Rigorous numerics for piecewise-smooth systems: a functional analytic approach based on Chebyshev series

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Abstract

In this paper, a rigorous computational method to compute solutions of piecewise-smooth systems using a functional analytic approach based on Chebyshev series is introduced. A general theory, based on the radii polynomial approach, is proposed to compute crossing periodic orbits for continuous and discontinuous (Filippov) piecewise-smooth systems. Explicit analytic estimates to carry the computer-assisted proofs are presented. The method is applied to prove existence of crossing periodic orbits in a model nonlinear Filippov system and in the Chua's circuit system. A general formulation to compute rigorously crossing connecting orbits for piecewise-smooth systems is also introduced.

Keywords

Rigorous numerics · Piecewise smooth systems · Periodic orbits · Contraction mapping theorem · Chebyshev series · Filippov

Mathematics Subject Classification (2010)

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1 Introduction

In this paper, we introduce a rigorous computational method for the study of piecewise-smooth (PWS) systems, which are described by a finite set of ODEs

$$\dot{u} = g^{(i)}(u), \quad u \in \mathcal{R}_i \subset \mathbb{R}^n \tag{1}$$

where $\mathcal{R}_1, \ldots, \mathcal{R}_N$ are open non-overlapping regions separated by $(n-1)$-dimensional manifolds $\Sigma_{ij} := \partial\mathcal{R}_i \cap \partial\mathcal{R}_j$ for $i \neq j$. When non empty, the set Σ_{ij} is the common boundary of the two adjacent regions $\mathcal{R}_i$ and $\mathcal{R}_j$, and we refer to it as a *switching manifold*. Given $\Sigma_{ij} \neq \emptyset$, assume the existence of $H^{(i,j)} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Sigma_{ij} = \left\{ u \in \mathbb{R}^n : H^{(i,j)}(u) = 0 \right\}. \tag{2}$$

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Assume that the functions $g^{(i)}$ and $H^{(i,j)}$ are smooth, and that the union of all regions and switching manifolds covers the entire state space.

Let us introduce some definitions, following closely the presentation of [1].

A PWS system is called *continuous* if, for all couple $i, j \in \{1, \ldots, N\}$ such that $\Sigma_{ij} \neq \emptyset$, $g^{(i)}(u) = g^{(j)}(u)$ at any point $u \in \Sigma_{ij}$, where in this case, we implicitly considered the continuous extension of the $g^{(i)}$'s to the closure of $\mathcal{R}_i$. For continuous PWS systems the tangent vectors $\dot{u}$ are uniquely defined at any point of the state space, and orbits in region $\mathcal{R}_i$ approaching transversally Σ_{ij}, cross it and enter into the adjacent region $\mathcal{R}_j$. Therefore, in continuous PWS systems, all orbits entering the switching manifold transversally undergo crossing. We refer to such orbits as *crossing orbits*.

The situation is different for discontinuous PWS systems, which are often called *Filippov systems*. In this case, two different tangent vectors $g^{(i)}(u)$ and $g^{(j)}(u)$ can be assigned to a point $u \in \Sigma_{ij}$. If the transversal components of $g^{(i)}(u)$ and $g^{(j)}(u)$ have the same sign, that is if

$$\left(\left(\nabla H^{(i,j)}(u)\right)^T \cdot g^{(i)}(u)\right)\left(\left(\nabla H^{(i,j)}(u)\right)^T \cdot g^{(j)}(u)\right) > 0, \tag{3}$$

then the orbit crosses the switching manifold Σ_{ij} with a discontinuity in its tangent vector at u. An orbit which visits some switching manifolds in a way that (3) holds at any point of the visited switching manifolds is also referred to as a *crossing orbit*. If on the other hand the transversal components of $g^{(i)}(u)$ and $g^{(j)}(u)$ have different signs at $u \in \Sigma_{ij}$, that is if

$$\left(\left(\nabla H^{(i,j)}(u)\right)^T \cdot g^{(i)}(u)\right)\left(\left(\nabla H^{(i,j)}(u)\right)^T \cdot g^{(j)}(u)\right) < 0, \tag{4}$$

then the two vector fields are pushing in opposite directions, and the solution remains on the switching manifold and slides on it for some time. While there are different ways of defining the motion of the solution on a switching manifold, the convexification method proposed by Filippov in [2] is perhaps the most natural. In the present paper, we do not discuss Filippov convexification's method and refer instead to [1, 2, 3] for details. The approach of Filippov leads to a classification of other type of orbits, namely *crossing and sliding orbits*, and *sliding orbits*. In this paper, we consider only crossing orbits.

An important class of crossing orbits in the study of PWS systems is given by *crossing periodic orbits* (CPOs), which are periodic orbits with isolated points in common with the switching manifolds they visit. Another important class of crossing orbits in the study of PWS systems is given by *crossing connecting orbits* (CCOs) (which connect two equilibria) with isolated points in common with the switching manifolds they visit.

The goal of this paper is to adapt the recently developed rigorous computational methods of [4, 5, 6, 7] for the study of PWS systems, with a particular emphasis on the study of CPOs and CCOs. We expand the solutions using Chebyshev series, and we obtain computer-assisted proofs in a Banach space of fast decaying Chebyshev coefficients.

A rigorous computational method goes beyond a standard a posteriori analysis of numerical computations. More explicitly, the field of rigorous numerics aims at developing mathematical theorems formulated in such a way that the assumptions can be rigorously verified by a computer. The approach requires an a priori setup that allows analysis and numerics to work together: the choice of function space, the choice of the basis functions, the Galerkin projection, the analytic estimates, and the computational parameters must all work hand in hand to bound the errors due to approximation, rounding and truncation, and this needs to be sufficiently tight for the proof to go through.

The first step of our approach is to setup an equivalent formulation of the form $F(x) = 0$, where $F : X \to Y$ with X and Y two infinite dimensional Banach spaces, whose solution $x \in X$ corresponds to the targeted dynamical object of interest (in our case a CPO or a CCO). Setting up the operator F requires expanding the solution using a spectral Chebyshev method. The next step is to consider a finite dimensional Galerkin projection of F, to apply Newton's method on it and to obtain a numerical approximation $\bar{x}$ to a solution of $F(x) = 0$. We then construct, with the help of the computer, an injective approximate inverse A of $DF(\bar{x})$ so that $AF : X \to X$. We define a Newton-like operator $T : X \to X$ by $T(x) = x - AF(x)$, and we aim at obtaining

(a) the existence of $\tilde{x} \in X$ such that $T(\tilde{x}) = \tilde{x}$, or equivalently (since A is injective) such that $F(\tilde{x}) = 0$;

(b) the existence of an explicit and small $r > 0$ such that $\|\tilde{x} - \bar{x}\|_X \leq r$.

The existence of the solution $\tilde{x} \in X$ and of the explicit error bound r is obtained by applying a modified version of Newton-Kantorovich theorem, namely the radii polynomial approach. The radii polynomials provide an efficient mean of determining a closed ball $\overline{B_r(\bar{x})}$ of radius r centered at the numerical approximation $\bar{x}$ on which the Newton-like operator $T(x) = x - AF(x)$ is a contraction. We present carefully this whole process in general in the context of computing CPOs.

It is important to mention that this work is by no means the first attempt to study PWS systems within the field of rigorous numerics. A by now classical example that has been studied rigorously with the help of the computer is Chua's circuit system [8, 9]. The existence of a homoclinic orbit for some unknown parameter value within a certain range of the Chua circuit was shown in [10], and existence of chaos was therefore obtained. In his study of the Chua's system, Galias introduced rigorous integration for piecewise-linear (PWL) systems [11, 12, 13]. He computed rigorously CPOs in [12], and *sliding periodic orbits* in [13]. Note that the Chua's circuit system is a continuous PWL systems, and therefore it is not a Filippov system.

Moreover, it is important to note that Chebyshev series have been used before to obtain computer-assisted proofs of existence of connecting orbits [5, 14], of solutions of boundary value problems [15] and to study Cauchy problem [5].

While we focus our attention on the computation of CPOs and CCOs, a very similar approach could be developed for initial value problems and more general boundary value problems, as considered for instance in [5].

The paper is organized as follows. In Section 2, we present the method in its full generality to obtain computer-assisted proofs of existence of CPOs for general PWS systems (continuous and Filippov). In Section 3, we modify the method to the context of studying CCOs, where we limit essentially the presentation to the general formulation of the operator $F(x) = 0$. Then, we present two applications of computer-assisted proofs of existence of CPOs. The first example is presented in Section 4 and is a proof of existence of CPOs for a nonlinear planar Filippov system. The second example is presented in Section 5, we present some computer-assited proofs of existence of CPOs in the piecewise-linear three-dimensional continuous Chua's circuit system. In Section 6, we present some possible future directions of studies.

2 Rigorous numerics for crossing periodic orbits

2.1 Setting up $F(x) = 0$ for crossing periodic orbits

A piecewise-smooth parameterization of a crossing periodic orbit Γ with M segments is given by

$$\Gamma = \bigcup_{j=1}^{M} \Gamma^{(j)} = \bigcup_{j=1}^{M} \left\{ \gamma^{(j)}(t) : t \in [-L_j, L_j] \right\}. \tag{5}$$

The parameterization (5) is globally continuous if (1) is a Filippov system and it is globally differentiable if (1) is a continuous PWS system. In both case, we have that $\gamma^{(j)}(L_j) = \gamma^{(j+1)}(-L_{j+1})$ for all $j = 1, \dots, M-1$ and $\gamma^{(M)}(L_M) = \gamma^{(1)}(-L_1)$.

Given a crossing periodic orbit (5), define the *itinerary* of the periodic orbit $\sigma = \sigma(\Gamma)$ to be a vector $\sigma = (\sigma_1, \dots, \sigma_M) \in \{1, \dots, N\}^M$ defined component-wise by

$$\sigma_j = \ell, \text{ if } \Gamma^{(j)} \subset \mathcal{R}_\ell. \tag{6}$$

Consider a periodic orbit Γ with parameterization given by (5) with itinerary $\sigma = (\sigma_1, \dots, \sigma_M)$. Then, for each $j = 1, \dots, M$, we have that $\gamma^{(j)}(t)$ is a solution of $\dot{u} = g^{(\sigma_j)}(u)$. For each $j = 1, \dots, M$, we rescale the ODE by the factor L_j so that each $\gamma^{(j)}$ is now re-parameterized over the time interval $[-1, 1]$ and satisfies

$$\frac{d}{dt}\gamma^{(j)} = L_j g^{(\sigma_j)}(\gamma^{(j)}), \quad t \in [-1, 1]. \tag{7}$$

We use the notation $\gamma^{(j)}$ to denote the parameterizations of the same object over the intervals $[-L_j, L_j]$ and $[-1, 1]$.

Remark 2.1. The reason for considering a parameterization over the time interval $[-1, 1]$ is because we will later on expand $\gamma^{(j)}$ using Chebyshev series. The basis functions are in this case the Chebyshev polynomials which are defined on $[-1, 1]$.

Denote by $\Sigma^{(\sigma_j)}$ the switching manifold from which $\gamma^{(j)}$ begins its journey in the region $\mathcal{R}_{\sigma_j}$. We now make two important assumptions.

($\mathcal{A}_1$) Each vector field $g^{(i)}$ is real analytic in the region $\mathcal{R}_i$.

($\mathcal{A}_2$) For each $j = 1, \dots, M$, assume we have a parameterization of the switching manifold $\Sigma^{(\sigma_j)}$ given by

$$P^{(\sigma_j)} : \mathbb{R}^{n-1} \to \mathbb{R}^n : \theta^{(j)} \mapsto P^{(\sigma_j)}(\theta^{(j)}). \tag{8}$$

Integrating each ODE of (7) from -1 to t, and using the initial condition $\gamma^{(j)}(-1) = P^{(\sigma_j)}(\theta^{(j)})$ (with $\theta^{(j)} \in \mathbb{R}^{n-1}$ to be uniquely determined) yields

$$\hat{F}^{(j)} := P^{(\sigma_j)}(\theta^{(j)}) + L_j \int_{-1}^{t} g^{(\sigma_j)}(\gamma^{(j)}(s))\, ds - \gamma^{(j)}(t) = 0, \tag{9}$$

for each $j = 1, \dots, M$ and for all $t \in [-1, 1]$. The fact that Γ is a periodic orbit implies that the following extra equations are satisfied

$$\begin{cases} \eta^{(j)} := & \gamma^{(j)}(1) - P^{(\sigma_{j+1})}(\theta^{(j+1)}) = 0, \quad j = 1, \dots, M-1, \\ \eta^{(M)} := & \gamma^{(M)}(1) - P^{(\sigma_1)}(\theta^{(1)}) = 0. \end{cases}$$

By construction, the problem of looking for crossing periodic orbits of (1) reduces to the equivalent problem of looking for solutions of $(\eta^{(1)}, \dots, \eta^{(M)}, \hat{F}^{(1)}, \dots, \hat{F}^{(M)}) = 0$. Instead of solving this problem in state space, we will solve it rigorously with Chebyshev spectral Galerkin method in a Banach space consisting of fast decaying Chebyshev coefficients.

