Modèles de dépendance avec copule Archimédienne: Fondements basés sur la construction par mélange, méthodes de calcul et applications

Mémoire

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Résumé

Le domaine de l'assurance est basé sur la loi des grands nombres, un théorème stipulant que les caractéristiques statistiques d'un échantillon aléatoire suffisamment grand convergent vers les caractéristiques de la population complète. Les compagnies d'assurance se basent sur ce principe afin d'évaluer le risque associé aux évènements assurés. Cependant, l'introduction d'une relation de dépendance entre les éléments de l'échantillon aléatoire peut changer drastiquement le profil de risque d'un échantillon par rapport à la population entière. Il est donc crucial de considérer l'effet de la dépendance lorsqu'on agrège des risques d'assurance, d'où l'intérêt porté à la modélisation de la dépendance en science actuarielle.

Dans ce mémoire, on s'intéresse à la modélisation de la dépendance à l'intérieur d'un portefeuille de risques dans le cas où une variable aléatoire (v.a.) mélange introduit de la dépendance entre les différents risques. Après avoir introduit l'utilisation des mélanges exponentiels dans la modélisation du risque en actuariat, on démontre comment cette construction par mélange nous permet de définir les copules Archimédiennes, un outil puissant pour la modélisation de la dépendance.

Dans un premier temps, on démontre comment il est possible d'approximer une copule Archimédienne construite par mélange continu par une copule construite par mélange discret. Puis, nous dérivons des expressions explicites pour certaines mesures d'intérêt du risque agrégé. Nous développons une méthode de calcul analytique pour évaluer la distribution d'une somme de risques aléatoires d'un portefeuille sujet à une telle structure de dépendance. On applique enfin ces résultats à des problèmes d'agrégation, d'allocation du capital et de théorie de la ruine. Finalement, une extension est faite aux copules Archimédiennes hiérarchiques, une généralisation de la dépendance par mélange commun où il existe de la dépendance entre les risques à plus d'un niveau.

Abstract

The law of large numbers, which states that statistical characteristics of a random sample will converge to the characteristics of the whole population, is the foundation of the insurance industry. Insurance companies rely on this principle to evaluate the risk of insured events. However, when we introduce dependencies between each component of the random sample, it may drastically affect the overall risk profile of the sample in comparison to the whole population. This is why it is essential to consider the effect of dependency when aggregating insurance risks from which stems the interest given to dependence modeling in actuarial science.

In this thesis, we study dependence modeling in a portfolio of risks for which a mixture random variable (rv) introduces dependency. After introducing the use of exponential mixtures in actuarial risk modeling, we show how this mixture construction can define Archimedean copulas, a powerful tool for dependence modeling.

First, we demonstrate how an Archimedean copula constructed via a continuous mixture can be approximated with a copula constructed by discrete mixture. Then, we derive explicit expressions for a few quantities related to the aggregated risk. The common mixture representation of Archimedean copulas is then at the basis of a computational strategy proposed to compute the distribution of the sum of risks in a general setup. Such results are then used to investigate risk models with respect to aggregation, capital allocation and ruin problems. Finally, we discuss an extension to nested Archimedean copulas, a general case of dependency via common mixture including different levels of dependency.

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The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom.

Isaac Asimov

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Avant-propos

Ce mémoire traite de la modélisation de la dépendance dans un modèle de risque actuariel. Plus particulièrement, il s'attarde aux situations où la dépendance est induite par une copule Archimédienne et les méthodes étudiées reposent sur la construction des copules Archimédiennes par mélange commun. Une généralisation aux copules Archimédiennes hiérarchiques ainsi que plusieurs exemples d'applications pratiques y sont également discutés.

Ce mémoire est composé de quatre chapitres. Le premier étant une introduction générale et le dernier une conclusion. Les chapitres 2 et 3 sont présentés sous forme d'articles scientifiques.

Le Chapitre 2 est constitué de l'article intitulé *Risk models defined with multivariate mixtures of exponential distributions* soumis en juillet 2018 pour publication dans la revue *Astin Bulletin*. Il s'agit d'un article co-écrit avec mes directeurs de recherche Hélène Cossette et Etienne Marceau, tous deux professeurs à l'École d'actuariat de l'Université Laval, ainsi qu'avec l'étudiante au doctorat en actuariat Itre Mtalai, aussi de l'Université Laval. Cet article utilise la méthode de construction par mélange d'exponentielles des copules Archimédiennes pour démontrer qu'il est possible d'approximer la relation de dépendance induite par certaines copules connues à partir de copules construites par mélanges discrets.

Le Chapitre 3 présente différentes méthodes de calculs analytiques afin de résoudre des modèles de risque où la dépendance entre les risques est induite par une copule Archimédienne construite par mélange discret. Différentes applications y sont développées et une généralisation aux copules Archimédiennes hiérarchiques est présentée. Le chapitre repose sur l'article *Dependent risk models with Archimedean copulas : a computational strategy based on common mixtures and applications* publié en novembre 2017 dans la revue avec comité de lecture *Insurance : Mathematics and Economics*. Cet article a également été co-écrit avec Hélène Cossette, Etienne Marceau et Itre Mtalai.

Introduction

Introduction

Au début du 18^e siècle, Jacques Bernoulli publiait l'ouvrage *Ars Conjectandi* (Bernoulli (1713)) dans lequel il énoncait entre autres la loi des grands nombres. L'assurance telle qu'on la connaît de nos jours est grandement basée sur cette loi. En effet, les compagnies d'assurance utilisent des données de populations entières afin d'établir les différentes hypothèses requises pour évaluer adéquatement les risques d'assurance qu'elles couvrent. Elles se fient ensuite qu'à partir d'un nombre d'assurés suffisamment élevé le comportement de ceux-ci s'approchera du comportement de la population complète, tel que le prévoit la loi des grands nombres.

En général, si les conditions habituelles sont satisfaites, cette méthode fonctionne bien. Par contre, il est important que l'échantillon retenu, le portefeuille d'assurance, soit représentatif de la population utilisée dans l'établissement des hypothèses actuarielles. En pratique, l'échantillon est souvent moins bien diversifié et il existe des relations de dépendance entre les assurés. Même lorsque ces relations de dépendance sont faibles, elles peuvent changer le profil de risque d'un portefeuille d'assurance de manière importante. C'est pourquoi il est important de considérer l'effet de la dépendance dans un bon modèle d'évaluation du risque.

Dans un contexte d'assurance, une relation de dépendance peut être due à plusieurs facteurs. En pratique, les compagnies oeuvrent normalement dans un secteur spécifique de la population que ce soit démographiquement ou géographiquement. Cela les expose à des risques de concentration et se traduit par une relation de dépendance entre les risques individuels. Par exemple, les assurés d'une compagnie d'assurance automobile habitant dans une même région seront exposés aux mêmes conditions météorologiques et aux mêmes catastrophes naturelles. Si l'assureur n'a pas adéquatement considéré cette dépendance entre les assurés, il risque de subir des pertes importantes si un tel évènement systématique touche un secteur dans lequel il a une forte exposition.

L'exemple ci-dessous illustre bien l'effet important que la dépendance peut avoir sur un portefeuille de risques. L'effet est d'autant plus important lorsque les risques sont plus volatiles ou plus nombreux.

Exemple 1. (Illustration de l'effet de la dépendance)

Soit deux risques assurés, Y_1 et Y_2, où le montant de chaque sinistre suit une distribution lognormale de paramètres $\mu_i = 0$ et $\sigma_i = 1$ pour $i = 1, 2$. On s'intéresse à la distribution de la somme des risques, soit $S = Y_1 + Y_2$. On sait que $E[S] = E[Y_1 + Y_2] = 3.30$ peu importe la relation de dépendance entre Y_1 et Y_2. Bien qu'on ne connaisse pas la distribution résultante de la convolution de deux variables aléatoires (v.a.) lognormales, on est en mesure de simuler la distribution de S à partir de la construction suivante d'un v.a. lognormale.

Soit X_i des v.a. normales de paramètres $\mu = 0$ et $\sigma = 1$. On a que $Y_i = e^{X_i} \sim$ lognormale(μ_i, σ_i), pour $i = 1, 2$. On peut facilement introduire une relation de dépendance entre les v.a. X_1 et X_2 et construire une loi normale bivariée où la dépendance entre X_1 et X_2 est définie par la matrice de corrélation suivante :

$$\begin{bmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{bmatrix},$$

où ρ_{X_1,X_2} est le facteur de corrélation entre X_1 et X_2. De plus, on peut facilement démontrer que le coefficient de corrélation résultant entre les v.a. Y_1 et Y_2 est donné par

$$\rho_{Y_1,Y_2} = \frac{e^{\sigma_1 \sigma_2 \rho_{X_1,X_2}} - 1}{\sqrt{\left(e^{\sigma_1^2} - 1\right)\left(e^{\sigma_2^2} - 1\right)}}. \tag{1}$$

À partir de la distribution normale bivariée, on peut alors facilement simuler un échantillon $\{X_1^{(i)}, X_2^{(i)}\}$ que l'on transforme en échantillon $\{Y_1^{(i)} = e^{X_1^{(i)}}, Y_2^{(i)} = e^{X_2^{(i)}}\}$ de lois lognormales corrélées.

On considère trois cas ; le cas indépendant (pour lequel $\rho_{Y_1,Y_2} = 0$), le cas faiblement corrélé (pour lequel $\rho_{Y_1,Y_2} = 0.5$) et le cas comonotone (pour lequel $\rho_{Y_1,Y_2} = 1$). Notons qu'à partir de (1), on détermine ρ_{X_1,X_2} de telle sorte que $\rho_{Y_1,Y_2} = 0, 0.5$ ou 1 et on obtient $\rho_{X_1,X_2} = 0, 0.62011451$ et 1 respectivement avec $\sigma_1 = \sigma_2 = 1$.

L'illustration 0.1 présente les fonctions de densité résultantes.

On remarque que plus la corrélation entre Y_1 et Y_2 est forte, plus la queue de la distribution est lourde. En analysant les quantiles de la distribution (voir tableaux 0.1 et 0.2), on remarque la même chose et on en déduit que l'inclusion de dépendance entre les différents éléments d'un portefeuille introduit un niveau de risque supplémentaire.

Notons en passant que le coefficient de corrélation, aussi connu sous le nom de coefficient de corrélation de Pearson, est une mesure simple mais qui ne mesure que le niveau de corrélation linéaire. En effet, bien que de façon générale $-1 \leq \rho \leq 1$, les cas $\rho = -1$ et $\rho = 1$ ne sont pas toujours possibles. L'exemple ci-dessus est un cas particulier où un coefficient de corrélation de 1 correspond bel et bien au cas comonotone. Par contre, dans le même exemple, on aurait aussi pu

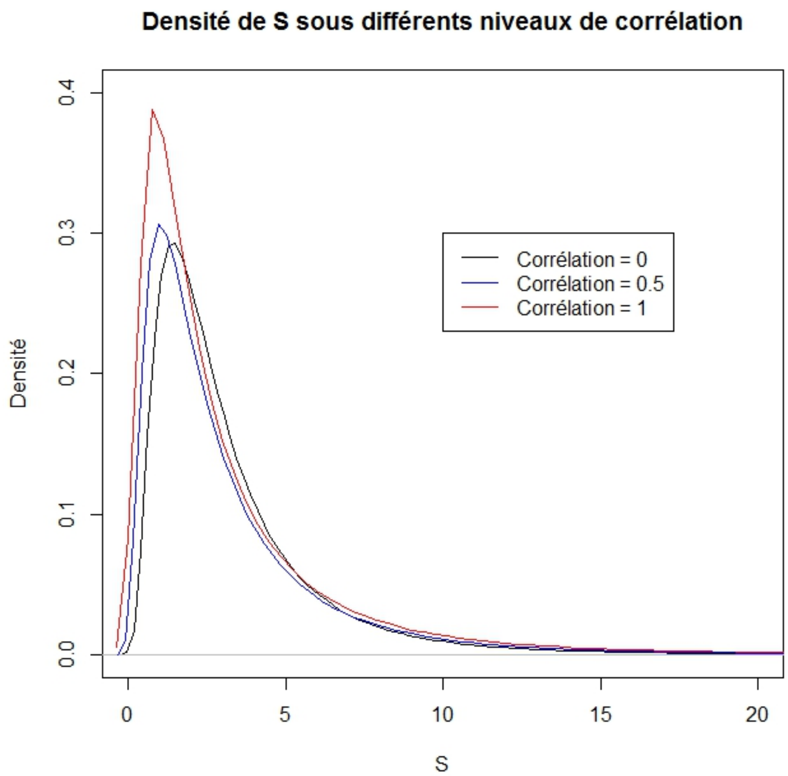

Illustration 0.1 – Fonctions de densité de S, la somme de deux risques aléatoires, selon différents niveaux de corrélation

ρ_{Y_1,Y_2}	$VaR_{0.01}(S)$	$VaR_{0.05}(S)$	$VaR_{0.10}(S)$	$VaR_{0.25}(S)$
0	0.444	0.728	0.950	1.482
0.5	0.264	0.490	0.682	1.181
1	0.195	0.385	0.554	1.017

Tableau 0.1 – Quantiles de la queue gauche de la distribution de S, la somme de deux risques aléatoires définis à l'exemple 1, selon différents niveaux de corrélation

ρ_{Y_1,Y_2}	$VaR_{0.75}(S)$	$VaR_{0.90}(S)$	$VaR_{0.95}(S)$	$VaR_{0.99}(S)$
0	4.064	6.484	8.630	15.072
0.5	4.024	6.986	9.703	18.149
1	3.927	7.204	10.360	20.562

Tableau 0.2 – Quantiles de la queue droite de la distribution S, la somme de deux risques aléatoires définis à l'exemple 1, selon différents niveaux de corrélation

s'intéresser à une relation de dépendence négative ($\rho \leq 0$) et même au cas limite antimonotone. On peut alors démontrer que le cas antimonotone correspond à $\rho_{X_1,X_2} = -1$ et, à partir de l'équation (1), on obtient $\rho_{Y_1,Y_2} = e^{-1} = -0.36787944$ et non pas $\rho_{Y_1,Y_2} = -1$. Pour des mesures de dépendance plus adéquates, on devrait plutôt utiliser le rho de Spearman (Spearmanl (1904)) ou le tau de Kendall (Kendall (1938)).

Il existe plusieurs méthodes pour tenir compte d'une relation de dépendance entre les risques d'un portefeuille, l'une d'entre elles étant l'utilisation d'une copule afin de créer la distribution multivariée des risques. L'utilisation d'une distribution normale multivariée, comme dans l'exemple ci-dessus, est en fait un cas spécifique de copule où la copule utilisée est dite Gaussienne. Celle-ci était fréquemment utilisée avant la crise financière de 2008 dans l'évaluation du risque d'obligations adossées à des actifs (*"Collateralized Debt Obligations" (CDO)*). Plusieurs médias en ont d'ailleurs fait mention suite à la crise financière, parfois pour critiquer cette approche (Salmon (2009)) ou pour la défendre (The Economist (2009)). Cela a certainement contribué à stimuler la recherche sur la modélisation de la dépendance à l'aide de copules dans le secteur de la finance.

Depuis le milieu des années 1990, la théorie des copules est un thème de recherche important tant en actuariat qu'en finance quantitative, en statistique ou en économétrie. Notamment, la théorie des copules est maintenant primordiale en modélisation des risques en actuariat et en gestion quantitative des risques. Elle permet de modéliser des relations de dépendance spécifiques entre des variables aléatoires et de construire des modèles multivariés plus flexibles et généraux. Entre autres, l'utilisation de copules permet d'introduire différents degrés de dépendance selon le niveau de sévérité d'un risque. Par exemple, une copule peut introduire une dépendence faible lorsqu'on considère les valeurs du support autour de la moyenne de la distribution, mais une forte dépendance dans les valeurs se trouvant dans la queue de la distribution, c'est-à-dire lorsqu'on regarde les valeurs plus extrêmes de celle-ci.

Peu de formules analytiques ont été développées par rapport aux applications concrètes qu'on peut faire des copules en modélisation des risques en actuariat. Diverses méthodes de simulation ont plutôt été proposées, voir par exemple Hofert (2010), McNeil and Nešlehová (2009) ou McNeil (2008). Dans ce travail, nous fournissons justement quelques expressions analytiques exactes pour des problèmes actuariels courants pouvant être modélisés à l'aide de la théorie des copules. En particulier, on s'intéresse aux copules Archimédiennes, une famille de copules où la relation de dépendance est introduite par une construction par mélange de la v.a. du risque (voir Marshall and Olkin (1988) pour les détails).

Dans le reste de ce chapitre, nous établissons quelques notions de base sur les distributions mélanges, les copules et, en particulier, la famille des copules Archimédiennes. On introduit également la méthode de la transformée de Fourier rapide afin de résoudre des problèmes de convolution et on termine en expliquant quelques contextes pratiques dans lesquels la théorie des copules a été appliquée. Le chapitre 2 traite de l'utilisation de mélanges de distributions exponentielles dans des modèles de risque actuariels et démontre comment approximer quelques copules Archimédiennes construites par mélange continu à partir de copules construites par mélange discret. Finalement, le chapitre 3 présente des méthodes de calcul analytiques à partir de la construction présentée au chapitre 2. On y développe également plusieurs applications pertinentes à l'actuariat et on généralise les concepts étudiés dans un cas où la structure de dépendance est hiérarchique.

Notions de base

Distributions mélanges univariées

En actuariat, il est fréquent de travailler avec des distributions construites par mélange. Dans ce mémoire, on utilise une méthode de construction de copules basée sur les mélanges communs. On introduit donc ici les distributions mélanges dans le cas univarié.

Soit une variable aléatoire X telle que sa distribution f_X est fonction d'un paramètre inconnu θ qui lui même suit une distribution f_Θ quelconque. On dit de cette v.a. qu'elle obéit à une distribution mélange où Θ est la v.a. mélange et f_Θ est la distribution mélange. Bref, la distribution de X conditionnellement à la v.a. mélange Θ est connue et on obtient la distribution inconditionnelle de X en intégrant la loi conditionnelle sur tout le domaine de Θ.

Exemple 2. (Quelques exemples de distributions mélanges)
Soit X la v.a. obéissant à une distribution mélange et Θ, la v.a. mélange. On dit de X qu'elle obéit à une distribution mélange si la distribution de X conditionnelle à une v.a. de mélange Θ suit une autre distribution connue. Sans considérer les distributions inconditionnelles résultantes, quelques exemples de mélanges seraient

1. $(X|\Theta = \theta) \sim$ Binomiale(n, θ) avec $\Theta \sim$ Unif$(0, 1)$;
2. $(X|\Theta = \theta) \sim$ Exponentielle(θ) avec $\Theta \sim$ Gamma(α, β) ;
3. $(X|\Theta = \theta) \sim$ Pareto(α, θ) avec $\Theta \sim$ Géométrique(q) ; ou
4. $(X|\Theta = \theta) \sim$ Normale(θ, σ) avec

$$\Theta = \begin{cases} -0.05 & \text{, avec probabilité } 0.25 \\ 0.00 & \text{, avec probabilité } 0.50 \\ 0.05 & \text{, avec probabilité } 0.25 \end{cases} .$$

Formellement, on a donc la v.a. mélange Θ pouvant être soit discrète ($\Theta \in A = \{\theta_1, \theta_2, \cdots\}$) ou continue ($\Theta \in \Re$). Dans le cas discret, on définit le fonction de masse de probabilité (fmp) comme

$$\Pr(\Theta = \theta) = f_\Theta(\theta), \tag{2}$$

pour $\theta \in A$. De plus, si la distribution de Θ est continue, on utilise aussi la notation $f_\Theta(\theta)$ pour $\theta \in A \subset \Re$ pour exprimer la fonction de densité de Θ.

Alors, on dit d'une distribution X qu'elle est une distribution mélange si on peut exprimer sa fonction de répartition F_X sous la forme

$$F_X(x) = \int_A F_{X|\Theta=\theta}(x) dF_\Theta(\theta), \tag{3}$$

où $F_{X|\Theta=\theta}(x)$, fonction de répartition de X conditionnelle à $\Theta = \theta$, est une loi connue avec au moins un paramètre qui est fonction de θ.

Dans le cas d'un mélange discret, l'équation précédente devient

$$F_X(x) = \sum_A F_{X|\Theta=\theta}(x) f_\Theta(\theta). \tag{4}$$

À partir de la formule de l'espérance totale, on développe des expressions pour l'espérance, l'espérance tronquée, la fonction génératrice des moments (fgm) ou la fmp de la v.a. X.

Selon la formule de l'espérance totale on a, pour une fonction $g(x)$ quelconque,

$$E[g(X)] = \int_{\theta\in A} E\left[g(X)|\Theta = \theta\right] dF_\Theta(\theta). \tag{5}$$

Donc, on peut exprimer l'espérance, l'espérance tronquée et l'espérance limitée de la distribution mélange X respectivement par

$$E[X] = \int_{\theta\in A} E\left[X|\Theta = \theta\right] dF_\Theta(\theta)\,; \tag{6}$$

$$E[X\ 1_{\{X>d\}}] = \int_{\theta\in A} E\left[X\ 1_{\{X>d\}}|\Theta = \theta\right] dF_\Theta(\theta)\,; \text{ et} \tag{7}$$

$$E[\min(X;d)] = \int_{\theta\in A} E\left[\min(X;d)|\Theta = \theta\right] dF_\Theta(\theta). \tag{8}$$

De plus, la fgm peut s'exprimer comme suit :

$$M_X(t) = \int_{\theta\in A} M_{X|\Theta=\theta}(t) dF_\Theta(\theta). \tag{9}$$

La fonction de densité est donnée par

$$f_X(t) = \int_{\theta\in A} f_{X|\Theta=\theta}(t) dF_\Theta(\theta). \tag{10}$$

.

Pour le cas discret, l'équation (5) devient

$$E[g(X)] = \sum_{\theta\in A} E\left[g(X)|\Theta = \theta\right] f_\Theta(\theta). \tag{11}$$

À partir de l'équation (11), les équations analogues aux équations (6) à (10) dans le cas discret deviennent triviales.

Exemple 3. (Illustration du mélange)

Reprenons l'example du mélange discret de trois lois normales décrit ci-haut. Soit $(X|\Theta = \theta) \sim$ Normale$(\theta, \sigma = 1)$ avec

$$\Theta = \begin{cases} -0.05 & \text{, avec probabilité } 0.25 \\ 0.00 & \text{, avec probabilité } 0.50 \\ 0.05 & \text{, avec probabilité } 0.25 \end{cases}$$

On s'intéresse à la distribution inconditionnelle de X et on illustre l'effet du mélange. On a tout d'abord les densités des trois lois normales du mélange à l'illustration (0.2).

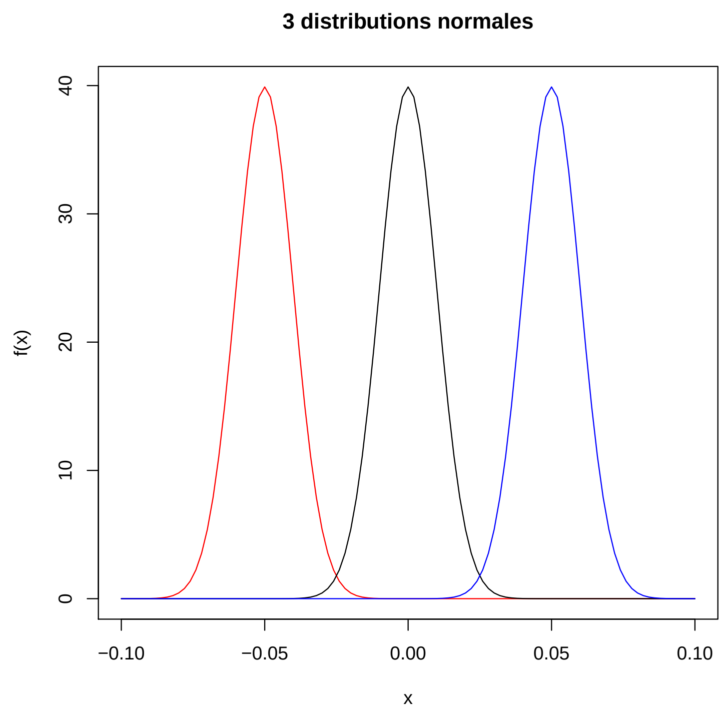

Illustration 0.2 – Fonction de densité des trois distributions normales définies à l'exemple 3

Ensuite, l'illustration (0.3) présente la fonction de densité de la distribution mélange. Contrairement aux lois normales de départ, on se retrouve avec une distribution multimodale.

Puis, dans l'illustration (0.4) on a la fonction de répartition dans laquelle on peut distinguer l'apport de chacune des distribution normales sous-jacentes à la fonction de répartition.

Lois multivariées construites par mélange commun

Marshall and Olkin (1988) et Oakes (1989) ont introduit le mode de construction de lois multivariées par mélange commun. On définit une suite de v.a. $(X_1, \cdots, X_m)$ dont le comportement

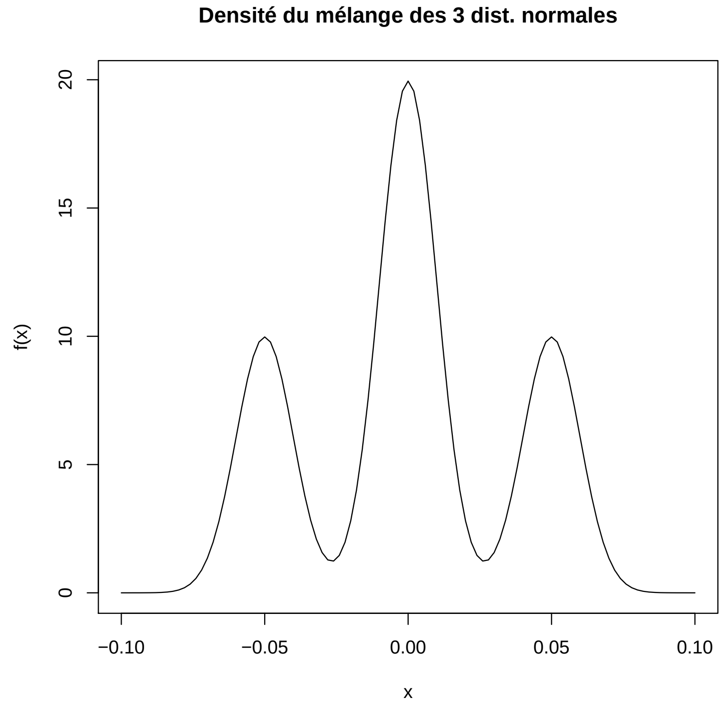

Illustration 0.3 – Fonction de densité de la distribution mélange de trois lois normales définies à l'exemple 3

aléatoire est influencé par une même v.a. Θ, strictement positive, dite de mélange. La v.a. de mélange Θ peut être discrète ou continue dont la transformée de Laplace-Stieljes est donnée par $\mathcal{L}_\Theta(t) = E\left[e^{-t\Theta}\right], t > 0$. En conditionnant par rapport à Θ, les v.a. X_i sont indépendantes entre elles, c'est-à-dire :

$$(X_i|\Theta = \theta) \perp (X_j|\Theta = \theta), \forall i \neq j.$$

Soit les v.a. indépendantes $Y_i, i = 1, \cdots, m$, dont les fonctions de survie sont notées F_{Y_i}. On fait l'hypothèse que les distributions de X_i, conditionnellement à Θ sont définies par leurs fonctions de survie $\overline{F}_{X_i|\Theta=\theta}$. Alors on peut démontrer qu'il existe une v.a. Y_i telle que les fonctions de survie de X_i s'écrivent

$$\begin{aligned} \overline{F}_{X_i|\Theta=\theta}(x_i) &= \Pr(X_i > x_i|\Theta = \theta) \\ &= \left(\overline{F}_{Y_i}(x_i)\right)^\theta. \end{aligned} \tag{12}$$

Y_i est une v.a. sous-jacente connue à laquelle on applique le mélange. Comme les v.a. Y_i pour $i = 1, \cdots, m$ sont indépendantes entre elles, on déduit que la fonction de survie conditionnelle multivariée est

$$\begin{aligned} \overline{F}_{X_1,\cdots,X_m|\Theta=\theta}(x_1, \cdots, x_m) &= \Pr\left(X_1 > x_1, \cdots, X_m > x_m|\Theta = \theta\right) \\ &= \left(\overline{F}_{Y_1}(x_1) \times \cdots \times \overline{F}_{Y_m}(x_m)\right)^\theta. \end{aligned} \tag{13}$$

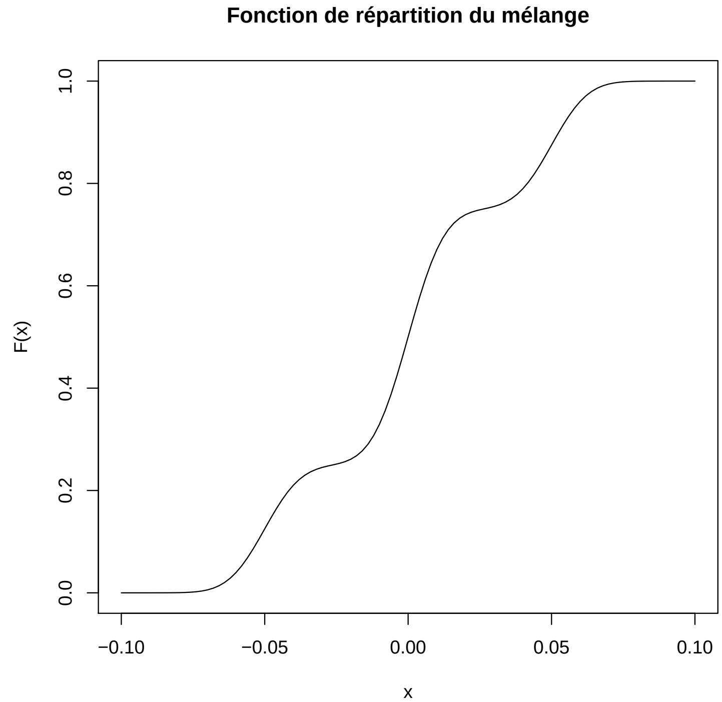

Illustration 0.4 – Fonction de répartition de la distribution mélange de trois lois normales définies à l'exemple 3

On précise que le support de la v.a. mélange Θ est $A_\Theta \in \Re^+$. Alors, on peut développer l'expression de la fonction de survie inconditionnelle de X_i comme étant

$$\begin{aligned}
\overline{F}_{X_i}(x_i) &= \int_{\theta \in A_\Theta} \overline{F}_{X_i|\Theta=\theta}(x_i) \mathrm{d}F_\Theta(\theta) \\
&= \int_{\theta \in A_\Theta} \left(\overline{F}_{Y_i}(x_i)\right)^\theta \mathrm{d}F_\Theta(\theta) \\
&= L_\Theta\left(\ln\left(\overline{F}_{Y_i}(x_i)\right)\right), \qquad (14)
\end{aligned}$$

où $L_\Theta(t)$ est la transformée de Laplace-Stieljes de la v.a. Θ.

De façon similaire, on développe l'expression de la fonction de survie multivariée de $(X_1, \cdots, X_m)$ par

$$\begin{aligned}
\overline{F}_{X_1,\cdots,X_m}(x_1, \cdots, x_m) &= \int_{\theta \in A_\Theta} \overline{F}_{X_1,\cdots,X_m|\Theta=\theta}(x_1, \cdots, x_m) \mathrm{d}F_\Theta(\theta) \\
&= \int_{\theta \in A_\Theta} \left(\overline{F}_{Y_1}(x_1) \times \cdots \times \overline{F}_{Y_m}(x_m)\right)^\theta \mathrm{d}F_\Theta(\theta) \\
&= L_\Theta\left(-\ln\left(\overline{F}_{Y_1}(x_1) \times \cdots \times \overline{F}_{Y_m}(x_m)\right)\right). \qquad (15)
\end{aligned}$$

On justifie le passage à la dernière ligne en exprimant la transformée de Laplace-Stieljes par son intégrale

$$\begin{aligned}\mathcal{L}_Y(t) &= E\left[e^{-Yt}\right] \\ &= \int_{y\in A_y} e^{-yt}\mathrm{d}F_Y(y).\end{aligned}$$

Copules

Tel que mentionné, les copules peuvent être utilisées pour modéliser la relation de dépendance entre une série de v.a. et construire une distribution multivariée de celles-ci. Ce problème était déjà étudié au début des années cinquante alors que Fréchet (Fréchet (1951)) a établit que toute distribution conjointe de v.a. $X_1, \cdots, X_m$ faisait nécessairement partie de la classe de Fréchet $\Gamma\left(F_{X_1}, \cdots, F_{X_m}\right)$. La classe de Fréchet borne en fait la relation de dépendance pouvant exister entre les v.a. par la borne supérieure de Fréchet (F^+) et la borne inférieure de Fréchet (F^-). Ces bornes sont définies par

$$F^+_{X_1,\cdots,X_m}(x_1,\cdots,x_m) = \min\left(F_{X_1}(x_1);\cdots;F_{X_m}(x_m)\right) \tag{16}$$

et

$$F^-_{X_1,\cdots,X_m}(x_1,\cdots,x_m) = \max\left(\sum_{i=1}^m F_{X_i}(x_i) - (m-1); 0\right). \tag{17}$$

Concrètement, on a que toute distribution conjointe de $(X_1, \cdots, X_m)$ sera bornée par les bornes de Fréchet de la manière suivante

$$F^-_{X_1,\cdots,X_m}(x_1,\cdots,x_m) \leq F_{X_1,\cdots,X_m}(x_1,\cdots,x_m) \leq F^+_{X_1,\cdots,X_m}(x_1,\cdots,x_m). \tag{18}$$

Bien quelle définisse les relations de dépendance maximale possibles, la classe de Fréchet ne précise pas quelles sont les autres distributions conjointes plausibles à l'intérieur de ces bornes. L'utilisation des copules permet de modéliser des distributions conjointes avec des relations de dépendance différentes des bornes de Fréchet. C'est en 1959 que Sklar a introduit la notion de copule pour la première fois (voir Sklar (1959) pour les détails). Il a énoncé le théorème suivant.

Theorem 1. *(Théorème de Sklar)*
D'une part, soit $F_{X_1,\cdots,X_m}(x_1,\cdots,x_m)$*, une fonction de distribution conjointe des lois marginales* $F_{X_i}(x_i)$ *avec fonctions de répartition inverses* $F^{-1}_{X_i}(u_i)$ *pour* $i = 1, \cdots, m$*. Alors, il existe une copule* $C(u_1,\cdots,u_m)$ *telle que*

$$C(u_1,\cdots,u_m) = F_{X_1,\cdots,X_m}\left(F^{-1}_{X_1}(u_1),\cdots,F^{-1}_{X_m}(u_m)\right).$$

D'autre part, soit une copule $C(u_1,\cdots,u_m)$ *et les fonctions de répartition* $F_{X_i}(x_i)$*,* $i = 1, \cdots, m$*. Alors, il existe une fonction de répartition conjointe* $F_{X_1,\cdots,X_m}(x_1,\cdots,x_m)$ *telle que*

$$F_{X_1,\cdots,X_m}(x_1,\cdots,x_m) = C\left(F_{X_1}(x_1),\cdots,F_{X_m}(x_m)\right).$$

Bref, les copules sont un mode de construction de distributions multivariées permettant d'introduire différents niveaux de dépendance entre des lois marginales. La copule définit entièrement la relation de dépendance entre les lois marginales et celles-ci n'ont pas besoin de suivre le même type de distribution. Il est donc possible d'utiliser une copule pour modéliser la dépendance entre deux risques se comportant très différemment.

Dans les chapitres 2 et 3 de ce mémoire, on se concentre sur les copules Archimédiennes, une famille de copules qui ont la particularité d'être construites par mélange commun. Ceci permet de développer des méthodes d'évaluation particulièrement intéressantes.

Copules Archimédiennes

La famille des copules Archimédiennes est un cas particulier des copules. Une copule Archimédienne est complètement définie par son générateur ψ et, dans le cas bivariée, elle peut être exprimée par

$$C(u_1, u_2) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2)\right), \tag{19}$$

où C représente la copule, ψ^{-1} est l'inverse du générateur et $u_i \in [0,1], i = 1, 2$ sont les points auxquels la copule est évaluée.

En remplaçant u_i par $F_{X_i}(x_i)$, on obtient

$$C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right) = F_{X_1,X_2}(x_1, x_2),$$

la fonction de répartition conjointe de X_1 et X_2.

De manière similaire, dans le cas multivarié à m dimensions on a

$$C(u_1, \cdots, u_m) = \psi\left(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_m)\right), \tag{20}$$

et

$$C\left(F_{X_1}(x_1), \cdots, F_{X_m}(x_m)\right) = F_{X_1,\cdots,X_m}(x_1, \cdots, x_m). \tag{21}$$

De plus, le générateur d'une copule Archimédienne doit posséder les propriétés suivantes :

— $\psi : [0,1] \rightarrow [0, \infty)$;

— ψ est une fonction continue, strictement décroissante et convexe ; et

— $\lim_{t\rightarrow 1} \psi(t) = 0$.

La transformée de Laplace-Stieljes d'une v.a. Θ est définie comme suit

$$\begin{aligned} \mathcal{L}_\Theta(t) &= E\left[e^{-t\Theta}\right] \\ &= \int_0^\infty e^{-t\Theta} dG_\Theta(\theta), \end{aligned} \tag{22}$$

où $G_\Theta(\theta)$ est la fonction de répartition de la v.a. Θ.

Dans cet ouvrage, on utilise le générateur ψ comme étant la transformée de Laplace-Stieltjes de la variable aléatoire mélange (Θ) sous-jacente à la copule. Par exemple, pour la copule AMH on sait que la v.a. mélange Θ suit une distribution géométrique tronquée à zéro (voir par exemple Joe (2014)). En utilisant la transformée de Laplace-Stieljes, toutes les propriétés désirées sont respectées et on retrouve les formes connues des copules avec lesquelles on travaillera.

On a montré qu'une fonction de survie multivariée construite par mélange commun peut s'exprimer selon l'équation (15). Si, en plus, on suppose que les v.a. indépendantes Y_i obéissent à des lois exponentielles de moyenne 1, alors on a que

$$\overline{F}_{Y_i}(x_i) = e^{-x_i},$$

pour $i = 1, \cdots, m$ d'où, selon l'équation (15),

$$\begin{aligned} \overline{F}_{X_1,\cdots,X_m}(x_1, \cdots, x_m) &= \mathcal{L}_\Theta \left(-\ln\left(e^{-x_1} \times \cdots \times e^{-x_m}\right)\right) \\ &= \mathcal{L}_\Theta \left(x_1 + \cdots + x_m\right) \\ &= \mathcal{L}_\Theta \left(\sum_{i=0}^{m} x_i\right). \end{aligned} \tag{23}$$

De la même façon, de l'équation (14) on déduit que les fonctions de survie marginales $\overline{F}_{X_i}$ sont

$$\begin{aligned} \overline{F}_{X_i}(x_i) &= \mathcal{L}_\Theta(-\ln(\overline{F}_{Y_i}(x_i))) \\ &= \mathcal{L}_\Theta(x_i), \end{aligned} \tag{24}$$

pour $i = 1, \cdots, m$.

Le résultat précédent nous permet d'écrire que

$$x_i = \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right), \tag{25}$$

pour $i = 1, \cdots, m$.

En introduisant (25) dans (23), on obtient

$$\overline{F}_{X_1,\cdots,X_m}(x_1, \cdots, x_m) = \mathcal{L}_\Theta \left(\sum_{i=0}^{m} \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)\right),$$

qui, en remplaçant $\overline{F}_{X_i}(x_i)$ par $u_i \in [0, 1]$, devient

$$\overline{F}_{X_1,\cdots,X_m}(x_1, \cdots, x_m) = \mathcal{L}_\Theta \left(\mathcal{L}_\Theta^{-1}(u_1) + \cdots + \mathcal{L}_\Theta^{-1}(u_m)\right),$$

qui est analogue à (20).

On montre ci-dessous que les copules Archimédiennes sont construites par mélange commun. Pour un vecteur de m v.a. uniformes $(U_1, \cdots, U_m)$ conditionnellement indépendantes sachant Θ, on a que

$$F_{U_i|\Theta=\theta}(u_i) = e^{-\theta\times\mathcal{L}_\Theta^{-1}(u_i)}, \tag{26}$$

pour $i = 1, \cdots, m$ et $u_i \in [0, 1]$.

Donc, on obtient

$$F_{U_1,\cdots,U_m}(u_1, \cdots, u_m) = \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(u_1) + \cdots + \mathcal{L}_\Theta^{-1}(u_m)\right) \tag{27}$$

qui correspond à une copule Archimédienne avec générateur $\mathcal{L}_\Theta$, soit la transformée de Laplace-Stieljes de la distribution mélange Θ.

Mesures d'association

Rho de Pearson

Le rho de Pearson, ou coefficient de corrélation, est une mesure d'association bien connue permettant de mesurer la corrélation linéaire existant entre deux v.a. Soit deux risques X et Y. On définit le rho de Pearson $\rho_{X,Y}$ par

$$\rho_{X,Y} = \frac{cov\,(X,Y)}{\sigma_X\sigma_Y}.$$

Pour un échantillon de n paires d'observations X et Y, $\{(X_1, Y_1), \cdots, (X_n, Y_n)\}$, le rho de Pearson se définit comme

$$\rho_{X,Y} = \frac{\sum_{i=1}^n \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum_{i=1}^n \left(X_i - \overline{X}\right)^2}\sqrt{\sum_{i=1}^n \left(Y_i - \overline{Y}\right)^2}}, \tag{28}$$

où $\overline{X} = \frac{\sum_{i=1}^n X_i}{n}$ et $\overline{Y} = \frac{\sum_{i=1}^n Y_i}{n}$.

Notons que $\rho_{X,Y} = 1$ signifie une dépendance linéaire parfaite, $\rho_{X,Y} = 0$ signifie une dépendance linéaire nulle et $\rho_{X,Y} = -1$ signifie que les v.a. sont parfaitement inversement corrélées linéairement. Toutefois, cette mesure a le désavantage de ne mesurer que la dépendance linéaire entre les v.a. Ainsi, bien que deux v.a. soient comonotones (ou antimonotone), le coefficient de corrélation résultant ne sera pas nécessairement de 1 (ou de -1) si la relation de dépendance entre ces v.a. n'est pas linéaire.

Des mesures de rang comme le tau de Kendall ou le rho de Spearman dont il est question dans les prochaines sous-sections sont des mesures de dépendance plus adéquates, car elles permettent de capturer la corrélation non-linéaire entre les risques.

Rho de Spearman

Une mesure de rang mesure le niveau de correspondance entre les rangs de deux séries d'observations de v.a. On définit le rang comme le classement des observations ordonnées de la plus petite à la plus élevée.

Soit deux v.a. continues X et Y avec les observations $X_1, \cdots, X_n$ et $Y_1, \cdots, Y_n$. On définit R_i comme le rang de l'observation X_i parmi l'échantillon $X_1, \cdots, X_n$ et S_i comme le rang de l'observation Y_i parmi l'échantillon $Y_1, \cdots, Y_n$. Ainsi, $R_i = 1$ si X_i est la plus petite observation de la série $X_1, \cdots, X_n$ et $R_i = n$ si X_i est la plus grande observation de la série $X_1, \cdots, X_n$.

Alors, le rho de Spearman, r_s, est simplement défini comme le coefficient de corrélation linéaire entre les paires d'observations (R_i, S_i), soit

$$r_{sX,Y} = \frac{\sum_{i=1}^{n} \left(R_i - \overline{R}\right)\left(S_i - \overline{S}\right)}{\sqrt{\sum_{i=1}^{n} \left(R_i - \overline{R}\right)^2}\sqrt{\sum_{i=1}^{n} \left(S_i - \overline{S}\right)^2}}.$$

Notons que $-1 \leq r_s \leq 1$.

Genest and Favre (2007) démontrent que pour une copule $C(u_1, u_2)$, il est possible de calculer r_s avec

$$r_s = 12 \int_{[0,1]^2} u_1 u_2 dC(u_1, u_2) - 3 \tag{29}$$

$$= 12 \int_{[0,1]^2} C(u_1, u_2)\, du_1 du_2 - 3. \tag{30}$$

Tau de Kendall

Le tau de Kendall, τ, quant-à-lui, est défini par l'équation

$$\tau_{X,Y} = \frac{2}{n(n-1)}\left(P_n - Q_n\right), \tag{31}$$

où P_n et Q_n représentent le nombre de paires d'observations concordantes et discordantes respectivement. Pour deux v.a. continues X et Y, on dit que deux paires d'observations (X_i, Y_i) et (X_j, Y_j) sont concordantes lorsque $(X_i - X_j)(Y_i - Y_j) > 0$ et qu'elles sont discordantes lorsque $(X_i - X_j)(Y_i - Y_j) < 0$.

Pour une copule $C(u_1, u_2)$ donnée, Genest and Favre (2007) démontrent que le tau de Kendall peut être calculer par

$$\tau = 4 \int_{[0,1]^2} C(u_1, u_2)\, dC(u_1, u_2) - 1. \tag{32}$$

Tant pour le tau de Kendall que pour le rho de Spearman, on pourrait démontrer que pour deux v.a. X et Y, si $\tau_{X,Y} = 1$ ou $r_{sX,Y} = 1$, alors la fonction de répartition conjointe de X et Y, $F_{X,Y}$, sera donnée par $F_{X,Y}(x, y) = F^{+}_{X,Y}(x, y)$, où F^{+} correspond à la borne supérieure de

Fréchet. Également, si $\tau_{X,Y} = -1$ ou $r_{sX,Y} = -1$, alors $F_{X,Y} = F^-_{X,Y}(X, Y)$, où F^- correspond à la borne inférieure de Fréchet. Les bornes supérieures et inférieures de Fréchet sont définies en détails dans la section sur les copules. On note également que si $\tau_{X,Y} = 0$ ou $r_{sX,Y} = 0$, alors les v.a. X et Y sont indépendantes et donc leur fonction de répartition conjointe sera donnée par $F_{X,Y}(x, y) = F_X(x)F_Y(y)$.

À partir des mesures d'association ci-dessus, il est possible de définir les paramètres d'une copule de sorte à ce que le rho de Spearman ou le tau de Kendall soit égal à celui d'un échantillon de d'observations donné.

Algorithmes de calcul numérique

Les méthodes de calcul proposées dans le reste du mémoire reposent sur la décomposition de la distribution du risque total en éléments conditionnellement indépendants. Un fois dans le cas indépendant, on effectue numériquement les calculs à l'aide d'un produit de convolution puis on agrège sur l'univers des valeurs possibles de la v.a. mélange. Dans le cas de v.a. indépendantes, il existe plusieurs techniques connues afin d'effectuer le produit de convolution, pensons par exemple à l'algorithme de DePril ou à l'alglorithme de Panjer (voir De Pril (1986), Panjer (1981), Panjer and Willmot (1986) et Klugman et al. (2009) pour les détails). Dans notre cas, on utilise surtout la méthode de la transformée de Fourier rapide (FFT) (Cooley and Tukey (1965)) que nous décrivons ci-dessous.

Transformée de Fourier rapide (FFT)

La FFT est un algorithme très efficace pour trouver la transformée de Fourier, aussi nommée la fonction caractéristique, d'une v.a. discrète. On peut donc, à partir de la FFT, trouver la fonction caractéristique de la convolution de deux ou plusieurs v.a. discrètes pour ensuite trouver la distribution de la v.a. sous-jacente à cette fonction caractéristique.

