma \subset \mathcal{B}.$$

This fact is a direct consequences of Corollary 2.14 and of the following proposition.

Proposition 2.21. *Let γ be a nontrivial periodic solution of* $(\mathbf{W_q})$*, and let $\mathcal{U}$ be its unstable manifold. Then*

$$LJL^{-1}\mathcal{U} \subset \mathcal{B}.$$

Proof. Let $\mathbf{w}_0 \in LJL^{-1}\mathcal{U}$, i.e. $LJL^{-1}\mathbf{w}_0 \in \mathcal{U}$, and let $\Gamma = \mathcal{O}(\gamma)$. By definition of unstable manifold, we have that $\mathbf{\Psi}(s, LJL^{-1}\mathbf{w}_0)$ approaches Γ as $s \to -\infty$, and hence it is bounded for $s \in (-\infty, 0]$. By (2.19), we immediately have that $\mathbf{\Phi}(r, JL^{-1}\mathbf{w}_0)$ vanishes as $r \to -\infty$, and hence, by (2.4), $\mathbf{\Phi}(r, L^{-1}\mathbf{w}_0)$ vanishes as $r \to +\infty$. The claim follows by (2.18). □

2.4.1 An explicit estimate on the blow-up time

Here we provide an explicit lower bound on the blow-up time for solutions of $(\mathbf{U_q})$ by constructing an open set (a "ball" in an appropriate metric) contained in $\mathcal{B}$.

First, note that A, defined in (2.28), is a *companion* matrix. Consequently, its eigenvectors are given by the columns of the following *Vandermonde* matrix

$$V := \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{bmatrix}, \tag{2.31}$$

where the λ_i's are the eigenvalues of A arranged in decreasing order.

From now on, we denote with $\|\cdot\|_1$ the 1-norm defined as

$$\|\mathbf{w}\|_1 := \sum_{j=0}^{3} |w_j| \ ,$$

and let

$$B_1(r) := \{\mathbf{z} \in \mathbb{R}^4 \ : \ \|\mathbf{z}\|_1 < r\}$$

the ball of radius r in 1-norm.

We have the following

Proposition 2.22. $VB_1\left(\left(\frac{3}{q-1}\right)^{\frac{1}{q-1}}\right) \subset \mathcal{B}$.

Proof. Let V be as in (2.31). We can write

$$V^{-1}AV = \begin{bmatrix} -\eta & 0 & 0 & 0 \\ 0 & -\eta-1 & 0 & 0 \\ 0 & 0 & -\eta-2 & 0 \\ 0 & 0 & 0 & -\eta-3 \end{bmatrix} =: \Lambda. \tag{2.32}$$

Now define $\mathbf{z}(s) := V^{-1}\mathbf{w}(s)$. If $\mathbf{w}(\cdot)$ solves $(\mathbf{W_q})$, then

$$\mathbf{z}' = \Lambda\mathbf{z} - V^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ |w_0|^{q-1}\, w_0 \end{bmatrix} = \Lambda\mathbf{z} - |w_0|^{q-1}\, w_0\, \mathbf{c}_4\,, \tag{2.33}$$

where $\mathbf{c}_4 = \begin{bmatrix} 1/6 & -1/2 & 1/2 & -1/6 \end{bmatrix}^T$ is the last column of V^{-1}. Next, since $\mathbf{w} = V\mathbf{z}$, denoting by $\mathbf{r}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ the first row of V, we have that $w_0 = \mathbf{r}_1 \cdot \mathbf{z}$, and therefore (2.33) can be written as

$$\mathbf{z}' = \Lambda\mathbf{z} - |\mathbf{r}_1 \cdot \mathbf{z}|^{q-1}\, (\mathbf{r}_1 \cdot \mathbf{z})\, \mathbf{c}_4. \tag{2.34}$$

Here below we show that, if $\mathbf{z}$ is a solution of (2.34) with initial data $\mathbf{z}(0) = \mathbf{z}_0$ such that $\|\mathbf{z}_0\|_1 < \left(\frac{3}{q-1}\right)^{1/(q-1)}$, then $\mathbf{z} \to 0$ as $s \to +\infty$, from which the claim immediately follows.

Let $p > 1$ and denote with $\|z\|_p$ the p-norm of z, that is $\|z\|_p := \left(\sum_{j=0}^{3} |z_j|^p\right)^{1/p}$. In the following, we denote by $\mathbf{z}^{p-1}$ the vector with entries $(|z_j|^{p-2}\, z_j)_{j=0,1,2,3}$. By direct computation, we have

$$\frac{d}{ds}\frac{1}{p}\, ||\mathbf{z}||_p^p = \mathbf{z}^{p-1} \cdot \mathbf{z}' = \mathbf{z}^{p-1} \cdot \Lambda\mathbf{z} - |\mathbf{r}_1 \cdot \mathbf{z}|^{q-1}\, (\mathbf{r}_1 \cdot \mathbf{z})\, (\mathbf{z}^{p-1} \cdot \mathbf{c}_4). \tag{2.35}$$

Applying Hölder's inequality with exponents p and p', we obtain

$$\begin{aligned} \frac{d}{ds}\frac{1}{p}\, ||\mathbf{z}||_p^p &\le -\eta\, ||\mathbf{z}||_p^p + \|\mathbf{r}_1\|_{p'}^q\, \|\mathbf{z}\|_p^q\, \left\|\mathbf{z}^{p-1}\right\|_{p'} \|\mathbf{c}_4\|_p = \\ &= \|\mathbf{z}\|_p^p \left(-\eta + \|\mathbf{r}_1\|_{p'}^q\, \|\mathbf{z}\|_p^{q-1}\, \|\mathbf{c}_4\|_p\right). \end{aligned}$$

Therefore, if

$$\|\mathbf{z}\|_p < R_p := \left(\frac{\eta}{\|\mathbf{r}_1\|_{p'}^q\, \|\mathbf{c}_4\|_p}\right)^{\frac{1}{q-1}},$$

then $\|\mathbf{z}\|_p^p$ is decreasing, and by Gronwall's Lemma we obtain that $\|\mathbf{z}\|_p \to 0$ as $s \to +\infty$.

Writing $B_p(r) := \{\mathbf{z} \in \mathbb{R}^4 \,:\, \|\mathbf{z}\|_p < r\}$, we have that

$$VB_p(R_p) \subset \mathcal{B}.$$

Finally, let $\|\mathbf{z}_0\|_1 < \left(\frac{3}{q-1}\right)^{\frac{1}{q-1}}$. Since, as $p \to 1$, $R_p \to \left(\frac{3}{q-1}\right)^{1/(q-1)}$ and $\|\cdot\|_p \to \|\cdot\|_1$, we have that, for p sufficiently close to 1, $\mathbf{z}_0 \in B_p(R_p)$, and this concludes the proof. □

Proposition 2.22 translates into the following estimate for the blow-up time of solutions of $(\mathbf{U_q})$.

Corollary 2.23. *Let $\mathbf{u}_0 \in \mathbb{R}^4$. If $\alpha > 0$ is such that*

$$D(\alpha)\mathbf{u}_0 \in L^{-1} V B_1\left(\left(\frac{3}{q-1}\right)^{\frac{1}{q-1}}\right), \tag{2.36}$$

then $R_+(\mathbf{u}_0) \geq \alpha$. Therefore, setting

$$B^* = B_1\left(\left(\frac{3}{q-1}\right)^{\frac{1}{q-1}}\right),$$

we have

$$R_+(\mathbf{u}_0) \geq \mathrm{gauge}_{L^{-1}VB^*}(\mathbf{u}_0).$$

We point out that $\mathrm{gauge}_{L^{-1}VB^*}(\mathbf{u}_0)$ provides an explicit –computable–, although certainly not sharp, lower bound of the blow-up time of the solution of $(\mathbf{U_q})$ going through $\mathbf{u}_0$.

2.5 Main results for the auxiliary equation

The main result of this section is the following theorem, which establishes existence and stability properties of a periodic solution of equation $(\mathbf{W_q})$ when $q = 3$.

Theorem 2.24. *Let $q = 3$. Then,*

A) there exists a (nontrivial) τ-periodic solution $\boldsymbol{\gamma}$ of equation $(\mathbf{W_q})$, with

$$\tau \in [1.908097232050663, 1.908097232051545];$$

B) the periodic solution $\boldsymbol{\gamma}$ enjoys the following properties:

  i) $\boldsymbol{\gamma}(s + \tau/2) = -\boldsymbol{\gamma}(s)$, for all $s \in \mathbb{R}$,

  ii) each of the first two components of $\boldsymbol{\gamma}$ has exactly two distinct zeros inside $[0, \tau)$ and they are simple;

C) the periodic solution γ possesses a 3-dimensional stable manifold $\mathcal{S}$, such that any solution $\boldsymbol{\Psi}(\cdot, \mathbf{w}_0)$, with $\mathbf{w}_0 \in \mathcal{S}$, approaches asymptotically $\Gamma = \mathcal{O}(\gamma)$ in forward time. More precisely, for any $\mathbf{w}_0 \in \mathcal{S}$, there exist $s_0 \in [0, \tau)$ and $c > 0$ such that

$$\|\boldsymbol{\Psi}(s, \mathbf{w}_0) - \boldsymbol{\gamma}(s - s_0)\| \leq c e^{-\lambda s}, \text{ for all } s \geq 0,$$

where $\lambda = 7 - 4.395973655577130 \times 10^{-10}$;

D) the periodic orbit γ possesses a 2-dimensional unstable manifold $\mathcal{U}$, such that any solution $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$, with $\mathbf{w}_0 \in \mathcal{U}$, approaches asymptotically $\Gamma = \mathcal{O}(\gamma)$ in backward time. More precisely, for any $\mathbf{w}_0 \in \mathcal{U}$, there exist $s_0 \in [0, \tau)$ and $c > 0$ such that

$$\|\mathbf{\Psi}(s, \mathbf{w}_0) - \boldsymbol{\gamma}(s - s_0)\| \leq ce^{s}, \text{ for all } s \leq 0\,;$$

Moreover, $\mathcal{U} = L\mathcal{D}L^{-1}\Gamma$;

E) $\mathcal{S}$ and $\mathcal{U}$ intersect transversally along Γ.

The proofs of some facts stated in the above theorem rely on some results given in the next section.

Proof. Part A) is essentially contained in Theorem 4.14. Part B) follows from Theorems 4.14, 4.15 and 4.16. Parts C) and E) are standard consequences of the hyperbolicity of γ proved in Theorem 4.17. In particular, the value of λ is related to the smallest interval that is guaranteed to contain the largest negative Floquet exponent of $\boldsymbol{\gamma}$, see end of Section 4. The last thing we need to show is D). It follows from Theorem 2.13 that the unstable manifold of $\boldsymbol{\gamma}$ contains $L\mathcal{D}L^{-1}\Gamma$. But, by virtue of Corollary 2.11, $L\mathcal{D}L^{-1}\Gamma$ is a 2-dimensional manifold that contains Γ, Therefore $\mathcal{U}$ and $L\mathcal{D}L^{-1}\Gamma$ must coincide. □

The next theorem shows that the results of Theorem 2.24 hold essentially unchanged for q in a neighborhood of 3.

Theorem 2.25. *There exists a neighborhood (q_-, q_+) of 3 such that, for any $q \in (q_-, q_+)$, the following holds.*

A) there exists a (nontrivial) τ_q-periodic solution $\boldsymbol{\gamma}_q$ of equation $(\mathbf{W_q})$;

B) the periodic solution $\boldsymbol{\gamma}_q$ enjoys the following properties:

- *i) $\boldsymbol{\gamma}_q(s + \tau_q/2) = -\boldsymbol{\gamma}_q(s)$, for all $s \in \mathbb{R}$,*
- *ii) each of the first two components of $\boldsymbol{\gamma}_q$ has exactly two distinct zeros inside $[0, \tau_q)$ and they are simple;*

C) the periodic solution $\boldsymbol{\gamma}_q$ possesses a 3-dimensional manifold $\mathcal{S}_q$, such that any solution $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$, with $\mathbf{w}_0 \in \mathcal{S}_q$, approaches asymptotically $\Gamma_q = \mathcal{O}(\gamma_q)$ in forward time. More precisely, for any $\mathbf{w}_0 \in \mathcal{S}_q$, there exist $s_0 \in [0, \tau_q)$ and $c > 0$ such that

$$\left\|\mathbf{\Psi}(s, \mathbf{w}_0) - \boldsymbol{\gamma}_q(s - s_0)\right\| \leq ce^{-\lambda_q s}, \text{ for all } s \geq 0\,,$$

where λ_q belongs to an appropriate neighborhood of λ given in C) of Theorem 2.24.

D) the periodic solution $\boldsymbol{\gamma}_q$ possesses 2-dimensional unstable manifold, such that any solution $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$, with $\mathbf{w}_0 \in \mathcal{U}_q$, approaches asymptotically $\Gamma_q = \mathcal{O}(\gamma_q)$ in backward time. More precisely, for any $\mathbf{w}_0 \in \mathcal{U}_q$, there exist $s_0 \in [0, \tau_q)$ and $c > 0$ such that

$$\left\|\mathbf{\Psi}(s, \mathbf{w}_0) - \boldsymbol{\gamma}_q(s - s_0)\right\| \leq ce^{s}, \text{ for all } s \leq 0\,;$$

Moreover, $\mathcal{U}_q = L\mathcal{D}L^{-1}\Gamma_q$;

E) $\mathcal{S}_q$ *and* $\mathcal{U}_q$ *intersect transversally along* Γ_q*.*

Finally, τ_q*,* $\boldsymbol{\gamma}_q$*,* $\mathcal{S}_q$*,* $\mathcal{U}_q$ *and* λ_q *depend smoothly on* q*.*

Proof. We first note that the map $\mathbf{w} \mapsto \mathbf{g}(\mathbf{w}) = \mathbf{g}_q(\mathbf{w})$ defined in (2.7) is $\mathcal{C}^1$ with respect to q. Because of the hyperbolicity and symmetry *B), i)* of the periodic solution $\boldsymbol{\gamma}$ of Theorem 2.24, we are in the position to apply Theorem A.1 in the Appendix, which renders a family of periodic solutions $\boldsymbol{\gamma}_q$ that satisfy *B), i)*. The remaining part follows from the smoothness of $\boldsymbol{\gamma}_q$ with respect to q, and from the considerations in Remark A.3.□

Remark 2.26. *We point out that, by continuity of* τ_q *and* λ_q *with respect to* q*, for* q *close to* 3 *we have that* τ_q *and* λ_q *are close, respectively, to the values of* τ *and* λ *given in Theorem 2.24.*

Remark 2.27. *Note that* B), i) *of Theorem 2.25 implies that each* $\Gamma_q = \mathcal{O}(\gamma_q)$ *is symmetric with respect to the origin, i.e.* $\Gamma_q = -\Gamma_q$*.*

3 Main theorem

This section contains the main results of this paper, mainly concerned with the behavior of solutions $u = u(r)$ of $(\mathbf{U_q})$ as r approaches the endpoints of the maximal interval of existence of u. The first theorem provides a detailed characterization of the blow-up profile for solutions of $(\mathbf{U_q})$ having initial condition inside a certain subset of $\mathbb{R}^4$ when q is in an appropriate neighborhood of 3.

Theorem 3.1. *Let* $D = D(\alpha)$*,* L *and* $\mathcal{D}$ *be as in, respectively,* (2.8), (2.14) *and* (2.20)*. Let* $q \in (q_-, q_+)$*, let* γ_q *be the* τ_q*-periodic function and* $\mathcal{S}_q \subset \mathbb{R}^4$ *be the 3-dimensional manifold established in Theorem 2.25. Set*

$$\Omega_q := \mathcal{D}L^{-1}\mathcal{S}_q. \tag{3.1}$$

Then,

A) *the set* Ω_q *is unbounded, arc-connected, symmetric with respect to the origin, with non-empty interior, and invariant with respect to* $(\mathbf{U_q})$*;*

Moreover, if $u = u(r)$ *is the solution of* $(\mathbf{U_q})$ *with initial condition* $\mathbf{u}(0) = \mathbf{u}_0 \in \Omega_q$*, then*

B) u *blows up at* $0 < T < +\infty$*, where* T *is such that* $LD(T)\mathbf{u}_0 \in \mathcal{S}_q$*,*

C) *there exists an asymptotically periodic function* $w : [0, +\infty) \to \mathbb{R}$ *such that*

$$u(r) = (T - r)^{-\frac{4}{q-1}}\, w\left(\ln\left(\frac{T}{T - r}\right)\right); \tag{3.2}$$

D) w *approaches exponentially* γ_q *with asymptotic phase, that is there exist* $s_0 \in [0, \tau_q)$ *and* $\lambda_q > \dfrac{4}{q-1}$ *such that*

$$|\gamma_q(s - s_0) - w(s)| \leq ce^{-\lambda_q s}; \tag{3.3}$$

E) let $(z_j)_j$ be the sequence of positive consecutive zeros of u. Then, for j sufficiently large, z_j is simple,

$$\frac{T - z_{j+1}}{T - z_j} \to e^{-\tau_q/2} \text{ as } j \to +\infty, \tag{3.4}$$

$$\frac{z_{j+1} - z_j}{z_j - z_{j-1}} \to e^{-\tau_q/2} \text{ as } j \to +\infty, \tag{3.5}$$

and there exist $c_2 > c_1 > 0$ such that

$$c_1 \le |z_{j+1} - z_j|^{\frac{4}{q-1}} M_j \le c_2, \tag{3.6}$$

where $M_j := \max\{|u(r)| : z_j < r < z_{j+1}\}$.

Remark 3.2. *We point out that, in light of Remark 2.26, for q close to 3, we have τ_q belongs to a neighborhood of 1.90809723205 and λ_q belongs to a neighborhood of 7.*

Remark 3.3. *Combining C) and D) of the theorem above, we have that, if u solves $(\mathbf{U_q})$ with initial condition $\mathbf{u}_0 \in \Omega_q$, then there exists $s_0 \in [0, \tau_q)$ such that*

$$\left| u(r) - (T-r)^{-\frac{4}{q-1}} \gamma \left(\ln \left(\frac{T}{T-r} \right) - s_0 \right) \right| < c(T-r)^a, \text{ for all } r \in [0, T). \tag{3.7}$$

Note that, while T, s_0 and c depend on the initial condition $\mathbf{u}_0$, $a > 0$ does –not– depend on $\mathbf{u}_0$. In fact, we have $a < \min\{|\lambda|_1, |\lambda|_2\} - \eta$ where the λ_j's are the negative Floquet exponents associated to $\mathcal{S}_q$ (see Section 4); for q close to 3, one can choose a close to 5. The inequality above shows that solutions whose initial condition lies in Ω_q blow-up according to a precise profile (modulo the phase-shift s_0). Figure 2 provides a visual elucidation this fact.

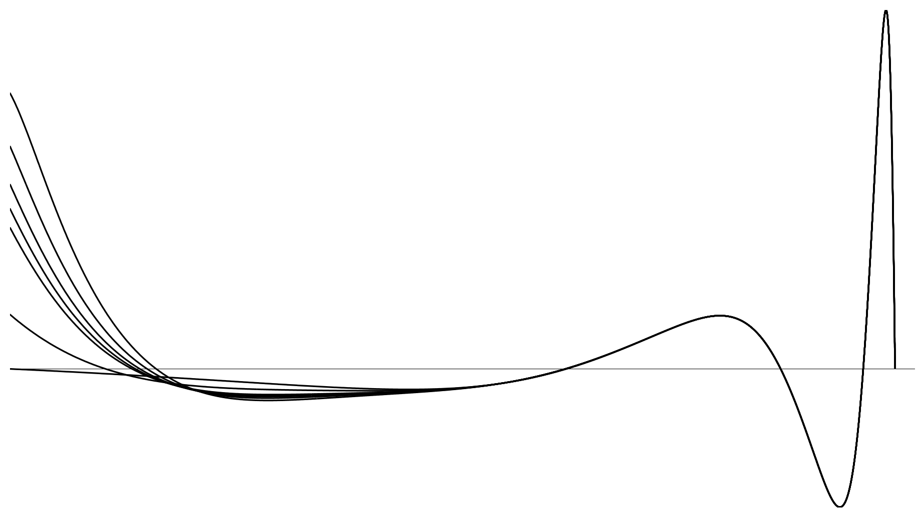

Figure 2: Convergence of some solutions of $(\mathbf{U_q})$ to their blow-up profile.

Proof. The proof relies to large extent on Theorem 2.25, which establishes, for all $q \in (q_-, q_+)$, the existence of a τ_q-periodic solution γ_q for $(\mathbf{W_q})$ which possesses a 3-dimensional stable manifold $\mathcal{S}_q$.

B), C), D) Let $u(r) = \phi(r, \mathbf{u}_0)$, with $\mathbf{u}_0 \in \Omega_q$. By definition, there exists $T > 0$ such that $LD(T)\mathbf{u}_0 \in \mathcal{S}_q$. Now, let $w(s) = \phi(s, LD(T)\mathbf{u}_0)$. Then, (3.3) follows directly from *C)* of Theorem 2.25, while (3.2) follows by virtue of (2.19) and Remark 2.2. The fact that $w(s)$ approaches a non-trivial periodic function as $s \to +\infty$ implies that u blows-up at T (see Remark 2.1).

A) The set $\Omega_q = \mathcal{D}L^{-1}\mathcal{S}_q$ is clearly unbounded since each "fiber" $\mathcal{D}L^{-1}\mathbf{w}_0$, with $\mathbf{w}_0 \in \mathcal{S}_q$, is such. Since $\mathcal{S}_q$ is connected, and hence $L^{-1}\mathcal{S}_q$ is connected, the arc-connectedness follows from the fact that each fiber is an arc connecting a point not in $L^{-1}\mathcal{S}_q$ to $L^{-1}\mathcal{S}_q$. The symmetry of Ω_q with respect to the origin comes from the analogous property of $\mathcal{S}_q$, which, in turn, is a consequence of Remark 2.27 and of the fact that, if w solves $(\mathbf{W_q})$ with initial condition $\mathbf{w}_0$, then $-w$ solves $(\mathbf{W_q})$ with initial condition $-\mathbf{w}_0$. The invariance of Ω_q with respect to $(\mathbf{U_q})$ is a consequence of Proposition 2.9, where the hypothesis that $R_+(L^{-1}\mathbf{w}_0) \leq 1$ is satisfied for all $\mathbf{w}_0 \in \mathcal{S}_q$ by virtue of *v)* in Remark 2.1. Finally, lets show that the interior of Ω_q is not empty. Let $\Gamma_q = \mathcal{O}(\gamma_q)$ be the orbit of γ_q in $\mathbb{R}^4$. For each point in Γ_q, the tangent bundle of $\mathcal{S}_q$ and that of the unstable manifold $\mathcal{U}_q$ span the whole $\mathbb{R}^4$. By continuity, they must remain transverse in a small neighborhood of Γ_q. Since the unstable manifold is given by $L\mathcal{D}L^{-1}\Gamma_q$ (see Corollary 2.14), and L is invertible, also $\mathcal{D}L^{-1}\Gamma_q$ (which is contained in Ω_q) and $L^{-1}\mathcal{S}_q$ are transverse along $L^{-1}\Gamma_q$, from which the claim follows.

E) Throughout the rest of the proof, we let

$$\varphi(r) := \ln\left(\frac{T}{T-r}\right), \text{ for } 0 \leq r < T, \tag{3.8}$$

whose inverse is given by

$$\varphi^{-1}(s) := T(1 - e^{-s}), \text{ for } s \geq 0. \tag{3.9}$$

Consider γ_q and assume, without loss of generality, that $\gamma_q(0) = 0$. Then, by virtue of *B)* of Theorem 2.25, the sequence of consecutive zeros of γ_q is given by $(j\tau_q/2)_j$, and all are simple.

From (2.11), we have that $u(z_j) = 0$ if and only if $w(s_j) = 0$, where $s_j := \varphi(z_j)$; furthermore, $(s_j)_j$ is a sequence of consecutive zeros of w. Since $w(s) - \gamma_q(s - s_0) \to 0$ as $s \to +\infty$, we have that $\gamma_q(s_j - s_0) \to 0$ as $j \to +\infty$ which implies that there exists $\nu \in \mathbb{Z}$ such that

$$s_j - s_0 - (j + \nu)\tau_q/2 \to 0 \text{ as } j \to +\infty. \tag{3.10}$$

By direct computation, one obtains

$$u'(z_j) = (T - z_j)^{-\eta-1}\left[\eta w(s_j) + w'(s_j)\right] = (T - z_j)^{-\eta-1} w'(s_j).$$

Combining $w'(s_j) - \gamma_q'(s_j - s_0) \to 0$ as $j \to +\infty$ with (3.10), we obtain that $w'(s_j) - \gamma_q'((j+\nu)\tau_q/2) \to 0$. Hence, recalling the symmetry property of γ_q (see *B), i)* of Theorem 2.25), we conclude that $|w'(s_j)| \to \left|\gamma_q'(0)\right| \neq 0$ as $j \to +\infty$. This implies that $|w'(s_j)| \neq 0$ for j sufficiently large, from which it follows that z_j is a simple zero of u for j sufficiently large.

Relation (3.4) comes from

$$s_{j+1} - s_j = \varphi(z_{j+1}) - \varphi(z_j) = \ln\left(\frac{T - z_j}{T - z_{j+1}}\right),$$

and

$$s_{j+1} - s_j - \tau_q/2 \to 0 \text{ as } j \to +\infty, \tag{3.11}$$

the last one being a direct consequence of (3.10).

Taking into account (3.11), we have

$$\begin{aligned} \frac{z_{j+1} - z_j}{z_j - z_{j-1}} e^{\tau_q/2} &= \frac{\varphi^{-1}(s_{j+1}) - \varphi^{-1}(s_j)}{\varphi^{-1}(s_j) - \varphi^{-1}(s_{j-1})} e^{\tau_q/2} = \frac{e^{-s_j} - e^{-s_{j+1}}}{e^{-s_{j-1}} - e^{-s_j}} e^{\tau_q/2} = & (3.12) \\ &= e^{s_{j-1} - s_j + \tau_q/2} \frac{1 - e^{s_j - s_{j+1}}}{1 - e^{s_{j-1} - s_j}} \to 1 \text{ as } j \to +\infty, & (3.13) \end{aligned}$$

which is (3.5).

In order to prove (3.6), we argue as follows. For any j, let $m_j \in (z_j, z_{j+1})$ be such that $|u(m_j)| = M_j$, $k_j := \varphi(m_j) \in (s_j, s_{j+1})$, and $\alpha_j := k_j - s_j$. Notice that, because of (3.11), $(\alpha_j)_j$ is bounded.

Now, observe that

$$\frac{z_{j+1} - z_j}{T - m_j} = \frac{\varphi^{-1}(s_{j+1}) - \varphi^{-1}(s_j)}{T - \varphi^{-1}(k_j)} = \frac{e^{-s_j} - e^{-s_{j+1}}}{e^{-k_j}} = e^{\alpha_j}\left(1 - e^{s_j - s_{j+1}}\right),$$

which, taking into account (3.11), assures that there exist two constants $\tilde{c}_2 > \tilde{c}_1 > 0$ such that

$$\tilde{c}_1 < \frac{z_{j+1} - z_j}{T - m_j} < \tilde{c}_2, \text{ for } j \text{ sufficiently large.}$$

Next, since

$$|z_{j+1} - z_j|^\eta M_j = |z_{j+1} - z_j|^\eta |u(m_j)| = \left|\frac{z_{j+1} - z_j}{T - m_j}\right|^\eta |w(k_j)|,$$

we have that

$$\tilde{c}_1 |w(k_j)| < |z_{j+1} - z_j|^\eta M_j < \tilde{c}_2 |w(k_j)|, \text{ for } j \text{ sufficiently large.}$$

To conclude the proof, it suffices to prove that the sequence $(w(k_j))_j$ is bounded and definitively away from zero.

The boundedness of $(w(k_j))_j$ is a consequence of the fact that w is asymptotically periodic.