Definition 2.2. The *Chebyshev polynomials* $T_k : [-1, 1] \to \mathbb{R}$ are defined by $T_0(t) = 1$, $T_1(t) = t$ and $T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t)$ for $k \geq 1$. Equivalently, $T_k(t) = \cos(k \arccos t)$.

By assumption $(\boldsymbol{\mathcal{A}_1})$, the solution $\gamma^{(j)}(t)$ of (7) is analytic. Each component $\gamma_i^{(j)}$ of $\gamma^{(j)}$ therefore admits a unique Chebyshev series representation

$$\gamma_i^{(j)}(t) = (a_i^{(j)})_0 + 2\sum_{k=1}^{\infty}(a_i^{(j)})_k T_k(t) \tag{10}$$

whose coefficients $a_i^{(j)} := \{(a_i^{(j)})_k\}_{k\geq 0}$ ($i = 1, \dots, n$ and $j = 1, \dots, M$) decay to zero exponentially fast [16]. Consider also $c_i^{(\sigma_j)} = \{(c_i^{(\sigma_j)})_k\}_{k\geq 0}$ ($i = 1, \dots, n$ and $j = 1, \dots, M$) the vector of Chebyshev coefficients of the i^{th} component of $g^{(\sigma_j)}(\gamma^{(j)}(t))$ given componentwise by

$$g_i^{(\sigma_j)}(\gamma^{(j)}(t)) = (c_i^{(\sigma_j)})_0 + 2\sum_{k=1}^{\infty}(c_i^{(\sigma_j)})_k T_k(t). \tag{11}$$

The exponential decay rate of the Chebyshev coefficients of each component of $\gamma^{(j)}$ motivates the following choice of Banach space. For any $\nu > 1$ we define the ν-weighted ℓ^1-norm on sequences of real numbers $a = \{a_n\}_{n=0}^{\infty}$ by

$$\|a\|_\nu := \sum_{n=0}^{\infty} |a_n|\nu^n, \tag{12}$$

and consider the weighted ℓ^1 Banach space

$$\ell_\nu^1 := \{a = \{a_n\}_{n=0}^{\infty} : a_k \in \mathbb{R} \quad \text{and} \quad \|a\|_\nu < \infty\}. \tag{13}$$

Remark 2.3. Assumption $(\boldsymbol{\mathcal{A}_1})$ guarantees that the decay rate of the Chebyshev coefficients of the solutions is geometric. This justifies working in the space ℓ_ν^1. If instead we assume that $g^{(i)}$ is C^k in region $\mathcal{R}_i$, then the decay rate of Chebyshev coefficients is algebraic, and hence different Banach spaces have to be considered. A weighted ℓ^∞ Banach space consisting of sequences with algebraic decay rates could be used in this case. Explicit truncation error estimates in such space are developed in [5, 15, 23, 24].

Given two sequences $a, b \in \ell_\nu^1$, denote by $a * b$ the discrete convolution

$$(a * b)_k = \sum_{\substack{k_1+k_2=k \\ k_i\in\mathbb{Z}}} a_{|k_1|}b_{|k_2|}. \tag{14}$$

An important property of ℓ_ν^1 is that it is a Banach space and an algebra under the discrete convolution (14). We have the important following property, whose proof is standard.

Lemma 2.4. *For $a, b \in \ell_\nu^1$, $\|a * b\|_\nu \leq 3\|a\|_\nu\|b\|_\nu$.*

This result is particularly interesting for our purpose, because we use Chebyshev series to prove existence of CPOs. Since Chebyshev series are in fact Fourier series *in disguise* as $T_k(t) = \cos(k \arccos t)$, then the product of two functions in state space will result in a discrete convolution product as in (14) in the space of Chebyshev coefficients. Therefore, Lemma 2.4 simplifies the nonlinear analysis.

The unknowns for the problem are given by

- $\theta = (\theta^{(1)}, \ldots, \theta^{(M)}) \in \mathbb{R}^{M(n-1)}$, where $\theta^{(j)} = \left(\theta_1^{(j)}, \ldots, \theta_{n-1}^{(j)}\right) \in \mathbb{R}^{n-1}$ is the parameter that defines the point $P^{(\sigma_j)}(\theta^{(j)})$ in the switching manifold $\Sigma^{(\sigma_j)}$ from which $\gamma^{(j)}$ begins its journey in the region $\mathcal{R}_{\sigma_j}$.
- $L = (L_1, \ldots, L_M) \in \mathbb{R}^M$, with $L_j \in \mathbb{R}$ provides the a priori unknown length of time $2L_j$ on which $\gamma^{(j)}$ is defined.
- $a = \left(a^{(1)}, \ldots, a^{(M)}\right) \in \left(\ell_\nu^1\right)^{Mn}$, where $a^{(j)}$ is the vector of the Chebyshev coefficients of all components of $\gamma^{(j)}$. The i^{th} component of $a^{(j)}$ is given by $a_i^{(j)} = \{(a_i^{(j)})_k\}_{k\geq 0} \in \ell_\nu^1$, $i = 1, \ldots, n$, for some $\nu > 1$.

All the above unknowns (variables) are collected in a single infinite dimensional vector of the form

$$x = (\theta, L, a) \in \mathbb{R}^{M(n-1)} \times \mathbb{R}^M \times \left(\ell_\nu^1\right)^{Mn}.$$

Define the Banach space

$$X := \mathbb{R}^{Mn} \times \left(\ell_\nu^1\right)^{Mn}, \tag{15}$$

endowed with the norm

$$\|x\|_X := \max\left\{ \max_{\substack{i=1,\ldots,n-1 \\ j=1,\ldots,M}} \{|\theta_i^{(j)}|\}, \max_{j=1,\ldots,M} \{|L_j|\}, \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,M}} \|a_i^{(j)}\|_\nu \right\}. \tag{16}$$

Remark 2.5. To simplify the presentation we use the same decay rate $\nu > 1$ for all $\gamma^{(j)}$. It is, of course, possible to use a different decay rate $\nu_j > 1$ for each $\gamma^{(j)}$. In this case we would have $a_i^{(j)} = \{(a_i^{(j)})_k\}_{k\geq 0} \in \ell_{\nu_j}^1$.

Following the approach of [5], we plug (10) in (9), use (11), compute the resulting Chebyshev coefficients and set up the new problem

$$F(x) = \begin{pmatrix} \eta^{(1)}(x) \\ \vdots \\ \eta^{(M)}(x) \\ f^{(1)}(x) \\ \vdots \\ f^{(M)}(x) \end{pmatrix} = 0, \tag{17}$$

where $\eta^{(j)} = \left(\eta_1^{(j)}, \ldots, \eta_n^{(j)}\right) \in \mathbb{R}^n$ is given component-wise by

$$\eta_i^{(j)}(x) = (a_i^{(j)})_0 + 2\sum_{k=1}^{\infty} (a_i^{(j)})_k - P_i^{(\sigma_{j+1})}(\theta^{(j+1)}), \quad j = 1, \ldots, M-1, \tag{18}$$

$$\eta_i^{(M)}(x) = (a_i^{(M)})_0 + 2\sum_{k=1}^{\infty} (a_i^{(M)})_k - P_i^{(\sigma_1)}(\theta^{(1)}), \tag{19}$$

and $f^{(j)} = \left(f_1^{(j)}, \dots, f_n^{(j)}\right)$ is given component-wise by

$$(f_i^{(j)}(x))_k := \begin{cases} P_i^{(\sigma_j)}(\theta^{(j)}) - (a_i^{(j)})_0 - 2\sum_{\ell=1}^{\infty}(-1)^\ell (a_i^{(j)})_\ell, & k=0, \\ 2k(a_i^{(j)})_k + L_j\left((c_i^{(\sigma_j)})_{k+1} - (c_i^{(\sigma_j)})_{k-1}\right), & k \geq 1. \end{cases} \tag{20}$$

Remark 2.6. It is important to realize that the setup of the operator $F(x) = 0$ given in (17) depends on knowing a priori the itinerary $\sigma = (\sigma_1, \dots, \sigma_k) \in \{1, \dots, N\}^k$ of the orbit, and the switching manifold from which the orbit begins its journey in a given region. This necessary information is obtained from numerical simulations. In general, it is difficult to obtain a numerical guess for the CPO, as we do not know ahead of time which manifolds it visits and how many crossings occur. The goal of this paper is however not to discuss how to obtain a numerical approximation but rather to assume that a good guess has been obtained and to prove the existence of a true CPO nearby.

Now that the operator F is identified, we introduce the radii polynomial approach, which provides an efficient way of proving existence of solutions close to numerical approximations. Before that, we need some basic results from elementary functional analysis.

2.2 The dual space and linear operators

When studying nonlinear maps on ℓ_ν^1 it is often necessary to estimate certain linear operators and functionals. The estimates are natural when viewed in the context of the Banach space dual of ℓ_ν^1. For an infinite sequence of real numbers $c = \{c_n\}_{n=0}^\infty$ define the ν-weighted supremum norm

$$\|c\|_\nu^\infty := \sup_{n \geq 0} \frac{|c_n|}{\nu^n},$$

and let

$$\ell_\nu^\infty := \left\{c = \{c_n\}_{n=0}^\infty : \|c\|_\nu^\infty < \infty\right\}.$$

It is classical result in the elementary theory of Banach spaces that for $\nu > 0$, the dual of ℓ_ν^1, denoted $(\ell_\nu^1)^*$, is isometrically isomorphic to ℓ_ν^∞. Moreover, for any $h \in (\ell_\nu^1)^*$, there is a unique $c \in \ell_\nu^\infty$, such that $h = h_c$, where for any $a \in \ell_\nu^1$,

$$h_c(a) = \sum_{n=0}^\infty c_n a_n \qquad \text{and} \qquad \|h_c\|_{(\ell_\nu^1)^*} = \|c\|_\nu^\infty.$$

Hence,

$$\sup_{\|a\|_\nu = 1} \left|\sum_{n=0}^\infty c_n a_n\right| = \|h_c\|_{(\ell_\nu^1)^*} = \|c\|_\nu^\infty = \sup_{n \geq 0} \frac{|c_n|}{\nu^n}. \tag{21}$$

This bound is used to estimate linear operators of the following type. Denote by $B(\ell_\nu^1, \ell_\nu^1)$ the space of bounded linear operators from ℓ_ν^1 to ℓ_ν^1 and by $\|\cdot\|_{B(\ell_\nu^1, \ell_\nu^1)}$ the operator norm.

Corollary 2.7. *Let A_F be an $m \times m$ matrix, $\{\mu_n\}_{n=m}^\infty$ be a sequence of numbers with*

$$|\mu_n| \leq |\mu_m|,$$

for all $n \geq m$, and $A\colon \ell^1_\nu \to \ell^1_\nu$ be the linear operator defined by

$$A(a) = \begin{pmatrix} A_F & & 0 & \\ & \mu_m & & \\ 0 & & \mu_{m+1} & \\ & & & \ddots \end{pmatrix} \begin{bmatrix} a_F \\ a_m \\ a_{m+1} \\ \vdots \end{bmatrix}.$$

Here $a_F = (a_0, a_1, \dots, a_{m-1})^T \in \mathbb{R}^m$. Then $A \in B(\ell^1_\nu, \ell^1_\nu)$ is a bounded linear operator and

$$\|A\|_{B(\ell^1_\nu,\ell^1_\nu)} \leq \max\{K, \mu_m\}, \tag{22}$$

where

$$K := \max_{0\leq n\leq m-1} \frac{1}{\nu^n} \sum_{k=0}^{m-1} |A_{k,n}|\nu^k.$$

Proof. See Corollary 6 in [14]. □

2.3 The radii polynomial approach for CPOs

Given an infinite dimensional vector $x = (\theta, L, a) \in \mathbb{R}^{M(n-1)} \times \mathbb{R}^M \times \left(\ell^1_\nu\right)^{Mn}$, consider the projection

$$\Pi x = (\theta, L, \Pi a^{(1)}, \dots, \Pi a^{(M)}) \in \mathbb{R}^{M(n-1)} \times \mathbb{R}^M \times (\mathbb{R}^m)^{Mn} = \mathbb{R}^{(Mn)(m+1)}$$

where $\Pi a^{(j)}$ is given component-wise by $\Pi a_i^{(j)} = \{(a_i^{(j)})_k\}_{k=0}^{m-1} \in \mathbb{R}^m$ $(i = 1, \dots, n)$.