On peut définir une v.a. par sa fonction caractéristique définie comme

$$\begin{aligned} \tilde{f}_X(t) &= E\left[e^{itX}\right] && (33) \\ &= E\left[\cos(tX)\right] + iE\left[\sin(tX)\right], && (34) \end{aligned}$$

où $i = \sqrt{-1}$ et où $\tilde{f}_X(t)$ prend des valeurs dans l'ensemble des nombres complexes. Notons que la fonction caractéristique existe pour toute v.a. X, car les deux espérances $E[\cos(tX)]$ et $E[\sin(tX)]$ correspondent à des intégrales de fonctions bornées. Ainsi, $|\tilde{f}_X(t)| \leq 1$.

Si la v.a. X est discrète et définie sur $\mathbb{N}$, alors on a

$$\tilde{f}_X(t) = \sum_{k=0}^{\infty} e^{itk} f_X(k) = \sum_{k=0}^{\infty} f_X(k)cos(tk) + i\sum_{k=0}^{\infty} f_X(k)sin(tk), \quad (35)$$

où $f_X(k)$, $k = 0, 1, \cdots$ est la fmp de la v.a. X.

La fonction caractéristique est donc une généralisation de la fgm. On peut également montrer que la fonction caractéristique associée à une v.a. S étant définie par

$$S = \sum_{k=1}^{m} X_k,$$

où $X_1, X_2, \cdots, X_m$ sont des v.a. indépendantes, peut s'exprimer comme suit

$$\begin{aligned}
\tilde{f}_S(t) &= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_m=0}^{\infty} e^{it(j_1+j_2+\cdots+j_m)} f_{X_1}(j_1) f_{X_2}(j_2) \cdots f_{X_m}(j_m) \\
&= \sum_{j_1=0}^{\infty} e^{itj_1} f_{X_1}(j_1) \times \sum_{j=0}^{\infty} e^{itj_2} f_{X_2}(j_2) \times \cdots \times \sum_{j_m=0}^{\infty} e^{itj_m} f_{X_m}(j_m) \\
&= \tilde{f}_{X_1}(t) \tilde{f}_{X_2}(t) \cdots \tilde{f}_{X_m}(t). \qquad (36)
\end{aligned}$$

Ainsi, à partir des fonctions caractéristiques, il est possible d'effectuer différentes opérations, telle que la somme finie ci-dessus, pour obtenir une fonction caractéristique d'une v.a. S et d'en déduire la distribution en l'inversant.

Soit les vecteurs de n composantes suivants :

$$\begin{aligned}
\underline{f}_X &= (f_X(0), f_X(1), \cdots, f_X(n-1)) \\
\tilde{\underline{f}}_X &= \left(\tilde{f}_X(t_0), \tilde{f}_X(t_1), \cdots, \tilde{f}_X(t_{n-1})\right)
\end{aligned}$$

avec $t_j = \frac{2j\pi}{n}$, pour $j = 0, 1, \cdots, n$.

On peut déterminer les valeurs de $\tilde{\underline{f}}_X$ à partir des valeurs de $\underline{f}_X$ avec la relation

$$\begin{aligned}
\tilde{f}_X\left(\frac{2j\pi}{n}\right) &= \sum_{k=0}^{n-1} f_X(k) e^{i\frac{2\pi jk}{n}} \qquad (37) \\
&= \sum_{k=0}^{n-1} f_X(k) \cos\frac{2\pi jk}{n} + i \sum_{k=0}^{n-1} f_X(k) \sin\frac{2\pi jk}{n}, \qquad (38)
\end{aligned}$$

pour $j = 0, 1, \cdots, n-1$.

Par inversion, on peut alors déterminer les valeurs de $\underline{f}_X$ à partir de $\tilde{\underline{f}}_X$ avec la relation

$$f_X(k) = \frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}_X\left(\frac{2\pi j}{n}\right) e^{i\frac{2\pi jk}{n}}, \qquad (39)$$

pour $k = 0, 1, \cdots, n-1$.

Ainsi, pour un vecteur $\underline{f} = (f_0, f_1, \cdots, f_{n-1})$ donné, la FFT retourne un autre vecteur $\tilde{\underline{f}} = (\tilde{f}_0, \cdots, \tilde{f}_{n-1})$ tel que

$$\tilde{f}_j = \sum_{k=0}^{n-1} e^{\frac{2\pi i}{n}jk} f_k, \qquad (40)$$

pour $j = 0, 1, \cdots, n-1$.

De la même façon, la transformée inverse de Fourier rapide (IFFT) sert à isoler le vecteur $\underline{f}$. Pour l'IFFT, on a plutôt l'équation

$$f_j = \frac{1}{n}\sum_{k=0}^{n-1} e^{-\frac{2\pi i}{n}jk}\tilde{f}_k, \tag{41}$$

pour $j = 0, 1, \cdots, n-1$.

On précise également qu'on doit choisir $n = 2^m$, où m est un entier assez grand de sorte à ce que $\sum_{j=0}^{n} f_j = 1$ et donc que $\underline{f}$ soit une fmp.

Les équations (36) et (40) représentent les mêmes résultats que la transformée de Fourier discrète, mais le fait de passer par la FFT est beaucoup plus efficient numériquement. De plus, la FFT est préprogrammée dans plusieurs langages de programmation dont R.

On utilise alors les algorithmes suivants pour trouver les fonctions caractéristiques du produit de convolution de v.a. indépendantes à l'aide de la FFT puis, avec la FFT inverse, on calcule les valeurs de la fmp résultante de la convolution à partir de sa fonction caractéristique.

Exemple 4. (Somme de deux v.a. discrètes indépendantes)

Pour une v.a. $S = X_1 + X_2$ où X_1 et X_2 sont des v.a. discrètes indépendantes avec fmp f_{X_1} et f_{X_2}, on s'intéresse à la fmp de S, f_S. On utilise l'algorithme suivant :

1. On contruit les vecteurs $\underline{f}_{X_1}$ et $\underline{f}_{X_2}$ de longueur 2^m en utilisant des masses de probabilité nulles pour compléter chacun des vecteurs si nécessaire ;
2. On utilise la fonction FFT pour produire les fonction caractéristiques $\underline{\tilde{f}}_{X_1}$ et $\underline{\tilde{f}}_{X_2}$;
3. On fait le produit des fonctions caractéristiques pour retrouver la fonction caractéristique de la somme de X_1 et de X_2, car de l'équation (22) on a que $\underline{\tilde{f}}_S = \underline{\tilde{f}}_{X_1} \times \underline{\tilde{f}}_{X_2}$, et ;
4. On utilise la fonction IFFT sur la fonction caractéristique de la somme $\underline{\tilde{f}}_S$ pour produire le vecteur $\underline{f}_S$, la fmp de S.

Le code R développé pour utiliser cet algorithme se trouve en annexe.

Exemple 5. (Somme de m v.a. discrètes indépendantes)

Pour une v.a. $S = \sum_{i=1}^{m} X_i$, où X_i, $i \in \{1, 2, \cdots, m\}$ sont des v.a. indépendantes identiquement distribuées (iid) avec fmp f_{X_i}, on s'intéresse à f_S, la fmp de S. On utilise l'algorithme suivant :

1. On contruit les vecteurs $\underline{f}_{X_1}, \cdots, \underline{f}_{X_m}$ de longueur 2^m en ajoutant suffisamment de 0 à chacun des vecteurs ;
2. On utilise la fonction FFT pour produire les fonction caractéristiques $\underline{\tilde{f}}_{X_1}, \cdots, \underline{\tilde{f}}_{X_m}$;
3. On fait le produit des fonctions caractéristiques pour retrouver la fonction caractéristique de la somme $\sum_{i=1}^{m} X_i$, car de l'équation (22) on a que $\underline{\tilde{f}}_S = \underline{\tilde{f}}_{X_1} \times \cdots \times \underline{\tilde{f}}_{X_m}$;

4. On utilise la fonction IFFT sur la fonction caractéristique de la somme $\tilde{\underline{f}}_S$ pour produire le vecteur $\underline{f}_S$, la fmp de S.

Le code R développé pour utiliser cet algorithme se trouve en annexe.

Exemple 6. (Somme aléatoire)

Nous avons introduit les distributions par mélange commun précédemment. Dans cet exemple, on illustre comment la méthode FFT s'appliquerait dans le cas d'une loi Poisson composée.

Pour une v.a. $S(t) = \sum_{i=1}^{N(t)} X_i$, pour $N(t) > 0$ et où X_i, $i \in \{1, 2, \cdots\}$ sont des v.a. iid avec fmp f_{X_i} et $N(t)$ est un processus de Poisson, on s'intéresse à $f_{S(t)}$, la fmp de $S(t)$. On utilise l'algorithme suivant :

1. On contruit le vecteur $\underline{f}_X$ de longueur 2^m en ajoutant des masses de probabilité nulles à chacun des vecteurs si nécessaire ;
2. On utilise la fonction FFT pour produire les fonctions caractéristiques $\tilde{\underline{f}}_X$;
3. À l'aide de la fonction génératrice des probabilités ($P_{N(t)}$), on évalue la fonction caractéristique $\tilde{\underline{f}}_{S(t)}$ par $\tilde{\underline{f}}_{S(t)} = P_{N(t)}(\tilde{\underline{f}}_X)$;
4. On utilise la fonction IFFT sur la fonction caractéristique de la somme $\tilde{\underline{f}}_{S(t)}$ pour produire le vecteur $\underline{f}_{S(t)}$, la fmp de $S(t)$.

Le code R développé pour utiliser cet algorithme se trouve en annexe.

Application des copules en pratique

On peut avoir recours à des copules pour modéliser une distribution conjointe dans plusieurs contextes de gestion des risques.

En finance, l'application la plus fréquente est certainement dans la modélisation du risque de crédit. Dans ce cas, l'utilisation de copules permet de modéliser la corrélation entre le risque de défaut de différents actifs (voir par exemple Frey and McNeil (2002) ou Schönbucher and Schubert (2001)). Des applications au risque de marché et au risque opérationnel ont également été développées, notamment pour la modélisation de la dépendance entre les rendements d'indices boursiers ou pour introduire de la dépendance entre la fréquence d'occurence de sinistres dans un contexte de risque opérationnel.

En actuariat, les applications sont d'autant plus variées. Parmi les sujets d'actualité en recherche actuarielle, on compte l'évaluation du capital économique ainsi que les méthodes d'allocation du capital. Ces sujets sont particulièrement importants pour les compagnies d'assurance, puisqu'elles doivent s'assurer d'avoir suffisamment de capital afin d'absorber des pertes significatives dans l'éventualité où des risques importants se réaliseraient.

De plus, les méthodes d'allocation du capital leur permettent de mieux évaluer quels risques ou quelles lignes d'affaire sont les plus rentables et ont le meilleur rendement par rapport au risque qu'ils entraînent. Pour être en mesure de prendre de meilleures décisions stratégiques et d'exploiter les opportunités de diversification naturelle entre ses risques, une compagnie devrait tenir compte des coûts ou des gains causés par la corrélation existant entre les risques auxquels elle est exposée. Ce type d'applications est donc très propice à l'utilisation de copules.

Dans cette optique, Côté and Genest (2015) et Arbenz et al. (2012) présentent des modèles d'agrégation des risques à partir de copules. À partir d'un modèle de risques avec copules, Bargès et al. (2009) présentent une méthode d'allocation du capital. De son côté, Furman and Zitikis (2008) proposent des méthodes d'allocation du capital générales qu'on réutilise au chapitre 3 dans un contexte où la dépendance entre les risques est modélisée par une copule Archimédienne.

Un autre sujet d'intérêt en actuariat est la théorie de la ruine qui étudie la probabilité qu'une compagnie devienne insolvable. La théorie de la ruine s'intéresse entre autres à la distribution du surplus dans le temps, au temps de la ruine ainsi qu'au montant de déficit au moment où la ruine survient. Les modèles de ruine peuvent être utilisés pour fixer une cible de solvabilité précise sur un horizon de temps prédéfini et donc d'établir le montant de capital minimal à maintenir par la société en fonction de cette cible. Pour plus de détails sur la théorie de la ruine et pour différents exemples de modèles, voir Albrecher et al. (2011), Cossette et al. (2010), Cossette et al. (2003), ou Willmot (1993).

Un bon modèle de ruine doit modéliser adéquatement les prestations totales auxquelles l'assureur est exposé. Il devrait donc tenir compte de la dépendance entre les montants de sinistres, entre les probabilités d'occurence de ceux-ci ou même les deux. Pour ce faire, des copules peuvent être utilisées. Dans Constantinescu et al. (2011), des copules Archimédiennes hiérarchiques sont utilisées dans un modèle de ruine. Au chapitre 3, on utilise aussi des copules Archimédiennes dans de tels modèles.

Finalement, un exemple actuariel très concret d'utilisation des copules dans un contexte d'assurance de dommage est présenté dans Abdallah et al. (2015). Les auteurs ont utilisé des copules pour calculer des réserves pour prestations encourues mais non rapportées ($IBRN$). Ils ont appliqué des copules Archimédiennes hiérarchiques sur la méthode des triangles de développement afin de modéliser deux niveaux de dépendance ; un premier niveau pour modéliser la dépendance entre les observations d'un même portefeuille et un second pour la relation de dépendance existant entre deux lignes d'affaire.

Dans le reste de ce mémoire, on reprend plusieurs de ces applications, on les applique dans un contexte où la dépendance est induite par des copules Archimédiennes, puis on développe des méthodes de calcul analytiques lorsque possible. Toutefois, ces expressions analytiques ne sont possibles que lorsque la copule Archimédienne est construite via mélange discret. Alors, au prochain chapitre, on s'intéresse à la construction des copules Archimédiennes et on démontre qu'il est possible d'approximer des copules Archimédiennes construites par mélange continu à partir de copules Archimédiennes construites par mélange discret.

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Chapitre 1

Risk models defined with multivariate mixtures of exponential distributions

1.1 Résumé

Il est fréquent d'utiliser des distributions exponentielles mélange en modélisation du risque en actuariat. Les distribution obtenues par mélange permettent plus de flexibilité dans la modélisation des montants de sinistres en assurance de dommage. Plusieurs travaux de recherche ont étudié les distributions mélange exponentielles tant dans un contexte univarié que multivarié. Dans cet article, on souligne l'utilité de telles distributions et on explique les techniques de mélange qui les sous-tendent. On y explique aussi les liens entre les différents travaux qui ont été effectués. De plus, trois cas particuliers sont traités en détails et de nouvelles distributions univariées et multivariées sont dérivées. Afin d'illustrer les utilités variées et les propriétés utiles de ces distributions, des applications en actuariat sont présentées à travers l'article.

Mots clés : Modèles de risque avec dépendance ; Mélange de distributions exponentielles ; Distribution de mélange discret ; Copules Archimédiennes

1.2 Abstract

Mixed exponential distributions are frequently used in actuarial risk modeling. Distributions obtained through mixtures allow greater flexibility in the modeling of non-life insurance loss amounts. Several research works have studied mixed exponential distributions in univariate and multivariate settings. The present paper highlights the usefulness of such distributions and lays the story of the mixing technique behind them. It also explains the underlying link between all these works. In addition, a comprehensive study of three special cases of mixing distributions is considered. Applications in actuarial science of these distributions are presented throughout the paper highlighting their many uses and useful properties.

Keywords: Risk models with dependence; Multivariate mixture of exponential distributions; Discrete mixing distribution; Archimedean copulas.

1.3 Introduction

Mixed exponential distributions are frequently used in actuarial risk modeling. Distributions obtained through mixtures allow greater flexibility in the modeling of loss amounts. Often, the use of mixtures leads to heavy tailed distributions such as the Pareto distribution which arises as a mixed exponential-gamma distribution. More specifically, mixed exponential distributions have a mode of zero and a coefficient of variation greater than one which makes them good candidates for modeling claim amounts and inter-claim times. Properties of univariate mixtures of exponential distributions and their stop-loss transforms are provided in Hesselager et al. (1998). Lee et al. (2012) use univariate mixed exponential distributions to model claim sizes. Applications of univariate mixed exponential distributions, both in the modeling of claim sizes and interclaim times, in the context of surplus analysis can be found in Willmot and Woo (2018). Albrecher et al. (2011a) and Cossette et al. (2018) derived explicit ruin formulas in the context of actuarial risk models with either dependent inter-claim times or dependent claim amounts. Sarabia et al. (2018), Dacorogna et al. (2017), and Cossette et al. (2018) examine aggregation of dependent risks modelled with multivariate mixtures of exponential distributions. An interesting survey on univariate mixed exponential distributions and their applications can be found in Cai (2006).

Mixed exponential distributions are also frequently used as life-time distributions in the context of failure data analysis. Several mixed exponential distributions have been proposed in the literature in such a context, see, e.g., Adamidis and Loukas (1998), Adamidis et al. (2005), Kuş (2007), Tahmasbi and Rezaei (2008), Chahkandi and Ganjali (2009), Barreto-Souza and Bakouch (2013), Hajebi et al. (2013), Gui et al. (2014) and Yilmaz et al. (2016). Mixed exponential distributions have a decreasing failure rate (DFR) which makes them very useful in reliability theory (see, e.g., Proschan (1963) for a proof of the DFR property of mixtures of distributions with a constant hazard rate).

Multivariate distributions can also be constructed through mixtures. Marshall and Olkin (1988) and Oakes (1989) were among the first ones to use such a construction technique to obtain new multivariate distributions. In such multivariate models, the dependence between risks is induced via a common random variable (rv) (or several common rvs) representing common economic, geographical or climate conditions, or any other common external factor. Such a mixing technique can also be used to define a multivariate dependence structure defined with an Archimedean copula (see, e.g., Marshall and Olkin (1988)).

As previously stated, several research works have studied mixed exponential distributions in univariate and multivariate settings. The present paper highlights the usefulness of such distributions and lays the story of the mixing technique behind them. It also explains the underlying link between all these works. In addition, a comprehensive study of three special cases of mixing distributions is considered. Applications in actuarial science of these distributions are presented throughout the paper highlighting their many uses and useful properties.

The outline of the paper is as follows. Section 1.4 presents the univariate mixed exponential distribution, discusses some of its interesting properties and studies three special cases. A generalization of this distribution to the multivariate case is treated in Section 1.5. The link with Archimedean copulas is also established in this section. Applications of univariate and multivariate mixed exponential distributions in the context of actuarial risk models are presented in Sections 1.4 and 1.5. Finally, Section 1.6 is devoted to the investigation of ruin problems in portfolios with exchangeable inter-claim times.

1.4 Univariate mixed exponential distributions

Let Θ be a strictly positive mixing rv with cdf F_Θ (i.e., $F_\Theta(0) = 0$) and LST $\mathcal{L}_\Theta$, where

$$\mathcal{L}_\Theta(t) = \int \mathrm{e}^{-\theta t} dF_\Theta(\theta) = E[\mathrm{e}^{-\Theta t}]. \tag{1.1}$$

A rv X follows a mixed exponential distribution if, given $\Theta = \theta$, the conditional distribution of $(X|\Theta = \theta)$ is exponential with mean $\frac{1}{\theta}$ and its survival function is $\bar{F}_{X|\Theta=\theta}(x) = \mathrm{e}^{-\theta x}$ $(x \in \mathbb{R}^+)$. Then, from (1.1), it implies that the survival function of the unconditional rv X is given by

$$\overline{F}_X(x) = \int \mathrm{e}^{-\theta x} dF_\Theta(\theta) = \mathcal{L}_\Theta(x). \tag{1.2}$$

A brief review of applications of mixed exponential distributions in actuarial science and queueing theory is provided in, e.g., Cai (2006) and references therein. Also, mixed exponential distributions can be notably used to fit long tail distributions such as the Pareto distribution (see, e.g., Feldmann and Whitt (1997)).

In the following proposition, we list different general properties of a univariate mixed exponential distribution.

Proposition 7. *Let X a positive rv with univariate mixed exponential distribution and survival function as given in (1.2). Properties of such a distribution can be written in terms of the mixing rv Θ as follows (assuming that the expectations exist):*

1. *LST $\mathcal{L}_X(t) = E_\Theta\left[\frac{\Theta}{\Theta+t}\right]$, $t > 0$;*
2. *probability density function $f_X(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_\Theta(x)$, $x \in \mathbb{R}^+$;*

3. *failure rate* $h_X(x) = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_\Theta(x)}{\mathcal{L}_\Theta(x)}$, $x \in \mathbb{R}^+$;
4. *moments* $E[X^n] = \Gamma(n+1) \times E\left[\Theta^{-n}\right]$, *if it exists;*
5. *stop-loss function* $\pi_d(X) = E\left[\max\left(X - d; 0\right)\right] = E_\Theta\left[\Theta^{-1}\mathrm{e}^{-d\,\Theta}\right]$, *if* $E[X] < \infty$;
6. $VaR_\kappa(X) = F_X^{-1}(\kappa) = \inf\left\{x \in \mathbb{R}, F_X(x) \geq \kappa\right\} = \mathcal{L}_\Theta^{-1}\left(1-\kappa\right)$, $\kappa \in (0,1)$;
7. $TVaR_\kappa(X) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u\left(X\right)du = \frac{E_\Theta\left[\left(\frac{1}{\Theta}+VaR_\kappa(X)\right)\mathrm{e}^{-VaR_\kappa(X)\,\Theta}\right]}{1-\kappa}$, $\kappa \in (0,1)$, *if* $E[X] < \infty$.

Proof. All properties are obtained by simply conditioning with respect to Θ and using the fact that $(X|\Theta=\theta) \sim Exp(\theta)$. For example, the expression for the TVaR in [7] can be obtained as follows:

$$\begin{aligned} TVaR_\kappa(X) &= \frac{E\left[X \times 1_{\{X>VaR_\kappa(X)\}}\right]}{1-\kappa} \\ &= \frac{E\left[E\left[X \times 1_{\{X>VaR_\kappa(X)\}}|\Theta\right]\right]}{1-\kappa}. \end{aligned}$$

Since $(X|\Theta=\theta) \sim Exp(\theta)$, then

$$TVaR_\kappa(X) = \frac{E_\Theta\left[\left(\Theta^{-1}+VaR_\kappa(X)\right)\mathrm{e}^{-VaR_\kappa(X)\,\Theta}\right]}{1-\kappa},$$

if the expectation exists. We proceed similarly to obtain expressions for the other properties. □

Several known continuous distributions arise as mixed exponential distributions. For example, if the mixing rv is gamma distributed, then, the resulting mixture is a Pareto distribution or a Lomax distribution (also called Pareto of type II). Another well-known result is the Weibull distribution which arises from a mixed exponential-stable distribution.

Several other distributions can also be constructed via the exponential mixture method by taking different discrete mixing distributions. For example, Adamidis and Loukas (1998) considered a geometric mixing distribution, whereas Chahkandi and Ganjali (2009) proposed exponential-power series distributions which include binomial, Poisson (Kuş (2007)) and logarithmic (Tahmasbi and Rezaei (2008)) mixing distributions.

Later, Hajebi et al. (2013) studied an exponential-negative binomial distribution using another construction method. Let Θ be a strictly positive discrete rv. Then, a mixed exponential distribution can be represented as the distribution of the random minimum $X = \min(Y_1, \ldots, Y_\Theta)$ where $Y_1, \ldots, Y_\Theta$ is a random sample from an exponential distribution and Θ a discrete rv independent of $(Y_1, Y_2, \ldots)$. This construction method provides an alternative approach to the mixing representation of $\bar{F}_X(x) = \Pr(X > x)$ provided in (1.2), i.e.,

$$\begin{aligned}
\Pr(X > x) &= \Pr\left(\min\left(Y_1, \ldots, Y_\Theta\right) > x\right) \\
&= \sum_k \Pr\left(\min\left(Y_1, \ldots, Y_k\right) > x\right) \Pr(\Theta = k) \\
&= \sum_k \Pr(Y_1 > x) \ldots \Pr(Y_k > x) \Pr(\Theta = k) \\
&= \sum_k (\mathrm{e}^{-x})^k \Pr(\Theta = k) \\
&= \mathcal{L}_\Theta(x).
\end{aligned}$$

1.4.1 Univariate mixed Exponential - Negative Binomial Distribution

Let Θ be a discrete rv following a negative binomial distribution (i.e., $\Theta \sim NB(r, q)$) with pmf

$$\Pr(\Theta = k) = \binom{k-1}{r-1} q^r \, (1-q)^{k-r}, \;\; \forall\, k \in \{r, r+1, \ldots\}, \tag{1.3}$$

and LST given by

$$\mathcal{L}_\Theta(t) = \left(\frac{q}{\mathrm{e}^t - (1-q)}\right)^r. \tag{1.4}$$

Combining (1.2) and (1.4) leads to the following unconditional survival function of X

$$\overline{F}_X(x) = \left(\frac{q}{\mathrm{e}^x - (1-q)}\right)^r,$$

and probability density function (pdf) of X

$$f_X(x) = \frac{r\, q^r\, \mathrm{e}^{-rx}}{(1 - (1-q)\, \mathrm{e}^{-x})^{r+1}}, \; x \in \mathbb{R}^+.$$

Note that this formula is a special case of a formula that can be found in Hajebi et al. (2013) using scale parameter $\beta = 1$. Also, since the geometric distribution is a special case of the negative binomial distribution with $r = 1$, the mixed exponential-geometric distribution introduced by Adamidis and Loukas (1998) is a special case of the mixed exponential-negative binomial distribution presented here. A multivariate extension of such a distribution is presented in Section 1.5.1.

While some properties of the mixed exponential-negative binomial distribution are already given in Hajebi et al. (2013) such as the expression of the hazard rate, the moments, order statistics, extreme values and parameter estimators, other interesting properties have not been studied yet. The aim here is to consider a more comprehensive study of the mixed exponential-negative binomial distribution, including a detailed survey of limit cases and to discuss some of its characteristics in relation with actuarial science and quantitative risk management.

We first briefly recall the representation of the Pareto distribution as a mixed exponential-gamma distribution. Indeed, let the mixing rv $\Theta_{(\alpha)}^{Ga}$ follow a gamma distribution, i.e., $\Theta_{(\alpha)}^{Ga} \sim Gamma(\frac{1}{\alpha}, 1)$ with LST

$$\mathcal{L}_{\Theta_{(\alpha)}^{Ga}}(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}. \tag{1.5}$$

Then, combining (1.2) and (1.5) leads to

$$\bar{F}_{X_{(\alpha)}^{Pa}}(x) = \left(\frac{1}{1+x}\right)^{\frac{1}{\alpha}}, \, x \in \mathbb{R}^+, \tag{1.6}$$

i.e., the rv $X_{(\alpha)}^{Pa}$ follows a Pareto distribution with shape parameter $\frac{1}{\alpha}$ and scale parameter 1.

In order to discuss the limit cases of the univariate mixed exponential-negative binomial distribution, a new parametrization is needed. Let M be a discrete rv which follows a negative binomial distribution with parameters r ($r \in \mathbb{R}^+$) and $q = 1 - \mathrm{e}^{-h}$ ($h \in \mathbb{R}^+$), pmf as defined in (1.3) and LST given by

$$\mathcal{L}_M(t) = \left(\frac{1-\mathrm{e}^{-h}}{\mathrm{e}^{t} - \mathrm{e}^{-h}}\right)^r. \tag{1.7}$$

Let the mixing rv $\Theta_{(h,r)}$ be defined as $\Theta_{(h,r)} = h \times M$. Then, it is said that $\Theta_{(h,r)}$ follows a negative binomial distribution with, using (1.7), LST given by

$$\mathcal{L}_{\Theta_{(h,r)}}(t) = E[\mathrm{e}^{-t\,h \times M}] = \left(\frac{\mathrm{e}^{-th} - \mathrm{e}^{-(t+1)h}}{1 - \mathrm{e}^{-(t+1)h}}\right)^r = \left(\frac{1-\mathrm{e}^{-h}}{\mathrm{e}^{t\,h} - \mathrm{e}^{-h}}\right)^r. \tag{1.8}$$

Proposition 8. *Let $\Theta_{(r)}^{Ga} \sim Gamma\left(\frac{1}{r}, 1\right)$ and $\Theta_{(h,r)} \sim NB\left(r, 1 - \mathrm{e}^{-h}\right)$ with LST given in (1.5) and (1.8) respectively. Then,*

$$\Theta_{(h,r)} \xrightarrow{\mathcal{D}} \Theta_{(r)}^{Ga},$$

as $h \to 0$.

Proof. Clearly, the LST of the discrete rv $\Theta_{(h,r)}$ in (1.8) tends to the LST of the continuous rv $\Theta_{(r)}^{Ga}$ in (1.5) as $h \to 0$, i.e.,

$$\lim_{h \to 0} \left(\frac{\mathrm{e}^{-th} - \mathrm{e}^{-(t+1)h}}{1 - \mathrm{e}^{-(t+1)h}}\right)^r = \left(\frac{1}{1+t}\right)^r. \tag{1.9}$$

By Lévy's continuity theorem, (1.9) implies that $\Theta_{(h,r)}$ converges in distribution to $\Theta_{(r)}^{Ga}$. □

In Figures 1.1 and 1.2, the convergence of the negative binomial distribution to the gamma distribution (as $h \to 0$) is clearly illustrated. The negative binomial distribution can therefore be seen as a discrete version of the continuous gamma distribution.

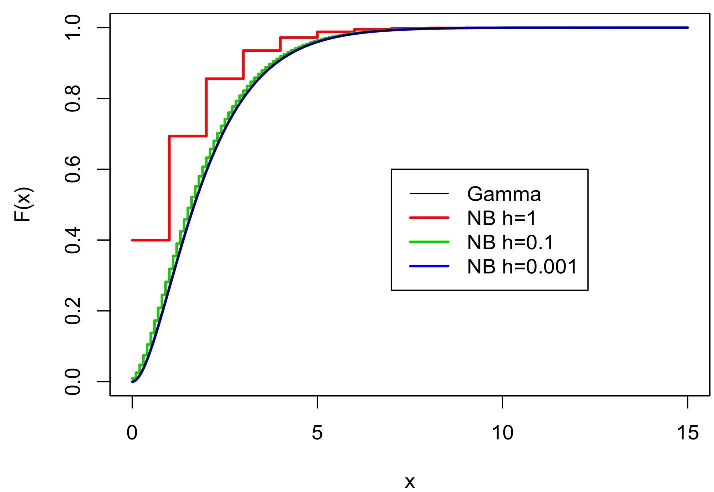

Figure 1.1 – Illustration of the convergence of the cdf of the negative binomial distribution to the cdf of the gamma distribution with $r = 2$.

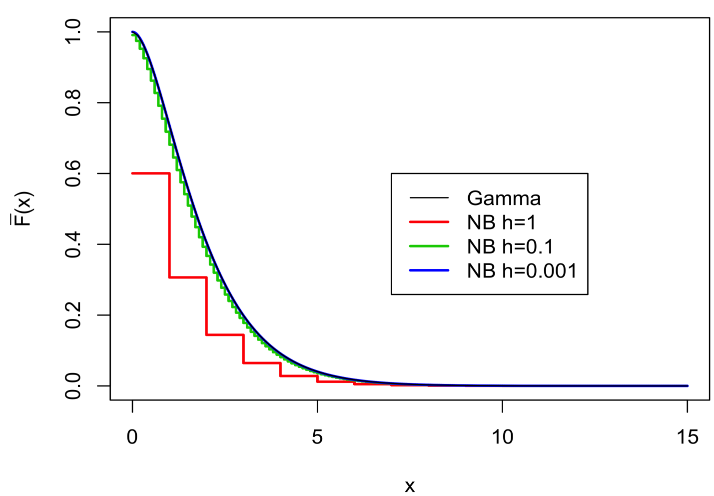

Figure 1.2 – Illustration of the convergence of the survival function of the negative binomial distribution to the survival function of the gamma distribution with $r = 2$.

Let the rv $X_{(h,r)}$ follow a univariate mixed exponential-negative binomial distribution, i.e $(X_{(h,r)}|\Theta_{(h,r)} = \theta) \sim Exp(\theta)$ where $\Theta_{(h,r)} \sim NB(r, 1 - \mathrm{e}^{-h})$. Then, combining (1.2) and (1.8), the unconditional survival function of $X_{(h,r)}$ is given by

$$\overline{F}_{X_{(h,r)}}(x) = \mathcal{L}_{\Theta_{(h,r)}}(x) = \left(\frac{1-\mathrm{e}^{-h}}{\mathrm{e}^{xh}-\mathrm{e}^{-h}}\right)^r. \tag{1.10}$$

Moments and other risk related quantities of the mixed exponential-negative binomial distribution are defined in terms of generalized hypergeometric functions which allow to effectively perform numerical calculations. It requires the use of the rising factorial, also called Pochammer symbol, defined as $(q)_j = q \times (q+1) \times \ldots \times (q+j-1)$, for $j > 0$ and $(q)_0 = 1$.

Definition 9. For $x \in \mathbb{R}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $b_j \in \mathbb{R}^+$, $j = 1, \ldots, m$, the generalized hypergeometric function, denoted by ${}_nF_m$, is defined as

$${}_nF_m\left([a_1, \ldots, a_n], [b_1, \ldots, b_m], x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \times \ldots \times (a_n)_k}{(b_1)_k \times \ldots \times (b_m)_k} \frac{x^k}{k!}, \tag{1.11}$$

where $n \leq m+1$. Special care must be taken when $b_j \in \mathbb{Z}^-$ for $j = 1, \ldots, m$. For more details see, e.g., Abramowitz and Stegun (1964).

Proposition 10. *Let $X_{(h,r)}$ follow a univariate mixed exponential-negative binomial distribution with survival function as given in (1.10). Then, the following properties hold:*

1. $X_{(h,r)} \xrightarrow{\mathcal{D}} X_{(r)}^{Pa}$, *as* $h \to 0$, *where* $X_{(r)}^{Pa} \sim Pareto\left(\frac{1}{r}, 1\right)$;
2. $\mathcal{L}_{X_{(h,r)}}(t) = \frac{1}{hr+t}\left(hr\left(1-\mathrm{e}^{-h}\right)^r {}_2F_1\left(\left[1+r, \frac{hr+t}{h}\right], \left[\frac{h+hr+t}{h}\right], \mathrm{e}^{-h}\right)\right)$, $t > -hr$;
3. $f_{X_{(h,r)}}(x) = rh\mathrm{e}^{xh}\frac{\left(1-\mathrm{e}^{-h}\right)^r}{\left(\mathrm{e}^{xh}-\mathrm{e}^{-h}\right)^{r+1}}$, $x>0$;
4. *Failure rate* $h_{X_{(h,r)}}(x) = \frac{f_{X_{(h,r)}}(x)}{\overline{F}_{X_{(h,r)}}(x)} = \frac{rh\mathrm{e}^{xh}}{\mathrm{e}^{xh}-\mathrm{e}^{-h}}$, $x > 0$;
5. $E[X_{(h,r)}^n] = \frac{n!}{h^n r^n} \times \left(1-\mathrm{e}^{-h}\right)^r \times {}_{n+1}F_n\left([r, \ldots, r], [1+r, \ldots, 1+r], \mathrm{e}^{-h}\right)$;
6. $\pi_d\left(X_{(h,r)}\right) = E\left[\max(X_{(h,r)} - d; 0)\right] = \frac{\left(1-\mathrm{e}^{-h}\right)^r {}_2F_1\left([r,r],[r+1],\mathrm{e}^{-h(d+1)}\right)}{hr\mathrm{e}^{hrd}}$, $d > 0$;
7. $VaR_\kappa(X_{(h,r)}) = \inf\{x > 0 : F_X(x) \geq \kappa\} = \frac{1}{h}\ln\left(\frac{1-\mathrm{e}^{-h}}{(1-\kappa)^{\frac{1}{r}}} + \mathrm{e}^{-h}\right)$, $\kappa \in (0,1)$;
8. $TVaR_\kappa(X_{(h,r)}) = \frac{\left(\frac{1}{hr}+\xi\right)\left(1-\mathrm{e}^{-h}\right)^r}{\mathrm{e}^{hr\xi}(1-\kappa)} \times {}_3F_2\left(\left[r, r, \frac{1+h(1+r)\xi}{h\xi}\right], \left[1+r, \frac{1+hr\xi}{h\xi}\right], \mathrm{e}^{-h(\xi+1)}\right)$, *where* $\xi = VaR_\kappa(X_{(h,r)})$.

Proof. For Property 1, clearly, $\lim_{h\to 0}\overline{F}_{X_{(h,r)}}(x) = \left(\frac{1}{x+1}\right)^r$, which corresponds to the survival function provided in (1.6) of the Pareto distribution (with $r = \frac{1}{\alpha}$). It implies that $X_{(h,r)} \xrightarrow{\mathcal{D}} X_{(r)}^{Pa}$ as $h \to 0$.
For Properties 3, 4, 5, and 7, see Hajebi et al. (2013) for similar results. Finally, the expressions given in Properties 2, 6, and 8 are obtained directly with Proposition 7. □

Remark. *Figures 1.3 and 1.4 show that the exponential-negative binomial mixture indeed converges to the Pareto distribution as h gets smaller.*

For a fixed $r > 0$ *and for any* $h_1 > h_2 > 0$*, note that* $\overline{F}_{X_{(h_1,r)}}(x) \leq \overline{F}_{X_{(h_2,r)}}(x) \leq \overline{F}_{X^{Pa}_{(r)}}(x)$ *for all* $x > 0$*. It implies that* $X_{(h_1,r)} \preccurlyeq_{sd} X_{(h_2,r)} \preccurlyeq_{sd} X^{Pa}_{(r)}$*, where "*$\preccurlyeq_{sd}$*" denotes the usual stochastic dominance order (see, e.g., Müller and Stoyan (2002), Denuit et al. (2005), Shaked and Shanthikumar (2007) for details on the usual stochastic dominance order and its properties). Consequently,*

$$VaR_\kappa(X_{(h_1,r)}) \leq VaR_\kappa(X_{(h_2,r)}) \leq VaR_\kappa(X^{Pa}_{(r)}) \tag{1.12}$$

for all $\kappa \in (0, 1)$*.*

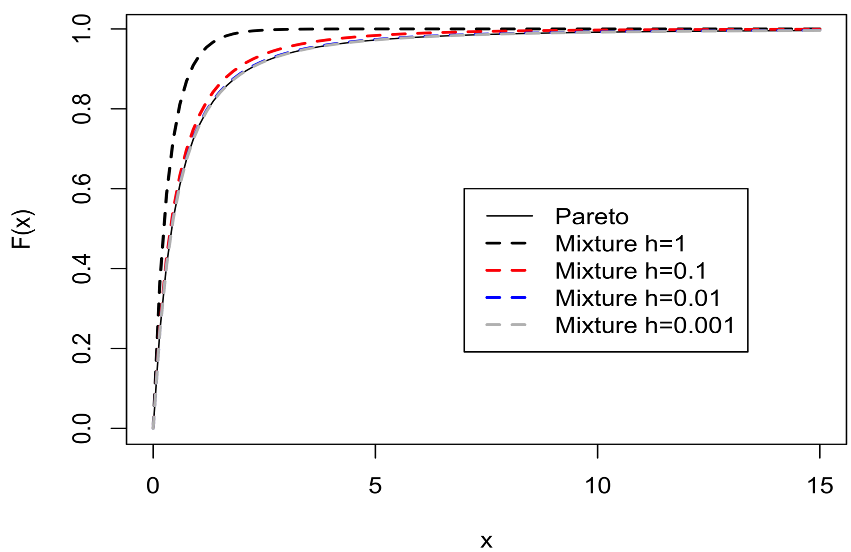

Figure 1.3 – Illustration of the convergence of the mixed exponential-negative binomial distribution to the Pareto distribution with $r = 2$.

Remark. *As expected, for a mixed exponential distribution (see, e.g., Cai (2006)), the failure rate function for the mixed exponential-negative binomial distribution given in Proposition 10 is decreasing in* x *which is very useful in reliability theory and in modeling lifetime data (see, e.g., Barlow et al. (1963) for properties of distributions with a monotone hazard rate).*

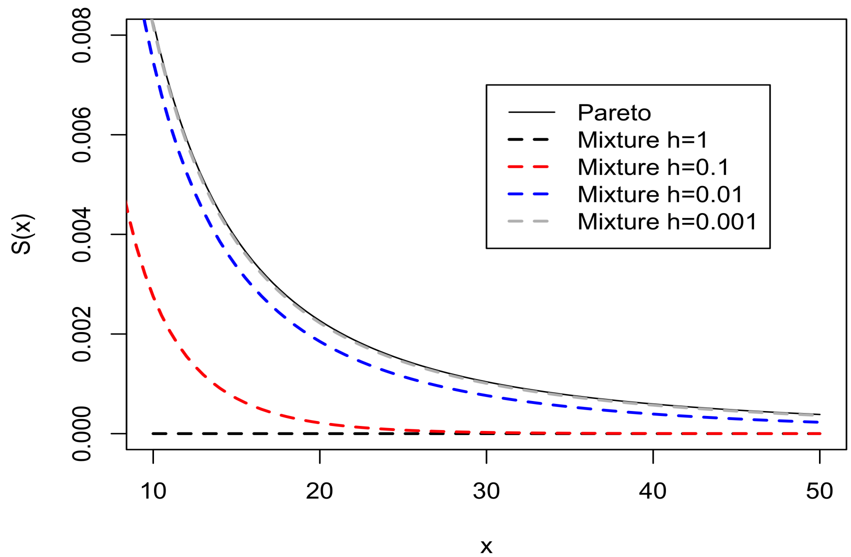

Figure 1.4 – Illustration of the convergence of the mixed exponential-negative binomial distribution to the Pareto distribution with $r = 2$.

1.4.2 Univariate mixed Exponential - Discrete Stable distribution

The discrete stable distribution was introduced by Steutel and Van Harn (1979). The properties of this discrete distribution are discussed notably in Devroye (1993), Christoph and Schreiber (1998) and Rémillard et al. (2000). Also, Devroye (1993) established a sampling algorithm to generate observations from the discrete stable distribution.

Let Θ be a rv with a discrete stable distribution with parameters $\alpha \in (0,1]$ and $\lambda > 0$, i.e., $\Theta \sim Stable(\lambda;\alpha)$. Its pgf P_N is given by

$$P_{\Theta}(t) = \exp\left\{-\lambda\left(1-t\right)^{\alpha}\right\}, \ |t| \leq 1. \tag{1.13}$$

Clearly, if $\alpha = 1$, then (1.13) becomes the pgf of a Poisson distribution with parameter λ, i.e., $\Theta \sim Poisson(\lambda)$. For this reason, we can consider the discrete stable distribution as an extension of the Poisson distribution. As explained in Christoph and Schreiber (1998), the following explicit expression for the pmf of the discrete stable distribution is obtained from (1.13)

$$\Pr\left(\Theta = k\right) = (-1)^k \sum_{j=0}^{\infty} \binom{\alpha\, j}{k} \frac{(-1)^j \lambda^j}{j!}, \tag{1.14}$$

for $k = 0, 1, \ldots$ Note that the sum in (1.14) is absolutely convergent.

The rv $\Theta \sim Stable(\lambda, \alpha)$ can be represented as a random sum as follows

$$\Theta \stackrel{D}{=} Z_1 + \ldots + Z_M, \tag{1.15}$$

where $M \sim Poisson(\lambda)$ and $\{Z_k, k = 1, 2, \ldots\}$ is a sequence of iid rvs where $Z_k \sim Z \sim Sibuya(\alpha)$, for $k = 1, 2, \ldots$, with pgf $\mathcal{P}_Z(t) = 1 - (1-t)^\alpha$. The rvs Z_k, $k = 1, 2, ...$, are also independent of the rv M. This representation can be used to recursively compute the exact values of the pmf of the discrete stable distribution.

In order to construct the univariate mixed exponential - discrete stable distribution, let Θ' be a truncated discrete stable rv with LST given by

$$\mathcal{L}_{\Theta'}(t) = \frac{\exp\{-\lambda\left(1 - \mathrm{e}^{-t}\right)^\alpha\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}. \tag{1.16}$$

In this case, combining (1.2) and (1.16), the unconditional survival function of a rv X with an exponential-discrete stable distribution is given by

$$\overline{F}_X(x) = \frac{\exp\{-\lambda\,(1 - \mathrm{e}^{-x})^\alpha\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}, \tag{1.17}$$

where $x \in \mathbb{R}^+$.

As for the negative binomial distribution presented in the previous section, we use a transformation of the rv Θ to explore the limit cases of the mixed exponential-discrete stable distribution. More precisely, let us define the rv $\Theta_{(h,\alpha)} = h \times \Theta'$. Then, using (1.16), the LST of $\Theta_{(h,\alpha)}$ is given by

$$\mathcal{L}_{\Theta_{(h,\alpha)}}(t) \quad = \quad \frac{\exp\{-\lambda\left(1 - \mathrm{e}^{-t\,h}\right)^\alpha\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}. \tag{1.18}$$

Proposition 11. *Let $\Theta^{St}_{(\alpha)}$ follow a continuous positive stable distribution with LST $\mathcal{L}_{\Theta^{St}_{(\alpha)}}(t) = \mathrm{e}^{-t^\alpha}$ and $\Theta_{(h,\alpha)}$ be a discrete rv following a discrete stable distribution with LST given in (1.18). Then,*

$$\Theta_{(h,\alpha)} \stackrel{\mathcal{D}}{\to} \Theta^{St}_{(\alpha)},$$

as $h \to 0$.

Proof. Using the Taylor series expansion of the exponential function, (1.18) becomes

$$
\begin{aligned}
\mathcal{L}_{\Theta_{(h,\alpha)}}(t) &= \frac{\exp\{-\lambda\,\left(1-\mathrm{e}^{-t\,h}\right)^{\alpha}\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}} \\
&= \frac{\exp\left\{-\lambda\,\left(1-\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(t\,h)^n}{n!}\right)\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}} \\
&= \frac{\exp\left\{-\lambda\,\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(t\,h)^n}{n!}\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}. \qquad (1.19)
\end{aligned}
$$

Letting $h=\frac{1}{\lambda^{\frac{1}{\alpha}}}$, (1.19) becomes

$$
\mathcal{L}_{\Theta_{(h,\alpha)}}(t) = \frac{\exp\left\{-t^{\alpha}\,\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(t\,h)^n}{(n+1)!}\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}. \qquad (1.20)
$$

Clearly, as $h\to 0$ (or equivalently $\lambda\to+\infty$), the LST of the mixing rv $\Theta_{(h,\alpha)}$ given in (1.20) tends to the LST of a continuous rv $\Theta^{St}_{(\alpha)}$ with a stable distribution with parameter α with LST $\mathcal{L}_{\Theta^{St}_{(\alpha)}}(t)=\mathrm{e}^{-t^{\alpha}}$, i.e.,

$$
\lim_{h\to 0}\frac{\exp\left\{-t^{\alpha}\,\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(t\,h)^n}{n!}\right)^{\alpha}\right\}-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}=\mathrm{e}^{-t^{\alpha}}. \qquad (1.21)
$$

Therefore, from (1.21), we can deduce the convergence in distribution of $\Theta_{(h,\alpha)}$ to $\Theta^{St}_{(h)}$ by using Lévy's continuity theorem. □

Proposition 12. *Let the rv $X_{(\alpha)}$ follow a mixed exponential-discrete stable distribution with mixing rv $\Theta_{(\alpha)}$ and LST as in (1.18). Then, the following properties hold:*

1. $X_{(h,\alpha)}\xrightarrow{\mathcal{D}}X^{W}$, *as $h\to 0$, where $X^{W}_{(\alpha)}\sim Weibull(\alpha,1)$;*
2. $f_{X_{(h,\alpha)}}(x)=\dfrac{\lambda\,\left(1-\mathrm{e}^{-xh}\right)^{\alpha-1}\,\alpha h\,\mathrm{e}^{-xh-\lambda\,\left(1-\mathrm{e}^{-xh}\right)^{\alpha}}}{1-\mathrm{e}^{-\lambda}},\ x\geq 0;$
3. $h_{X_{(h,\alpha)}}(x)=\dfrac{\lambda\,\left(1-\mathrm{e}^{-xh}\right)^{\alpha-1}\,\alpha h\,\mathrm{e}^{-xh-\lambda\,\left(1-\mathrm{e}^{-xh}\right)^{\alpha}}}{\mathrm{e}^{-\lambda(1-\mathrm{e}^{-hx})^{\alpha}}-\mathrm{e}^{-\lambda}},\ x\geq 0;$
4. $VaR_{\kappa}(X)=-\frac{1}{h}\ln\left(1-\left(\frac{\lambda-\ln\left((1-\kappa)(\mathrm{e}^{\lambda}-1)+1\right)}{\lambda}\right)^{\frac{1}{\alpha}}\right),\ \kappa\in(0,1).$

Proof. For property 1, combining (1.2) and (1.20) leads to the unconditional survival function of $X_{(h,\alpha)}$ written as

$$
\overline{F}_{X_{(h,\alpha)}}(x)=\frac{\exp\left\{-x^{\alpha}\,\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(x\,h)^n}{n!}\right)^{\alpha}\right\}-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}},\ x\geq 0.
$$

Clearly, as $h \to 0$, the survival function of $X_{(h,\alpha)}$ tends to the survival function of a continuous rv $X^W_{(\alpha)}$ with a Weibull distribution with scale parameter 1 and shape parameter α, i.e.,

$$\lim_{h \to 0} \overline{F}_{X_{(h,\alpha)}}(x) = \mathrm{e}^{-x^\alpha}, \ x \geq 0.$$

For properties 2, 3, and 4, the expressions are obtained directly from their definitions. □

Note that other properties of the mixed exponential-discrete stable distribution such as expectation, variance, moments and TVaR can only be obtained numerically.