Arguing by contradiction, assume that there is a subsequence $(k_{j_i})_i$ such that $w(k_{j_i}) \to 0$ as $i \to +\infty$. Up to a subsequence, we can choose the sequence $(j_i)_i$ so that the $(j_i + \nu)$'s are all even or all odd. Assume that each $j_i + \nu$ is even (the case "$j_i + \nu$ odd" is similar, and its details will be omitted). Direct computation yields

$$0 = u'(m_j) = (T - m_j)^{-\eta-1}\left[\eta w(k_j) + w'(k_j)\right].$$

Since $(w(k_{j_i}))_i$ vanishes as $i \to +\infty$, necessarily $(w'(k_{j_i}))_i$ must do the same. From the fact that, for $\beta = 0, 1$, $w^{(\beta)}(s) - \gamma_q^{(\beta)}(s - s_0) \to 0$ as $s \to +\infty$ we get that $\gamma_q^{(\beta)}(k_{j_i} - s_0) \to 0$ as $i \to +\infty$. Up to a subsequence, we can assume $\alpha_{j_i} = k_{j_i} - s_{j_i}$ to be convergent, say $\alpha_{j_i} \to d$ as $i \to +\infty$. Using the fact that γ_q is τ_q-periodic and that each $j_i + \nu$ is even, we can write

$$\gamma_q^{(\beta)}(k_{j_i} - s_0) = \gamma_q^{(\beta)}(s_{j_i} + \alpha_{j_i} - s_0) = \gamma_q^{(\beta)}(s_{j_i} - s_0 - (j_i + \nu)\tau_q/2 + \alpha_{j_i}),$$

for $\beta = 0, 1$. Letting $i \to +\infty$, and taking into account (3.10), we have that

$$0 = \gamma_q^{(\beta)}(d), \text{ for } \beta = 0, 1.$$

That is, γ_q has a non-simple zero. This is a contradiction, and concludes the proof. □

Next, we turn our attention to solutions of $(\mathbf{U_q})$ whose lifespan is unbounded. We know that any non-trivial solution $u = u(r)$ of $(\mathbf{U_q})$ cannot be globally defined (see Theorem 1.1). It could, however, be the case that blow-up occurs only in forward (or backward) time. If u is a non-trivial solution of $(\mathbf{U_q})$ that is defined, say, for all $r \geq 0$, then Gazzola and Pavani show (see [12]) that u must vanish for $r \to +\infty$. The next theorem not only shows that there is no lack of those solutions, but also provides a precise characterization of their asymptotic behavior.

Theorem 3.4. *Let (q_-, q_+) and $\mathcal{U}_q$ be as in Theorem 2.25 and let $q \in (q_-, q_+)$. For any $\mathbf{u}_0 \in \mathbb{R}^4$, let $u = u(r)$ be the solution $\phi(r, \mathbf{u}_0)$ of $(\mathbf{U_q})$. Then:*

A) if $\mathbf{u}_0 \in L^{-1}\mathcal{U}_q$, $\mathbf{u}_0 \neq 0$, then $R_-(\mathbf{u}_0) = -\infty$, $T := R_+(\mathbf{u}_0) < +\infty$ and

$$\left|u^{(j)}(r)\right| < \frac{C}{(T - r)^{\frac{4}{q-1} + j}}, \text{ for all } r \leq 0 \text{ and } j = 0, 1, 2, 3\,,$$

in particular, $\mathbf{\Phi}(r, \mathbf{u}_0) \to 0$ as $r \to -\infty$;

B) if $\mathbf{u}_0 \in JL^{-1}\mathcal{U}_q$, $\mathbf{u}_0 \neq 0$, then $R_+(\mathbf{u}_0) = +\infty$, $T = R_-(\mathbf{u}_0) > -\infty$ and

$$\left|u^{(j)}(r)\right| < \frac{C}{(T + r)^{\frac{4}{q-1} + j}}, \text{ for all } r \geq 0 \text{ and } j = 0, 1, 2, 3\,;$$

in particular, $\mathbf{\Phi}(r, \mathbf{u}_0) \to 0$ as $r \to +\infty$.

Proof. First, note that *A)* and *B)* are specular, i.e. one follows from the other by reversing the direction of r (this is the role played by J, see Remark 2.4). Therefore, we only prove *A)*.

We recall that $\mathcal{U}_q$ is the 2-dimensional unstable manifold of the τ_q-periodic solution γ_q of $(\mathbf{W_q})$ established in Theorem 2.25, and that $\mathcal{U}_q = L\mathcal{D}L^{-1}\Gamma_q$, where $\Gamma_q = \mathcal{O}(\gamma_q)$. Let $\mathbf{u}_0 \in L^{-1}\mathcal{U}_q = \mathcal{D}L^{-1}\Gamma_q$, $\mathbf{u}_0 \neq 0$. By definition, there exists $T > 0$ such that $LD(T)\mathbf{u}_0 \in \Gamma_q$. This means that $\phi(s, LD(T)\mathbf{u}_0) = \gamma_q(s - s_0)$ for some $s_0 \in [0, \tau_q)$. Then, from (2.19) and Remark 2.2, we deduce that

$$u(r) = (T - r)^{-\frac{4}{q-1}}\, \gamma_q\left(\ln\left(\frac{T}{T - r}\right) - s_0\right). \tag{3.14}$$

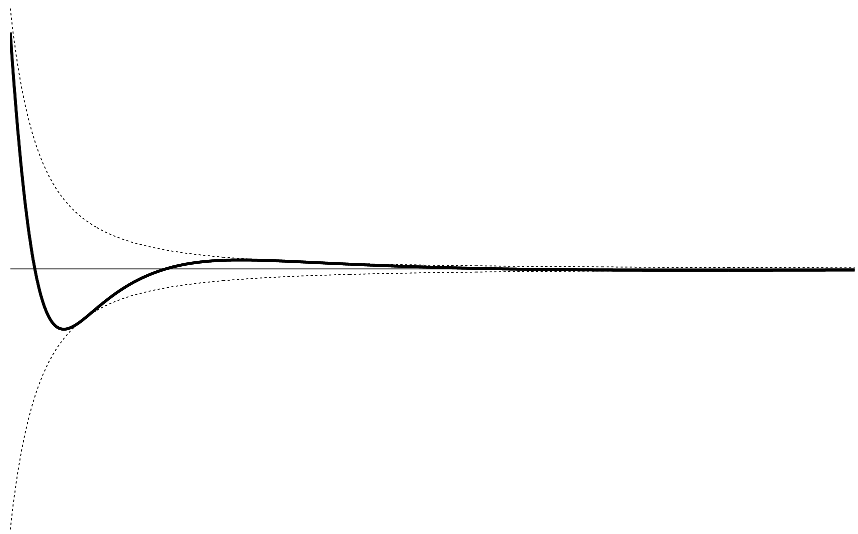

Figure 3: A solution (solid line) that vanishes in forward time enclosed in its envelope (dotted line).

This yields the desired inequality for $j = 0$. Differentiating, we get the claim for $j = 1, 2, 3$. □

From (3.14) we see that solutions that vanish as $s \to \pm\infty$ do so by means of damped oscillations. Figure 3 depicts one of those solutions.

3.1 Preserved quantities

As observed in [18], equation ($\mathbf{U_q}$) has a first integral, or energy identity,

$$\mathcal{E}(u) := \frac{1}{2}u''^2 - u'u''' - F(u(r)) = \text{ constant } = E_u, \tag{3.15}$$

where $F' = f$. Clearly the constant E_u depends on the solution u, and hence on the initial condition: $E_u = E_u(\mathbf{u}_0)$.

Furthermore defining

$$\mathcal{H} := u'u'' - uu''', \tag{3.16}$$

one can recognize that $\mathcal{H}$ along the nontrivial solutions of ($\mathbf{U_q}$) is increasing. Indeed, since

$$\mathcal{H}'(r) = u''^2 + u\,u'''' = u''^2 + u\,f(u) \geq 0$$

we have that $\mathcal{H}$ is non decreasing. Writing $\mathcal{H}$ as

$$\mathcal{H}(r) = \mathcal{H}(0) + \int_0^r u''^2(s)ds + \int_0^r u(s)\,f(u(s))ds.$$

one recognizes that, if u is non-trivial, then $\mathcal{H}$ is strictly increasing (see also [11]). Combining this last fact with [12, Lemma 9], one obtains the following proposition, the proof of which is straightforward and, therefore, omitted.

Proposition 3.5. *A solution u of $(\mathbf{U_q})$ blows up at $R_+ < \infty$ if and only if there exists r_0 such that $\mathcal{H}(r_0) > 0$. Moreover, if $\mathcal{H}(r_0) > 0$ for some r_0, then $\mathcal{H}(r) \to +\infty$ as $r \to R_+$.*

A consequence of the proposition above is that the set of initial conditions that lead to blow-up in finite forward time is open.

Theorem 3.6. *The sets $\{\mathbf{u}_0 \in \mathbb{R}^4 : R_+(\mathbf{u}_0) < +\infty\}$, $\{\mathbf{u}_0 \in \mathbb{R}^4 : R_-(\mathbf{u}_0) > -\infty\}$ and $\{\mathbf{u}_0 \in \mathbb{R}^4 : -\infty < R_-(\mathbf{u}_0) < R_+(\mathbf{u}_0) < +\infty\}$ are open.*

Proof. Clearly it suffices to prove the claim for the first set. Let $\mathbf{u}_0 \in \mathbb{R}^4$ be such that $R_+(\mathbf{u}_0) < +\infty$. Throughout this proof, to highlight the dependence of $\mathcal{H}$ on $\mathbf{u}_0$, we will write $\mathcal{H}(r) = \mathcal{H}(\mathbf{\Phi}(r, \mathbf{u}_0))$. By Proposition 3.5, we have that $\mathcal{O}(\mathbf{u}_0)$ must eventually enter the region where $\mathcal{H} > 0$, i.e. $\mathcal{H}(\mathbf{\Phi}(r_0, \mathbf{u}_0)) > 0$ for some r_0. Observing that $\mathcal{H}(\mathbf{\Phi}(r_0, \cdot))$ is well defined and continuous in a neighborhood of $\mathbf{u}_0$, we have that $\mathcal{H}(\mathbf{\Phi}(r_0, \cdot))$ must remain positive in a suitable neighborhood of $\mathbf{u}_0$. The claim follows appealing again to Proposition 3.5. □

Proposition 3.5 also states that, if blow-up occurs at $R_+ < +\infty$, then $\mathcal{H}$ diverges as r approaches R_+. The following result provides, under suitable hypotheses on the initial condition, an estimate on the order of infinity of $\mathcal{H}$.

Proposition 3.7. *Assume that $(\mathbf{W_q})$ admits a periodic solution possessing a stable manifold $\mathcal{S}$. Let $\mathbf{u} = \mathbf{\Phi}(\cdot, \mathbf{u}_0)$ be a solution of $(\mathbf{U_q})$and let $\mathcal{H}$ be defined as in (3.16). If $\mathbf{u}_0 \in \mathcal{D}L^{-1}\mathcal{S}$, then there exist $c_2 > c_1 > 0$ such that*

$$c_1(R_+ - r)^{-\frac{3q+5}{q-1}} < \mathcal{H}(r) < c_2(R_+ - r)^{-\frac{3q+5}{q-1}}, \text{ for } r \text{ close to } R_+. \tag{3.17}$$

This result is based on a counterpart of $\mathcal{E}$ and $\mathcal{H}$ for solutions of $(\mathbf{W_q})$ that we present below. In what follows, we denote with A and B the following matrices

$$A := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and notice that the energies $\mathcal{E}$ and $\mathcal{H}$ can be written, respectively, as

$$\mathcal{E} = \frac{1}{2}\mathbf{u}^T B\mathbf{u} - \frac{1}{q+1}|u_0|^{q+1} = E_u(\mathbf{u}_0), \quad \mathcal{H} = \frac{1}{2}\mathbf{u}^T A\mathbf{u}. \tag{3.18}$$

Proposition 3.8. *Let $\mathbf{w} = \mathbf{\Psi}(\cdot, \mathbf{w}_0)$ be a solution of $(\mathbf{W_q})$. Then*

A) There exists a constant $E_w = E_w(\mathbf{w}_0) \in \mathbb{R}$ such that the identity

$$\frac{1}{2}\mathbf{w}^T(s){L^{-1}}^T BL^{-1}\mathbf{w}(s) - \frac{1}{q+1}|w_0(s)|^{q+1} = E_w e^{-4\frac{q+1}{q-1}s} \tag{3.19}$$

holds for any $s \in (S_-, S_+)$.[1] *Moreover,*

[1] The identity (3.19) can be written explicitly as

$$\frac{1}{2}{w_2}^2 - \frac{\eta^2}{2}\left(3 + 4\eta + \eta^2\right){w_0}^2 - \eta\left(2\left(1+\eta\right)w_2 + \left(3 + 2\eta^2 + 6\eta\right)w_1 + w_3\right)w_0 +$$
$$-\frac{1}{2}w_1\left(2(\eta+2)w_2 + \left(2\eta^2 + 3 + 8\eta\right)w_1 + 2w_3\right) - \frac{1}{q+1}|w_0|^{q+1} = E_w e^{-(2\eta+4)s}$$

i) the constants $E_w(\mathbf{w}_0)$ *and* $E_u(\mathbf{u}_0)$ *coincide provided that* $\mathbf{w}_0 = L\mathbf{u}_0$,

ii) if $\mathbf{w}$ *is periodic, then* $E_w(\mathbf{w}_0) = 0$.

B) If $\mathbf{w}_0 \neq 0$, *the quantity*

$$\frac{1}{2}[\mathbf{w}(s)^T {L^{-1}}^T A L^{-1}\mathbf{w}(s)]e^{\frac{3q+5}{q-1}s} =: h(s)e^{\frac{3q+5}{q-1}s} \tag{3.20}$$

is strictly increasing[2]. *Moreover,*

i) if $\mathbf{\Phi}(\cdot, L^{-1}\mathbf{w}_0)$ *blows up at* $R_+(L^{-1}\mathbf{w}_0) \leq 1$, *then*

$$\mathbf{w}(s)^T {L^{-1}}^T A L^{-1}\mathbf{w}(s) > 0, \text{ for } s \text{ close to } S_+(\mathbf{w}_0),$$

ii) if $\mathbf{w}$ *is periodic then*

$$\mathbf{w}^T {L^{-1}}^T A L^{-1}\mathbf{w} > c > 0, \text{ for all } s \in \mathbb{R}.$$

iii) let γ *be a periodic solution of* $(\mathbf{W_q})$; *if* $\mathbf{\Psi}(\cdot, \mathbf{w}_0)$ *approaches* γ *with asymptotic phase as* $s \to +\infty$, *then*

$$\mathbf{w}(s)^T {L^{-1}}^T A L^{-1}\mathbf{w}(s) > c > 0 \text{ for } s \text{ sufficiently large.}$$

Proof. Combining (3.18), (2.19), and the fact that

$$D(\alpha)BD(\alpha) = \alpha^{2\eta+4}B,$$

we obtain (3.19) and the claim *A), i)*. The claim *A), ii)* follows observing that the left hand side of (3.19) is periodic, while the right hand side is strictly monotone for $E_w \neq 0$.

To prove the claims in *B)*, first we notice that, since $\varphi^{-1}(s)$ is increasing, also $\mathcal{H}(\varphi^{-1}(s))$ is increasing. Combining (3.18), (2.19), and the fact that

$$D(\alpha)AD(\alpha) = \alpha^{2\eta+3}A,$$

we obtain that (3.20) is increasing. Proposition 3.5 yields

$$\lim_{r\to R_+} \mathcal{H}(r) = +\infty$$

and hence

$$\lim_{s\to S_+} \mathcal{H}(\varphi^{-1}(s)) = +\infty.$$

From which *B), i)* follows. The claim *B), ii)* is a direct consequences of *B), i)* and of the periodicity of $\mathbf{w}$. The claim *B), iii)* follows from *B), ii)* since $\mathbf{w}(s)$ gets arbitrarily close to γ for s large enough. □

Proof of Proposition 3.7. Without loss of generality, we may assume that $R_+ = 1$. We have that $\mathcal{H}(r) = h(s)e^{\frac{3q+5}{q-1}s}$, where $h(s)$ is bounded from below as stated in Proposition 3.8, and from above since $\mathbf{w}(s)$ approaches the periodic solution as $s \to +\infty$. The claim follows noticing that $e^s = (R_+ - 1)^{-1}$. □

[2] Or, explicitly, the quantity

$$e^{\frac{3q+5}{q-1}s}\left[\ w_2 w_1 + {w_1}^2 + 2\eta\, {w_1}^2 - 2\eta\,(1+\eta)\, {w_0}^2 - (4\eta\, w_1 + w_3 + (2\,\eta+3)\, w_2 + 2\, w_1)\, w_0\ \right]$$

is increasing.

Remark 3.9. *From* A), ii) *we have that the orbit corresponding to any possible periodic solution of* $(\mathbf{W_q})$ *must lie on the manifold given by*

$$\frac{1}{2}\mathbf{w}^T(s){L^{-1}}^T BL^{-1}\mathbf{w}(s) - \frac{1}{q+1}\left|w_0(s)\right|^{q+1} = 0.$$

4 Rigorous numerics for the periodic solution and the Floquet exponents

In this section we show that equation $(\mathbf{W_q})$ with $q = 3$, that is

$$w'''' + 14w''' + 71w'' + 154w' + 120w + w^3 = 0, \tag{4.1}$$

has a nontrivial symmetric periodic solution (see Theorem 4.14). The proof is obtained using the general method introduced in [15]. Once the periodic solution is obtained, we adapt slightly the ideas of [15] to show that the corresponding periodic orbit is hyperbolic, and that it has two stable Floquet exponents and one unstable Floquet exponent (see Theorem 4.17).

Denote by $w(t)$ an a priori unknown $\frac{2\pi}{\omega}$-periodic solution to this system. Denote its Fourier expansion as

$$w(t) = \sum_{k\in\mathbb{Z}} a_k e^{\mathbf{i}\omega k t}. \tag{4.2}$$

The unknowns for this problem are the frequency ω and the sequence of Fourier coefficients $a = \{a_k\}_{k\in\mathbb{Z}}$ of the periodic solution $w(t)$. Since the differential equation (4.1) is analytic, any periodic solution is analytic. This implies that the Fourier coefficients of $w(t)$ decay exponentially fast to 0. Therefore, the infinite dimensional vector of unknowns $x \stackrel{\text{def}}{=} (\omega, a)$ is an element of the infinite dimensional space

$$X \stackrel{\text{def}}{=} \mathbb{C} \times \ell^1_\nu, \tag{4.3}$$

where

$$\ell^1_\nu \stackrel{\text{def}}{=} \left\{a = \{a_k\}_{k\in\mathbb{Z}} \,|\, a_k \in \mathbb{C} \ \text{ and } \ \|a\|_\nu < \infty\right\}, \tag{4.4}$$

with

$$\|a\|_\nu \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} |a_k|\nu^{|k|}, \tag{4.5}$$

for some fixed weight $\nu \geq 1$. Note that with $\nu = 1$ the space ℓ^1_ν is the classical Wiener algebra.

We denote by $x = (\omega, a)$ an element of X and endow the space with the norm

$$\|x\|_X = \max\left(|\omega|, \|a\|_\nu\right). \tag{4.6}$$

Lemma 4.1. *The function space* ℓ^1_ν *as defined in* (4.4) *is a Banach algebra under discrete convolution, that is it is a Banach space and for any* $a, b \in \ell^1_\nu$, $a * b = \{(a*b)_k\}_{k\in\mathbb{Z}}$ *defined by*

$$(a*b)_k \stackrel{\text{def}}{=} \sum_{k_1+k_2=k} a_{k_1} b_{k_2} \tag{4.7}$$

satisfy $a * b \in \ell^1_\nu$ *and* $\|a*b\|_\nu \leq \|a\|_\nu \|b\|_\nu$.

Proof. We omit the proof that X is a Banach space. Let $a, b \in \ell_\nu^1$, that is $\|a\|_\nu, \|b\|_\nu < \infty$. Consider $a * b$ defined component-wise by (4.7). Then,

$$
\begin{aligned}
\|a*b\|_\nu &= \sum_{k\in\mathbb{Z}} |(a*b)_k|\nu^{|k|} = \sum_{k\in\mathbb{Z}} \left| \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} a_{k_1} b_{k_2} \right| \nu^{|k|} \\
&\le \sum_{k\in\mathbb{Z}} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} |a_{k_1}||b_{k_2}|\nu^{|k|} \le \sum_{k\in\mathbb{Z}} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} |a_{k_1}|\nu^{|k_1|}|b_{k_2}|\nu^{|k_2|} \\
&\le \left(\sum_{k_1\in\mathbb{Z}} |a_{k_1}|\nu^{|k_1|} \right) \left(\sum_{k_2\in\mathbb{Z}} |b_{k_2}|\nu^{|k_2|} \right) = \|a\|_\nu \|b\|_\nu .
\end{aligned}
$$

That shows that X is a Banach algebra. □

Plugging (4.2) in the differential equation (4.1), we obtain that

$$\sum_{k\in\mathbb{Z}} \left(\mu_k(\omega) a_k + (a^3)_k \right) e^{\mathbf{i}\omega k t} = 0, \tag{4.8}$$

where

$$\mu_k(\omega) \stackrel{\text{def}}{=} \omega^4 k^4 - 14\mathbf{i}\omega^3 k^3 - 71\omega^2 k^2 + 154\mathbf{i}\omega k + 120, \tag{4.9}$$

and

$$(a^3)_k \stackrel{\text{def}}{=} \sum_{k_1+k_2+k_3=k} a_{k_1} a_{k_2} a_{k_3}. \tag{4.10}$$

Therefore, given $x \in X$, define $F_1(x) = \{(F_1(x))_k\}_{k\in\mathbb{Z}}$ component-wise by

$$(F_1(x))_k \stackrel{\text{def}}{=} \mu_k(\omega) a_k + (a^3)_k. \tag{4.11}$$

Throughout this section, we denote by $\operatorname{conj}(z)$ the complex conjugate of the complex number $z \in \mathbb{C}$. From the previous discussion, we have the following result.

Lemma 4.2. *Fix an exponential decay rate $\nu > 1$. If $x = (\omega, a) \in \mathbb{R} \times \ell_\nu^1$ solves $F_1(x) = 0$, then (4.8) is satisfied for all $t \in \mathbb{R}$, that is the function $w(t)$ as defined by (4.2) is a $\frac{2\pi}{\omega}$-periodic solution of (4.1). Moreover, if the sequence of Fourier coefficients $a = (a_k)_{k\in\mathbb{Z}}$ of (4.2) satisfy $a_{-k} = \operatorname{conj}(a_k)$, then the periodic solution $w(t)$ is real.*

The idea of the rigorous numerical method is to compute a numerical approximation $\bar{x}$ of $F_1 = 0$ and then use the Contraction Mapping Theorem (CMT) to show that close to $\bar{x}$, there exists $\tilde{x} \in X$ such that $F_1(\tilde{x}) = 0$. The CMT requires to have a locally isolated solution. Note that periodic solutions of the form $x = (\omega, a)$ given by $w(t)$ represented by (4.2) are not isolated as any time shift of the form $w(t+\tau)$ is a periodic solution. In order to isolate the periodic solution in the space X, we introduce a *phase condition.* We impose that the solution u given by (4.2) satisfies

$$F_0(x) \stackrel{\text{def}}{=} \sum_{|k|\le 3} a_k = 0. \tag{4.12}$$

Note that $F_0(x)$ is a rough approximation for the value $w(0) = \sum_{k\in\mathbb{Z}} a_k$. Given any $x \in X$, recall (4.12) and (4.11), and set

$$F(x) \stackrel{\text{def}}{=} \begin{pmatrix} F_0(x) \\ F_1(x) \end{pmatrix}. \tag{4.13}$$

Before proceeding any further, let us recall some basis tools from functional analysis which will be useful to perform the computer-assisted proofs based on the radii polynomial approach. The following content is essentially taken from the discussions in [15].

4.1 Basic functional analytic background

Recall the classical fact that the dual space of ℓ_1^1, which is denoted $(\ell_1^1)^*$, is the space ℓ^∞. Similarly if $\nu > 1$ then the dual of ℓ_ν^1 is a weighted "ell-infinity" space which we define now. For a bi-infinite sequence of complex numbers $c = \{c_k\}_{k\in\mathbb{Z}}$, the ν-weighted supremum norm is defined by

$$\|c\|_\nu^\infty \stackrel{\text{def}}{=} \sup_{k\in\mathbb{Z}} \frac{|c_k|}{\nu^{|k|}}. \tag{4.14}$$

Let

$$\ell_\nu^\infty = \{c = \{c_k\}_{k\in\mathbb{Z}} \mid c_k \in \mathbb{C} \ \forall\ k \in \mathbb{Z}, \text{ and } \|c\|_\nu^\infty < \infty\}. \tag{4.15}$$

The key to the proof that $\ell_\nu^\infty = (\ell_\nu^1)^*$ is the following bound which is itself useful in the sequel.

Lemma 4.3. *Suppose that $a \in \ell_\nu^1$ and $c \in \ell_\nu^\infty$. Then*

$$\left|\sum_{k\in\mathbb{Z}} c_k a_k\right| \le \sum_{k\in\mathbb{Z}} |c_k||a_k| \le \|c\|_\nu^\infty \|a\|_\nu.$$

The following results states that ℓ_ν^∞ is the dual of ℓ_ν^1, in the sense of isometric isomorphism. It follows that any linear functional on ℓ_ν^1 can be represented as an element of ℓ_ν^∞, and that the operator norm can be computed by taking the weighted "ell-infinity" norm of the corresponding sequence.

Theorem 4.4. *For any $\nu \ge 1$ we have that $(\ell_\nu^1)^* \cong \ell_\nu^\infty$.*

A related result, which is not usually stated but which is useful in the work to follow, is the following isometric isomorphism theorem for linear maps from $\mathbb{C}$ into ℓ_ν^1.

Lemma 4.5. *The set $B(\mathbb{C}, \ell_\nu^1)$ of bounded linear maps from $\mathbb{C}$ into ℓ_ν^1 is isometrically isomorphic to ℓ_ν^1. Specifically $l \in B(\mathbb{C}, \ell_\nu^1)$ if and only if there exists $a \in \ell_\nu^1$ so that $l(z) = za$, for all $z \in \mathbb{C}$. Moreover $\|l\|_{B(\mathbb{C},\ell_\nu^1)} = \|a\|_\nu$.*

The following result is a consequence of Lemma 4.3, and provides a useful and explicit bound on the norm of an "eventually diagonal" linear operator on ℓ_ν^1. The proof is omitted.

Corollary 4.6. *Let $A^{(m)}$ be an $(2m-1) \times (2m-1)$ matrix with complex valued entries, $\{\delta_k\}_{|k|\ge m}$ a bi-infinite sequence of complex numbers and $\delta_m > 0$ a real number such that*

$$|\delta_k| \le \delta_m, \quad \text{for all } |k| \ge m.$$

Given $a = (a_k)_{k\in\mathbb{Z}} \in \ell^1_\nu$*, denote by* $a^{(m)} = (a_{-m+1}, \dots, a_{-1}, a_0, a_1, \dots, a_{m-1}) \in \mathbb{C}^{2m-1}$*. Define the map* $A\colon \ell^1_\nu \to \ell^1_\nu$ *by*

$$[A(a)]_k = \begin{cases} [A^{(m)}a^{(m)}]_k, & |k| < m \\ \delta_k a_k, & |k| \geq m. \end{cases}$$

Then A *is a bounded linear operator and*

$$\|A\|_{B(\ell^1_\nu,\ell^1_\nu)} \leq \max(K, \delta_m),$$

where

$$K \stackrel{\text{def}}{=} \max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |A_{k,n}|\nu^{|k|}. \tag{4.16}$$

4.2 Rigorous computation of the periodic solution

We are ready to present the rigorous computational method to find a nontrivial zero of (4.13). This approach requires first the computation of a numerical approximation which is obtained by consider a finite dimensional projection. Given $a = (a_k)_{k\in\mathbb{Z}} \in \ell^1_\nu$ denote by $a_F = (a_k)_{|k|<m} \in \mathbb{C}^{2m-1}$ a finite part of a of size $2m-1$. Consider a finite dimensional projection $F^{(m)}$ of (4.13) given by

$$F^{(m)}(\omega, a_F) = \begin{pmatrix} F_0(\omega, a_F) \\ F_1^{(m)}(\omega, a_F) \end{pmatrix}, \tag{4.17}$$

where $F_1^{(m)}(\omega, a_F) \in \mathbb{C}^{2m-1}$ corresponds to the finite part of F_1 of size $2m-1$, that is $F_1^{(m)} = \{(F_1^{(m)})_k\}_{|k|<m}$. More explicitly,

$$(F_1)_k^{(m)}(x^{(m)}) = \mu_k(\omega)a_k + \sum_{\substack{k_1+k_2+k_3=k \\ |k_i|<m}} a_{k_1}a_{k_2}a_{k_3}.$$

We have that $F^{(m)} : \mathbb{C} \times \mathbb{C}^{2m-1} \to \mathbb{C} \times \mathbb{C}^{2m-1}$, and we seek a numerical solution of the finite dimensional problem $F^{(m)} = 0$ using Newton's method. Let $\bar{x} = (\bar{\omega}, \bar{a}) \in \mathbb{C} \times \mathbb{C}^{2m-1}$ be the approximate solution of $F^{(m)} = 0$.

We would now like to use a Newton-Kantorovich argument to establish the existence of a solution of $F = 0$ in X near $\bar{x}$. Note that F does not map X into itself. This is because a differential operator is in general unbounded on ℓ^1_ν. In order to overcome this problem we look for an injective linear smoothing operator A such that

$$AF(x) \in X, \tag{4.18}$$

for all $x \in X$. The choice of the approximate inverse A is presented in Section 4.2.2. For now we take A as given and define the *Newton-like operator* by

$$T(x) = x - AF(x). \tag{4.19}$$

The injectivity of A implies that x is a solution of $F(x) = 0$ if and only if it is a fixed point of T. Moreover since T now maps X back into itself we study (4.19) via the contraction mapping theorem applied on closed balls centered at the approximation $\bar{x}$.