Using this finite dimensional projection, consider a finite dimensional Galerkin projection of (17),

$$F^{(m)} : \mathbb{R}^{(Mn)(m+1)} \to \mathbb{R}^{(Mn)(m+1)} \tag{23}$$

defined by $F^{(m)}(\Pi x) = \Pi F(\Pi x)$.

Assume that using Newton's method, we compute a numerical approximation $\bar{x} \in \mathbb{R}^{(Mn)(m+1)}$ such that $F^{(m)}(\bar{x}) \approx 0$. Consider $\overline{B_r(0)} = \{x \in X : \|x\| \leq r\}$ the closed ball of radius r centered at 0 in the Banach space X, and consider $\overline{B_r(\bar{x})} = \bar{x} + \overline{B_r(0)} \subset X$, the closed ball of radius r centered at $\bar{x}$. We now consider an approximate inverse of $DF(\bar{x})$, that we denote by A. To simplify the presentation, a point $x = (\theta, L, a) \in X$ is denoted by $x = (x_1, \dots, x_{2Mn})$, where $(x_1, \dots, x_{M(n-1)}) = \theta \in \mathbb{R}^{M(n-1)}$, $(x_{M(n-1)+1}, \dots, x_{Mn}) = L \in \mathbb{R}^M$, and $(x_{Mn+1}, \dots, x_{2Mn}) = a \in \left(\ell^1_\nu\right)^{Mn}$. Given a sequence $b \in \ell^1_\nu$, we denote its finite dimensional projection into the first m components by b_F, that is, $b_F = (b_0, b_1, \dots, b_{m-1})^T \in \mathbb{R}^m$.

In order to define A, we compute (with the help of the computer) a matrix $A^{(m)}$ such that $A^{(m)} \approx \left(DF^{(m)}(\bar{x})\right)^{-1}$. In other words, $A^{(m)}$ is an approximate inverse of the Jacobian matrix $DF^{(m)}(\bar{x})$. To have a proof that $A^{(m)}$ is invertible, we show with interval arithmetic that

$$\|I - DF^{(m)}A^{(m)}\|_{\mathbb{R}^{2Mm}} \leq \delta < 1. \tag{24}$$

The $(Mn)(m+1)\times(Mn)(m+1)$ matrix $A^{(m)}$ is expressed by

$$A^{(m)} = \begin{bmatrix} A^{(m)}_{1,1} & \dots & A^{(m)}_{1,Mn} & A^{(m)}_{1,Mn+1} & \dots & A^{(m)}_{1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A^{(m)}_{Mn,1} & \dots & A^{(m)}_{Mn,Mn} & A^{(m)}_{Mn,Mn+1} & \dots & A^{(m)}_{Mn,2Mn} \\ A^{(m)}_{Mn+1,1} & \dots & A^{(m)}_{Mn+1,Mn} & A^{(m)}_{Mn+1,Mn+1,} & \dots & A^{(m)}_{Mn+1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A^{(m)}_{2Mn,1} & \dots & A^{(m)}_{2Mn,Mn} & A^{(m)}_{2Mn,Mn+1} & \dots & A^{(m)}_{2Mn,2Mn} \end{bmatrix}.$$

Based on the computation of $A^{(m)}$, we can explicitly define A (see [14]). We express A as a $2Mn\times 2Mn$ matrix of linear operators of the form

$$A = \begin{bmatrix} A_{1,1} & \dots & A_{1,Mn} & A_{1,Mn+1} & \dots & A_{1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A_{Mn,1} & \dots & A_{Mn,Mn} & A_{Mn,Mn+1} & \dots & A_{Mn,2Mn} \\ A_{Mn+1,1} & \dots & A_{Mn+1,Mn} & A_{Mn+1,Mn+1,} & \dots & A_{Mn+1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A_{2Mn,1} & \dots & A_{2Mn,Mn} & A_{2Mn,Mn+1} & \dots & A_{2Mn,2Mn} \end{bmatrix}, \tag{25}$$

where

- $A_{i,j} = A^{(m)}_{ij} \in \mathbb{R}$, for $1 \le i, j, \le Mn$,
- $A_{i,j} \in (\ell^1_\nu)^*$, for $1 \le i \le Mn, Mn+1 \le j \le 2Mn$,
  for $x_j \in \ell^1_\nu, A_{ij}x_j = A^{(m)}_{ij}\cdot (x_j)_F \in \mathbb{R}$,
- $A_{i,j} \in \ell^1_\nu$, for $Mn+1 \le i \le 2Mn, 1 \le j \le Mn$,
  for $x_j \in \mathbb{R}, A_{ij}x_j = (A^{(m)}_{ij}x_j, 0_\infty) \in \ell^1_\nu$,
- $A_{i,j} \in B\left(\ell^1_\nu, \ell^1_\nu\right)$, for $Mn+1 \le i, j \le 2Mn$, for $x_j \in \ell^1_\nu$,

  $$(A_{ij}x_j)_k = \begin{cases} (A^{(m)}_{ij}(x_j)_F)_k, & k = 0,\dots,m-1, \\ \delta_{i,j}\dfrac{1}{2k}(x_j)_k, & k \ge m, \end{cases}$$

  where $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise.

Since the tail of A is invertible we conclude from (24) that A is an injective linear operator. Moreover, it acts on $x = (x_1,\dots,x_{2Mn}) \in X$ component-wise as

$$(Ax)_i = \sum_{j=1}^{2Mn} A_{ij}x_j,$$

with $(Ax)_i \in \mathbb{R}$ for $i = 1,\dots,Mn$ and $(Ax)_i \in \ell^1_\nu$, for $i = Mn+1,\dots,2Mn$.

Recalling the linear operator A in (25), define

$$T(x) = x - AF(x). \tag{26}$$

Proposition 2.8. $T : X \to X$.

Proof. The proof is similar to Proposition 8 in [14]. □

The injectivity of A implies that x is a solution of $F = 0$ if and only if it is a fixed point of T. Moreover since T now maps X back into itself we study (26) via the contraction mapping theorem applied on closed balls of the form $\overline{B_r(\bar{x})}$ centered at the numerical approximation $\bar{x}$.

Given $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{2Mn})$, define the bounds

$$\begin{aligned} Y &= (Y_1, \ldots, Y_{Mn}, Y_{Mn+1}, \ldots, Y_{2Mn}) \\ Z(r) &= (Z_1(r), \ldots, Z_{Mn}(r), Z_{Mn+1}(r), \ldots, Z_{2Mn}(r)) \end{aligned} \tag{27}$$

with $Y_j, Z_j(r) \in \mathbb{R}$ satisfying

$$\left|(T(\bar{x}) - \bar{x})_j\right| \leq Y_j \quad \text{and} \quad \sup_{b,c\in\overline{B_r(0)}} |DT_j(\bar{x}+b)c| \leq Z_j(r), \quad \text{for} \quad j = 1, \ldots, Mn \tag{28}$$

$$\left\|(T(\bar{x}) - \bar{x})_j\right\|_\nu \leq Y_j \quad \text{and} \quad \sup_{b,c\in\overline{B_r(0)}} \|DT_j(\bar{x}+b)c\|_\nu \leq Z_j(r) \quad \text{for} \quad j = Mn+1, \ldots, 2Mn.$$

Remark 2.9 (The *Z* bound as a polynomial in *r*). The computation of the Z bound requires estimating each component of $DT(\bar{x}+b)c$ for all $b, c \in \overline{B_r(0)}$. This is equivalent to estimating each component of $DT(\bar{x}+ur)vr$ for all $v, r \in \overline{B_1(0)}$. If the nonlinearities of the differential equation are polynomials of order less or equal to q, then F in (17) will consists of discrete convolutions with power at most q. Since $T(x) = x - AF(x)$ and $DT(\bar{x}+ur)vr \in X = \mathbb{R}^{Mn} \times \left(\ell^1_\nu\right)^{Mn}$, then each component of $DT(\bar{x}+ur)vr$ can be expanded as a q^{th} order polynomial in r with the coefficients being either in $\mathbb{R}$ or in ℓ^1_ν.

Proposition 2.10. *Consider the bounds Y and $Z(r)$ as* (27) *and satisfying the component-wise inequalities* (28)*. If*

$$\max_{i=1,\ldots,2Mn} \{Y_i + Z_i(r)\} < r, \tag{29}$$

then (a) $T\left(\overline{B_r(\bar{x})}\right) \subset \overline{B_r(\bar{x})}$, and (b) T is a contraction on $\overline{B_r(\bar{x})}$. Therefore, by the contraction mapping theorem, there exists a unique $\tilde{x} \in \overline{B_r(\bar{x})}$ such that $T(\tilde{x}) = \tilde{x}$. By injectivity of the approximate inverse A, we obtain that $F(\tilde{x}) = 0$. Moreover, we obtain the rigorous bound

$$\|\tilde{x} - \bar{x}\|_X \leq r. \tag{30}$$

Proof. See the proof of Proposition 1 in [4]. □

The previous remark justifies the following definition.

Definition 2.11. *Given the bounds Y and $Z(r)$ satisfying* (28) *we define the* radii polynomials *$\{p_j\}_{j=1,\ldots,2Mn}$ by*

$$p_j(r) := Z_j(r) - r + Y_j, \qquad \text{for} \quad j = 1, \ldots, 2Mn. \tag{31}$$

The next result shows that the radii polynomials provide an efficient strategy for obtaining sets on which the corresponding Newton-like operator T is a contraction mapping.

Proposition 2.12. *Fix $\nu > 1$ an exponential decay rate and construct the radii polynomials $p_j = p_j(r)$, for $j = 1, \ldots, 2Mn$, of Definition 2.11. If*

$$\mathcal{I} := \bigcap_{j=1}^{2Mn} \{r > 0 \mid p_j(r) < 0\} \neq \emptyset, \tag{32}$$

then $\mathcal{I}$ is an open interval. Moreover, there exists $\tilde{x} \in X$ such that $\tilde{x}$ is the unique solution of $F(\tilde{x}) = 0$ within the ball $\overline{B_r(\bar{x})}$ for all $r \in \mathcal{I}$.

Proof. Assume that the degree of the polynomial nonlinearity of the original differential equation is q. Fix $\nu > 1$ and $j \in \{1, \ldots, 2Mn\}$. From Remark 2.9, the coefficients of the radii polynomials will be of the form

$$p_j(r) = a_q^{(j)} r^n + a_{q-1}^{(j)} r^{n-1} + \cdots + a_1^{(j)} r - r + a_0^{(j)},$$

with $a_i^{(j)} \geq 0$ for all $i = 0, \ldots, q$. Since $\mathcal{I} \neq \emptyset$, then $a_1^{(j)} - 1 < 0$. Otherwise we would not be able to find $r > 0$ such that $p_j(r) < 0$. By Descartes' rule of signs and since $\mathcal{I} \neq \emptyset$, each radii polynomial p_j has exactly two positive real zeros that we denote by $r_-^{(j)} < r_+^{(j)}$. Defining $\mathcal{I}_j = (r_-^{(j)}, r_+^{(j)})$, we obtain that $\mathcal{I} = \cap_{j=1}^{2Mn} \mathcal{I}_j$. This implies that $\mathcal{I}$ is an open interval. Consider now $r \in \mathcal{I}$ so that $p_j(r) < 0$ for all $j = 1, \ldots, 2Mn$. Then (29) holds, and the result follows from Proposition 2.10. $\square$

If hypothesis (32) of Proposition 2.12 holds, then there exists

$$\tilde{x} = (\tilde{\theta}, \tilde{L}, \tilde{a}) \in X = \mathbb{R}^{M(n-1)} \times \mathbb{R}^M \times \left(\ell_\nu^1\right)^{Mn} \text{ such that } F(\tilde{x}) = 0,$$

such that $\tilde{x}$ is the unique solution of $F = 0$ in $\overline{B_r(\bar{x})} \subset X$ for all $r \in \mathcal{I}$. From the theory of Section 2.1, the solution $(\tilde{\theta}, \tilde{L}, \tilde{a})$ defines a periodic orbit

$$\Gamma = \bigcup_{j=1}^{M} \Gamma^{(j)} = \bigcup_{j=1}^{M} \left\{ \tilde{\gamma}^{(j)}(t) : t \in [-1, 1] \right\}$$

where

$$\tilde{\gamma}_i^{(j)}(t) = (\tilde{a}_i^{(j)})_0 + 2 \sum_{k=1}^{\infty} (\tilde{a}_i^{(j)})_k T_k(t), \quad \text{for } i = 1, \ldots, n,$$

where $\frac{d}{dt}\tilde{\gamma}^{(j)}(t) = \tilde{L}_j g^{(\sigma_j)}(\tilde{\gamma}^{(j)}(t))$ for $t \in [-1, 1]$, and where $\tilde{\gamma}^{(j)}(1) = \tilde{\gamma}^{(j+1)}(-1)$ for all $j = 1, \ldots, M-1$ and $\tilde{\gamma}^{(M)}(1) = \tilde{\gamma}^{(1)}(-1)$.