Remark. *The convergence of an exponential-discrete stable distribution to a Weibull distribution was expected since a Weibull distribution arises from a mixed exponential-stable distribution. In Figures 1.5 and 1.6, an illustration of the convergence of the exponential-discrete stable distribution to the Weibull distribution is provided.*

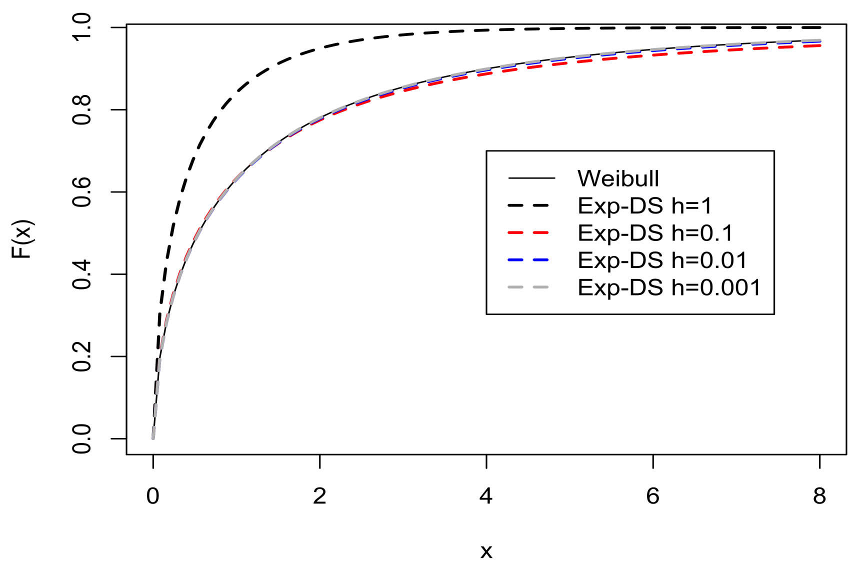

Figure 1.5 – Illustration of the convergence of the mixed exponential-discrete stable distribution to the Weibull distribution with $\alpha = 0.9$.

1.4.3 Univariate mixed Exponential - Discrete Linnik distribution

The discrete Linnik distribution was proposed by Devroye (1993) as a specific mixed Poisson distribution. This distribution is a discrete version of the Linnik distribution, originally introduced by Yu. V. Linnik in 1953 (see Linnik Yu (1963)), with characteristic function

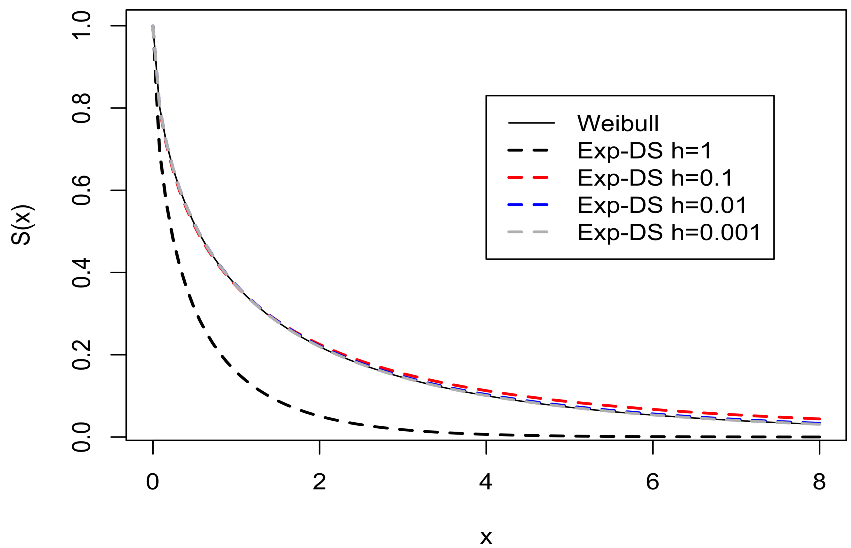

Figure 1.6 – Illustration of the convergence of the mixed exponential-discrete stable distribution to the Weibull distribution with $\alpha = 0.9$.

$\left(1+|t|^{\delta}\right)^{\beta}$. For more details about the continuous Linnik and its generalizations see, e.g., Arnold (1973), Devroye (1990), Anderson (1992), Anderson and Arnold (1993), Kozubowski (1998) and Pakes (1998).

Let M be a discrete rv with positive discrete Linnik distribution proposed by Devroye (1993) with support $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Its pgf can be written as

$$\mathcal{P}_M(t) = \left(1+\beta\,(1-t)^{\alpha}\right)^{-\gamma},\ |t| \leq 1, \tag{1.22}$$

where $\alpha \in (0,1]$, $\gamma > 0$ and $\beta > 0$.

Devroye (1993) showed that a discrete Linnik rv, with parameters α, γ and β and with pgf as in (1.22), is a conditional Poisson rv with parameter $G^{\frac{1}{\alpha}} \times S_\alpha$, where $G \sim Gamma\left(\gamma, \frac{1}{\beta}\right)$ and $S_\alpha \sim Stable(\alpha)$ with LST $\mathcal{L}_{S_\alpha}(t) = \mathrm{e}^{-t^{\alpha}}$, $t > 0$, meaning that

$$M \sim Poisson\left(G^{\frac{1}{\alpha}} \times S_\alpha\right).$$

Note that, thanks to this mixture representation of the discrete Linnik distribution, a simulation algorithm can be derived to generate observations from the discrete Linnik distribution. Also,

S_α can be generated using Kanter's method (see, e.g., Kanter et al. (1975) and Devroye (1993)).

Inspired from the results of Pakes (1998) obtained in the continuous case of a generalized Linnik distribution, Bouzar (2002) proposed an interesting generalization of the discrete Linnik distribution. He showed that such a distribution is derived from the discrete Stable distribution and proved that any discrete Linnik distribution is a mixture of negative binomial distributions.

A discrete Linnik distribution can also be represented as a compound distribution. Let $M \sim Linnik(\alpha, \beta)$. We can represent M as follows

$$M = \begin{cases} \sum_{k=1}^{N} Z_k, & N > 0 \\ 0, & N = 0 \end{cases},$$

where the primary distribution follows a negative binomial distribution, i.e, $N \sim NB(\gamma, \beta)$ with pgf $\mathcal{P}_N(t) = (1-\beta(t-1))^{-\gamma}$ and where the rvs Z_k, $k = 1, 2, \ldots$, have a Sibuya distribution with parameter α and pgf $\mathcal{P}_{Z_k}(t) = 1 - (1-t)^\alpha$. See, e.g., Cossette et al. (2001) for more details. Note that the parametrization of the negative binomial distribution here is different from the one used in Section 1.4.1.

To construct the univariate mixed exponential-discrete Linnik distribution, a transformation of the rv M with pgf in (1.22) is made here. Let the mixing rv Θ be defined as follows

$$\Theta = Z_1 + Z_2 + \ldots + Z_{N^*},$$

where $N^* = N + \gamma$ with $N \sim NB(\gamma, \beta)$ and $Z_k \sim Sibuya(\alpha)$, for $k = 1, 2, \ldots$, as defined just below.

Then, the LST of Θ can be written as

$$\mathcal{L}_\Theta(t) = \left(\frac{1 - (1 - \mathrm{e}^{-t})^\alpha}{1 + \beta\,(1 - \mathrm{e}^{-t})^\alpha} \right)^\gamma, \; t > 0. \tag{1.23}$$

Note that, if $\gamma = 1$, then Θ is said to follow a discrete Mittag-Leffler distribution (see, e.g., Pillai and Jayakumar (1995)).

Combining (1.2) and (1.23), the unconditional survival function of a rv X with an exponential-discrete Linnik distribution is given by

$$\overline{F}_X(x) = \left(\frac{1 - (1 - \mathrm{e}^{-x})^\alpha}{1 + \beta\,(1 - \mathrm{e}^{-x})^\alpha} \right)^\gamma, \; x > 0. \tag{1.24}$$

Proposition 13. *Let the rv X follow a mixed exponential-discrete Linnik distribution with univariate survival function as in (1.24). Then, the following properties hold:*

1. $f_X(x) = \dfrac{\gamma\,\alpha\,(1+\beta)\left((1-\mathrm{e}^{-x})^{2\,\alpha}\,\beta + (1-\mathrm{e}^{-x})^{\alpha}\right)}{(1-(1-\mathrm{e}^{-x})^{\alpha})\,(1+\beta\,(1-\mathrm{e}^{-x})^{\alpha})^2\,(\mathrm{e}^{x}-1)}\left(\dfrac{1-(1-\mathrm{e}^{-x})^{\alpha}}{1+\beta\,(1-\mathrm{e}^{-x})^{\alpha}}\right)^{\gamma},\ x>0.$
2. *Failure rate* $h_X(x) = \dfrac{\gamma\,\alpha\,(1+\beta)\left((1-\mathrm{e}^{-x})^{2\,\alpha}\,\beta + (1-\mathrm{e}^{-x})^{\alpha}\right)}{(1-(1-\mathrm{e}^{-x})^{\alpha})\,(1+\beta\,(1-\mathrm{e}^{-x})^{\alpha})^2\,(\mathrm{e}^{x}-1)},\ x>0.$
3. $VaR_\kappa(X) = -\ln\left(1-\left(\dfrac{1-s^{\frac{1}{\gamma}}}{1+s^{\frac{1}{\gamma}}\beta}\right)^{\frac{1}{\alpha}}\right).$

Proof. The expressions follow directly from their definitions. □

Remark. *As expected, the failure rate of a mixed exponential-discrete Linnik distribution is decreasing in x.*

Note that the expectation, the moments in addition to other properties given in Proposition 7, can only be obtained numerically.

In order to study limit cases, we need to define a family of distributions introduced by Joe (2014).

Definition 14. Let Z be a positive rv with LST $\mathcal{L}_Z$ and let Θ be a positive rv such that, given $Z=z$, the conditional LST of Θ is written as

$$\mathcal{L}_{\Theta|Z=z}(t) = \left(\mathcal{L}_W(t)\right)^z,\ z>0 \textit{ and } t>0,$$

where W is a positive rv with LST $\mathcal{L}_W$.

Then, the unconditional distribution of Θ is referred to as "the distribution of Z"-stopped-"the distribution of W" with LST

$$\mathcal{L}_\Theta(t) = E\left[\left(\mathcal{L}_W(t)\right)^Z\right] = \mathcal{L}_Z\left(-\ln\left(\mathcal{L}_W(t)\right)\right).$$

For example, if $Z \sim Gamma(\gamma,1)$ and $W \sim Sibuya(\alpha)$, then Θ follows a gamma-stopped-Sibuya distribution with parameters α and γ and LST given by

$$\mathcal{L}_\Theta(t) = \left(1-\ln\left(1-\left(1-\mathrm{e}^{-t}\right)^{\alpha}\right)\right)^{-\gamma}.$$

Also, if $Z \sim BN(\gamma,\beta)$ and $W \sim Sibuya(\alpha)$, then Θ follows a negative binomial-stopped-Sibuya distribution with parameters α, β and γ and LST given by

$$\mathcal{L}_\Theta(t) = \left(\frac{1-(1-\mathrm{e}^{-t})^{\alpha}}{1+\beta\,(1-\mathrm{e}^{-t})^{\alpha}}\right)^{\gamma},$$

which is exactly the form of the LST of a discrete Linnik distribution given in (1.23). Therefore, the discrete Linnik distribution coincides with the negative binomial-stopped-Sibuya distribution. Since there is a link between negative binomial distributions and gamma distributions, we use the same representation of the negative binomial distribution as in Section 1.4.1 to verify if such a link still holds when these two distributions are involved in distributions of the form "the distribution of Z"-stopped-"the distribution of W". To do so, let $\beta = \frac{1-q}{q}$, where $q = 1 - \mathrm{e}^{-h}$. Then, the mixing rv $\Theta_{(h,\alpha,\gamma)}$ can be represented as follows

$$\Theta_{(h,\alpha,\gamma)} = Z_1 + Z_2 + \ldots + Z_{N^*}, \tag{1.25}$$

where $N^* = h \times (N + \gamma)$ with $N \sim NB(\gamma, \frac{1}{\mathrm{e}^h - 1})$. Since $\Theta_{(h,\alpha,\gamma)}$ follows a negative binomial-stopped-Sibuya distribution with parameters α, β and γ, its LST is given by

$$\mathcal{L}_{\Theta_{(h,\alpha,\gamma)}}(t) = E\left[(\mathcal{L}_Z(t))^{N^*}\right] = \mathcal{L}_{N^*}\left(-\ln\left(\mathcal{L}_Z(t)\right)\right), \tag{1.26}$$

where $Z \sim Sibuya(\alpha)$.

Proposition 15. *Let $X_{(h,\alpha,\gamma)}$ follow a univariate mixed exponential-discrete Linnik distribution with mixing rv $\Theta_{(h,\alpha,\gamma)}$ as defined in (1.25) and (1.26). Then,*

$$\Theta_{(h,\alpha,\gamma)} \overset{\mathcal{D}}{\to} \Theta^{GsS},$$

where Θ^{GsS} follows a gamma-stopped-Sibuya distribution with parameters α and γ. Also, when $h \to 0$

$$X_{(h,\alpha,\gamma)} \overset{\mathcal{D}}{\to} X^{GsS},$$

where X^{GsS} follows a mixed exponential distribution for which the mixing rv Θ^{GsS} follows a gamma-stopped-Sibuya distribution with parameters α and γ.

Proof. We know that $N^* \overset{\mathcal{D}}{\to} Y$, where $Y \sim Gamma(\gamma, 1)$ when $h \to 0$, and $-\ln\left(\mathcal{L}_Z(t)\right) > 0$, $\forall t > 0$. Then,

$$\begin{aligned} \lim_{h\to 0} \mathcal{L}_{\Theta_{(h,\alpha,\gamma)}}(t) &= \lim_{h\to 0} \mathcal{L}_{N^*}\left(-\ln\left(\mathcal{L}_Z(t)\right)\right) \\ &= \mathcal{L}_Y\left(-\ln\left(\mathcal{L}_Z(t)\right)\right). \end{aligned}$$

We conclude that, $\Theta_{(h,\alpha,\gamma)} \overset{\mathcal{D}}{\to} \Theta^{GsS}$ and then $X_{(h,\alpha,\gamma)} \overset{\mathcal{D}}{\to} X^{GsS}$. □

Another interesting limit case is discussed in the following Proposition.

Proposition 16. *Let $\Theta_{h,\alpha,\gamma} = h \times \Theta$ where Θ follows a discrete Linnik distribution with LST as given in (1.23) and let Θ^{CL} be a positive rv with continuous positive Linnik distribution (with parameters $\alpha \in (0,1]$, $\lambda > 0$ and $\gamma > 0$) and LST given by*

$$\mathcal{L}_{\Theta^{CL}}(t) = (1 + \lambda t^{\alpha})^{-\gamma}, \ t > 0. \tag{1.27}$$

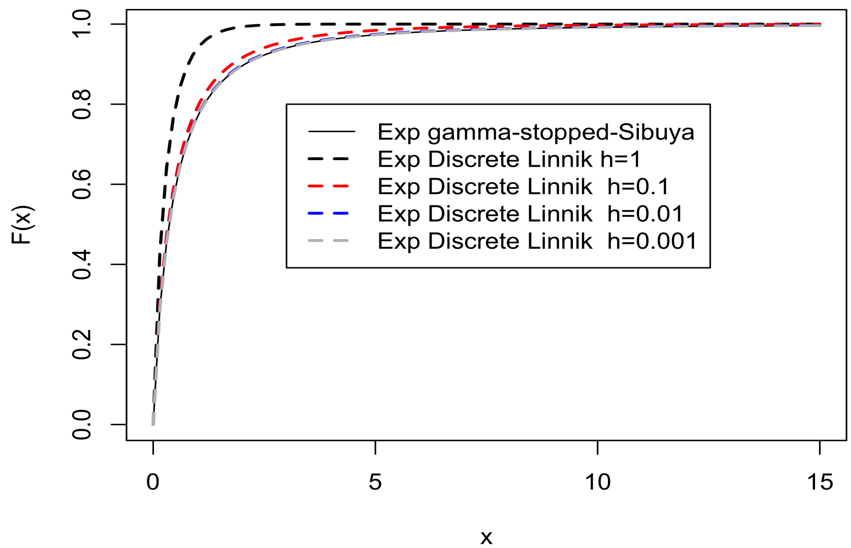

Figure 1.7 – Illustration of the convergence of the mixed exponential-discrete Linnik distribution to the Gamma-stopped-Sibuya distribution with $\alpha = 0.9$ and $\gamma = 2$.

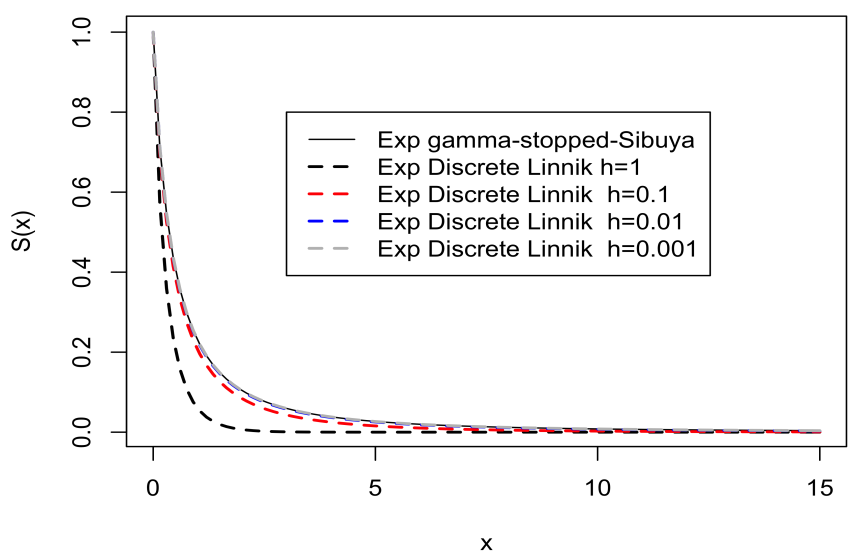

Figure 1.8 – Illustration of the convergence of the mixed exponential-discrete Linnik distribution to the Gamma-stopped-Sibuya distribution with $\alpha = 0.9$ and $\gamma = 2$.

Then, when $h \to 0$, i.e., $\beta \to \infty$, we have

$$\Theta_{h,\alpha,\gamma} \stackrel{\mathcal{D}}{\to} \Theta^{CL}.$$

Let $X_{h,\alpha,\gamma}$ and X^{CL} two positive rvs following mixed exponential distributions with mixing rv $\Theta_{h,\alpha,\gamma}$ and Θ^{CL} respectively. Then, when $h \to 0$, we have

$$X_{h,\alpha,\gamma} \stackrel{\mathcal{D}}{\to} X^{CL}.$$

Proof. We have

$$\begin{aligned}
\mathcal{L}_{\Theta_{h,\alpha,\gamma}}(t) &= \left(\frac{1-(1-\mathrm{e}^{-th})^{\alpha}}{1-\beta(1-\mathrm{e}^{-th})^{\alpha}}\right)^{\gamma} \\
&= \left(1-(1-\mathrm{e}^{-th})^{\alpha}\right)^{\gamma} \times \left(1-\beta\left(1-\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}(ht)^n\right)\right)^{\alpha}\right)^{-\gamma} \\
&= \left(1-(1-\mathrm{e}^{-th})^{\alpha}\right)^{\gamma} \times \left(1-\beta\, h^{\alpha}\, t^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}(ht)^n\right)^{\alpha}\right)^{-\gamma}.
\end{aligned}$$

By letting $h = \left(\frac{\lambda}{\beta}\right)^{\frac{1}{\alpha}}$, we obtain

$$\mathcal{L}_{\Theta_{h,\alpha,\gamma}}(t) = \left(1-(1-\mathrm{e}^{-th})^{\alpha}\right)^{\gamma} \times \left(1-\lambda\, t^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}(ht)^n\right)^{\alpha}\right)^{-\gamma}.$$

Then,

$$\lim_{h\to 0}\mathcal{L}_{\Theta_{h,\alpha,\gamma}}(t) = (1+\lambda t^{\alpha})^{-\gamma} = \mathcal{L}_{\Theta^{CL}}(t).$$

Therefore, we conclude that $\Theta_{h,\alpha,\gamma} \stackrel{\mathcal{D}}{\to} \Theta^{CL}$. □

Remark. *Note that Proposition 16 still holds when $\gamma = 1$. In this case, $\Theta_{h,\alpha,\gamma}$ and Θ^{CL} follow, respectively, discrete and continuous Mittag-leffler distributions.*

1.4.4 Classical renewal risk model with mixed exponential interclaim times

We present in this section another motivation for the class of univariate mixed exponential distributions. More precisely, we consider the classical Sparre-Andersen renewal risk model in which inter-claim times are assumed to follow a mixed exponential distribution. For an insurance portfolio, the surplus process is defined by $\underline{U} = \{U(t), t \geq 0\}$ where the surplus level at time t, $U(t)$, is given by

$$U(t) = u + ct - S(t),$$

where $U(0) = u$ is the initial surplus and c is the premium rate. The aggregate claim amount process, denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ is a compound renewal process with iid inter-claim times. The claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a renewal process where the inter-claim times $\underline{W} = \{W_j, j \in \mathbb{N}\}$ form a sequence of iid and strictly positive real-valued rvs. The time between the $(j-1)$th and the jth claim $(j = 2, ...)$ is defined by the rv W_j with W_1 the time of the first claim. The rvs $\{W_j, j \in \mathbb{N}\}$, are identically distributed as the canonical rv W, have pdf f_W, cdf F_W, and survival function $\overline{F}_W$. The univariate rv W follows a univariate mixed exponential distribution.

The time of arrival of the jth claim is denoted $T_j = W_1 + \cdots + W_j$. The claim amount rvs $\underline{X} = \{X_j, j \in \mathbb{N}\}$, where X_j corresponds to the amount of the jth claim, are assumed to be a sequence of strictly positive and iid rvs with pdf f_X and cdf F_X. The sequences $\underline{W}$ and $\underline{X}$ are independent.

The time of ruin is defined by the rv $\tau_u = \inf\{t \geq 0 : U(t) < 0\}$ with $\tau_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The infinite-time ruin probability is $\zeta(u) = \Pr(\tau_u < \infty | U(0) = u)$. Throughout this section, we assume the positive security loading condition $E[cW - X] > 0$ to be verified which ensures that ruin will not occur almost surely.

We limit our analysis to exponentially distributed claim amounts with parameter β, which implies (see, e.g., Asmussen and Albrecher (2010))

$$\zeta(u) = \frac{\beta - \rho}{\beta} \mathrm{e}^{-\rho u}, \; u \geq 0, \tag{1.28}$$

where ρ is the adjustment coefficient which is the smallest strictly positive solution to the Lundberg relation

$$E\left[\mathrm{e}^{r(X-cW)}\right] = E\left[\mathrm{e}^{rX}\right] \times E\left[\mathrm{e}^{-rcW}\right] = 1. \tag{1.29}$$

Let us look at a numerical example in which we compare the adjustment coefficients obtained for different choices of mixed exponential distributions for the inter-claim times

Example 17. Let W follows a univariate mixed exponential distribution with expectation 1 and let $X \sim Exp(1)$. We consider a scale parameter β_s such that $W = \beta_s Y$ where Y follows a univariate mixed exponential distribution with expectation $1/\beta_s$ and with $\overline{F}_W$ as given in (1.2). Let the mixing rv Θ follow one of the distributions considered in this paper (i.e., negative binomial, gamma, Sibuya, discrete stable, stable, Linnik, and logarithmic distributions), and let $c = 1.25$. Table 1.1 lists the values of the adjustment coefficient ρ for different mixing distributions. As we can see, the smaller the h becomes, the closer become the adjustment coefficients related to both the exponential negative binomial distribution and the Pareto distribution (exponential gamma distribution). Such a result is expected since the exponential-negative binomial distribution converges to the Pareto distribution. Also, the values of ρ differ from one distribution to another.

Θ's distribution	Parameters	ρ
Negative binomial	$r = 2,\ h = 0.01,\ \beta_s = 1.04335$	0.09051169
	$r = 2,\ h = 0.001,\ \beta_s = 1.006456$	0.08110895
	$r = 2,\ h = 0.0001,\ \beta_s = 1.000872$	0.07934719
Gamma	$\alpha = 2,\ \beta_s = 1$	0.07903681
Sibuya	$\alpha = 0.5,\ \beta_s = 1.629446$	0.4909645
Logarithmic	$\alpha = 0.1,\ \beta_s = 0.97527775$	0.1570996
Discrete stable	$\alpha = 0.5,\ h = 0.01,\ \beta_s = 0.4653859$	0.0858715
	$\alpha = 0.5,\ h = 0.001,\ \beta_s = 0.4969561$	0.09494302
	$\alpha = 0.5,\ h = 0.001,\ \beta_s = 0.4996998$	0.09565799
Discrete Linnik	$\alpha = 0.5,\ \gamma = 2\ h = 0.01,\ \beta_s = 1.647551$	0.1164167
	$\alpha = 0.5,\ \gamma = 2\ h = 0.001,\ \beta_s = 1.612658$	0.1163437
	$\alpha = 0.5,\ \gamma = 2\ h = 0.0001,\ \beta_s = 1.60928$	0.1163387
Stable	$\alpha = 2,\ \beta_s = 0.8862269$	0.2996294

Table 1.1 – Adjustment coefficients for different distributions.

1.5 From multivariate mixed exponential distributions to Archimedean copulas

A multivariate mixed exponential distribution can be constructed with the elegant procedure introduced by Oakes (1989) and Marshall and Olkin (1988). Let $\underline{X} = (X_1, \ldots, X_n)$ be an n-dimensional random vector. The dependence between the rvs $X_1, \ldots, X_n$ is introduced via the common mixing strictly positive rv Θ which can be discrete or continuous. Given $\Theta = \theta$, the conditional rvs $(X_1|\Theta = \theta), \ldots, (X_n|\Theta = \theta)$ are conditionally independent with $(X_i|\Theta = \theta) \sim Exp(\theta)$, for $i = 1, \ldots, n$. It implies that

$$\bar{F}_{\underline{X}|\Theta=\theta}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathrm{e}^{-\theta\, x_i} = \exp\left\{-\theta(x_1 + \ldots + x_n)\right\}. \tag{1.30}$$

Then, using (1.1) with (1.30), the joint survival function of $\underline{X}$ is given by

$$\bar{F}_{\underline{X}}(x_1, \ldots, x_n) = \mathcal{L}_{\Theta}\left(x_1 + \ldots + x_n\right). \tag{1.31}$$

Multivariate mixed exponential distributions have been frequently used in actuarial science, quantitative risk management and reliability theory, see, e.g., Sarabia et al. (2017), Dacorogna et al. (2016), and Albrecher et al. (2011a).

A prominent problem in actuarial science and quantitative risk management is to determine the share C of the total capital to be allocated to each risk. Different capital allocation methods were proposed in the literature. We provide in Proposition 18 explicit formulas in terms of Θ for different capital allocation methods proposed in Furman and Zitikis (2008).

Proposition 18 lists several general properties of a multivariate mixed exponential distribution. The conditional independence underlying this multivariate dependence model allows to better study the relationship between risks and to calculate different expectations of the form $E\left[g(X_i, X_j)\right]$, for $i, j \in \{1, \ldots, n\}$, where g is a bivariate function. It also allows to derive different capital allocation formulas without knowing the distribution of the global risk $S = X_1 + \ldots + X_n$ (see properties 6 - 8).

Proposition 18. *Consider $\underline{X}$ to be an n-dimensional vector with multivariate mixed exponential distribution and joint survival function as given in (1.31). Let $S = X_1 + \ldots + X_n$ be the aggregated risk. Then, properties of such a distribution can be written in terms of the mixing rv Θ as follows (provided that the expectations exist):*

1. *Joint density function* $f_{\underline{X}}(x_1, \cdots, x_m) = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}x_1 \ldots \mathrm{d}x_m} \mathcal{L}_\Theta(x)$;
2. *Joint moments* $E[X_1^{n_1} \ldots X_m^{n_m}] = \prod_{i=1}^{m} \Gamma(n_i + 1) \times E\left[\Theta^{-(n_1 + \ldots + n_m)}\right]$;
3. *Covariance* $Cov\left(X_1, X_2\right) = E\left[\Theta^{-2}\right] - E^2\left[\Theta^{-1}\right]$;
4. *Pearson's correlation coefficient* $\rho\left(X_1, X_2\right) = \frac{E\left[\Theta^{-2}\right] - E^2\left[\Theta^{-1}\right]}{2\,E\left[\Theta^{-2}\right] - E^2\left[\Theta^{-1}\right]}$;
5. *Kendall's tau* $\tau\left(X_1, X_2\right) = 1 + 4\int_0^1 \frac{\mathcal{L}_\Theta^{-1}(t)}{\left(\mathcal{L}_\Theta^{-1}(t)\right)'} \mathrm{d}t = 1 - 4\int_0^\infty t \times \left(\mathcal{L}'_\Theta(t)\right)^2 \mathrm{d}t$;
6. *MCov capital allocation* $C_\omega(X_i, S) = \frac{n+1}{n} \times \frac{E\left[\Theta^{-2}\right]}{E\left[\Theta^{-1}\right]}$, $i \in \{1, 2, \ldots, n\}$;
7. *Esscher's capital allocation* $C_\omega(X_i, S) = \frac{E\left[\Theta(\Theta - t)^{-2}\right] E\left[\left(\frac{\Theta}{\Theta - t}\right)^{n-1}\right]}{E\left[\left(\frac{\Theta}{\Theta - t}\right)^{n}\right]}$, $i \in \{1, 2, \ldots, n\}$;
8. *Kamps's capital allocation* $C_\omega(X_i, S) = \frac{E\left[\Theta^{-1}\right] - E\left[\Theta(\Theta + t)^{-2}\right] \times E\left[\left(\frac{\Theta}{\Theta + t}\right)^{n-1}\right]}{1 - E\left[\left(\frac{\Theta}{\Theta + t}\right)^{n}\right]}$, $i \in \{1, 2, \ldots, n\}$.

Proof. Properties 1 - 5 are known results (see, e.g., Sarabia et al. (2017)). Expressions of capital allocations given in properties 6 - 8 are obtained by first conditioning with respect to Θ and then using the fact that $X|\Theta = \theta \sim Exp(\theta)$. We use also the fact that $S|\Theta = \theta \sim Gamma(n, \theta)$. For example, we have

For MCov capital allocation method, we have

$$
\begin{aligned}
C_\omega(X_i, S) &= E[X_i] + \frac{Cov(X_i, S)}{E[S]} \\
&= E\left[\Theta^{-1}\right] + \frac{E\left[X_i \times S\right] - n \times E\left[\Theta^{-1}\right]^2}{n \times E\left[\Theta^{-1}\right]} \\
&= E\left[\Theta^{-1}\right] + \frac{E\left[X_i \times (S_{-i} + X_i)\right] - n \times E\left[\Theta^{-1}\right]^2}{n \times E\left[\Theta^{-1}\right]},
\end{aligned}
$$

where $i = 1, \ldots, n$ and $S_{-i} = X_1 + \ldots + X_{i-1} + X_{i+1} + \ldots + X_n$. Since $S_{-i}|\Theta = \theta \sim Gamma(n-1, \theta)$, we have

$$
\begin{aligned}
E\left[X_i \times (S_{-i}+X_i)\right] &= E\left[E\left[X_i \times (S_{-i}+X_i)|\Theta\right]\right] \\
&= E\left[E\left[X_i \times S_{-i}|\Theta\right]\right] + E\left[E\left[X_i^2|\Theta\right]\right] \\
&= E\left[E\left[X_i|\Theta\right] \times E\left[S_{-i}|\Theta\right]\right] + E\left[E\left[X_i^2|\Theta\right]\right] \\
&= E\left[\Theta^{-1} \times (n-1)\,\Theta^{-1}\right] + 2\,E\left[\Theta^{-2}\right] \\
&= (n+1)\,E\left[\Theta^{-2}\right].
\end{aligned}
$$

Then, the MCov capital allocation can be written as

$$
C_\omega(X_i,S) = \frac{n+1}{n} \times \frac{E\left[\Theta^{-2}\right]}{E\left[\Theta^{-1}\right]}.
$$

For Esscher's capital allocation, we have

$$
\begin{aligned}
C_\omega(X_i,S) &= \frac{E\left[X_i\,\mathrm{e}^{t\,S}\right]}{E\left[\mathrm{e}^{t\,S}\right]} \\
&= \frac{E\left[E\left[X_i\,\mathrm{e}^{t\,S}|\Theta\right]\right]}{E\left[E\left[\mathrm{e}^{t\,S}|\Theta\right]\right]} \\
&= \frac{E\left[E\left[X_i\,\mathrm{e}^{t\,(S_{-i}+X_i)}|\Theta\right]\right]}{E\left[\mathcal{L}_S(-t)\right]} \\
&= \frac{E\left[E\left[X_i\,\mathrm{e}^{t\,X_i}|\Theta\right] \times E\left[\mathrm{e}^{t\,S_{-i}}|\Theta\right]\right]}{E\left[\mathcal{L}_S(-t)\right]} \\
&= \frac{E\left[E\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t\,X_i}\right] \times \mathcal{L}_{S_{-i}}(-t)\right]}{E\left[\mathcal{L}_S(-t)\right]} \\
&= \frac{E\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Theta}{\Theta-t}\right)\right] \times E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n-1}\right]}{E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n}\right]} \\
&= \frac{E\left[\Theta\,(\Theta-t)^{-2}\right] \times E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n-1}\right]}{E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n}\right]},
\end{aligned}
$$

for $i \in \{1,2,\ldots,n\}$. The procedure is exactly the same for property 8. □

The multivariate model defined in (1.30) and (1.31) can also be constructed with completely monotone marginals joined by a dependence structure defined with an Archimedean copula with generator $\mathcal{L}_\Theta$. The equivalence of these two models is shown notably in Marshall and Olkin (1988), Oakes (1989) and Albrecher et al. (2011a). In other words, an Archimedean copula $\overline{C}$ with generator $\mathcal{L}_\Theta$ can be constructed, using Sklar's theorem, from a multivariate mixed exponential distribution as defined in Section 1.5 and marginals with univariate mixed exponential distribution described in Section 1.4, i.e.,

$$
\overline{F}^{ME}_{\underline{X}}(x_1,\ldots,x_n) = \overline{C}\left(\overline{F}^{ME}_{X_1}(x_1),\ldots,\overline{F}^{ME}_{X_n}(x_n)\right)
$$

and

$$\overline{C}\left(u_1,...,u_n\right)=\mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(u_1\right)+...+\mathcal{L}_\Theta^{-1}\left(u_n\right)\right), \tag{1.32}$$

where $\overline{F}_{\underline{X}}^{ME}$ and $\overline{F}_{X_i}^{ME}$, $i=1,\ldots,n$, are as given in (1.31) and (1.2) respectively.

This method of construction, based on Marshall and Olkin (1988), provides the natural sampling algorithm given below based on the idea that, given Θ, all rvs X_i, $i=1,\ldots,n$, are iid and exponentially distributed with mean 1. Such an Algorithm will be used in the examples of the following section.

Algorithm 19. *Let C be a d-dimensional Archimedean copula with generator $\mathcal{L}_\Theta$.*

1. *Generate Θ with LST $\mathcal{L}_\Theta$;*
2. *Generate $R_i\sim Exp(1)$ for $i=1,\ldots,d$;*
3. *Produce $U_i=\mathcal{L}_\Theta\left(\frac{R_i}{\Theta}\right)$, for $i=1,\ldots,d$;*
4. *Return $\underline{U}=(U_1,\ldots,U_d)$.*

Since in a multivariate model incorporating dependence using a copula, the dependence relationship between the rvs of this model is fully described by the copula, it would be interesting to see how this dependence relationship behaves as a result of a variation of the parameter of dependence. For this purpose, we need to introduce the concept of concordance ordering as defined in Joe (1997) page 37.

Definition 20. Let C_1 and C_2 be two d-dimensional copulas with respective Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. C_2 is more **concordant** that C_1, written $C_1\prec_c C_2$, if

$$C_1\left(\underline{u}\right)\leq C_2\left(\underline{u}\right) \text{ and } \overline{C_1}\left(\underline{u}\right)\leq\overline{C_2}\left(\underline{u}\right),$$

for $\underline{u}\in[0,1]^d$. As a result of such a concordance ordering, $\tau_1\leq\tau_2$, $\rho_S^{(1)}\leq\rho_S^{(2)}$, and $\lambda_1\leq\lambda_2$ (see Joe (1997)).

If C_1 and C_2 are Archimedean copulas with respective generators $\mathcal{L}_1$ and $\mathcal{L}_2$, Theorems 4.1 and 4.7 in Joe (1997) prove that the condition $\left(\mathcal{L}_1^{-1}\circ\mathcal{L}_2\right)'$ is a completely monotone function guarantees that $C_1\prec_c C_2$ is verified.

For an Archimedean copula with mixing rv Θ such that $F_\Theta(0)=0$ and generator $\mathcal{L}_\Theta$, if the lower and upper tail-dependence coefficients exist, then, according to Joe and Hu (1996), λ_L and λ_U can be written in terms of $\mathcal{L}_\Theta$ as follows

$$\lambda_L = \lim_{t\to\infty} \frac{\mathcal{L}_\Theta(2t)}{\mathcal{L}_\Theta(t)} = 2\lim_{t\to\infty} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}$$

and

$$\lambda_U = 2 - \lim_{t\to 0} \frac{1-\mathcal{L}_\Theta(2t)}{1-\mathcal{L}_\Theta(t)} = 2 - 2\lim_{t\to 0} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}.$$

For the special case of discrete mixing rvs, the related Archimedean copulas cannot have lower tail dependence, i.e., $\lambda_L = 0$. Also, if $E[\Theta]$ is finite, then $\lambda_U = 0$ (see Hofert (2010) page 62 for proof).

1.5.1 Multivariate mixed Exponential - Negative Binomial Distribution

Once again, to investigate the limit cases of the multivariate exponential-negative binomial distribution, we will use the representation and the parametrization of the negative binomial distribution provided in Section 1.4.1. Consequently, let the mixing rv Θ follow a negative binomial distribution with LST given in (1.8). Consider $\underline{X} = (X_1, \ldots, X_n)$ a vector of n rvs for which the dependence is introduced via the mixing rv Θ. In this case, combining (1.8) and (1.31), the joint survival function of $\underline{X}$ can be written as

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \left(\frac{1-\mathrm{e}^{-h}}{\mathrm{e}^{\sum_{i=1}^n x_i h} - \mathrm{e}^{-h}}\right)^r, \; x_i \in \mathbb{R}^+, \; i = 1, \ldots, n. \tag{1.33}$$

Proposition 21. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential negative binomial distribution with joint survival function given in (1.33). Then, the following properties hold:*

1. *$\underline{X} \overset{\mathcal{D}}{\to} \underline{Y}$, as $h \to 0$, where $\underline{Y}$ follows a multivariate Pareto of the second kind;*
2. *Joint moments $E[X_1^{m_1} \ldots X_n^{m_n}] = \frac{(1-\mathrm{e}^{-h})^r}{h^m r^m} \, {}_{m+1}F_m\left([r, \ldots, r], [1+r, \ldots, 1+r], \mathrm{e}^{-h}\right) \times \prod_{i=1}^n \Gamma(m_i+1)$, where $m = m_1 + \ldots + m_n$;*
3. *$Cov(X_1, X_2) = \frac{-\left(1-\mathrm{e}^{-h}\right)^{2r} \left({}_2F_1([r,r],\,[r+1],\,\mathrm{e}^{-h})\right)^2 + \left(1-\mathrm{e}^{-h}\right)^r {}_3F_2([r,r,r],\,[r+1,r+1],\,\mathrm{e}^{-h})}{h^2\, r^2}$;*
4. *Pearson's correlation coefficient $\rho_P(X_1, X_2) = \frac{\left(1-\mathrm{e}^{-h}\right)^r \left({}_2F_1(r,r;\,r+1;\,\mathrm{e}^{-h})\right)^2 - {}_3F_2(r,r,r;\,r+1,r+1;\,\mathrm{e}^{-h})}{\left(1-\mathrm{e}^{-h}\right)^r \left({}_2F_1(r,r;\,r+1;\,\mathrm{e}^{-h})\right)^2 - 2\,{}_3F_2(r,r,r;\,r+1,r+1;\,\mathrm{e}^{-h})}$.*

Proof. For property 1, clearly, the joint survival function of $\underline{X}$ given in (1.33) tends to the joint survival function of a Pareto of the second kind distribution, i.e.,

$$\lim_{h\to 0} \left(\frac{1-\mathrm{e}^{-h}}{\mathrm{e}^{\sum_{i=1}^n x_i h} - \mathrm{e}^{-h}}\right)^r = \left(1 + \sum_{i=1}^n x_i\right)^{-r}$$

where $x_i \in \mathbb{R}^+$, $i = 1, \ldots, n$. See, e.g., Arnold (1983) and Arnold (2015) for more details concerning the Pareto of the second kind distribution.

Properties 2, 3, and 4 are directly obtained from their definitions. □

Let us now consider a special case where $r = 1$. Then, the resulting multivariate distribution related to the survival function in (1.33) is a multivariate extension of the mixed exponential-geometric distribution proposed by Adamidis and Loukas (1998). In this case, the mixing rv Θ follows a geometric distribution with parameter q and pmf given by

$$\Pr(\Theta = k) = q \times (1-q)^{k-1},\ k \in \mathbb{N}.$$

Since the limit case, when $h \to 0$, of such a distribution is already discussed in Proposition 21, there is no need for the representation using $q = 1 - \mathrm{e}^{-h}$. Then, the joint survival function in (1.33) becomes

$$\bar{F}_{\underline{X}}(\underline{x}) = \frac{q}{\exp\left(\sum_{i=1}^{m} x_i\right) - (1-q)}, \tag{1.34}$$

where $\underline{X}$ now follows a multivariate mixed exponential-geometric distribution with parameter q.

Since it is easier to work with a geometric distribution than a negative binomial one, more explicit formulas of different properties of the resulting multivariate mixed distribution can be found as shown in the following Proposition.

Proposition 22. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential geometric distribution with joint survival function given in (1.34). Then, the following properties hold:*

1. *Joint density function*
$$f_{\underline{X}}(\underline{x}) = \sum_{k=0}^{m-1} (-1)^{2m-k} (m-k)!\, S_2(m, m-k) \frac{\left(\mathrm{e}^{x_1+\ldots+x_m}\right)^{m-k}}{\left(\mathrm{e}^{x_1+\ldots+x_m} - (1-q)\right)^{m+1-k}};$$
2. *Joint moments* $E[X_1^{n_1} \ldots X_m^{n_m}] = \prod_{i=1}^{m} \Gamma(n_i+1) \times \frac{q\, Li_d(1-q)}{1-q}$*, where* $d = n_1 + \ldots + n_m$*;*
3. *Covariance* $Cov\,(X_1, X_2) = \frac{q\, Li_2(1-q)}{1-q} - \frac{q^2\,(\ln(q))^2}{(1-q)^2}$*;*
4. *Pearson's correlation coefficient* $\rho\,(X_1, X_2) = \frac{\frac{q\, Li_2(1-q)}{1-q} - \frac{q^2 (\ln(q))^2}{(1-q)^2}}{2\frac{q\, Li_2(1-q)}{1-q} - \frac{q^2 (\ln(q))^2}{(1-q)^2}}$*;*
5. *Kendall's tau* $\tau\,(X_1, X_2) = \frac{3(1-q)^2 - 2(1-q) - 2q^2 \ln(q)}{3\,(1-q)^2}$,

where "Li" denotes the general polylogarithm function defined as $Li_a(z) = \sum_{d=1}^{\infty} \frac{z^d}{d^a}$*. Note that the polylogarithm function is already built in R, Maple, Matlab.*

Proof. For property 5, see, e.g., Hofert (2010). For all other properties, the expressions follow directly from their definitions. □

Now, we use the link between Archimedean copulas and multivariate mixed exponential distributions to construct a multivariate Archimedean copula using a mixing rv with negative binomial distribution. We also show that such a copula can be seen as an extension of the well-known Clayton copula.

Proposition 23. *Let $\Theta_{(h,r)}$ be a discrete rv with negative binomial distribution with LST as given in (1.8). The associated multivariate Archimedean copula with generator $\mathcal{L}_{\Theta_{(h,r)}}$, called the multivariate negative binomial copula, is given by*

$$C_{r,q}(u_1, \cdots, u_n) = \left(q \left(\prod_{i=1}^{n} \left(q u_i^{\frac{-1}{r}} + (1-q) \right) - (1-q) \right)^{-1} \right)^{r}, \tag{1.35}$$

where $q = 1 - \mathrm{e}^{-h}$.

Proof. The result follows by combining (1.8) and (1.32). □

Remark. *The negative binomial copula has some interesting limit cases. When $r = 1$, it corresponds to an AMH copula, i.e. we have $C_{1,q}(\underline{u}) = C_{1-q}^{AMH}(\underline{u})$. If $r = 0$ or $r \to \infty$, we get the independence copula. The comonotonic copula is obtained when $r \to 0$ and $q = 0$. Finally, the most important limit case is the one when h tends to zero, i.e., $q \to 0$, which leads to the Clayton copula. The negative binomial copula can be seen as an approximation of the Clayton copula with dependence parameter $\alpha = \frac{1}{r}$.*

Proposition 24. *Let C be an n-dimensional negative binomial copula as defined in (1.35). Then, if $q > 0$, C cannot capture lower and upper tail-dependence, i.e.,*

$$\lambda_L = \lambda_U = 0.$$

If $q \to 0$, then $\lambda_L = 2^{-r}$ and $\lambda_U = 0$

Proof. When $q > 0$, we have that Θ is a strictly positive and discrete rv, therefore $\lambda_L = 0$. Also, $E[\Theta] = r + \frac{r(1-q)}{q} < \infty$, then $\lambda_U = 0$. When $q \to 0$, $\lambda_L = 2^{-r}$ and $\lambda_U = 0$ since the resulting copula is the Clayton copula with parameter $\frac{1}{r}$. □

We give in Figure 1.9 scatterplots of random points simulated, using Algorithm 19, from a negative binomial copula with $r = \frac{1}{3}$ and different values of h and a Clayton copula with $\alpha = 3$. One sees that the dependence introduced by a negative binomial copula is indeed really similar to the dependence structure of a Clayton copula.