Recall the definition of the norm on X in (4.6), denote by $B(r) = \{x : \|x\|_X \leq r\} \subset X$ the closed ball of radius r in X and denote

$$B_{\bar{x}}(r) \stackrel{\text{def}}{=} \bar{x} + B(r). \tag{4.20}$$

Given $\bar{x} = (\bar{\omega}, \bar{a})$, with $\bar{a} = (\bar{a}_{-m+1}, \ldots, \bar{a}_{-1}, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{m-1})$, define the bounds

$$\begin{aligned} Y &= (Y_0, Y_1) \\ Z(r) &= (Z_0(r), Z_1(r)) \end{aligned} \tag{4.21}$$

with $Y_0, Z_0(r) \in \mathbb{R}$ and $Y_1 = ((Y_1)_k)_{k\in\mathbb{Z}}, Z_1(r) = ((Z_1(r))_k)_{k\in\mathbb{Z}}$ satisfying

$$\begin{aligned} &|[T(\bar{x}) - \bar{x}]_0| \leq Y_0 \ \text{ and } \sup_{b,c\in B(r)} |DT_0(\bar{x} + b)c| \leq Z_0(r) \\ &|([T(\bar{x}) - \bar{x}]_1)_k| \leq (Y_1)_k \ \text{ and } \sup_{b,c\in B(r)} |[D(T_1)_k(\bar{x} + b)c]| \leq (Z_1(r))_k. \end{aligned} \tag{4.22}$$

The proof of the following result can be found in [15].

Proposition 4.7. *Consider the bounds Y and $Z(r)$ as in* (4.21) *and satisfying the component-wise inequalities* (4.22)*. If $\|Y\|_X + \|Z(r)\|_X < r$, then $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction. Moreover, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $F(\tilde{x}) = 0$.*

Consider an upper bound $\mathbf{Y}_1$ such that $\|Y_1\|_\nu \leq \mathbf{Y}_1$ and an upper bound $\mathbf{Z}_1(r)$ such that $\|Z_1(r)\|_\nu \leq \mathbf{Z}_1(r)$.

Definition 4.8. Given the bounds Y and $Z(r)$ satisfying (4.22) we define the *radii polynomials* p_0 and p_1 by

$$p_0(r) \stackrel{\text{def}}{=} Z_0(r) - r + Y_0 \tag{4.23}$$
$$p_1(r) \stackrel{\text{def}}{=} \mathbf{Z}_1(r) - r + \mathbf{Y}_1. \tag{4.24}$$

The next result (for the proof see [15]) shows that the radii polynomials provide an efficient strategy for obtaining sets on which the corresponding Newton-like operator T as defined in (4.19) is a contraction mapping.

Proposition 4.9. *Fix $\nu \geq 1$ an exponential decay rate and construct the radii polynomials $p_0(r)$ and $p_1(r)$ of Definition 4.8. Define*

$$\mathcal{I} \stackrel{\text{def}}{=} \{r > 0 \mid p_0(r) < 0\} \bigcap \{r > 0 \mid p_1(r) < 0\}. \tag{4.25}$$

If $\mathcal{I} \neq \emptyset$, then $\mathcal{I}$ is an open interval, and for any $r \in \mathcal{I}$, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $F(\tilde{x}) = 0$.

We will demonstrate the existence of a periodic solution of (4.1) by constructing the radii polynomials of Definition 4.8, and by applying Proposition 4.9. The construction of the polynomials will use the results of Section 4.1 and will require some computations using MATLAB and the interval arithmetic toolbox INTLAB [20].

4.2.1 Symmetry of the fixed points of T

We are interested in showing the existence of a periodic solution u given by (4.2) that is real and that satisfies the symmetry property

$$a_{2j} = 0, \quad \forall\, j \in \mathbb{Z}. \tag{4.26}$$

To do this, we design the method so that fixed points of T are in the *symmetry* space

$$X_{sym} \stackrel{\text{def}}{=} \mathbb{R} \times \tilde{\ell}^1_\nu, \tag{4.27}$$

where

$$\tilde{\ell}^1_\nu \stackrel{\text{def}}{=} \left\{ a \in \ell^1_\nu \,|\, a_{-k} = \operatorname{conj}(a_k) \ \ \forall\, k \in \mathbb{Z}, \ \text{ and } \ a_{2j} = 0 \ \ \forall\, j \in \mathbb{Z} \right\}. \tag{4.28}$$

Remark 4.10. *The condition $a_{-k} = \operatorname{conj}(a_k)$ is imposed in the function space ℓ^1_ν because we want u to be a real periodic solution, that is $\operatorname{conj}(w(t)) = w(t)$.*

Lemma 4.11. *Assume that $\bar{x} \in X_{sym}$ and consider the closed ball $B_{\bar{x}}(r) \subset X$ as in (4.20). Define T as in (4.19) and assume that the approximate inverse A satisfies*

$$AF : X_{sym} \to X_{sym}. \tag{4.29}$$

Assume that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction, and let $\tilde{x} \in X$ the unique fixed point of T in $B_{\bar{x}}(r)$ which exists by the contraction mapping theorem. Then, $\tilde{x} \in X_{sym}$.

Proof. By (4.29), $T : X_{sym} \to X_{sym}$. Using that $\bar{x} \in X_{sym} \cap B_{\bar{x}}(r)$, and that X_{sym} is a closed subset of X, we obtain that

$$\tilde{x} = \lim_{n\to\infty} T^n(\bar{x}) \in X_{sym}.$$

□

We now introduce an approximate inverse operator A that satisfies (4.29).

4.2.2 Definition of the approximate inverse operator A

In this section, we define an approximate inverse A for $DF(\bar{x})$ so that (4.29) holds. We begin the process assuming the existence of $\bar{x} = (\bar{\omega}, \bar{a}) \in X_{sym}$ so that $F(\bar{x}) \approx 0$. The Fréchet derivative $DF(\bar{x})$ can be visualized as

$$DF(\bar{x}) = \begin{bmatrix} 0 & D_aF_0(\bar{x}) \\ \partial_\omega F_1(\bar{x}) & D_aF_1(\bar{x}) \end{bmatrix},$$

since $\partial_\omega F_0(\bar{x}) = 0$, and where

$$\begin{cases} \partial_\omega F_1(\bar{x}) : \mathbb{R} \to \ell^1_\nu, \\ D_aF_0(\bar{x}) : \ell^1_\nu \to \mathbb{R} \text{ is a linear functional} \\ D_aF_1(\bar{x}) : \ell^1_\nu \to \ell^1_{\nu'} \text{ is a linear operator with } \nu' < \nu. \end{cases}$$

We first approximate $DF(\bar{x})$ with the operator

$$A^\dagger \stackrel{\text{def}}{=} \begin{bmatrix} 0 & A^\dagger_{a,0} \\ A^\dagger_{\omega,1} & A^\dagger_{a,1} \end{bmatrix},$$

which acts on $b=(b_0,b_1)$ component-wise as

$$\begin{aligned}(A^\dagger b)_0 &= A^\dagger_{a,0}\cdot b_1 \stackrel{\text{def}}{=} D_{a^{(m)}}F_0(\bar{x})\cdot b_1^{(m)}\\ (A^\dagger b)_1 &= A^\dagger_{\omega,1}b_0 + A^\dagger_{a,1}b_1 \in \ell^1_{\nu'},\end{aligned}$$

where $A^\dagger_{\omega,1}=\partial_\omega F_1^{(m)}(\bar{x})$ and $A^\dagger_{a,1}b_1\in\ell^1_{\nu'}$ is defined component-wise by

$$\left(A^\dagger_{a,1}b_1\right)_k = \begin{cases}\left(D_{a^{(m)}}F_1^{(m)}(\bar{x})b_1^{(m)}\right)_k, & |k|<m\\ \mu_k(\bar{\omega})(b_1)_k, & |k|\geq m.\end{cases}$$

Let $A^{(m)}$ a finite dimensional approximate inverse of $DF^{(m)}(\bar{x})$ which is obtained numerically and which has the decomposition

$$A^{(m)} = \begin{bmatrix} A^{(m)}_{\omega,0} & A^{(m)}_{a,0}\\ A^{(m)}_{\omega,1} & A^{(m)}_{a,1}\end{bmatrix}\in\mathbb{C}^{2m\times 2m},$$

where $A^{(m)}_{\omega,0}\in\mathbb{R}$, $A^{(m)}_{a,0}\in\mathbb{C}^{1\times(2m-1)}$, $A^{(m)}_{\omega,1}\in\mathbb{C}^{(2m-1)\times 1}$ and $A^{(m)}_{a,1}\in\mathbb{C}^{(2m-1)\times(2m-1)}$. Assume moreover that $A^{(m)}$ satisfies the following *symmetry* assumptions:

$$\begin{aligned}&1.\ (A^{(m)}_{a,0})_{-j} = \operatorname{conj}\left((A^{(m)}_{a,0})_j\right),\quad j=-m+1,\dots,m-1,\\ &2.\ (A^{(m)}_{\omega,1})_{-k} = \operatorname{conj}\left((A^{(m)}_{\omega,1})_k\right),\quad k=-m+1,\dots,m-1,\\ &3.\ (A^{(m)}_{a,1})_{-k,-j} = \operatorname{conj}\left((A^{(m)}_{a,1})_{k,j}\right),\quad k,j=-m+1,\dots,m-1,\\ &4.\ (A^{(m)}_{\omega,1})_{2k}=0,\quad \forall\ 2k\in\{-m+1,\dots,m-1\},\\ &5.\ (A^{(m)}_{a,1})_{2k,2j+1}=0,\quad \forall\ 2k,2j+1\in\{-m+1,\dots,m-1\}.\end{aligned}\tag{4.30}$$

A consequence of assumption 1. of (4.30) is that $(A^{(m)}_{a,0})_0\in\mathbb{R}$ while a consequence of assumption 2. is that $(A^{(m)}_{\omega,1})_0\in\mathbb{R}$.

We define the approximate inverse A of the infinite dimensional operator $DF(\bar{x})$ by

$$A \stackrel{\text{def}}{=} \begin{bmatrix} A_{\omega,0} & A_{a,0}\\ A_{\omega,1} & A_{a,1}\end{bmatrix},$$

where A acts on $b=(b_0,b_1)\in X=\mathbb{C}\times\ell^1_\nu$ component-wise as

$$\begin{aligned}(Ab)_0 &= A^{(m)}_{\omega,0}b_0 + A^{(m)}_{a,0}b_1^{(m)}\\ (Ab)_1 &= A^{(m)}_{\omega,1}b_0 + A_{a,1}b_1,\end{aligned}$$

where $A^{(m)}_{\omega,1}\in\mathbb{C}^{(2m-1)\times 1}$ is understood to be an element of ℓ^1_ν by *padding* the tail with zeros, and $A_{a,1}b_1\in\ell^1_\nu$ is defined component-wise by

$$(A_{a,1}b_1)_k = \begin{cases}\left(A^{(m)}_{a,1}b_1^{(m)}\right)_k, & |k|<m\\ \dfrac{1}{\mu_k(\bar{\omega})}(b_1)_k, & |k|\geq m.\end{cases}$$

Let us now verify that (4.18) holds.

Lemma 4.12. *Let $x \in X$. Then $AF(x) \in X$.*

Proof. Consider $x = (\omega, a) \in X$ and let $F(x) = (F_0(x), F_1(x))$, with $F_0(x)$ given in (4.12) and $F_1(x)$ given in (4.11). For sake of simplicity of the presentation, we denote $F_0 = F_0(x)$ and $F_1 = F_1(x)$.

We need to show that $\|(AF(x))_1\|_\nu < \infty$. Since

$$\lim_{k\to\pm\infty} \frac{\mu_k(\omega)}{\mu_k(\bar\omega)} = \left(\frac{\omega}{\bar\omega}\right)^4 < \infty,$$

there exists $C < \infty$ such that

$$\left|\frac{\mu_k(\omega)}{\mu_k(\bar\omega)}\right|, \left|\frac{1}{\mu_k(\bar\omega)}\right| < C, \quad \text{for all } |k| \geq m.$$

Then,

$$\begin{aligned}
\|(AF(x))_1\|_\nu &= \sum_{k\in\mathbb{Z}} |((AF(x))_1)_k| \, \nu^{|k|} = \sum_{k\in\mathbb{Z}} \left| \left(A^{(m)}_{\omega,1} F_0 + A_{a,1} F_1\right)_k \right| \nu^{|k|} \\
&\leq \sum_{|k|<m} \left| \left(A^{(m)}_{\omega,1}\right)_{k,1} \right| |F_0| \nu^{|k|} + \sum_{|k|<m} \left| \left(A^{(m)}_{a,1} F^{(m)}_1\right)_k \right| \nu^{|k|} \\
&\quad + \sum_{|k|\geq m} \left| \frac{1}{\mu_k(\bar\omega)} (\mu_k(\omega) a_k + (a^3)_k) \right| \nu^{|k|} \\
&\leq \sum_{|k|<m} \left| \left(A^{(m)}_{\omega,1}\right)_{k,1} \right| |F_0| \nu^{|k|} + \sum_{|k|<m} \left| \left(A^{(m)}_{a,1} F^{(m)}_1\right)_k \right| \nu^{|k|} \\
&\quad + C \sum_{|k|\geq m} |a_k| \, \nu^{|k|} + C \sum_{|k|\geq m} \left|(a^3)_k\right| \nu^{|k|} \\
&\leq \sum_{|k|<m} \left| \left(A^{(m)}_{\omega,1}\right)_{k,1} \right| |F_0| \nu^{|k|} + \sum_{|k|<m} \left| \left(A^{(m)}_{a,1} F^{(m)}_1\right)_k \right| \nu^{|k|} \\
&\quad + C\|a\|_\nu + C\|a\|_\nu^3 < \infty,
\end{aligned}$$

where we used the fact that $\|a^3\|_\nu \leq \|a\|_\nu^3$, because ℓ^1_ν is a Banach algebra. □

Let us show that the operator A that we defined above satisfies the symmetry assumption (4.29).

Lemma 4.13. *Let $x \in X_{sym}$. Then $AF(x) \in X_{sym}$.*

Proof. Let $x = (\omega, a) \in X_{sym} = \mathbb{R} \times \tilde\ell^1_\nu$, with $\tilde\ell^1_\nu$ as defined in (4.28). This implies that $a_{-k} = \text{conj}(a_k)$ and $a_{2k} = 0$ for all $k \in \mathbb{Z}$. Denote $F_0 = F_0(x)$ and $F_1 = F_1(x)$.

We begin the proof by showing that the operator F preserves the symmetry conditions, that is we show that $F_0 \in \mathbb{R}$, $(F_1)_{-k} = \text{conj}((F_1)_k)$ and $(F_1)_{2k} = 0$.

Recalling the definition of the Poincaré phase condition (4.12),

$$F_0 = \sum_{|k|\leq 3} a_k = a_{-3} + a_{-1} + a_1 + a_3 = \text{conj}(a_3) + \text{conj}(a_1) + a_1 + a_3 \in \mathbb{R}.$$

Also, from (4.9), we see that $\mu_{-k}(\omega) = \text{conj}\,(\mu_k(\omega))$. Then,

$$\begin{aligned}
(F_1)_{-k} &= \mu_{-k}(\omega)a_{-k} + \sum_{k_1+k_2+k_3=-k} a_{k_1}a_{k_2}a_{k_3} \\
&= \text{conj}\,(\mu_k(\omega)a_k) + \sum_{k_1+k_2+k_3=k} a_{-k_1}a_{-k_2}a_{-k_3} \\
&= \text{conj}\,(\mu_k(\omega)a_k) + \sum_{k_1+k_2+k_3=k} \text{conj}(a_{k_1})\text{conj}(a_{k_2})\text{conj}(a_{k_3}) \\
&= \text{conj}\,((F_1)_k)\,.
\end{aligned}$$

Now,

$$(F_1)_{2k} = \mu_{2k}(\omega)a_{2k} + (a^3)_{2k} = \mu_{2k}(\omega)(0) + \sum_{k_1+k_2+k_3=2k} a_{k_1}a_{k_2}a_{k_3} = 0, \tag{4.31}$$

since the condition $k_1 + k_2 + k_3 = 2k$ implies that there exists $i \in \{1, 2, 3\}$ such that k_i is even.

The second part of the proof is to show that AF preserves the symmetry conditions, that is $(AF(x))_0 \in \mathbb{R}$, $((AF(x))_1)_{-k} = \text{conj}\,(((AF(x))_1)_k)$ and $((AF(x))_1)_{2k} = 0$.

Combining that $A^{(m)}_{\omega,0}, F_0 \in \mathbb{R}$, $(F_1)_{-k} = \text{conj}\,((F_1)_k)$ and assumption 1 of (4.30), we have that

$$\begin{aligned}
(AF(x))_0 &= A^{(m)}_{\omega,0}F_0 + A^{(m)}_{a,0}F^{(m)}_1 \\
&= A^{(m)}_{\omega,0}F_0 + \sum_{k=-m+1}^{m-1} (A^{(m)}_{a,0})_k(F_1)_k \\
&= A^{(m)}_{\omega,0}F_0 + \sum_{k=-m+1}^{-1} (A^{(m)}_{a,0})_k(F_1)_k + (A^{(m)}_{a,0})_0(F_1)_0 + \sum_{k=1}^{m-1} (A^{(m)}_{a,0})_k(F_1)_k \\
&= A^{(m)}_{\omega,0}F_0 + (A^{(m)}_{a,0})_{1,0}(F_1)_0 + \sum_{k=1}^{m-1} \left(\text{conj}\left((A^{(m)}_{a,0})_k(F_1)_k\right) + (A^{(m)}_{a,0})_k(F_1)_k\right) \in \mathbb{R}.
\end{aligned}$$

By assumptions 2 and 3 of (4.30), for $|k| < m$, we have

$$\begin{aligned}
((AF(x))_1)_{-k} &= \left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_{-k} \\
&= \left(A^{(m)}_{\omega,1}\right)_{-k} F_0 + \left(A^{(m)}_{a,1}F^{(m)}_1\right)_{-k} \\
&= \left(A^{(m)}_{\omega,1}\right)_{-k} F_0 + \sum_{j=-m+1}^{m-1} (A^{(m)}_{a,1})_{-k,j}(F_1)_j \\
&= \left(A^{(m)}_{\omega,1}\right)_{-k} F_0 + \sum_{j=-m+1}^{m-1} (A^{(m)}_{a,1})_{-k,-j}(F_1)_{-j} \\
&= \text{conj}\left(\left(A^{(m)}_{\omega,1}\right)_k F_0\right) + \sum_{j=-m}^{m} \overline{(A^{(m)}_{a,1})_{k,j}(F_1)_j} \\
&= \text{conj}\,(((AF(x))_1)_k)\,,
\end{aligned}$$

and for $|k| \geq m$, we have

$$\begin{aligned}((AF(x))_1)_{-k} &= \left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_{-k} \\ &= (A_{a,1}F_1)_{-k} = \frac{1}{\mu_{-k}(\bar{\omega})}(F_1)_{-k} = \operatorname{conj}\left(((AF(x))_1)_k\right).\end{aligned}$$

That shows that $((AF(x))_1)_{-k} = \operatorname{conj}\left(((AF(x))_1)_k\right)$ for all k. It remains to show that $((AF(x))_1)_{2k} = 0$.

By assumptions 4 and 5 in (4.30), and using (4.31), we get that for $|k| < m$,

$$\begin{aligned}((AF(x))_1)_{2k} &= \left(A^{(m)}_{\omega,1}F_0 + A_{a,1}F_1\right)_{2k} = \left(A^{(m)}_{\omega,1}\right)_{2k} F_0 + \left(A^{(m)}_{a,1}F^{(m)}_1\right)_{2k} \\ &= \sum_{j=-m+1}^{m-1} (A^{(m)}_{a,1})_{2k,j}(F_1)_j = \sum_{\substack{j=-m+1 \\ j \text{ odd}}}^{m-1} (A^{(m)}_{a,1})_{2k,j}(F_1)_j = 0,\end{aligned}$$

and for $|k| \geq m$,

$$((AF(x))_1)_{2k} = \frac{1}{\mu_{2k}(\bar{\omega})}(F_1)_{2k} = 0.$$

□

Having defined A satisfying (4.29) we can now define the Newton-like operator as in (4.19) and use the radii polynomial approach to prove existence of a non trivial fixed point of T, by applying Proposition 4.9.

4.2.3 Construction of the radii polynomials for the periodic solution

We begin the construction of the radii polynomials (4.23) and (4.24) by constructing the bounds $Y_0, \mathbf{Y}_1$ such that

$$\begin{aligned}|(T(\bar{x}) - \bar{x})_0| = |(AF(\bar{x}))_0| \leq Y_0 \\ ||(T(\bar{x}) - \bar{x})_1||_\nu = ||(AF(\bar{x}))_1||_\nu \leq \mathbf{Y}_1.\end{aligned}$$

The upper bound Y_0 can be obtained by computing the finite sum

$$|(AF(\bar{x}))_0| = \left|A^{(m)}_{\omega,0}F_0(\bar{x}) + A^{(m)}_{a,0}F^{(m)}_1(\bar{x})\right|$$

with interval arithmetic. To obtain $\mathbf{Y}_1$, realize that

$$\begin{aligned}||(T(\bar{x}) - \bar{x})_1||_\nu &= ||(AF(\bar{x}))_1||_\nu \\ &= ||A_{\omega,1}F_0(\bar{x}) + A_{a,1}F_1(\bar{x})||_\nu \\ &= \sum_{k\in\mathbb{Z}} |[A_{\omega,1}F_0(\bar{x})]_k + [A_{a,1}F_1(\bar{x})]_k|\, \nu^{|k|} \\ &= \sum_{k=-m+1}^{m-1} \left|[A^{(m)}_{\omega,1}F_0(\bar{x})]_k + [A^{(m)}_{a,1}F^{(m)}_1(\bar{x})]_k\right| \nu^{|k|} + \sum_{|k|\geq m} |[A_{a,1}F_1(\bar{x})]_k|\, \nu^{|k|},\end{aligned}$$

where the first summand is finite and the second summand, since $\bar{a}_k = 0$ for $|k| \geq m$, satisfies

$$\sum_{|k|\geq m} |[A_{a,1}F_1(\bar{x})]_k| \, \nu^{|k|} = \sum_{|k|\geq m} \left| \frac{1}{\mu_k(\bar{\omega})} \left(\mu_k(\bar{\omega})\bar{a}_k + (\bar{a}^3)_k\right) \right| \nu^{|k|}$$
$$= \sum_{m\leq|k|<3m-2} \frac{1}{|\mu_k(\bar{\omega})|} \left|(\bar{a}^3)_k\right| \nu^{|k|}.$$

We are done by setting

$$Y_0 \stackrel{\text{def}}{=} \left|A^{(m)}_{\omega,0}F_0(\bar{x}) + A^{(m)}_{a,0}F^{(m)}_1(\bar{x})\right| \tag{4.32}$$

$$\mathbf{Y}_1 \stackrel{\text{def}}{=} \sum_{k=-m+1}^{m-1} \left|[A^{(m)}_{\omega,1}F_0(\bar{x})]_k + [A^{(m)}_{a,1}F^{(m)}_1(\bar{x})]_k\right| \nu^{|k|} \tag{4.33}$$
$$+ \sum_{m\leq|k|<3m-2} \frac{1}{|\mu_k(\bar{\omega})|} \left|(\bar{a}^3)_k\right| \nu^{|k|}.$$

The next step in the construction of the radii polynomials (4.23) and (4.24) is to construct the bounds $Z_0(r), \mathbf{Z}_1(r)$. Let $b, c \in B(r) \subset X$. Then

$$DT(\bar{x}+b)c = [I - ADF(\bar{x}+b)]c = [I - AA^\dagger]c - A[DF(\bar{x}+b) - A^\dagger]c. \tag{4.34}$$

We first bound the quantities involved in the first term of (4.34). Let $B \stackrel{\text{def}}{=} I - AA^\dagger$, which we express as

$$B = \begin{bmatrix} B_{\omega,0} & B_{a,0} \\ B_{\omega,1} & B_{a,1} \end{bmatrix}.$$

By definition of B, $[(Bc)_1]_k = 0$ for $|k| \geq m$ and $c \in B(r) \subset \mathbb{R} \times \ell^1_\nu$. Define

$$Z^{(0)}_0 \stackrel{\text{def}}{=} |B_{\omega,0}| + \left(\max_{|k|<m} \frac{|(B_{a,0})_k|}{\nu^{|k|}}\right) \tag{4.35}$$

$$Z^{(0)}_1 \stackrel{\text{def}}{=} \sum_{|k|<m} |(B_{\omega,1})_k|\nu^{|k|} + \max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |(B_{a,1})_{k,n}|\nu^{|k|}. \tag{4.36}$$

Now, recalling (4.14) and Lemma 4.3, we have that

$$|(Bc)_0| = \left|B_{\omega,0}c_0 + \sum_{k\in\mathbb{Z}} (B_{a,0})_k(c_1)_k\right| \leq \left(|B_{\omega,0}| + \|B_{a,0}\|^\infty_\nu\right) r = Z^{(0)}_0 r.$$

Recalling Lemma 4.5, Corollary 4.6 and (4.16), we get that

$$\|(Bc)_1\|_\nu = \|B_{\omega,1}c_0 + B_{a,1}c_1\|_\nu \leq \left(\|B_{\omega,1}\|_\nu + \|B_{a,1}\|_{B(\ell^1_\nu,\ell^1_\nu)}\right) r \leq Z^{(0)}_1 r.$$

We therefore have all estimates to bound the first term of (4.34). Next, we bound the quantities involved in the second term. Denote $b = (b_0, b_1) \in B(r) \subset X = \mathbb{C} \times \ell^1_\nu$. For $j = 0, 1$, let $z_j \stackrel{\text{def}}{=} \left([DF(\bar{x}+b) - A^\dagger]c\right)_j$ and set $z \stackrel{\text{def}}{=} (z_0, z_1)$. Recalling (4.12), we get that if $m > 3$, then $z_0 = 0$. Set $\tilde{b} = (\tilde{b}_0, \tilde{b}_1)$ and $\tilde{c} = (\tilde{c}_0, \tilde{c}_1)$ such that $b = (\tilde{b}_0 r, \tilde{b}_1 r)$ and $c = (\tilde{c}_0 r, \tilde{c}_1 r)$. Hence, $\tilde{b}, \tilde{c} \in B_0(1) \subset X$. Denote $\bar{x} = (\bar{\omega}, \bar{a})$. For $j = 1$,

$$z_1(r) = \sum_{i=1}^{5} z_{1,i} r^i$$

where each component of $z_{1,i} = ((z_{1,i})_k)_{k\in\mathbb{Z}}$ is given component-wise by

$$(z_{1,1})_k = \begin{cases} 3(\bar{a}^2 \tilde{c}_1^I)_k, & |k| < m \\ 3(\bar{a}^2 \tilde{c}_1)_k, & |k| \geq m \end{cases}$$

and

$$\begin{aligned}
(z_{1,2})_k &= \left(12k^4\bar{\omega}^2 - 84\mathbf{i}k^3\bar{\omega} - 142k^2\right)\tilde{b}_0\tilde{c}_0\bar{a}_k + 6(\bar{a}\tilde{b}_1\tilde{c}_1)_k \\
&\quad + \left(4k^4\bar{\omega}^3 - 42\mathbf{i}k^3\bar{\omega}^2 - 142k^2\bar{\omega} + 154\mathbf{i}k\right)\left(\tilde{c}_0(\tilde{b}_1)_k + \tilde{b}_0(\tilde{c}_1)_k\right) \\
(z_{1,3})_k &= \left(12k^4\bar{\omega} - 42\mathbf{i}k^3\right)\tilde{b}_0^2\tilde{c}_0\bar{a}_k + 3(\tilde{b}_1^2\tilde{c}_1)_k + \left(12k^4\bar{\omega}^2 - 84\mathbf{i}k^3\bar{\omega} - 142k^2\right)\tilde{b}_0\tilde{c}_0(\tilde{b}_1)_k \\
&\quad \left(6k^4\bar{\omega}^2 - 42\mathbf{i}k^3\bar{\omega} - 71k^2\right)\tilde{b}_0^2(\tilde{c}_1)_k \\
(z_{1,4})_k &= \left(4k^4\bar{\omega} - 14\mathbf{i}k^3\right)\tilde{b}_0^3(\tilde{c}_1)_k + \left(12k^4\bar{\omega} - 42\mathbf{i}k^3\right)\tilde{b}_0^2\tilde{c}_0(\tilde{b}_1)_k + 4k^4\tilde{b}_0^3\tilde{c}_0\bar{a}_k \\
(z_{1,5})_k &= k^4\tilde{b}_0^4(\tilde{c}_1)_k + 4k^4\tilde{b}_0^3\tilde{c}_0(\tilde{b}_1)_k.
\end{aligned}$$

Recalling that $z_0 = 0$, the second term of (4.34) is $A[DF(\bar{x} + b) - A^\dagger]c = Az$ given component-wise ($j = 0, 1$) by

$$\left(A[DF(\bar{x} + b) - A^\dagger]c\right)_j = (Az)_j = A_{a,j} z_1.$$

Defining the following vectors will be useful when constructing upper bounds for $|(Az)_j|$ for the cases $j = 0, 1$.