However, one must make the final check that the itinerary of the periodic orbit Γ is indeed $\sigma = (\sigma_1, \ldots, \sigma_M) \in \{1, \ldots, N\}^M$. In other words, one must check that

$$\Gamma^{(j)} \subset \mathcal{R}_{\sigma_j}, \qquad \text{for all } j = 1, \ldots, M. \tag{33}$$

For this, we will need to control the C^0 errors of $\tilde{\gamma}_i^{(j)}$ and $\frac{d}{dt}\tilde{\gamma}_i^{(j)}$.

Let $\bar{\gamma}^{(j)}(t)$ the numerical approximation given by

$$\bar{\gamma}_i^{(j)}(t) = (\bar{a}_i^{(j)})_0 + 2 \sum_{k=1}^{m-1} (\bar{a}_i^{(j)})_k T_k(t), \quad \text{for } i = 1, \ldots, n.$$

Since $\tilde{x} \in \overline{B_r(\bar{x})}$ and $|T_k(t)| \leq 1$, we get that for each $i = 1, \ldots, n$

$$\begin{aligned}
\|\tilde{\gamma}_i^{(j)} - \bar{\gamma}_i^{(j)}\|_\infty &\leq \sup_{t \in [-1,1]} \left| \tilde{\gamma}_i^{(j)}(t) - \bar{\gamma}_i^{(j)}(t) \right| \\
&\leq |(\tilde{a}_i^{(j)})_0 - (\bar{a}_i^{(j)})_0| + 2 \sum_{k=1}^{\infty} \left| (\tilde{a}_i^{(j)})_k - (\bar{a}_i^{(j)})_k \right| \\
&\leq 2 \|\tilde{a}_i^{(j)} - \bar{a}_i^{(j)}\|_\nu \\
&\leq 2r.
\end{aligned} \tag{34}$$

It is known that the Chebyshev polynomials satisfy $|T_k'(t)| \leq k^2$ for all $t \in [-1,1]$. Hence, since $\nu > 1$,

$$\begin{aligned}
\| \frac{d}{dt}\tilde{\gamma}_i^{(j)} - \frac{d}{dt}\bar{\gamma}_i^{(j)} \|_\infty &\leq \sup_{t\in[-1,1]} \left| \frac{d}{dt}\tilde{\gamma}_i^{(j)}(t) - \frac{d}{dt}\bar{\gamma}_i^{(j)}(t) \right| \\
&\leq 2\sum_{k=1}^{\infty} \left| (\tilde{a}_i^{(j)})_k - (\bar{a}_i^{(j)})_k \right| |T_k'(t)| \\
&\leq 2\sum_{k=1}^{\infty} \frac{\|\tilde{a}_i^{(j)} - \bar{a}_i^{(j)}\|_\nu}{\nu^k} k^2 \\
&\leq 2r\sum_{k=1}^{\infty} \frac{k^2}{\nu^k} \\
&= \frac{2\nu(\nu+1)}{(\nu-1)^3} r. \qquad (35)
\end{aligned}$$

Having now general C^0 bounds on $\tilde{\gamma}_i^{(j)}$ and $\frac{d}{dt}\tilde{\gamma}_i^{(j)}$, we may be able to verify that (33) holds. We postpone this explicit verification to the applications. Moreover, while also postponing the full construction of the radii polynomials to each application presented in Section 4 and in Section 5, we provide here a general guidance of how to proceed with their construction. The computation of the bounds Y does not require much analysis. It is obtained by computing finite sums with interval arithmetic. Hence, we present some ideas of how in general we compute the Z bound.

2.3.1 Guidance of how to compute the bound Z

In order to simplify the computation of the bound Z, we introduce the bounded linear operator $A^\dagger$ defined component-wise by

$$A^\dagger = \begin{bmatrix} A^\dagger_{1,1} & \dots & A^\dagger_{1,Mn} & A^\dagger_{1,Mn+1} & \dots & A^\dagger_{1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A^\dagger_{Mn,1} & \dots & A^\dagger_{Mn,Mn} & A^\dagger_{Mn,Mn+1} & \dots & A^\dagger_{Mn,2Mn} \\ A^\dagger_{Mn+1,1} & \dots & A^\dagger_{Mn+1,Mn} & A^\dagger_{Mn+1,Mn+1,} & \dots & A^\dagger_{Mn+1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A^\dagger_{2Mn,1} & \dots & A^\dagger_{2Mn,Mn} & A^\dagger_{2Mn,Mn+1} & \dots & A^\dagger_{2Mn,2Mn} \end{bmatrix}, \qquad (36)$$

where

- $A^\dagger_{i,j} = DF^{(m)}_{ij}(\bar{x}) \in \mathbb{R}$, for $1 \leq i,j, \leq Mn$,
- $A^\dagger_{i,j} \in (\ell^1_\nu)^*$, for $1 \leq i \leq Mn, Mn+1 \leq j \leq 2Mn$.
  For $x_j \in \ell^1_\nu$, $A^\dagger_{ij} x_j = DF^{(m)}_{ij}(\bar{x}) \cdot (x_j)_F \in \mathbb{R}$.
- $A^\dagger_{i,j} \in \ell^1_\nu$, for $Mn+1 \leq i \leq 2Mn, 1 \leq j \leq Mn$.
  For $x_j \in \mathbb{R}$, $A^\dagger_{ij} x_j = (DF^{(m)}_{ij}(\bar{x}) x_j, 0_\infty) \in \ell^1_\nu$.

- $A^{\dagger}_{i,j} \in B\left(\ell^1_\nu, \ell^1_\nu\right)$, for $Mn+1 \le i,j \le 2Mn$. For $x_j \in \ell^1_\nu$,

$$(A^{\dagger}_{ij} x_j)_k = \begin{cases} \left(DF^{(m)}_{ij}(\bar{x})(x_j)_F\right)_k, & k = 0,\dots,m-1, \\ \delta_{i,j} 2k (x_j)_k, & k \ge m, \end{cases}$$

  where $\delta_{i,j}$ equals 1 if $i=j$ and 0 otherwise.

Considering $b = (b_1, \dots, b_{2Mn})$, $c = (c_1, \dots, c_{2Mn}) \in \overline{B_r(0)}$ and recalling the definition of the Newton-like operator (26), notice that

$$DT(\bar{x}+b)c = [I - ADF(\bar{x}+b)]c = [I - AA^{\dagger}]c - A[DF(\bar{x}+b)c - A^{\dagger}c]. \tag{37}$$

The objective is to bound each component in the right-hand side of (37). Consider $u = (u_1, \dots, u_{2Mn})$, $v = (v_1, \dots, v_{2Mn}) \in \overline{B_1(0)}$ such that $b = ur$ and $c = vr$. Let $B := I - AA^{\dagger}$, which is denoted by

$$B = \begin{bmatrix} B_{1,1} & \dots & B_{1,Mn} & B_{1,Mn+1} & \dots & B_{1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ B_{Mn,1} & \dots & B_{Mn,Mn} & B_{Mn,Mn+1} & \dots & B_{Mn,2Mn} \\ B_{Mn+1,1} & \dots & B_{Mn+1,Mn} & B_{Mn+1,Mn+1,} & \dots & B_{Mn+1,2Mn} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ B_{2Mn,1} & \dots & B_{2Mn,Mn} & B_{2Mn,Mn+1} & \dots & B_{2Mn,2Mn} \end{bmatrix},$$

Note that by definition of the diagonal tails of A_{ij} and $A^{\dagger}_{ij}$, the tails of B_{ij} vanish, i.e., all B_{ij}, $Mn+1 \le i,j \le 2Mn$ are represented by $m \times m$ matrices. Let

$$Z^{(0)}_i := \begin{cases} \displaystyle\sum_{j=1}^{Mn} |B_{ij}| + \sum_{j=Mn+1}^{2Mn} \left(\max_{0 \le k \le m-1} \frac{\left|(B_{ij})_k\right|}{\nu^k} \right), & i = 1,\dots,Mn \\ \displaystyle\sum_{j=1}^{Mn} \left(\sum_{k=0}^{m-1} \left|(B_{ij})_k\right| \nu^k \right) + \sum_{j=Mn+1}^{2Mn} \left(\max_{0 \le n \le m-1} \frac{1}{\nu^n} \sum_{k=0}^{m-1} \left|(B_{ij})_{k,n}\right| \nu^k \right), & i = Mn+1,\dots,2Mn. \end{cases} \tag{38}$$

Using (21), one gets that for every $c \in \overline{B_r(0)}$ and for $i = 1,\dots,Mn$,

$$\begin{aligned} \left|[(I - AA^{\dagger})c]_i\right| &= \left|[(I - AA^{\dagger})v]_i\right| r \le \sup_{\|v\|_X = 1} \left|[(I - AA^{\dagger})v]_i\right| r \\ &\le \sum_{j=1}^{Mn} |B_{ij}| r + \sum_{j=Mn+1}^{2Mn} \|B_{ij}\|^{\infty}_{\nu} r \\ &\le \left(\sum_{j=1}^{Mn} |B_{ij}| + \sum_{j=Mn+1}^{2Mn} \left(\max_{0 \le k \le m-1} \frac{\left|(B_{ij})_k\right|}{\nu^k} \right) \right) r = Z^{(0)}_i r. \end{aligned} \tag{39}$$

Furthermore, using Corollary 2.7, for every $c \in \overline{B_r(0)}$ and for $i = Mn+1, \ldots, 2mn$,

$$\begin{aligned}
\left\|[(I - AA^\dagger)c]_i\right\|_\nu &= \left\|[(I - AA^\dagger)v]_i\right\|_\nu r \leq \sup_{\|v\|_X=1} \left\|[(I - AA^\dagger)v]_i\right\|_\nu r \\
&\leq \sum_{j=1}^{Mn} \|B_{ij}\|_\nu r + \sum_{j=Mn+1}^{2Mn} \|B_{ij}\|_{B(\ell^1_\nu,\ell^1_\nu)} r \\
&\leq \left(\sum_{j=1}^{Mn} \left(\sum_{k=0}^{m-1} \left|(B_{ij})_k\right| \nu^k \right) + \sum_{j=Mn+1}^{2Mn} \left(\max_{0\leq n\leq m-1} \frac{1}{\nu^n} \sum_{k=0}^{m-1} \left|(B_{ij})_{k,n}\right| \nu^k \right) \right) r \\
&= Z_i^{(0)} r. \qquad (40)
\end{aligned}$$

The next step is to bound the components of $A[DF(\bar{x}+b)c - A^\dagger c]$ the second term of (37). The computation for the bound $A[DF(\bar{x}+b)c - A^\dagger c]$ requires estimating each of its component for all $b, c \in \overline{B_r(0)}$. This is equivalent to estimating each component of $A[DF(\bar{x}+ur)vr - A^\dagger vr]$ for all $v, r \in \overline{B_1(0)}$. If the nonlinearities of each ODE in the original PWS system are polynomials of order less or equal to n, then F will consists of discrete convolutions with power at most n. Then each component of $A[DF(\bar{x}+ur)vr - A^\dagger vr]$ can be expanded as a polynomial of order n in r with the coefficients being either in $\mathbb{R}$ or in ℓ^1_ν. A useful approach is to let

$$z := DF(\bar{x}+ur)vr - A^\dagger vr, \qquad (41)$$

and to compute a polynomial expansion of each of its components. Then we bound the terms $|(Az)_i|$, for $i = 1, \ldots, Mn$ and the terms $\|(Az)_i\|_\nu$, for $i = Mn+1, \ldots, 2Mn$.

We postpone the construction of the bound for $A[DF(\bar{x}+ur)vr - A^\dagger vr]$ to each application presented in Section 4 and in Section 5.

3 Rigorous numerics for crossing connecting orbits

In this Section, we introduce the setting up to obtain the problem $F(x) = 0$ whose solutions correspond to CCOs. Once this operator is defined, we can use the radii polynomial approach as presented in Section 2 to prove existence of crossing connecting orbits. As mentioned previously, two examples are presented.

We being by assuming the existence of $u_0 \in \mathcal{R}_{\sigma_1} \subset \mathbb{R}^n$ and $u_1 \in \mathcal{R}_{\sigma_M} \subset \mathbb{R}^n$ such that

$$g^{(\sigma_1)}(u_0) = g^{(\sigma_M)}(u_1) = 0.$$

In other words, u_0 (resp. u_1) is a steady state solution of the vector field $g^{(\sigma_1)}$ (resp. $g^{(\sigma_M)}$) in the open region $\mathcal{R}_{\sigma_1}$ (resp. $\mathcal{R}_{\sigma_M}$).