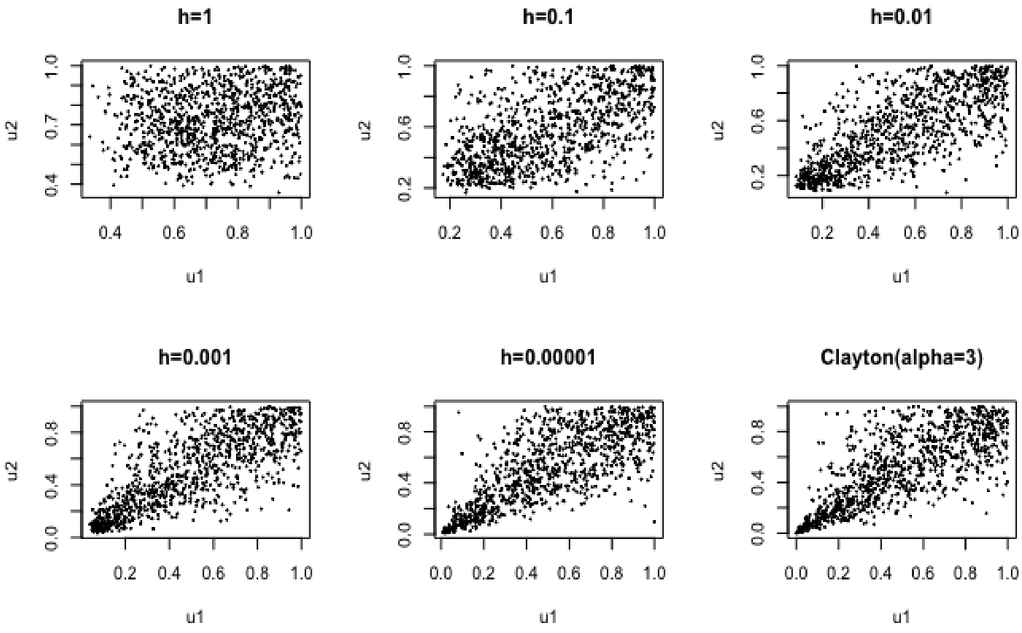

Figure 1.9 – Illustration of the approximation of a Clayton copula by an negative binomial Copula.

The bivariate negative binomial copula can be written as

$$C_{r,q}(u_1,u_2) = \frac{u_1u_2}{\left(1-(1-q)\left(1-u_1^{\frac{1}{r}}\right)\left(1-u_2^{\frac{1}{r}}\right)\right)^r}.$$

This bivariate copula was first introduced by Fang et al. (2000) and was given different names such as the Fang-Fang copula as in Fang et al. (2000), the bivariate Lomax copula as in Balakrishnan and Lai (2009) and the BB10 copula as in Joe (1997). Also, Balakrishnan and Lai (2009) shows that such a copula can also be constructed via the bivariate Lomax distribution defined by its survival function $\bar{F}(x,y) = (1+\alpha_1 x+\alpha_2 y+\alpha_{1,2}xy)^{-\eta}$. Such a copula was studied by several researchers. For example, Genest and Rivest (2001) and Balakrishnan and Lai (2009) developed interesting characteristics of this copula and Jovanovic (2011) showed that it provided a better fit than the Gaussian, Clayton or Frank copulas in a context of stock market uncertainty.

Spearman's correlation coefficent and Kendall's tau between U_1 and U_2 are respectively given by

$$\rho_S(U_1,U_2) = 3\times\Big({}_3F_2\big([1,1,r],[1+2r,1+2r],1-q\big)-1\Big)$$

and

$$\tau(U_1, U_2) = \frac{2r(1-q)}{(2r+1)^2} \times {}_2F_1\big([1,1], [2+2r], 1-q\big). \tag{1.36}$$

See Fang et al. (2000) for proof and more details.

Proposition 25. *Let C be a negative binomial copula with Kendall's tau τ as given in (1.36). Then*

$$0 \leq \tau \leq \tau_{Cl},$$

where $\tau_{Cl} = \frac{1}{2r+1}$ is the Kendall's tau of a Clayton copula with parameter $\frac{1}{r}$.

Proof. The construction of the negative binomial copula from a multivariate mixed exponential distribution implies that $\tau \geq 0$. To demonstrate the second inequality, let $g : [0,1] \to \mathbb{R}$, such that $g(x) = \frac{2rx}{(2r+1)^2} \times {}_2F_1\big([1,1], [2+2r], x\big)$. We can easily show that g is an increasing function and hence $g(1-q)$ is a decreasing function, i.e.,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2rx}{(2r+1)^2} \times {}_2F_1\big([1,1], [2+2r], x\big)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2rx}{(2r+1)^2} \times \sum_{k=0}^{\infty} \frac{1}{(2+2r)_k} \frac{x^k}{k!}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2r}{(2r+1)^2} \times \sum_{k=0}^{\infty} \frac{1}{(2+2r)_k} \frac{x^{k+1}}{k!}\right) \\
&= \frac{2r}{(2r+1)^2} \times \sum_{k=0}^{\infty} \frac{k+1}{(2+2r)_k} \frac{x^k}{k!} > 0.
\end{aligned}$$

Then,

$$0 \leq \tau \leq \tau_{Cl},$$

where

$$\begin{aligned}
\tau_{Cl} &= \lim_{q \to 0} \frac{2r(1-q)}{(2r+1)^2} \times {}_2F_1\big([1,1], [2+2r], 1-q\big) \\
&= \frac{2r}{(2r+1)^2} \times {}_2F_1\big([1,1], [2+2r], 1\big) \\
&= \frac{2r}{(2r+1)^2} \times \frac{2\,r+1}{2\,r} \\
&= \frac{1}{2\,r+1}.
\end{aligned}$$

As expected, when q tends to zero (or equivalently $h \to 0$), the Kendall's tau related to the negative binomial copula tends to the Kendall's tau of a Clayton copula with dependence parameter $\alpha = \frac{1}{r}$. $\square$

In order to analyze the dependence strength of such a copula, we recourse to concordance ordering as defined in Definition 20.

Proposition 26. *Let C_{r,q_1} and C_{r,q_2} be two d-dimensional negative binomial copulas with the same first parameter r and respective second parameters q_1, q_2, Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. Then*

$$C_{r,q_1} \prec_c C_{r,q_2} \text{ if } q_2 \leq q_1.$$

In this case, we have $\tau_1 \leq \tau_2$, $\rho_S^{(1)} \leq \rho_S^{(2)}$, and $\lambda_1 \leq \lambda_2$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the generators of C_{r,q_1} and C_{r,q_2} respectively. As discussed before, in order to show that $C_{r,q_1} \prec_c C_{r,q_2}$ we only have to show that $\left(\mathcal{L}_1^{-1} \circ \mathcal{L}_2\right)'$ is a completely monotone function. Then, for $x \in (0, \infty)$, we have

$$\begin{aligned}
\mathcal{L}_1^{-1} \circ \mathcal{L}_2 &= \ln\left(\frac{q_1 \mathrm{e}^x - q_1 + q_2}{q_2}\right) \\
&= -\ln\left(\frac{\frac{q_2}{q_1}}{\mathrm{e}^x - \left(1 - \frac{q_2}{q_1}\right)}\right) \\
&= -\ln\left(\mathcal{L}_N(x)\right),
\end{aligned}$$

where $N \sim Geo\left(\frac{q_2}{q_1}\right)$ if $q_2 \leq q_1$. It is well known that $(-\ln(\mathcal{L}_N))'$ is completely monotone if and only if $\mathcal{L}_N^m$ is the LST of a positive rv for all $m \in [0, \infty)$ (see Joe (1997)). Since $\mathcal{L}_N^m$, given by

$$\mathcal{L}_N^m(x) = \left(\frac{\frac{q_2}{q_1}}{\mathrm{e}^x - \left(1 - \frac{q_2}{q_1}\right)}\right)^m,$$

is clearly the LST of a negative binomial distribution with parameters m and $\frac{q_2}{q_1}$, we conclude that $C_{r,q_1} \prec_c C_{r,q_2}$ where $q_2 \leq q_1$. □

Remark. *Proposition 26 also implies that the Clayton copula is more concordant than the negative binomial copula, i.e., $C_{r,q} \prec_c C^{Cl}_{\frac{1}{r}}$. The result of Proposition 25 is then, an implication of such an ordering.*

Remark. *Note that Fang et al. (2000) also proposed a multivariate extension of the Lomax copula which is given by*

$$C(u_1, \cdots, u_n) = \left(\prod_{i=1}^{n} u_i\right) \times \left(1 - (1-q)\prod_{i=1}^{n}\left(1 - u_i^{\frac{1}{r}}\right)\right)^{-r}.$$

This version of the multivariate Lomax copula is not Archimedean and does not correspond to the multivariate negative binomial copula we presented in Proposition 23.

1.5.2 Multivariate mixed Exponential - Discrete Stable distribution

Consider the strictly positive mixing rv Θ with $\Theta \sim \mathit{Stable}(\lambda, \alpha)$ and LST as in (1.16). Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n rvs such that, given $\Theta = \theta$, X_i, $i = 1, \ldots, n$ are conditionally independent and distributed as $\mathit{Exp}(\theta)$. Therefore, combining (1.16) and (1.31), the joint survival function of $\underline{X}$ is given by

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \frac{\mathrm{e}^{-\lambda\left(1-\mathrm{e}^{-\sum_{i=1}^{n} x_i}\right)^{\alpha}} - \mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}, \; x_i \in \mathbb{R}^+, \; i = 1, \ldots, n. \tag{1.37}$$

Note that, as in the univariate case, properties related to the multivariate mixed exponential-discrete stable distribution with joint survival function given in (1.37) can only be obtained numerically.

Once again, in order to investigate limit cases, we will opt for the parameterization given in Section 1.4.2. In this case, combining (1.20) and (1.31), the joint survival function of $\underline{X}$ becomes

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \frac{\exp\left\{-\left(\sum_{i=1}^{n} x_i\right)^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n (h\sum_{i=1}^{n} x_i)^n}{n!}\right)^{\alpha}\right\} - \mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}} \tag{1.38}$$

where $x_i \in \mathbb{R}^+$ for $i = 1, \ldots, n$.

Proposition 27. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential-Discrete stable distribution with joint survival function given in (1.38). Then,*

$$\underline{X} \xrightarrow{\mathcal{D}} \underline{Y},$$

as $h \to 0$, where $\underline{Y}$ follows a multivariate Weibull distribution proposed by Hougaard (1986).

Proof. Clearly, when $(h \to 0)$, (1.38) becomes

$$\lim_{h\to 0} \frac{\exp\left\{-\left(\sum_{i=1}^{n} x_i\right)^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n (h\sum_{i=1}^{n} x_i)^n}{n!}\right)^{\alpha} - \mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}\right\}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}} = \exp\left\{-\left(\sum_{i=1}^{n} x_i\right)^{\alpha}\right\},$$

which is the joint survival function of a special case of the multivariate Weibull distribution proposed by Hougaard (1986) and defined by its survival function $\overline{F}(x_1, \ldots, x_n) = \exp\left\{\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \ldots + \left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}}\right\}^{\alpha}$. Here $\lambda_i = 1$ and $\gamma_i = \alpha$ for all $i = 1, \ldots, n$. See, e.g., Rinne (2008) and Lee and Wen (2006) for more details concerning this multivariate distribution. □

As for the negative binomial copula presented in the previous section, we will make use of the link between Archimedean copulas and exponential mixture models to construct a new multivariate copula with a discrete stable mixing distribution.

Proposition 28. *Let $\Theta_{(h,\alpha)}$ be a discrete rv with discrete stable distribution with LST as given in (1.18). The associated multivariate Archimedean copula with generator $\mathcal{L}_{\Theta_{(h,r)}}$, called the multivariate discrete stable copula, is given by*

$$C_{\lambda,\alpha}\left(u_1,\ldots,u_n\right)=\frac{\exp\left\{-\lambda\left(1-\prod_{i=1}^{n}\left(1-\left(\frac{\lambda-\ln\left(\left(\mathrm{e}^{\lambda}-1\right)u_i+1\right)}{\lambda}\right)^{\frac{1}{\alpha}}\right)\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}, \quad (1.39)$$

where $\lambda=h^{-\alpha}$.

Proof. The result is obtained by combining (1.18) and (1.32). □

The bivariate discrete stable copula is given by

$$C_{(h,\alpha)}\left(u_1,u_2\right) = \frac{\exp\left\{-\left(x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}}-\left(\lambda^{-1}\,x\,y\right)^{\frac{1}{\alpha}}\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}, \quad (1.40)$$

where $x=\lambda-\ln\big((\mathrm{e}^{\lambda}-1)u_1+1\big)$ and $y=\lambda-\ln\big((\mathrm{e}^{\lambda}-1)u_2+1\big)$.

The discrete stable copula has some interesting limit cases. When $h\to 0$, i.e., $\lambda\to+\infty$, the copula associated to the mixed exponential-discrete stable distribution can be seen as an approximation of the Gumbel copula with dependence parameter $\frac{1}{\alpha}$, i.e.,

$$\begin{aligned}\lim_{\lambda\to\infty} C_{(h,\alpha)}\left(u_1,u_2\right) &= \exp\left\{-\left((-\ln(u_1))^{\frac{1}{\alpha}}+(-\ln(u_2))^{\frac{1}{\alpha}}\right)^{\alpha}\right\}\\ &= C^{Gu}(u_1,u_2),\end{aligned}$$

since,

$$\begin{aligned}\lim_{\lambda\to+\infty}\Big(\lambda-\ln\Big((\mathrm{e}^{\lambda}-1)u_i+1\Big)\Big) &= \lim_{\lambda\to+\infty}\ln\left(\frac{\mathrm{e}^{\lambda}}{(\mathrm{e}^{\lambda}-1)u_i+1}\right)\\ &= \lim_{\lambda\to+\infty}\ln\left(\frac{\frac{\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda}-1}}{u_i+\frac{1}{\mathrm{e}^{\lambda}-1}}\right)\\ &= -\ln(u_i),\end{aligned}$$

for $i=1,2$.

When $\lambda\to+\infty$ and $\alpha=0$, we obtain the comonotonic copula and when $\lambda\to+\infty$ and $\alpha=1$, we get the independence copula.

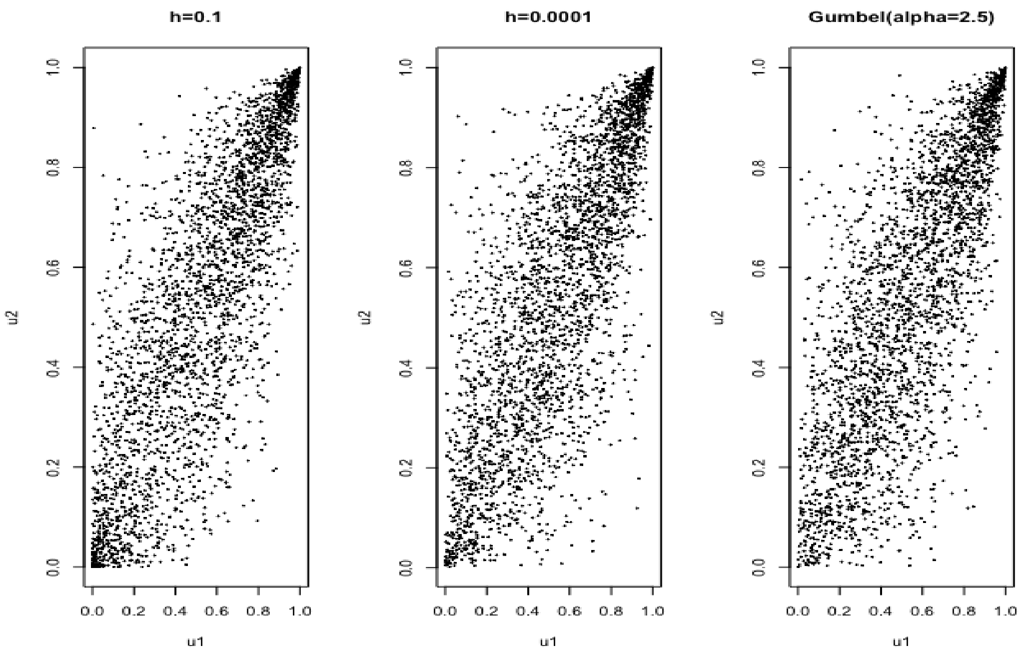

Figure 1.10 – Illustration of the approximation of a Gumbel copula by a discrete stable Copula with $\alpha = 0.4$.

Remark. *In order to generate realizations from the discrete stable copula, we can use Algorithm 19 where Θ can be sampled using the representation of a discrete stable distribution as a random sum as given in (1.15) where M follows a truncated Poisson distribution. Figure 1.10 represents scaterplots of 3000 realizations generated, using Algorithm 19, from the discrete stable copula with parameter $\alpha = 0.4$ and different values of h. We can see the convergence of the discrete stable copula to the Gumbel copula when h tends to zero.*

Proposition 29. *Let C be an n-dimensional discrete stable copula as defined in (1.39). Then, the lower and upper tail-dependence coefficients are given by*

$$\lambda_L = 0 \text{ and } \lambda_U = 2 - 2^{\alpha}.$$

Proof. $\lambda_L = 0$ since Θ is a strictly positive and discrete rv. Concerning the upper tail-dependence coefficient, we have

$$
\begin{aligned}
\lambda_U &= 2 - 2\lim_{t\to 0}\frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)} \\
&= 2 - 2\lim_{t\to 0}\frac{(1-\mathrm{e}^{-2\,t})^{\alpha-1}\mathrm{e}^{-t-\lambda(1-\mathrm{e}^{-2\,t})^{\alpha}+\lambda(1-\mathrm{e}^{-t})^{\alpha}}}{(1-\mathrm{e}^{-t})^{\alpha-1}} \\
&= 2 - 2\lim_{t\to 0}\frac{\left(1-\mathrm{e}^{-t}\right)^{1-\alpha}}{\left(1-\mathrm{e}^{-2t}\right)^{1-\alpha}} \\
&= 2 - 2\times 2^{\alpha-1} = 2 - 2^{\alpha}.
\end{aligned}
$$

This result still holds when $\lambda \to \infty$, i.e., $\lambda_L = 0$ and $\lambda_U = 2 - 2^\alpha$, since the resulting copula is the Gumbel copula with parameter $\frac{1}{\alpha}$. □

Once again, we will use concordance ordering as defined in Definition 20 to analyze the dependence strength of the discrete stable copula.

Proposition 30. *Let C_{λ,α_1} and C_{λ,α_2} be two d-dimensional discrete stable copulas with the same first parameter λ and respective second parameters α_1, α_2, Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. Then*

$$C_{\lambda,\alpha_1} \prec_c C_{\lambda,\alpha_2} \text{ if } \alpha_2 \le \alpha_1.$$

In this case, we have $\tau_1 \le \tau_2$, $\rho_S^{(1)} \le \rho_S^{(2)}$, and $\lambda_1 \le \lambda_2$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the generators of C_{λ,α_1} and C_{λ,α_2} respectively. In order for $C_{\lambda,\alpha_1} \prec_c C_{\lambda,\alpha_2}$ to be true, we have to show that $\left(\mathcal{L}_1^{-1}\circ\mathcal{L}_2\right)'$ is a completely monotone function. Then, for $x \in (0,\infty)$, we have

$$
\begin{aligned}
\mathcal{L}_1^{-1}\circ\mathcal{L}_2(t) &= -\ln\left(1-\left(1-\mathrm{e}^{-t}\right)^{\frac{\alpha_2}{\alpha_1}}\right) \\
&= -\ln\left(\mathcal{L}_N(t)\right),
\end{aligned}
\tag{1.41}
$$

where $N \sim Sibuya\left(\frac{\alpha_2}{\alpha_1}\right)$ if $\alpha_2 \le \alpha_1$. It is well known that $(-\ln(\mathcal{L}_N))'$ is completely monotone if and only if $\mathcal{L}_N^m$ is the LST of a positive rv for all $m \in [0,\infty)$ (see Joe (1997)). We have

$$
\begin{aligned}
\mathcal{L}_N^m(t) &= \left(1-(1-e^{-t})^{\alpha}\right)^m \\
&= \sum_{k=0}^{+\infty}\binom{m}{k}(-1)^k(1-e^{-t})^{\alpha k} \\
&= \sum_{k=0}^{+\infty}\binom{m}{k}(-1)^k\sum_{j=0}^{+\infty}\binom{\alpha k}{j}(-1)^j e^{-jt} \\
&= \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\binom{\alpha k}{j}\binom{m}{k}(-1)^{k+j}e^{-jt} \\
&= \sum_{j=0}^{+\infty} p_j e^{-jt},
\end{aligned}
$$

with $\alpha = \frac{\alpha_2}{\alpha_1}$, $p_j = \sum_{k=1}^{\infty} \binom{m}{k}\binom{\alpha k}{j}(-1)^{k+j}$, $p_j \geq 0$, $j \in \{0, 1, \ldots, \}$, and $\sum_{j=0}^{\infty} p_j = 1$ (see, e.g., Hofert (2010) page 95 for proof). Then, $\mathcal{L}_N^m$ is the LST of a positive rv for all $m \in [0, \infty)$. We conclude that $C_{\lambda,\alpha_1} \prec_c C_{\lambda,\alpha_2}$ where $\alpha_2 \leq \alpha_1$. □

1.5.3 Multivariate mixed Exponential - Discrete Linnik distribution

Consider the positive and discrete mixing rv $\Theta \sim Linnik(\alpha, \beta, \gamma)$ with LST as in (1.23). Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n rvs such that, given $\Theta = \theta$, X_i, $i = 1, \ldots, n$ are conditionally independent and distributed as $Exp(\theta)$. Then, combining (1.23) and (1.31), the joint survival function of $\underline{X}$ is given by

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \left(\frac{1 - \left(1 - e^{-\sum_{i=1}^{n} x_i}\right)^{\alpha}}{1 + \beta\left(1 - e^{-\sum_{i=1}^{n} x_i}\right)^{\alpha}} \right)^{\gamma}, \; x_i > 0, \; i = 1, \ldots, n. \tag{1.42}$$

Similarly to the univariate case, properties related to this multivariate mixed distribution can only be obtained numerically.

When $\gamma = 1$, we obtain the multivariate mixed exponential-discrete Mittag-Leffler distribution.

We will construct a new multivariate Archimedean copula using a mixed exponential-discrete Linnik distribution.

Proposition 31. *Let Θ be a rv with a discrete Linnik distribution with LST as given in (1.23). The associated multivariate Archimedean copula with generator $\mathcal{L}_\Theta$, called the multivariate discrete Linnik copula, is given by*

$$C_{\alpha,\beta,\gamma}(u_1, \ldots, u_n) = \left(\frac{1 - (1 - \prod_{i=1}^{n} x_i)^{\alpha}}{1 - \beta(1 - \prod_{i=1}^{n} x_i)^{\alpha}} \right)^{\gamma}, \tag{1.43}$$

where $x_i = \left(1 - \left(\frac{1 - u_i^{\frac{1}{\gamma}}}{1 + \beta u_i^{\frac{1}{\gamma}}} \right)^{\frac{1}{\alpha}} \right)$.

Proof. The result is obtained by combining (1.23) and (1.32). □

Proposition 32. *Let C be an n-dimensional discrete Linnik copula as defined in (1.43). Then, the lower and upper tail-dependence coefficients are given by*

$$\lambda_L = 0 \text{ and } \lambda_U = 2 - 2^{\alpha}.$$

Proof. Since the mixing rv underlying the Archimedean copula C is a strictly positive discrete rv, then $\lambda_L = 0$. For the upper tail dependence coefficient, we have

$$
\begin{aligned}
\lambda_U &= 2 - 2\lim_{t\to 0}\frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)} \\
&= 2 - 2\lim_{t\to 0}\frac{\left(1-e^{-2\,x}\right)^{\alpha} e^{-2\,x}\left(1+\beta\,\left(1-e^{-x}\right)^{\alpha}\right)\left(-1+e^{-x}\right)\left(-1+\left(1-e^{-x}\right)^{\alpha}\right)}{\left(1+\beta\,\left(1-e^{-2\,x}\right)^{\alpha}\right)\left(-1+e^{-2\,x}\right)\left(-1+\left(1-e^{-2\,x}\right)^{\alpha}\right)\left(1-e^{-x}\right)^{\alpha} e^{-x}}\frac{\left(\frac{1-\left(1-e^{-2\,x}\right)^{\alpha}}{1+\beta\,(1-e^{-2\,x})^{\alpha}}\right)^{\gamma}}{\left(\left(\frac{1-(1-e^{-x})^{\alpha}}{1+\beta\,(1-e^{-x})^{\alpha}}\right)^{\gamma}\right)} \\
&= 2 - 2\lim_{t\to 0}\frac{\left(1-\mathrm{e}^{-x}\right)^{1-\alpha}}{\left(1-\mathrm{e}^{-2x}\right)^{1-\alpha}} \\
&= 2 - 2\lim_{t\to 0}\left(\frac{\frac{1-\mathrm{e}^{-2x}}{1+\mathrm{e}^{-x}}}{1-\mathrm{e}^{-2x}}\right)^{1-\alpha} \\
&= 2 - 2\times 2^{\alpha} - 1 = 2 - 2^{\alpha}.
\end{aligned}
$$

□

Proposition 33. *Let C_{λ,α_1} and C_{λ,α_2} be two d-dimensional discrete Linnik copulas with the same last parameters β and γ and respective first parameters α_1, α_2, Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. Then*

$$C_{\alpha_1,\beta,\gamma} \prec_c C_{\alpha_2,\beta,\gamma} \textit{ if } \alpha_2 \leq \alpha_1.$$

In this case, we have $\tau_1 \leq \tau_2$, $\rho_S^{(1)} \leq \rho_S^{(2)}$, and $\lambda_1 \leq \lambda_2$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the generators of $C_{\alpha_1,\beta,\gamma}$ and $C_{\alpha_2,\beta,\gamma}$ respectively. In order for $C_{\alpha_1,\beta,\gamma} \prec_c C_{\alpha_1,\beta,\gamma}$ to be true, we have to show that $\left(\mathcal{L}_1^{-1}\circ\mathcal{L}_2\right)'$ is a completely monotone function. Then, for $t \in (0,\infty)$, we have

$$\mathcal{L}_1^{-1}\circ\mathcal{L}_2(t) \quad = \quad -\ln\left(1-\left(1-\mathrm{e}^{-t}\right)^{\frac{\alpha_2}{\alpha_1}}\right),$$

which is of the same form as (1.41). Therefore, $C_{\alpha_1,\beta,\gamma} \prec_c C_{\alpha_1,\beta,\gamma}$ if $\alpha_2 \leq \alpha_1$. □

Joe's copula with parameter $\frac{1}{\alpha}$ is obtained if $\beta \to 0$ and $\gamma \to 1$. Depending on the chosen parameterization, several particular cases can be found. For example, using the result of Proposition 15, we can obtain, as a limit case, the Archimedean copula based on the gamma-stopped-Sibuya LST (see Joe (2014) page 216 for more details). Using Proposition 16, copulas based on the Mittag-leffler LST can be derived as limit cases of the discrete Linnik copula (see, e.g., BB1 copula in Joe (2014) page 190 for more details).

1.5.4 Compound Poisson risk models with exchangeable claim amounts

We consider a variant of the risk model described in Section 1.4.4 in which we assume that the univariate rv W follows a univariate mixed exponential distribution. This means that $\underline{N}$ is here a compound Poisson process. We also assume that $(X_1, X_2, ..., X_k)$ follows a multivariate mixed exponential distribution as defined in Section 1.5, for $k \in \{2, 3, ...\}$ and $x_1, ..., x_k \geq 0$. The joint survival function of $(W_1, W_2, ..., W_k)$ is defined as in (1.30) and (1.31), for $k \in \{2, 3, ...\}$. See Section 2 of Albrecher et al. (2011a) for details on this class of risk models. In this context, let us define as before a strictly positive rv Θ such that, given $\Theta = \theta$, the X_j's ($j \geq 1$) are exponentially distributed with parameter θ and conditionally independent with joint conditional distribution (1.30). Let ζ_θ be the conditional ruin probability associated to the corresponding classical compound Poisson risk model which assumes independent and exponentially distributed claim amounts. Then, ζ_θ can be written as

$$\zeta_\theta(u) = \min\left\{\frac{\lambda}{\theta\, c}\exp\left\{-\left(\theta - \frac{\lambda}{c}\right)u\right\};1\right\},$$

for $u \geq 0$. The unconditional ruin probability can therefore be represented as a mixture with mixing rv Θ as follows

$$\zeta(u) = \int_0^\infty \zeta_\theta(u)\mathrm{d}F_\Theta(\theta). \tag{1.44}$$

The net profit condition is violated when the mixing rv Θ takes a value smaller that $\theta_0 > 0$. We define θ_0 such that $\zeta_\theta(u) = 1$ for all $u \geq 0$ ($\theta_0 = \frac{\lambda}{c}$). Then (1.44) becomes

$$\zeta(u) = F_\Theta(\theta_0) + \int_{\theta_0}^\infty \zeta_\theta(u)\mathrm{d}F_\Theta(\theta). \tag{1.45}$$

In particular, we have

$$\lim_{u\to\infty}\zeta(u) = F_\Theta(\theta_0). \tag{1.46}$$

Note that in Section 2 of Albrecher et al. (2011a) examples of explicit formulas for the ruin probability as in (1.44) are given. In all the three provided examples, they have considered the mixing rv Θ to be continuous (gamma, Pareto and a continuous distribution with a Pareto-type tail). In the following examples, we propose explicit ruin formulas based on a discrete mixing rv Θ. In this case, the ruin probability in (1.45) becomes

$$\zeta(u) = F_\Theta(\theta_0) + \sum_{\theta=\theta_0+1}^\infty \zeta_\theta(u)\Pr(\Theta = \theta). \tag{1.47}$$

Example 34. (Multivariate mixed Exponential - Negative Binomial distribution)
Let $(X_1, X_2, ..., X_k)$ ($k \in \{2, 3, ...\}$) follow a k-dimensional multivariate mixed Exponential - Negative Binomial distribution as described in Section 1.5.1 with joint survival function as given in (1.33). The corresponding model in this case is the negative binomial copula proposed in Section 1.5.1 with specific univariate mixed Exponential - Negative Binomial distribution marginals as defined in Section 1.4.1. From (1.47), we obtain

$$\zeta(u) = \begin{cases} \xi + \frac{\theta_0}{\theta_0+1}\mathrm{e}^{-u}q^r p^{\theta_0+1-r}\binom{\theta_0}{r-1}{}_3F_2\left([1,\theta_0+1,\theta_0+1];[\theta_0+2,\theta_0+2-r];p\,\mathrm{e}^{-u}\right), & u > 0 \\ \xi + \frac{\theta_0}{\theta_0+1}q^r p^{\theta_0+1-r}\binom{\theta_0}{r-1}{}_3F_2\left([1,\theta_0+1,\theta_0+1];[\theta_0+2,\theta_0+2-r];p\right), & u = 0 \end{cases},$$

where $p = 1 - q$ and $\xi = F_\Theta(\theta_0) = 1 - \binom{\theta_0}{r-1} q^r (1-q)^{\theta_0+1-r} {}_2F_1\left([1, \theta_0 + 1]; [2 + \theta_0 - r]; 1 - q\right)$. Also,

$$\lim_{u \to \infty} \zeta(u) = 1 - \binom{\theta_0}{r-1} q^r p^{\theta_0+1-r} {}_2F_1\left([1, \theta_0 + 1]; [2 + \theta_0 - r]; p\right),$$

with $\theta_0 = \frac{\lambda}{c}$.

□

Example 35. (Multivariate mixed Exponential - Geometric distribution)

Let $(X_1, X_2, ..., X_k)$ $(k \in \{2, 3, ...\})$ follow a k-dimensional multivariate mixed Exponential - Geometric distribution as described in Section 1.5.1 (see Proposition 22) with joint survival function as given in (1.34). In this case, the corresponding dependence structure is the AMH copula (with parameter $\alpha = 1 - q$) and the marginal distributions, given by

$$\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = \frac{q}{e^x - (1-q)}, \; i = 1, \ldots, n,$$

correspond to univariate mixed Exponential-Geometric distributions (see Section 1.4.1 and Adamidis and Loukas (1998) for more details).

From (1.47), it follows that, for this model, the ruin probability is given by

$$\zeta(u) = \begin{cases} 1 - (1-q)^{\theta_0} + q\, e^{-u}\, \theta_0\, (1-q)^{\theta_0}\, \Phi\left((1-q)e^{-u}, 1, \theta_0 + 1\right), & u > 0 \\ 1 - (1-q)^{\theta_0} + q\, \theta_0\, (1-q)^{\theta_0}\, \Phi\left((1-q), 1, \theta_0 + 1\right), & u = 0 \end{cases},$$

where $\Phi(z, a, v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^a}$ denotes the Lerch Phi function (also called the Lerch transcendent function).

We can easily verify (also from (1.46)) that

$$\lim_{u \to \infty} \zeta(u) = 1 - (1-q)^{\theta_0},$$

where $\theta_0 = \frac{\lambda}{c}$.

□

Example 36. (Multivariate mixed Exponential - Sibuya distribution)

Let $(X_1, X_2, ..., X_k)$ $(k \in \{2, 3, ...\})$ follow a k-dimensional multivariate mixed Exponential - Sibuya distribution as described in Section 1.8.3 with joint survival function as given in (1.57). The corresponding model in this case is the Joe copula (with parameter $\frac{1}{\alpha}$) with specific mixed Exponential - Sibuya distribution marginals as defined in Section 1.8.1. Then, from (1.47), we obtain

$$\zeta(u) = \begin{cases} 1 - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha} + \frac{\theta_0(-1)^{\theta_0}\binom{\alpha}{\theta_0+1}\mathrm{e}^{-u}{}_3F_2\left([1,1+\theta_0,1+\theta_0-\alpha];[2+\theta_0,2+\theta_0];\mathrm{e}^{-u}\right)}{\theta_0+1}, & u > 0 \\ 1 - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha} + \frac{\theta_0(-1)^{\theta_0}\binom{\alpha}{\theta_0+1}{}_3F_2([1,1+\theta_0,1+\theta_0-\alpha];[2+\theta_0,2+\theta_0];1)}{\theta_0+1}, & u = 0 \end{cases}.$$

Also,

$$\lim_{u\to\infty} \zeta(u) = 1 - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha},$$

where $\theta_0 = \frac{\lambda}{c}$.

□

Example 37. (Multivariate mixed Exponential - Logarithmic distribution)

Let $(X_1, X_2, ..., X_k)$ $(k \in \{2, 3, ...\})$ follow a k-dimensional multivariate mixed Exponential - Logarithmic distribution as described in Section 1.8.4 with joint survival function as given in (1.59) (see Proposition 44 for more details concerning such a distribution). As discussed before, the dependence structure underlying this multivariate distribution corresponds to the Frank copula with parameter $\alpha = -\ln(1-p)$. In this case, the marginal distribution of X_i, $i = 1, \ldots, n$, given by

$$\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = \frac{\ln(1 - p\,\mathrm{e}^{-x})}{\ln(1-p)},\ i = 1, \ldots, n,$$

corresponds to the univariate mixed Exponential - Logarithmic distribution proposed by Tahmasbi and Rezaei (2008).

From (1.47), we obtain

$$\zeta(u) = \begin{cases} 1 + \frac{B(p,\theta_0+1,0)}{\ln(1-p)} - \frac{\theta_0\,\mathrm{e}^{-u}\,p^{\theta_0+1}}{\ln(1-p)}\,\Phi\left(p\,\mathrm{e}^{-u}, 2, \theta_0+1\right), & u > 0 \\ 1 + \frac{B(p,\theta_0+1,0)}{\ln(1-p)} - \frac{\theta_0\,p^{\theta_0+1}}{\ln(1-p)}\,\Phi\left(p, 2, \theta_0+1\right), & u = 0 \end{cases},$$

where B is the incomplete beta function. Clearly,

$$\lim_{u\to\infty} \zeta(u) = 1 + \frac{B(p, \theta_0+1, 0)}{\ln(1-p)},$$

with $\theta_0 = \frac{\lambda}{c}$.

□

1.6 Renewal risk models with exchangeable inter-claim times

In this section, we consider the class of renewal risk models with exchangeable inter-claim times, which can be seen as extensions to the classical renewal risk model described in Section 1.4.4. Albrecher et al. (2011a) and Cossette et al. (2018) also discussed the continuous-time renewal risk model with exchangeable inter-claim times. In this class of renewal risk models, the claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a mixed renewal process where the inter-claim times $\underline{W} = \{W_j, j \in \mathbb{N}\}$ form a sequence of exchangeable and strictly positive real-valued rvs.

In the present class of risk models, $(W_1, W_2, ..., W_k)$ follows a multivariate mixed exponential distribution as defined in Section 1.5, for $k \in \{2, 3, ...\}$ and $x_1, ..., x_k \geq 0$. The joint survival function of $(W_1, W_2, ..., W_k)$ is defined as in (1.31), for $k \in \{2, 3, ...\}$. Due to this assumption on the inter-claim times, the claim number process $\underline{N}$ can be also seen as a mixed Poisson process, which has found many applications in actuarial science (see, e.g., Grandell (1997) and Albrecher et al. (2017)).

From Cossette et al. (2018), the expression for the ruin probability $\zeta(u)$ is given by

$$\zeta(u) = \overline{F}_\Theta(\theta_0) + \int_0^{\theta_0} \zeta_\theta(u) \mathrm{d}F_\Theta(\theta), \tag{1.48}$$

where $u \geq 0$ and $\theta_0 = \frac{c}{E[X_i]}$. The security loading is violated when $\theta > \theta_0$. In particular, we have

$$\lim_{u \to \infty} \zeta(u) = \overline{F}_\Theta(\theta_0). \tag{1.49}$$

When the mixing rv Θ follows a discrete distribution as in Section 1.4, the expression for the ruin probability in (1.48) becomes

$$\zeta(u) = \overline{F}_\Theta(\theta_0) + \sum_{\theta=1}^{\theta_0} \zeta_\theta(u) \Pr(\Theta = \theta). \tag{1.50}$$

As in Section 1.4.4, we suppose that the claim amount is exponentially distributed with parameter β. Then, $\theta_0 = c\beta$ and the conditional ruin probability given in (1.50) becomes

$$\zeta_\theta(u) = \min\left\{\frac{\theta}{\beta c} \exp\left\{-\left(\beta - \frac{\theta}{c}\right) u\right\}; 1\right\}, \; u \geq 0. \tag{1.51}$$

Example 38. We consider the following multivariate mixed exponential distributions for $(W_1, W_2, ..., W_k)$, for $k \in \{2, 3, ...\}$.

(a) Multivariate mixed Exponential - Negative Binomial distribution:

$$\psi(u) = \begin{cases} \xi + \frac{r\,q^r\,\mathrm{e}^{-(\theta_0+1)\frac{u}{c}}}{\theta_0\left(\mathrm{e}^{-\frac{u}{c}}-1+q\right)^{r+1}} - \frac{q^r(\theta_0+1)(1-q)^{n-r+1}\binom{\theta_0}{r-1}\mathrm{e}^{\frac{u}{c}}\,{}_2F_1\left([1,\theta_0+2];[2+\theta_0-r];\mathrm{e}^{\frac{u}{c}}(1-q)\right)}{\theta_0}, & u > 0 \\ \xi + \frac{r}{q\,\theta_0} - \frac{q^r(\theta_0+1)(1-q)^{n-r+1}\binom{\theta_0}{r-1}{}_2F_1([1,\theta_0+2];[2+\theta_0-r];1-q)}{\theta_0}, & u = 0 \end{cases},$$

where $\xi = \overline{F}_\Theta(\theta_0) = \binom{\theta_0}{r-1} q^r (1-q)^{\theta_0+1-r}\,{}_2F_1\left([1,\theta_0+1];[2+\theta_0-r];1-q\right)$. Also,

$$\lim_{u\to\infty}\psi(u) = \binom{\theta_0}{r-1} q^r (1-q)^{\theta_0+1-r}\,{}_2F_1\left([1,\theta_0+1];[2+\theta_0-r];1-q\right).$$

(b) Multivariate mixed Exponential - Geometric distribution (i.e., $\Theta \sim Geometric(q)$):

$$\psi(u) = \begin{cases} (1-q)^{\theta_0} - \frac{\left(\left(\theta_0(q-1)\mathrm{e}^{\frac{u}{c}}+\theta_0+1\right)(1-q)^{\theta_0}\mathrm{e}^{\frac{(\theta_0+1)u}{c}}-\mathrm{e}^{\frac{u}{c}}\right)q\mathrm{e}^{-\frac{\theta_0 u}{c}}}{\theta_0\left(1+(q-1)\mathrm{e}^{\frac{u}{c}}\right)^2}, & u > 0 \\ (1-q)^{\theta_0} - \frac{\left((\theta_0(q-1)+\theta_0+1)(1-q)^{\theta_0}-1\right)q}{\theta_0(1+(q-1))^2}, & u = 0 \end{cases},$$

and

$$\lim_{u\to\infty}\psi(u) = (1-q)^{\theta_0}.$$

(c) Multivariate mixed Exponential - Sibuya distribution:

$$\psi(u) = \begin{cases} \xi + \frac{\alpha}{\theta_0}\left(1-\mathrm{e}^{\frac{u}{c}}\right)^{\alpha-1}\mathrm{e}^{-\frac{u(\theta_0-1)}{c}} - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}\mathrm{e}^{\frac{u}{c}}\,{}_2F_1\left([1,\theta_0+1-\alpha];[\theta_0+1];\mathrm{e}^{\frac{u}{c}}\right)}{\theta_0}, & u > 0 \\ \xi - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}{}_2F_1([1,\theta_0+1-\alpha];[\theta_0+1];1)}{\theta_0}, & u = 0 \end{cases},$$

where $\xi = \overline{F}_\Theta(\theta_0) = \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha}$. In particular, we have

$$\lim_{u\to\infty}\psi(u) = \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha}.$$

(d) Multivariate mixed Exponential - Logarithmic distribution:

$$\psi(u) = \begin{cases} -\frac{B(p,\theta_0+1,0)}{\ln(1-p)} + \frac{p\,\mathrm{e}^{\frac{u(1-\theta_0)}{c}}-p^{\theta_0+1}\,\mathrm{e}^{\frac{u}{c}}}{\theta_0\ln(1-p)\left(p\mathrm{e}^{\frac{u}{c}}-1\right)}, & u > 0 \\ -\frac{B(p,\theta_0+1,0)}{\ln(1-p)} + \frac{p-p^{\theta_0+1}}{\theta_0\ln(1-p)(p-1)}, & u = 0 \end{cases},$$

and

$$\lim_{u\to\infty}\psi(u) = -\frac{B(p,\theta_0+1,0)}{\ln(1-p)}.$$

□

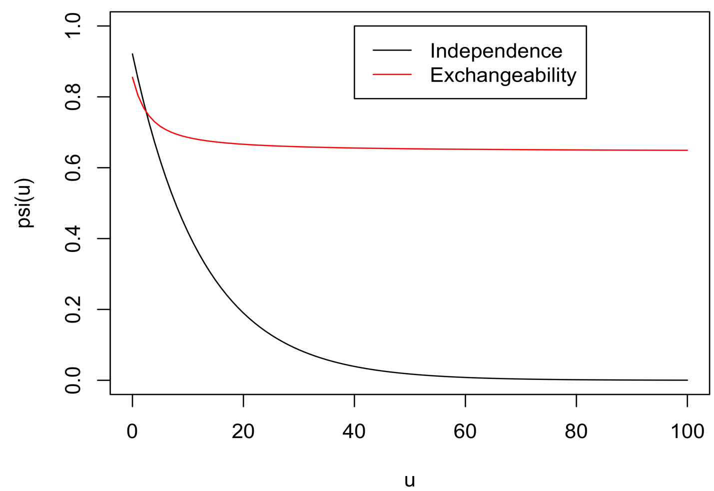

Figure 1.11 – Ruin probability for an exp-negative binomial distribution with $r = 2$ and $h = 0.0001$

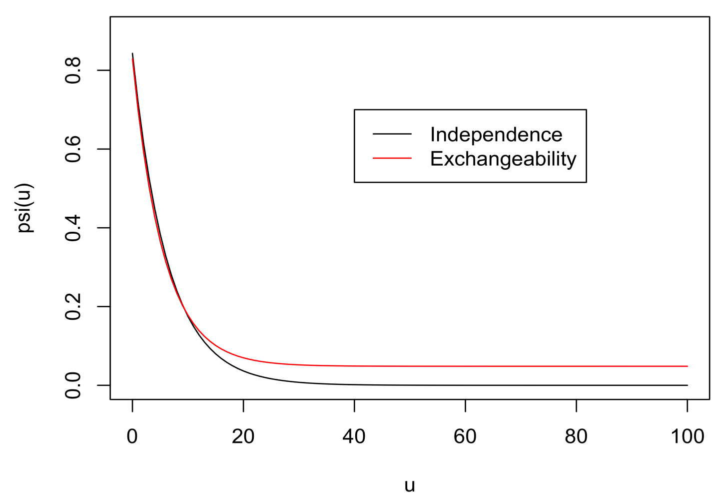

Figure 1.12 – Ruin probability for an exp-logarithmic distribution with $\alpha = 0.1$

Example 39. Let $X \sim Exp(1)$ and $(W_1, \ldots, W_k)$, for $k = 1, 2, \ldots$, follow a multivariate mixed exponential distribution with $E[W_k] = 1$, for $k = 1, 2, \ldots$. We consider three cases for the mixing distribution, namely, the negative binomial distribution, the Sibuya distribution, and the logarithmic distribution. Let the parameters be the same as in Example 17 and $c = 1.25$. Figures 1.11, 1.12, and 1.13 illustrate the significant impact of the dependence relationship

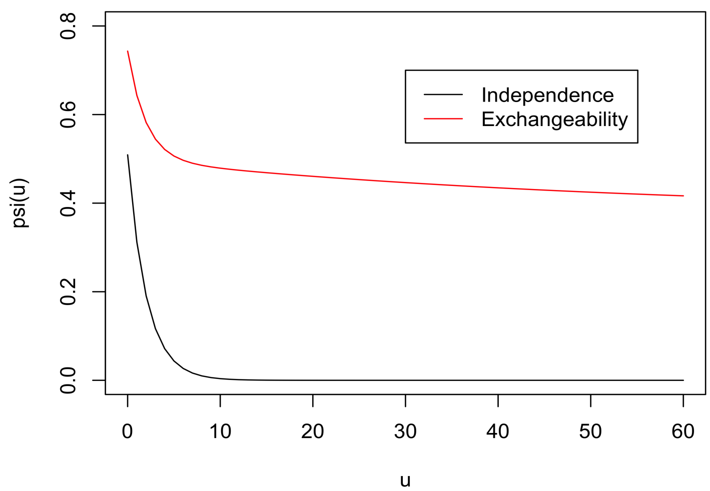

Figure 1.13 – Ruin probability for an exp-Sibuya distribution with $\alpha = 0.5$

between inter-claim times on the overall portfolio.