$$\begin{aligned}
\tilde{A}_{a,0}^{(m)} &= \left\{\left(12k^4\bar{\omega}^2 - 84\mathbf{i}k^3\bar{\omega} - 142k^2\right)(A_{a,0}^{(m)})_k\right\}_{|k|<m} \\
\tilde{B}_{a,0}^{(m)} &= \left\{\left(4k^4\bar{\omega}^3 - 42\mathbf{i}k^3\bar{\omega}^2 - 142k^2\bar{\omega} + 154\mathbf{i}k\right)(A_{a,0}^{(m)})_k\right\}_{|k|<m} \\
\tilde{C}_{a,0}^{(m)} &= \left\{\left(12k^4\bar{\omega} - 42\mathbf{i}k^3\right)(A_{a,0}^{(m)})_k\right\}_{|k|<m} \\
\tilde{D}_{a,0}^{(m)} &= \left\{\left(6k^4\bar{\omega}^2 - 42\mathbf{i}k^3\bar{\omega} - 71k^2\right)(A_{a,0}^{(m)})_k\right\}_{|k|<m} \\
\tilde{E}_{a,0}^{(m)} &= \left\{\left(4k^4\bar{\omega} - 14\mathbf{i}k^3\right)(A_{a,0}^{(m)})_k\right\}_{|k|<m} \\
\tilde{F}_{a,0}^{(m)} &= \left\{k^4(A_{a,0}^{(m)})_k\right\}_{|k|<m}.
\end{aligned}$$

Case 1: a bound on $|(\boldsymbol{Az})_0| = |\boldsymbol{A_{a,0} z_1}| = |\boldsymbol{A^{(m)}_{a,0} z^{(m)}_1}|$.

$$
\begin{aligned}
A^{(m)}_{a,0} z^{(m)}_1 &= \sum_{i=1}^{5} \left(A^{(m)}_{a,0} z^{(m)}_{1,i} \right) r^i \\
&= A^{(m)}_{a,0} \left\{ 3(\bar{a}^2 \tilde{c}^I_1)_k \right\}_{|k|<m} r \\
&\quad + \left(\tilde{A}^{(m)}_{a,0} \left\{ \tilde{b}_0 \tilde{c}_0 \bar{a}_k \right\}_{|k|<m} + 6 A^{(m)}_{a,0} \left\{ (\bar{a} \tilde{b}_1 \tilde{c}_1)_k \right\}_{|k|<m} + \tilde{B}^{(m)}_{a,0} \left\{ \left(\tilde{c}_0 (\tilde{b}_1)_k + \tilde{b}_0 (\tilde{c}_1)_k \right) \right\}_{|k|<m} \right) r^2 \\
&\quad + \left(\tilde{C}^{(m)}_{a,0} \left\{ \tilde{b}^2_0 \tilde{c}_0 \bar{a}_k \right\}_{|k|<m} + 3 A^{(m)}_{a,0} \left\{ (\tilde{b}^2_1 \tilde{c}_1)_k \right\}_{|k|<m} + \tilde{A}^{(m)}_{a,0} \left\{ \tilde{b}_0 \tilde{c}_0 (\tilde{b}_1)_k \right\}_{|k|<m} \right. \\
&\qquad \left. + \tilde{D}^{(m)}_{a,0} \left\{ \tilde{b}^2_0 (\tilde{c}_1)_k \right\}_{|k|<m} \right) r^3 \\
&\quad + \left(\tilde{E}^{(m)}_{a,0} \left\{ \tilde{b}^3_0 (\tilde{c}_1)_k \right\}_{|k|<m} + \tilde{C}^{(m)}_{a,0} \left\{ \tilde{b}^2_0 \tilde{c}_0 (\tilde{b}_1)_k \right\}_{|k|<m} + 4 \tilde{F}^{(m)}_{a,0} \left\{ \tilde{b}^3_0 \tilde{c}_0 \bar{a}_k \right\}_{|k|<m} \right) r^4 \\
&\quad + \left(\tilde{F}^{(m)}_{a,0} \left\{ \tilde{b}^4_0 (\tilde{c}_1)_k \right\}_{|k|<m} + 4 \tilde{F}^{(m)}_{a,0} \left\{ \tilde{b}^3_0 \tilde{c}_0 (\tilde{b}_1)_k \right\}_{|k|<m} \right) r^5.
\end{aligned}
$$

Let $\omega^I \stackrel{\text{def}}{=} \left\{ \nu^{-k} \right\}_{|k|\geq m}$, and let

$$
\begin{aligned}
Z^{(1)}_0 &\stackrel{\text{def}}{=} 3|A^{(m)}_{a,0}| \left\{ (|\bar{a}|^2 |\omega^I|)_k \right\}_{|k|<m} && (4.37) \\
Z^{(2)}_0 &\stackrel{\text{def}}{=} \|\tilde{A}^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu + 6\|A^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu + 2\|\tilde{B}^{(m)}_{a,0}\|^\infty_\nu && (4.38) \\
Z^{(3)}_0 &\stackrel{\text{def}}{=} \|\tilde{C}^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu + 3\|A^{(m)}_{a,0}\|^\infty_\nu + \|\tilde{A}^{(m)}_{a,0}\|^\infty_\nu + \|\tilde{D}^{(m)}_{a,0}\|^\infty_\nu && (4.39) \\
Z^{(4)}_0 &\stackrel{\text{def}}{=} \|\tilde{E}^{(m)}_{a,0}\|^\infty_\nu + \|\tilde{C}^{(m)}_{a,0}\|^\infty_\nu + 4\|\tilde{F}^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu && (4.40) \\
Z^{(5)}_0 &\stackrel{\text{def}}{=} 5\|\tilde{F}^{(m)}_{a,0}\|^\infty_\nu. && (4.41)
\end{aligned}
$$

Note that $|\tilde{b}_0|, \|\tilde{b}_1\|_\nu, |\tilde{c}_0|, \|\tilde{c}_1\|_\nu \leq 1$. Using Lemma 4.3, we get that

$$
\begin{aligned}
|(Az)_0| &= \left| A^{(m)}_{a,0} z^{(m)}_1 \right| \\
&\leq 3|A^{(m)}_{a,0}| \left\{ (|\bar{a}|^2 |\omega^I|)_k \right\}_{|k|<m} r \\
&\quad + \left(\|\tilde{A}^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu + 6\|A^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu + 2\|\tilde{B}^{(m)}_{a,0}\|^\infty_\nu \right) r^2 \\
&\quad + \left(\|\tilde{C}^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu + 3\|A^{(m)}_{a,0}\|^\infty_\nu + \|\tilde{A}^{(m)}_{a,0}\|^\infty_\nu + \|\tilde{D}^{(m)}_{a,0}\|^\infty_\nu \right) r^3 \\
&\quad + \left(\|\tilde{E}^{(m)}_{a,0}\|^\infty_\nu + \|\tilde{C}^{(m)}_{a,0}\|^\infty_\nu + 4\|\tilde{F}^{(m)}_{a,0}\|^\infty_\nu \|\bar{a}\|_\nu \right) r^4 \\
&\quad + 5\|\tilde{F}^{(m)}_{a,0}\|^\infty_\nu r^5 \\
&= \sum_{i=1}^{5} Z^{(i)}_0 r^i.
\end{aligned}
$$

Case 2: a bound on $\|(\boldsymbol{Az})_1\|_\nu = \|\boldsymbol{A_{a,1} z_1}\|_\nu$.
Define the linear functionals

$$
\begin{aligned}
\tilde{A}_{a,1} &= \left\{12k^4\bar{\omega}^2 - 84\mathbf{i}k^3\bar{\omega} - 142k^2\right\}_{k\in\mathbb{Z}} \\
\tilde{B}_{a,1} &= \left\{4k^4\bar{\omega}^3 - 42\mathbf{i}k^3\bar{\omega}^2 - 142k^2\bar{\omega} + 154\mathbf{i}k\right\}_{k\in\mathbb{Z}} \\
\tilde{C}_{a,1} &= \left\{12k^4\bar{\omega} - 42\mathbf{i}k^3\right\}_{k\in\mathbb{Z}} \\
\tilde{D}_{a,1} &= \left\{6k^4\bar{\omega}^2 - 42\mathbf{i}k^3\bar{\omega} - 71k^2\right\}_{k\in\mathbb{Z}} \\
\tilde{E}_{a,1} &= \left\{4k^4\bar{\omega} - 14\mathbf{i}k^3\right\}_{k\in\mathbb{Z}} \\
\tilde{F}_{a,1} &= \left\{k^4\right\}_{k\in\mathbb{Z}}.
\end{aligned}
$$

Let

$$
Z_1^{(1)} \stackrel{\text{def}}{=} 3\sum_{k=-m+1}^{m-1}\left(|A_{a,1}^{(m)}|(|\bar{a}|^2\omega^I)_F\right)_k \nu^{|k|} + \frac{3}{|\mu_m(\bar{\omega})|}\|\bar{a}\|_\nu^2 \tag{4.42}
$$

$$
Z_1^{(2)} \stackrel{\text{def}}{=} \left\|A_{a,1}^{(m)}\right\|_{B(\ell_\nu^1,\ell_\nu^1)}\Big(\sum_{k=-m+1}^{m-1}\left|12k^4\bar{\omega}^2 - 84\mathbf{i}k^3\bar{\omega} - 142k^2\right||\bar{a}_k|\nu^{|k|} \tag{4.43}
$$

$$
+6\|\bar{a}\|_\nu + 2\|\tilde{B}_{a,1}\|_\nu^\infty\Big) \tag{4.44}
$$

$$
Z_1^{(3)} \stackrel{\text{def}}{=} \left\|A_{a,1}^{(m)}\right\|_{B(\ell_\nu^1,\ell_\nu^1)}\Big(\sum_{k=-m+1}^{m-1}\left|12k^4\bar{\omega} - 42\mathbf{i}k^3\right||\bar{a}_k|\nu^{|k|} \tag{4.45}
$$

$$
+3 + \|\tilde{A}_{a,1}\|_\nu^\infty + \|\tilde{D}_{a,1}\|_\nu^\infty\Big) \tag{4.46}
$$

$$
Z_1^{(4)} \stackrel{\text{def}}{=} \left\|A_{a,1}^{(m)}\right\|_{B(\ell_\nu^1,\ell_\nu^1)}\left(\|\tilde{E}_{a,1}\|_\nu^\infty + \|\tilde{C}_{a,1}\|_\nu^\infty + 4\sum_{k=-m+1}^{m-1}k^4|\bar{a}_k|\nu^{|k|}\right) \tag{4.47}
$$

$$
Z_1^{(5)} \stackrel{\text{def}}{=} \left\|A_{a,1}^{(m)}\right\|_{B(\ell_\nu^1,\ell_\nu^1)}\left(5\|\tilde{F}_{a,1}\|_\nu^\infty\right) \tag{4.48}
$$

First, we have that

$$
\begin{aligned}
\|A_{a,1}z_{1,1}\|_\nu &= \sum_{|k|<m}|(A_{a,1}z_{1,1})_k|\,\nu^{|k|} + \sum_{|k|\geq m}|(A_{a,1}z_{1,1})_k|\,\nu^{|k|} \\
&\leq 3\sum_{k=-m+1}^{m-1}\left(|A_{a,1}^{(m)}|(|\bar{a}|^2|\tilde{c}_1^I|)_F\right)_k\nu^{|k|} + 3\sum_{|k|\geq m}\frac{1}{|\mu_k(\bar{\omega})|}\left|(\bar{a}^2\tilde{c}_1)_k\right|\nu^{|k|} \\
&\leq 3\sum_{k=-m+1}^{m-1}\left(|A_{a,1}^{(m)}|(|\bar{a}|^2\omega^I)_F\right)_k\nu^{|k|} + \frac{3}{|\mu_m(\bar{\omega})|}\|\bar{a}\|_\nu^2 = Z_1^{(1)}.
\end{aligned}
$$

which is a finite sum that is evaluated using interval arithmetic. Now,

$$
\begin{aligned}
\|A_{a,1}z_{1,2}\|_\nu &\le \left\|A_{a,1}^{(m)}\right\|_{B(\ell^1_\nu,\ell^1_\nu)} \|z_{1,2}\|_\nu \\
&\le \left\|A_{a,1}^{(m)}\right\|_{B(\ell^1_\nu,\ell^1_\nu)} \Big(\sum_{k=-m+1}^{m-1} \left|12k^4\bar\omega^2 - 84\mathbf{i}k^3\bar\omega - 142k^2\right| |\bar a_k|\nu^{|k|} \\
&\qquad\qquad + 6\|\bar a\|_\nu + 2\|\tilde B_{a,1}\|_\nu^\infty \Big) = Z_1^{(2)}, \\
\|A_{a,1}z_{1,3}\|_\nu &\le \left\|A_{a,1}^{(m)}\right\|_{B(\ell^1_\nu,\ell^1_\nu)} \Big(\sum_{k=-m+1}^{m-1} \left|12k^4\bar\omega - 42\mathbf{i}k^3\right| |\bar a_k|\nu^{|k|} \\
&\qquad\qquad + 3 + \|\tilde A_{a,1}\|_\nu^\infty + \|\tilde D_{a,1}\|_\nu^\infty \Big) = Z_1^{(3)}, \\
\|A_{a,1}z_{1,4}\|_\nu &\le Z_1^{(4)} \\
\|A_{a,1}z_{1,5}\|_\nu &\le Z_1^{(5)},
\end{aligned}
$$

where using Corollary 4.6, we get that

$$
\left\|A_{a,1}^{(m)}\right\|_{B(\ell^1_\nu,\ell^1_\nu)} \le \max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} \left| \left(A_{a,1}^{(m)}\right)_{k,n} \right| \nu^{|k|}.
$$

Hence, we get that

$$
\|(Az)_1\|_\nu = \|A_{a,1}z_1\|_\nu \le \sum_{i=1}^{5} \|A_{a,1}z_{1,i}\|_\nu r^i = \sum_{i=1}^{5} Z_1^{(i)} r^i.
$$

Combining (4.35), (4.36), (4.37), (4.38), (4.39), (4.40), (4.41), (4.42), (4.43), (4.45), (4.47), (4.48), we set

$$
Z_0(r) \stackrel{\text{def}}{=} Z_0^{(5)}r^5 + Z_0^{(4)}r^4 + Z_0^{(3)}r^3 + Z_0^{(2)}r^2 + \left(Z_0^{(1)} + Z_0^{(0)}\right) r \tag{4.49}
$$

$$
\mathbf{Z}_1(r) \stackrel{\text{def}}{=} Z_1^{(5)}r^5 + Z_1^{(4)}r^4 + Z_1^{(3)}r^3 + Z_1^{(2)}r^2 + \left(Z_1^{(1)} + Z_1^{(0)}\right) r \tag{4.50}
$$

Using (4.32), (4.33), (4.49) and (4.50), we can define the two radii polynomials as defined in Definition 4.8.

4.2.4 Proof of existence of the symmetric periodic solution $\gamma(t)$

Using the radii polynomial approach, we have proved the following result.

Theorem 4.14. *There exists a periodic solution $\gamma(t) \neq 0$ of (4.1) with period τ that satisfies*

$$
\tau \in [1.908097232050663\ ,\ 1.908097232051545].
$$

The periodic solution has a Fourier expansion

$$
\gamma(t) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \tilde a_k e^{\mathbf{i}\tilde\omega k t}, \qquad t \in [0,\tau], \tag{4.51}
$$

where $\tilde\omega$ is its frequency, $\tilde a_{-k} = \mathrm{conj}(\tilde a_k)$, and $\tilde a_{2k} = 0$ for all $k \in \mathbb{Z}$, that is γ is a real τ-periodic symmetric solution of (4.1). We refer to Figure 4 for the graph of the solution.

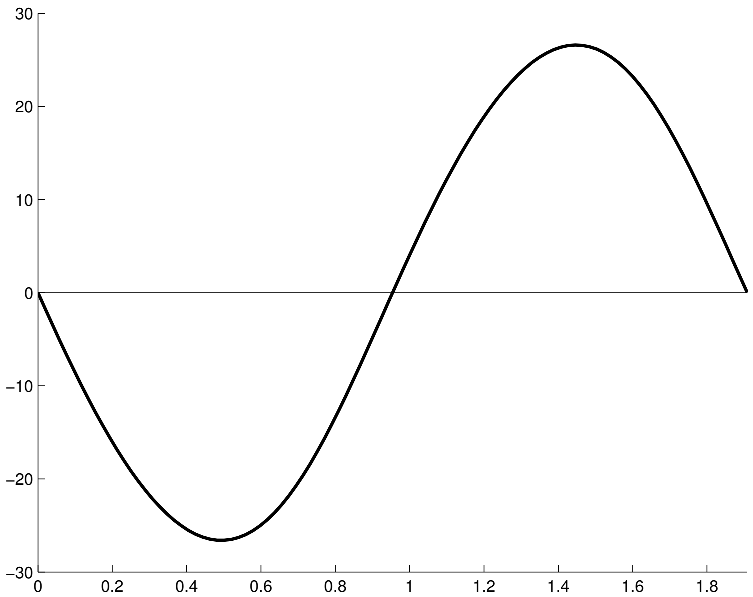

Figure 4: Graph of the periodic solution γ given in (4.51).

Proof.

Fix $\nu = 1.3$ and let $\bar{x} = (\bar{\omega}, \bar{a}) \in \mathbb{R} \times \mathbb{C}^{31}$ as given in Figure 5.

	$\bar{\omega}$
	3.292906253223528
k	$\bar{a}_k$
1	$1.780548377519241 \times 10^{-1} + (1.334498530100925 \times 10^{-1})\mathbf{i}$
3	$-1.780548377519304 \times 10^{-1} + (6.415134398797947 \times 10^{-2})\mathbf{i}$
5	$-1.116766962688650 \times 10^{-3} - (5.237561088492284 \times 10^{-4})\mathbf{i}$
7	$-1.723996326464334 \times 10^{-6} - (6.620322617252511 \times 10^{-6})\mathbf{i}$
9	$2.006586787990499 \times 10^{-8} - (2.909959121074274 \times 10^{-8})\mathbf{i}$
11	$1.732979780563067 \times 10^{-10} - (2.102402650424280 \times 10^{-11})\mathbf{i}$
13	$6.243643874701474 \times 10^{-13} + (5.489211652414192 \times 10^{-13})\mathbf{i}$
15	$3.239760646543639 \times 10^{-17} + (3.859717081491257 \times 10^{-15})\mathbf{i}$
≥ 16	0

Figure 5: We show $\bar{a}_k$ for $k \geq 1$, as $\bar{a}_{-k} = \text{conj}(\bar{a}_k)$. Note that all even coefficients are 0.

In the separate computer MATLAB program *script_proof_gamma.m* available at [6] which uses the Matlab toolbox INTLAB for reliable computing [20], we construct the two quintic radii polynomials $p_0(r)$ and $p_1(r)$ of Definition 4.8, and then apply Proposition 4.9 to show that

$$\mathcal{I} = \{r > 0 \mid p_0(r) < 0\} \bigcap \{r > 0 \mid p_1(r) < 0\} \neq \emptyset.$$

The program verifies with interval arithmetic that $\mathcal{I} \subset [r_- \; r_+]$ where

$$r_- = 7.595549832526767 \times 10^{-13} \quad \text{and} \quad r_+ = 1.275811117241133 \times 10^{-3}.$$

Moreover, for any $r \in \mathcal{I}$, there exists a unique $\tilde{x} = (\tilde{\omega}, \tilde{a}) \in B_{\bar{x}}(r)$ such that $T(\tilde{x}) = \tilde{x}$. We also have that

$$\|x - \tilde{x}\|_X = \max\left(|\bar{\omega} - \tilde{\omega}|, \|\bar{a} - \tilde{a}\|_\nu\right) \leq r_-. \tag{4.52}$$

Since A is an injective linear operator, we get that $\tilde{x}$ is the unique zero of F in $B_{\bar{x}}(r)$. Moreover, $\bar{x} \in X_{sym}$, and then by Lemma 4.11, $\tilde{x} \in X_{sym}$. With interval arithmetic, the program verifies that

$$\tau \stackrel{\text{def}}{=} \frac{2\pi}{\tilde{\omega}} \in [1.908097232050663\ ,\ 1.908097232051545].$$

Since $\tilde{x} \in X_{sym}$, then $\tilde{\omega} \in \mathbb{R}$, $\tilde{a}_{-k} = \text{conj}(\tilde{a}_k)$, and $\tilde{a}_{2k} = 0$ for all $k \in \mathbb{Z}$. By Lemma 4.2, we get that

$$\gamma(t) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \tilde{a}_k e^{\mathbf{i}\tilde{\omega}kt}, \qquad t \in [0,\tau]$$

is a real τ-periodic symmetric solution of (4.1). □

4.2.5 Studying the zeroes of $\gamma(t)$ and $\gamma'(t)$

In this short section, we demonstrate that $\gamma(t)$ and $\gamma'(t)$ only vanish twice on the interval $[0,\tau]$. More precisely, we show the following two results.

Theorem 4.15. *Consider the periodic solution γ given by* (4.51)*. Then there exist $t_0, t_1 \in [0,\tau)$ with $t_0 < t_1$ such that $\gamma(t_0) = \gamma(t_1) = 0$, $\gamma'(t_0) \neq 0$ and $\gamma'(t_1) \neq 0$. Moreover, $\forall\ t \in [0,\tau) \setminus \{t_0, t_1\}$, $\gamma(t) \neq 0$.*

Theorem 4.16. *Consider the periodic solution γ given by* (4.51)*. Then there exist $t_2, t_3 \in [0,\tau)$ with $t_2 < t_3$ such that $\gamma'(t_2) = \gamma'(t_3) = 0$, $\gamma''(t_2) \neq 0$ and $\gamma''(t_3) \neq 0$. Moreover, $\forall\ t \in [0,\tau) \setminus \{t_2, t_3\}$, $\gamma'(t) \neq 0$.*

To simplify the proof of Theorem 4.15 and Theorem 4.16, let us rescale time using the transformation

$$\tilde{t} = \frac{\tilde{\omega}}{2\pi} t.$$

Hence, a new parameterization for the periodic solution (4.51) is given by

$$\gamma(\tilde{t}) = \sum_{k\in\mathbb{Z}} \tilde{a}_k e^{\mathbf{i}2\pi k\tilde{t}}, \qquad \tilde{t} \in [0,1].$$

For the proof of the theorems, we will bound the values of γ, γ' and γ'' on some time intervals.

Bound on γ: Since $\nu = 1.3 \geq 1$ and by (4.52), we have that

$$\left| \sum_{k\in\mathbb{Z}} (\tilde{a}_k - \bar{a}_k) e^{\mathbf{i}2\pi k\tilde{t}} \right| \leq \|\tilde{a} - \bar{a}\|_1 \leq \|\tilde{a} - \bar{a}\|_\nu \leq r_- = 7.595549832526767 \times 10^{-13}.$$

Therefore, the value of $\gamma(\tilde{t})$ can be enclosed using

$$\gamma(\tilde{t}) = \sum_{k\in\mathbb{Z}} (\tilde{a}_k - \bar{a}_k) e^{\mathbf{i}2\pi k\tilde{t}} + \sum_{k\in\mathbb{Z}} \bar{a}_k e^{\mathbf{i}2\pi k\tilde{t}} \in [-r_-, r_-] + \sum_{|k|\leq 16} \bar{a}_k e^{\mathbf{i}2\pi k\tilde{t}}, \tag{4.53}$$

where the second quantity is easily evaluated using interval arithmetic.

Bound on γ': Now, since $\|\tilde{a}-\bar{a}\|_\nu \leq r_-$, then for any $k \in \mathbb{Z}$, $|\tilde{a}_k-\bar{a}_k|\nu^{|k|} \leq \|\tilde{a}-\bar{a}\|_\nu \leq r_-$, and therefore

$$|\tilde{a}_k - \bar{a}_k| \leq \frac{r_-}{\nu^{|k|}}.$$

Defining

$$C_1 \stackrel{\text{def}}{=} 4\pi \left(\frac{\nu}{(\nu-1)^2} \right) r_-,$$

we get that

$$\left| \sum_{k \in \mathbb{Z}} \mathbf{i}2\pi k(\tilde{a}_k - \bar{a}_k) e^{\mathbf{i}2\pi k \tilde{t}} \right| \leq 4\pi \left(\sum_{k \geq 1} \frac{k}{\nu^k} \right) r_- = C_1.$$

Therefore, the value of $\gamma'(\tilde{t})$ can be enclosed using

$$\begin{aligned} \gamma'(\tilde{t}) &= \sum_{k \in \mathbb{Z}} \mathbf{i}2\pi k(\tilde{a}_k - \bar{a}_k) e^{\mathbf{i}2\pi k \tilde{t}} + \sum_{k \in \mathbb{Z}} \mathbf{i}2\pi k \bar{a}_k e^{\mathbf{i}2\pi k \tilde{t}} \\ &\in [-C_1, C_1] + \sum_{|k| \leq 16} \mathbf{i}2\pi k \bar{a}_k e^{\mathbf{i}2\pi k \tilde{t}}, \end{aligned} \tag{4.54}$$

where the second quantity is easily evaluated using interval arithmetic.
Bound on γ'': Letting

$$C_2 \stackrel{\text{def}}{=} 8\pi^2 \left(\sum_{k \geq 1} \frac{k^2}{\nu^k} \right) r_- = 8\pi^2 \frac{\nu(\nu+1)}{(\nu-1)^3} r_-,$$

we get that

$$\left| \sum_{k \in \mathbb{Z}} -(2\pi k)^2 (\tilde{a}_k - \bar{a}_k) e^{\mathbf{i}2\pi k \tilde{t}} \right| \leq 8\pi^2 \left(\sum_{k \geq 1} \frac{k^2}{\nu^k} \right) r_- = C_2.$$

Then, the value of $\gamma''(\tilde{t})$ can be enclosed using

$$\begin{aligned} \gamma''(\tilde{t}) &= \sum_{k \in \mathbb{Z}} -(2\pi k)^2 (\tilde{a}_k - \bar{a}_k) e^{\mathbf{i}2\pi k \tilde{t}} + \sum_{k \in \mathbb{Z}} -(2\pi k)^2 \bar{a}_k e^{\mathbf{i}2\pi k \tilde{t}} \\ &\in [-C_2, C_2] + \sum_{|k| \leq 16} -(2\pi k)^2 \bar{a}_k e^{\mathbf{i}2\pi k \tilde{t}}, \end{aligned} \tag{4.55}$$

where the second quantity is easily evaluated using interval arithmetic.

Using the above estimates, we can prove the two above theorems.

Proof. [Proof of Theorem 4.15] The proof is computer-assisted and is performed by running the MATLAB program *script_zeroes_of_gamma.m* available at [6] which uses the Matlab toolbox INTLAB for reliable computing [20]. The program has 4 parts.

First, consider the mesh

$$s_0 = 0, s_1 = .0001, s_2 = .001, s_3 = .01, s_4 = .05, s_5 = .2, s_6 = .4, s_7 = .49, s_8 = .499$$

of the interval $[0,\ .499]$. For $j = 0, \ldots, 7$, let $\mathbf{s}_j = [s_j,\ s_{j+1}]$. Then we use (4.53) and interval arithmetic to show that

$$\gamma(\mathbf{s}_j) \in [-r_-, r_-] + \sum_{|k| \leq 16} \bar{a}_k e^{\mathbf{i}2\pi k \mathbf{s}_j} \subset (-\infty, 0),$$

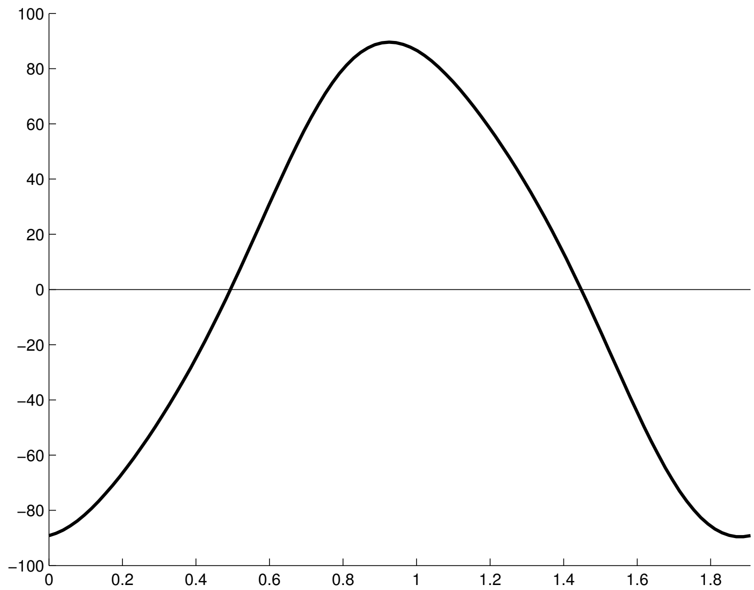

Figure 6: Graph of the derivative $\gamma'(t)$ of the periodic solution.

for each $j = 0, \ldots, 7$. Second, let $\mathbf{s}_8 = [s_8, s_9] = [.499,\ .5]$, use (4.54) and interval arithmetic to show that

$$\gamma'(\mathbf{s}_8) \in [-C_1, C_1] + \sum_{|k|\le 16} \mathbf{i}2\pi k\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_8} \subset (0, \infty).$$

Third, consider the mesh

$$s_9 = .5, s_{10} = .5001, s_{11} = .501, s_{12} = .51, s_{13} = .59,$$

$$s_{14} = .78, s_{15} = .95, s_{16} = .997, s_{17} = .999$$

of the interval $[.5,\ .999]$. For $j = 9, \ldots, 16$, let $\mathbf{s}_j = [s_j,\ s_{j+1}]$. Then we use (4.53) and interval arithmetic to show that

$$\gamma(\mathbf{s}_j) \in [-r_-, r_-] + \sum_{|k|\le 16} \bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_j} \subset (0, \infty),$$

for each $j = 9, \ldots, 16$. Finally, let $\mathbf{s}_{17} = [s_{17}, s_{18}] = [.999,\ 1]$, use (4.54) and interval arithmetic to show that

$$\gamma'(\mathbf{s}_{17}) \in [-C_1, C_1] + \sum_{|k|\le 16} \mathbf{i}2\pi k\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_{17}} \subset (-\infty, 0).$$

Combining the above, we conclude that $\gamma > 0$ on $[0,\ .499]$, $\gamma' > 0$ on $[.499,\ .5]$, $\gamma < 0$ on $[.5,\ .999]$ and $\gamma' < 0$ on $[.999,\ 1]$. That shows that γ has exactly two distinct zeroes. □

Proof. [Proof of Theorem 4.16] The proof is computer-assisted and is performed by running the MATLAB program *script_zeroes_of_gamma_prime.m* available at [6]. This program requires the Matlab toolbox INTLAB for reliable computing [20]. The program has 5 parts.