Since $u_0 \in \mathcal{R}_{\sigma_1}$ with $\mathcal{R}_{\sigma_1}$ open, there exists an open ball $\mathcal{B}_{u_0}$ in which the vector field $\dot{u} = g^{(\sigma_1)}(u)$ is defined. Since in $\mathcal{B}_{u_0}$, the vector field is smooth, by the classical theory of ODEs, there exists a local unstable manifold $W^u_{\text{loc}}(u_0) \subset \mathcal{B}_{u_0}$. Denote by n_u the dimension of $W^u_{\text{loc}}(u_0)$. Similarly, there exists an open ball $\mathcal{B}_{u_1} \subset \mathcal{R}_{\sigma_M}$ containing a local stable manifold $W^u_{\text{loc}}(u_1) \subset \mathcal{B}_{u_1}$. Denote by n_s the dimension of $W^s_{\text{loc}}(u_1)$.

A piecewise-smooth parameterization of a crossing connecting orbit $\Gamma(u_0, u_1)$ with M segments is given by

$$\Gamma = \bigcup_{j=1}^{M} \Gamma^{(j)} = \bigcup_{j=1}^{M} \left\{ \gamma^{(j)}(t) : t \in [-L_j, L_j] \right\}. \qquad (42)$$

Definition 3.1. The *itinerary* of a crossing connecting orbit as given in (42) is denoted by $\sigma = \sigma(\Gamma(u_0, u_1))$ and is defined to be a vector $\sigma = (\sigma_1, \ldots, \sigma_M) \in \{1, \ldots, N\}^M$ defined component-wise by

$$\sigma_j = \ell, \text{ if } \Gamma^{(j)} \subset \mathcal{R}_\ell. \tag{43}$$

As before denote by $\Sigma^{(\sigma_j)}$, $j = 2, \ldots, M$, the switching manifold from which $\gamma^{(j)}$ begins its journey in the region $\mathcal{R}_{\sigma_j}$.

We now make the following important assumptions.

($\mathcal{A}_1$) Each vector field $g^{(i)}$ is real analytic in the region $\mathcal{R}_i$.

($\mathcal{A}_2$) For each $j = 2, \ldots, M$, assume we have a parameterization of the switching manifold $\Sigma^{(\sigma_j)}$ given by

$$P^{(\sigma_j)} : \mathbb{R}^{n-1} \to \mathbb{R}^n : \theta^{(j)} \mapsto P^{(\sigma_j)}(\theta^{(j)}). \tag{44}$$

($\mathcal{A}_3$) Assume we have a parameterization of a local unstable manifold $W^u_{\text{loc}}(u_0)$ given by

$$P^{(u)} : B_{\rho_0} \subset \mathbb{R}^{n_u} \to \mathbb{R}^n : \theta^{(u)} \mapsto P^{(u)}(\theta^{(u)}), \tag{45}$$

where B_{ρ_0} is the open ball in $\mathbb{R}^{n_u}$ with a small enough radius $\rho_0 > 0$ so that $P^{(u)}(B_{\rho_0}) \subset \mathcal{B}_{u_0}$.

($\mathcal{A}_4$) Assume we have a parameterization of a local stable manifold $W^s_{\text{loc}}(u_1)$ given by

$$P^{(s)} : B_{\rho_1} \subset \mathbb{R}^{n_s} \to \mathbb{R}^n : \theta^{(s)} \mapsto P^{(s)}(\theta^{(s)}), \tag{46}$$

where B_{ρ_1} is the open ball in $\mathbb{R}^{n_s}$ with a small enough radius $\rho_1 > 0$ so that $P^{(s)}(B_{\rho_1}) \subset \mathcal{B}_{u_1}$.

Remark 3.2. The assumptions **($\mathcal{A}_1$)** and **($\mathcal{A}_2$)** are similar than the ones in Section 2.1. The assumptions **($\mathcal{A}_3$)** and **($\mathcal{A}_4$)** can be verified by combining the Parameterization Method introduced in [17, 18, 19] with the recent results [7, 20, 21, 6, 14] which allow computing rigorously stable and unstable manifolds of equilibria of vector fields.

For sake of simplicity of the presentation, we let $P^{(\sigma_1)} := P^{(u)}$ the local parameterization (45) of $W^u_{\text{loc}}(u_0)$ and $\theta^{(1)} := \theta^{(u)}$.

Integrating each ODE of (7) from -1 to t, and using the initial condition $\gamma^{(j)}(-1) = P^{(\sigma_j)}(\theta^{(j)})$ (with $\theta^{(1)} \in \mathbb{R}^{n_u}$, $\theta^{(j)} \in \mathbb{R}^{n-1}$ for $j = 2, \ldots, M$ to be uniquely determined) yields

$$\hat{F}^{(j)} := P^{(\sigma_j)}(\theta^{(j)}) + L_j \int_{-1}^{t} g^{(\sigma_j)}(\gamma^{(j)}(s))\, ds - \gamma^{(j)}(t) = 0, \tag{47}$$

for each $j = 1, \ldots, M$ and for all $t \in [-1, 1]$. The fact that $\Gamma(u_0, u_1)$ is a connecting orbit between u_0 and u_1 implies that the following extra equations are satisfied

$$\begin{cases} \eta^{(j)} := & \gamma^{(j)}(1) - P^{(\sigma_{j+1})}(\theta^{(j+1)}) = 0, \quad j = 1, \ldots, M-1, \\ \eta^{(M)} := & \gamma^{(M)}(1) - P^{(s)}(\theta^{(s)}) = 0. \end{cases}$$

By construction, the problem of looking for crossing connecting orbits of (1) reduces to the equivalent problem of looking for solutions of $(\eta^{(1)}, \ldots, \eta^{(M)}, \hat{F}^{(1)}, \ldots, \hat{F}^{(M)}) = 0$. Instead of solving this problem in state space, we can solve it rigorously with Chebyshev spectral Galerkin method in a Banach space consisting of fast decaying Chebyshev coefficients. Since the idea is to apply a contraction mapping argument, we need the solutions to be isolated. However, this is not the case now as the phase in the parameterization of the local stable and unstable manifolds is not fixed. To take care of that, we can follow the setup introduced in [6] and impose that the orbit $\gamma^{(1)}$ leaves the local unstable manifold of u_0 at a parameter $\theta^{(u)}$ such that $\|\theta^{(u)}\| = \rho_0$, and that the orbit $\gamma^{(M)}$ enters the local stable manifold of u_1 at a parameter $\theta^{(s)}$ such that $\|\theta^{(s)}\| = \rho_1$. That way, the dimension of the parameterization of the local unstable manifold goes down by one and is now parameterized as $P^{(u)}\left(\theta^{(u)}(\psi^{(u)})\right)$ with $\psi^{(u)} = \left(\psi_1^{(u)}, \ldots, \psi_{n_u-1}^{(u)}\right) \in \mathbb{R}^{n_u-1}$. Similarly, the dimension of the parameterization of the local stable manifold goes down by one and is now parameterized as $P^{(s)}\left(\theta^{(s)}(\psi^{(s)})\right)$ with $\psi^{(s)} = \left(\psi_1^{(s)}, \ldots, \psi_{n_s-1}^{(s)}\right) \in \mathbb{R}^{n_s-1}$.

Following a similar procedure as in Section 2.1, we expand each solution segment with Chebyshev series, and we obtain the operator

$$F(x) = \begin{pmatrix} \eta^{(1)}(x) \\ \vdots \\ \eta^{(M)}(x) \\ f^{(1)}(x) \\ \vdots \\ f^{(M)}(x) \end{pmatrix} = 0, \tag{48}$$

where $\eta^{(j)} = \left(\eta_1^{(j)}, \ldots, \eta_n^{(j)}\right) \in \mathbb{R}^n$ is given component-wise by

$$\eta_i^{(j)}(x) = (a_i^{(j)})_0 + 2\sum_{k=1}^{\infty}(a_i^{(j)})_k - P_i^{(\sigma_{j+1})}(\theta^{(j+1)}), \quad j = 1, \ldots, M-1, \tag{49}$$

$$\eta_i^{(M)}(x) = (a_i^{(M)})_0 + 2\sum_{k=1}^{\infty}(a_i^{(M)})_k - P_i^{(s)}(\theta^{(s)}(\psi^{(s)})), \tag{50}$$

where $f^{(1)} = \left(f_1^{(1)}, \ldots, f_n^{(1)}\right)$ is given component-wise by

$$(f_i^{(1)}(x))_k := \begin{cases} P_i^{(u)}(\theta^{(u)}(\psi^{(u)})) - (a_i^{(1)})_0 - 2\displaystyle\sum_{\ell=1}^{\infty}(-1)^{\ell}(a_i^{(1)})_{\ell}, & k = 0, \\ 2k(a_i^{(1)})_k + L_1\left((c_i^{(\sigma_1)})_{k+1} - (c_i^{(\sigma_1)})_{k-1}\right), & k \geq 1. \end{cases}$$

and for $j = 2, \ldots, M$, $f^{(j)} = \left(f_1^{(j)}, \ldots, f_n^{(j)}\right)$ is given component-wise by

$$(f_i^{(j)}(x))_k := \begin{cases} P_i^{(\sigma_j)}(\theta^{(j)}) - (a_i^{(j)})_0 - 2\displaystyle\sum_{\ell=1}^{\infty}(-1)^{\ell}(a_i^{(j)})_{\ell}, & k = 0, \\ 2k(a_i^{(j)})_k + L_j\left((c_i^{(\sigma_j)})_{k+1} - (c_i^{(\sigma_j)})_{k-1}\right), & k \geq 1. \end{cases}$$

Similarly as in Section 2.1, the unknowns for the problem are given by

- $\theta = (\psi^{(u)}, \theta^{(2)}, \ldots, \theta^{(M)}, \psi^{(s)}) \in \mathbb{R}^{n_u-1} \times \mathbb{R}^{(M-1)(n-1)} \times \mathbb{R}^{n_s-1}$.
- $L = (L_1, \ldots, L_M) \in \mathbb{R}^M$.

- $a = \left(a^{(1)}, \dots, a^{(M)}\right) \in \left(\ell^1_\nu\right)^{Mn}$, where $a^{(j)}$ is the vector of the Chebyshev coefficients of all components of $\gamma^{(j)}$. The i^{th} component of $a^{(j)}$ is given by $a_i^{(j)} = \{(a_i^{(j)})_k\}_{k\geq 0} \in \ell^1_\nu$ $(i = 1, \dots, n)$ for some $\nu > 1$.

Remark 3.3. In order for problem (48) to be well-conditioned, we need the non degeneracy condition

$$n_u + n_s = n + 1, \tag{51}$$

which ensures that the dimensions of (θ, L) and $\eta = \left(\eta^{(1)}, \dots, \eta^{(M)}\right)$ coincide. Note that (51) is a standard non degeneracy condition in the study of intersection of stable and unstable manifolds in the classical theory of ODEs (e.g. see [22]).

4 CPOs in a model nonlinear problem

Consider

$$\dot{u} = \begin{cases} g^{(1)}(u) := \begin{pmatrix} \beta & 1 \\ -1 & \beta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \varepsilon \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, & u \in \mathcal{R}_1, \\ g^{(2)}(u) := \begin{pmatrix} -1 & 1/\alpha \\ -\alpha & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \varepsilon \begin{pmatrix} u_1^2 \\ u_1^2 + u_2^2 \end{pmatrix}, & u \in \mathcal{R}_2, \end{cases} \tag{52}$$

where $\mathcal{R}_1 = \{u = (u_1, u_2) : u_2 < 1\}$ and $\mathcal{R}_2 = \{u = (u_1, u_2) : u_2 > 1\}$, and where α, β and ε are parameters. The PWS system (52) is a (discontinuous) nonlinear Filippov system which is a slight modification of the model nonlinear problem considered in [3].