1.7 Acknowledgment

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1.8 Appendix

1.8.1 Univariate mixed Exponential - Sibuya distribution

The Sibuya distribution was first introduced by Masaaki Sibuya in Sibuya (1979) and later studied by several reserchers (see, e.g., Devroye (1993), Christoph and Schreiber (1998) and Christoph and Schreiber (2000)). This discrete distribution has some interesting properties such as self-decomposability and infinite divisibility and can be generalized to a distribution with two or three parameters, often called scaled Sibuya or generalized Sibuya distribution (see, e.g., Christoph and Schreiber (2000) and Huillet (2016)). Also, Devroye (1993) derived a distributional representation of the Sibuya distribution useful for sampling observations from such a distribution.

Let Θ be a discrete rv with a Sibuya distribution with parameter $\alpha \in (0,1]$, i.e., $\Theta \sim Sibuya(\alpha)$, and pgf given by

$$P_\Theta(t) = 1 - (1-t)^\alpha. \tag{1.52}$$

The pmf and LST of Θ are respectively given by

$$\Pr(\Theta = k) = (-1)^{k-1}\binom{\alpha}{k},\ k \in \mathbb{N},$$

and

$$\mathcal{L}_\Theta(t) = 1 - \left(1 - \mathrm{e}^{-t}\right)^\alpha. \tag{1.53}$$

Let X be a positive rv with univariate mixed exponential-Sibuya distribution derived from a discrete mixing rv Θ with LST as in (1.53). Combining (1.2) and (1.53), the univariate survival function of X can be written as

$$\bar{F}_X(x) = 1 - \left(1 - \mathrm{e}^{-x}\right)^\alpha. \tag{1.54}$$

Proposition 40. *Let the rv X follow a mixed exponential-Sibuya distribution with univariate survival function as defined in (1.54). Then, the following properties hold:*

1. $\mathcal{L}_X(t) = \dfrac{\alpha\Gamma(1+t)\Gamma(\alpha)}{\Gamma(t+1+\alpha)}$, $t > 0$;
2. $f_X(x) = \alpha\left(1-\mathrm{e}^{-x}\right)^{\alpha-1}\mathrm{e}^{-x}$, $x \in \mathbb{R}^+$;
3. $h_X(x) = \dfrac{\alpha\mathrm{e}^{-x}}{(1-(1-\mathrm{e}^{-x})^\alpha)}\left(1-\mathrm{e}^{-x}\right)^{\alpha-1}$, $x \in \mathbb{R}^+$;
4. $E[X] = \alpha\ {}_3F_2(1,1,1-\alpha;\ 2,2;\ 1)$;
5. $E[X^n] = \alpha\Gamma(n+1)\,{}_{n+2}F_{n+1}(1,\ldots,1,1-\alpha;\ 2,\ldots,2;\ 1)$, $n = 1,2,\ldots$;
6. $VaR_\kappa(X) = -\ln\left(1-\kappa^{\frac{1}{\alpha}}\right)$, $\kappa \in (0,1)$;

7. $TVaR_\kappa(X) = \frac{\alpha \ln(\kappa_\alpha)\kappa_\alpha - \kappa_\alpha}{(\kappa - 1)} {}_4F_3\left(1, 1, 1-\alpha, \frac{2\ln(\kappa_\alpha) - 1}{\ln(\kappa_\alpha)}; 2, 2, \frac{\ln(\kappa_\alpha) - 1}{\ln(\kappa_\alpha)}; \kappa_\alpha\right)$, *where* $\kappa_\alpha = 1 - \kappa^{\frac{1}{\alpha}}$, $\kappa \in (0, 1)$.

Proof. All expressions follow directly from Proposition 7. □

1.8.2 Univariate mixed Exponential - logarithmic distribution

Let Θ be a discrete rv with a logarithmic distribution with parameter $p \in (0, 1]$, i.e., $\Theta \sim Log(p)$, and pmf

$$\Pr(\Theta = k) = \frac{-p^k}{k \ln(1-p)}, \; k \in \mathbb{N}.$$

Note that we can also consider another parametrization using $\alpha = -\ln(1-p)$.

The LST of Θ is given by

$$\mathcal{L}_\Theta(t) = \frac{\ln\left(1 - p\mathrm{e}^{-t}\right)}{\ln(1-p)}, \; x > 0. \tag{1.55}$$

The resulting univariate mixed distribution, first introduced by Tahmasbi and Rezaei (2008), is called the mixed exponential-logarithmic distribution. Let X be a positive rv with univariate mixed exponential-logarithmic distribution derived from a discrete mixing rv Θ with LST as in (1.55). Combining (1.2) and (1.55), the univariate survival function of X can be written as

$$\bar{F}_X(x) = \frac{\ln(1 - p\mathrm{e}^{-x})}{\ln(1-p)}, \; x > 0. \tag{1.56}$$

Proposition 41. *Let the rv X follow a mixed exponential-Sibuya distribution with univariate survival function as defined in (1.56). Then, the following properties hold:*

1. $\mathcal{L}_X(t) = \frac{-(\Phi(p, 1, t) - 1/t)}{\ln(1-p)}$ $t > 0$;
2. $f_X(x) = \frac{-p\mathrm{e}^{-x}}{\ln(1-p)(1-p\mathrm{e}^{-x})}$, $x \in \mathbb{R}^+$;
3. $h_X(x) = \frac{-p\mathrm{e}^{-x}}{(1-p\mathrm{e}^{-x})\ln(1-p\mathrm{e}^{-x})}$, $x \in \mathbb{R}^+$;
4. $E[X] = \frac{-Li_2(p)}{\ln(1-p)}$;
5. $E[X^n] = \frac{-\Gamma(n+1)Li_{n+1}(p)}{\ln(1-p)}$, $n = 1, 2, \ldots$;
6. $VaR_\kappa(X) = -\ln\left(\frac{p - 1 + (1-p)^\kappa}{p}\right) + \kappa \ln(1-p)$, $\kappa \in (0, 1)$;
7. $TVaR_\kappa(X) = \frac{p\,\mathrm{e}^{-d}(d+1)\,{}_4F_3(1, 1, 1, (2d+1)/d; 2, 2, (d+1)/d; p\mathrm{e}^{-d})}{\ln(1-p)(\kappa-1)}$, *where* $d = VaR_\kappa(X)$ *and* $\kappa \in (0, 1)$.

Proof. All expressions follow directly from Proposition 7. □

1.8.3 Multivariate mixed Exponential - Sibuya distribution

Consider Θ to be a positive distrete rv with a Sibuya distribution, i.e., $\Theta \sim Sibuya(\alpha)$, and LST as in (1.23). Let $\underline{X} = (X_1, \ldots, X_n)$ follow a multivariate mixed exponential distribution with mixing rv Θ. Then, combining (1.23) and (1.31), the joint survival function of $\underline{X}$ can be written as

$$\bar{F}_{\underline{X}}(\underline{x}) = 1 - \left(1 - \mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)^{\frac{1}{\alpha}}. \tag{1.57}$$

Proposition 42. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential-Sibuya distribution with joint survival function given in (1.57). Then, the following properties hold:*

1. $$f_{\underline{X}}(\underline{x}) = \sum_{k=1}^{m} \left(1 - \mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)^{\frac{1}{\alpha}} \left(\frac{\mathrm{e}^{-\sum_{i=1}^{m} x_i}}{\alpha\left(1 - \mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)}\right)^{k} S_2(m,k) \sum_{j=1}^{k} (-1)^{k-1} S_1(k, k+1-j)\, \alpha^{j-1}.$$
2. *Joint moments* $E[X_1^{n_1} \ldots X_m^{n_m}] = \prod_{i=1}^{m} \frac{\Gamma(n_i+1)}{\alpha} \; {}_{d+2}F_{d+1}\left(1, \ldots, 1, \frac{-1+\alpha}{\alpha}; 2, \ldots, 2; 1\right)$ *with* $d = n_1 + \ldots, n_m$.
3. $Cov\,(X_1, X_2) = \frac{1}{\alpha^2}\left({}_4F_3\left(1,1,1,\frac{-1+\alpha}{\alpha}; 2,2,2; 1\right)\alpha - \left({}_3F_2\left(1,1,\frac{-1+\alpha}{\alpha}; 2,2; 1\right)\right)^2\right).$
4. $\rho_P\,(X_1, X_2) = \dfrac{{}_4F_3(1,1,1,\frac{\alpha-1}{\alpha}; 2,2,2; 1)\alpha - \left({}_3F_2(1,1,\frac{\alpha-1}{\alpha}; 2,2; 1)\right)^2}{2\,{}_4F_3(1,1,1,\frac{\alpha-1}{\alpha}; 2,2,2; 1)\alpha - \left({}_3F_2(1,1,\frac{\alpha-1}{\alpha}; 2,2; 1)\right)^2}.$
5. *Kendall's tau* $\tau\,(X_1, X_2) = 1 - 4\sum_{k=1}^{\infty} \left(k\,(\alpha k + 2)(\alpha(k-1)+2)\right)^{-1}$.

where S_1 and S_2 denote the Stirling number of the first and the second kinds, respectively.

Proof. For property 5, see, e.g., Hofert (2010). For all other properties, the expessions are derived directly from their definitions. □

Using the link between multivariate mixed exponential-Sibuya distribution and Archimedean copulas, the resulting copula is the well known Joe copula.

Proposition 43. *Let Θ be a discrete rv with Sibuya distribution with LST as given in (1.23). The associated multivariate Archimedean copula with generator $\mathcal{L}_\Theta$, called the Joe copula with parameter $\frac{1}{\alpha}$, is given by*

$$C_{\frac{1}{\alpha}}\,(u_1, \ldots, u_n) = 1 - \left(1 - \prod_{i=1}^{n}\left(1 - (1-u_i)^{\frac{1}{\alpha}}\right)\right)^{\alpha}, \tag{1.58}$$

for $u_i \in [0,1]$, $\forall i \in \{1, \ldots, n\}$, and $\alpha \in [0,1]$.

Proof. See Joe (2014) for proof and more details concerning such a copula. □

1.8.4 Multivariate mixed Exponential - Logarithmic distribution

In this section, we consider the mixing rv Θ to follow a logarithmic ditribution with parameter p and pmf

$$\Pr(\Theta = k) = \frac{-p^k}{k\ \ln(1-p)},\ \ k \in \mathbb{N}.$$

Note that we can also consider another parametrization using $\alpha = -\ln(1-p)$.

The resulting univariate mixed distribution, first introduced by Tahmasbi and Rezaei (2008), is called the mixed exponential-logarithmic distribution. In this section, we propose a multivariate extension of such a distribution. Let $\underline{X} = (X_1, \ldots, X_n)$ be an n-dimensional random vector. The dependence between the rvs $X_1, \ldots, X_n$ is introduced via the mixing rv Θ as described in the begining of Section 1.5. Then, from (1.31), the joint survival function of $\underline{X}$ can be written as

$$\bar{F}_{\underline{X}}(\underline{x}) = \frac{\ln\left(1 - p\mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)}{\ln(1-p)}. \tag{1.59}$$

Proposition 44. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential logarithmic distribution with joint survival function given in (1.59). Then, the following properties hold:*

1. *Joint density function* $f_{\underline{X}}(x_1, \cdots, x_m) = \sum_{k=1}^{m} \frac{1}{k} \left(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} i^m \right) \frac{-p^k \mathrm{e}^{-k\sum_{i=1}^{m} x_i}}{\ln(1-p)\ \left(1 - p\mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)^k}$.
2. *Joint moments* $E[X_1^{n_1} \ldots X_m^{n_m}] = \prod_{i=1}^{m} \Gamma(n_i+1) \times \frac{Li_d\,(p)}{-\ln(1-p)}$, *where* $d = n_1 + \ldots + n_m + 1$.
3. *Covariance* $Cov\,(X_1, X_2) = \frac{-Li_3\,(p) \ln(1-p) - (Li_2\,(p))^2}{\ln(1-p)^2}$.
4. *Pearson's correlation coefficient* $\rho\,(X_1, X_2) = \frac{Li_3\,(p) \ln(1-p)\,(Li_2\,(p))^2}{2\,Li_3\,(p) \ln(1-p)\,(Li_2\,(p))^2}$.
5. *Kendall's tau* $\tau = 1 + 4\frac{D_1(-\ln(1-p))-1}{-\ln(1-p)}$, $D_1(\alpha) = \frac{1}{\alpha}\int_0^\alpha t\,(\mathrm{t}-1)^{-1}\mathrm{d}t$*: "The Debye function of order one".*

Proof. For property 5, see, e.g., Hofert (2010). For all other properties, the expressions are derived directly from their definitions. □

Using the link between multivariate mixed exponential-logarithmic distribution and Archimedean copulas, the resulting copula is the well known Frank copula.

Proposition 45. *Let Θ be a discrete rv with logarithmic distribution with parameter p. The associated multivariate Archimedean copula with generator $\mathcal{L}_\Theta$, called the Frank copula with parameter $\delta = -\ln(1-p)$, is given by*

$$C_{\frac{1}{\alpha}}(u_1, \ldots, u_n) = -\delta^{-1} \ln \left(1 - \frac{\prod_{i=1}^n \left(1 - \mathrm{e}^{-\delta u_i}\right)}{\left(1 - \mathrm{e}^{-\delta}\right)^{n-1}} \right), \tag{1.60}$$

for $u_i \in [0,1]$, $\forall i \in \{1, \ldots, n\}$.

Proof. See, e.g., Joe (2014) for proof and more details concerning such a copula. □

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Chapitre 2

Dependent risk models with Archimedean copulas : a computational strategy based on common mixtures and applications

2.1 Résumé

Dans cet article, nous étudions les modèles de risque pour lesquels la structure de dépendance est donnée par une copule Archimédienne. En utilisant une telle structure avec des distributions marginales spécifiques, nous dérivons des expressions explicites pour la fonction de densité et d'autres mesures d'intérêt pour l'évaluation du risque agrégé. Basée sur la représentation des copules Archimédiennes par mélange commun, nous proposons une méthode de calcul pour trouver les valeurs exactes ou approximées de la distribution d'une somme de risques aléatoires. Ces résultats sont alors utilisés pour étudier des problèmes d'agrégation, d'allocation du capital et de théorie de la ruine sur différents modèles de risque. Une extension aux copules archimédiennes hiérarchiques est également discutée.

Mots clés : Copules Archimédiennes ; Représentation pas mélange commun ; Méthode d'agrégation ; Mesures de risque ; Allocation du capital ; Théorie de la ruine ; Copules Archimédiennes hiérarchiques

2.2 Abstract

In this paper, we investigate dependent risk models in which the dependence structure is defined by an Archimedean copula. Using such a structure with specific marginals, we derive explicit expressions for the pdf of the aggregated risk and other related quantities. The common mixture representation of Archimedean copulas is at the basis of a computational strategy proposed to find exact or approximated values of the distribution of the sum of risks in a general setup. Such results are then used to investigate risk models in regard to aggregation, capital allocation and ruin problems. An extension to nested Archimedean copulas is also discussed.

Keywords: Archimedean Copulas; Common Mixture Representation; Aggregation Strategy; Risk Measures; Capital Allocation; Ruin Theory; Nested Archimedean Copulas

2.3 Introduction

The present paper deals with risk models incorporating dependent components whose dependence structure is induced via an Archimedean copula. Copula theory provides a flexible approach for modelling the dependency relationship between risks. A copula is a multivariate distribution function for which the marginals are standard uniformly distributed. See e.g. Joe (1997) or Nelsen (2007) for further details. One important class of copulas is the class of Archimedean copulas.

A d-dimensional copula C is said to be an *Archimedean copula* if

$$C(u_1,\ldots,u_d) = \psi\left(\psi^{-1}(u_1)+\ldots+\psi^{-1}(u_d)\right), \text{ for } (u_1,\ldots,u_d)\in[0,1]^d. \tag{2.1}$$

The continuous and strictly decreasing function ψ is called the generator of the copula, where $\psi:[0,\infty)\to[0,1]$, $\psi(0)=1$ and $\lim_{t\to\infty}\psi(t)=0$. In the same manner, $\psi^{-1}:[0,1]\to[0,\infty)$, for which $\psi^{-1}(0)=\inf\{t:\psi(t)=0\}$, where ψ^{-1} is the inverse of the generator ψ. McNeil and Nešlehová (2009) show that (2.1) is a d-dimensional copula if and only if ψ is a d-monotone function. In this paper, we consider a specific class of Archimedean copulas with completely monotone generators. By using Bernstein's theorem (see e.g. Feller (1971)), it has been shown that such generators correspond to the Laplace-Stieltjes Transform (LST) of a strictly positive rv Θ with cumulative distribution function (cdf) F_Θ, where the LST of the rv Θ is given by

$$\mathcal{L}_\Theta(t) = \int_0^\infty \mathrm{e}^{-t\,\theta}\,\mathrm{d}F_\Theta(\theta) = E\left[\mathrm{e}^{-t\Theta}\right]. \tag{2.2}$$

Then, the expression in (2.1) becomes

$$C\left(u_1,\ldots,u_d\right) = \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(u_1\right)+\ldots+\mathcal{L}_\Theta^{-1}\left(u_d\right)\right). \tag{2.3}$$

The strictly positive rv Θ, which can be either discrete or continuous, corresponds to a latent mixing rv, and there is a one-to-one relation between its distribution and the expression of an Archimedean copula C. This special class of Archimedean copulas defined in (2.3) is intimately linked to common mixtures. As explained in Denuit et al. (2006) and Embrechts et al. (2005), the representation of an Archimedean copula C as a common mixture allows us to identify the conditional univariate cumulative distribution functions (cdfs) of $(U_1|\Theta=\theta)$,..., $(U_d|\Theta=\theta)$, where $\underline{U}=(U_1,...,U_d)$ and $U_i\sim Unif\,(0,1)$, $i=1,2,...,d$. Using the definition in (2.2) in combination with (2.3), we have the following representation of an Archimedean copula C as a common mixture:

$$
\begin{aligned}
C\left(u_1,\ldots,u_d\right) &= \psi\left(\psi^{-1}\left(u_1\right)+\ldots+\psi^{-1}\left(u_d\right)\right) \\
&= \mathcal{L}_{\Theta}\left(\mathcal{L}_{\Theta}^{-1}\left(u_1\right)+\ldots+\mathcal{L}_{\Theta}^{-1}\left(u_d\right)\right) \\
&= \int_0^{\infty}\prod_{i=1}^{d}\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(u_i)}\mathrm{d}F_{\Theta}\left(\theta\right)
\end{aligned}
$$

which becomes

$$
C\left(u_1,\ldots,u_d\right)=F_{\underline{U}}\left(u_1,...,u_d\right)=\int_0^{\infty}\prod_{i=1}^{d}F_{U_i|\Theta=\theta}\left(u_i\right)\mathrm{d}F_{\Theta}\left(\theta\right). \tag{2.4}
$$

Then, (2.4) implies that the conditional cdf of $(U_i|\Theta=\theta)$ is $F_{U_i|\Theta=\theta}\left(u_i\right)=\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(u_i)}$, for $i=1,\ldots,d$, $u_i\in[0,1]$, $\theta>0$, which leads to the useful representation of the copula C as a common mixture. Examples of the most popular Archimedean copulas (Θ discrete and continuous) are provided in the Appendix section (see e.g. Nelsen (2007) for an extensive list of Archimedean copulas).

The above representation provides a natural sampling algorithm (see e.g. Marshall and Olkin (1988), Hofert (2008a), and references therein) useful for actuarial science and quantitative risk management purposes. Mixture representations are frequently involved in risk models. For example, in a credit-risk context, Vasicek (1987) was the first to develop the idea of the conditional independence of all defaults upon some market factor. Later, this idea was used by Li (2000), Schönbucher and Schubert (2001) and Frey and McNeil (2001) and put into a copula setup. Among the papers that exploit the conditional independence technique with Archimedean and nested Archimedean copulas in a credit risk context one finds for example Schönbucher (2002) and Hofert and Scherer (2011). Copulas in a credit risk setting is also treated with much detail in Cherubini et al. (2004) and references therein. Several researchers have also used the conditional representation of Archimedean and nested Archimedean copulas to derive their sampling algorithms see e.g. Marshall and Olkin (1988), McNeil (2008) and Hofert (2012). Archimedean risk models are also involved in the context of collective risk models with dependence. Thanks to the latter representation, Albrecher et al. (2011) were able to establish explicit formulas for the ruin probability with dependence among claim sizes and among claim inter-occurrence times modeled by Archimedean copulas. Also, a recent work of Jordanova et al. (2017), considered dependent inter-arrival times and exploited Archimedean copulas as treated in Albrecher et al. (2011).

Our objective here is to explore in more depth the mixture representation of an Archimedean copula and its advantages. More precisely, we propose a new strategy relying on (2.4), to tackle risk assessment problems such as risk aggregation for finite and random sums, capital allocation, ruin problems and so on. We show that this representation allows to avoid entirely or partially

Monte-Carlo (MC) simulations. Furthermore, this methodology is accurate, exact in specific cases, very flexible, and most importantly, naturally applicable in high dimensions.

The outline of the paper is as follows. We propose in Section 2.4, a computational methodology based on the common mixture representation of Archimedean copulas in different settings. This strategy allows to derive the distribution of the aggregated risks which can be later used in different applications. Analytic expressions related to the aggregated risks are also derived for some special cases using specific marginal distributions. Section 2.5 deals with capital allocation issues involving the strategy proposed in Section 2.4. Random sums are then considered in Section 2.6. Sections 2.7 and 2.8 are devoted to the investigation of ruin problems. Finally, Section 2.9 discusses the application of the mixture-based strategy in the case of a hierarchical dependence structure based on Archimedean copulas.

2.4 Computational Methodology based on the common mixture representation of Archimedean copulas

2.4.1 Common mixture representation

Let $\underline{X} = (X_1, ..., X_d)$ be a vector of rvs with multivariate distribution defined in terms of a d-dimensional Archimedean copula C given in (2.3). The multivariate cdf $F_{\underline{X}}$ of $\underline{X}$ can be defined with the copula C and marginal univariate cdfs F_{X_1}, ..., F_{X_d} as

$$F_{\underline{X}}(x_1, ..., x_d) = C\left(F_{X_1}(x_1), ..., F_{X_d}(x_d)\right). \tag{2.5}$$

The common mixture representation of $F_{\underline{X}}$ is given by

$$F_{\underline{X}}(x_1, ..., x_d) = \int_0^\infty \prod_{i=1}^d F_{X_i|\Theta=\theta}(x_i)\, \mathrm{d}F_\Theta(\theta) = \int_0^\infty \prod_{i=1}^d \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(x_i)\right)} \mathrm{d}F_\Theta(\theta), \tag{2.6}$$

where

$$F_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(x_i)\right)} \ (i = 1, 2, ..., d). \tag{2.7}$$

Similarly, we can define the multivariate distribution of $\underline{X}$ through its multivariate survival function with the copula C and marginal univariate survival functions $\overline{F}_{X_1}$, ..., $\overline{F}_{X_d}$, i.e.,

$$\overline{F}_{\underline{X}}(x_1, ..., x_d) = C\left(\overline{F}_{X_1}(x_1), ..., \overline{F}_{X_d}(x_d)\right). \tag{2.8}$$

Then, the common mixture representation of $\overline{F}_{\underline{X}}$ is given by

$$\overline{F}_{\underline{X}}(x_1, ..., x_d) = \int_0^\infty \prod_{i=1}^d \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)} \mathrm{d}F_\Theta(\theta) = \int_0^\infty \prod_{i=1}^d \overline{F}_{X_i|\Theta=\theta}(x_i)\, \mathrm{d}F_\Theta(\theta), \tag{2.9}$$

where

$$\overline{F}_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)} \ (i = 1, 2, ..., d). \tag{2.10}$$

Our main objective in what follows is to find $E[\phi(S)]$, for any integrable univariate function ϕ of the rv $S = X_1 + ... + X_d$, or to find $E[\varphi(X_1, ..., X_d)]$, for any integrable d-variate function φ. The required steps to derive these quantities are as follows:

1. Find the expressions of the conditional cdfs (2.7) or the conditional survival functions (2.10) of $X_i|\Theta = \theta$.
2. Derive the conditional expectations $E[\phi(S)|\Theta = \theta]$ or $E[\varphi(X_1, ..., X_d)|\Theta = \theta]$.
3. Find $E[\phi(S)]$ and $E[\varphi(X_1, ..., X_d)]$ with

$$E[\phi(S)] = E_\Theta[E[\phi(S)|\Theta]] = \int_0^\infty E[\phi(S)|\Theta = \theta]\mathrm{d}F_\Theta(\theta),$$

   and

$$E[\varphi(X_1, ..., X_d)] = E_\Theta[E[\varphi(X_1, ..., X_d)|\Theta]] = \int_0^\infty E[\varphi(X_1, ..., X_d)|\Theta = \theta]\mathrm{d}F_\Theta(\theta),$$

   where $(S|\Theta = \theta) = \sum_{i=1}^d (X_i|\Theta = \theta)$.

We present below three examples with specific marginals but any Archimedean copula with generator $\mathcal{L}_\Theta$ in which the required steps just detailed to find the pdf of S are easily performed and allow explicit expressions.

Example 46. Let $\underline{X} = (X_1, ..., X_d)$ be a vector of exchangeable Bernoulli rvs where $X_i \sim Bern(q)$ $(i = 1, ..., d)$ and

$$F_{\underline{X}}(k_1, ..., k_d) = C(F_{X_1}(k_1), ..., F_{X_d}(k_d)),$$

for $k_1, ..., k_d \in \{0, 1\}$. From (2.7), we find that

$$(X_i|\Theta = \theta) \sim Bern\left(1 - \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(1-q)}\right),$$

for $i = 1, 2, ..., d$. Therefore, $(S|\Theta = \theta)$ follows a binomial distribution with

$$f_{S|\Theta=\theta}(k) = \binom{d}{k}\left(1 - \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(1-q)}\right)^k \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(1-q)(d-k)} = \binom{d}{k}\sum_{j=0}^k \binom{k}{j}(-1)^j \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(1-q)(j+d-k)},$$

and we conclude

$$f_S(k) = \binom{d}{k}\sum_{j=0}^k \binom{k}{j}(-1)^j \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(1-q)(j+d-k)\right),$$

for $k = 0, 1, 2, ..., d$.

Example 47. Let $\underline{X} = (X_1, ..., X_d)$ be a vector of exchangeable Bernoulli rvs where $X_i \sim Bern(q)$ $(i = 1, ..., d)$ and

$$\overline{F}_{\underline{X}}(k_1, ..., k_d) = C\left(\overline{F}_{X_1}(k_1), ..., \overline{F}_{X_d}(k_d)\right)$$

for $k_1, ..., k_d \in \{0, 1\}$. It follows from (2.10) that

$$(X_i|\Theta = \theta) \sim Bern\left(\mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(q)}\right)$$

for $i = 1, 2, ..., d$. Since

$$f_{S|\Theta=\theta}(k) = \binom{d}{k}\mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(q)k}\left(1 - \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(q)}\right)^{n-k} = \binom{d}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\,\mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(q)(k+j)},$$

we obtain

$$f_S(k) = \binom{d}{k}\sum_{j=0}^{d-k}\binom{d-k}{j}(-1)^j\,\mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(q)(k+j)\right),$$

for $k = 0, 1, 2, ..., d$. This expression can be found e.g., in Cossette et al. (2002) or on page 321 in Marceau (2013).

Example 48. Let Θ be a strictly positive rv with LST $\mathcal{L}_\Theta$. Given $\Theta = \theta$, let $(X_1|\Theta = \theta)$, ..., $(X_n|\Theta = \theta)$ conditionally independent rvs where $(X_i|\Theta = \theta)$ follows a geometric distribution with

$$f_{X_i|\Theta=\theta}(k) = \Pr(X_i = k|\Theta = \theta) = \mathrm{e}^{-\theta rk}\left(1 - e^{-r\theta}\right)$$

and

$$\overline{F}_{X_i|\Theta=\theta}(k_i) = \Pr(X_i > k_i|\Theta = \theta) = \mathrm{e}^{-\theta r(k_i+1)}\;(i = 1, 2, ..., n)$$

for $k, k_i = 0, 1, 2, ...$ and $r > 0$. Then, $\underline{X} = (X_1, ..., X_n)$ follows a multivariate mixed geometric distribution where

$$\overline{F}_{\underline{X}}(k_1, ..., k_n) = \mathcal{L}_\Theta(r(k_1 + 1) + ... + r(k_n + 1))$$

with

$$\overline{F}_{X_i}(k_i) = \mathcal{L}_\Theta(r(k_i + 1))\;(i = 1, 2, ..., n).$$

It implies that

$$\begin{aligned}\overline{F}_{\underline{X}}(k_1, ..., k_n) &= \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_1}(k_1)\right) + ... + \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_n}(k_n)\right)\right)\\ &= C\left(\overline{F}_{X_1}(k_1), ..., \overline{F}_{X_n}(k_n)\right)\end{aligned}$$

where C is an Archimedean copula defined with LST $\mathcal{L}_\Theta$.

Now, we are in position to derive the closed-form expression of the pmf of $S_n = \sum_{i=1}^n X_n$. Observe that $(S_n|\Theta = \theta)$ follows a negative binomial distribution with

$$\begin{aligned}f_{X_i|\Theta=\theta}(k) &= \Pr(X_i = k|\Theta = \theta)\\ &= \binom{k+n-1}{k}\mathrm{e}^{-\theta rk}\left(1 - e^{-r\theta}\right)^n\\ &= \binom{k+n-1}{k}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\,\mathrm{e}^{-\theta r(j+k)},\end{aligned}$$

for $k = 0, 1, 2, \ldots$. Then, it follows that

$$f_S(k) = \binom{k+n-1}{k} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \mathcal{L}_\Theta (r(j+k)),$$

for $k = 0, 1, 2, \ldots$.

In the following subsection, we consider the family of multivariate mixed exponential distributions which is equivalent to choosing specific combinations of Archimedean copulas and marginals. For this class of multivariate distributions, we derive analytic expressions for the pdf of the sum of risks and other related quantities of interest.

2.4.2 Closed-form expressions for multivariate mixed exponential distributions

Let $\underline{X}$ follow a multivariate mixed exponential distribution which belongs to the class of multivariate distributions constructed by common frailty as explained in e.g. Marshall and Olkin (1988). Briefly, given $\Theta = \theta$, the conditional distribution of the rv X_i is exponential with mean $\frac{\lambda_i}{\theta}$, i.e.,

$$\overline{F}_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{\frac{-\theta x_i}{\lambda_i}}, \tag{2.11}$$

for $i = 1, \ldots, d$. It implies that the marginal survival function of X_i is given by

$$\overline{F}_{X_i}(x_i) = \mathcal{L}_\Theta \left(\frac{x_i}{\lambda_i} \right), \tag{2.12}$$

for $i = 1, \ldots, d$. Also, the multivariate survival function of $\underline{X}$ is given by

$$\overline{F}_{\underline{X}}(x_1, \ldots, x_d) = \mathcal{L}_\Theta \left(\frac{x_1}{\lambda_1} + \ldots + \frac{x_d}{\lambda_d} \right),$$

which implies that $\overline{F}_{\underline{X}}$ satisfies (2.8) with Archimedean copula C given in (2.3). Clearly, in this setting, the required steps 1, 2, and 3 of the methodology described in Section 2.4.1 are easily performed. The expression in (2.10) is clearly given in (2.11). We first consider the case where $0 < \lambda_1 < \ldots < \lambda_d$. Here, $(S|\Theta = \theta)$ follows a generalized Erlang distribution with pdf

$$f_{S|\Theta=\theta}(x) = \sum_{i=1}^{d} \left(\prod_{j=1,\, j\neq i}^{d} \frac{\lambda_i}{\lambda_i - \lambda_j} \right) \frac{\theta}{\lambda_i} \mathrm{e}^{-\frac{\theta}{\lambda_i} x}. \tag{2.13}$$

Then, using (2.13), the unconditional pdf of S is given by

$$f_S(x) = \int_0^\infty f_{S|\Theta=\theta}(x) \mathrm{d}F_\Theta(\theta) = \sum_{i=1}^{d} \frac{1}{\lambda_i} \left(\prod_{j=1,\, j\neq i}^{d} \frac{\lambda_i}{\lambda_i - \lambda_j} \right) \left((-1) \frac{\mathrm{d}\mathcal{L}_\Theta(t)}{\mathrm{d}t} \Bigg|_{t=\frac{x}{\lambda_i}} \right).$$

The bivariate case ($d = 2$) is detailed on page 295 of Marceau (2013). Sarabia et al. (2017) consider the subclass of multivariate mixed exponential distributions in which $\lambda_1 = ... = \lambda_d = 1$. In such a case, $(S|\Theta = \theta)$ follows an Erlang distribution with

$$f_{S|\Theta=\theta}(x) = \frac{\theta^d x^{d-1}}{\Gamma(d)} \mathrm{e}^{-\theta x}.$$

Sarabia et al. (2017) find the following closed-form expressions for the pdf and the survival function of the rv S:

$$f_S(x) = \frac{x^{d-1}}{\Gamma(d)} \left\{ (-1)^d \frac{\mathrm{d}^d}{\mathrm{d}x^d} \mathcal{L}_\Theta(x) \right\} \tag{2.14}$$

and

$$\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{k!} \left\{ (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathcal{L}_\Theta(x) \right\}, \ x \in \mathbb{R}^+.$$

From (2.14), one can deduce the following expression for the TVaR of the rv S:

$$\begin{aligned} TVaR_\kappa(S) &= \frac{E\left[S \times 1_{\{S > VaR_\kappa(S)\}}\right]}{1-\kappa} \\ &= \sum_{j=0}^{d} \frac{d \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)} \left\{ (-1)^{j-1} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \mathcal{L}_\Theta(x) \bigg|_{x=VaR_\kappa(S)} \right\}, \end{aligned}$$

where $VaR_\kappa(S)$ is the solution of $F_S(x) = \kappa$, with $\kappa \in [0, 1)$. The stop-loss function of S can also be expressed in terms of $\mathcal{L}_\Theta$ as follows

$$\begin{aligned} \Pi_S(y) &= E\left[\max(S - y; 0)\right] \\ &= d \sum_{j=0}^{d} \frac{y^j}{j!} \left\{ (-1)^{j-1} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \mathcal{L}_\Theta(x) \bigg|_{x=y} \right\} - y \sum_{j=0}^{d-1} \frac{y^j}{j!} \left\{ (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathcal{L}_\Theta(x) \bigg|_{x=y} \right\}, \ y \in \mathbb{R}^+. \end{aligned}$$

Sarabia et al. (2017) give explicit formulas for the pdf of S for some multivariate mixed exponential distributions for which the dependence structure is an Archimedean copula with continuous common factor Θ such as the Clayton and Gumbel copulas. In the same context, Dacorogna et al. (2016) use similar explicit formulas to compute analytically risk measures and the associated diversification benefit.

Since there are several much used copulas generated from a discrete mixing rv Θ such as the Ali-Mikhail-Haq (AMH) copula, the Frank copula and the Joe copula, we provide the two following examples to complete those provided in Sarabia et al. (2017). Also, since the derivatives of the generators are known for different Archimedean copulas with discrete or continuous mixing rv (see, e.g., Hofert et al. (2012)), the results provided in the following examples follow directly from the definitions just presented above.

Example 49. Let $\underline{X} = (X_1, \ldots, X_n)$ follow a multivariate mixed exponential-geometric distribution, i.e., $\Theta \sim Geo(q)$ with probability mass function (pmf) $f_\Theta(k) = q\,(1-q)^{k-1}$, for $k \in \mathbb{N}$. Clearly, the dependence structure underlying this multivariate distribution is the AMH copula with dependence parameter $\alpha = 1-q$. Then, the following properties hold:

1. $\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = \frac{q}{\mathrm{e}^x - (1-q)},\ x > 0,\ i = 1, \ldots, d.$
2. $\bar{F}_{\underline{X}}(\underline{x}) = \frac{q}{\exp\left(\sum_{i=1}^d x_i\right) - (1-q)},\ x > 0.$
3. $f_S(x) = \dfrac{x^{d-1} q}{(1-q)\,\Gamma(d)} Li_{-d}\left((1-q)\mathrm{e}^{-x}\right),\ x > 0.$
4. $\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k q}{(1-q)\times k!} Li_{-k}\left((1-q)\mathrm{e}^{-x}\right),\ x > 0.$
5. $TVaR_\kappa(S) = \sum_{j=0}^{d} \frac{dq \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)(1-q)} Li_{1-j}\left((1-q)\mathrm{e}^{-VaR_\kappa(S)}\right),\ \kappa \in [0,1).$
6. $\Pi_S(y) = d\sum_{j=0}^{d} \frac{y^j\, q}{j!\,(1-q)} Li_{1-j}\left((1-q)\mathrm{e}^{-y}\right) - y\sum_{j=0}^{d-1} \frac{q\, y^j}{j!\,(1-q)} Li_{-j}\left((1-q)\mathrm{e}^{-y}\right),\ y \in \mathbb{R}^+.$

We denote by "Li" the general polylogarithm function defined as $Li_a(z) = \sum_{d=1}^{\infty} \frac{z^d}{d^a}$ (see e.g. Lewin (1981) for more details).

Example 50. Let $\underline{X} = (X_1, \ldots, X_n)$ be an n-dimensional vector with multivariate mixed exponential-logarithmic distribution, i.e., $\Theta \sim Log(1-\mathrm{e}^{-\alpha})$ with pmf $f_\Theta(k) = \frac{\left(1-e^{-\alpha}\right)^k}{k\alpha}$, for $k \in \mathbb{N}$. The corresponding dependence structure is the Frank copula with dependence parameter α. Then, the following properties hold:

1. $\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = -\dfrac{\ln\left(1 - (1-\mathrm{e}^{-\alpha})\,\mathrm{e}^{-x}\right)}{\alpha},\ x > 0,\ i = 1, \ldots, d.$
2. $\bar{F}_{\underline{X}}(\underline{x}) = -\dfrac{\ln\left(1 - (1-\mathrm{e}^{-\alpha})\,\mathrm{e}^{-\sum_{i=1}^d x_i}\right)}{\alpha},\ x > 0.$
3. $f_S(x) = \dfrac{x^{d-1}}{\Gamma(d)\,\alpha} Li_{1-d}\left(\dfrac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^x}\right),\ x > 0.$
4. $\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{\alpha \times k!} Li_{1-k}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^x}\right),\ x \in \mathbb{R}^+.$
5. $TVaR_\kappa(S) = \sum_{j=0}^{d} \frac{d \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)\alpha} Li_{2-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^{VaR_\kappa(S)}}\right),\ \kappa \in [0,1).$
6. $\Pi_S(y) = d\sum_{j=0}^{d} \frac{y^j}{j!\,\alpha} Li_{2-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^y}\right) - y\sum_{j=0}^{d-1} \frac{y^j}{j!\,\alpha} Li_{1-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^y}\right),\ y \in \mathbb{R}^+.$

In this subsection, we have discussed multivariate mixed exponential distributions which, as highlighted in the examples, require specific combinations of the mixing rv Θ and the marginal distributions of X_i ($i = 1, ..., d$). The dependence structure defined by the Archimedean copula is governed by Θ and hence only one choice of marginal distribution can be linked through such a dependency framework. This is very limitative since it does not allow to choose the dependence construction without any regard to the specification of the marginals. This is somewhat counterintuitive with copulas being used as flexible tools to build dependence risk models.

In the next three subsections, we provide insight on how to benefit from the mixture representation of an Archimedean copula for any multivariate distribution defined through an Archimedean copula with a discrete mixing rv Θ and discrete marginals. Furthermore, we provide a strategy that can be used when one or both of these rvs are continuous.

2.4.3 Discrete mixing rv Θ and discrete marginals

Our strategy is to use the conditional independence assumption to identify the conditional distribution of X_i through (2.7) or (2.10). This step is usually more difficult for continuous rvs X_i than for discrete ones which is the basis of the computational strategy presented in Section 2.4.3. We have recourse to discretization methods in the continuous case. The conditional distributions of S given $\Theta = \theta$ are also easier to identify for discrete distributions.

The proposed strategy has many advantages. First, it is easy to implement regardless of the portfolio's dimension. Second, it yields the exact values of F_S for discrete risks X_i, $i = 1, \ldots, d$ and it gives an accurate approximation for continuous ones. The proposed strategy can also be used in the context of many actuarial risk models involving Archimedean dependence which will be discussed in Sections 2.6, 2.7 and 2.8.

Our strategy is to make use of the conditional independence representation as well as the convolution of independent rvs to derive a simple computation approach for the distribution of S and any integrable function of $\underline{X}$. Given these techniques, risk aggregation problems for a portfolio of dependent risks with a multivariate joint distribution defined with an Archimedean copula or a nested Archimedean copula, as well as other related quantities become easier to deal with.

Let $\underline{X} = (X_1, \ldots, X_d)$ be a vector of discrete rvs such that $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, ..., d)$. The univariate pmf of X_i, the univariate cdf of X_i, the multivariate pmf of $\underline{X}$, and the multivariate cdf of $\underline{X}$ are respectively denoted by $f_{X_i}(k_i h) = \Pr(X_i = k_i h)$, $F_{X_i}(k_i h) = \Pr(X_i \leq k_i h)$,

$$f_{\underline{X}}(k_1 h, ..., k_d h) = \Pr(X_1 = k_1 h, ..., X_d = k_d h),$$

and

$$F_{\underline{X}}(k_1 h, ..., k_d h) = \Pr(X_1 \leq k_1 h, ..., X_d \leq k_d h),$$

for $h > 0$ and $k_1, ..., k_n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In theory, exact values of f_S and $E[\varphi(X_1, ..., X_d)]$ can be found with

$$f_S(kh) = \sum_{k_1=0}^{k} \cdots \sum_{k_{d-1}=0}^{k-(k_1+\ldots+k_{d-2})} f_{\underline{X}}\left(k_1 h, \ldots, k_{d-1} h, \left(k - \sum_{j=1}^{d-1} k_j\right) h\right) \tag{2.15}$$

and

$$E[\varphi(X_1, ..., X_d)] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \varphi(k_1 h, ..., k_d h) f_{\underline{X}}(k_1 h, \ldots, k_d h), \tag{2.16}$$

where

$$f_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = \sum_{i_1=0,1} \cdots \sum_{i_d=0,1} (-1)^{i_1+\ldots+i_d} \times F_{\underline{X}}\left(\left(k_1-i_1\right) h, \ldots,\left(k_d-i_d\right) h\right). \tag{2.17}$$

The computation of (2.15), (2.16), and (2.17) is feasible when the dimension d of $\underline{X}$ is small (e.g. $d = 2$, 3 or 4). However, it rapidly becomes impracticable when d gets larger.

Assume that the dependence structure of $\underline{X}$ is induced via an Archimedean copula C. In such a case, the multivariate cdf of $\underline{X}$ (or its survival function) is defined with the copula C and the univariate cdfs (or the univariate survival functions) of $X_1, \ldots, X_d$, i.e, (2.5) and (2.8) become

$$F_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = C\left(F_{X_1}\left(k_1 h\right), \ldots, F_{X_d}\left(k_d h\right)\right) \tag{2.18}$$

or

$$\overline{F}_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = C\left(\overline{F}_{X_1}\left(k_1 h\right), \ldots, \overline{F}_{X_d}\left(k_d h\right)\right), \tag{2.19}$$

for $k_i \in \mathbb{N}$ and $i = 1, \ldots, d$.

In the following, let the copula C in either (2.18) or (2.19) be an Archimedean copula defined by a discrete mixing rv Θ such that $E[\Theta]$ is finite. Then, we can take advantage of the common mixture representation to compute the exact values of f_S and $E\left[\varphi\left(X_1, ..., X_d\right)\right]$.

Assuming discrete marginals and a discrete mixing rv Θ and using (2.6), the expression for $F_{\underline{X}}$ in (2.18) becomes

$$F_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} F_{X_i|\Theta=\theta}\left(k_i h\right) f_{\Theta}(\theta), \tag{2.20}$$

and, from (2.7), we have

$$F_{X_i|\Theta=\theta}\left(k_i h\right) = \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}\left(k_i h\right)\right)}, \tag{2.21}$$

for $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, d$, and $\theta \in \mathbb{N}$. For $i = 1, 2, ..., d$ and for each $\theta \in \mathbb{N}$, we can easily find the values of $f_{X_i|\Theta=\theta}\left(k_i h\right)$ with

$$f_{X_i|\Theta=\theta}\left(k_i h\right) = \begin{cases} \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}\left(k_i h\right)\right)} - \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}\left(\left(k_i-1\right) h\right)\right)} & , k_i \in \mathbb{N} \end{cases}. \tag{2.22}$$

Similarly, using (2.9), the expression for $\overline{F}_{\underline{X}}$ in (2.19) turns into

$$\overline{F}_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} \overline{F}_{X_i|\Theta=\theta}\left(k_i h\right) f_{\Theta}(\theta), \tag{2.23}$$

and, from (2.10), we have

$$\overline{F}_{X_i|\Theta=\theta}\left(k_i h\right) = \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_i}\left(k_i h\right)\right)}, \tag{2.24}$$

for $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, d$, and $\theta \in \mathbb{N}$. It follows that

$$f_{X_i|\Theta=\theta}(k_i h) = \begin{cases} 1 - \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}((k_i-1)h)\right)} - \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(k_i h)\right)} & , k_i \in \mathbb{N} \end{cases}, \tag{2.25}$$

for $i = 1, 2, ..., d$ and for each $\theta \in \mathbb{N}$.

Then, from either (2.22) or (2.25), the expression for $f_{\underline{X}}(k_1 h, \ldots, k_d h)$ is now given by

$$f_{\underline{X}}(k_1 h, \ldots, k_d h) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} f_{X_i|\Theta=\theta}(k_i h) f_\Theta(\theta). \tag{2.26}$$

Let $(S|\Theta = \theta) = \sum_{i=1}^{d}(X_i|\Theta = \theta)$ be the sum of conditionally independent rvs and $f_{S|\Theta=\theta}$ be the corresponding pmf. Due to the representation of $f_{\underline{X}}$ in (2.26), the expression for the pmf of S can be written as follows

$$f_S(kh) = \sum_{\theta=1}^{\infty} f_{S|\Theta=\theta}(kh) f_\Theta(\theta), \ k \in \mathbb{N}_0. \tag{2.27}$$

Since $f_{S|\Theta=\theta}$ corresponds to the convolution product of $f_{X_1|\Theta=\theta}$, ..., $f_{X_d|\Theta=\theta}$, traditional aggregation tools (e.g. FFT and DePril algorithm see e.g. Panjer et al. (2008)) from actuarial science can be applied to find values of $f_{S|\Theta=\theta}$ for each θ. It is important to note here that our strategy leads to exact values of f_S contrarily to MC simulation methods. This is also true even when d is large.

Given the representation of $f_{\underline{X}}$ in (2.26), $E[\varphi(X_1, ..., X_d)]$ becomes

$$E[\varphi(X_1, ..., X_d)] = \sum_{\theta=1}^{\infty} E[\varphi(X_1, ..., X_d)|\Theta = \theta] f_\Theta(\theta), \tag{2.28}$$

where

$$E[\varphi(X_1, ..., X_d)|\Theta = \theta] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \varphi(k_1 h, ..., k_d h) \prod_{i=1}^{d} f_{X_i|\Theta=\theta}(k_i h). \tag{2.29}$$

Applications of (2.28) and (2.29) are provided in Section 2.5.

The procedure to compute the values of $f_S(kh)$ is summarized in the following algorithm.

Algorithm 2. ***Computation of the values of*** f_S.