First, consider the mesh

$$s_0 = 0, s_1 = .16, s_2 = .23, s_3 = .25$$

of the interval $[0,\ .25]$. For $j = 0, 1, 2$, let $\mathbf{s}_j = [s_j,\ s_{j+1}]$. Then we use (4.54) and interval arithmetic to show that

$$\gamma'(\mathbf{s}_j) \in [-C_1, C_1] + \sum_{|k|\le 16} \mathbf{i}2\pi k\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_j} \subset (-\infty, 0),$$

for each $j = 0, 1, 2$. Second, let $\mathbf{s}_3 = [s_3, s_4] = [.25,\ .28]$, use (4.55) and interval arithmetic to show that

$$\gamma''(\mathbf{s}_3) \in [-C_2, C_2] + \sum_{|k|\le 16} -(2\pi k)^2\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_3} \subset (0, \infty).$$

Third, consider the mesh

$$s_4 = .28, s_5 = .38, s_6 = .56, s_7 = .69, s_8 = .74,$$

of the interval $[.28,\ .74]$. For $j = 4, 5, 6, 7$, let $\mathbf{s}_j = [s_j,\ s_{j+1}]$. Then we use (4.54) and interval arithmetic to show that

$$\gamma'(\mathbf{s}_j) \in [-C_1, C_1] + \sum_{|k|\le 16} \mathbf{i}2\pi k\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_j} \subset (0, \infty),$$

for each $j = 4, 5, 6, 7$. Fourth, let $\mathbf{s}_8 = [s_8, s_9] = [.74,\ .78]$, use (4.55) and interval arithmetic to show that

$$\gamma''(\mathbf{s}_8) \in [-C_2, C_2] + \sum_{|k|\le 16} -(2\pi k)^2\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_8} \subset (-\infty, 0).$$

Finally, consider the mesh

$$s_9 = .78, s_{10} = .88, s_{11} = 1$$

of the interval $[.78,\ 1]$. For $j = 9, 10$, let $\mathbf{s}_j = [s_j,\ s_{j+1}]$. Then we use (4.54) and interval arithmetic to show that

$$\gamma'(\mathbf{s}_j) \in [-C_1, C_1] + \sum_{|k|\le 16} \mathbf{i}2\pi k\bar{a}_k e^{\mathbf{i}2\pi k\mathbf{s}_j} \subset (-\infty, 0),$$

for $j = 9, 10$.

Combining the above, we conclude that $\gamma' < 0$ on $[0,\ .25]$, $\gamma'' > 0$ on $[.25,\ .28]$, $\gamma' > 0$ on $[.28,\ .74]$, $\gamma'' < 0$ on $[.74,\ .78]$ and that $\gamma' < 0$ on $[.78,\ 1]$. That shows that γ' has exactly two distinct zeroes. □

4.3 Rigorous computation of the Floquet exponents

Let $\gamma(t)$ the real symmetric periodic solution of (4.1) given by (4.51). In this section, we compute rigorously its Floquet exponents. More precisely, we prove the following result.

Theorem 4.17. *Let γ the τ-periodic solution given by* (4.51). *Define*

$$\Gamma = \{\gamma(t) \ : \ t \in [0,\tau]\}. \tag{4.56}$$

Then Γ is a hyperbolic periodic orbit, it has two stable Floquet exponents and one unstable Floquet exponent. Therefore, attached to Γ, there exist a three-dimensional stable manifold $\mathcal{S} = \mathcal{S}(\Gamma)$ and a two-dimensional unstable manifold $\mathcal{U} = \mathcal{U}(\Gamma)$.

The rest of the section presents the proof of Theorem 4.17.

Let us first re-write the fourth order differential equation as a vector field

$$\mathbf{w}' = g(\mathbf{w}) \stackrel{\text{def}}{=} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ -120w_0 - w_0^3 - 154w_1 - 71w_2 - 14w_3 \end{pmatrix},$$

where $\mathbf{w} = (w_0, w_1, w_2, w_3) = (w, w', w'', w''')$. Let

$$A(t) \stackrel{\text{def}}{=} Dg(\gamma(t)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -120 - 3\gamma(t)^2 & -154 & -71 & -14 \end{pmatrix}$$

and consider the linear system with τ-periodic coefficients

$$\dot{\Phi}(t) = A(t)\Phi(t). \tag{4.57}$$

An invariant bundle $v(t)$ of the periodic orbit associated to the eigenvalue λ satisfies the equation

$$\Phi(t)v(0) = e^{\lambda t}v(t). \tag{4.58}$$

Note that $v(t) \in \mathbb{R}^4$ is a periodic function with the same period as $\gamma(t)$. Differentiating equation (4.58) and using (4.57), we obtain the *invariance equation*

$$\dot{v}(t) + \lambda v(t) - A(t)v(t) = 0. \tag{4.59}$$

Remark 4.18. *Let (λ, v) a solution of the eigenvalue problem* (4.59). *Then λ is a Floquet exponent associated to the periodic orbit Γ given by* (4.56). *The corresponding number $\mu \stackrel{\text{def}}{=} e^{\tau\lambda}$ is a Floquet multiplier associated to Γ. We know that one Floquet multiplier is always equal to one.*

Based on the above remark, the proof of Theorem 4.17 is to find three solutions $(\lambda_j, v_j(t))$ $(j = 1,2,3)$ of (4.59) with the radii polynomial approach, and then to determine the stability of the periodic orbit by studying wether each $|e^{\tau\lambda_j}|$ in inside or outside the unit circle in the complex plane.

Denote $v(t) = (v_1(t), v_2(t), v_3(t), v_4(t))$, and expand each of its component as

$$v_j(t) = \sum_{k\in\mathbb{Z}} (a_j)_k e^{\mathbf{i}\tilde{\omega}kt}, \tag{4.60}$$

where $\tilde{\omega}$ is the frequency of the periodic solution γ as given in Theorem 4.14. Plugging (4.60) in the invariance equation (4.59), we obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left(\mathbf{i}\tilde{\omega}k(a_1)_k + \lambda(a_1)_k - (a_2)_k\right) e^{\mathbf{i}\tilde{\omega}kt} = 0\\
&\sum_{k\in\mathbb{Z}} \left(\mathbf{i}\tilde{\omega}k(a_2)_k + \lambda(a_2)_k - (a_3)_k\right) e^{\mathbf{i}\tilde{\omega}kt} = 0\\
&\sum_{k\in\mathbb{Z}} \left(\mathbf{i}\tilde{\omega}k(a_3)_k + \lambda(a_3)_k - (a_4)_k\right) e^{\mathbf{i}\tilde{\omega}kt} = 0\\
&\sum_{k\in\mathbb{Z}} \left(\mathbf{i}\tilde{\omega}k(a_4)_k + \lambda(a_4)_k + 120(a_1)_k + 154(a_2)_k + 71(a_3)_k + 14(a_4)_k + 3(\tilde{a}^2a_1)_k\right) e^{\mathbf{i}\tilde{\omega}kt} = 0,
\end{aligned}$$

where $\tilde{a} = (\tilde{a}_k)_{k\in\mathbb{Z}}$ is the infinite dimensional vector of Fourier coefficients of γ as given in (4.51). The unknowns for this problem are λ and $a_j = ((a_j)_k)_{k\in\mathbb{Z}}$ for $j = 1, 2, 3, 4$. Let $x = (\lambda, a_1, a_2, a_3, a_4)$, and

$$\begin{aligned}
f_1(x) &\stackrel{\text{def}}{=} \mathbf{i}\tilde{\omega}k(a_1)_k + \lambda(a_1)_k - (a_2)_k\\
f_2(x) &\stackrel{\text{def}}{=} \mathbf{i}\tilde{\omega}k(a_2)_k + \lambda(a_2)_k - (a_3)_k\\
f_3(x) &\stackrel{\text{def}}{=} \mathbf{i}\tilde{\omega}k(a_3)_k + \lambda(a_3)_k - (a_4)_k\\
f_4(x) &\stackrel{\text{def}}{=} \mathbf{i}\tilde{\omega}k(a_4)_k + \lambda(a_4)_k + 120(a_1)_k + 154(a_2)_k + 71(a_3)_k + 14(a_4)_k + 3(\tilde{a}^2a_1)_k
\end{aligned}$$

If (λ, v) is a solution of (4.59), then (λ, cv) is also a solution of (4.59) for any $c \in \mathbb{C}$. Hence, we have to impose a phase condition in order to apply a contraction mapping argument. We therefore fix the length of the eigenvector at time $t = 0$ to be approximately equal to 1 by imposing the condition

$$f_0(x) \stackrel{\text{def}}{=} \sum_{j=1}^{4} ((a_j)_{-1} + (a_j)_0 + (a_j)_1)^2 - 1 = 0.$$

Finally, we define the operator $f = (f_0, f_1, f_2, f_3, f_4)$, and look for solutions of

$$f(x) = 0. \tag{4.61}$$

We look for solutions of (4.61) in the space

$$X^\beta \stackrel{\text{def}}{=} \mathbb{C} \times \left(\ell^1_\nu\right)^4. \tag{4.62}$$

endowed with the norm

$$\|x\|_X^\beta \stackrel{\text{def}}{=} \max\left(|\lambda|, \|a_1\|_\nu, \|a_2\|_\nu, \|a_3\|_\nu, \beta\|a_4\|_\nu\right), \tag{4.63}$$

where β is a weight to be fixed when performing the computer-assisted proof. We denote by $x = (\lambda, a_1, a_2, a_3, a_4)$ an element of X^β.

As in Section 4, we need a good approximate inverse to apply the radii polynomial approach on problem (4.61).

4.3.1 Definition of the approximate inverse operator A

Assume that using a finite dimensional projection $f^{(m)} : \mathbb{C} \times (\mathbb{C}^{2m-1})^4 \to \mathbb{C} \times (\mathbb{C}^{2m-1})^4$ of (4.61), we applied Newton's method to find a numerical solution $\bar{x} = (\bar{\lambda}, \bar{a}_1^{(m)}, \bar{a}_2^{(m)}, \bar{a}_3^{(m)}, \bar{a}_4^{(m)}) = (\bar{\lambda}, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) \in \mathbb{C} \times (\mathbb{C}^{2m-1})^4$ such that $f^{(m)}(\bar{x}) \approx 0$.

Denote

$$Df(\bar{x}) = \begin{bmatrix} \partial_\lambda f_0(\bar{x}) & D_{a_1} f_0(\bar{x}) & D_{a_2} f_0(\bar{x}) & D_{a_3} f_0(\bar{x}) & D_{a_4} f_0(\bar{x}) \\ \partial_\lambda f_1(\bar{x}) & D_{a_1} f_1(\bar{x}) & D_{a_2} f_1(\bar{x}) & D_{a_3} f_1(\bar{x}) & D_{a_4} f_1(\bar{x}) \\ \partial_\lambda f_2(\bar{x}) & D_{a_1} f_2(\bar{x}) & D_{a_2} f_2(\bar{x}) & D_{a_3} f_2(\bar{x}) & D_{a_4} f_2(\bar{x}) \\ \partial_\lambda f_3(\bar{x}) & D_{a_1} f_3(\bar{x}) & D_{a_2} f_3(\bar{x}) & D_{a_3} f_3(\bar{x}) & D_{a_4} f_3(\bar{x}) \\ \partial_\lambda f_4(\bar{x}) & D_{a_1} f_4(\bar{x}) & D_{a_2} f_4(\bar{x}) & D_{a_3} f_4(\bar{x}) & D_{a_4} f_4(\bar{x}) \end{bmatrix},$$

where each component of $Df(\bar{x})$ is a linear operator having that

$$\begin{cases} \partial_\lambda f_0(\bar{x}) : \mathbb{R} \to \mathbb{R} \\ \partial_\lambda f_j(\bar{x}) : \mathbb{R} \to \ell^1_\nu \text{ for } j = 1,2,3,4, \\ D_{a_i} f_0(\bar{x}) : \ell^1_\nu \to \mathbb{R} \text{ are linear functionals } (i = 1,2,3,4) \\ D_{a_i} f_j(\bar{x}) : \ell^1_\nu \to \ell^1_{\nu'} \text{ are linear operators for } i,j = 1,2,3,4 \text{ with } \nu' < \nu. \end{cases}$$

We first approximate $Df(\bar{x})$ with the operator

$$A^\dagger \stackrel{\text{def}}{=} \begin{bmatrix} A^\dagger_{\lambda,0} & A^\dagger_{a_1,0} & A^\dagger_{a_2,0} & A^\dagger_{a_3,0} & A^\dagger_{a_4,0} \\ A^\dagger_{\lambda,1} & A^\dagger_{a_1,1} & A^\dagger_{a_2,1} & A^\dagger_{a_3,1} & A^\dagger_{a_4,1} \\ A^\dagger_{\lambda,2} & A^\dagger_{a_1,2} & A^\dagger_{a_2,2} & A^\dagger_{a_3,2} & A^\dagger_{a_4,2} \\ A^\dagger_{\lambda,3} & A^\dagger_{a_1,3} & A^\dagger_{a_2,3} & A^\dagger_{a_3,3} & A^\dagger_{a_4,3} \\ A^\dagger_{\lambda,4} & A^\dagger_{a_1,4} & A^\dagger_{a_2,4} & A^\dagger_{a_3,4} & A^\dagger_{a_4,4} \end{bmatrix},$$

which acts on $b = (b_0, b_1, b_2, b_3, b_4)$ component-wise as

$$\begin{aligned} (A^\dagger b)_0 &= A^\dagger_{\lambda,0} b_0 + \sum_{i=1}^4 A^\dagger_{a_i,0} b_i \stackrel{\text{def}}{=} \partial_\lambda f_0(\bar{x}) b_0 + \sum_{i=1}^4 D_{a_i} f_0(\bar{x}) \cdot b_i \\ (A^\dagger b)_j &= A^\dagger_{\lambda,j} b_0 + \sum_{i=1}^4 A^\dagger_{a_i,j} b_i \in \ell^1_{\nu'}, \qquad (j = 1,2,3,4), \end{aligned}$$

where for $j = 1,2,3,4$, $A^\dagger_{\lambda,j} = \partial_\lambda f_j^{(m)}(\bar{x})$ and $A^\dagger_{a_i,j} b_i \in \ell^1_{\nu'}$ is defined component-wise by

$$\left(A^\dagger_{a_i,j} b_i \right)_k = \begin{cases} \left(D_{a_i} f_j^{(m)}(\bar{x}) b_i^{(m)} \right)_k, & |k| < m \\ \delta_{i,j} (\mathbf{i} \tilde{\omega} k)(b_i)_k, & |k| \geq m. \end{cases}$$

Let $A^{(m)}$ a finite dimensional approximate inverse of $Df^{(m)}(\bar{x})$ which is obtained numerically. Define the decomposition

$$A^{(m)} = \begin{bmatrix} A^{(m)}_{\lambda,0} & A^{(m)}_{a_1,0} & A^{(m)}_{a_2,0} & A^{(m)}_{a_3,0} & A^{(m)}_{a_4,0} \\ A^{(m)}_{\lambda,1} & A^{(m)}_{a_1,1} & A^{(m)}_{a_2,1} & A^{(m)}_{a_3,1} & A^{(m)}_{a_4,1} \\ A^{(m)}_{\lambda,2} & A^{(m)}_{a_1,2} & A^{(m)}_{a_2,2} & A^{(m)}_{a_3,2} & A^{(m)}_{a_4,2} \\ A^{(m)}_{\lambda,3} & A^{(m)}_{a_1,3} & A^{(m)}_{a_2,3} & A^{(m)}_{a_3,3} & A^{(m)}_{a_4,3} \\ A^{(m)}_{\lambda,4} & A^{(m)}_{a_1,4} & A^{(m)}_{a_2,4} & A^{(m)}_{a_3,4} & A^{(m)}_{a_4,4} \end{bmatrix} \in \mathbb{C}^{(8m-3)\times(8m-3)},$$

where $A^{(m)}_{\lambda,0} \in \mathbb{R}$, $A^{(m)}_{a_i,0} \in \mathbb{C}^{1\times(2m-1)}$, $A^{(m)}_{\lambda,j} \in \mathbb{C}^{(2m-1)\times 1}$ and $A^{(m)}_{a_i,j} \in \mathbb{C}^{(2m-1)\times(2m-1)}$. By *approximate inverse* we mean that for some ϵ with $0 < \epsilon \ll 1$,

$$\left|\left| I_{\mathbb{C}^{(8m-3)\times(8m-3)}} - A^{(m)} Df^{(m)}(\bar{x}) \right|\right| \leq \epsilon.$$

We define the approximate inverse A of the infinite dimensional operator $Df(\bar{x})$ by

$$A \stackrel{\text{def}}{=} \begin{bmatrix} A_{\lambda,0} & A_{a_1,0} & A_{a_2,0} & A_{a_3,0} & A_{a_4,0} \\ A_{\lambda,1} & A_{a_1,1} & A_{a_2,1} & A_{a_3,1} & A_{a_4,1} \\ A_{\lambda,2} & A_{a_1,2} & A_{a_2,2} & A_{a_3,2} & A_{a_4,2} \\ A_{\lambda,3} & A_{a_1,3} & A_{a_2,3} & A_{a_3,3} & A_{a_4,3} \\ A_{\lambda,4} & A_{a_1,4} & A_{a_2,4} & A_{a_3,4} & A_{a_4,4} \end{bmatrix}.$$

A acts on $b = (b_0, b_1, b_2, b_3, b_4) \in X = \mathbb{C} \times (\ell^1_\nu)^4$ component-wise as

$$\begin{aligned} (Ab)_0 &= A^{(m)}_{\lambda,0} b_0 + \sum_{i=1}^{4} A^{(m)}_{a_i,0} b^{(m)}_i \\ (Ab)_j &= A^{(m)}_{\lambda,j} b_0 + \sum_{i=1}^{4} A_{a_i,j} b_i \in \ell^1_\nu, \qquad (j = 1,2,3,4), \end{aligned}$$

where $A^{(m)}_{\lambda,j} \in \mathbb{C}^{(2m-1)\times 1}$ is understood to be an element of ℓ^1_ν by *padding* the tail with zeros, and $A_{a_i,j} b_i \in \ell^1_\nu$ is defined component-wise by

$$(A_{a_i,j} b_i)_k = \begin{cases} \left(A^{(m)}_{a_i,j} b^{(m)}_i \right)_k, & |k| < m \\ \dfrac{\delta_{i,j}}{\mathbf{i}\tilde{\omega} k} (b_i)_k, & |k| \geq m. \end{cases}$$

Having defined A piece by piece, we can now define the Newton-like operator by

$$T(x) = x - AF(x).$$

We show existence of fixed points of T again with the radii polynomial approach.

4.3.2 Construction of the radii polynomials for the Floquet problem

Define the bounds

$$\begin{aligned} Y &= (Y_0, Y_1, Y_2, Y_3, Y_4) \\ Z(r) &= (Z_0(r), Z_1(r), Z_2(r), Z_3(r), Z_4(r)) \end{aligned} \tag{4.64}$$

with $Y_0, Z_0(r) \in \mathbb{R}$ and $Y_j = ((Y_j)_k)_{k\in\mathbb{Z}}, Z_j(r) = ((Z_j(r))_k)_{k\in\mathbb{Z}} \in \ell^1_\nu$ $(j = 1,2,3,4)$ satisfying

$$\begin{aligned}
&|(T(\bar{x})-\bar{x})_0| \le Y_0 \quad \text{and} \quad \sup_{b,c\in B(r)} |DT_0(\bar{x}+b)c| \le Z_0(r) \\
&\left|((T(\bar{x})-\bar{x})_j)_k\right| \le (Y_j)_k \quad \text{and} \quad \sup_{b,c\in B(r)} \left|(DT_j(\bar{x}+b)c)_k\right| \le (Z_j(r))_k, \text{ for } j=1,2,3,4.
\end{aligned} \tag{4.65}$$

We begin by showing the following result.

Proposition 4.19. *Consider the bounds Y and $Z(r)$ as (4.64) and satisfying the component-wise inequalities (4.65). If $\|Y\|_X^\beta + \|Z(r)\|_X^\beta < r$, then $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction. Moreover, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $f(\tilde{x}) = 0$.*

Proof. The ball of radius r centered at 0 in X^β is given by

$$B(r) = \left\{ x = (\lambda, a_1, a_2, a_3, a_4) \;:\; \|x\|_X^\beta = \max\left(|\lambda|, \|a_1\|_\nu, \|a_2\|_\nu, \|a_3\|_\nu, \beta\|a_4\|_\nu\right) \le r \right\}.$$

First, let $x \in B_{\bar{x}}(r) = \bar{x} + B(r)$. Then, $y \overset{\text{def}}{=} x - \bar{x} \in B(r)$. There exists $\xi_0 \in [0,1]$ such that

$$\begin{aligned}
|(T(x)-\bar{x})_0| &= |T_0(x) - \bar{\lambda}| \\
&\le |T_0(x) - T_0(\bar{x})| + |T_0(\bar{x}) - \bar{\lambda}| \\
&= |DT_0(\bar{x}+\xi_0 y)y| + |T_0(\bar{x}) - \bar{\lambda}| \\
&\le Z_0(r) + Y_0.
\end{aligned}$$

Similarly, for each $j = 1,2,3,4$ and each $k \in \mathbb{Z}$, there exists $\xi = \xi(k,j) \in [0,1]$ such that

$$\begin{aligned}
|\left((T(x)-\bar{x})_j\right)_k| &= |(T_j(x))_k - (\bar{a}_j)_k| \\
&\le |\left(T_j(x) - T_j(\bar{x})\right)_k| + |(T_j(\bar{x}))_k - (\bar{a}_j)_k| \\
&= |(DT_j(\bar{x}+\xi y)y)_k| + |(T_j(\bar{x}))_k - (\bar{a}_j)_k| \\
&\le (Z_j(r))_k + (Y_j)_k,
\end{aligned}$$

and then

$$\|(T(x)-\bar{x})_j\|_\nu = \sum_{k\in\mathbb{Z}} |((T(x)-\bar{x})_j)_k| \nu^{|k|} \le \sum_{k\in\mathbb{Z}} ((Z_j(r))_k + (Y_j)_k)\nu^{|k|} = \|Y_j\|_\nu + \|Z_j(r)\|_\nu.$$

Therefore,

$$\begin{aligned}
\|T(x)-\bar{x}\|_X^\beta &= \max\left(|(T(x)-\bar{x})_0|, \|(T(x)-\bar{x})_1\|_\nu, \|(T(x)-\bar{x})_2\|_\nu,\right. \\
&\qquad \left.\|(T(x)-\bar{x})_3\|_\nu, \beta\|(T(x)-\bar{x})_4\|_\nu\right) \\
&\le \max\left(Y_0 + Z_0(r), \|Y_1\|_\nu + \|Z_1(r)\|_\nu, , \|Y_2\|_\nu + \|Z_2(r)\|_\nu,\right. \\
&\qquad \left.\|Y_3\|_\nu + \|Z_3(r)\|_\nu, \beta(\|Y_4\|_\nu + \|Z_4(r)\|_\nu)\right) \\
&= \|Y\|_X^\beta + \|Z(r)\|_X^\beta < r.
\end{aligned}$$

That shows that $T(x) \in B_{\bar{x}}(r)$, that is $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$. Let us now show that T is a contraction. Consider $x, y \in B_{\bar{x}}(r)$ such that $x \neq y$. There exists $\xi_0 \in [0,1]$ such that

$$\begin{aligned}
|(T(x) - T(y))_0| &= |DT_0(\xi_0 x + (1-\xi_0)y)(x-y)| \\
&= \left| DT_0(\xi_0 x + (1-\xi_0)y)(x-y)\left(\frac{r}{\|x-y\|_X^\beta}\right)\right| \frac{\|x-y\|_X^\beta}{r} \\
&\leq \frac{Z_0(r)}{r}\|x-y\|_X^\beta.
\end{aligned}$$

Similarly, for each $j = 1,2,3,4$ and each $k \in \mathbb{Z}$, there exists $\xi = \xi(k,j) \in [0,1]$ such that

$$\begin{aligned}
|((T(x) - T(y)_j)_k| &= |(DT_j(\xi x + (1-\xi)y)(x-y))_k| \\
&= \left|\left(DT_j(\xi x + (1-\xi)y)(x-y)\left(\frac{r}{\|x-y\|_X^\beta}\right)\right)_k\right| \frac{\|x-y\|_X^\beta}{r} \\
&\leq \frac{(Z_j(r))_k}{r}\|x-y\|_X^\beta,
\end{aligned}$$

and moreover

$$\|(T(x)-T(y))_j\|_\nu = \sum_{k\in\mathbb{Z}} |((T(x)-T(y))_j)_k|\nu^{|k|} \leq \sum_{k\in\mathbb{Z}} \frac{(Z_j(r))_k}{r}\|x-y\|_X^\beta \nu^{|k|} = \frac{\|Z_j(r)\|_\nu}{r}\|x-y\|_X^\beta.$$

Since $\|Y\|_X^\beta + \|Z(r)\|_X^\beta < r$, then

$$\kappa \stackrel{\text{def}}{=} \frac{\|Z(r)\|_X^\beta}{r} < 1. \tag{4.66}$$

Therefore,

$$\begin{aligned}
\|T(x) - T(y)\|_X^\beta &= \max\left(|(T(x) - T(y))_0|, \|(T(x) - T(y))_1\|_\nu, \|(T(x) - T(y))_2\|_\nu,\right. \\
&\qquad \left.\|(T(x) - T(y))_3\|_\nu, \beta\|(T(x) - T(y))_4\|_\nu\right) \\
&\leq \max\left(\frac{Z_0(r)}{r}\|x-y\|_X^\beta, \frac{\|Z_1(r)\|_\nu}{r}\|x-y\|_X^\beta, \frac{\|Z_2(r)\|_\nu}{r}\|x-y\|_X^\beta\right. \\
&\qquad \left.\frac{\|Z_3(r)\|_\nu}{r}\|x-y\|_X^\beta, \beta\frac{\|Z_4(r)\|_\nu}{r}\|x-y\|_X^\beta\right) \\
&= \frac{\|Z(r)\|_X^\beta}{r}\|x-y\|_X^\beta \\
&= \kappa\|x-y\|_X^\beta.
\end{aligned}$$

This implies that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction with contraction constant $\kappa < 1$ defined by (4.66). By the contraction mapping theorem, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $T(\tilde{x}) = \tilde{x} = \tilde{x} - Af(\tilde{x})$. A is injective since $\kappa < 1$. It follows that there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $f(\tilde{x}) = 0$. □

Consider bounds $\mathbf{Y}_j$ such that $\|Y_j\|_\nu \leq \mathbf{Y}_j$ and bounds $\mathbf{Z}_j(r)$ such that $\|Z_j(r)\|_\nu \leq \mathbf{Z}_j(r)$, for $j = 1,2,3,4$.