There is only one switching manifold Σ given by $u_2 = 1$. Its parameterization is given by

$$P : \mathbb{R} \to \mathbb{R}^2 : \theta \mapsto P(\theta) = \begin{pmatrix} \theta \\ 1 \end{pmatrix}. \tag{53}$$

We consider the simplest possible case of periodic orbit, that is with itinerary $\sigma = (\sigma_1, \sigma_2) = (1, 2)$. In this case, $P^{(\sigma_1)} = P^{(\sigma_2)} = P$, with P given by (53). Let

$$\gamma_i^{(1)}(t) = (a_i)_0 + 2\sum_{k=1}^{\infty}(a_i)_k T_k(t), \qquad \gamma_i^{(2)}(t) = (b_i)_0 + 2\sum_{k=1}^{\infty}(b_i)_k T_k(t)$$

$$g_i^{(1)}\left(\gamma^{(1)}(t)\right) = (c_i)_0 + 2\sum_{k=1}^{\infty}(c_i)_k T_k(t), \qquad g_i^{(2)}\left(\gamma^{(2)}(t)\right) = (d_i)_0 + 2\sum_{k=1}^{\infty}(d_i)_k T_k(t)$$

In this case, the operator (17) becomes

$$F(x) = \begin{pmatrix} \eta^{(1)}(x) \\ \eta^{(2)}(x) \\ f^{(1)}(x) \\ f^{(2)}(x) \end{pmatrix} = 0, \tag{54}$$

where $\eta^{(j)} = \left(\eta_1^{(j)}, \eta_2^{(j)}\right)^T \in \mathbb{R}^2$, $j = 1, 2$, is given component-wise by

$$\eta_i^{(1)}(x) = (a_i)_0 + 2\sum_{k=1}^{\infty}(a_i)_k - P_i(\theta_1), \quad \eta_i^{(2)}(x) = (b_i)_0 + 2\sum_{k=1}^{\infty}(b_i)_k - P_i(\theta_2),$$

and $f^{(1)}$, $f^{(2)}$ are given component-wise by

$$(f_i^{(1)}(x))_k := \begin{cases} P_i(\theta_2) - (a_i)_0 - 2\sum_{\ell=1}^{\infty}(-1)^\ell (a_i)_\ell, & k=0, \\ 2k(a_i)_k + L_1\left((c_i)_{k+1} - (c_i)_{k-1}\right), & k \geq 1. \end{cases}$$

and

$$(f_i^{(2)}(x))_k := \begin{cases} P_i(\theta_1) - (b_i)_0 - 2\sum_{\ell=1}^{\infty}(-1)^\ell (b_i)_\ell, & k=0, \\ 2k(b_i)_k + L_2\left((d_i)_{k+1} - (d_i)_{k-1}\right), & k \geq 1, \end{cases}$$

where

$$\begin{aligned} (c_1)_k &:= \beta(a_1)_k + (a_2)_k + \varepsilon(a_1)_k \quad , \quad (c_2)_k := -(a_1)_k + \beta(a_2)_k + \varepsilon(a_2)_k \\ (d_1)_k &:= -(b_1)_k + \frac{1}{\alpha}(b_2)_k + \varepsilon(b_1^2)_k \quad , \quad (d_2)_k := -\alpha(b_1)_k - (b_2)_k + \varepsilon(b_1^2 + b_2^2)_k. \end{aligned}$$

The unknown is $x = (\theta_1, \theta_2, L_1, L_2, a_1, a_2, b_1, b_2)$ and the Banach space is given by

$$X = \mathbb{R}^4 \times \left(\ell_\nu^1\right)^4,$$

with the norm $\|x\|_X = \max\left\{|\theta_1|, |\theta_2|, |L_1|, |L_2|, \|a_1\|_\nu, \|a_2\|_\nu, \|b_1\|_\nu, \|b_2\|_\nu\right\}$.

We now apply the radii polynomial approach to problem (52) with the theory presented in Section 2.3. In this case, $n = 2$ (the dimension of the state space) and $M = 2$ (the number of segments of the periodic orbit). Consider a finite dimensional projection $F^{(m)} : \mathbb{R}^{4(m+1)} \to \mathbb{R}^{4(m+1)}$, and assume that using Newton's method, we compute an numerical approximation $\bar{x} \in \mathbb{R}^{4(m+1)}$ such that $F^{(m)}(\bar{x}) \approx 0$.

We prove the existence of a CPO of (52) by constructing the radii polynomials of Definition 2.11 and by verifying the hypothesis (32) of Proposition 2.12. For this, we need to construct the bounds Y and Z satisfying (27).

4.1 The bound Y for the nonlinear model problem

Denote

$$F(\bar{x}) = (F_i(\bar{x}))_{i=1}^8 = \left(\eta_1^{(1)}(\bar{x}), \eta_2^{(1)}(\bar{x}), \eta_1^{(2)}(\bar{x}), \eta_2^{(2)}(\bar{x}), f_1^{(1)}(\bar{x}), f_2^{(1)}(\bar{x}), f_1^{(2)}(\bar{x}), f_2^{(2)}(\bar{x})\right).$$

For $i = 1, \ldots, 4$, we can use interval arithmetic and compute Y_i such that

$$|(T(\bar{x}) - \bar{x})_i| = |(AF(\bar{x}))_i| = \left|\sum_{j=1}^{4} A_{ij}^{(m)} F_j^{(m)}(\bar{x}) + \sum_{j=5}^{8} A_{ij}^{(m)} \cdot F_j^{(m)}(\bar{x})\right| \leq Y_i.$$

For $i = 5, \ldots, 8$, we now compute bounds Y_i

$$\begin{aligned} ||(T(\bar{x}) - \bar{x})_i||_\nu &\leq \left\|\sum_{j=1}^{4} A_{ij}^{(m)} F_j^{(m)}(\bar{x}) + \sum_{j=5}^{8} A_{ij} F_j^{(m)}(\bar{x})\right\|_\nu + \sum_{k=m}^{2m-1} \frac{1}{2k} \left|(F_i(\bar{x}))_k\right| \nu^k \\ &\leq Y_i. \end{aligned}$$

4.2 The bound Z for the nonlinear model problem

We have already obtained in general a component-wise bound (38) for the first term $[I - AA^\dagger]c$ of the splitting (37). To simplify the computation of the bound of the second term $A[DF(\bar{x}+b)c - A^\dagger c]$, recall (41) and let $z = (z_1, \ldots, z_8)$ such that

$$z = DF(\bar{x}+ur)vr - A^\dagger vr.$$

The formulas for each z_j can be found in Appendix A. Using these formulas and Lemma 2.4, and using (21), for $i = 1, \ldots, 8$, we get upper bounds for $\|(Az)_i\|$, where $\|(Az)_i\| = |(Az)_i|$ for $i = 1, \ldots, 4$ and where $\|(Az)_i\| = \|(Az)_i\|_\nu$ for $i = 5, \ldots, 8$. For $i = 1, \ldots, 8$, let

$$Z_i^{(3)} = 18\varepsilon\left(\nu + \frac{1}{\nu}\right)(\|A_{i,7}\| + 2\|A_{i,8}\|)\, r^3 \tag{55}$$

$$\begin{aligned} Z_i^{(2)} = 4\left(\nu + \frac{1}{\nu}\right)\Bigg[&(\beta+\varepsilon)(\|A_{i,5}\| + \|A_{i,6}\|) + \|A_{i,7}\|\left(1 + \frac{1}{\alpha} + 6\varepsilon\|\bar{b}_1\|_\nu + 3\varepsilon|\bar{\theta}_2|\right) \\ &+\|A_{i,8}\|\left(\alpha + 1 + 3\varepsilon(\|\bar{b}_1\|_\nu + \|\bar{b}_2\|_\nu) + 3|\bar{\theta}_2|(\varepsilon + \frac{1}{2})\right)\Bigg] \end{aligned} \tag{56}$$

and let

$$\begin{aligned} Z_i^{(1)} = \;& \frac{2}{\nu^m}\sum_{j=1}^{4}\|A_{ij}\| + \frac{2}{\nu^m}\sum_{j=5}^{8}\|(A_{ij})_{:,0}\| \\ &+(\delta_{i,5}+\delta_{i,6})\left[\left(\frac{\beta+\varepsilon}{m}\right)\left(|(\bar{a}_1)_{m-1}| + |(\bar{a}_2)_{m-1}| + |\bar{L}_1|\left(\nu + \frac{1}{\nu}\right)\right)\right] \\ &+\delta_{i,7}\left[\frac{1}{m}\left(|(\bar{b}_1)_{m-1}| + \frac{1}{\alpha}|(\bar{b}_2)_{m-1}| + |\bar{\theta}_2|\left(\nu+\frac{1}{\nu}\right)\left(1 + \frac{1}{\alpha} + 6\varepsilon\|\bar{b}_1\|_\nu\right)\right)\right. \\ &\qquad\left. + 2\varepsilon\left(\nu + \frac{1}{\nu}\right)\sum_{k=m}^{2m-1}\frac{1}{2k}|(\bar{b}_1^2)_k|\nu^k\right] \\ &+\delta_{i,8}\left[\frac{1}{m}\left(\alpha|(\bar{b}_1)_{m-1}| + |(\bar{b}_2)_{m-1}| + |\bar{\theta}_2|\left(\nu+\frac{1}{\nu}\right)(\alpha + 1 + 6\varepsilon(\|\bar{b}_1\|_\nu + \|\bar{b}_2\|_\nu))\right)\right. \\ &\qquad\left. + 2\varepsilon\left(\nu + \frac{1}{\nu}\right)\sum_{k=m}^{2m-1}\frac{1}{2k}\left(|(\bar{b}_1^2)_k| + |(\bar{b}_2^2)_k|\right)\nu^k\right] \\ &+4|\bar{\theta}_2|\varepsilon\left(\nu+\frac{1}{\nu}\right)\left(\left\||A_{i,7}|(|\bar{b}_1|\omega^I)_F\right\| + \left\||A_{i,8}|(|\bar{b}_1|\omega^I)_F\right\| + \left\||A_{i,8}|(|\bar{b}_2|\omega^I)_F\right\|\right), \end{aligned} \tag{57}$$

where $\|(A_{ij})_{:,0}\| = |(A_{ij})_{0,0}|$ if $i = 1, \ldots, 4$ and $j = 5, \ldots, 8$, and $\|(A_{ij})_{:,0}\|$ is the ν-norm of the first column of A_{ij} if $i, j = 5, \ldots, 8$, and where $\omega^I = (0, \ldots, 0, \nu^{-m}, \nu^{-(m+1)}, \ldots)$.

Combining the bounds (57), (56) and (55), we obtain that

$$\|(A\left(DF(\bar{x}+ur)vr - A^\dagger vr\right))_i\| \le Z_i^{(3)}r^3 + Z_i^{(2)}r^2 + Z_i^{(1)}r.$$

Hence, combining the computation of Section 4.1, the general formula (38), the bounds (57), (56) and (55), we obtain that the radii polynomials of Definition 2.11, as defined in equation (31).

$(\bar{a}_1)_k$	$(\bar{a}_2)_k$	$(\bar{b}_1)_k$	$(\bar{b}_2)_k$
−1.1336e+01	−6.4451e+00	−1.7926e+01	1.4762e+00
−1.2073e+01	−2.6237e+00	1.2806e+01	−6.9345e-02
−6.4206e+00	3.0150e+00	−2.8090e+00	−2.2993e-01
−9.9022e-01	2.5751e+00	8.7892e-02	6.9121e-02
3.4518e-01	7.2733e-01	5.8278e-02	−8.2052e-04
1.7960e-01	5.4531e-02	−1.1106e-02	2.3462e-04
3.1031e-02	−1.9069e-03	9.0176e-04	6.1550e-05
9.9974e-04	−5.9691e-03	−1.8769e-05	−1.0369e-05
−5.6192e-04	−6.7641e-04	−3.9380e-06	8.4906e-07
−1.1350e-04	−1.6708e-06	5.3976e-07	−2.9546e-08
−8.6913e-06	9.9105e-06	−3.5712e-08	−3.1600e-09
2.0464e-07	1.3760e-06	6.5237e-10	7.0890e-10
1.1442e-07	7.0787e-08	1.6670e-10	−6.9803e-11
1.1414e-08	−3.7362e-09	−2.4756e-11	3.2914e-12
3.7205e-10	−9.2311e-10	1.8040e-12	1.2967e-13
−3.6156e-11	−6.7949e-11	−4.6891e-14	−3.9911e-14
−5.4584e-12	−1.1579e-12	−5.9380e-15	3.8026e-15
−2.9973e-13	2.3468e-13	9.3346e-16	−1.6719e-16
−8.0271e-16	2.4513e-14	−6.6614e-17	−6.9056e-18
1.1188e-15	9.9302e-16	1.7269e-18	2.0251e-18

Table 1: The Chebyshev coefficients (in order) of $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$ and $\bar{b}_2$.

4.3 Results

Consider the nonlinear model Filippov system (52), and fix the parameters to be $\varepsilon = 5 \times 10^{-5}$, $\beta = 0.83061$ and $\alpha = 0.1$.