1. *Let* θ^* *be chosen such that* $F_\Theta(\theta^*) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*).*
2. *Fix* $\theta = 1$.
3. *For* $i = 1, ..., d$, *calculate either* $F_{X_i|\Theta=\theta}(k_i h)$ *with (2.21) or* $\overline{F}_{X_i|\Theta=\theta}(k_i h)$ *with (2.24), for* $k_i \in \mathbb{N}_0$.
4. *For* $i = 1, ..., d$, *calculate* $f_{X_i|\Theta=\theta}(k_i h)$ *with either (2.22) ou (2.25).*

5. *Using e.g. FFT or DePril's Algorithm, compute* $f_{S|\Theta=\theta}(kh)$ *for* $k \in \mathbb{N}_0$.
6. *Repeat steps 3, 4, and 5 for* $\theta = 2, ..., \theta^*$.
7. *Compute* $f_S(kh) = \sum_{\theta=1}^{\theta^*} f_{S|\Theta=\theta}(kh) f_\Theta(\theta)$*, for* $k \in \mathbb{N}_0$.

Some remarks must be made in regard to the proposed methodology of this section and the tail dependence of an Archimedean copula C. Let us recall that if the lower and upper-tail dependence coefficients exist for an Archimedean copula C with mixing rv Θ, then according to Joe and Hu (1996), these coefficients λ_L and λ_U can be written in terms of $\mathcal{L}_\Theta$ as follows

$$\lambda_L = \lim_{t\to\infty} \frac{\mathcal{L}_\Theta(2t)}{\mathcal{L}_\Theta(t)} = 2 \lim_{t\to\infty} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}$$

and

$$\lambda_U = 2 - \lim_{t\to 0} \frac{1-\mathcal{L}_\Theta(2t)}{1-\mathcal{L}_\Theta(t)} = 2 - 2\lim_{t\to 0} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}.$$

For the special case of discrete mixing rvs Θ, the underlying Archimedean copulas cannot have lower tail dependence, i.e. $\lambda_L = 0$. This implies that our proposed methodology does not allow lower tail dependence. Also, if $E[\Theta]$ is finite, then $\lambda_U = 0$ (see Hofert (2010) page 62 for proof). This means that it is possible to find a finite θ^* for which the methodology described in Algorithm 2 leads to exact values of the pmf of the rv S. In such a case, the methodology works well for notably the AMH and Frank copulas. Note that the proposed methodology cannot be applied when $E[\Theta]$ is infinite (e.g. for Joe's copula) since Algorithm 2 suggests to truncate the distribution of Θ at θ^*. Such a truncation leads to an Archimedean copula with no upper tail dependence which violates the initial assumption.

In the following two examples, we illustrate the accuracy of our computational methodology. More precisely, we present a first example which considers a small portfolio. This allows us to compare the values of f_S obtained with Algorithm 2 and (2.15). As expected, both results coincide. The second example is somewhat similar but illustrates the applicability of Algorithm 2 to large portfolios.

Example 51. Let $F_{\underline{X}}$ be defined as in (2.18) with the Frank copula as given in the Appendix and $X_i \sim Bin(10, q_i)$, where $q_i = 0.1i$, for $i = 1, 2, 3, 4$. In this case, the mixing rv Θ follows a logarithmic distribution. While values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ are given in Table 2.1, Table 2.2 provides the exact values of f_S obtained with (2.15) and Algorithm 2.

Example 52. Let $X_i \sim Bin(10, q_i)$, with $q_i = 0.1$, for $i = 1, ..., 100$. The joint cdf $F_{\underline{X}}$ is defined as in (2.18) with the AMH copula. In Figure 2.1, we depict the exact values of F_S obtained with Algorithm 2 ($\alpha = 0.5$ and 0.9). For comparison purposes, the exact values of $F_{S^\perp}$, where $S^\perp$ is the sum of the independent rvs $X_1^\perp$, ..., $X_{100}^\perp$ and $X_i^\perp \sim X_i$ for $i = 1, 2, ..., 100$, are also provided. It is well known that the AMH copula introduces a low to moderate positive dependence relation between the rvs X_1, ..., X_{100}. However, the impact is clearly significant when the number of

	$\alpha = 1$	$\alpha = 3$	$\alpha = 6$
$E[S]$	10	10	10
$Var(S)$	9.99256	15.15425	19.90096
$VaR_{0.9}(S)$	14	15	16
$VaR_{0.999}(S)$	20	21	23
$TVaR_{0.9}(S)$	15.82535	17.11038	18.04888
$TVaR_{0.999}(S)$	20.88055	22.39553	23.41423

Table 2.1 – Values of the expectation, variance, VaR and TVaR of $S = X_1 + ... + X_4$ where $F_{X_1,...,X_4}$ is defined with the Frank copula.

		Exact values of $f_S(s)$	Exact values of $f_S(s)$
α	s	(with (2.15))	(with Algorithm 2)
1	0	1.992354e-05	1.992354e-05
	5	0.03925912	0.03925912
	10	0.1199567	0.1199567
3	0	0.0001013881	0.0001013881
	5	0.0619347088	0.0619347088
	10	0.0887858345	0.0887858345
6	0	0.0003887516	0.0003887516
	5	0.0691872490	0.0691872490
	10	0.0767524930	0.0767524930

Table 2.2 – Values of the pmf of $S = X_1 + ... + X_4$ where $F_{X_1,...,X_4}$ is defined with the Frank copula.

	$\alpha = 0$ (independence)	$\alpha = 0.5$	$\alpha = 0.9$
$E[S]$	100	100	100
$Var(S)$	90	1454.027	2793.690
$VaR_{0.9}(S)$	112	156	172
$VaR_{0.999}(S)$	130	225	242
$TVaR_{0.9}(S)$	116.934	176.206	192.651
$TVaR_{0.999}(S)$	133.277	233.651	250.590

Table 2.3 – Values of the expectation, variance, VaR and TVaR of $S = X_1 + ... + X_{100}$ where $F_{X_1,...,X_{100}}$ is defined with the AMH copula.

risks of the portfolio becomes large as illustrated in Figure 2.1. In Table 2.3, we provide the values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$ and $TVaR_\kappa(S)$. Note that the computation time increases as the dependence parameter becomes larger. Indeed, $\theta^* = F_\Theta^{-1}(1-\varepsilon;\alpha)$ increases with α. For example, if $\varepsilon = 10^{-10}$, $\theta^* = 34$ and $\theta^* = 219$ for $\alpha = 0.5$ and $\alpha = 0.9$ respectively.

Example 52 allows us to better understand the impact of the dependence between individual risks on the overall exposure evaluation. As shown in Figure 2.1, using a portfolio of 100 risks highlights this significant impact. Indeed, one cannot neglect the dependence, even moderate, for a portfolio of large dimension. Note that the proposed algorithm allowed us to make such a

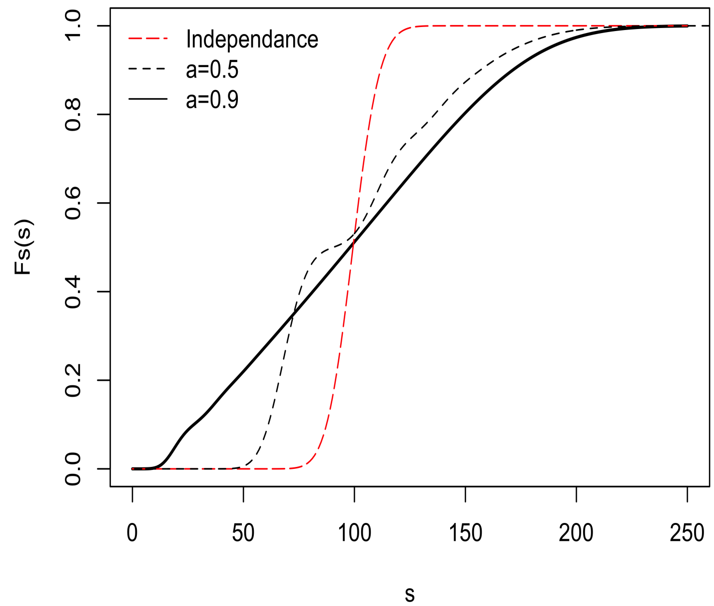

Figure 2.1 – The cdf of $S = X_1 + ... + X_{100}$ where $F_{X_1,...,X_{100}}$ is defined by the AMH copula with $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 0.9$.

conclusion, which is not possible using the convolution method in (2.15), since it only applies to small portfolios.

2.4.4 Discrete mixing rv Θ and continuous marginals

Let us assume that $\underline{X} = (X_1, ..., X_d)$ is a vector of continuous positive rvs where the multivariate cdf is defined by an Archimedean copula C as in (2.5) and the mixing rv Θ is discrete. In this case, the computation of $F_{S|\Theta=\theta}(x)$ or eventually $E[\varphi(X_1, ..., X_d)|\Theta = \theta])$ becomes more difficult. Since $(X_1|\Theta = \theta)$, ..., $(X_d|\Theta = \theta)$ are conditionally independent rvs, one can apply the tools at hand to compute an accurate (as possible) approximation of $F_{S|\Theta=\theta}(x)$ or $E[\varphi(X_1, ..., X_d)|\Theta = \theta]$ for the sum or any function of independent rvs. Note that the approach described here can be easily adapted when the multivariate survival function instead of the multivariate cdf is defined with an Archimedean copula C as in (2.8) and when the mixing rv Θ is discrete.

Inspired by Bargès et al. (2009), we propose an approximation based on discretization methods and the application of Algorithm 2. This approximation leads to numerical bounds as accurate as one may desire. Indeed, we approximate $\underline{X}$ by $\underline{\widetilde{X}} = \left(\widetilde{X}_1, ..., \widetilde{X}_d\right)$, a vector of discrete rvs with $\widetilde{X}_i \in A = \{0, 1h, 2h, \ldots\}$, for $i = 1, ..., d$ and a discretization step $h > 0$. The multivariate cdf of $\underline{\widetilde{X}}$ is defined with the same copula C and with marginals $F_{\widetilde{X}_1}$, ..., $F_{\widetilde{X}_d}$ obtained with a

	$\widetilde{S}^{(u,1)}$	$\widetilde{S}^{(u,0.1)}$	$\widetilde{S}^{(l,0.1)}$	$\widetilde{S}^{(l,1)}$	$\widetilde{S}^{(m,1)}$	$\widetilde{S}^{(m,0.1)}$	MC	95%-Confidence Int.
Expectation	380.333	398.003	402.003	420.333	400	400	399.807	[398.830 ; 400.783]
Variance	24667.624	24749.656	24749.656	24667.624	24796.995	24750.952	24817.951	[24601.844 ; 25036.932]
$\text{VaR}_{0.9}$	606	624	628.4	646	627	626.4	627.309	[624.872 ; 629.670]
$\text{VaR}_{0.999}$	975	993.2	997.2	1015	995	995.2	989.248	[981.237 ; 1000.459]
$\text{TVaR}_{0.9}$	705.605	723.321	727.319	745.605	725.772	725.329	726.247	[725.724 ; 726.771]
$\text{TVaR}_{0.999}$	1027.286	1043.722	1047.716	1067.286	1047.454	1045.747	1039.461	[1039.115 ; 1039.806]

Table 2.4 – Approximated values of the expectation, variance, VaR and TVaR of $S = X_1 + ... + X_{40}$ where $F_{X_1,...,X_{40}}$ is defined with the AMH copula and continuous marginals. The discretization steps is either $h = 1$ or $h = 0.1$. For the MC method, the number of simulations is 100000.

discretization method, i.e., $F_{\underline{\widetilde{X}}}(k_1 h, ..., k_d h) = C\left(F_{\widetilde{X}_1}(k_1 h), ..., F_{\widetilde{X}_d}(k_d h)\right)$, for $k_1, ..., k_d \in \mathbb{N}_0$. In this article, we consider the upper, lower and mean preserving discretization methods (see e.g. Müller and Stoyan (2002) or Bargès et al. (2009) for details).

Example 53. Let $X_i \sim Exp(0.1)$, for $i = 1, 2, ..., d$ $(d = 40)$. The multivariate cdf of $\underline{X}$, $F_{\underline{X}}$, is defined as in (2.5) with the AMH copula ($\alpha = 0.5$). For $S = X_1 + \ldots + X_{40}$, we compute the upper and lower bounds to F_S with $h = 1$ and $h = 0.1$. Values of the expectation, variance, VaR and TVaR of the rvs $\widetilde{S}^{(u,1)}$, $\widetilde{S}^{(u,0.1)}$, $\widetilde{S}^{(l,1)}$, $\widetilde{S}^{(l,0.1)}$, $\widetilde{S}^{(m,1)}$, and $\widetilde{S}^{(m,0.1)}$ are given in Table 2.4. Note that "u", "l", and "m" in the superscripts refer respectively to the upper, lower and mean preserving discretization methods. Also, the approximated values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ ($\kappa = 0.9, 0.999$) obtained using 100000 MC simulations are provided (with 95%-level confidence intervals given in the last column). Clearly, as the discretization step h goes to 0, the difference between the bounds also tends to 0. Also, notice that the approximated values $\widetilde{VaR}^{MC}_{0.999}(S)$ and $\widetilde{TVaR}^{MC}_{0.999}(S)$ of $VaR_{0.999}(S)$ and $TVaR_{0.999}(S)$ obtained by 100000 MC simulations are outside the interval defined by the upper and the lower bounds. This is also illustrated in Figure 2.2. In the left panel, we provide the values for the cdfs of the rvs $\widetilde{S}^{(u,1)}$, $\widetilde{S}^{(u,0.1)}$, $\widetilde{S}^{(l,1)}$, and $\widetilde{S}^{(l,0.1)}$ and the approximated values of F_S, obtained with 100000 MC simulations. The close-up on the on Figure 2.2 clearly shows that the approximated values of F_S obtained with MC simulations lie outside of the upper and lower bounds $F_{\widetilde{S}^{(u,0.1)}}$ and $F_{\widetilde{S}^{(l,0.1)}}$. The proposed approximation has the advantage to allow to control the precision of the approximation. On the left panel of Figure 2.2, the exact values of $F_{S^\perp}$, where $S^\perp$ is the sum of the independent rvs $X_1^\perp$, ..., $X_{100}^\perp$ and $X_i^\perp \sim X_i$ for $i = 1, 2, ..., 100$, are also depicted.

2.4.5 Continuous mixing rv Θ

The common mixture representation in (2.6) and (2.9) leads to a two-step natural sampling procedure for $\underline{X}$. The first step is to simulate a sampled value of Θ. Then, in the second step, the sampled value of $\underline{X}$ is computed via the conditional distribution of $\underline{X}$ given the sampled value of Θ using either (2.7) or (2.10). See, e.g., Marshall and Olkin (1988) or Hofert (2008b)) for details. We examine two alternatives to this approach: one based on the simulation of the mixing rv Θ and another one on the approximation of Θ by a discrete rv.

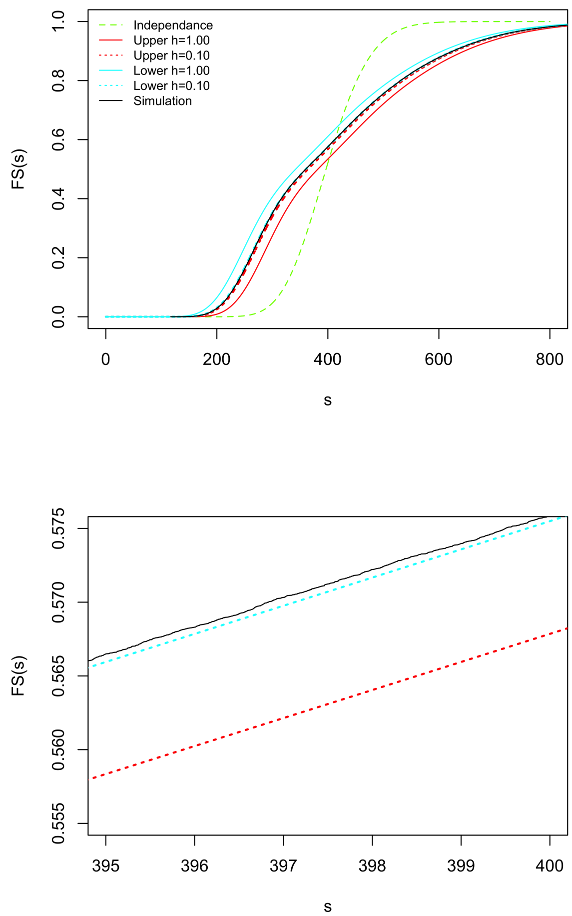

Figure 2.2 – Approximated values of the cdf of $S = X_1 + ... + X_{40}$ and zoom in for values of s between 395 and 400 to illustrate that simulated values go outside the lower and upper bounds.

We consider a vector of discrete (or discretized) rvs $\underline{X} = (X_1, \ldots, X_d)$ where $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, ..., d)$. Let the rv Z be the estimator of $\varphi(X_1, .., X_d)$ under a MC simulation approach, which is constructed using the two-step sampling procedure of $\underline{X}$. The standard error of the estimator Z is given by $\sqrt{Var(Z)}$. We can use a variance reduction technique, namely the conditional MC simulation method (see, e.g., Lemieux (2009) or Kroese et al. (2013) for details on this topic) to reduce this standard error. First, one generates sampled values of the rv Θ and then applies Algorithm 2 as follows. We produce m sampled values of Θ, denoted by $\Theta^{(1)}$, ..., $\Theta^{(m)}$. Then, (2.27) becomes $f_S(kh) \simeq \sum_{j=1}^{m} f_{S|\Theta=\Theta^{(j)}}(kh)\frac{1}{m}$, $k \in \mathbb{N}_0$, where the values of $f_{S|\Theta=\Theta^{(j)}}(kh)$ are computed using e.g. FFT or DePril's Algorithm. Then, $E[Z|\Theta]$ corresponds to the resulting conditional MC approximation of $\varphi(X_1, .., X_d)$. The standard error of $E[Z|\Theta]$ is given by $\sqrt{Var(E[Z|\Theta])}$. Since

$$Var(Z) = E[Var(Z|\Theta)] + Var(E[Z|\Theta]), \tag{2.30}$$

it is clear that

$$\sqrt{Var(E[Z|\Theta])} \leq \sqrt{Var(Z)}. \tag{2.31}$$

The magnitude of the difference $\sqrt{Var(Z)} - \sqrt{Var(E[Z|\Theta])}$ is analyzed in the following example.

Example 54. Let $\underline{X}^{(C,s,\alpha)} = \left(X_1^{(C,s,\alpha)}, ..., X_{50}^{(C,s,\alpha)}\right)$ and $\underline{X}^{(G,s,\alpha)} = \left(X_1^{(G,s,\alpha)}, ..., X_{50}^{(G,s,\alpha)}\right)$ be two vectors of rvs with $X_i^{(C,s,\alpha)} \sim X_i^{(G,s,\alpha)} \sim Bin\,(100, 0.2)$, for $i = 1, ..., 50$, where their joint survival function is defined either with the Clayton copula or the Gumbel copula according to (2.8). Similarly, let $\underline{X}^{(C,f,\alpha)} = \left(X_1^{(C,f,\alpha)}, ..., X_{50}^{(C,f,\alpha)}\right)$ and $\underline{X}^{(G,f,\alpha)} = \left(X_1^{(G,f,\alpha)}, ..., X_{50}^{(G,f,\alpha)}\right)$ be two vectors of rvs with $X_i^{(C,f,\alpha)} \sim X_i^{(G,f,\alpha)} \sim Bin\,(100, 0.2)$, for $i = 1, ..., 50$, where their joint cdf is defined either with the Clayton copula or the Gumbel copula according to (2.5). For each copula, the dependence parameter α of the copula is fixed such that Kendall's tau is equal to 0.2, 0.5, or 0.8. The exact values of the covariances between pairs of rvs for each vector of rvs are provided in Tables 2.5 and 2.6 for the three values of the dependence parameter. We define $S^{(C,s,\alpha)} = \sum_{i=1}^{50} X_i^{(C,s,\alpha)}$, $S^{(G,s,\alpha)} = \sum_{i=1}^{50} X_i^{(G,s,\alpha)}$, $S^{(C,f,\alpha)} = \sum_{i=1}^{50} X_i^{(C,f,\alpha)}$, and $S^{(G,f,\alpha)} = \sum_{i=1}^{50} X_i^{(G,f,\alpha)}$. Clearly, $E[S^{(C,s,\alpha)}] = E[S^{(G,s,\alpha)}] = E[S^{(C,f,\alpha)}] = E[S^{(G,f,\alpha)}] = 1000$, whatever the values of the dependence parameters. In Tables 2.7 and 2.8, we also give the exact values of $Var(S^{(C,s,\alpha)})$, $Var(S^{(G,s,\alpha)})$, $Var(S^{(C,f,\alpha)})$, and $Var(S^{(G,f,\alpha)})$ for the three values of their dependence parameter. In Tables 2.9 to 2.14, we provide the approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$, $\overline{F}_{S^{(G,s,\alpha)}}$, $\overline{F}_{S^{(C,f,\alpha)}}$, and $\overline{F}_{S^{(G,f,\alpha)}}$. Those values are computed using both the conditional MC and the MC approaches with $m = 100000$ simulations. In parentheses, we indicate the values of the standard deviation for each approximation. As expected from (2.31), we observe that the standard error of the approximation based on the conditional MC approach is lower than the corresponding one for the approximation based on the MC approach. For a given multivariate distribution, we observe that the improvement is more significant as the dependence parameter α decreases. The improvement is also more significant for large values of x. However, for a specific value of Kendall's tau, the improvement differs from one multivariate distribution to another. Notably, we observe that the improvement is the least significant for the results associated to $\underline{X}^{(C,s,\alpha)}$ and $\underline{X}^{(G,f,\alpha)}$, suggesting that the improvement observed with the conditional MC approach is less significant with multivariate distributions having a non-zero right tail dependence.

α	$Cov(X_i^{(G,s,\alpha)}, X_j^{(G,s,\alpha)})$	$Cov(X_i^{(G,f,\alpha)}, X_j^{(G,f,\alpha)})$
1.25	4.893995	5.11945
2	11.01781	11.2806
5	15.01042	15.09308

Table 2.5 – Values of $Cov(X_i^{(G,s,\alpha)}, X_j^{(G,s,\alpha)})$ and $Cov(X_i^{(G,f,\alpha)}, X_j^{(G,f,\alpha)})$, $i \neq j \in \{1, 2, ..., 50\}$.

Let us now consider the case of the Clayton copula for which Θ is gamma distributed. To obtain f_S, we can approximate the Clayton copula with the shifted negative binomial copula

α	$Cov(X_i^{(C,s,\alpha)}, X_j^{(C,s,\alpha)})$	$Cov(X_i^{(C,f,\alpha)}, X_j^{(C,f,\alpha)})$
0.5	5.20697	4.859871
2	11.14602	10.62849
8	14.72897	14.44488

Table 2.6 – Values of $Cov(X_i^{(C,s,\alpha)}, X_j^{(C,s,\alpha)})$ and $Cov(X_i^{(C,f,\alpha)}, X_j^{(C,f,\alpha)})$, $i \neq j \in \{1, 2, ..., 50\}$.

α	$Var(S^{(G,s,\alpha)})$	$Var(S^{(G,f,\alpha)})$
1.25	12790.29	13342.65
2	27793.64	28437.48
5	37575.52	37778.05

Table 2.7 – Values of $Var(S^{(G,s,\alpha)})$ and $Var(S^{(G,f,\alpha)})$, for $\alpha = 1.5, 2, 5$.

α	$Var(S^{(C,s,\alpha)})$	$Var(S^{(C,f,\alpha)})$
0.5	13557.08	12706.68
2	28107.76	26839.79
8	36885.98	36189.97

Table 2.8 – Values of $Var(S^{(C,s,\alpha)})$ and $Var(S^{(C,f,\alpha)})$, for $\alpha = 0.5, 2, 8$.

allowing us to use Algorithm 2. This family includes the AMH copula as a special case and the Clayton copula as a limit case (see Cossette et al. (2017) for more details). The idea here is to approximate the rv Θ by a discrete rv $\widetilde{\Theta}$ and apply the proposed methodology of Section 2.4.3. However, we need to be careful. We cannot blindly apply the three discretization methods used in the previous section because we aim to approximate the copula generated from Θ and not only the distribution of Θ. Thus, finding an appropriate discrete rv $\widetilde{\Theta}$ is not an easy task as we will see in the special case of the Clayton copula.

The multivariate shifted negative binomial copula is defined by

$$C_{\alpha,q_h}^{SNB}(u_1, ..., u_d) = \left(q_h \left(\prod_{i=1}^{d} \left(q_h u_i^{-\alpha} + (1-q_h) \right) - (1-q_h) \right)^{-1} \right)^{\frac{1}{\alpha}}. \tag{2.32}$$

The two parameters of the copula are $\alpha > 0$ and $q_h = 1 - \mathrm{e}^{-h}$, where $h > 0$ can be seen as a discretization parameter.

The underlying mixing rv, associated to (2.32) and denoted by $\Theta_{(h)}^{SNB(\alpha)}$, follows a shifted negative binomial distribution, i.e., the rv $\Theta_{(h)}^{SNB(\alpha)}$ is defined as $\Theta_{(h)}^{SNB(\alpha)} = h\left(M_{(h)}^{NB(\alpha)} + \alpha\right)$ where $M_{(h)}^{NB(\alpha)}$ follows a negative binomial distribution, i.e., $M_{(h)}^{NB(\alpha)} \sim NB\left(\frac{1}{\alpha}, q_h\right)$, with $f_{M_{(h)}^{NB(\alpha)}}(k) = \binom{\frac{1}{\alpha}+k-1}{k} (q_h)^{\frac{1}{\alpha}} (1-q_h)^k$, $k \in \mathbb{N}_0$, and $E\left[M_{(h)}^{NB(\alpha)}\right] = \frac{1-q_h}{\alpha q_h}$ with $q_h = 1 - \mathrm{e}^{-h}$, $h > 0$. The LST

i	x_i	CMC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$
1	1216	0.050527249 (0.198427516)	0.050140000 (0.218234866)	1157	0.050705395 (0.174562788)	0.050730000 (0.219446915)
2	1354	0.010079740 (0.090022495)	0.010160000 (0.100283972)	1201	0.010331348 (0.070988687)	0.010450000 (0.101690220)
3	1332	0.013160376 (0.102769800)	0.013100000 (0.113703647)	1243	0.001055487 (0.018413807)	0.001150000 (0.033892315)
4	1698	0.000108547 (0.009061601)	0.000090000 (0.009486453)	1273	0.000107720 (0.004440821)	0.000060000 (0.007745773)
5	1840	0.000010584 (0.002904670)	0.000020000 (0.004472114)	1296	0.000010881 (0.000807741)	0.000010000 (0.003162278)

Table 2.9 – Approximated values of $\overline{F}_{S(C,s,\alpha)}$ and $\overline{F}_{S(C,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 0.5$ ($\tau = 0.2$). The values in parentheses correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$
1	1313	0.049968514 (0.211556800)	0.050030000 (0.218007969)	1232	0.050956074 (0.192522672)	0.050430000 (0.218831657)
2	1468	0.010020219 (0.096953570)	0.010000000 (0.099499241)	1289	0.010297759 (0.081601626)	0.010130000 (0.100137323)
3	1641	0.001008032 (0.030668361)	0.000990000 (0.031448844)	1340	0.001000416 (0.021596196)	0.000870000 (0.029483076)
4	1798	0.000100325 (0.009958315)	0.000100000 (0.009999550)	1374	0.000104887 (0.006010938)	0.000130000 (0.011401070)
5	1958	0.000010000 (0.003162278)	0.000010000 (0.003162278)	1397	0.000012494 (0.001157376)	0.000000000 (-)

Table 2.10 – Approximated values of $\overline{F}_{S(C,s,\alpha)}$ and $\overline{F}_{S(C,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parentheses correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$
1	1337	0.050074019 (0.215637266)	0.050050000 (0.218049244)	1303	0.050640550 (0.203357022)	0.050360000 (0.218687788)
2	1489	0.010064309 (0.098773129)	0.010130000 (0.100137323)	1374	0.010921753 (0.090974062)	0.011040000 (0.104490323)
3	1649	0.001000266 (0.030807217)	0.001010000 (0.031764603)	1436	0.001052248 (0.024805081)	0.001090000 (0.032997315)
4	1799	0.000111861 (0.010159259)	0.000120000 (0.010953849)	1472	0.000103602 (0.006558965)	0.000080000 (0.008943959)
5	1960	0.000010000 (0.003162278)	0.000010000 (0.003162278)	1497	0.000010341 (0.001357239)	0.000020000 (0.004472114)

Table 2.11 – Approximated values of $\overline{F}_{S(C,s,\alpha)}$ and $\overline{F}_{S(C,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 8$ ($\tau = 0.8$). The values in parentheses correspond to the standard errors.

of $\Theta_{(h)}^{SNB\left(\frac{1}{\alpha}\right)}$ is

$$\mathcal{L}_{\Theta_{(h)}^{SNB(\alpha)}}(t) = E\left[\mathrm{e}^{-t\Theta_{(h)}^{SNB(\alpha)}}\right] = \left(\frac{\mathrm{e}^{-th} - \mathrm{e}^{-(t-1)h}}{1 - \mathrm{e}^{-(t-1)h}}\right)^{\frac{1}{\alpha}}.$$

The shifted negative binomial copula is an Archimedean copula. Indeed, (2.32) can be represented as

$$C_{\alpha,q_h}^{SNB}(u_1, ..., u_d) = \mathcal{L}_{\Theta_{(h)}^{SNB(\alpha)}}\left(\mathcal{L}^{-1}_{\Theta_{(h)}^{SNB(\alpha)}}(u_1) + ... + \mathcal{L}^{-1}_{\Theta_{(h)}^{SNB(\alpha)}}(u_d)\right).$$

When $d = 2$, $q_h = 1 - \beta$ and $\gamma = \frac{1}{\alpha}$, (2.32) becomes

$$\begin{aligned} C_{\alpha,q_h}^{SNB}(u_1, u_2) &= \left((1-\beta)\left(\prod_{i=1}^{2}\left((1-\beta)\, u_i^{-\frac{1}{\gamma}} + \beta\right) - \beta\right)^{-1}\right)^{\gamma} \\ &= \frac{u_1 u_2}{\left(1 - \beta\left(1 - u_1^{\frac{1}{\gamma}}\right)\left(1 - u_2^{\frac{1}{\gamma}}\right)\right)^{\gamma}}, \end{aligned}$$

which corresponds to the so-called bivariate Lomax copula in Balakrishnan and Lai (2009), bivariate Fang-Fang-Rosen copula in Fang et al. (2000) and Genest and Rivest (2001), and bivariate BB10 copula in Joe (2014). Note that the multivariate copula provided in (2.2) of Fang et al. (2000) does not correspond to the Archimedean copula in (2.32). The copula in (2.32) is constructed via the approach proposed by Marshall and Olkin (1988) as for the BB10 copula.

i	x_i	CMC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$
1	1141	0.050095336 (0.138900179)	0.050220000 (0.218399699)	1227	0.050329739 (0.208588438)	0.050500000 (0.218975408)
2	1179	0.009979292 (0.047470342)	0.009950000 (0.099252688)	1388	0.010100157 (0.095770936)	0.009950000 (0.099252688)
3	1218	0.001031343 (0.009745777)	0.001070000 (0.032693513)	1568	0.001077327 (0.031529811)	0.001040000 (0.032232418)
4	1249	0.000105875 (0.001984137)	0.000110000 (0.010487564)	1745	0.000093758 (0.009441001)	0.000090000 (0.009486453)
5	1276	0.000010431 (0.000391711)	0.000010000 (0.003162278)	1900	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 2.12 – Approximated values of $\overline{F}_{S(G,s,\alpha)}$ and $\overline{F}_{S(G,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 1.25$ ($\tau = 0.2$). The values in parentheses correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$
1	1252	0.050799403 (0.194516995)	0.051030000 (0.220060045)	1307	0.049717809 (0.211499142)	0.049840000 (0.217615367)
2	1336	0.010132678 (0.082516449)	0.010190000 (0.100430398)	1458	0.010201015 (0.097931511)	0.010140000 (0.100186230)
3	1425	0.001002746 (0.022945213)	0.001040000 (0.032232418)	1630	0.001026755 (0.031203817)	0.001040000 (0.032232418)
4	1495	0.000100023 (0.007070299)	0.000070000 (0.008366349)	1793	0.000100527 (0.009880428)	0.000100000 (0.009999550)
5	1572	0.000010349 (0.001879823)	0.000010000 (0.003162278)	1960	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 2.13 – Approximated values of $\overline{F}_{S(G,s,\alpha)}$ and $\overline{F}_{S(G,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parentheses correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$
1	1320	0.050086643 (0.210804858)	0.050100000 (0.218152391)	1330	0.050803544 (0.217050585)	0.050620000 (0.219221569)
2	1450	0.010204031 (0.095399818)	0.010190000 (0.100430398)	1475	0.010269781 (0.099412094)	0.010220000 (0.100576601)
3	1600	0.001057080 (0.029683037)	0.001040000 (0.032232418)	1648	0.001008499 (0.031358794)	0.001020000 (0.031921306)
4	1730	0.000102036 (0.009012274)	0.000110000 (0.010487564)	1788	0.000109967 (0.010484446)	0.000110000 (0.010487564)
5	1908	0.000010152 (0.002586474)	0.000020000 (0.004472114)	1920	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 2.14 – Approximated values of $\overline{F}_{S(G,s,\alpha)}$ and $\overline{F}_{S(G,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 5$ ($\tau = 0.8$). The values in parentheses correspond to the standard errors.

Clearly, when $h \to 0$, we have

$$\lim_{h \to 0} E\left[\mathrm{e}^{-t\Theta_{(h)}^{SNB(\alpha)}}\right] = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}, \tag{2.33}$$

where $\left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$ is the LST of the rv $\Theta^{Ga\left(\frac{1}{\alpha},1\right)}$ which follows a gamma distribution, i.e., $\Theta^{\left(\frac{1}{\alpha},1\right)} \sim Gamma\left(\frac{1}{\alpha},1\right)$. Then, given (2.33), $\Theta_{(h)}^{SNB\left(\frac{1}{\alpha}\right)} \overset{\mathcal{D}}{\to} \Theta^{Ga\left(\frac{1}{\alpha},1\right)}$, as the discretization parameter $h \to 0$, where "$\overset{\mathcal{D}}{\to}$" corresponds to the convergence in distribution.

As a special case, when $\alpha = 1$, the shifted negative binomial copula in (2.32) becomes the AMH copula. For a fixed $\alpha > 0$, when $h \to 0$ (i.e., $q_h \to 1$), the limit of the shifted negative binomial copula corresponds to the Clayton copula with parameter α, i.e.,

$$\lim_{h \to 0} C_{\alpha,q_h}^{SNB}(u_1, ..., u_n) = \left(u_1^{-\alpha} + ... + u_n^{-\alpha} - (n-1)\right)^{-\frac{1}{\alpha}} = C_{\alpha}^{CLAY}(u_1, ..., u_n).$$

In the following example, we examine the efficiency of the approximation of the Clayton copula with the shifted negative binomial copula in a risk aggregation context.

Let $\underline{X}^{(C,f,\alpha)} = \left(X_1^{(C,s,\alpha)}, ..., X_{50}^{(C,s,\alpha)}\right)$ and $\underline{X}^{(G,s,\alpha)} = \left(X_1^{(G,s,\alpha)}, ..., X_{50}^{(G,s,\alpha)}\right)$ be two vectors of rvs with $X_i^{(C,s,\alpha)} \sim X_i^{(G,s,\alpha)} \sim Bin(100, 0.2)$, for $i = 1, ..., 50$, where their joint survival function is defined either with the Clayton copula or the Gumbel copula according to (2.8). Similarly, let $\underline{X}^{(C,f,\alpha)} = \left(X_1^{(C,f,\alpha)}, ..., X_{50}^{(C,f,\alpha)}\right)$ and $\underline{X}^{(G,f,\alpha)} = \left(X_1^{(G,f,\alpha)}, ..., X_{50}^{(G,f,\alpha)}\right)$ be two vectors of rvs with $X_i^{(C,f,\alpha)} \sim X_i^{(G,f,\alpha)} \sim Bin\,(100, 0.2)$, for $i = 1, ..., 50$, where their joint cdf is defined either with the Clayton copula or the Gumbel copula according to (2.5). For each copula, the dependence parameter α of the copula is fixed such that Kendall's tau is equal to 0.2, 0.5, or 0.8. The exact values of the covariances between pairs of rvs for each vector of rvs are provided in Tables 2.5 and 2.6 for the three values of the dependence parameter. We define $S^{(C,s,\alpha)} = \sum_{i=1}^{50} X_i^{(C,s,\alpha)}$, $S^{(G,s,\alpha)} = \sum_{i=1}^{50} X_i^{(G,s,\alpha)}$, $S^{(C,f,\alpha)} = \sum_{i=1}^{50} X_i^{(C,f,\alpha)}$, and $S^{(G,f,\alpha)} = \sum_{i=1}^{50} X_i^{(G,f,\alpha)}$. Clearly, $E[S^{(C,s,\alpha)}] = E[S^{(G,s,\alpha)}] = E[S^{(C,f,\alpha)}] = E[S^{(G,f,\alpha)}] = 1000$, whatever the values of the dependence parameters. In Tables 2.7 and 2.8, we also give the exact values of $Var(S^{(C,s,\alpha)})$, $Var(S^{(G,s,\alpha)})$, $Var(S^{(C,f,\alpha)})$, and $Var(S^{(G,f,\alpha)})$ for the three values of their dependence parameter.

Example 55. In this example, we compare the performance of the approximation of the Clayton copula by the shifted negative binomial copula. Let $\underline{X}^{(C,\alpha)} = \left(X_1^{(C,\alpha)}, ..., X_4^{(C,\alpha)}\right)$ and $\underline{X}^{(SNB,\alpha,h)} = \left(X_1^{(SNB,\alpha,h)}, ..., X_4^{(SNB,\alpha,h)}\right)$ be two vectors of rvs with $X_i^{(C,\alpha)} \sim X_i^{(SNB,\alpha,h)} \sim Bin\,(10, 0.2)$, for $i = 1, \ldots, 4$, where their joint cdf is defined either with the Clayton copula (with $\alpha = 0.5$, 2, 8) or the shifted negative binomial copula (with $\alpha = 0.5$, 2, 8 and $h = 0.001$, 0.0001) according to (2.5). We define $S^{(C,\alpha)} = \sum_{i=1}^{4} X_i^{(C,\alpha)}$ and $S^{(SNB,\alpha,h)} = \sum_{i=1}^{4} X_i^{(SNB,\alpha,h)}$. Both the exact values of $\overline{F}_{S^{(C,\alpha)}}$ resulting from (2.15) and $\overline{F}_{S^{(SNB,\alpha,h)}}$ using the shifted negative binomial copula are given in Tables 2.15, 2.16 and 2.17. The results of both the conditional and the full MC simulation methods are also provided. The approximation using the shifted negative binomial copula is good and more significant as h decreases and for Kendall taus below 0.5. The stronger is the dependence relationship, the more the calculation time increases. Once again, the efficiency of the approximation of a copula with non-zero lower tail dependence by another one with $\lambda_L = 0$ is less significant when the related dependence relationship is strong.

i	x_i	$\overline{F}_{S^{(C,0.5)}}(x_i)$	$\overline{F}_{S^{(SNB,0.5,0.001)}}(x_i)$	$\overline{F}_{S^{(SNB,0.5,0.001)}}(x_i)$	CMC approx.$\overline{F}_{S^{(C,0.5)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,0.5)}}(x_i)$
1	13	0.046898732	0.046896056	0.046898466	0.046743618 (0.076386143)	0.045940000 (0.209356048)
2	14	0.022517508	0.022516026	0.022517365	0.022429261 (0.042487275)	0.022260000 (0.147528675)
3	17	0.001310321	0.001310236	0.001310341	0.001301674 (0.003570962)	0.001290000 (0.035893576)
4	19	0.000113916	0.000113946	0.000113956	0.000112915 (0.000370698)	0.000090000 (0.009486453)
5	20	0.000028520	0.000028559	0.000028562	0.000028239 (0.000099466)	0.000020000 (0.004472114)
6	26	0.000000001	0.000000044	0.000000044	0.000000001 (0.000000003)	0.000000000 (-)

Table 2.15 – Approximated values of $\overline{F}_{S^{(C,0.5)}}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 0.5$ ($\tau = 0.2$). The values in parentheses correspond to the standard errors.

i	x_i	$\overline{F}_{S^{(C,2)}}(x_i)$	$\overline{F}_{S^{(SNB,2,0.001)}}(x_i)$	$\overline{F}_{S^{(SNB,2,0.001)}}(x_i)$	CMC approx.$\overline{F}_{S^{(C,2)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,2)}}(x_i)$
1	14	0.057715752	0.057710844	0.057715261	0.057828623 (0.134180425)	0.057780000 (0.233328130)
2	16	0.014628680	0.014627145	0.014628527	0.014636998 (0.049205147)	0.014280000 (0.118643257)
3	18	0.002313893	0.002313606	0.002313864	0.002314054 (0.011151686)	0.002190000 (0.046746398)
4	20	0.000220129	0.000220099	0.000220126	0.000220007 (0.001443237)	0.000200000 (0.014140792)
5	22	0.000012529	0.000012527	0.000012529	0.000012508 (0.000103554)	0.000000000 (-)
6	28	1.145583e-10	3.685083e-10	3.685204e-10	1.139485e-10 (0.000000001)	0.000000000 (-)

Table 2.16 – Approximated values of $\overline{F}_{S^{(C,2)}}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parentheses correspond to the standard errors.

i	x_i	$\overline{F}_{S(C,8)}(x_i)$	$\overline{F}_{S(SNB,8,0.001)}(x_i)$	$\overline{F}_{S(SNB,8,0.001)}(x_i)$	CMC approx.$\overline{F}_{S(C,8)}(x_i)$	MC approx.$\overline{F}_{S(C,8)}(x_i)$
1	16	0.042577614	0.042573043	0.042577157	0.043076480 (0.145774109)	0.042560000 (0.201863949)
2	18	0.011940884	0.011939463	0.011940742	0.012148560 (0.061046175)	0.011880000 (0.108346587)
3	20	0.001991095	0.001990824	0.001991068	0.002048498 (0.015964780)	0.002080000 (0.045559789)
4	22	0.000182527	0.000182499	0.000182525	0.000189279 (0.002184336)	0.000290000 (0.017027002)
5	24	0.000008969	0.000008967	0.000008969	0.000009308 (0.000143841)	0.000000000 (-)
6	29	3.483774e-10	3.672044e-10	3.672627e-10	3.57897e-10 (0.000000008)	0.000000000 (-)

Table 2.17 – Approximated values of $\overline{F}_{S(C,8)}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 8$ ($\tau = 0.8$). The values in parentheses correspond to the standard errors.

To summarize, if the mixing rv Θ is continuous, we can proceed either by conditional MC simulation or by the approximation of the rv Θ with a discrete one. The difference between both approximation methods depends on several elements such as the value of the chosen dependence parameter, the size of the portfolio, the number of simulations, etc. For example, the approximation of the Clayton copula with the shifted negative binomial copula is more accurate compared to the simulation method when the dependence parameter is small. In the latter case, execution times of both methods are comparable. However, in the event of a greater degree of dependence or a greater number of risks, the conditional MC simulation executes faster. Note that the aim here was to only present different approaches to solve this problem, rather than putting them through a comprehensive comparison.

2.4.6 Portfolio of exchangeable risks

Exchangeability plays an important role in the analysis of homogeneous portfolios in actuarial science and in quantitative risk management, notably in credit risk modelling (e.g. McNeil et al. (2015) and the references therein). Based on DeFinetti's Theorem (see De Finetti (1957)) and its extension in Bühlmann (1960), an infinite exchangeable sequence of rvs can be represented as a mixture over a common parameter rv of an infinite sequence of iid rvs (see e.g. Feller (1971) for more details). Let $\underline{X} = \{X_n, n \in \mathbb{N}\}$ be a sequence of n positive and exchangeable rvs. Define the sequence $\underline{W} = \{W_n, n \in \mathbb{N}\}$, where $W_n = \frac{X_1+...+X_n}{n}$, for $n \in \mathbb{N}$. As discussed in, e.g., Aldous (1985), the common parameter rv drives the asymptotic behavior of $\underline{W}$. In the context of credit risk, Frey and McNeil (2001) and Frey and McNeil (2002) have studied the asymptotic behavior of $\underline{W}$ with the distribution of $(X_1, ..., X_n)$ defined in terms of exchangeable Bernoulli mixture models, for $n = 2, 3, ...$ (see e.g. Proposition 3.2 in Frey and McNeil (2002)). Their result can be generalized to other dependence models. In the context of Section 2.4.3, we consider the joint distribution of $(X_1, ..., X_n)$ to be defined with an Archimedean copula C with either (2.18) or (2.19) for $n = 2, 3, \ldots$. Let Z be a discrete rv with $Z \in \{z_\theta, \theta \in \mathbb{N}\}$ where $z_\theta = E[X|\Theta = \theta] < \infty$ and $f_Z(z_\theta) = f_\Theta(\theta)$, $\theta \in \mathbb{N}$. Clearly, the sequence $\underline{W}$ converges in distribution to the rv Z, i.e.,

$$W_n \overset{\mathcal{D}}{\to} Z, \ \text{as} n \lim \infty. \tag{2.34}$$

Indeed, we have

$$\mathcal{L}_{W_n}(t) = \sum_{\theta=1}^{\infty} \mathcal{L}_{W_n|\Theta=\theta}(t) f_\Theta(\theta) = \sum_{\theta=1}^{\infty} \left(\mathcal{L}_{X|\Theta=\theta}\left(\frac{t}{n}\right)\right)^n f_\Theta(\theta).$$

	$\alpha = 0.5$	$\alpha = 0.9$		$\alpha = 0.5$	$\alpha = 0.9$
$E[W_n]$	5	5	$E[Z]$	5	5
$Var(W_n)$	0.920999	1.886689	$Var(Z)$	0.879797	1.855241
$VaR_{0.9}(W_n)$	6.39	6.74	$VaR_{0.9}(Z)$	6.426813	6.702153
$VaR_{0.99}(W_n)$	7.34	7.71	$VaR_{0.99}(Z)$	7.265279	7.660453
$VaR_{0.9999}(W_n)$	8.31	8.66	$VaR_{0.9999}(Z)$	8.241060	8.579895
$TVaR_{0.9}(W_n)$	6.833850	7.191685	$TVaR_{0.9}(Z)$	6.786601	7.165780
$TVaR_{0.99}(W_n)$	7.599597	7.958826	$TVaR_{0.99}(Z)$	7.535144	7.910267
$TVaR_{0.9999}(W_n)$	8.448255	8.795515	$TVaR_{0.9999}(Z)$	8.347715	8.704634

Table 2.18 – Values of the expectation, variance, VaR and TVaR of W_n and Z where $F_{X_1,\ldots,,X_n}$ is defined with the AMH copula and Poisson marginals.

Near the origin, $\mathcal{L}_{X|\Theta=\theta}(t) = 1 - z_\theta t + o(t)$ which implies that

$$\lim_{n\to\infty} \mathcal{L}_{W_n}(t) = \lim_{n\to\infty} \sum_{\theta=1}^{\infty} f_\Theta(\theta) \left(1 - z_\theta \frac{t}{n} + o\left(\frac{t}{n}\right)\right)^n = \sum_{\theta=1}^{\infty} f_\Theta(\theta) \, \mathrm{e}^{-z_\theta t} = \mathcal{L}_Z(t),$$

leading to (2.34).

The result in (2.34) shows that for a large portfolio of exchangeable risks, the distribution of W_n can be approximated by the distribution of Z. One of the interesting features of the strategy described in Section 2.4.3 is that the exact values of z_θ ($\theta \in \mathbb{N}$) can be computed as well as its pmf. If the rv X is continuous, discretization methods presented in Section 2.4.4 are used.

Let us look at a simple example that illustrates the convergence of W_n to Z for a portfolio of 100 exchangeable risks.