Definition 4.20. Given the bounds Y and $Z(r)$ satisfying (4.65) we define the five *radii polynomials* p_0, p_1, p_2, p_3, p_4 by

$$\begin{aligned} p_0(r) &\stackrel{\text{def}}{=} Y_0 + Z_0(r) - r && (4.67)\\ p_j(r) &\stackrel{\text{def}}{=} \mathbf{Y}_j + \mathbf{Z}_j(r) - r, \quad j = 1,2,3 && (4.68)\\ p_4(r) &\stackrel{\text{def}}{=} \mathbf{Y}_4 + \mathbf{Z}_4(r) - \frac{r}{\beta}. && (4.69)\end{aligned}$$

Proposition 4.21. *Fix $\nu \geq 1$ an exponential decay rate and construct the five radii polynomials $p_0(r), \ldots, p_4(r)$ of Definition 4.20. Define*

$$\mathcal{I} \stackrel{\text{def}}{=} \bigcap_{j=0}^{4} \{r > 0 \mid p_j(r) < 0\}. \tag{4.70}$$

If $\mathcal{I} \neq \emptyset$, then $\mathcal{I}$ is an open interval, and for any $r \in \mathcal{I}$, there exists a unique $\tilde{x} \in B_{\bar{x}}(r)$ such that $f(\tilde{x}) = 0$.

Proof. Let $r \in \mathcal{I} \neq \emptyset$. Now since $p_j(r) < 0$ for $j = 0,1,2,3,4$, we have that

$$\begin{aligned} \|Y\|_X^\beta + \|Z(r)\|_X^\beta &= \max\left(|Y_0 + Z_0(r)|, \|Y_1\|_\nu + \|Z_1(r)\|_\nu, , \|Y_2\|_\nu + \|Z_2(r)\|_\nu,\right.\\ &\qquad \left.\|Y_3\|_\nu + \|Z_3(r)\|_\nu, \beta(\|Y_4\|_\nu + \|Z_4(r)\|_\nu)\right)\\ &\leq \max\left(Y_0 + Z_0(r), \mathbf{Y}_1 + \mathbf{Z}_1(r), \mathbf{Y}_2 + \mathbf{Z}_2(r),\right.\\ &\qquad \left.\mathbf{Y}_3 + \mathbf{Z}_3(r), \beta(\mathbf{Y}_4 + \mathbf{Z}_4(r))\right) < r. \end{aligned}$$

□

Computation of the bounds Y.

The computation of the bound Y_0 is easy, as one realizes that

$$|(Af(\bar{x}))_0| = \left| A^{(m)}_{\lambda,0} f_0(\bar{x}) + \sum_{i=1}^{4} A^{(m)}_{a_i,0} f_i^{(m)}(\bar{x}) \right|.$$

For the computation of the bound $\mathbf{Y}_j$ for $j = 1,2,3,4$, we first need to elaborate on the computation of $f_4(\bar{x})$. Notice that

$$f_4(\bar{x}) = i\bar{\omega}k(\bar{a}_4)_k + \lambda(\bar{a}_4)_k + 120(\bar{a}_1)_k + 154(\bar{a}_2)_k + 71(\bar{a}_3)_k + 14(\bar{a}_4)_k + 3(\tilde{a}^2\bar{a}_1)_k,$$

where $\tilde{a} = (\tilde{a}_k)_{k\in\mathbb{Z}}$ is the infinite vector of nonzero Fourier coefficients of the periodic solution $\gamma(t)$ given in (4.51). From (4.52), we get that

$$\|\tilde{a} - \bar{a}^{(\gamma)}\|_\nu \leq r_\gamma, \qquad r_\gamma \stackrel{\text{def}}{=} 7.595549832526767 \times 10^{-13},$$

where $\bar{a}^{(\gamma)}$ is the numerical data given in Figure 5. Let $\alpha \stackrel{\text{def}}{=} \tilde{a} - \bar{a}^{(\gamma)} \in B_0(r_\gamma) \subset \ell^1_\nu$. Then, $\|\alpha\|_\nu \leq r_\gamma$. Let $\tilde{\alpha}$ such that $\alpha = \tilde{\alpha} r_\gamma$ so that $\|\tilde{\alpha}\|_\nu \leq 1$ that is $\tilde{\alpha} \in B_0(1) \subset \ell^1_\nu$. We have that

$$3(\tilde{a}^2\bar{a}_1)_k = 3((\bar{a}^{(\gamma)} + \tilde{\alpha}r_\gamma)^2\bar{a}_1)_k = 3((\bar{a}^{(\gamma)})^2\bar{a}_1)_k + 6(\bar{a}^{(\gamma)}\tilde{\alpha}\bar{a}_1)_k r_\gamma + 3(\tilde{\alpha}^2\bar{a}_1)_k r_\gamma^2.$$

Denote

$$\begin{aligned} g_4(\bar{x}) &\stackrel{\text{def}}{=} i\tilde{\omega}k(\bar{a}_4)_k + \lambda(\bar{a}_4)_k + 120(\bar{a}_1)_k + 154(\bar{a}_2)_k + 71(\bar{a}_3)_k + 14(\bar{a}_4)_k + 3((\bar{a}^{(\gamma)})^2\bar{a}_1)_k \\ h_4(\bar{x}) &\stackrel{\text{def}}{=} 6(\bar{a}^{(\gamma)}\tilde{\alpha}\bar{a}_1)_k r_\gamma + 3(\tilde{\alpha}^2\bar{a}_1)_k r_\gamma^2, \end{aligned}$$

so that $f_4(\bar{x}) = g_4(\bar{x}) + h_4(\bar{x})$. The term $g_4(\bar{x})$ is evaluated using interval arithmetic and the term $h_4(\bar{x})$ is controlled using analytic estimates.

Assume that $m \geq 3m_\gamma - 2 = 46$ and assume that

$$\bar{a}_k^{(\gamma)} = (\bar{a}_1)_k = 0, \quad \text{for } |k| \geq m_\gamma = 16.$$

Then, $((\bar{a}^{(\gamma)})^2\bar{a}_1)_k = 0$, for all $|k| \geq m \geq 3m_\gamma - 2$. Hence for $j = 1, 2, 3, 4$, we have that

$$\begin{aligned} ||[T(\bar{x}) - \bar{x}]_j||_\nu &= ||[Af(\bar{x})]_j||_\nu \\ &= ||A_{\lambda,j}f_0(\bar{x}) + A_{a_1,j}f_1(\bar{x}) + A_{a_2,j}f_2(\bar{x}) + A_{a_3,j}f_3(\bar{x}) + A_{a_4,j}f_4(\bar{x})||_\nu \\ &= \sum_{k\in\mathbb{Z}} |[A_{\lambda,j}f_0(\bar{x})]_k + [A_{a_1,j}f_1(\bar{x})]_k + [A_{a_2,j}f_2(\bar{x})]_k + [A_{a_3,j}f_3(\bar{x})]_k + [A_{a_4,j}f_4(\bar{x})]_k|\,\nu^{|k|} \\ &\leq \sum_{|k|<m} \Big|[A^{(m)}_{\lambda,j}f_0^{(m)}(\bar{x})]_k + [A^{(m)}_{a_1,j}f_1^{(m)}(\bar{x})]_k + [A^{(m)}_{a_2,j}f_2^{(m)}(\bar{x})]_k \\ &\qquad + [A^{(m)}_{a_3,j}f_3^{(m)}(\bar{x})]_k + [A^{(m)}_{a_4,j}g_4^{(m)}(\bar{x})]_k\Big|\,\nu^{|k|} + 3\sum_{|k|\geq m}\left|\frac{\delta_{4,j}}{\mathbf{i}\tilde{\omega}k}((\bar{a}^{(\gamma)})^2\bar{a}_1)_k\right|\nu^{|k|} \\ &\quad + 6\sum_{|k|\geq m}\left|\frac{\delta_{4,j}}{\mathbf{i}\tilde{\omega}k}(\bar{a}^{(\gamma)}\tilde{\alpha}\bar{a}_1)_k\right|\nu^{|k|}r_\gamma + 3\sum_{|k|\geq m}\left|\frac{\delta_{4,j}}{\mathbf{i}\tilde{\omega}k}(\tilde{\alpha}^2\bar{a}_1)_k\right|\nu^{|k|}r_\gamma^2 \\ &\leq \sum_{|k|<m} \Big|[A^{(m)}_{\lambda,j}f_0^{(m)}(\bar{x})]_k + [A^{(m)}_{a_1,j}f_1^{(m)}(\bar{x})]_k + [A^{(m)}_{a_2,j}f_2^{(m)}(\bar{x})]_k \\ &\qquad + [A^{(m)}_{a_3,j}f_3^{(m)}(\bar{x})]_k + [A^{(m)}_{a_4,j}g_4^{(m)}(\bar{x})]_k\Big|\,\nu^{|k|} \\ &\quad + \frac{6}{\tilde{\omega}m}\delta_{4,j}\|\bar{a}^{(\gamma)}\|_\nu\|\bar{a}_1\|_\nu r_\gamma + \frac{3}{\tilde{\omega}m}\delta_{4,j}\|\bar{a}_1\|_\nu r_\gamma^2, \end{aligned}$$

which is evaluated using interval arithmetic.

Computation of the bounds $\boldsymbol{Z}$.

The next step in the construction of the radii polynomials is to construct the bounds $Z_0(r), \mathbf{Z}_1(r), \mathbf{Z}_2(r), \mathbf{Z}_3(r), \mathbf{Z}_4(r)$. Let $b, c \in B(r) \subset \mathbb{C} \times (\ell^1_\nu)^4$. Then

$$DT(\bar{x}+b)c = [I - A \cdot Df(\bar{x}+b)]c = [I - AA^\dagger]c - A[Df(\bar{x}+b) - A^\dagger]c. \tag{4.71}$$

We first bound the quantities involved in the first term of (4.71). Let $B \stackrel{\text{def}}{=} I - AA^\dagger$, which we express as

$$B = \begin{bmatrix} B_{\lambda,0} & B_{a_1,0} & B_{a_2,0} & B_{a_3,0} & B_{a_4,0} \\ B_{\lambda,1} & B_{a_1,1} & B_{a_2,1} & B_{a_3,1} & B_{a_4,1} \\ B_{\lambda,2} & B_{a_1,2} & B_{a_2,2} & B_{a_3,2} & B_{a_4,2} \\ B_{\lambda,3} & B_{a_1,3} & B_{a_2,3} & B_{a_3,3} & B_{a_4,3} \\ B_{\lambda,4} & B_{a_1,4} & B_{a_2,4} & B_{a_3,4} & B_{a_4,4} \end{bmatrix}.$$

Due to the structure of B, we have that $[(Bc)_j]_k = 0$ for $|k| \geq m$, $j = 1,2,3,4$ and $c \in B(r) \subset \mathbb{C} \times (\ell^1_\nu)^4$. Define

$$Z_0^{(0)} \stackrel{\text{def}}{=} |B_{\lambda,0}| + \sum_{i=1}^{4} \left(\max_{|k|<m} \frac{|(B_{a_i,0})_k|}{\nu^{|k|}} \right) \tag{4.72}$$

$$Z_j^{(0)} \stackrel{\text{def}}{=} \sum_{|k|<m} |(B_{\lambda,j})_k| \nu^{|k|} + \sum_{i=1}^{4} \left(\max_{|n|<m} \frac{1}{\nu^{|n|}} \sum_{|k|<m} |(B_{a_i,j})_{k,n}| \nu^{|k|} \right), \tag{4.73}$$

for $j = 1,2,3,4$. Now, recalling (4.14) and Lemma 4.3, we have that

$$|(Bc)_0| = \left| B_{\lambda,0} c_0 + \sum_{i=1}^{3} \sum_{k \in \mathbb{Z}} (B_{a_i,0})_k (c_i)_k \right| \leq \left(|B_{\lambda,0}| + \sum_{i=1}^{4} \|B_{a_i,0}\|_\nu^\infty \right) r = Z_0^{(0)} r.$$

Thus, for $j = 1,2,3,4$, recalling Lemma 4.5, Corollary 4.6 and (4.16), we get that

$$\|(Bc)_j\|_\nu = \left\| B_{\lambda,j} c_0 + \sum_{i=1}^{4} B_{a_i,j} c_i \right\|_\nu \leq \left(\|B_{\lambda,j}\|_\nu + \sum_{i=1}^{4} \|B_{a_i,j}\|_{B(\ell^1_\nu,\ell^1_\nu)} \right) r \leq Z_j^{(0)} r,$$

which bounds the first term of (4.71).

Next, we bound the quantities involved in the second term of (4.71). Denote $b = (b_0, b_1, b_2, b_3, b_4), c = (c_0, c_1, c_2, c_3, c_4) \in B(r) \subset X = \mathbb{C} \times (\ell^1_\nu)^4$. For $j = 0,1,2,3,4$, let $z_j \stackrel{\text{def}}{=} \left([Df(\bar{x}+b) - A^\dagger]c\right)_j$ and set $z \stackrel{\text{def}}{=} (z_0, z_1, z_2, z_3, z_4)$. Then

$$\begin{aligned}
z_0 &= 2 \sum_{j=1}^{4} ((b_j)_{-1} + (b_j)_0 + (b_j)_1)((c_j)_{-1} + (c_j)_0 + (c_j)_1) \\
z_1 &= \left\{ \bar{\lambda}(c_1)_k \right\}_{|k| \geq m} - \{(c_2)_k\}_{|k|\geq m} + c_0 \{(b_1)_k\}_{k \in \mathbb{Z}} + b_0 \{(c_1)_k\}_{k \in \mathbb{Z}} \\
z_2 &= \left\{ \bar{\lambda}(c_2)_k \right\}_{|k| \geq m} - \{(c_3)_k\}_{|k|\geq m} + c_0 \{(b_2)_k\}_{k \in \mathbb{Z}} + b_0 \{(c_2)_k\}_{k \in \mathbb{Z}} \\
z_3 &= \left\{ \bar{\lambda}(c_3)_k \right\}_{|k| \geq m} - \{(c_4)_k\}_{|k|\geq m} + c_0 \{(b_3)_k\}_{k \in \mathbb{Z}} + b_0 \{(c_3)_k\}_{k \in \mathbb{Z}} \\
z_4 &= \left\{ \bar{\lambda}(c_4)_k \right\}_{|k| \geq m} + 120 \{(c_1)_k\}_{|k|\geq m} + 154 \{(c_2)_k\}_{|k|\geq m} + 71 \{(c_3)_k\}_{|k|\geq m} \\
&\quad + 14 \{(c_4)_k\}_{|k|\geq m} + 3 \left\{ (\tilde{a}^2 c_1^I)_k \right\}_{|k|<m} + 3 \left\{ (\tilde{a}^2 c_1)_k \right\}_{|k|\geq m} \\
&\quad + c_0 \{(b_4)_k\}_{k \in \mathbb{Z}} + b_0 \{(c_4)_k\}_{k \in \mathbb{Z}}
\end{aligned}$$

The second term of (4.71) is $A[Df(\bar{x}+b) - A^\dagger]c = Az$ given component-wise by

$$\left(A[Df(\bar{x}+b) - A^\dagger]c \right)_j = (Az)_j = A_{\lambda,j} z_0 + \sum_{i=1}^{4} A_{a_i,j} z_i.$$

Consider $\tilde{b} = (\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4), \tilde{c} = (\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4) \in B(1)$ such that $b = \tilde{b} r$ and $c = \tilde{c} r$ for $r > 0$. We now construct an upper bound for $|(Az)_j|$ for the cases $j = 0,1,2,3,4$.

<u>Case 1</u>: a bound on $|(Az)_0|$. Note first that since $\|\tilde{b}_j\|_\nu, \|\tilde{c}_j\|_\nu \leq 1$ for $j = 1,2,3$ and $\|\tilde{b}_4\|_\nu, \|\tilde{c}_4\|_\nu \leq \frac{1}{\beta}$, then for $j = 1,2,3$, $|(\tilde{b}_j)_k|, |(\tilde{c}_j)_k| \leq \frac{1}{\nu^{|k|}}$ for all $k \in \mathbb{Z}$ and $|(\tilde{b}_4)_k|, |(\tilde{c}_4)_k| \leq$

$\frac{1/\beta}{\nu^{|k|}}$. Hence,

$$\begin{aligned}
|z_0| &\leq 2\sum_{j=1}^{4}\left(\left|(\tilde{b}_j)_{-1}\right| + \left|(\tilde{b}_j)_0\right| + \left|(\tilde{b}_j)_1\right|\right)\left(|(\tilde{c}_j)_{-1}| + |(\tilde{c}_j)_0| + |(\tilde{c}_j)_1|\right) r^2 \\
&\leq 2\sum_{j=1}^{3}\left(\frac{1}{\nu}+1+\frac{1}{\nu}\right)\left(\frac{1}{\nu}+1+\frac{1}{\nu}\right) r^2 + 2\frac{1}{\beta}\left(\frac{1}{\nu}+1+\frac{1}{\nu}\right)\left(\frac{1}{\nu}+1+\frac{1}{\nu}\right) r^2 \\
&= \left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 r^2.
\end{aligned}$$

As before, let $\alpha \stackrel{\text{def}}{=} \tilde{a} - \bar{a}^{(\gamma)} \in B_0(r_\gamma) \subset \ell^1_\nu$. Then, $\|\alpha\|_\nu \leq r_\gamma$. Let $\tilde{\alpha}$ such that $\alpha = \tilde{\alpha} r_\gamma$ so that $\|\tilde{\alpha}\|_\nu \leq 1$. We have that

$$(\tilde{a}^2 c_1^I)_F = ((\bar{a}^{(\gamma)} + \tilde{\alpha} r_\gamma)^2 c_1^I)_F = ((\bar{a}^{(\gamma)})^2 c_1^I)_F + 2(\bar{a}^{(\gamma)}\tilde{\alpha} c_1^I)_F r_\gamma + (\tilde{\alpha}^2 c_1^I)_F r_\gamma^2.$$

Letting

$$Z_0^{(1)} \stackrel{\text{def}}{=} 3|A_{a_4,0}\cdot(|\bar{a}^{(\gamma)}|^2\omega^I)_F| + 6\|A_{a_4,0}\|_\nu^\infty \|\bar{a}^{(\gamma)}\|_\nu r_\gamma + 3\|A_{a_4,0}\|_\nu^\infty r_\gamma^2 \tag{4.74}$$

$$\begin{aligned}
Z_0^{(2)} &\stackrel{\text{def}}{=} |A_{\lambda,0}|\left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 + 2\|A_{a_1,0}\|_\nu^\infty + 2\|A_{a_2,0}\|_\nu^\infty \\
&\quad +2\|A_{a_3,0}\|_\nu^\infty + \frac{2}{\beta}\|A_{a_4,0}\|_\nu^\infty,
\end{aligned} \tag{4.75}$$

we get that

$$\begin{aligned}
|(Az)_0| &\leq |A_{\lambda,0}z_0| + \sum_{i=1}^{4}|A_{a_i,0}z_i| \\
&\leq |A_{\lambda,0}|\left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 r^2 + 2\|A_{a_1,0}\|_\nu^\infty r^2 + 2\|A_{a_2,0}\|_\nu^\infty r^2 + 2\|A_{a_3,0}\|_\nu^\infty r^2 \\
&\quad +\frac{2}{\beta}\|A_{a_4,0}\|_\nu^\infty r^2 + 3|A_{a_4,0}\cdot(\tilde{a}^2 c_1^I)_F| \\
&\leq \left(|A_{\lambda,0}|\left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 + 2\|A_{a_1,0}\|_\nu^\infty + 2\|A_{a_2,0}\|_\nu^\infty + 2\|A_{a_3,0}\|_\nu^\infty + \frac{2}{\beta}\|A_{a_4,0}\|_\nu^\infty\right) r^2 \\
&\quad + \left(3|A_{a_4,0}\cdot(|\bar{a}^{(\gamma)}|^2\omega^I)_F| + 6\|A_{a_4,0}\|_\nu^\infty\|\bar{a}^{(\gamma)}\|_\nu r_\gamma + 3\|A_{a_4,0}\|_\nu^\infty r_\gamma^2\right) r \\
&= Z_0^{(1)} r + Z_0^{(2)} r^2.
\end{aligned}$$

Case 2: a bound on $\|(Az)_j\|_\nu$, $j = 1,2,3,4$.

For $j = 1$, letting

$$\begin{aligned}
Z_1^{(1)} &\stackrel{\text{def}}{=} \frac{|\bar{\lambda}|+1}{\tilde{\omega}m} + 3\|A_{a_4,1}^{(m)}(|\bar{a}^{(\gamma)}|^2\omega^I)_F\|_\nu + 6\|A_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)}\|\bar{a}^{(\gamma)}\|_\nu r_\gamma \\
&\quad +3\|A_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} r_\gamma^2
\end{aligned} \tag{4.76}$$

$$\begin{aligned}
Z_1^{(2)} &\stackrel{\text{def}}{=} \|A_{\lambda,1}^{(m)}\|_\nu\left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 + 2\|A_{a_1,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} \\
&\quad +2\|A_{a_2,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} + 2\|A_{a_3,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \frac{2}{\beta}\|A_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)}
\end{aligned} \tag{4.77}$$

we get that

$$
\begin{aligned}
\|(Az)_1\|_\nu &\le \|A^{(m)}_{\lambda,1}\|_\nu |z_0| + \|A_{a_1,1}z_1\|_\nu + \|A_{a_2,1}z_2\|_\nu + \|A_{a_3,1}z_3\|_\nu + \|A_{a_4,1}z_4\|_\nu \\
&\le \|A^{(m)}_{\lambda,1}\|_\nu \left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 r^2 + \sum_{|k|\ge m}\left|\frac{\bar\lambda}{\mathbf{i}\tilde\omega k}(c_1)_k\right|\nu^{|k|} + \sum_{|k|\ge m}\left|\frac{1}{\mathbf{i}\tilde\omega k}(c_2)_k\right|\nu^{|k|} \\
&\quad +2\|A^{(m)}_{a_1,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} r^2 + 2\|A^{(m)}_{a_2,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} r^2 + 2\|A^{(m)}_{a_3,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} r^2 \\
&\quad +3\|A^{(m)}_{a_4,1}(\tilde a^2 c_1^I)_F\|_\nu + \frac{2}{\beta}\|A^{(m)}_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} r^2 \\
&\le \|A^{(m)}_{\lambda,1}\|_\nu \left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 r^2 + \left(\frac{|\bar\lambda|+1}{\tilde\omega m}\right) r \\
&\quad +2\left(\|A^{(m)}_{a_1,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \|A^{(m)}_{a_2,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \|A^{(m)}_{a_3,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \frac{1}{\beta}\|A^{(m)}_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)}\right) r^2 \\
&\quad + \left(3\|A^{(m)}_{a_4,1}(|\bar a^{(\gamma)}|^2\omega^I)_F\|_\nu + 6\|A^{(m)}_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)}\|\bar a^{(\gamma)}\|_\nu r_\gamma + 3\|A^{(m)}_{a_4,1}\|_{B(\ell^1_\nu,\ell^1_\nu)} r_\gamma^2\right) r \\
&= Z_1^{(1)} r + Z_1^{(2)} r^2.
\end{aligned}
$$

Similarly, for $j=2$, letting

$$
\begin{aligned}
Z_2^{(1)} &\stackrel{\text{def}}{=} \frac{|\bar\lambda|+1}{\tilde\omega m} + 3\|A^{(m)}_{a_4,2}(|\bar a^{(\gamma)}|^2\omega^I)_F\|_\nu + 6\|A^{(m)}_{a_4,2}\|_{B(\ell^1_\nu,\ell^1_\nu)}\|\bar a^{(\gamma)}\|_\nu r_\gamma \\
&\quad +3\|A^{(m)}_{a_4,2}\|_{B(\ell^1_\nu,\ell^1_\nu)} r_\gamma^2
\end{aligned}
\tag{4.78}
$$

$$
\begin{aligned}
Z_2^{(2)} &\stackrel{\text{def}}{=} \|A^{(m)}_{\lambda,2}\|_\nu \left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 + 2\|A^{(m)}_{a_1,2}\|_{B(\ell^1_\nu,\ell^1_\nu)} \\
&\quad +2\|A^{(m)}_{a_2,2}\|_{B(\ell^1_\nu,\ell^1_\nu)} + 2\|A^{(m)}_{a_3,2}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \frac{2}{\beta}\|A^{(m)}_{a_4,2}\|_{B(\ell^1_\nu,\ell^1_\nu)},
\end{aligned}
\tag{4.79}
$$

we get that

$$\|(Az)_2\|_\nu \le Z_2^{(1)} r + Z_2^{(2)} r^2.$$

For $j=3$, we let

$$
\begin{aligned}
Z_3^{(1)} &\stackrel{\text{def}}{=} \frac{|\bar\lambda|+1/\beta}{\tilde\omega m} + 3\|A^{(m)}_{a_4,3}(|\bar a^{(\gamma)}|^2\omega^I)_F\|_\nu + 6\|A^{(m)}_{a_4,3}\|_{B(\ell^1_\nu,\ell^1_\nu)}\|\bar a^{(\gamma)}\|_\nu r_\gamma \\
&\quad +3\|A^{(m)}_{a_4,3}\|_{B(\ell^1_\nu,\ell^1_\nu)} r_\gamma^2
\end{aligned}
\tag{4.80}
$$

$$
\begin{aligned}
Z_3^{(2)} &\stackrel{\text{def}}{=} \|A^{(m)}_{\lambda,3}\|_\nu \left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 + 2\|A^{(m)}_{a_1,3}\|_{B(\ell^1_\nu,\ell^1_\nu)} \\
&\quad +2\|A^{(m)}_{a_2,3}\|_{B(\ell^1_\nu,\ell^1_\nu)} + 2\|A^{(m)}_{a_3,3}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \frac{2}{\beta}\|A^{(m)}_{a_4,3}\|_{B(\ell^1_\nu,\ell^1_\nu)}
\end{aligned}
\tag{4.81}
$$

to get that

$$\|(Az)_3\|_\nu \le Z_3^{(1)} r + Z_3^{(2)} r^2.$$

Finally, for $j = 4$, letting

$$Z_4^{(1)} \stackrel{\text{def}}{=} \frac{|\bar\lambda|/\beta + 120 + 154 + 71 + 14/\beta}{\tilde\omega m} + 3\|A^{(m)}_{a_4,4}(|\bar a^{(\gamma)}|^2\omega^I)_F\|_\nu \tag{4.82}$$
$$+6\|A^{(m)}_{a_4,4}\|_{B(\ell^1_\nu,\ell^1_\nu)}\|\bar a^{(\gamma)}\|_\nu r_\gamma + 3\|A^{(m)}_{a_4,4}\|_{B(\ell^1_\nu,\ell^1_\nu)} r_\gamma^2$$
$$+\frac{1}{\tilde\omega m}\left(\|\bar a^{(\gamma)}\|_\nu^2 + 2\|\bar a^{(\gamma)}\|_\nu r_\gamma + r_\gamma^2\right)$$

$$Z_4^{(2)} \stackrel{\text{def}}{=} \|A^{(m)}_{\lambda,4}\|_\nu\left(6+\frac{2}{\beta}\right)\left(\frac{\nu+2}{\nu}\right)^2 + 2\|A^{(m)}_{a_1,4}\|_{B(\ell^1_\nu,\ell^1_\nu)} \tag{4.83}$$
$$+2\|A^{(m)}_{a_2,4}\|_{B(\ell^1_\nu,\ell^1_\nu)} + 2\|A^{(m)}_{a_3,4}\|_{B(\ell^1_\nu,\ell^1_\nu)} + \frac{2}{\beta}\|A^{(m)}_{a_4,4}\|_{B(\ell^1_\nu,\ell^1_\nu)}$$

we have that

$$\|(Az)_4\|_\nu \le Z_4^{(1)} r + Z_4^{(2)} r^2.$$

Definition of the radii polynomials for the Floquet problem.

Using the estimates of Section 4.3.2, we can compute bounds Y_0, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$, such that $|(T(\bar x)-\bar x)_0| \le Y_0$ and that $\|(T(\bar x)-\bar x)_j\|_\nu \le \mathbf{Y}_j$ for $j = 1,2,3,4$.

Recall (4.72), (4.74) and (4.75), so that we can define the first radii polynomial by

$$p_0(r) = Y_0 + \left(Z_0^{(0)} + Z_0^{(1)} - 1\right) r + Z_0^{(2)} r^2. \tag{4.84}$$

For $j = 1,2,3$, recall (4.73), (4.76), (4.77), (4.78), (4.79), (4.80) and (4.81) to define

$$p_1(r) = Y_1 + \left(Z_1^{(0)} + Z_1^{(1)} - 1\right) r + Z_1^{(2)} r^2 \tag{4.85}$$
$$p_2(r) = Y_2 + \left(Z_2^{(0)} + Z_2^{(1)} - 1\right) r + Z_2^{(2)} r^2 \tag{4.86}$$
$$p_3(r) = Y_3 + \left(Z_3^{(0)} + Z_3^{(1)} - 1\right) r + Z_3^{(2)} r^2. \tag{4.87}$$

Finally, recalling (4.73), (4.82) and (4.83),

$$p_4(r) = Y_4 + \left(Z_4^{(0)} + Z_4^{(1)} - \frac{1}{\beta}\right) r + Z_4^{(2)} r^2. \tag{4.88}$$

4.3.3 Proof of Theorem 4.17

Fix the geometric decay rate to be $\nu = 1.3$ and the weight β in the Banach space X^β as defined in (4.62) to be $\beta = 1/20$.