We let $m = 20$ and considered a finite dimensional Galerkin projection $F^{(20)} : \mathbb{R}^{84} \to \mathbb{R}^{84}$, and computed using Newton's method an approximation $\bar{x} \in \mathbb{R}^{84}$. The graph of the periodic orbit can be found in Figure 2. We fix the exponential decay rate to be $\nu = 1.1$. We use the Y bounds of Section 4.1 and the Z bounds of Section 4.2 to compute the eight cubic radii polynomials defined by

$$p_j(r) := Z_j^{(3)} r^3 + Z_j^{(2)} r^2 + \left(Z_j^{(1)} + Z_j^{(0)} - 1 \right) r + Y_j, \qquad \text{for} \quad j = 1, \dots, 8. \tag{58}$$

The coefficients of the polynomials can be found in Figure 1. Hence, we obtained the following result.

```
P =
  6.4056e-01  7.5974e+03 -3.9024e-01  6.1218e-15
  1.2258e-02  1.7929e+02 -9.3807e-01  8.0949e-15
  4.5177e-03  5.5091e+01 -9.9480e-01  1.8831e-15
  1.1586e-03  1.5062e+01 -9.9597e-01  2.4368e-16
  4.3073e-01  5.1093e+03 -9.5746e-01  2.2569e-14
  3.4376e-02  4.1066e+02 -9.5747e-01  2.2143e-14
  5.5974e-02  6.3931e+02 -2.6044e-01  2.6968e-14
  2.1865e-04  4.7352e+00 -9.2525e-01  2.2841e-14
```

Figure 1: The eight radii polynomials generated with $\bar{x}$ given in Table 1. The first column represent the vector $\left(Z_i^{(3)}\right)_{i=1}^{8}$, the second $\left(Z_i^{(2)}\right)_{i=1}^{8}$, the third $\left(Z_i^{(1)} + Z_i^{(0)} - 1\right)_{i=1}^{8}$ and the fourth $(Y_i)_{i=1}^{8}$.

Theorem 4.1. *Let $r = 2 \times 10^{-13}$. Then there exists a unique $\tilde{x} \in \overline{B_r(\bar{x})}$ such that $F(\tilde{x}) = 0$, with F given in* (54)*. That corresponds to a crossing periodic orbit of the Filippov system* (52) *with period $\tau \in [5.156727575035736\ ,\ 5.156727575037338]$.*

Proof. The existence of $\tilde{x} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{L}_1, \tilde{L}_2, \tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2) \in \overline{B_r(\bar{x})}$ such that $F(\tilde{x}) = 0$ follows from an application of Proposition 2.12. Let

$$\tilde{\gamma}^{(1)}(t) = \begin{pmatrix} \tilde{\gamma}_1^{(1)}(t) \\ \tilde{\gamma}_2^{(1)}(t) \end{pmatrix} = \begin{pmatrix} (\tilde{a}_1)_0 + 2\displaystyle\sum_{k=1}^{\infty} (\tilde{a}_1)_k T_k(t) \\ (\tilde{a}_2)_0 + 2\displaystyle\sum_{k=1}^{\infty} (\tilde{a}_2)_k T_k(t) \end{pmatrix}$$

$$\tilde{\gamma}^{(2)}(t) = \begin{pmatrix} \tilde{\gamma}_1^{(2)}(t) \\ \tilde{\gamma}_2^{(2)}(t) \end{pmatrix} = \begin{pmatrix} (\tilde{b}_1)_0 + 2\displaystyle\sum_{k=1}^{\infty} (\tilde{b}_1)_k T_k(t) \\ (\tilde{b}_2)_0 + 2\displaystyle\sum_{k=1}^{\infty} (\tilde{b}_2)_k T_k(t) \end{pmatrix}$$

the corresponding orbit segments which satisfy $\frac{d}{dt}\tilde{\gamma}^{(j)}(t) = \tilde{L}_j g^{(j)}(\tilde{\gamma}^{(j)}(t))$ for all $t \in [-1,1]$ and $j = 1, 2$, and $\tilde{\gamma}^{(1)}(1) = \tilde{\gamma}^{(2)}(-1)$ and $\tilde{\gamma}^{(2)}(1) = \tilde{\gamma}^{(1)}(-1)$.

Recalling (33), we now show that

$$\Gamma^{(j)} = \left\{ \tilde{\gamma}^{(j)}(t) : t \in [-1,1] \right\} \subset \mathcal{R}_j, \qquad \text{for } j = 1, 2.$$

Since $\mathcal{R}_1 = \{u = (u_1, u_2) : u_2 < 1\}$ and $\mathcal{R}_2 = \{u = (u_1, u_2) : u_2 > 1\}$, this means that we need to show, for all $t \in [-1,1]$, that

$$\tilde{\gamma}_2^{(1)}(t) = (\tilde{a}_2)_0 + 2\sum_{k=1}^{\infty} (\tilde{a}_2)_k T_k(t) < 1 \quad \text{and} \quad \tilde{\gamma}_2^{(2)}(t) = (\tilde{b}_2)_0 + 2\sum_{k=1}^{\infty} (\tilde{b}_2)_k T_k(t) > 1.$$

Consider a uniform mesh of the interval $[-1,1]$

$$\Delta : -1 = t_1 < \cdots < t_N = 1$$

of size $h = 0.0005$ with $N = 4001$. Denote by $I_n = [t_n, t_{n+1}] \subset [-1,1]$, for $n = 1, \dots, N-1$. Then by (34),

$$\tilde{\gamma}_2^{(1)}(I_n) = \bar{\gamma}_2^{(1)}(I_n) + \left(\tilde{\gamma}_2^{(1)}(I_n) - \bar{\gamma}_2^{(1)}(I_n) \right) \in (\bar{a}_2)_0 + 2\sum_{k=1}^{m-1} (\bar{a}_2)_k T_k(I_n) + [-2r, 2r],$$

where $T_k(I_n)$ can be evaluated with interval arithmetic using that

$$T_k(I_n) = T_k([t_n, t_{n+1}]) = \cos\left(k\left[\cos^{-1}(t_{n+1}), \cos^{-1}(t_n)\right]\right).$$

With interval arithmetic, we show that $\tilde{\gamma}_2^{(1)}(I_n) < 1$ for all $n = N_1, \dots, N_2 - 1$ with $N_1 = 18$ and $N_2 = 4000 < N$. It remains to show that $\tilde{\gamma}_2^{(1)}(t) < 1$ for all $t \in [-1, t_{N_1}] \cup [t_{N_2}, 1] = [-1, -0.991] \cup [0.9995, 1]$. We do this by showing that $\frac{d}{dt}\tilde{\gamma}_2^{(1)}(t) \neq 0$ for all $t \in [-1, -0.991] \cup [0.9995, 1]$. Indeed, from the proof, it is known already that $\tilde{\gamma}_2^{(1)}(-1) = \tilde{\gamma}_2^{(1)}(1) = 1$. Assume that there exists $t^* \in [-1, t_{N_1}]$ such that $\tilde{\gamma}_2^{(1)}(t^*) > 1$. Since $\gamma_2^{(1)}$

is differentiable (in fact analytic), then there exists $\bar{t} \in [-1, t_{N_1}]$ such that $\frac{d}{dt}\tilde{\gamma}_2^{(1)}(\bar{t}) = 0$. Hence, if we can show that $\frac{d}{dt}\tilde{\gamma}_2^{(1)} \neq 0$ on $[-1, -0.991] \cup [0.9995, 1]$, then we are done showing that $\Gamma^{(1)} =\subset \mathcal{R}_1$.

Let $J_1 = [-1, -0.991]$ and $J_2 = [0.9995, 1]$. Then, by (35) and for $\ell = 1, 2$,

$$\begin{aligned}
\frac{d}{dt}\tilde{\gamma}_2^{(1)}(J_\ell) &= \frac{d}{dt}\bar{\gamma}_2^{(1)}(J_\ell) + \left(\frac{d}{dt}\tilde{\gamma}_2^{(1)}(J_\ell) - \frac{d}{dt}\bar{\gamma}_2^{(1)}(J_\ell)\right) \\
&\in 2\sum_{k=1}^{m-1} (\bar{a}_2)_k T_k'(J_\ell) + \left[-\frac{2\nu(\nu+1)}{(\nu-1)^3}r, \frac{2\nu(\nu+1)}{(\nu-1)^3}r\right] \\
&= 2\sum_{k=1}^{m-1} (\bar{a}_2)_k k U_{k-1}(J_\ell) + \left[-\frac{2\nu(\nu+1)}{(\nu-1)^3}r, \frac{2\nu(\nu+1)}{(\nu-1)^3}r\right],
\end{aligned}$$

since for $k \geq 1$, $\frac{d}{dt}T_k(t) = kU_{k-1}(t)$, where the U_k are the Chebyshev polynomials of the second kind given recursively by $U_0(t) = 1$, $U_1(t) = 2t$ and $U_{k+1}(t) = 2tU_k(t) - U_{k-1}(t)$. We use the above and interval arithmetic to rigorously enclose the value of $\frac{d}{dt}\tilde{\gamma}_2^{(1)}(J_\ell)$ and to show that it does not contain 0. This implies that $\Gamma^{(1)} \subset \mathcal{R}_1$.

To show that $\Gamma^{(2)} = \left\{\tilde{\gamma}^{(2)}(t) : t \in [-1, 1]\right\} \subset \mathcal{R}_2$, we proceed using the same approach. □

Remark 4.2 (Choosing the parameters m and ν). We now discuss how to tune the parameters for the proof. First, the dimension parameter m is chosen large enough so that

$$|(\bar{a}_1)_{m-1}|, |(\bar{a}_2)_{m-1}|, |(\bar{b}_1)_{m-1}|, |(\bar{b}_2)_{m-1}| \leq 10^{-14},$$

that is the last Chebyshev coefficient of each solution is less than 10^{-14} (see Figure 1 for the coefficients). The criterium for tuning m and ν is the control of the terms involving $\frac{2}{\nu^m}$ appearing in (57). Clearly, choosing a larger m and a larger ν improves control of these terms. However, one must take extra care in not choosing m and ν too large as the computations of the ν-norms involved in the computation Y bounds (as defined in Section 4.1) and the Z bounds (as defined in (57) and (56)) will blow up. This will result in the failure of the proof. In the end, choosing the proper m and ν follows from numerical experimentations on the success or failure of the proofs.

5 The Chua circuit

In this section we consider

$$\begin{cases} C_1\dot{u}_1 = (u_2 - u_1)/R - g(u_1) \\ C_2\dot{u}_2 = (u_1 - u_2)/R + u_3 \\ C_3\dot{u}_3 = -u_2 - R_0 u_3 \end{cases} \tag{59}$$

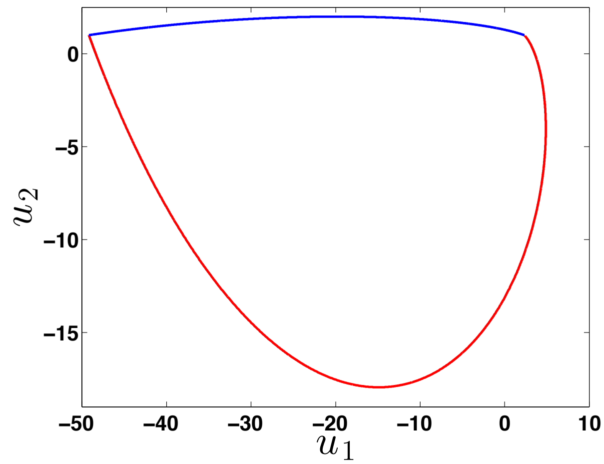

Figure 2: The graph of the periodic solution of Theorem 4.1. The period of the solution satisfies $\tau \in [5.156727575035736\ ,\ 5.156727575037338]$.

where $g(u_1) = G_b u_1 + \frac{1}{2}(G_a - G_b)\left(|u_1+1| - |u_1-1|\right)$ is a piecewise linear function. This system can be written as a three-part piecewise linear system

$$\dot{u} = \begin{cases} g^{(1)}(u) := \begin{pmatrix} -\frac{1}{RC_1} + G_b & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{C_3} & \frac{R_0}{C_3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} G_b - G_a \\ 0 \\ 0 \end{pmatrix}, & u \in \mathcal{R}_1, \\ g^{(2)}(u) := \begin{pmatrix} -\frac{1}{RC_1} + G_a & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{C_3} & \frac{R_0}{C_3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, & u \in \mathcal{R}_2, \\ g^{(3)}(u) := \begin{pmatrix} -\frac{1}{RC_1} + G_b & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{C_3} & \frac{R_0}{C_3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} G_a - G_b \\ 0 \\ 0 \end{pmatrix}, & u \in \mathcal{R}_3, \end{cases} \tag{60}$$

where $\mathcal{R}_1 = \{u = (u_1, u_2, u_3) : u_1 < -1\}$, $\mathcal{R}_2 = \{u = (u_1, u_2, u_3) : |u_1| < 1\}$, and $\mathcal{R}_3 = \{u = (u_1, u_2, u_3) : u_1 > 1\}$. We consider system (60) with the following parameter values $C_1 = 1$, $C_2 = 7.65$, $C_3 = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$, $G_a = -3.4429$, and $G_b = -2.1849$.