Example 56. Let $(X_1, \ldots, X_{100})$ be a vector of 100 exchangeable rvs with $X_i \sim X \sim Pois(\lambda = 5)$, for $i = 1, \ldots, 100$. The multivariate cdf of $(X_1, \ldots, X_{100})$ is defined with an AMH copula with dependence parameter $\alpha = 0.5$ and $\alpha = 0.9$. As $n = 100$ is large, we show that, in this case, Z would be an appropriate approximation of the distribution of W_n.

Table 2.18 and illustration 2.3 demonstrates that, for large portfolios, the distribution of Z is a good candidate to approximate the behavior of W_n. This result is very important in terms of computation time for very large portfolios since using Z is a faster, more efficient and easier to handle tool in comparison to directly computing W_n

2.5 Capital Allocation

Capital allocation is fundamental in actuarial science and quantitative risk management. It describes how the capital needed for the whole portfolio can be divided and allocated between risks of the portfolio. It is crucial for an insurance company or a financial institution to evaluate the overall capital charge for a portfolio of risks in order to protect itself from large rare events. The amount of capital needed for the entire portfolio is determined with a chosen risk measure ρ.

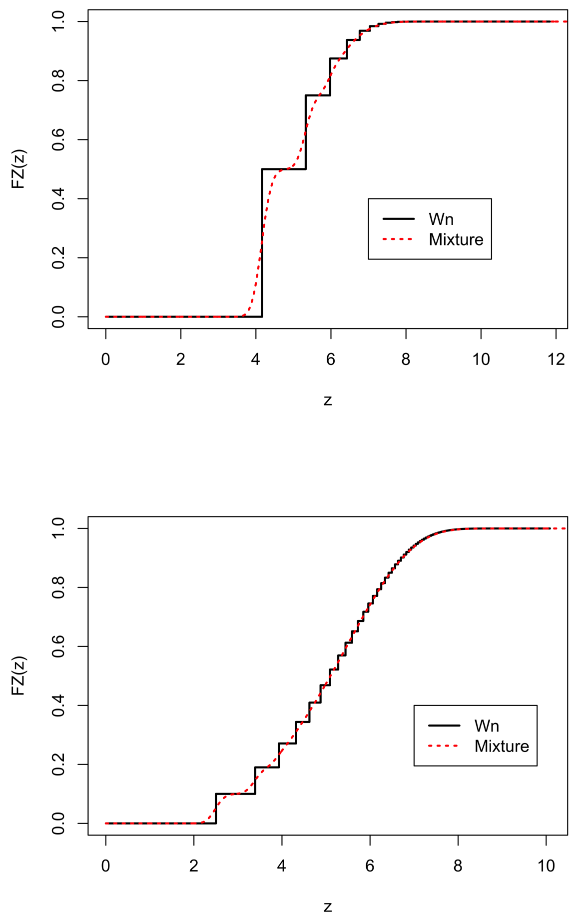

Figure 2.3 – The cdf of Z and W_n where $F_{X_1,\ldots,X_{100}}$ is defined by Poisson marginals and AMH copula with $\alpha = 0.5$ (left) and $\alpha = 0.9$ (right).

Among the desired properties for a capital allocation rule (see e.g. Furman and Zitikis (2008)), one has to be fully additive, i.e.,

$$\rho(S) = \sum_{i}^{n} C_i, \tag{2.35}$$

where $S = X_1 + \ldots + X_n$ and C_i is the contribution of the i^{th} risk to the aggregate risk of the portfolio.

In the present section, we show how to compute the values (exact or approximated) of the contributions C_i with the proposed methodology of Section 2.4. We consider Euler's capital allocation principle and the weighted risk capital allocation principle using different risk measures.

Let us assume here that we are in the context of Section 2.4.3, meaning that Θ is a discrete rv defined on $\mathbb{N}$ and that $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, ..., n)$. Also, assume that the joint cdf of $\underline{X}$ or its joint survival function is defined with the Archimedean copula C as in (2.18) or (2.19).

If the rvs X_i are continuous, the procedure described in Section 2.4.4 will be used to find the contributions. For a continuous Θ, we refer to Section 2.4.5.

To apply Euler's capital allocation rule, we need to assume that the risk measure ρ is positive homogeneous. Let us define

$$L\left(\underline{\lambda}\right) = \sum_{i=1}^{n} \lambda_i X_i,$$

where $\underline{\lambda} = (\lambda_1, ..., \lambda_n)$. For a given risk measure ρ, the contribution C_i allocated to risk i is given by

$$C_i = \rho\left(L; X_i\right) = \lambda_i \frac{\partial}{\partial \lambda_i} \rho\left(L\left(\underline{\lambda}\right)\right)\Big|_{\underline{\lambda}=\underline{1}}, \text{ for } i = 1, 2, ..., n,$$

where $\underline{1} = (1, ..., 1)$. We apply the Euler allocation principle with three different risk measures: covariance, VaR and TVaR (see Table 2.19 for their expressions). For more details, see e.g. Tasche (1999), Embrechts et al. (2005) or Rosen et al. (2011).

The second capital allocation principle we use is the one proposed by Furman and Zitikis (2008) in which the capital amount required and the contributions are based on weighted risk measures. Furman and Zitikis (2008) propose to compute the capital amount with weighted risk measures and to use weighted allocation methods for determining the contribution of each risk i, $i = 1, 2, ..., n$. Let ω be a weight function. Then, the capital amount corresponds to ρ_ω which is defined by $\rho_\omega\left(S\right) = \frac{E[S\omega(S)]}{E[\omega(S)]}$, assuming that both expectations exist. Since

$$\frac{E\left[S\omega\left(S\right)\right]}{E\left[\omega\left(S\right)\right]} = \frac{E\left[\left(\sum_{i=1}^{n} X_i\right)\omega\left(S\right)\right]}{E\left[\omega\left(S\right)\right]} = \sum_{i=1}^{n} \frac{E\left[X_i\omega\left(S\right)\right]}{E\left[\omega\left(S\right)\right]}, \tag{2.36}$$

the share of the capital allocated to X_i is given by

$$C_i = \rho_\omega\left(X_i; S\right) = \frac{E\left[X_i\omega(S)\right]}{E\left[\omega(S)\right]},$$

for $i = 1, 2, ..., n$. Clearly, due to (2.36), property (2.35) is satisfied.

Among the several weight functions that can be used to calculate the capital allocation, we consider the following three methods: Esscher with $\omega(s) = \mathrm{e}^{\eta s}$, Kamp with $\omega(s) = 1 - \mathrm{e}^{-\eta s}$ and size-biased with $\omega(s) = s^\eta$.

Let us address the computation of C_i for the considered Euler capital allocation rules and weighted risk capital allocation rules given in Table 2.19. One encounters frequently the evaluation of the expectation $E[X_iS]$ (or a variation of it) which poses problem given that X_i and S are dependent. To circumvent this, we rewrite the product of these two rvs as

$$E\left[X_iS\right] = E_\Theta\left[E\left[X_i\left(X_i + S_{-i}\right)|\Theta\right]\right],$$

where $S_{-i} = X_1 + \ldots + X_{i-1} + X_{i+1} + \ldots + X_n$. The conditional independence of X_i and S_{-i} given Θ simplifies the evaluation of the quantities of interest.

Allocation rule	Contribution C_i
Euler-Covariance	$E\left[X_i\right]+\frac{Cov(X_i,S)}{Var(S)}\left\{\xi(S)-E\left[S\right]\right\}$, with ξ is is a chosen risk measure
Euler-VaR	$E\left[X_i\vert S=VaR_\kappa\left(S\right)\right]$
Euler-TVaR	$\frac{1}{1-\kappa}\left\{E\left[X_i\times 1_{S>VaR_\kappa(S)}\right]+\beta E\left[X_i 1_{S=VaR_\kappa(S)}\right]\right\}$, with $\beta=F_S\left(VaR_\kappa(S)\right)-\kappa$
Weighted-Esscher	$\frac{E\left[X_i \mathrm{e}^{\eta S}\right]}{E\left[\mathrm{e}^{\eta S}\right]}$
Weighted-Kamps	$\frac{E\left[X_i\left(1-\mathrm{e}^{-\eta S}\right)\right]}{E\left[1-\mathrm{e}^{-\eta S}\right]}$
Weighted-size-biased	$\frac{E\left[X_i S^\eta\right]}{E\left[S^\eta\right]}$

Table 2.19 – Contributions under considered allocation rules.

Note that with the VaR risk measure, it is slightly more tedious. In this case, the contribution of risk i is given by

$$C_i=E\left[X_i\vert S=VaR_\kappa\left(S\right)\right]=E\left[X_i\vert S=k_0h\right]=\frac{E\left[X_i\times 1_{\{S=k_0h\}}\right]}{\Pr\left(S=k_0h\right)},$$

assuming that $\Pr\left(S=k_0h\right)>0$. Then, using our approach, we have

$$\begin{aligned}
E\left[X_i\times 1_{\{S=k_0h\}}\right] &= E_\Theta\left[E\left[X_i\times 1_{\{S=k_0h\}}\vert\Theta\right]\right]\\
&= \sum_{\theta=1}^{\infty}\left(\sum_{j=1}^{k_0} j f_{X_i\vert\Theta=\theta}(jh) f_{S_{-i}\vert\Theta=\theta}\left(\left(k_0-j\right)h\right)\right) f_\Theta\left(\theta\right)\\
&= \sum_{\theta=1}^{\infty} E\left[X_i\vert\Theta=\theta\right]\left(\sum_{j=1}^{k_0} f^*_{X_i\vert\Theta=\theta}(jh) f_{S_{-i}\vert\Theta=\theta}\left(\left(k_0-j\right)h\right)\right) f_\Theta(\theta),
\end{aligned}$$

with $f^*_{X_i\vert\Theta=\theta}(jh)=\frac{jf_{X_i\vert\Theta}(j)}{E[X_i\vert\Theta=\theta]}$, for $j\in\mathbb{N}$. Finally, defining

$$g_{i\vert\Theta=\theta}(k_0h)=\sum_{j=1}^{k_0} f^*_{X_i\vert\Theta=\theta}(jh) f_{S_{-i}\vert\Theta=\theta}\left(\left(k_0-j\right)h\right)$$

as a result of a convolution product, we obtain

$$E\left[X_i\times 1_{\{S=k_0h\}}\right]=\sum_{\theta=1}^{\infty} E\left[X_i\vert\Theta=\theta\right] g_{i\vert\Theta=\theta}(k_0h) f_\Theta(\theta), \quad (2.37)$$

where the values of $g_{i\vert\Theta=\theta}\left(kh\right)$ can be easily obtained using classic aggregation methods such as FFT.

Example 57. We consider a portfolio of 10 risks, where $X_i-1\sim NB\left(r_i=\frac{1+i}{2}, q_i=\frac{1}{1+\frac{8}{1+i}}\right)$, such that $E\left[X_i\right]=5$, for $i=1,2,...,10$. It implies $E\left[S\right]=50$. The multivariate cdf of $\underline{X}$ is defined as in (2.18) with the AMH copula. For $\alpha=0$, 0.3 and 0.8, the variance of S is 104.6361, 185.2821 and 343.7994 respectively. In Table 2.20, we provide the relative contributions of X_1, ..., X_{10} (i.e., $\frac{C_i}{\rho(S)}$) for the methods based on Euler's capital allocation rule and the ones under the weighted risk allocation approach assuming a dependence parameter $\alpha=0.8$. As shown in Table 2.20, we are able to find exact contributions based on different allocation methods. Consistent results are obtained for $\alpha=0$ and $\alpha=0.3$.

	Covariance	VaR	TVaR	$Esscher$ $(\eta=0.1)$	$Kamps$ $(\eta=10^{-6})$	$Size-biased$ $(\eta=10)$
i	$\frac{C_i}{\rho_{0.99}(S)}$	$\frac{VaR_{0.99}(X_i;S)}{VaR_{0.99}(S)}$	$\frac{TVaR_{0.99}(X_i;S)}{TVaR_{0.99}(S)}$	$\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$	$\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$	$\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$
1	11.53	28.52	31.92	14.23	10.55	14.25
2	10.75	11.88	11.65	11.52	10.27	11.58
3	10.32	9.17	8.80	10.45	10.12	10.48
4	10.04	8.15	7.75	9.85	10.01	9.87
5	9.84	7.61	7.21	9.47	9.94	9.47
6	9.69	7.28	6.88	9.21	9.89	9.20
7	9.58	7.05	6.65	9.01	9.85	8.99
8	9.49	6.89	6.49	8.86	9.81	8.84
9	9.41	6.77	6.37	8.75	9.79	8.71
10	9.35	6.68	6.28	8.65	9.77	8.61
$\rho(S)$	76.2266	96	103.0490	89.2213	57.3053	94.1052

Table 2.20 – Contributions of X_i ($i = 1, 2, ..., 10$) in % under the 6 allocation methods. The multivariate cdf of $\underline{X}$ is defined with an AMH copula ($\alpha = 0.8$).

2.6 Random sum of exchangeable risks

Random sums are essential in the description of many fundamental risk models in actuarial science. The so-called frequency-severity model is based on random sums. In this section, the computational methodology exposed in Section 2.4.3 is used to analyze the distribution of the aggregate claim amount rv S which is defined as the random sum of exchangeable individual claim amounts. Under this risk model, the rv S is defined by $S = \sum_{j=1}^{N} X_j$, where $\underline{X} = \{X_j, j \in \mathbb{N}\}$ forms a sequence of exchangeable rvs independent of the counting positive discrete rv N. The components of $\underline{X}$ are such that $X_j \in A = \{0, 1h, 2h, \ldots\}$ ($j \in \mathbb{N}$). The notation for the univariate pmf of X_i and the univariate cdf of X_i is given in Section 2.4. Also, we assume that the joint distribution of $(X_1, ..., X_j)$ is defined via either (2.18) or (2.19) with $d = j$.

The risk model considered in this section can be seen as an extension of the risk model described and studied in Section 2 of Albrecher et al. (2011). Indeed, they consider a risk model defined with a compound Poisson process where the vector of claim amounts follows a multivariate mixed exponential distribution as defined in Section 2.4.2. Here, the multivariate distribution of the vector of claim amounts can be defined with any Archimedean copula and any marginal distribution for the claim amounts.

To apply the computational methodology, assume that the mixing rv Θ is a strictly positive discrete rv defined on $\mathbb{N}$. Clearly, the aggregate claim amount rv $S \in A$ and the objective is to compute the values of $f_S(kh) = \Pr(S = kh)$, $k \in \mathbb{N}_0$. Given the common mixture representation of either $F_{X_1,...,X_j}$ or $\overline{F}_{X_1,...,X_j}$, we have $f_S(kh) = \sum_{\theta=1}^{\infty} f_{S|\Theta=\theta}(kh) f_\Theta(\theta)$, where $(S|\Theta = \theta) = \sum_{j=1}^{N} (X_j|\Theta = \theta)$. The usual aggregation algorithms (see e.g. Panjer et al. (2008)) such as FFT or Panjer's recursive algorithm can be used to compute the values of $f_{S|\Theta=\theta}(kh)$ for $k \in \mathbb{N}_0$ and for each $\theta = 1, 2, ..., \theta^*$ where θ^* is chosen such that $F_\Theta(\theta^*) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$). Again, note that we just need the values of $f_{X|\Theta=\theta}(kh)$, which are

	$\alpha = 0.8$ (simul)	$IC_{0.05}$	$\alpha = 0.8$ (upper)	$\alpha = 0.8$ (lower)	$\alpha = 0$ (exact)
$E[S]$	400.0359	[399.8034 ; 400.2685]	399.950	400.050	400
$Var(S)$	140775.7887	[140652.4769 ; 140899.2633]	140912.018	140942.018	80000
$VaR_{0.9}(S)$	906.9635	[906.3475 ; 907.5886]	907.000	907.200	873.8748
$VaR_{0.99}(S)$	1644.4705	[1642.6493 ; 1646.4017]	1645.350	1645.600	1470.9808
$VaR_{0.999}(S)$	2317.5620	[2312.0092 ; 2322.2396]	2323.350	2323.650	1992.0052
$VaR_{0.9999}(S)$	2959.6858	[2942.0263 ; 2975.0607]	2967.250	2967.600	2473.3833
$TVaR_{0.9}(S)$	1230.7730	[1230.5806 ; 1230.9654]	1231.206	1231.333	1138.1220
$TVaR_{0.99}(S)$	1939.1160	[1938.9381 ; 1939.2938]	1941.278	1941.494	1699.2458
$TVaR_{0.999}(S)$	2596.5160	[2596.3469 ; 2596.6851]	2603.626	2603.935	2202.1856
$TVaR_{0.9999}(S)$	3223.8834	[3223.7239 ; 3224.0429]	3236.634	3237.032	2672.1090

Table 2.21 – Values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ where S is defined as a random sum of dependent rvs.

given in (2.22) or (2.25). Also, the LST of the rv S is given by

$$\mathcal{L}_S(t) = E\left[e^{-tS}\right] = \sum_{\theta=1}^{\infty} E\left[e^{-tS}|\Theta=\theta\right] f_\Theta(\theta) = \sum_{\theta=1}^{\infty} \mathcal{L}_{S|\Theta=\theta}(t) f_\Theta(\theta),$$

with $\mathcal{L}_{S|\Theta=\theta}(t) = P_N\left(\mathcal{L}_{X|\Theta=\theta}(t)\right)$ where

$$\mathcal{L}_{X|\Theta=\theta}(t) = E\left[e^{-tX}|\Theta=\theta\right] = \sum_{k=0}^{\infty} e^{-tkh} f_{X|\Theta=\theta}(kh)$$

and $P_N(s) = E\left[s^N\right]$ is the probability generating function of the positive discrete rv N.

Example 58. Let $N \sim Pois(\lambda = 2)$ and $X \sim Gamma(\alpha = 2, \beta = 0.01)$ such that $E[X] = 200$. It implies that $E[S] = 400$. Also, we assume that the joint distribution of $(X_1, ..., X_j)$, $j \in \mathbb{N}$, is defined as in (2.18) with an AMH copula with dependence parameter $\alpha = 0.8$. To obtain the desired results provided in Table 2.21, we use the upper and lower discretization methods with $h = \frac{1}{20}$. Different MC simulation studies (with 10 million simulations) have been performed. We present results from one of them in the second column of Table 2.21. From one study to the next, we have observed results that may or may not fall between the upper and lower bounds given in the third and fourth columns of Table 2.21, which highlights the advantage of the proposed methodology.

2.7 Renewal risk models with exchangeable inter-claim times

In this section, we consider a general class of continuous-time renewal risk models with exchangeable inter-claim times. This class is an extension of the class discussed in Section 3 of Albrecher et al. (2011). For an insurance portfolio, the surplus process is defined by $\underline{U} = \{U(t), t \geq 0\}$ where the surplus level at time t, $U(t)$, is given by

$$U(t) = u + ct - S(t),$$

where $U(0) = u$ is the initial surplus and c is the premium rate. The aggregate claim amount process, denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ is a mixed compound renewal process with exchangeable inter-claim times. The claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a mixed renewal process where the inter-claim times $\underline{W} = \{W_j, j \in \mathbb{N}\}$ form a sequence of exchangeable and strictly positive real-valued rvs. The time between the $(j-1)$th and the jth claim $(j = 2, ...)$ is defined by the rv W_j with W_1 the time of the first claim. The rvs $\{W_j, j \in \mathbb{N}\}$, are identically distributed as the canonical rv W, have pdf f_W, cdf F_W, and survival function $\overline{F}_W$.

To simplify the presentation, the joint survival function of $(W_1, W_2, ..., W_k)$ is defined with an Archimedean copula as in (2.8), i.e.,

$$\overline{F}_{W_1,W_2,...,W_k}(x_1, ..., x_k) = C\left(\overline{F}_W(x_1), ..., \overline{F}_W(x_k)\right), \tag{2.38}$$

for $k \in \{2, 3, ...\}$ and $x_1, ..., x_k \geq 0$. The multivariate distribution of $(W_1, \ldots, W_k)$ can also be defined with joint cdf as in (2.5). The time of arrival of the jth claim is denoted $T_j = W_1 + \cdots + W_j$.

The claim amount rvs $\underline{X} = \{X_j, j \in \mathbb{N}\}$, where X_j corresponds to the amount of the jth claim, are assumed to be a sequence of strictly positive and iid rvs with pdf f_X and cdf F_X. The sequences $\underline{W}$ and $\underline{X}$ are independent.

The time of ruin is defined by the rv $\tau_u = \inf\{t \geq 0 : U(t) < 0\}$ with $\tau_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The infinite-time ruin probability is $\zeta(u) = \Pr(\tau_u < \infty | U(0) = u)$. Throughout this section, we assume the positive security loading condition $E[cW - X] > 0$ to be verified which ensures that ruin will not occur almost surely. Due to the common mixture representation, (2.38) is given by

$$\overline{F}_{W_1,W_2,...,W_k}(x_1, ..., x_k) = C\left(\overline{F}_W(x_1), ..., \overline{F}_W(x_k)\right) = \int_0^\infty \overline{F}_{W|\Theta=\theta}(x_1) \times ... \times \overline{F}_{W|\Theta=\theta}(x_k)\, \mathrm{d}F_\Theta(\theta),$$

where

$$\overline{F}_{W|\Theta=\theta}(x) = \mathrm{e}^{-\theta\psi^{-1}\left(\overline{F}_W(x)\right)}$$

for $x \geq 0$. As mentioned in Section 3 of Albrecher et al. (2011), $\overline{F}_{W|\Theta=\theta}$ is the canonical survival function of the inter-claim time rvs for an ordinary renewal process. Let ζ_θ be the conditional ruin probability associated to the corresponding renewal process. It implies that the ruin probability ζ can be represented as a mixture, where Θ is the mixing rv, i.e.,

$$\zeta(u) = \int_0^\infty \zeta_\theta(u)\, \mathrm{d}F_\Theta(\theta). \tag{2.39}$$

The security loading condition is violated when the mixing rv Θ takes a value larger that $\theta_0 > 0$. We define θ_0 such that $\zeta_\theta(u) = 1$, for $\theta > \theta_0 > 0$. There exists a θ_0 such that $c \times E[W|\Theta = \theta] > E[X]$, for $\theta \in \{1, 2, \ldots, \theta_0\}$, and $c \times E[W|\Theta = \theta] < E[X]$, for $\theta \in \{\theta_0 + 1, \ldots\}$. Then, (2.39) becomes

$$\zeta(u) = \int_0^{\theta_0} \zeta_\theta(u)\, \mathrm{d}F_\Theta(\theta) + \overline{F}_\Theta(\theta_0), \tag{2.40}$$

for $u \geq 0$. Assuming that ζ_θ could be computed for each $\theta \in \{1, 2, \ldots, \theta_0\}$ of the ordinary renewal process associated to $\overline{F}_{W|\Theta=\theta}$, the value of $\zeta(u)$ can be computed using (2.40).

In the examples of Section 3 of Albrecher et al. (2011), the authors assume that $(W_1, W_2, ..., W_k)$ follows a multivariate mixed exponential distribution as defined in Section 2.4.2, where the LST of the mixing rv corresponds to the generator of an Archimedean copula. It means that the univariate marginal distribution of the inter-claim time is a univariate mixed exponential distribution. Indeed, the authors consider specific examples of mixed Poisson risk models.

In this section, we show that it is possible to consider any multivariate distribution for $\underline{W}$ defined with any Archimedean copula and given marginal distributions. First, let Θ be a strictly positive discrete rv defined on $\mathbb{N}$. As in Albrecher et al. (2011), we limit our analysis to exponentially distributed claim amounts with parameter β. It implies that

$$\zeta_\theta(u) = \frac{\beta - \rho_\theta}{\beta} \mathrm{e}^{-\rho_\theta u},\ u \geq 0, \tag{2.41}$$

where ρ_θ is the adjustment coefficient which is the smallest strictly positive solution to the Lundberg relation

$$E\left[\mathrm{e}^{r(X-cW)}|\Theta = \theta\right] = E\left[\mathrm{e}^{rX}\right] \times E\left[\mathrm{e}^{-rcW}|\Theta = \theta\right] = 1 \tag{2.42}$$

with $E\left[\mathrm{e}^{-rcW}|\Theta = \theta\right] = \int_0^\infty \mathrm{e}^{-rcx} f_{W|\Theta=\theta}(x)\,\mathrm{d}x$ and $f_{W|\Theta=\theta}(x) = -\frac{\mathrm{d}\overline{F}_{W|\Theta=\theta}(x)}{\mathrm{d}x}$. Given that Θ is a discrete rv and with (2.41), (2.40) becomes

$$\zeta(u) = \sum_{\theta=1}^{\theta_0} \Pr(\Theta = \theta) \frac{\beta - \rho_\theta}{\beta} \mathrm{e}^{-\rho_\theta u} + \overline{F}_\Theta(\theta_0). \tag{2.43}$$

The expression in (2.43) is illustrated in the following example.

Example 59. Let $X \sim Exp(1)$ and $\overline{F}_{W_1, W_2, ..., W_k}(x_1, ..., x_k)$ be defined as in (2.38) where C is an AMH copula. It means that Θ follows a geometric distribution with parameter $q = 1 - \alpha$ and

$$\mathcal{L}_\Theta(t) = \frac{q\mathrm{e}^{-t}}{1 - (1-q)\,\mathrm{e}^{-t}} \text{ and } \mathcal{L}_\Theta^{-1}(u) = -\ln\left(\frac{1}{\frac{q}{t} + 1 - q}\right).$$

Then,

$$\overline{F}_{W|\Theta=\theta}(x) = \left(\frac{1}{q\mathrm{e}^x + 1 - q}\right)^\theta \text{ and } f_{W|\Theta=\theta}(x) = \frac{\theta q \mathrm{e}^x}{(q\mathrm{e}^x + 1 - q)^{\theta+1}}$$

for $x \geq 0$, $\theta \in \mathbb{N}$ and $0 < q < 1$. Note that $\overline{F}_{W|\Theta=1}(x) = \frac{1}{q\mathrm{e}^x + 1 - q}$ corresponds to the univariate survival function of the exponential distribution with tilt as defined in Marshall and Olkin (2007).

In this case, we have

$$E[W|\Theta = \theta] = \int_0^\infty \left(\frac{1}{q\,\mathrm{e}^x + 1 - q}\right)^\theta \mathrm{d}x = \lim_{t\to\infty} \int_0^t \left(\frac{1}{q\,\mathrm{e}^x + 1 - q}\right)^\theta \mathrm{d}x. \tag{2.44}$$

Let $u = q\,\mathrm{e}^x + 1 - q$, then (2.44) becomes

$$\int_0^\infty \left(\frac{1}{q\,\mathrm{e}^x + 1 - q}\right)^\theta \mathrm{d}x = \lim_{t\to\infty}\int_0^t \frac{1}{u^\theta\,(u - 1 + q)}\mathrm{d}u.$$

Using a partial fraction decomposition, we obtain

$$\int_0^\infty \left(\frac{1}{q\,\mathrm{e}^x + 1 - q}\right)^\theta \mathrm{d}x = -\frac{\ln(q)}{(1-q)^\theta} - \sum_{k=1}^{\theta-1}\frac{1}{(\theta-k)(1-q)^k}.$$

Also, in order to calculate the adjustment coefficient ρ_θ, we develop the expression of $E\left[\mathrm{e}^{-t\,W}|\Theta=\theta\right]$ as follows

$$E\left[\mathrm{e}^{-t\,W}|\Theta=\theta\right] = \int_0^\infty \mathrm{e}^{-t\,x}\frac{\theta q\mathrm{e}^x}{(q\mathrm{e}^x + 1 - q)^{\theta+1}}\mathrm{d}x. \tag{2.45}$$

Let $u = q\mathrm{e}^x + 1 - q$, then (2.45) becomes

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-t\,x}\frac{\theta q\mathrm{e}^x}{(q\mathrm{e}^x + 1 - q)^{\theta+1}}\mathrm{d}x &= \int_1^\infty \frac{\theta q^t (u+q-1)^{-t}}{u^{\theta+1}}\mathrm{d}u \\
&= \frac{\theta q^t}{(q-1)^t}\int_1^\infty \frac{1}{\left(\frac{u}{q-1}+1\right)^t u^{\theta+1}}\mathrm{d}u \\
&= \frac{\theta q^t}{(q-1)^t} \times \frac{(q-1)^t\,{}_2F_1\left([t, t+\theta]; [t+\theta+1]; 1-q\right)}{t+\theta} \\
&= \frac{\theta q^t\,{}_2F_1\left([t, t+\theta]; [t+\theta+1]; 1-q\right)}{t+\theta},
\end{aligned}$$

where ${}_nF_m$ denotes the generalized hypergeometric function defined as follows

$${}_nF_m\left([a_1,\ldots,a_n];[b_1,\ldots,b_m];z\right) = \sum_{k=0}^\infty \frac{(a_1)_k\ldots(a_n)_k}{(b_1)_k\ldots(b_m)_k}\frac{z^k}{k!},$$

with $(x)_k = x(x+1)\ldots(x+k-1)$.

Let $\lambda = 1$, $\beta = 1$, and $c = 1.2$. The parameter of the AMH copula is $\alpha = 1 - q = 0.9$. We find that $\theta_0 = 5$ and the expression in (2.43) for $\zeta(u)$ becomes

$$\zeta(u) = \sum_{\theta=1}^5 0.9^{\theta-1}\times 0.1 \times (1-\rho_\theta)\mathrm{e}^{-\rho_\theta u} + 0.9^5,$$

where the values of ρ_θ for $\theta = 1, 2, \ldots, 5$ are given in the following table:

θ	1	2	3	4	5
ρ_θ	0.7666945	0.5591448	0.3649751	0.1794426	0.0001673811

The values of ρ_θ are computed by numerical optimization using (2.42) for $\theta = 1, 2, \ldots, 5$. □

We have defined here the multivariate distribution of $\underline{W}$ in terms of the multivariate survival function as in (2.19). Our strategy, based on common mixtures, allows us however to also do our investigation of ruin models with the multivariate cdf as in (2.18). Also, it is possible to adapt the strategy discussed in Section 2.4.5 with a continuous distribution for the mixing rv Θ. The content of this section clearly demonstrates that using our common mixture representation methodology allows to analyze a variety of risk models which opens the door to further research.

2.8 Classical discrete-time risk models with exchangeable losses

In this section, we consider a discrete-time risk model with exchangeable losses. Let $\underline{X} = \{X_k, k \in \mathbb{N}\}$ be a sequence of exchangeable rvs, where X_k is the aggregate loss for a portfolio in period $k \in \mathbb{N}$ with $X_k \sim X$, $k \in \mathbb{N}$. Let $\underline{U} = \{U_k, k \in \mathbb{N}_0\}$ be the surplus process of the portfolio, where U_k corresponds to the surplus level at period $k \in \mathbb{N}$. For $k = 0$, $U_0 = u$ corresponds to the initial amount of capital allocated to the portfolio. Then, at period $k \in \mathbb{N}$, $U_k = U_{k-1} + \pi - X_k = u - \sum_{j=1}^{k} (X_j - \pi)$. The time of ruin $\tau_u = \inf\{k \in \mathbb{N}, U_k < 0\}$, if $\underline{U}$ goes below 0 at least once, or ∞, if $\underline{U}$ never goes below 0. We define the infinite-time ruin probability by $\zeta(u) = \Pr(\tau_u < \infty)$. To prevent ruin with certainty (i.e., $\zeta(u) = 1$, $u \geq 0$), we assume that the net profit condition is satisfied, i.e., $E[X - \pi] < 0$, where π is the premium income per period with $\pi = (1+\eta)E[X]$. For simplification purposes, we assume $X \in \mathbb{N}_0$, $u \in \mathbb{N}_0$, and $\pi = 1$ with $\pi > E[X]$. Then, with these additional assumptions, the classical discrete-time risk model with exchangeable losses corresponds to an extension of the compound binomial classical risk model (see e.g. Gerber (1988), Shiu (1989), Willmot (1993), De Vylder and Marceau (1996), Dickson (1992) for details on the compound binomial risk model).

For $j = 2, 3, ...$, the multivariate distribution of $(X_1, ..., X_j)$ is defined with an Archimedean copula C with either (2.18) or (2.19) as in Section 2.4.3. Let $\zeta_\theta(u)$ be the conditional infinite-time ruin probability given $\Theta = \theta$. There exists a θ_0 such that $E[X|\Theta = \theta] < 1$, for $\theta \in \{1, 2, ..., \theta_0\}$, and $E[X|\Theta = \theta] > 1$, for $\theta \in \{\theta_0 + 1, \theta_0 + 2, \ldots\}$. Then, when $\theta = \theta_0 + 1, \theta_0 + 2, ...$, the solvency condition is not satisfied and the conditional infinite-time ruin probability $\zeta_\theta(u) = 1$, for all $u \in \mathbb{N}_0$. For $\theta = 1, 2, ..., \theta_0$, adapting expressions from Cossette et al. (2003), we have

$$\zeta_\theta(u) = \frac{\zeta_\theta(u-1) - \sum_{j=1}^{u} \zeta_\theta(u-j) \times f_{X|\Theta=\theta}(j) - \overline{F}_{X|\Theta=\theta}(u)}{f_{X|\Theta=\theta}(0)} \quad , \quad \text{for } u \in \mathbb{N}_0 \ ,$$

with initial value $\zeta_\theta(0) = \frac{E[X|\Theta=\theta] - \Pr(X>0|\Theta=\theta)}{f_{X|\Theta=\theta}(0)}$. The unconditional infinite-time ruin probability $\zeta(u)$ is given by

$$\zeta(u) = \sum_{\theta=1}^{\infty} \zeta_\theta(u) f_\Theta(\theta) = \sum_{\theta=1}^{\theta_0} \zeta_\theta(u) f_\Theta(\theta) + \overline{F}_\Theta(\theta_0), \ u \in \mathbb{N}_0. \tag{2.46}$$

Example 60. Let X be a non-negative discrete rv with

$$f_X(0) = (1-\delta) + \delta \times f_B(0) \quad \text{and} \quad f_X(k) = \delta \times f_B(k), \quad k \in \mathbb{N}, \delta \in (0,1),$$

and $B-1 \sim NB(r,q)$ ($r \in \mathbb{R}^+$, $q \in (0,1)$) with $E[B] = 1 + r \times \frac{1-q}{q}$. We fix the different parameters as follows: $\delta = 0.1$, $r = 2$, and $q = \frac{1}{5}$, and $E[X] = 0.9 < 1$. Finally, $F_{X_1,\ldots,X_j}$ is defined with an AMH copula as in (2.18). The ruin probability is calculated for several values of initial capital and different values of dependence parameter α. Results are presented in Figure 2.4 from which we can see that for a small dependency parameter, the ruin probability tends to zero. Otherwise, the greater the parameter becomes, the more likely the probability of ruin tends to a value close to 0.35. Once again, we emphasize the significant impact of a low to moderate dependence relation between rvs on the overall portfolio.

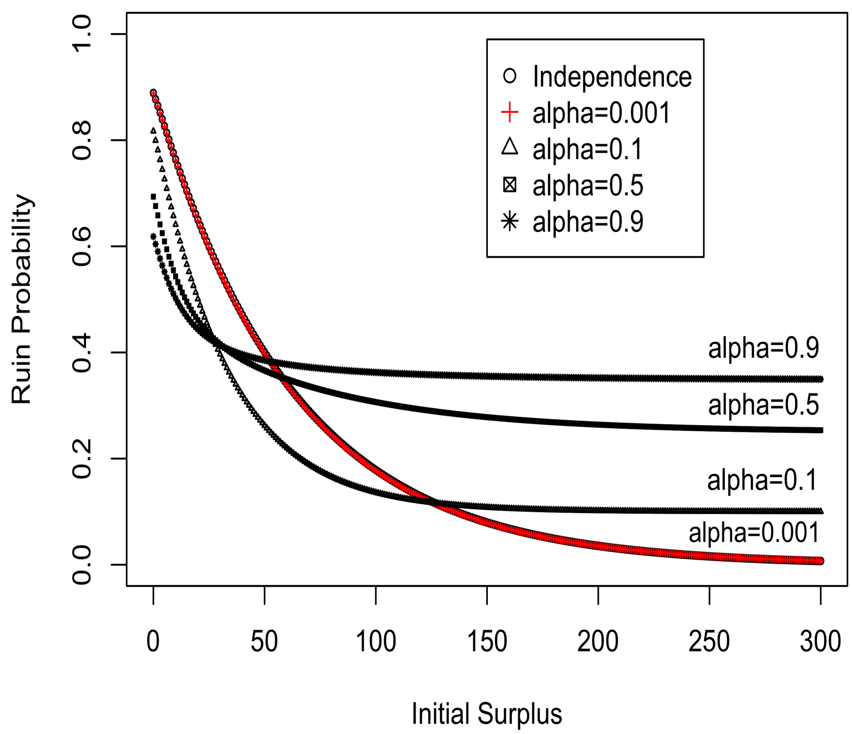

Figure 2.4 – Ultimate ruin probability for various degrees of dependence.

In the following example we provide an analytical expression of ζ for a specific loss distribution.

Example 61. Let X be a non-negative discrete rv with

$$f_X(0) = (1-\delta) \quad \text{and} \quad f_X(2) = \delta, \quad k \in \mathbb{N}, \delta \in (0, 0.5).$$

Assume that $F_{X_1,\ldots,X_j}$ is defined by (2.18) with copula C. Using (2.22), we find

$$f_{X|\Theta=\theta}(0) = 1 - \delta_\theta = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}(f_X(0))}.$$

From Example 3.1 of Willmot (1993), we find

$$\zeta_\theta(u) = \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1}, \tag{2.47}$$

for $u \in \mathbb{N}_0$ and $\theta = 1, 2, ..., \theta_0$. Replacing (2.47) in (2.46), the expression $\zeta(u)$ becomes

$$\zeta(u) = \sum_{\theta=1}^{\theta_0} f_\Theta(\theta) \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1} + \overline{F}_\Theta(\theta_0), \; u \in \mathbb{N}_0.$$

For illustration purposes, we fix the different parameters as follows: $\delta = 0.4$, implying $E[X] = 0.8 < 1$. Finally, $F_{X_1,...,X_j}$ is defined by (2.18) where C is a Frank copula with $\alpha = 4$. We obtain that $\theta_0 = 9$ and

$$\zeta(u) = \sum_{\theta=1}^{9} \frac{\left(1-\mathrm{e}^{-4}\right)^\theta}{4\,\theta} \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1} + \overline{F}_\Theta(9), \; u \in \mathbb{N}_0,$$

where the values of δ_θ are given in the following table:

θ	1	2	3	4	5	6	7	8	9
δ_θ	0.92625	0.85793	0.79466	0.73604	0.68176	0.63148	0.58491	0.54177	0.50181

2.9 Partially nested Archimedean copulas

In the present section, we generalize the proposed strategy of Section 2.4 and adapt it to hierarchical structures for which at least one of the arguments is an Archimedean copula. More precisely, we consider in detail nested Archimedean copulas. Since there are several ways of nesting copulas and the computational methodology based on the mixing representation is the same for any chosen structure, we will only present a detailed example where we apply the strategy to a partially nested Archimedean copula. See Joe (1997), McNeil (2008) and Hofert (2011) for a general introduction to nested Archimedean copulas.

Let C be a one level partially nested Archimedean copula with d children, i.e., C is of the form

$$\begin{aligned} C(\underline{u}) &= C(\underline{u}; \psi_0, \psi_1, ..., \psi_n) \\ &= C\left(C\left(\underline{u_1}; \psi_1\right), \ldots, C\left(\underline{u_d}; \psi_d\right); \psi_0\right) \\ &= C\left(C\left(u_{1,1}, \ldots, u_{1,n_1}; \psi_1\right), \ldots, C\left(u_{d,1}, \ldots, u_{d,n_d}; \psi_d\right); \psi_0\right), \end{aligned} \tag{2.48}$$

with $\underline{u} = \left(\underline{u_1}, \ldots, \underline{u_d}\right)$, where $\underline{u_i} = (u_{i,1}, \ldots, u_{i,n_i})$ for $i = 1, \ldots, d$.

An example of the partially nested Archimedean copula with arguments $\underline{u}$ is defined as follows in terms of a multivariate parent copula of dimension d and d child copulas, where the dimension of the i^{th} copula is denoted by n_i:

$$\begin{aligned} C(\underline{u}) &= C(\underline{u};\psi_0,\psi_1,...,\psi_d) \\ &= C\left(C\left(\underline{u_1};\psi_1\right),...,C\left(\underline{u_d};\psi_d\right);\psi_0\right) \\ &= C\left(C\left(u_{1,1},...,u_{1,n_1};\psi_1\right),...,C\left(u_{d,1},...,u_{d,n_d};\psi_d\right);\psi_0\right) \\ &= \psi_0\left(\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right)\right)\right)\right). \end{aligned} \tag{2.49}$$

Since ψ_0 is the LST of a strictly positive rv Θ_0, (2.49) develops into

$$\begin{aligned} C(\underline{u}) &= \int_0^\infty e^{-\theta_0\times\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)\right)}dF_{\Theta_0}(\theta_0) \\ &= \int_0^\infty \prod_{i=1}^{d} e^{-\theta_0\times\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)\right)}dF_{\Theta_0}(\theta_0) \end{aligned}$$

which becomes

$$\begin{aligned} C(\underline{u}) &= \int_0^\infty \prod_{i=1}^{d}\psi_{0,i}\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right);\theta_0\right)dF_{\Theta_0}(\theta_0) \\ &= \int_0^\infty \prod_{i=1}^{d}\left(\int_0^\infty\prod_{j=1}^{n_i}e^{-\theta_{0,i}\times\psi_i^{-1}(u_{i,j})}dF_{\Theta_{0,i}}(\theta_{0,i})\right)dF_{\Theta_0}(\theta_0), \end{aligned}$$

where $\psi_{0,i}(t) = e^{-\theta_0\times\psi_0^{-1}\circ\psi_i(t)}$. As mentioned notably in Hofert (2010), $\psi_0^{-1}\circ\psi_i$ must be completely monotone in order to verify the nesting condition.

Let $\underline{k} = (k_{1,1},\ldots,k_{1,n_1},\ldots,k_{d,1},\ldots,k_{d,n_d})$. As in (2.18), let the multivariate distribution of $\underline{X} = \left(\underline{X_1},\ldots,\underline{X_d}\right)$ with $\underline{X_i} = (X_{i,1},\ldots,X_{i,n_i})$, for $i=1,\ldots,d$ be defined in terms of its joint cdf as follows

$$\begin{aligned} F_{\underline{X}}(\underline{k}h) &= C\left(F_{X_{1,1}}(k_{1,1}h),\ldots,F_{X_{1,n_1}}(k_{1,n_1}h),\ldots,F_{X_{d,1}}(k_{d,1}h),\ldots,F_{X_{d,n_d}}(k_{d,n_d}h)\right) \\ &= \int_0^\infty\prod_{i=1}^{d}\left(\int_0^\infty\prod_{j=1}^{n_i}F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h)\,dF_{\Theta_{0,i}}(\theta_{0,i})\right)dF_{\Theta_0}(\theta_0), \end{aligned} \tag{2.50}$$

where $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}) = e^{-\theta_{0,i}\times\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$ for $j=1,2,\ldots,n_i$ and $i=1,\ldots,d$. Note that the multivariate distribution of $\underline{X}$ can also be defined with its joint survival function as in (2.19).

To apply our methodology in a risk aggregation context, we need to assume that we can identify the distribution of the rv $\Theta_{0,i}$ from $\psi_{0,i}$, i.e., we should be able to write $\psi_{0,i}(t) = \mathcal{L}_{\Theta_{0,i}}(t)$. Also, we assume that $\Theta_0,\Theta_{0,1},...,\Theta_{0,d}$ are strictly positive discrete rvs defined on $\mathbb{N}$ with pmf

$f_{\Theta_{0,i}}(\theta_{0,i}) = \Pr(\Theta_{0,i} = \theta_{0,i})$ and cdf $F_{\Theta_{0,i}}(\theta_{0,i}) = \Pr(\Theta_{0,i} \leq \theta_{0,i}) = \sum_{j=1}^{\theta_{0,i}} f_{\Theta_{0,i}}(j)$ for $\theta_{0,s} \in \mathbb{N}$ and $i = 1, 2, \ldots, d$. Then, (2.50) becomes

$$F_{\underline{X}}\left(\underline{k_1}h, ..., \underline{k_d}h\right) = \sum_{\theta_0=1}^{\infty} \prod_{i=1}^{d} \left(\sum_{\theta_{0,i}=1}^{\infty} \prod_{j=1}^{n_i} F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) f_{\Theta_{0,i}}(\theta_{0,i}) \right) f_{\Theta_0}(\theta_0). \quad (2.51)$$

We define $S = \sum_{i=1}^{d} \sum_{j=1}^{n_i} X_{i,j} = \sum_{i=1}^{d} S_i$ where $S_i = \sum_{j=1}^{n_i} X_{i,j}$, for $i = 1, \ldots, d$. Then, similarly to (2.27), we have

$$f_{S_i|\Theta_0=\theta_0}(kh) = \sum_{\theta_{0,i}=1}^{\infty} f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(kh) f_{\Theta_{0,i}}(\theta_{0,i}), \text{for } i = 1, \ldots, d,$$

and

$$f_S(kh) = \sum_{\theta_0=1}^{\infty} f_{S|\Theta_0=\theta_0}(kh) f_{\Theta_0}(\theta_0),$$

where

$$(S|\Theta_0 = \theta_0) = \sum_{i=1}^{d} (S_i|\Theta_0 = \theta_0),$$

and

$$(S|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i}) = \sum_{j=1}^{n_i} (X_{i,j}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i}).$$

Note that $(S_1|\Theta_0 = \theta_0)$, ..., $(S_d|\Theta_0 = \theta_0)$ are conditionally independent. Within each class $i = 1, \ldots, d$, $(X_{i,1}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$,...,$(X_{i,n_i}|\Theta_0 = \theta_0, \Theta_{0,s} = \theta_{0,i})$ are also conditionally independent. It implies that

$$f_{S|\Theta_0=\theta_0}(kh) = f_{S_1|\Theta_0=\theta_0} * \ldots * f_{S_d|\Theta_0=\theta_0}(kh), \quad (2.52)$$

and

$$f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(kh) = f_{X_{i,1}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}} * \ldots * f_{X_{i,n_i}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(kh), \quad (2.53)$$

for $k \in \mathbb{N}_0$ and $i = 1, \ldots, d$ and where "$*$" denotes the convolution product. Values of (2.52) and (2.53) are computed using the same tools (e.g. DePril, FFT, etc) as the ones mentioned in Section 2.4.

Algorithm 3. *Let* $\theta_0 \in \{1, 2, ..., \theta_0^*\}$.

1. *Begin with* $\theta_0 = 1$.
2. *For each child copula* C_i *where* $i \in \{1, \ldots, d\}$, *let* $\theta_{0,i} \in \left\{1, 2, ..., \theta_{0,i}^*\right\}$ *and proceed as follows:*

    a) *Begin with* $\theta_{0,i} = 1$.

b) For $j = 1, ..., n_i$, calculate $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) = \mathrm{e}^{-\theta_{0,i}\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$, for $k_{i,j} \in \mathbb{N}_0$.

c) For $j = 1, \ldots, n_i$, calculate

$$f_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) = \begin{cases} \mathrm{e}^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)} & ,k_{i,j} = 0 \\ \mathrm{e}^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)} - \mathrm{e}^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}((k_{i,j}-1)h)\right)} & ,k_{i,j} \in \mathbb{N} \end{cases}.$$

d) Using e.g. FFT or DePril's Algorithm, compute $f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_ih)$ for $k_i \in \mathbb{N}_0$.

e) Repeat steps (2b), (2c), and (2d) for $\theta_{0,i} = 2, ..., \theta_{0,i}^*$ where $\theta_{0,i}^*$ is chosen such that $F_{\Theta_{0,i}}\left(\theta_{0,i}^*\right) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).

f) Compute $f_{S_i|\Theta_0=\theta_0}(k_ih) = \sum_{\theta_{0,i}=1}^{\theta_{0,s}^*} f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_ih)\, f_{\Theta_{0,i}}(\theta_{0,i})$, for $k_i \in \mathbb{N}_0$.