We computed each solution of (4.61) using a finite dimensional reduction of dimension $8(16)-3 = 125$, that is we used 16 Fourier coefficients to compute each v_i. Using Newton's method, we computed four numerical approximations $\bar x_1$, $\bar x_2$, $\bar x_3, \bar x_4 \in \mathbb{C}^{125}$ of $f(x) = 0$ given by (4.61). Denote by $\bar\lambda_j$ the first component of $\bar x_j$ for $j = 1,2,3,4$. Note that these (approximate) Floquet exponents are given by

$$\bar\lambda_1 = 1, \quad \bar\lambda_2 = 0, \quad \bar\lambda_3 = -7, \quad \bar\lambda_4 = -8.$$

For the proof, we fixed $m = 56 > 3m_\gamma - 2 = 46$. In a separate computer program in MATLAB that can be found at [6] and which uses the Matlab toolbox INTLAB for reliable computing [20], we construct the five quadratic radii polynomials $p_0(r), \ldots, p_4(r)$ as defined in Section 4.3.2, and then apply Proposition 4.21 to show that

$$\mathcal{I} \stackrel{\text{def}}{=} \bigcap_{k=0}^{4} \{r > 0 \mid p_k(r) < 0\} \neq \emptyset.$$

For each approximate solution $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3, \bar{x}_4$, the program verifies with interval arithmetic that $\mathcal{I} \subset [r_- \; r_+]$. These radii are shown in Figure 7.

approximation	r_-	r_+
$\bar{x}_1$	$8.418031060851105 \times 10^{-10}$	$6.065683532607337 \times 10^{-3}$
$\bar{x}_2$	$2.723686358612173 \times 10^{-9}$	$2.861864807487595 \times 10^{-3}$
$\bar{x}_3$	$4.395973655577131 \times 10^{-10}$	$7.100507274414346 \times 10^{-4}$
$\bar{x}_4$	$1.007563239163118 \times 10^{-10}$	$2.325334375579078 \times 10^{-3}$

Figure 7: For each approximate solution, this figure shows the radii r_-, r_+ for which it is proved that $\mathcal{I} \subset [r_- \; r_+]$.

Moreover, for each $j = 1, 2, 3, 4$, there exists a unique $\tilde{x}_j \in B_{\bar{x}_j}(r_-)$ such that $T(\tilde{x}_j) = \tilde{x}_j$. Denote by $\tilde{\lambda}_j$ the first component of $\tilde{x}_j$ for $j = 1, 2, 3, 4$ We then have that

$$\begin{aligned}
|\bar{\lambda}_1 - \tilde{\lambda}_1| &\leq 8.418031060851105 \times 10^{-10} \\
|\bar{\lambda}_2 - \tilde{\lambda}_2| &\leq 2.723686358612173 \times 10^{-9} \\
|\bar{\lambda}_3 - \tilde{\lambda}_3| &\leq 4.395973655577131 \times 10^{-10} \\
|\bar{\lambda}_4 - \tilde{\lambda}_4| &\leq 1.007563239163118 \times 10^{-10}.
\end{aligned}$$

Let τ be the period of the periodic solution γ rigorously computed in Section 4. Note that we have a proof that $\tau \in [1.908097232050663 \; , \; 1.908097232051545]$. Define the Floquet multipliers μ_1, μ_2, μ_3 and μ_4 by

$$\mu_j \stackrel{\text{def}}{=} e^{\tilde{\lambda}_j \tau}, \qquad j = 1, 2, 3, 4. \tag{4.89}$$

Note that, since the monodromy matrix associated with γ is real, and the balls around each $\tilde{\lambda}_j$ are disjoint, the Floquet exponents, as well as the Floquet multipliers, are real. Using this fact, by means of interval arithmetic we obtain that

$$\begin{aligned}
\mu_1 &\in [6.740251443896278 \; , \; 6.740251465555177] \\
\mu_2 &\in [9.999999948029414 \; , \; 1.000000005197059] \\
\mu_3 &\in [1.582221595847551 \times 10^{-6} \; , \; 1.582221598511634 \times 10^{-6}] \\
\mu_4 &\in [2.347422210428808 \times 10^{-7} \; , \; 2.347422211347963 \times 10^{-7}].
\end{aligned}$$

We therefore have the proof of Theorem 4.17. All the steps from this section are performed by running the computer program *script_proof_Floquet.m* available at [6].

5 Some extensions

In this section we will sketch an argument which allows us to study (1.1) by looking at it as a perturbation of $(\mathbf{U_q})$.

First, we notice that the invariance described in Remark 2.2, which holds for $(\mathbf{U_q})$, does not hold for more general forms of (1.1). For this reason, on (1.1) we perform the change of variables (2.11). With this choice, if u solves (1.1) with initial condition $\mathbf{u}_0$, then the function w solves the equation

$$w''''(s) + N(w)(s) + e^{-s(\eta+4)}T^{\eta+4} f\left(T^{-\eta} w(s) e^{\eta s}\right) + \kappa T^2 e^{-2s} M(w)(s) = 0, \tag{5.1}$$

where N is defined in (2.5) and M is defined as

$$M(w) := w'' + (2\eta + 1)w' + \eta(\eta + 1)w,$$

with initial condition

$$\mathbf{w}_0 = LD(T)\mathbf{u}_0. \tag{5.2}$$

In the particular case $f = \mu\,|t|^{p-1}\,t + |t|^{q-1}\,t$, (5.1) becomes

$$\begin{aligned} w''''(s) + N(w)(s) + |w|^{q-1}\,w(s) + \\ + \mu\; T^{4\frac{q-p}{q-1}}\; e^{-4\frac{q-p}{q-1}s}\,|w|^{p-1}\,w(s) + \kappa\; T^2\; e^{-2s} M(w)(s) = 0. \end{aligned} \tag{5.3}$$

This equation is autonomous if and only if $\mu = \kappa = 0$. Note that (5.3) does not admit a periodic solution if $\mu \neq 0$ or $\kappa \neq 0$ since the non-autonomous term is non-periodic. We are, therefore, forced to modify our approach. Yet, we will be able to do so building upon the knowledge obtained for the case $\kappa = \mu = 0$ in Sections 2 and 3. In what follows we shall make use of the following assumption

$$\begin{gathered} (\mathbf{W_q}) \text{ has a } \tau-\text{periodic solution } \gamma \\ \text{which possesses a 3-dimensional stable manifold } \mathcal{S} \\ \text{with corresponding Floquet exponents } \lambda_1 \text{ and } \lambda_2. \end{gathered} \tag{5.4}$$

For the sake of simplicity, we will provide a detailed proof only for the case $\mu = 0$, namely we will consider the equation

$$w''''(s) + N(w)(s) + |w|^{q-1}\,w(s) + \ell\; e^{-2s} M(w)(s) + |w|^{q-1}\,w(s) = 0. \tag{5.5}$$

The remaining cases ($\kappa = 0, \mu \neq 0$ and $\mu \neq 0 \neq \kappa$) can be handled similarly, and we will state the results below but leave the proof to the interested readers.

We begin with a result for (5.5), which will be then translated into a result for the corresponding form of equation (1.1).

Theorem 5.1. *Assume that* (5.4) *holds. Then there exists a 4-dimensional manifold* $\tilde{\mathcal{S}} \subset \mathbb{R}^5$ *containing* $\mathcal{S} \times \{0\}$ *and such that, if* $(\mathbf{w}_0, z_0) \in \tilde{\mathcal{S}}$*, then the solution* w *of* (5.5) *with* $\ell = z_0$ *and initial condition* $\mathbf{w}_0$ *approaches exponentially* γ *with asymptotic phase, i.e. there exist* $s_0 \in [0, \tau)$ *and* $c > 0$ *such that*

$$|\gamma(s - s_0) - w(s)| \leq c e^{-\lambda s}, \text{ for all } s > 0, \tag{5.6}$$

with $\lambda < \min\{2, |\Re(\lambda_1)|, |\Re(\lambda_2)|\}$.

The key idea behind our result is to recast the non-autonomous 4^{th} order equation (5.5) as an autonomous system, namely:

$$\begin{cases} w''''(s) + N(w)(s) + |w|^{q-1} w(s) + z(s)M(w)(s) + |w|^{q-1} w(s) = 0 \\ z'(s) = -2z(s) \end{cases}, \tag{5.7}$$

with initial conditions $\mathbf{w}(0) = \mathbf{w}_0$ and $z(0) = \ell$.

Relabeling the unknown $\tilde{\mathbf{w}} = (w_0, w_1, w_2, w_3, w_4) = (w_0, w_1, w_2, w_3, z) = (\mathbf{w}, z)$, we have that the system (5.7) can be stated as

$$\begin{aligned} \tilde{\mathbf{w}}'(s) &= \tilde{\mathbf{g}}(\tilde{\mathbf{w}}) := \begin{bmatrix} w_1(s) \\ w_2(s) \\ w_3(s) \\ -w_4(s)M(\mathbf{w})(s) - N(\mathbf{w})(s) - |w_0(s)|^{q-1} w_0(s) \\ -2w_4(s) \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{g}(\mathbf{w}(s)) \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -w_4(s)M(\mathbf{w})(s) \\ -2w_4(s) \end{bmatrix}, \end{aligned} \tag{5.8}$$

where $\mathbf{g}$ is defined in (2.7). Having assumed that the "unperturbed problem" $(\mathbf{W_q})$, i.e. (5.5) with $\ell = 0$, admits a periodic solution with a 3-dimensional stable manifold $\mathcal{S}$ (see (5.4)), we will appropriately "grow" $\mathcal{S}$ into a 4-dimensional invariant manifold and obtain asymptotically periodic solutions of (5.8) with $|w_4(0)|$ small.

Proof. We begin noticing that the function $\tilde{\boldsymbol{\gamma}} = [\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4] := [\boldsymbol{\gamma}, 0]$ is a periodic solution of (5.7), and that, if $\tilde{\mathbf{w}}$ is a solution of (5.7) with initial condition $\tilde{\mathbf{w}}(0) \in \mathcal{S} \times \{0\}$, then $\tilde{\mathbf{w}}(s)$ approaches $\tilde{\gamma}(s)$ with asymptotic phase as $s \to +\infty$, that is to say that $\tilde{\boldsymbol{\gamma}}$ has a stable manifold $\tilde{\mathcal{S}}$ which contains the 3-dimensional manifold $\mathcal{S} \times \{0\}$.

Since equation (5.5) and system (5.7) are equivalent, the claim follows by proving that $\tilde{\boldsymbol{\gamma}}$ has a 4-dimensional stable manifold $\tilde{\mathcal{S}}$.

In order to study the stability of the periodic orbit $\tilde{\boldsymbol{\gamma}}$ we need to study its Floquet exponents. Let $\tilde{X}(t)$ be the *principal matrix solution* for the variational equation associated with $\tilde{\boldsymbol{\gamma}}$, i.e. the solution of the Cauchy problem

$$\tilde{X}'(t) = A_0(t)\tilde{X}(t), \quad \tilde{X}(0) = I_5, \tag{5.9}$$

where

$$A_0(t) := \tilde{\mathbf{g}}'(\tilde{\boldsymbol{\gamma}}(t)),$$

and I_5 is the 5×5 identity matrix. Recalling that $\tilde{\gamma}_4 = 0$, we have that

$$\tilde{\mathbf{g}}'(\tilde{\boldsymbol{\gamma}}) = \begin{bmatrix} & & 0 \\ & \mathbf{g}'(\boldsymbol{\gamma}) & 0 \\ & & 0 \\ & & 0 \\ & 0 & 0 \end{bmatrix} + \begin{bmatrix} & 0 \\ & 0 \\ 0 & 0 \\ & -M(\boldsymbol{\gamma}) \\ 0 & -2 \end{bmatrix}. \tag{5.10}$$

Evaluating $\tilde{X}(t)$ at $t = \tau$ we obtain

$$\tilde{X}(\tau) = \left[\begin{array}{cc} & * \\ & * \\ X(\tau) & * \\ & * \\ 0 & e^{-2\tau} \end{array}\right]. \tag{5.11}$$

where $X(\tau)$ is the monodromy matrix associated with $\boldsymbol{\gamma}$.

From this fact we easily recognize that the Floquet exponents associated with $\tilde{\boldsymbol{\gamma}}$ are those associated with $\boldsymbol{\gamma}$ along with -2. Therefore, the stable manifold of $\tilde{\boldsymbol{\gamma}}$ is 4-dimensional. Moreover, because of (5.11), the tangent bundle of $\mathcal{S}$ and the vector $[0, 0, 0, 0, 1]$ are transverse along $\tilde{\boldsymbol{\gamma}}$. This concludes the proof. □

Remark 5.2. *Under the same hypotheses of Theorem 5.1, and using the same notations as in its proof, we notice that the 2-dimensional unstable manifold $\mathcal{U}$ of $\boldsymbol{\gamma}$ is contained in the unstable manifold $\tilde{\mathcal{U}}$ for the periodic solution $\tilde{\gamma}$. Actually, we have $\mathcal{U} \times \{0\} = \tilde{\mathcal{U}}$. Indeed, one inclusion is trivial, whereas the other follows by a dimensional argument.*

As a consequence we have that there is no solution of (5.5) *with $\ell \neq 0$ approaching $\boldsymbol{\gamma}$ as $s \to -\infty$.*

As a consequences of Theorem 5.1, we can characterize the blow up profile for some solutions of the equation

$$u''''(r) + \kappa u''(r) + |u|^{q-1} u(r) = 0. \tag{5.12}$$

Before presenting these results, we set

$$\tilde{D}(\alpha) := \begin{bmatrix} D(\alpha) & 0 \\ 0 & \alpha^2 \end{bmatrix} = \begin{bmatrix} \alpha^\eta & 0 & 0 & 0 & 0 \\ 0 & \alpha^{\eta+1} & 0 & 0 & 0 \\ 0 & 0 & \alpha^{\eta+2} & 0 & 0 \\ 0 & 0 & 0 & \alpha^{\eta+3} & 0 \\ 0 & 0 & 0 & 0 & \alpha^2 \end{bmatrix}, \tag{5.13}$$

and

$$\tilde{L} := \begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\eta & 1 & 0 & 0 & 0 \\ \eta^2 & -2\eta - 1 & 1 & 0 & 0 \\ -\eta^3 & 3\eta^2 + 3\eta + 1 & -3\eta - 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.14}$$

Theorem 5.3. *Assume that* (5.4) *holds, and let $\tilde{\mathcal{S}} \subset \mathbb{R}^5$ be the 4-dimensional manifold established in Theorem 5.1. Define $\tilde{\Omega} := \tilde{\mathcal{D}}\tilde{L}^{-1}\tilde{\mathcal{S}}$.*

The set $\tilde{\Omega}$, as subset of $\mathbb{R}^5$, is unbounded, arc-connected, symmetric with respect to the hyperplane $\kappa = 0$ (that is, if $(\mathbf{w}_0, \kappa) \in \tilde{\Omega}$ then $(-\mathbf{w}_0, \kappa) \in \tilde{\Omega}$) and with non-empty interior which contains $L^{-1}\gamma$.

Let $(\mathbf{u}_0, \kappa) \in \tilde{\Omega}$. Then, the solution u of (5.12) *with initial condition $\mathbf{u}_0$ blows up at $T > 0$, where T is such that $\tilde{L}\tilde{D}(T)(\mathbf{u}_0, \kappa) \in \tilde{\mathcal{S}}$, and u can be written as in* (2.11)*, that is*

$$u(r) = \left(\frac{1}{T - r}\right)^{\frac{4}{q-1}} w\left(\ln\left(\frac{T}{T - r}\right)\right), \tag{5.15}$$

where w approaches asymptotically the periodic solution γ as in (5.6)*, with $\lambda < \min\{2, |\Re(\lambda_1)|, |\Re(\lambda_2)|\}$.*

In particular, if $q > 3$ and $\frac{4}{q-1} < \min\{|\Re(\lambda_1)|, |\Re(\lambda_2)|\}$, then there exist $s_0 > 0$ and $c > 0$ such that

$$\left|u(r) - \left(\frac{1}{T-r}\right)^{\frac{4}{q-1}} \gamma\left(\ln\left(\frac{T}{T-r}\right) - s_0\right)\right| < c(T-r)^a, \quad \textit{for all } r \in [0, T), \tag{5.16}$$

where $a > 0$ does not depend on $\mathbf{u}_0$ nor on κ.

Proof. Let $\mathbf{u}_0$, κ, and u as in the hypotheses of the theorem. Let w be defined by (5.15), or equivalently by (2.10). As written above, w solves (5.5) with initial condition $\mathbf{w}_0 = LD(T)\mathbf{u}_0$ and $\ell = \kappa T^2$. That is, w solves (5.5) with $(\mathbf{w}_0, \ell) \in \tilde{\mathcal{S}}$. The first part of the thesis follows by Theorem 5.1.

The proof of the properties of $\tilde{\Omega}$ follows by the same arguments used in the proof of Theorem 3.1.

The estimate (5.16) is a consequence of (5.6), the transformation (2.11) and the choice $a = \lambda - \eta > 0$. □

We recall that assumption (5.4) holds for the values of q considered in Theorem 2.25. In which case, we have the following

Corollary 5.4. *Let $q \in (q_-, q_+)$ be as in Theorem 2.25. Then there exist a τ-periodic function γ and a 4-dimensional manifold $\tilde{\mathcal{S}} \subset \mathbb{R}^5$ such that, if $(\mathbf{u}_0, \kappa) \in \tilde{\mathcal{D}}\tilde{L}^{-1}\tilde{\mathcal{S}}$, the solution u of* (5.12) *with initial condition $\mathbf{u}_0$ blows up at $T > 0$, where T is such that $\tilde{L}\tilde{D}(T)(\mathbf{u}_0, \kappa) \in \tilde{\mathcal{S}}$, and u can be written as in* (5.15) *with w approaching asymptotically the periodic solution γ as in* (5.6)*, with $\lambda < \min\{2, |\Re(\lambda_1)|, |\Re(\lambda_2)|\}$.*

In particular if $q \in (3, q_+)$, then (5.16) *holds with $a > 0$ independent of $\mathbf{u}_0$ nor of κ and suitable $s_0 > 0$ and $c > 0$.*

Proof. We can always assume that the Floquet exponents of the stable manifold $\mathcal{S}_q$ for $q < q_+$ are larger than $2 > \frac{4}{q-1}$ (indeed they lie in neighborhoods of 8 and 7). The claim follows from Theorem 5.3. □

The proof of Theorem 1.5 follows from the following remark.

Remark 5.5. *Let $\tilde{\Gamma} = \mathcal{O}(\tilde{\gamma})$, that is $\tilde{\Gamma} = \Gamma \times \{0\}$ where $\Gamma = \mathcal{O}(\gamma)$. Since $\tilde{\Omega}$ contains an open set which in turn contains the 1-dimensional compact manifold $\mathcal{M} := \tilde{L}^{-1}\tilde{\Gamma}$, by compactness argument we have that there exists an open neighborhood of $\mathcal{M}$ with uniform radius $\kappa_0 > 0$. More precisely, for each $\mathbf{p} \in \mathcal{M}$, the ball $B_\infty(\mathbf{p}, \kappa_0) \subset \mathbb{R}^5$ (i.e. the ball in the infinity norm centered at $\mathbf{p}$ of radius κ_0) is contained in $\tilde{\Omega}$, i.e.*

$$\bigcup_{\mathbf{p} \in \mathcal{M}} B_\infty(\mathbf{p}, \kappa_0) \subset \tilde{\Omega}.$$

Let $\Omega_1 \subset \mathbb{R}^4$ be the open set obtained by intersecting $\bigcup_{\mathbf{p} \in \mathcal{M}} B_\infty(\mathbf{p}, \kappa_0)$ with the hyperplane $\mathbb{R}^4 \times \{\kappa\}$ with $|\kappa| < \kappa_0$, namely

$$\Omega_1 \times \{\kappa\} = \bigcup_{\mathbf{p} \in \mathcal{M}} B_\infty(\mathbf{p}, \kappa_0) \cap (\mathbb{R}^4 \times \{\kappa\}).$$

Note that the set Ω_1 does not depend on the particular choice of κ since we have used the ball in the infinity norm and each point $\mathbf{p} \in \mathcal{M}$ lies on $\mathbb{R}^4 \times \{0\}$.

Finally, we have that, if $\mathbf{u}_0 \in \Omega_1$ and $|\kappa| < \kappa_0$, then $(\mathbf{u}_0, \kappa) \in \tilde{\Omega}$ and hence, for the solution of (5.12) *with initial condition $\mathbf{u}_0$, the conclusions of Theorem 5.3 apply.*

Now we consider the problem

$$w''''(s) + N(w)(s) + |w|^{q-1}\, w(s) + \ell\, e^{-4\frac{q-p}{q-1}s}\, |w|^{p-1}\, w(s) = 0, \tag{5.17}$$

with $q > p \geq 1$. Arguing as for Theorem 5.1, we deduce the following results.

Theorem 5.6. *Assume that* (5.4) *holds. Then there exists a 4-dimensional manifold $\tilde{\mathcal{S}} \subset \mathbb{R}^5$ containing $\mathcal{S} \times \{0\}$, such that if $(\mathbf{w}_0, z_0) \in \tilde{\mathcal{S}}$, then the solution w of* (5.17) *with $\ell = z_0$ and initial condition $\mathbf{w}_0$ approaches exponentially γ as in* (5.6)*, with $\lambda < \min\{4\frac{q-p}{q-1}, |\Re(\lambda_1)|\,, |\Re(\lambda_2)|\}$.*

Analogously to the previous results, below we give the consequences of Theorem 5.6 for the equation

$$u''''(r) + |u|^{q-1}\, u(r) + \mu\, |u|^{p-1}\, u(r) = 0. \tag{5.18}$$

To this end we define

$$\tilde{D}_1(\alpha) := \begin{bmatrix} D(\alpha) & 0 \\ 0 & \alpha^{\eta(q-p)} \end{bmatrix} = \begin{bmatrix} \alpha^{\eta} & 0 & 0 & 0 & 0 \\ 0 & \alpha^{\eta+1} & 0 & 0 & 0 \\ 0 & 0 & \alpha^{\eta+2} & 0 & 0 \\ 0 & 0 & 0 & \alpha^{\eta+3} & 0 \\ 0 & 0 & 0 & 0 & \alpha^{\eta(q-p)} \end{bmatrix}, \tag{5.19}$$

Theorem 5.7. *Assume that* (5.4) *holds, and let $\tilde{\mathcal{S}} \subset \mathbb{R}^5$ be the 4-dimensional manifold established in the Theorem 5.6. Define $\tilde{\Omega} := \tilde{\mathcal{D}}\tilde{L}^{-1}\tilde{\mathcal{S}}$.*

The set $\tilde{\Omega}$, as subset of $\mathbb{R}^5$, is unbounded, arc-connected, symmetric with respect to the hyperplane $\mu = 0$ (that is if $(\mathbf{w}_0, \mu) \in \tilde{\Omega}$ then $(-\mathbf{w}_0, \mu) \in \tilde{\Omega}$) and with non-empty interior which contains $L^{-1}\gamma$.

Let $(\mathbf{u}_0, \mu) \in \tilde{\Omega}$. Then the solution u of (5.18) *with initial condition $\mathbf{u}_0$ blows up at $T > 0$, where T is such that $\tilde{L}\tilde{D}(T)(\mathbf{u}_0, \mu) \in \tilde{\mathcal{S}}$, and u can be written as in* (5.15) *with w approaching asymptotically the periodic solution γ as in* (5.6)*, with $\lambda < \min\{4\frac{q-p}{q-1}, |\Re(\lambda_1)|\,, |\Re(\lambda_2)|\}$.*

In particular if $q > p+1$ and $\frac{4}{q-1} < \min\{|\Re(\lambda_1)|\,, |\Re(\lambda_2)|\}$, then there exist $s_0 > 0$ and $c > 0$ such that (5.16) *holds with $a > 0$ independent of $\mathbf{u}_0$ and μ.*

In particular we obtain the following

Corollary 5.8. *Let $q \in (q_-, q_+)$ be as in Theorem 2.25. Then there exist a τ-periodic function γ and a 4-dimensional manifold $\tilde{\mathcal{S}} \subset \mathbb{R}^5$ such that if $(\mathbf{u}_0, \mu) \in \tilde{\mathcal{D}}_1\tilde{L}^{-1}\tilde{\mathcal{S}}$, then the solution u of* (5.18) *with initial condition $\mathbf{u}_0$ blows up at $T > 0$ where T is such that $\tilde{L}\tilde{D}(T)(\mathbf{u}_0, \mu) \in \tilde{\mathcal{S}}$, and u can be written as in* (5.15) *with w approaching asymptotically the periodic solution γ as in* (5.6)*, with $\lambda < \min\{4\frac{q-p}{q-1}, |\Re(\lambda_1)|\,, |\Re(\lambda_2)|\}$.*

In particular if $q > p+1$, then (5.16) *holds with $a > 0$ independent of $\mathbf{u}_0$ and μ, and suitable $s_0 > 0$ and $c > 0$.*

The proofs of the results above are very similar to the ones provided for Theorems 5.1, 5.3 and Corollary 5.4. The proof of Theorem 1.4 is based on arguments similar to those given in Remark 5.5. We leave them to the interested readers.

Through the same ideas exposed so far in this section, one can obtain several results by perturbing equation ($\mathbf{U_q}$) (and consequently ($\mathbf{W_q}$)) in different ways.

For instance, we consider as a last example the "doubly perturbed" equation:

$$u''''(r) + \kappa u''(r) + |u|^{q-1} u(r) + \mu |u|^{p-1} u(r) = 0. \tag{5.20}$$

In this case the idea is to embed the corresponding non-autonomous equation in w (see (5.3)) in an autonomous system of order 6.

In this case, defining, analogously to the previous case, $\tilde{\mathcal{D}}_2$ and $\tilde{L}_2$ as

$$\tilde{D}_2(\alpha) := \begin{bmatrix} D(\alpha) & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^{4\frac{q-p}{q-1}} \end{bmatrix} \quad \text{and} \quad \tilde{L}_2 := \begin{bmatrix} L & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.21}$$

we obtain the following result.

Theorem 5.9. *Assume that* (5.4) *holds. Then there exists a 5-dimensional manifold* $\tilde{\mathcal{S}} \subset \mathbb{R}^6$ *containing* $\mathcal{S} \times \{(0,0)\}$ *and such that if* $(\mathbf{u}_0, \kappa, \mu) \in \tilde{\mathcal{D}}_2 \tilde{L}_2^{-1} \tilde{\mathcal{S}}$, *then the solution* u *of* (5.20) *with initial condition* $\mathbf{u}_0$ *blows up at* $T > 0$ *where* T *is such that* $\tilde{L}\tilde{D}(T)(\mathbf{u}_0, \kappa, \mu) \in \tilde{\mathcal{S}}$, *and* u *can be written as in* (5.15) *with* w *approaching asymptotically the periodic solution* γ *as in* (5.6), *with* $\lambda < \min\{2, 4\frac{q-p}{q-1}, |\Re(\lambda_1)|, |\Re(\lambda_2)|\}$.

In particular if $q > p + 1$, $q > 3$ *and* $\frac{4}{q-1} < \min\{|\Re(\lambda_1)|, |\Re(\lambda_2)|\}$, *then there exist* $s_0 > 0$, $c > 0$ *such that* (5.16) *holds with* $a > 0$ *independent of* $\mathbf{u}_0$ *and* μ.

We leave the proof and the discussion on the convergence to the blow up profile to the interested readers.

Remark 5.10. *Finally, below we list some further perturbations that can be handled with the same methods presented in this section.*

1) The nonlinearity f *of* (1.1) *could be chosen as*

$$f(t) = |t|^{q-1} t + \mu_1 |t|^{p_1 - 1} t + \mu_2 |t|^{p_2},$$

with $q > p_j \geq 1$ $(j = 1, 2)$. *In this case the corresponding non-autonomous equation in* w *should be recast into an autonomous system of order 7.*

2) More linear perturbations could be obtained by adding further derivatives of u:

$$u''''(r) + \mu_3 u'''(r) + \mu_2 u''(r) + \mu_1 u'(r) + \mu u(r) + |u|^{q-1} u(r) = 0. \tag{5.22}$$

3) Further nonlinear perturbations can be studied similarly. For instance, for the equation

$$u''''(r) + \kappa u''(r) + |u|^{q-1} u(r) + G(u, u', u'', u''')(r) = 0, \tag{5.23}$$

with

- $G = \mu u\, u'$, *and* $q > 7/3$;
- $G = \mu u'^2$, *and* $q > 3$;
- $G = \mu u'\, u''$, *and* $q > 5$;
- $G = \mu u\, u''$, *and* $q > 3$;
- *more generally, if* G *is* C^1 *function homogeneous of degree* d_j *with respect to* $u^{(j)}$ *with* $d_j = 0$ *(that is* G *does not depend on* $u^{(j)}$*) or* $d_j \geq 1$*, and* $q > \frac{4d_0+3d_1+2d_2+d_3}{4-d_1-2d_2-3d_3} > 0$;

results similar to the previous ones could be obtained.