For this system we have two switching manifolds $\Sigma^{(1)}$ and $\Sigma^{(2)}$ given by $u_1 = -1$ and $u_1 = 1$, respectively. These manifolds can be parameterized by

$$\begin{aligned} P^{(1)} : \mathbb{R}^2 \to \mathbb{R}^3 : \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \mapsto P^{(1)} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ \theta_1 \\ \theta_2 \end{pmatrix}, \\ P^{(2)} : \mathbb{R}^2 \to \mathbb{R}^3 : \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \mapsto P^{(2)} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \theta_1 \\ \theta_2 \end{pmatrix}. \end{aligned} \tag{61}$$

The remaining bounds needed for the construction of the radii polynomials are the bounds for (41), which are given by

$$\|(Az)_i\| \leq Z_i^{(2)} r^2 + Z_i^{(1)} r,$$

where $Z_i^{(1)}$ and $Z_i^{(2)}$ are given below. For $i = 1, \dots, Mn$ we have

$$Z_i^{(1)} = \frac{2}{\nu^m} \sum_{j=1}^{Mn} \left| A_{ij}^{(m)} \right| + \frac{2}{\nu^m} \sum_{j=Mn+1}^{2Mn} \left| \left(A_{ij}^{(m)} \right)_0 \right|, \tag{62}$$

and

$$Z_i^{(2)} = 2 \left(\nu + \frac{1}{\nu} \right) \sum_{j=Mn+1}^{2Mn} \left(\sum_{\ell=1}^{3} \left| \alpha_\ell^{(k_i,\sigma_j)} \right| \right) \left\| A_{ij}^{(m)} \right\|_\nu^\infty , \tag{63}$$

where $\left(A_{ij}^{(m)} \right)_0$ denotes the first entry of the row vector $A_{ij}^{(m)}$, $k_i \in \{1, \dots, n\}$ denotes the component of the vector field corresponding to the entry $Z_i^{(2)}$, and $\alpha_\ell^{(k_i,\sigma_j)}$ is the (ℓ, k_i)-entry of the matrix corresponding to the linear part of the vector field corresponding to σ_j in system (60). For $i = Mn + 1, \dots, 2Mn$ we have

$$\begin{aligned} Z_i^{(1)} \quad = \quad & \frac{2}{\nu^m} \sum_{j=1}^{Mn} \left\| A_{ij}^{(m)} \right\|_\nu + \frac{2}{\nu^m} \sum_{j=Mn+1}^{2Mn} \left\| \left(A_{ij}^{(m)} \right)_{:,0} \right\|_\nu + \\ & \frac{1}{2m} \sum_{\ell=1}^{3} \left| (\bar{a}_\ell^{(\sigma_i)})_{m-1} \right| \left| \alpha_\ell^{(k_i,\sigma_j)} \right| + \frac{|\bar{L}_i|}{2m} \left(\nu + \frac{1}{\nu} \right) \sum_{\ell=1}^{3} \left| \alpha_\ell^{(k_i,\sigma_j)} \right|, \end{aligned} \tag{64}$$

and

$$Z_i^{(2)} = 2 \left(\nu + \frac{1}{\nu} \right) \sum_{j=Mn+1}^{2Mn} \left(\sum_{\ell=1}^{3} \alpha_\ell^{(k_i,\sigma_j)} \right) \left\| A_{ij}^{(m)} \right\|_{B(\ell_\nu^1, \ell_\nu^1)} , \tag{65}$$

where now $\left(A_{ij}^{(m)} \right)_{:,0}$ denotes the first column of the matrix $A_{ij}^{(m)}$.

We applied a similar analysis than the example of Section 4, and we have the following theorems. The proofs of these theorems are computer assisted and follow from an application of Proposition 2.12.

Theorem 5.1. *There is a CPO for system* (60) *which crosses the switching manifold* $\Sigma^{(2)}$ *exactly two times. This orbit was computed with* $m = 210$ *and we found*

$$\mathcal{I} = \left[1.08692 \times 10^{-12},\ 6.12802 \times 10^{-4} \right]$$

as the interval of radii given by (32). *This orbit is depicted in Figure 3.*

Proof. Follows from Proposition 2.12. □

Theorem 5.2. *There is a CPO for system* (60) *which crosses the switching manifold* $\Sigma^{(2)}$ *exactly four times. This orbit was computed with* $m = 410$ *and we found*

$$\mathcal{I} = \left[1.91205 \times 10^{-12},\ 1.52764 \times 10^{-4} \right]$$

as the interval of radii given by (32). *This orbit is depicted in Figure 3.*

Proof. Follows from Proposition 2.12. □

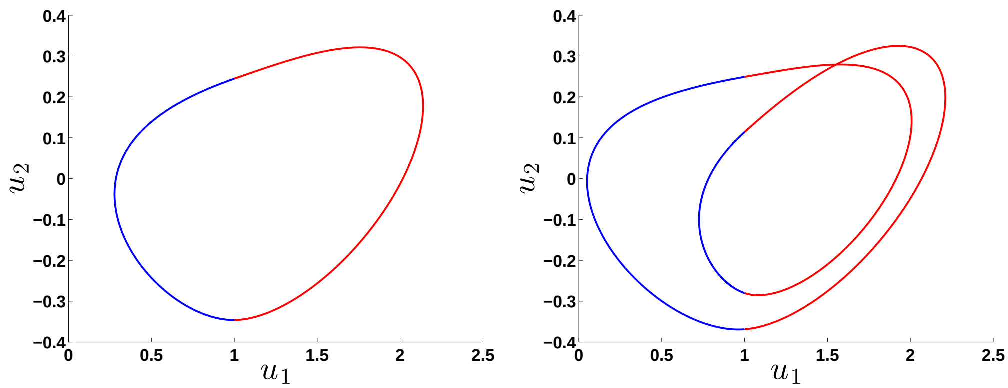

Figure 3: Plots of the two periodic solutions of Theorem 5.1 and Theorem 5.2. The periods of the solutions satisfies $\tau_1 \in [6.623074165809448,\ 6.625525376117043]$ and $\tau_2 \in [13.08983094730359,\ 13.09105306378421]$, where τ_1 is the period of the orbit on the left and τ_2 is the period of the orbit on the right.

6 Conclusion and future directions

In this paper, we introduced a rigorous numerical method to compute periodic orbits of PWS systems using a functional analytic approach based on Chebyshev series. We presented two applications. The results were quite successful, and we believe that this provides a new approach to obtain rigorous results about PWS systems.

However, we did not manage to prove existence of all the orbits we wished to prove. Indeed, system (52) seems to possess a much larger CPO at the same parameter values we considered. The radii polynomials seemed very sensitive to the dependency on the decay rate ν, and we failed in this case to verify hypothesis (32) of Proposition 2.12. Increasing the dimension of the Galerkin projection did not help, as the ν-norm of the quantities involved in the computation of the coefficients of the radii polynomials seem to blow-up. A similar situation occurred when we try to prove existence of longer orbits in the Chua's circuit system.

Based on the above remark, we believe that using a different function space with less instability with the computation of the norms could be useful. A weighed ℓ^∞ space could be for instance more numerically stable. In this regard, we believe that the estimates presented in [23, 24] could be helpful.

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A Formulas for the z_j in the model nonlinear problem

$$z_1 = 2\sum_{k\ge m}(v_5)_k r, \quad z_2 = 2\sum_{k\ge m}(v_6)_k r, \quad z_3 = 2\sum_{k\ge m}(v_7)_k r, \quad z_4 = 2\sum_{k\ge m}(v_8)_k r.$$

$$\begin{aligned}
(z_5)_n \;=\;& \{v_3\left[(\beta+\varepsilon)[\pm(u_5)_{n\pm1}] \pm (u_6)_{n\pm1}\right] + u_3\left[(\beta+\varepsilon)[\pm(v_5)_{n\pm1}] \pm (v_6)_{n\pm1}\right]\}_{n\ge1} r^2 \\
&+ \left\{v_3\left[(\beta+\varepsilon)[\pm(\bar a_1)_{n\pm1} \pm (\bar a_2)_{n\pm1}\right] + \bar L_1\left[(\beta+\varepsilon)[\pm(v_5)_{n\pm1}] \pm (v_6)_{n\pm1}\right]\right\}_{n>N} r \\
&+ \{-2\sum_{i\ge N+1}(-1)^i (v_5)_i\}_{n=0} r \\
(z_6)_n \;=\;& \{v_3\left[(\beta+\varepsilon)[\pm(u_6)_{n\pm1}] \mp (u_5)_{n\pm1}\right] + u_3\left[(\beta+\varepsilon)[\pm(v_6)_{n\pm1}] \mp (v_5)_{n\pm1}\right]\}_{n\ge1} r^2 \\
&+ \left\{v_3\left[(\beta+\varepsilon)[\pm(\bar a_2)_{n\pm1} \mp (\bar a_1)_{n\pm1}\right] + \bar L_1\left[(\beta+\varepsilon)[\pm(v_6)_{n\pm1}] \mp (v_5)_{n\pm1}\right]\right\}_{n>N} r \\
&+ \{-2\sum_{i\ge N+1}(-1)^i (v_6)_i\}_{n=0} r \\
(z_7)_n \;=\;& \Bigg\{v_4\left[\mp(u_7)_{n\pm1} + \frac{1}{\alpha}[\pm(u_8)_{n\pm1}] + 2\varepsilon[\pm(\bar b_1 u_7)_{n\pm1}]\right] + \bar\theta_2\left[2\varepsilon[\pm(u_7v_7)_{n\pm1}]\right] \\
&\quad + u_4\left[\mp(v_7)_{n\pm1} + \frac{1}{\alpha}[\pm(v_8)_{n\pm1}] + 2\varepsilon[\pm(\bar b_1 v_7)_{n\pm1}]\right]\Bigg\}_{n\ge1} r^2 \\
&+ \{v_4\left[\varepsilon[\pm(u_7u_7)_{n\pm1}]\right] + u_4\left[2\varepsilon[\pm(u_7v_7)_{n\pm1}]\right]\}_{n\ge1} r^3 \\
&+ \Bigg\{v_4\left[\mp(\bar b_1)_{n\pm1} + \frac{1}{\alpha}[\pm(\bar b_2)_{n\pm1}] + \varepsilon[\pm(\bar b_1\bar b_1)_{n\pm1}]\right] \\
&\quad + \bar\theta_2\left[\mp(v_7)_{n\pm1} + \frac{1}{\alpha}[\pm(v_8)_{n\pm1}] + 2\varepsilon[\pm(\bar b_1 v_7)_{n\pm1}]\right]\Bigg\}_{n>N} r \\
&+ \{-2\sum_{i\ge N+1}(-1)^i (v_7)_i\}_{n=0} r + \left\{2\bar\theta_2\varepsilon[\pm(\bar b_1 v_7)^I_{n\pm1}]\right\}_{1\le n\le N} r \\
(z_8)_n \;=\;& \{v_4\left[\alpha[\mp(u_7)_{n\pm1}] \mp (u_8)_{n\pm1} + 2\varepsilon[\pm(\bar b_1 u_7)_{n\pm1} \pm (\bar b_2 u_8)_{n\pm1}]\right] + \bar\theta_2\left[2\varepsilon[\pm(u_7v_7)_{n\pm1}\right. \\
&\quad \left.\pm (u_8v_8)_{n\pm1}]\right] + u_4\left[\alpha[\mp(v_7)_{n\pm1}] \mp (v_8)_{n\pm1} + 2\varepsilon[\pm(\bar b_1 v_7)_{n\pm1} \pm (\bar b_2 v_8)_{n\pm1}]\right]\}_{n\ge1} r^2 \\
&+ \{v_4\left[\varepsilon[\pm(u_7u_7)_{n\pm1} \pm (u_8u_8)_{n\pm1}]\right] + u_4\left[2\varepsilon[\pm(u_7v_7)_{n\pm1} \pm (u_8v_8)_{n\pm1}]\right]\}_{n\ge1} r^3 \\
&+ \{v_4\left[\alpha[\mp(\bar b_1)_{n\pm1}] \mp (\bar b_2)_{n\pm1}] + \varepsilon[\pm(\bar b_1\bar b_1)_{n\pm1} \pm (\bar b_2\bar b_2)_{n\pm1}]\right] \\
&\quad + \bar\theta_2\left[\alpha[\mp(v_7)_{n\pm1}] \mp (v_8)_{n\pm1} + 2\varepsilon[\pm(\bar b_1 v_7)_{n\pm1} \pm (\bar b_2 v_8)_{n\pm1}]\right]\}_{n>N} r \\
&+ \{-2\sum_{i\ge N+1}(-1)^i (v_8)_i\}_{n=0} r + \left\{2\bar\theta_2\varepsilon[\pm(\bar b_1 v_7)^I_{n\pm1} \pm (\bar b_2 v_8)^I_{n\pm1}]\right\}_{1\le n\le N} r
\end{aligned}$$

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