6 Numerics and conjectures

We begin this section by showing how applying the transformation (2.11) is also of great benefit in the numerical investigation of equations of the general type given in (1.1). Motivated by the results we have obtained in Sections 3 and 5, and by the numerical experiments reported in [11, 12], we expect many solutions of (1.1) to blow up in finite time through progressively wider oscillations, for several types of non-linearity f. This behavior poses several unavoidable difficulties in their numerical integration, which can be effectively mitigated by turning to an "auxiliary equation" obtained through (2.11). Below we illustrate this procedure, and prove its usefulness by means of a few examples.

Suppose we consider $\mathbf{u}_0 \in \mathbb{R}^4$, and are interested in gaining quantitative, as well as qualitative, information about the solution $u = u(t)$ of (1.1) having initial condition $\mathbf{u}_0$ (e.g., its blow-up time, its blow-up profile, or its sequence of consecutive zeros) by means of numerical experiments. Assume that u blows up at $T < +\infty$. Instead of working directly with u and (1.1), we perform the transformation (2.11) and obtain a new differential equation for w, namely (5.1), which we refer to as the *auxiliary equation*[3]. Let $w_\alpha = w_\alpha(s)$ be the solution of the auxiliary equation having initial condition $LD(\alpha)\mathbf{u}_0$, with $\alpha > 0$. It follows from (2.10) that, if $\alpha < T$, then w_α vanishes exponentially as $s \to +\infty$, whereas, if $\alpha > T$, then w_α blows up in finite time. Ideally, α should be chosen as close as possible to the blow-up time T (which, of course, is a priori unknown) in order to be able to recover from w_α, through the transformation (2.11), the sought information about u. What we do in practice is computing the largest floating-point number $\bar{\alpha} > 0$ such that $w_{\bar{\alpha}}(s) \to 0$ as $s \to +\infty$. We do this by applying the bisection method to the function

$$\rho : \alpha \in \mathbb{F} \mapsto \rho(\alpha) := \begin{cases} 1 & \text{if } w_\alpha(s) \to 0 \text{ as } s \to +\infty \\ -1 & \text{if } w_\alpha(s) \text{ blows up at } S_+ < +\infty \end{cases},$$

where $\mathbb{F}$ represents the set of floating point numbers in use.

Once we have obtained $\bar{\alpha}$, we integrate numerically the auxiliary equation with initial condition $LD(\bar{\alpha})\mathbf{u}_0$, and obtain the sought information about u through (2.11).

Some observations follow in order:

[3] We recall that this notion has been extensively used throughout the paper. For instance, $(\mathbf{W_q})$ is the auxiliary equation for $(\mathbf{U_q})$.

- If we restrict our attention to the equation $(\mathbf{U_q})$ and recall equation (2.29), we can interpret $\bar{\alpha}$ as the "numerical gauge" of $\mathbf{u}_0$ with respect to $L^{-1}\mathcal{B}$; in this case, by Proposition 2.15, it is legitimate to expect $\bar{\alpha}$ being a good approximation of the blow-up time T;

- For initial conditions $\mathbf{u}_0$ that satisfy the typical assumption of many of the theorems in Sections 3 and 5 (say, those in Ω_q of Theorem 3.1), $w_{\bar{\alpha}}$ is known to converge asymptotically to a periodic function γ. Numerically, this convergence cannot be observed as it is hampered by the instability of γ (this justifies the two alternatives for $\rho(\alpha)$ given above); yet, we expect the outlined procedure to yield results that are more accurate than the ones we could obtain by working with u;

- We point out that in –all– the experiments we have performed, we have always observed $w_{\bar{\alpha}}$ to approach asymptotically a periodic function (before instability would take over), even in the cases not currently covered by our theory; this suggests that results stronger than the ones we have been able to prove hold.

Below we present some examples in which, for a given value of $\mathbf{u}_0$, we have applied the procedure outlined above to compute an estimate of the blow-time of u, and of as many zeros u as we could compute.

All experiments reported have been performed in MATLAB, with floating-point numbers represented in the (default) IEEE Standard 754 double-precision format. Numerical integration of all differential equations has been performed through MATLAB's solver `ode45` with the tightest absolute/relative possible tolerances.

Example 6.1. *For several different combinations of f and κ, we have computed the blow-up time T and a few consecutive zeros z_j of the solution u of equation* (1.1) *having initial condition $\mathbf{u}_0 = [1, 0, 0, 0]$. The computations have been performed using the procedure outlined above, and the results are reported in Table 6. The second column of the table can be compared with [11, Numerical result 5]. We believe our results to be more accurate than the ones presented there because the auxiliary equation is easier to integrate numerically, and the transformation* (2.11) *preserves the relative error in mapping zeros of w (solution of the auxiliary equation) in zeros of u.*

Much of the main results of this paper, collected in Sections 3 and 5, hold under assumptions that we may recap as follows: (1) initial conditions belong to a specified subset of $\mathbb{R}^4$ (say, Ω_q of Theorem 3.1), and (2) q belongs to a certain neighborhood (q_-, q_+) of 3. These restrictions on Ω_q and q have much to do with the tools that have been used to prove them. Specifically, the size of Ω_q is limited by the fact that the stable manifold $\mathcal{S}_q$ is only guaranteed to be –locally– transversal to the "fibers" generated by $\mathcal{D}$, whereas the choice of q is influenced by the arguments in Section 4 that require analyticity of the non-linearity in $(\mathbf{W_q})$ (which only holds if q is an odd integer) and by the fact that the estimates required to carry out the computer assisted proofs in that section have been sought only for $q = 3$.

We believe that the situation is largely that same for all other values of $q > 1$, and that all non-trivial solutions blow-up according to the same profile. We conclude this section by formulating two conjectures. Both are supported my numerical experiments, and currently remain open.

	$f(t)=t^3,\kappa=0$	$f(t)=t+t^3,\kappa=0$	$f(t)=t+t^3,\kappa=2$	$f(t)=t+t^3,\kappa=-2$
T	5.951916305391872	5.270364367304759	6.536969879267078	4.620922759247079
z_1	2.235187278061087	1.873684837690039	2.008051005605458	1.774393022413332
z_2	4.524133986157715	3.967407791337167	4.366943378886131	3.659723928257193
z_3	5.401968847922506	4.768530324852534	5.642264262692262	4.255940618753756
z_4	5.740088413057987	5.077069010517033	6.188137669652720	4.480626289607646
z_5	5.870324771904173	5.195911169058382	6.402357081555895	4.566899868286294
z_6	5.920489007203656	5.241686601847287	6.485105553736973	4.600115236864337
z_7	5.939811188240718	5.259318310834631	6.516992037647651	4.612908203704770
z_8	5.947253675458953	5.266109664694681	6.529274798730553	4.617835728418425
z_9	5.950120360947892	5.268725547663634	6.534005897595732	4.619733701754597
z_{10}	5.951224546343381	5.269733129340024	6.535828217327979	4.620464759967336
z_{11}	5.951649854732570	5.270121228061048	6.536530135713155	4.620746347807399
z_{12}	5.951813674347151	5.270270715314329	6.536800499538011	4.620854809367000
z_{13}	5.951876774127930	5.270328294578786	6.536904637850143	4.620896586419438
z_{14}	5.951901078801974	5.270350472869060	6.536944749680997	4.620912678038606
z_{15}	5.951910440437991	5.270359015467930	6.536960199892745	4.620918876183099
z_{16}	5.951914046338187	5.270362305892689	6.536966150980939	4.620921263574620
z_{17}	5.951915435253137	5.270363573293309	6.536968443211705	4.620922183146321
z_{18}	5.951915970233207	5.270364061468682	6.536969326129535	4.620922537345497
z_{19}	5.951916176295987	5.270364249503301	6.536969666210434	4.620922673775378
z_{20}	5.951916255666924	5.270364321930179	6.536969797202256	4.620922726325224
z_{21}	5.951916286238895	5.270364349827450	6.536969847657481	4.620922746566292
z_{22}	5.951916298014560	5.270364360572876	6.536969867091745	4.620922754362714
z_{23}	5.951916302550292	5.270364364711783	6.536969874577403	4.620922757365727
z_{24}	5.951916304297358	5.270364366306000	6.536969877460718	4.620922758522423
z_{25}	5.951916304970290	5.270364366920059	6.536969878571308	4.620922758967957
z_{26}	5.951916305229489	5.270364367156581	6.536969878999084	4.620922759139567
z_{27}	5.951916305329327	5.270364367247684	6.536969879163854	4.620922759205668
z_{28}	5.951916305367781	5.270364367282775	6.536969879227319	4.620922759231129
z_{29}	5.951916305382594	5.270364367296292	6.536969879251766	4.620922759240935
z_{30}	5.951916305388299	5.270364367301497	6.536969879261181	4.620922759244713
z_{31}	5.951916305390498	5.270364367303503	6.536969879264808	4.620922759246167
z_{32}	5.951916305391344	5.270364367304275	6.536969879266205	4.620922759246728
z_{33}	5.951916305391669	5.270364367304573	6.536969879266743	4.620922759246945
z_{34}	5.951916305391795	5.270364367304688	6.536969879266950	4.620922759247027
z_{35}	5.951916305391843	5.270364367304731	6.536969879267030	4.620922759247059
z_{36}	5.951916305391863	5.270364367304748	6.536969879267060	4.620922759247072
z_{37}	5.951916305391869	5.270364367304754	6.536969879267073	4.620922759247076

Table 1: Outcome of the experiments illustrated in Example 6.1.

Conjecture 1. *For each $q > 1$, equation* $(\mathbf{W_q})$ *admits a periodic solution γ_q which has* *the following non-trivial Floquet exponents:*

$$\lambda_1(q) = -4\frac{q+1}{q-1}, \quad \lambda_2(q) = -\frac{3q+5}{q-1}, \quad \lambda_3(q) = 1.$$

Numerical evidence. To support our conjecture, we have sought numerical evidence for the existence of a periodic solution of $(\mathbf{W_q})$ for various values of q odd. Namely, for each $q = 5, 7, \ldots, 31$ we have constructed the finite Galerkin projection $F_q^{(m)}$ as in (4.17) (keeping 100 Fourier modes) and successfully computed an approximate zero of $F_q^{(m)}$ though Newton's method. Such an approximate zero of $F_q^{(m)}$ provides numerical evidence for the existence of a periodic solution γ_q of $(\mathbf{W_q})$. In Figure 8 we report on the computed period for each of those solutions, which seems to exhibits logarithmic growth with respect to q. Next, for each of those periodic solutions, we have computed an approximate monodromy matrix, and consequently computed the corresponding Floquet exponents. Figure 9 shows the remarkable agreement we have found between the expressions provided in the statement and those obtained from the computations. □

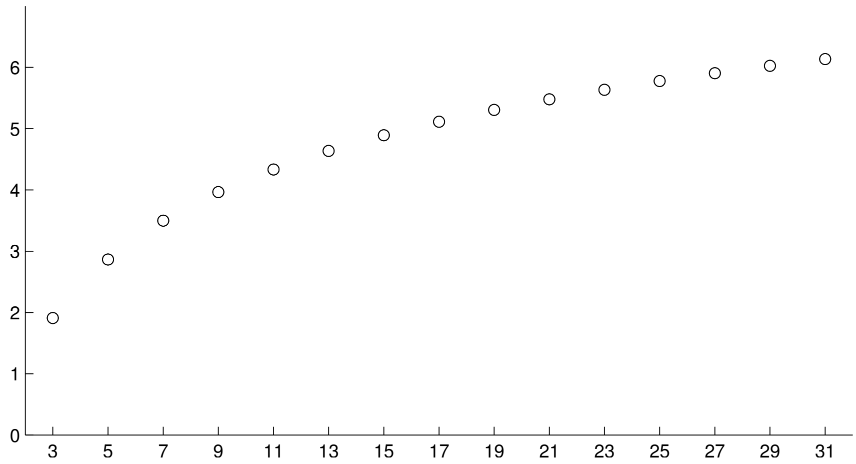

Figure 8: Period of the periodic solutions conjectured in Conjecture 1.

Remark 6.2. *We point out that we do not foresee any obstruction in obtaining, for the values of q reported in Conjecture 1, estimates analogous to the ones presented in Section 4 . Obtaining (and rigorously verifying) such estimates would turn this "numerical evidence" into computer assisted proofs of the existence of periodic solutions of* $(\mathbf{W_q})$ *for open neighborhoods of those values of q. Seeking such estimates is, however, out of the scope of this work. We also point out that the expressions of* λ_1 *and* λ_2 *are suggested by the exponents in* (3.19) *and* (3.20)*, while the expression for* λ_3 *is suggested by* (2.24).

According to Conjecture 1, each of the periodic solutions γ_q has a 3-dimensional stable manifold $\mathcal{S}_q$ associated to it (once again we recall that, for $q \in (q_-, q_+)$, this is true, see Theorem 2.25). Below we state the next conjecture, which relates $\mathcal{S}_q$ to the boundary of the basin of attraction of the origin for $(\mathbf{W_q})$.

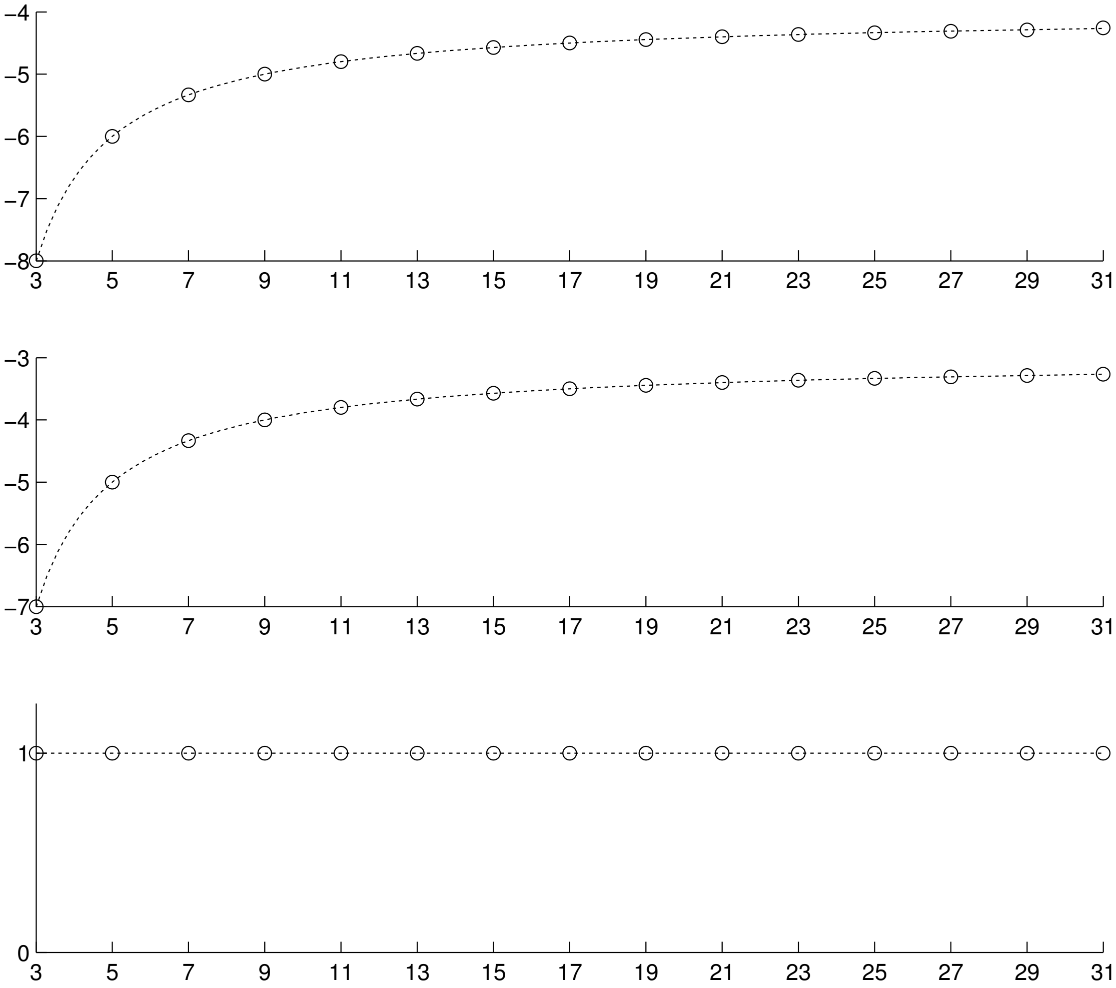

Figure 9: Numerical evidence supporting Conjecture 1. Dotted curves represent the conjectured expression for the Floquet exponents, circles represent the result of the computations. Curves appear in the same order (top to bottom) as the exponents in the conjecture.

Conjecture 2. *Let γ_q be as in Conjecture 1. Then, for each $q > 1$, the boundary of the basin of attraction of the origin for $(\mathbf{W_q})$ coincides with the stable manifold $\mathcal{S}_q$ associated to γ_q.*

Numerical evidence. For each $q > 1$, let $\mathcal{B}_q \subset \mathbb{R}^4$ be the basin of attraction of the origin for $(\mathbf{W_q})$. Recall that, because of Remark 2.19, we must have $\mathcal{S}_q \subset \partial\mathcal{B}_q$. The following experiments support our conjecture that $\partial\mathcal{B}_q \subset \mathcal{S}_q$. We have performed the experiment for $q = 3$. First, we generated 10000 uniformly distributed points on $\mathbb{S}^3$, the unit sphere in $\mathbb{R}^4$. Let us denote those points with $\mathbf{p}_j$, $j = 1, 2, \ldots, 10000$. For each $\mathbf{p}_j$, we have computed, using the procedure illustrated at eh beginning of this section, the largest floating-point number $\alpha_j > 0$ such that $D(\alpha_j)\mathbf{p}_j \in \mathcal{B}_q$. Each $D(\alpha_j)\mathbf{p}_j$ is, roughly speaking, as close as we can get to $\partial\mathcal{B}_q$. Finally, for each j, we integrated numerically $(\mathbf{W_q})$ starting from the initial condition $D(\alpha_j)\mathbf{p}_j \in \mathcal{B}_q$. Each numerical solution was observed to quickly convergence to γ_q (apart from a phase-shift) over a finite interval, and then vanish exponentially. □

Remark 6.3. *Conjecture 2 would also imply that each* γ_q *is the unique periodic solution of* $(\mathbf{W_q})$.

Remark 6.4. *Another consequence of Conjecture 2 would be the following global classification of solutions of* $(\mathbf{U_q})$*. Let* $\mathbf{u}_0 \in \mathbb{R}^4$*,* $\mathbf{u}_0 \neq 0$*, then:*

$$\begin{aligned} R_+(\mathbf{u}_0) = +\infty &\iff \mathbf{u}_0 \in \mathcal{D}JL^{-1}\Gamma_q\,, \\ R_-(\mathbf{u}_0) = -\infty &\iff \mathbf{u}_0 \in \mathcal{D}L^{-1}\Gamma_q\,, \\ -\infty < R_-(\mathbf{u}_0) < R_+(\mathbf{u}_0) < +\infty &\iff \mathbf{u}_0 \in \mathcal{D}L^{-1}(\mathcal{S}_q \setminus \Gamma_q)\,. \end{aligned}$$

Note that the set of initial conditions $\mathbf{u}_0 \in \mathbb{R}^4$ *for which the lifetime of the solution* $\phi(\cdot, \mathbf{u}_0)$ *is unbounded would be given by the union of two smooth 2-dimensional manifolds embedded in* $\mathbb{R}^4$*, which has zero Lebesgue measure.*

A Stability of symmetric periodic orbits

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function, and consider the family of differential equations

$$y'(t) = f(\epsilon, y(t)). \tag{A.1}$$

Suppose that $y'(t) = f(0, y(t))$ admits a τ-periodic solution γ. It is natural to ask whether (A.1) admits a periodic solution also for $|\varepsilon| > 0$ small enough. Several results in this direction are available in the literature, e.g. see [13]. A typical case in which the answer is affirmative is when the periodic orbit γ is hyperbolic. Here, however, we are interested in problems where the periodic solution γ enjoys some special symmetries, and our purpose is to investigate whether or not the periodic solutions of (A.1) for $|\varepsilon| > 0$ small inherit the symmetries of γ. In this appendix we present a result in this spirit. We believe the result to be of independent interest, and, to the best of our knowledge, missing in the literature.

As is customary, we will consider the linearized problem along the periodic solution γ of (A.1). In what follows, $X(t)$ stands for the *principal matrix solution* of the linear variational equation associated with γ , i.e. $X(t)$ solves the Cauchy problem

$$\begin{cases} X'(t) = A_0(t)X(t) \\ X(0) = I_n \end{cases} \tag{A.2}$$

where

$$A_0(t) := f_y(0, \gamma(t))$$

and I_n is the $n \times n$ identity matrix.

Theorem A.1. *Consider* (A.1)*, and suppose that* $f(\varepsilon, \cdot)$ *is odd for any* $|\varepsilon|$ *small. Assume that, for* $\varepsilon = 0$*,* (A.1) *has a* τ*-periodic solution* γ *which enjoys the following property:*

$$\gamma(t + \tau/2) = -\gamma(t), \text{ for any } t \in \mathbb{R}. \tag{A.3}$$

Furthermore, assume that

$$\lambda = 1 \text{ is a simple eigenvalue of } X(\tau). \tag{A.4}$$

Then, for any $|\varepsilon|$ *small,* (A.1) *admits a* $\tau^*(\epsilon)$*-periodic solution* $\gamma^*(\varepsilon,\cdot)$*, where* $\tau^*(0)=\tau$*,* $\gamma^*(\cdot,0)=\gamma$*,* $\tau^*(\cdot)$ *and* $\gamma^*(\cdot,\cdot)$ *are* $\mathcal{C}^1$ *functions of each of their arguments, and*

$$\gamma^*(\varepsilon, t+\tau^*(\varepsilon)/2) = -\gamma^*(\varepsilon,t), \text{ for any } t \in \mathbb{R}. \tag{A.5}$$

The main ingredients of the proof of Theorem A.1 will be the Implicit Function Theorem and the following lemma

Lemma A.2. *Assume that all the hypothesis of Theorem A.1 are satisfied. Then*

$$\lambda = -1 \text{ is a simple eigenvalue of } X(\tau/2),$$

with $\gamma'(0)$ *as an eigenvector.*

Proof. Set $v := \gamma'$, and observe that

$$v'(t) = \gamma''(t) = f_y(0,\gamma(t))\gamma'(t) = A_0(t)v(t).$$

It follows from (A.2) that $v(t) = X(t)v(0)$, for all $t \in \mathbb{R}$. Therefore, we have

$$X(\tau/2)v(0) = v(\tau/2) = \gamma'(\tau/2) = -\gamma'(0) = -v(0),$$

hence -1 is an eigenvalue of $X(\tau/2)$ with $v(0) = \gamma'(0)$ as an eigenvector.

Now, recall that (A.4) holds. We are going to show that

$$X(\tau) = X(\tau/2)^2, \tag{A.6}$$

from which, through the relation

$$X(\tau) - \lambda^2 I_n = X(\tau/2)^2 - \lambda^2 I_n = (X(\tau/2) - \lambda I_n)\,(X(\tau/2) + \lambda I_n),$$

we immediately obtain that -1 must be a simple eigenvalue of $X(\tau/2)$ (if not, then also 1 would not be a simple eigenvalue of $X(\tau)$, contradicting (A.4)).

Now, let us show that (A.6) holds. First, we note that A_0 is $\tau/2$-periodic. In fact, we have

$$A_0(t+\tau/2) = f_y(0,\gamma(t+\tau/2)) = f_y(0,-\gamma(t)) = f_y(0,\gamma(t)) = A_0(t),$$

where we have used (A.3) and the fact that $f_y(\varepsilon,\cdot)$ is even for $|\varepsilon|$ small. Making use of the semigroup property of fundamental matrix solutions, we can write $X(\tau) = Z(\tau)X(\tau/2)$, where Z solves

$$\begin{cases} Z'(t) = A_0(t)Z(t) \\ Z(\tau/2) = I_n \end{cases} \tag{A.7}$$

The conclusion follows by observing that, because of $A_0(t+\tau/2) = A_0(t)$, and through the change of variables $t \to t+\tau/2$, we obtain $Z(\tau) = X(\tau/2)$. □

Proof of Theorem A.1. Let $z := \gamma'(0) = f(0,\gamma(0))$, and recall that, as shown in Lemma A.2, z is an eigenvector of $X(\tau/2)$ associated to the –simple– eigenvalue -1. Being -1

simple, we can write $\mathbb{R}^n = \text{span}\{z\} \oplus V$, where V is an invariant subspace of $X(\tau/2)$. Now, let $w \in \mathbb{R}^n$ be such that $\text{span}\{w\} = V^\perp$, and set $\alpha := \gamma(0) \cdot w$. Denote by $y(\varepsilon, t, p)$ the solution of (A.1) with initial condition $p = y(\varepsilon, 0, p)$, and let $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ be the function defined as

$$F(\varepsilon, t, p) := (p \cdot w - \alpha,\ y(\varepsilon, t/2, p) + p) . \tag{A.8}$$

The conclusion of the theorem will follow as an application of the Implicit Function Theorem (IFT). To this end, first, we observe that

$$F(0, \tau, \gamma(0)) = (0, 0).$$

Next, we consider the Jacobian of F with respect to the variable (t, p), and write its action on a vector $(a, v) \in \mathbb{R} \times \mathbb{R}^n$ as

$$F_{(t,p)}(\varepsilon, t, p)(a, v) = \Big(v \cdot w,\ \frac{a}{2} y'(\varepsilon, t/2, p) + \nabla_p y(\varepsilon, t/2, p)(v) + v\Big) .$$

Evaluating at $(0, \tau, \gamma(0))$, we obtain

$$\frac{a}{2} y'(0, \tau/2, \gamma(0)) = \frac{a}{2} \gamma'(\tau/2) = -\frac{a}{2} \gamma'(0) = -\frac{a}{2} z$$

and

$$\nabla_p y(\varepsilon, t/2, p)\Big|_{(0,\tau,\gamma(0))} = X(\tau/2).$$

Hence, we have

$$F_{(t,p)}(0, \tau/2, \gamma(0))(a, v) = \Big(v \cdot w,\ -\frac{a}{2} z + (X(\tau/2) + I_n) v\Big) .$$

Last, we show that $F_{(t,p)}(0, \tau/2, \gamma(0))$ is invertible as a linear map from $\mathbb{R} \times \mathbb{R}^n$ onto itself. To do so, we will show that its kernel is trivial. Suppose that

$$F_{(t,p)}(0, \tau/2, \gamma(0))(a, v) = (0, 0). \tag{A.9}$$

Equating the first components in (A.9), we have that $v \in w^\perp = V$. Recalling that V is invariant under the action of $X(\tau/2)$, we obtain

$$(X(\tau/2) + I_n) v \in V. \tag{A.10}$$

Equating the second components in (A.9), we have $-\frac{a}{2} z + (X(\tau/2) + I_n) v = 0$, from which, because $\mathbb{R}^n = \text{span}\{z\} \oplus V$, it follows that $a = 0$ and $(X(\tau/2) + I_n) v = 0$. Since $X(\tau/2) + I_n$ is invertible from V onto itself (again, because -1 is a simple eigenvalue of $X(\tau/2)$, see Lemma A.2), we conclude that $v = 0$.

We are finally allowed to apply the IFT, and conclude that there exist $\varepsilon_0 > 0$ and $\mathcal{C}^1$ maps $p = p(\varepsilon)$, $\tau^* = \tau^*(\varepsilon)$ such that $p(0) = \gamma(0)$, $\tau^*(0) = \tau$, and

$$y(\varepsilon, \tau(\varepsilon)/2, p(\varepsilon)) = -p(\varepsilon), \tag{A.11}$$

for all $|\varepsilon| < \varepsilon_0$.

Using (A.11), and the fact that $f(\varepsilon, \cdot)$ is odd, we obtain

$$y(\varepsilon, \tau^*(\varepsilon), p(\varepsilon)) = y(\varepsilon, \tau^*(\varepsilon)/2, y(\varepsilon, \tau^*(\varepsilon)/2, p(\varepsilon))) = \\ = y(\varepsilon, \tau^*(\varepsilon)/2, -p(\varepsilon)) = -y(\varepsilon, \tau^*(\varepsilon)/2, p(\varepsilon)) = p(\varepsilon),$$

which shows that $y(\varepsilon, t, p(\varepsilon))$ is $\tau^*(\varepsilon)$-periodic.

We conclude the proof by setting $\gamma^*(\varepsilon, t) := y(\varepsilon, t, p(\varepsilon))$, for all $|\varepsilon| < \varepsilon_0$, and observing that (A.11) immediately translates into (A.11), while the smoothness of $\gamma^*(\cdot, \cdot)$ is a mere consequence of smoothness of $p(\varepsilon)$ and of solutions of (A.1) with respect to initial conditions. □

Remark A.3. *If $X(\tau)$ of Theorem A.1 has no eigenvalues on the unit circle in $\mathbb{C}$ besides $\lambda = 1$, then it is a standard fact that stable and unstable manifolds associated to the periodic solution $y^*(\varepsilon, \cdot)$ are smooth with respect to ε (hence have constant dimension independent of ε), and intersect transversally along the periodic orbit. See [13].*

Acknowledgements. The authors wish to thank Professor Luca Dieci and Professor Filippo Gazzola for helpful discussions and for their encouragement during the preparation of this paper.

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