3. *Convolute all $f_{S_i|\Theta=\theta}$ for $i = 1, \ldots, d$, to calculate $f_{S|\Theta=\theta}$.*
4. *Repeat steps (2) and (3) for $\theta_0 = 2, ..., \theta_0^*$ where θ_0^* is chosen such that $F_{\Theta_0}(\theta_0^*) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).*
5. *Compute $f_S(kh) = \sum_{\theta_0=1}^{\theta_0^*} f_{S|\Theta_0=\theta}(kh)\, f_{\Theta_0}(\theta_0)$, for $k \in \mathbb{N}_0$.*

As mentioned notably in Hofert (2010), the difficult task is to identify the distributions of $\Theta_{0,1}, \ldots, \Theta_{0,d}$. For example, if we assume that $C_{\alpha_0}, C_{\alpha_1}, \ldots, C_{\alpha_d}$ are AMH copulas, with respective parameters $\alpha_0, \alpha_1, \ldots, \alpha_d$ (with $\alpha_0 < \min(\alpha_1, \ldots, \alpha_d)$), then, it implies that $\Theta_0 \sim \text{Geo}(1 - \alpha_0)$ and $\Theta_{0,i} \sim \text{Shifted NB}(\alpha_0, \frac{1-\alpha_i}{1-\alpha_0})$, for $i = 1, \ldots, d$, as shown in Hofert (2010) (see details in Appendix).

In the following example, we consider a two-dimensional partially nested Archimedean copula as defined in (2.48), where all copulas involved are AMH copulas. The example illustrates the accuracy of the proposed strategy in comparison to the MC simulation method.

Example 62. Consider a portfolio of 80 risks $\underline{X} = (X_{1,1}, ..., X_{1,40}, X_{2,1}, ..., X_{2,40})$ with multivariate cdf defined as in (2.51) with $d = 2$ and $n_1 = n_2 = 40$. Assume C_{α_0}, C_{α_1} and C_{α_2} to be AMH copulas with $\alpha_0 = 0.2$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.4$. Let $X_{i,j} \sim Bin(10, q_{i,j})$ where $q_{i,j} = 0.05 \times i + 0.005j$, $i = 1, 2$ and $j = 1, 2, ..., 40$. It implies that $E[S] = 142$. Relevant measures of $S = \sum_{i=1}^{2}\sum_{j=1}^{40} X_{i,j}$ can be obtained with Algorithm 3 or with MC simulations. Values of the expectation, the variance, the VaR and the TVaR for both methods in addition to their confidence intervals (with a confidence level of 95%) are given in Table 2.22. We can also calculate the exact values of Pearson's correlation coefficient between different risks. For example $\rho_P(X_{1,1}, X_{1,2}) = 0.07548428$, $\rho_P(X_{2,1}, X_{2,2}) = 0.12173402$ and $\rho_P(X_{1,1}, X_{1,2}) = 0.05336731$. Note that the simulation results (1 million simulations) are very close to the exact values obtained with the proposed approach.

	Exact values	Simulated values	$\mathrm{IC}_{0.05}$
$E[S]$	142	141.9887	[141.9304 ; 142.0469]
$Var(S)$	883.6003	883.3098	[880.8666 ; 885.7633]
$VaR_{0.5}(S)$	133	133	[133 ; 134]
$VaR_{0.9}(S)$	186	186	[186 ; 186]
$VaR_{0.99}(S)$	225	225	[225 ; 226]
$VaR_{0.999}(S)$	250	249	[248 ; 250]
$VaR_{0.9999}(S)$	267	267	[266 ; 268]
$TVaR_{0.5}(S)$	165.3440	165.3444	[165.3386 ; 165.3984]
$TVaR_{0.9}(S)$	204.2611	204.4585	[204.4295 ; 204.4874]
$TVaR_{0.99}(S)$	236.2996	236.3021	[236.2140 ; 236.3066]
$TVaR_{0.999}(S)$	257.3535	257.5276	[257.1387 ; 257.5444]
$TVaR_{0.9999}(S)$	273.0259	273.5895	[272.7160 ; 273.805]

Table 2.22 – Values of the expectation, variance, VaR and TVaR of $S = X_{1,1} + ... + X_{1,40} + X_{2,1} + ... + X_{2,40}$ where the joint cdf $F_{X_{1,1},...,X_{1,40},X_{2,1},...,X_{2,40}}$ is as defined in Example 62.

In the following example, we present a specific five-dimensional partially nested Archimedean copula with two nesting levels.

Example 63. Assume a multivariate cdf of $\underline{X} = (X_1, X_2, X_3, X_4, X_5)$ defined with binomial marginals and the following five-dimensional partially nested Archimedean copula:

$$C(u_1, u_2, u_3, u_4, u_5) = C_{\alpha_0}\left(C_{\alpha_1}(u_1, u_2), C_{\alpha_2}(u_3, C_{\alpha_3}(u_4, u_5))\right),$$

where C_{α_j}, $j = 0, 1, 2, 3$, correspond to bivariate AMH copulas. Also, $X_i \sim Bin(10, 0.05i)$, $i = 1, 2, ..., 5$, and $\alpha_j = 0.1j + 0.2$, $j = 0, 1, 2, 3$. Let $S = \sum_{i=1}^{5} X_i$ which implies $E[S] = 7.5$. In Table 2.23, we provide the exact and the simulated (10 million MC simulations) values of $f_S(k)$, $k = 0, 1, 2, ..., 50$ in addition to their confidence intervals (with a confidence level of 95%). We noticed a fluctuation in the results for several distinct 10 million simulation paths, contrarily to our proposed approach.

k	$f_S(k)$ (exact values)	$f_S(k)$ (simulated values)	$\mathrm{IC}_{0.05}$
0	0.000808	0.000805	[0.000786 ; 0.000821]
1	0.005795	0.005785	[0.005714 ; 0.005808]
2	0.020111	0.020082	[0.019991 ; 0.020165]
3	0.045814	0.045699	[0.045598 ; 0.045856]
4	0.078337	0.078477	[0.078137 ; 0.078470]
5	0.108726	0.108730	[0.108657 ; 0.109043]
10	0.086310	0.086248	[0.086074 ; 0.086421]
15	0.006728	0.006732	[0.006667 ; 0.006768]

Table 2.23 – Values of the pmf of $S = \sum_{i=1}^{5} X_i$ where the multivariate cdf of $\underline{X} = (X_1, X_2, X_3, X_4, X_5)$ is as defined in Example 63.

	Exact values	Simulated values	$IC_{0.05}$
$E[S]$	7.5	7.49949	[7.49771 ; 7.50128]
$Var(S)$	8.31314	8.31013	[8.30286 ; 8.31742]
$VaR_{0.5}(S)$	7	7	[7 ; 7]
$VaR_{0.9}(S)$	11	11	[11 ; 11]
$VaR_{0.99}(S)$	15	15	[15 ; 15]
$VaR_{0.999}(S)$	17	17	[17 ; 17]
$VaR_{0.9999}(S)$	19	19	[19 ; 19]
$TVaR_{0.5}(S)$	9.81112	9.81008	[9.80771 ; 9.81239]
$TVaR_{0.9}(S)$	12.85623	12.85600	[12.85160 ; 12.86020]
$TVaR_{0.99}(S)$	15.75489	15.75197	[15.744530 ; 15.759540]
$TVaR_{0.999}(S)$	17.89402	17.89070	[17.86590 ; 17.91640]
$TVaR_{0.9999}(S)$	19.72388	19.72100	[19.6540 ; 19.7920]

Table 2.24 – Values of expectation, variance, VaR and TVaR of $S = \sum_{i=1}^{5} X_i$ as defined in Example 63.

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2.11 Appendix – Archimedean copulas

Archimedean copulas defined with a strictly positive discrete mixing rv Θ:

1. Ali-Mikhail-Haq (AMH) family:
   - Copula: $C_\alpha(u_1, ..., u_n) = (1-\alpha)\left(\prod_{i=1}^{n}\left((1-\alpha)u_i^{-1}+\alpha\right)-\alpha\right)^{-1}$
   - Parameter: $\alpha \in [0,1)$
   - Discrete distribution for Θ: Shifted Geometric$(1-\alpha)$
   - Pmf: $f_\Theta(k) = \alpha^{k-1}(1-\alpha)$, $k \in \mathbb{N}$
   - LST: $\mathcal{L}_\Theta(t) = \frac{1-\alpha}{e^t-\alpha}$
   - Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = \ln\left(\frac{1-\alpha}{u}+\alpha\right)$
   - Particular cases: as $\alpha \to 0$, $C_\alpha(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$
2. Frank family:
   - Copula: $C_\alpha(u_1, ..., u_n) = \frac{-1}{\alpha}\ln\left(1-\frac{\left(1-e^{-u_1\alpha}\right)\times ...\times\left(1-e^{-u_n\alpha}\right)}{\left(1-e^{-\alpha}\right)^{n-1}}\right)$
   - Parameter: $\alpha \in (0,\infty)$
   - Discrete distribution for Θ: Logarithmic$(1-e^{-\alpha})$
   - Pmf: $f_\Theta(k) = \frac{\left(1-e^{-\alpha}\right)^k}{k\alpha}$, $k \in \mathbb{N}$
   - LST: $\mathcal{L}_\Theta(t) = -\frac{1}{\alpha}\ln\left(1-\left(1-e^{-\alpha}\right)e^{-t}\right)$
   - Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = -\ln\left(\frac{1-e^{-\alpha u}}{1-e^{-\alpha}}\right)$
   - Particular cases: $C_{\alpha\to 0}(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$ and $C_{\alpha\to\infty}(u_1, ..., u_n) = C^+(u_1, ..., u_n)$
3. Shifted Negative Binomial family:
   - Copula: $C_{r,q}(u_1, ..., u_n) = \left(q\left(\prod_{i=1}^{n}\left(qu_i^{\frac{-1}{r}}+1-q\right)-(1-q)\right)^{-1}\right)^r$
   - Parameter: $r \in \mathbb{R}^+$ and $q \in (0,1)$
   - Discrete distribution for Θ: Shifted Negative Binomial(r,q)
   - $\Theta = M + r$ with $M \sim NB(r,q)$
   - Pmf: $f_\Theta(k) = \binom{k-1}{k-r}q^r(1-q)^{k-r}$, $k = r, r+1, \ldots$
   - LST: $\mathcal{L}_\Theta(t) = \left(\frac{qe^{-t}}{1-(1-q)e-t}\right)^r$
   - Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = \ln\left(qu^{-\frac{1}{r}}+(1-q)\right)$
   - Particular cases: as $r \to 0$ or $r \to \infty$, $C_{r,q}(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$. Also, $C_{1,q}(u_1, ..., u_n) = C_{1-q}^{AMH}(u_1, ..., u_n)$ and when $r \to 0$, $C_{0,0}(u_1, ..., u_n) = C^+(u_1, ..., u_n)$. The most important case is when $q \to 0$, $C_{r,0}(u_1, ..., u_n) = C_{1/r}^{Clay}(u_1, ..., u_n)$

Archimedean copulas defined with a strictly positive continuous mixing rv Θ:

1. Clayton family:

    — Copula: $C_\alpha(u_1, ..., u_n) = \left(u_1^{-\alpha} + ... + u_n^{-\alpha} - (n-1)\right)^{-\frac{1}{\alpha}}$

    — Parameter: $\alpha \in (0, \infty)$

    — Continuous distribution for Θ: Gamma($\frac{1}{\alpha}$,1)

    — LST: $\mathcal{L}_\Theta(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$

    — Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = u^{-\alpha} - 1$

    — Particular cases: $C_{\alpha \to 0}(u_1, ..., u_n) = C^{\perp}(u_1, ..., u_n)$ and $C_{\alpha \to \infty}(u_1, ..., u_n) = C^{+}(u_1, ..., u_n)$

2. Gumbel family:

    — Copula: $C_\alpha(u_1, ..., u_n) = \exp\left(-\left((-\ln(u_1))^\alpha + ... + (-\ln(u_n))^\alpha\right)^{\frac{1}{\alpha}}\right)$

    — Parameter: $\alpha \in [1, \infty)$

    — Continuous distribution for Θ: Positive Stable$\left(\frac{1}{\alpha}, 1, \cos^\alpha\left(\frac{\pi}{2\alpha}\right), 1_{\{\alpha=1\}}\right)$

    — LST: $\mathcal{L}_\Theta(t) = \mathrm{e}^{-t^{\frac{1}{\alpha}}}$

    — Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = (-\ln(u))^\alpha$

    — Particular cases: $C_{\alpha \to 1}(u_1, ..., u_n) = C^{\perp}(u_1, ..., u_n)$ and $C_{\alpha \to \infty}(u_1, ..., u_n) = C^{+}(u_1, ..., u_n)$

2.12 Appendix – Nested Archimedean Copula

1. **The Nested AMH family**
   - — $\psi_{0s}(t) = \mathcal{L}_{\Theta_{0s}}(t) = \left(\frac{1-q_s}{(1-q_0)(e^t-q_s)+q0(1-q_s)}\right)^{\Theta_0}$
   - — Distribution of Θ_{0s}: $(\Theta_{0s}|\Theta_0 = \theta) \sim SNB\left(\theta, \frac{1-q_s}{1-q_0}\right)$ with $q_0 \leq q_s$.
   - — pmf of Θ_{0s}: $f_{\Theta_{0s}|\Theta_0=\theta}(k) = \binom{k-1}{k-\theta}(q^*)^r(1-q^*)^{k-r}$ with $q^* = \frac{1-q_s}{1-q_0}$ and $k \in \{\theta, \theta+1, \cdots\}$
2. **The Nested Frank family**
   - — $\psi_{0s}(t) = \left(\frac{(1-e^{-\alpha_s})e^{-s}}{1-e^{-\alpha_0}}\right)^{\Theta_0}$
   - — Distribution of Θ_{0s}: $(\Theta_{0s}|\Theta_0 = \theta) \sim \sum_{i=1}^{\theta} V_i$, with $P(V_i = k) = p_k$ with $\alpha_0 \leq \alpha_s$
   - — with $p_k = \frac{(1-e^{-\alpha_s})^k}{(1-e^{-\alpha_s})^{\theta_0}} \sum_{j=0}^{\infty} \binom{\theta_0}{j}\binom{j\frac{\alpha_0}{\alpha_s}}{k}(-1)^{j+k}$ for $k \in \{1, 2, \cdots\}$

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Conclusion

Dans ce mémoire, nous nous sommes intéressés à la modélisation de la dépendance à l'aide de distributions construites par mélange et plus particulièrement à l'aide des copules Archimédiennes. Dans un premier temps, nous avons défini quelques mélanges exponentiels discrets et démontrés comment ceux-ci peuvent approximer des distributions continues connues. Nous avons utilisé ces résultats pour démontrer qu'il est également possible d'utiliser des copules Archimédiennes construites par mélange discret pour approximer des copules Archimédiennes construites par mélange continu.

L'intérêt de ces approximations prend son sens au chapitre 3, puisqu'on y a développé des méthodes de calcul analytiques exactes ou approximatives pour résoudre des problèmes d'agrégation avec dépendance lorsque la relation de dépendance est introduite par le biais d'une copule Archimédienne discrète. Ces méthodes de calcul reposent sur la construction par mélange des copules Archimédiennes. Nous avons utilisé ces méthodes pour résoudre des problèmes d'allocation du capital ou de théorie de la ruine. Alors que des méthodes de simulation Monte Carlo existent déjà pour résoudre de tels problèmes, les méthodes qu'on a proposées ont l'avantage de pouvoir borner le résultat exact d'un problème d'agrégation par des bornes supérieure et inférieure aussi précises que nécessaire. Enfin, nous avons développé une extension aux copules Archimédiennes hiérarchiques et avons démontré qu'il est aussi possible de trouver des méthodes de calcul exactes dans ce cas.

Les méthodes de calculs proposées dans ce mémoire ont l'avantage de produire des résultats exacts dans un contexte discret. Dans un contexte continu, elles permettent également de calculer des bornes supérieure et inférieure aussi précises que désirées pour le résultat exact. Toutefois, le temps de calcul requis pour obtenir un résultat exact est parfois considérablement plus long que d'utiliser une méthode par simulation Monte Carlo.

Notons qu'en pratique les compagnies d'assurance sont encore loin d'utiliser des méthodes aussi avancées pour modéliser la dépendance entre leurs portefeuilles de risques. Par contre, les cadres réglementaires canadiens font de plus en plus la promotion de méthodes de calcul basées sur des principes plutôt que des méthodes de calcul prescrites. Par exemple, en 2018, les exigences de capital des sociétés d'assurance de personne au Québec (norme ESCAP de l'AMF) et au Canada (norme TSAV du BSIF) ont changé. Elles accordent maintenant plus de liberté aux assureurs afin d'utiliser leurs propres modèles alors que les anciennes normes étaient basées sur une méthode de calcul par facteurs prescrits. Ces normes reconnaissent d'ailleurs une certaine dépendance entre les risques nécessitant de maintenir du capital et prescrivent l'utilisation d'une matrice de diversification pour agréger les différents risques. Bien que la méthode utilisée soit encore simple, il sera intéressant de voir si les organismes réglementaires recommanderont prochainement l'utilisation de méthodes plus complexes ou même l'évaluation explicite de la dépendance entre les différents risques des assureurs. Nous serions alors en mesure de nous questionner à savoir comment nos méthodes de calculs s'appliquent dans ce contexte.

En attendant, d'autres processus existent à l'international tel ORSA (*Own Risk and Solvency Assessment*) et visent à ce que les assureurs et les institutions financières se dotent de saines pratiques de gestion des riques. Afin de mieux gérer leurs risques, certaines compagnies bâtissent des modèles de capital économique qui sont plus complexes que les modèles réglementaires, mais qui ont aussi l'avantage de mieux refléter le risque réel. Pour une institution financière, de tels modèles peuvent être utilisés afin de gérer leur probabilité de ruine ou afin d'allouer leur capital entre leurs lignes d'affaire et ainsi de mieux évaluer la rentabilité de celles-ci. Nous pourrions poursuivre nos recherches en tentant d'appliquer les méthodes de calcul proposées dans ces cas réels.

Annexe A

Code informatique sélectionné

Cette annexe contient des exemples de code informatique pour certaines des applications présentées dans ce mémoire. Tous les extraits de code ont été réalisés à l'aide du logiciel R.

Par souci de simplicité, les lois marginales présentées dans ces extraits suivent majoritairement des lois géométriques. Le code est facilement adaptable à d'autres distributions.

A.1 Exemples FFT

A.1.1 Somme de deux v.a. discrètes indépendantes

```
# FFT -- Somme de deux v.a. discretes independantes

fft.directconvo<-function(m=16, fx, fy)
    {
       aa <- 2^m
       nx <- length(fx)
       ny <- length(fy)
       ftx <- fft(c(fx, rep(0, aa - nx)))
       fty <- fft(c(fy, rep(0, aa - ny)))
       fs <- Re(fft(ftx*fty, TRUE))/aa
       return(fs)
    }
```

A.1.2 Somme de m v.a. discrètes indépendantes

```
#FFT -- Somme de n v.a. discretes independantes
```

```
fft.nrisks<-function(matff,v.n,m=14)
{
aa <- 2^m
nbrisks<-dim(matff)[1]
fx<-matff[1,]
nx <- length(fx)
ftx <- fft(c(fx, rep(0, aa - nx)))
fts<-(ftx)^v.n[1]
   for (i in 2:nbrisks)
      {
      fx<-matff[i,]
      nx <- length(fx)
      ftx <- fft(c(fx, rep(0, aa - nx)))
      fts<-fts*(ftx^v.n[i])
   }
   ffs <- Re(fft(fts, TRUE))/aa
return(ffs)
}
```

A.1.3 Somme aléatoire

```
#FFT -- Somme aleatoire (loi composee)

  fft.poiscomposee<-function(lam, n, fx)
    {
    # 2**n = longueur du vecteur
    # prendre n eleve (ex: n=12 ou plus)
    # premiere masse de fx est Pr(X=0)
       aa <- 2^n
       nx <- length(fx)
       ftx <- fft(c(fx, rep(0, aa - nx)))
       fts<-exp(lam * (ftx - 1))
       fs <- Re(fft(fts, T))/aa
       return(fs)
    }
```

A.2 Exemples de copules simples

A.2.1 Copule de Frank

```
library("actuar")
library("hypergeom")

#Les deux elements suivants peuvent facilement changer
qi <- function(i) 1/40*i   #Geometriques tronquees moyenne 40/i
m=3 #Nombre de lois geo

psi.inverse <- function(alpha,u)
    {
        log((1-exp(-alpha))/(1-exp(-u*alpha)))
    }

fxi.theta <- function(ki,theta,alpha,i)
    {
        if(ki==0)
            {
                Fxi.theta(0,theta,alpha,i)
            }
        else
            {
                Fxi.theta(ki,theta,alpha,i)-Fxi.theta(ki-1,theta,alpha,i)
            }
    }

fmp.theta <- function(k,alpha)
    {
        if (k==0)
            {
                0
            }
        else
        {
            (1-exp(-alpha))^k/(k*alpha)
        }
    }
```

```
#FFT -- Somme de n v.a. discretes independantes

    fft.nrisks<-function(matff,v.n,m=14)
    {
    aa <- 2^m
    nbrisks<-dim(matff)[1]
    fx<-matff[1,]
    nx <- length(fx)
    ftx <- fft(c(fx, rep(0, aa - nx)))
    fts<-(ftx)^v.n[1]
       for (i in 2:nbrisks)
          {
          fx<-matff[i,]
          nx <- length(fx)
          ftx <- fft(c(fx, rep(0, aa - nx)))
          fts<-fts*(ftx^v.n[i])
       }
       ffs <- Re(fft(fts, TRUE))/aa
    return(ffs)
    }

fs.theta <- function(alpha,mm,ll)
{
    fs.theta <- matrix(0,ll,2^mm)
    for (theta in 1:ll)
        {
            vfxi <- matrix(0,m,2^mm)
            for (j in 1:m)
                {
                    vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
                }
            fs.theta[theta,] <- fft.nrisks(vfxi,rep(1,m),m=mm)
        }
    return(fs.theta)
}

#Soit fs la fmp de S inconditionnel
fs <- function(alpha,mm,ll)
    {
        fs.theta1 <- fs.theta(alpha,mm,ll)
```

```
        vfmp <- sapply(1:dim(fs.theta1)[1],fmp.theta,alpha)
        m.vfmp <- matrix(vfmp,nrow=2^mm,ncol=dim(fs.theta1)[1],byrow=FALSE)
        fs <- colSums(fs.theta1*vfmp)
        return(fs)
    }
```

A.2.2 Copule AMH

```
library("actuar")
library("hypergeom")

#Les deux elements suivants peuvent facilement changer
qi <- function(i) 1/40*i   #Geometriques tronquees moyenne 40/i
m=3 #Nombre de lois geo

psi.inverse <- function(alpha,u)
    {
        log((1-(1-alpha))/u+(1-alpha))
    }

fxi.theta <- function(ki,theta,alpha,i)
    {
        if(ki==0)
            {
                Fxi.theta(0,theta,alpha,i)
            }
        else
            {
                Fxi.theta(ki,theta,alpha,i)-Fxi.theta(ki-1,theta,alpha,i)
            }
    }

fmp.theta <- function(k,alpha)
    {
        if (k==0)
            {
                0
            }
        else
        {
```

```
            (1-alpha)*(alpha)^(k-1)
        }
    }

#FFT -- Somme de n v.a. discretes independantes

    fft.nrisks<-function(matff,v.n,m=14)
    {
    aa <- 2^m
    nbrisks<-dim(matff)[1]
    fx<-matff[1,]
    nx <- length(fx)
    ftx <- fft(c(fx, rep(0, aa - nx)))
    fts<-(ftx)^v.n[1]
       for (i in 2:nbrisks)
          {
          fx<-matff[i,]
          nx <- length(fx)
          ftx <- fft(c(fx, rep(0, aa - nx)))
          fts<-fts*(ftx^v.n[i])
       }
       ffs <- Re(fft(fts, TRUE))/aa
    return(ffs)
    }

fs.theta <- function(alpha,mm,ll)
{
    fs.theta <- matrix(0,ll,2^mm)
    for (theta in 1:ll)
        {
            vfxi <- matrix(0,m,2^mm)
            for (j in 1:m)
                {
                    vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
                }
            fs.theta[theta,] <- fft.nrisks(vfxi,rep(1,m),m=mm)
        }
    return(fs.theta)
}
```

```
#Soit fs la fmp de S inconditionnel
fs <- function(alpha,mm,ll)
    {
        fs.theta1 <- fs.theta(alpha,mm,ll)
        vfmp <- sapply(1:dim(fs.theta1)[1],fmp.theta,alpha)
        m.vfmp <- matrix(vfmp,nrow=2^mm,ncol=dim(fs.theta1)[1],byrow=FALSE)
        fs <- colSums(fs.theta1*vfmp)
        return(fs)
    }
```

A.3 Mélange exponentielle binomiale négative

```
library("actuar")
library(hypergeo)

EXHX <- function(r,h,lambda,t)
    {
        upper <- c(1+r,(h*r-t*lambda)/h,(h*r-t*lambda)/h)
        lower <- rep((h*r+h-t*lambda)/h,2)
        (1/(h*r-t*lambda)^2)*(h*r*lambda*(1-exp(-h))^r*genhypergeo(
            upper,lower,exp(-h),polynomial=TRUE,maxiter=500000))
    }


FGM <- function(r,h,lambda,t)
    {
        upper <- c(1+r,(h*r-t*lambda)/h)
        lower <- (h*r+h-t*lambda)/h
        (1/(h*r-t*lambda))*(h*r*(1-exp(-h))^r*genhypergeo(
            upper,lower,exp(-h),polynomial=TRUE,maxiter=200000))
    }

Prime.esscher <- function(r,h,lambda,t)
    {
        EXHX(r,h,lambda,t)/FGM(r,h,lambda,t)
    }

Prime.esscher.S <- function(n,r,h,lambda,t) #n=nb de loi iid
    {
```

```
        n*Prime.esscher(r,h,lambda,t)
    }


#Modele Poisson Compose:

Prime.ess.comp.2 <- function(lambda.poi,temps.pois,r,h,lambda,t)
    {
        lambda.poi*temps.pois*EXHX(r,h,lambda,t)
    }

#nieme moment
esp.n <- function(n,r,h,lambda)
    {
        lambda^n*gamma(n+1)/h^n/r^n*(1-exp(-h))^r*genhypergeo(
            rep(r,n+1),rep(r+1,n),exp(-h),polynomial=TRUE,maxiter=20000)
    }


######## Prob. de ruine, Borne exponentielle

FGM2 <- function(beta,r,h,lambda,t)
    {
        exp(beta*(FGM(r,h,lambda,t)-1))
    }

Coeff.ajust <- function(beta,r,h,lambda,c)
    {
        fun <- function(x) abs(FGM2(beta,r,h,lambda,x)*beta/(beta+c*x)-1)
        optimize(fun,c(0,0.02),tol=0.0000001)
    }


lundberg <- function(r,h,lambda,beta,coeff,c)
    {
        FGM2(beta,r,h,lambda,coeff)*beta/(beta+coeff*c)
    }

Sx <- function(x,r,h,lambda)
    {
```

```
        ((1-exp(-h))/(exp(x*h/lambda)-exp(-h)))^r
    }


fx <- function(x,r,h,lambda)
    {
        r*h/lambda*exp(x/lambda*h)*(1-exp(-h))^r/(exp(x*h/lambda)-exp(-h))^(r+1)
    }


hx <- function(x,r,h,lambda)   #Failure rate
    {
        r*h*exp(x*h/lambda)/(exp(x*h/lambda)-exp(-h))
    }


#nieme moment
esp.n <- function(n,r,h,lambda)
    {
        lambda^n*gamma(n+1)/h^n/r^n*(1-exp(-h))^r*genhypergeo(
            rep(r,n+1),rep(r+1,n),exp(-h),polynomial=TRUE,maxiter=20000)
    }


#E[X 1 {X>d}]
esp.tronqusup <- function(d,r,h,lambda)
    {
        d <- d/lambda
        lambda*(1/(h*r)+d)*(1-exp(-h))^r*genhypergeo(
            c(r,r,(1+h*d*(r+1))/(h*d)),c(r+1,(1+h*r*d)/(h*d)),
            exp(-h*(d+1)),polynomial=TRUE,maxiter=100000)/exp(h*r*d)
    }


#VaR
VaR.exp <- function(k,r,h,lambda)
    {
        lambda/h*log((1-exp(-h))/(1-k)^(1/r)+exp(-h))
    }
```

```
#TVaR
TVaR.exp <- function(k,r,h,lambda)
    {
        VaR.temp <- VaR.exp(k,r,h,lambda)
        output <- (esp.tronqusup(VaR.temp,r,h,lambda)+VaR.temp*
                   (1-Sx(VaR.temp,r,h,lambda)-k))/(1-k)
        return(output)
    }
```

A.4 Méthodes d'allocation du capital

A.4.1 Méthode de la covariance

```
Fxi.theta <- function(ki,theta,alpha,i)
    {
        exp(-theta*psi.inverse(alpha,pgeom(ki-1,qi(i))))
    }

#Fct calcule E[Xi*S|Theta=theta]
esp.fft.sk <- function(alpha,theta,mm)
    {
        mat.vfxi <- matrix(0,m,2^mm)
        for (j in 1:m)
            {
                mat.vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
            }
        mat.fxp <- matrix(0,m,2^mm)
        output <- matrix(0,m,2^(mm+1))
        for (i in 1:m)
            {
                mat.fxp[i,] <- (0:(2^mm-1))*mat.vfxi[i,]
                dummy <- sum(mat.fxp[i,])
                #v.fs <- (0:(2^mm-1))*fft.nrisks(mat.vfxi,rep(1,m),mm)
                if (m>2)
                    {
                        v.fs <- (0:(2^mm-1))*fft.nrisks(mat.vfxi[-i,],rep(1,m-1),mm)
                    }
                else
                    {
```

```
                    v.fs <- (0:(2^mm-1))*mat.vfxi[-i,]
                }
            dummy2 <- sum(v.fs)
            output[i,] <- fft.nrisks(rbind(mat.fxp[i,]/dummy,c(v.fs)/dummy2)
                                ,c(1,1),mm+1)*dummy*dummy2
        }
    EXi2 <- rowSums(mat.fxp*matrix(rep((0:(2^mm-1)),m),m,2^mm,byrow=TRUE))
    results <- rbind("Part risque i"=rowSums(output)+EXi2)
    return("E[Xi*S|Theta=theta]"=c(results))
}

Parts.cov <- function(alpha,ll,mm)
{
    vfmp <- sapply(1:ll,fmp.theta,alpha)
    m.vfmp <- matrix(rep(vfmp,m),m,ll,byrow=TRUE)
    #On calcule E[Xi*S|Theta=theta] pour tout theta
    mat <- sapply(1:ll,esp.fft.sk,alpha=alpha,mm=mm)
    EXS <- rowSums(mat*m.vfmp)
    EX <- 1/qi(1:m)
    ES <- sum(EX)
    covXS <- EXS-EX*ES
    var <- sum(covXS)
    return(c("Allocation pour le risque "=c(covXS/var)))
}
```

A.4.2 Méthode de la VaR

```
Fxi.theta <- function(ki,theta,alpha,i)
{
    exp(-theta*psi.inverse(1-alpha,pbinom(ki,n,qi(i))))
}

esp.fft.sk <- function(alpha,theta,mm)
{
    mat.vfxi <- matrix(0,m,2^mm)
    for (j in 1:m)
        {
            mat.vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
        }
```

```
        mat.fxp <- matrix(0,m,2^mm)
        output <- matrix(0,m,2^(mm+1))
        for (i in 1:m)
            {
                mat.fxp[i,] <- (0:(2^mm-1))*mat.vfxi[i,]
                if (m>2)
                    {
                        v.fs <- fft.nrisks(mat.vfxi[-i,],rep(1,m-1),mm)
                    }
                else
                    {
                        v.fs <- mat.vfxi[-i,]
                    }
                output[i,] <- fft.nrisks(rbind(mat.fxp[i,],c(v.fs)),c(1,1),mm+1)
            }
        v <- fft.nrisks(mat.vfxi,rep(1,m),mm+1)
        mat <- matrix(rep(v,m),nrow=m,ncol=2^(mm+1),byrow=TRUE)
        temp <- 1:(m*n)
        return(list("Ex[Xi 1{S=k}]"=rbind("VaR="=temp,output[,temp+1]/mat[,temp+1])
                    ,"fs.theta"=v))
    }

Parts.var.theta <- function(VaR,theta,alpha,mm)
    {
        res <- esp.fft.sk(alpha,theta,mm)
        vfs <- res[[2]] #Commence a 0
        Fs <- cumsum(vfs)
        output <- res[[1]][,VaR]
        return(c("VaR"=output[1],"E[Xi 1{S=VaR}] i= "=output[-1]))
    }

Parts.var <- function(k=0.95,alpha,ll=100,mm)
    {
        VaR <- which(cumsum(fs(alpha,mm,ll))>=k)[1]-1
        if (VaR==0) return(c("VaR"=0,"kappa"=k)) else
        vfmp <- matrix(sapply(1:ll,fmp.theta,alpha),m,ll,byrow=TRUE)
        test <- sum(vfmp[1,])>= max(k,0.999999)  #Environ 1, mais au moins kappa
        output <- sapply(X=1:ll,FUN=Parts.var.theta,VaR=VaR,alpha=alpha,mm=mm)

        if (test)
```

```
        {
            return(list("Info"=c(VaR=VaR,kappa=k),"Part.VaR.k(Xi)"=
                        rowSums(output[-1,]*vfmp)))
        }
    else
        {
            return("ll n'est pas assez grand, la fmp de Theta ne somme pas ")
        }
}
```

A.4.3 Méthode de la TVaR

```
Fxi.theta <- function(ki,theta,alpha,i)
    {
        exp(-theta*psi.inverse(1-alpha,pbinom(ki,n,qi(i))))
    }

esp.fft.sk <- function(alpha,theta,mm)
    {
        mat.vfxi <- matrix(0,m,2^mm)
        for (j in 1:m)
            {
                mat.vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
            }
        mat.fxp <- matrix(0,m,2^mm)
        output <- matrix(0,m,2^(mm+1))
        for (i in 1:m)
            {
                mat.fxp[i,] <- (0:(2^mm-1))*mat.vfxi[i,]
                if (m>2)
                    {
                        v.fs <- fft.nrisks(mat.vfxi[-i,],rep(1,m-1),mm)
                    }
                else
                    {
                        v.fs <- mat.vfxi[-i,]
                    }
                output[i,] <- fft.nrisks(rbind(mat.fxp[i,],c(v.fs)),c(1,1),mm+1)
            }
        v <- fft.nrisks(mat.vfxi,rep(1,m),mm+1)
```

```
        mat <- matrix(rep(v,m),nrow=m,ncol=2^(mm+1),byrow=TRUE)
        temp <- 1:(m*n)
        return(list("Ex[Xi 1{S=k}]"=rbind("VaR="
                        =temp,output[,temp+1]/mat[,temp+1]),"fs.theta"=v))
    }

Parts.var.theta <- function(VaR,theta,alpha,mm)
    {
        res <- esp.fft.sk(alpha,theta,mm)
        vfs <- res[[2]] #Commence
        Fs <- cumsum(vfs)
        #VaR <- which(Fs>=k)[1] #Donne directement VaRk(S)
        output <- res[[1]][,VaR]
        return(c("VaR"=output[1],"E[Xi 1{S=VaR}] i= "=output[-1]))
    }

Parts.tvar <- function(k=0.95,alpha,ll=100,mm)
    {
        vfs <- fs(alpha,mm,ll)
        Fs <- cumsum(vfs)
        VaR <- which(Fs>=k)[1]-1
        TVaR <- 1/(1-k)*(sum(((VaR+1):(m*n-1))*vfs[(VaR+2):(m*n)])+VaR*
                         (Fs[VaR+1]-k))
        vfmp <- matrix(sapply(1:ll,fmp.theta,alpha),m,ll,byrow=TRUE)
        mat.temp <- matrix(0,m,m*n-1-VaR)
        for (i in ((VaR+1):(m*n-1)))
            {
                mat.temp[,i-VaR] <- rowSums(sapply(X=1:ll,FUN=Parts.var.theta,
                                            VaR=i,alpha=alpha,mm=mm)[-1,]*vfmp)
            }
        p1 <- rowSums(mat.temp*matrix(vfs[(VaR+2):(m*n)],nrow=m,ncol=m*n-1-VaR,
                                       byrow=TRUE))
        p2 <- Parts.var(k,alpha,ll,mm)[[2]]
        beta <- (Fs[VaR+1]-k)
        parts <- p1+p2*beta
        return(list(c("TVaR"=TVaR,"kappa"=k),"Part.TVaR.k(Xi)"= parts/(1-k)))
    }
```

A.4.4 Méthode d'Esscher

```
Fxi.theta <- function(ki,theta,alpha,i)
    {
        exp(-theta*psi.inverse(alpha,pgeom(ki-1,qi(i))))
    }

esp.fft.sk <- function(alpha,theta,mm,h)
    {
        mat.vfxi <- matrix(0,m,2^mm)
        for (j in 1:m)
            {
                mat.vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
            }
        mat.fxp <- matrix(0,m,2^mm)
        output <- matrix(0,m,2^(mm+1))
        for (i in 1:m)
            {
                mat.fxp[i,] <- (0:(2^mm-1))*exp(h*(0:(2^mm-1)))*mat.vfxi[i,]
                dummy <- sum(mat.fxp[i,])
                if (m>2)
                    {
                        v.fs <- exp(h*(0:(2^mm-1)))*fft.nrisks(mat.vfxi[-i,],
                                                                rep(1,m-1),mm)
                    }
                else
                    {
                        v.fs <- exp(h*(0:(2^mm-1)))*mat.vfxi[-i,]
                    }
                dummy2 <- sum(v.fs)
                output[i,] <- fft.nrisks(rbind(mat.fxp[i,]/dummy,c(v.fs)/dummy2)
                                      ,c(1,1),mm+1)*dummy*dummy2
            }
        results <- rbind("Part risque i"=rowSums(output))
        return("Ex[Xi 1{S=k}]"=c("Part pour risque "=c(results)))
    }

Parts.esscher <- function(alpha,ll,mm,h)
    {
        vfmp <- sapply(1:ll,fmp.theta,alpha)
```

```
        m.vfmp <- matrix(rep(vfmp,m),ll,m)
        mat <- matrix(0,ll,m)
        for (i in 1:ll)
            {
                mat[i,] <- esp.fft.sk(alpha,i,mm,h)
            }
        results <- colSums(mat*m.vfmp)
        v <- fs(alpha,mm+1,ll)
        denom <- sum(exp(h*(0:(2^(mm+1)-1)))*v)
        prm <- sum(results[1:m])/denom
        prm2 <- sum((0:(2^(mm+1)-1))*exp(h*(0:(2^(mm+1)-1)))*v)/denom
        return(c("Prime Esscher"=prm, "Part risque "=c(results/denom),"Primes 2"=prm2))
    }
```

A.4.5 Méthode de Kamps

```
Fxi.theta <- function(ki,theta,alpha,i)
    {
        exp(-theta*psi.inverse(alpha,pgeom(ki-1,qi(i))))
    }

esp.fft.sk <- function(alpha,theta,mm,h)
    {
        mat.vfxi <- matrix(0,m,2^mm)
        for (j in 1:m)
            {
                mat.vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
            }
        mat.fxp <- matrix(0,m,2^mm)
        output <- matrix(0,m,2^(mm+1))
        for (i in 1:m)
            {
                mat.fxp[i,] <- (0:(2^mm-1))*exp(-h*(0:(2^mm-1)))*mat.vfxi[i,]
                dummy <- sum(mat.fxp[i,])
                if (m>2)
                    {
                        v.fs <- exp(-h*(0:(2^mm-1)))*fft.nrisks(mat.vfxi[-i,],
                                                                rep(1,m-1),mm)
                    }
                else
```

```
                {
                    v.fs <- exp(-h*(0:(2^mm-1)))*mat.vfxi[-i,]
                }
              dummy2 <- sum(v.fs)
              output[i,] <- fft.nrisks(rbind(mat.fxp[i,]/dummy,c(v.fs)/dummy2),
                                          c(1,1),mm+1)*dummy*dummy2
          }
      results <- rbind("Part risque i"=rowSums(output))
      return("Ex[Xi 1{S=k}]"=c("Part pour risque "=c(results)))
  }

Parts.kamps <- function(alpha,ll,mm,h)
  {
      vfmp <- sapply(1:ll,fmp.theta,alpha)
      m.vfmp <- matrix(rep(vfmp,m),ll,m)
      mat <- matrix(0,ll,m)
      for (i in 1:ll)
          {
              mat[i,] <- esp.fft.sk(alpha,i,mm,h)
          }
      EXi <- 1/qi(1:m)
      results <- EXi-colSums(mat*m.vfmp)
      v <- fs(alpha,mm+1,ll)
      denom <- sum((1-exp(-h*(0:(2^(mm+1)-1))))*v)
      prm <- sum(results[1:m])/denom
      prm2 <- sum((0:(2^(mm+1)-1))*(1-exp(-h*(0:(2^(mm+1)-1))))*v)/denom
      return(c("Prime Kamps"=prm, "Part risque "=c(results/denom),"Primes 2"=prm2))
  }
```

A.4.6 Méthode *Size-biased*

```
Fxi.theta <- function(ki,theta,alpha,i)
  {
      exp(-theta*psi.inverse(alpha,pgeom(ki-1,qi(i))))
  }

esp.fft.sk <- function(alpha,theta,mm,h)
  {
      mat.vfxi <- matrix(0,m,2^mm)
```

```
        for (j in 1:m)
            {
                mat.vfxi[j,] <- sapply(0:(2^mm-1),fxi.theta,theta,alpha,j)
            }
        v.fs <- matrix(0,m,2^mm)
        for (i in 1:m)
            {
                if (m>2)
                    {
                        v.fs[i,] <- fft.nrisks(mat.vfxi[-i,],rep(1,m-1),mm)
                    }
                else
                    {
                        v.fs[i,] <- mat.vfxi[-i,]
                    }
            }
        result <- matrix(0,nrow=m,ncol=1)
        for (i in (0:(2^mm-1)))
            {
                temp <- matrix(0,nrow=m,ncol=1)
                for (j in (0:(2^mm-1)))
                    {
                        temp <- temp+(j+i)^h*v.fs[,j+1]
                    }
                result <- result+i*mat.vfxi[,i+1]*temp
            }
        return("Ex[Xi S^t]"=result)
    }

Parts.size.biased <- function(alpha,ll,mm,h)
    {
        vfmp <- sapply(1:ll,fmp.theta,alpha)
        m.vfmp <- matrix(rep(vfmp,m),ll,m)
        mat <- matrix(0,ll,m)
        for (i in 1:ll)
            {
                mat[i,] <- esp.fft.sk(alpha,i,mm,h)
            }
        v <- fs(alpha,mm+1,ll)
        denom <- sum((0:(2^(mm+1)-1))^(h)*v)
```

```
        results <- colSums(mat*m.vfmp)
        prm <- sum(results[1:m])/denom
        prm2 <- sum((0:(2^(mm+1)-1))^(h+1)*v)/denom
        return(c("Prime Size-biased"=prm, "Part risque "=c(results/denom),
                 "Primes 2"=prm2))
    }
```

A.5 Calcul de la prime d'Esscher

```
fst <- function(alpha,lambda,mm,ll,t,qx,h)
    {
        fst <- numeric(0)
        fst <- dpois(0,lambda*t)
        st <- 0
        fst <- c(fst,dpois(1,lambda*t)*dgeom(0:2^mm,qx))
        st <- c(st,(1:(2^mm+1)))
        for (i in 2:(lambda*t*20))
            {
                fst <- c(fst,fs(alpha,mm,ll,i)*dpois(i,lambda*t))
                st <- c(st,0:(2^(mm+1)-1))
            }
        return(list(fst=fst,St=st))
    }

prime.ess <- function(alpha,lambda,mm,ll,t,qx,h)
    {
        vfst <- fst(alpha,lambda,mm,ll,t,qx)
        vfs <- vfst[[1]]
        vst <- vfst[[2]]
        esperance <- sum(vfs*vst)
        prime <- sum(vst*exp(h*vst)*vfs)/sum(exp(h*vst)*vfs)
        return(c("Esperance des pertes"=esperance,"Prime Esscher"=prime))
    }
```

A.6 Exemples de modèle de ruine

Cette section contient le code pour évaluer la probabilité de ruine selon le modèle présenté au chapitre 2 dans le cas général où une relation de dépendance s'applique à la probabilité d'occurence des sinistres ainsi que dans le cas particulier d'indépendance entre les probabilités d'occurence.

Le cas général avec dépendance entre les montants de sinistre est finalement présenté.

A.6.1 Dépendance entre les probabilité d'occurence

.

```
Fxi.theta <- function(ki,theta,alpha,i)
    {
        exp(-theta*psi.inverse(alpha,pbinom(ki,1,qi(i))))
    }

fmp.x <- function(k,qx)
    {
        dgeom(k-1,qx)
    }

prob.ruine <- function(u,alpha,ll,qx,prm,lx)
    {
        ruin.theta <- rep(0,ll)
        EX <- 1/qx #Geom
        vfmpx <- fmp.x(0:lx,qx)
        v.mm <- 2^(1:13)
        for (theta in 1:ll)
            {
                q.theta <- fxi.theta(1,theta,alpha,1)
                if((q.theta*EX)>prm)
                    {
                        ruin.theta[theta] <- 1
                    }
                else
                    {
                    ruin.u <- rep(0,u+1)
                    ruin.u[1] <- q.theta*(EX-1)/(1-q.theta)
                    if (u!=0)
                    {
                        for (i in 1:u)
                        {
                            mmm <- which(v.mm>i)[1]
                            temp <- sum(ruin.u)
```

```
                        temp2 <- sum(vfmpx[2:(i+1)])
                        matff <- rbind(c(0,vfmpx[2:(i+1)])/temp2,c(ruin.u[1:i],0)
                                       /temp)
                        prob.u <- fft.nrisks(matff,c(1,1),m=mmm)[i+1]*
                            temp*temp2
                        ruin.u[i+1] <- (ruin.u[i]-q.theta*prob.u-q.theta*
                                        sum(vfmpx[(i+2):lx]))/(1-q.theta)
                                    #i+2 car fct de survie ">"
                    }
                }
                else
                    {
                    }
                ruin.theta[theta] <- ruin.u[u+1]
            }
        }
    vfmp <- sapply(1:ll,fmp.theta,alpha)
    return(sum(ruin.theta*vfmp))
}
```

A.6.2 Indépendance entre les probabilité d'occurence

```
############## Cas independant #################
# Convolution directe (2 v.a. independantes)

    directconvo<-function(ff1,ff2)
     {
    # convolution de deux fns de masses de probabilite
       l1<-length(ff1)
       l2<-length(ff2)
       ffs<-ff1[1]*ff2[1]
       smax<-l1+l2-2
       ff1<-c(ff1,rep(0,smax-l1+1))
       ff2<-c(ff2,rep(0,smax-l2+1))
       for (i in 1:smax)
          {
          j<-i+1
```

```
        ffs<-c(ffs,sum(ff1[1:j]*ff2[j:1]))
      }
      return(ffs)
   }


q <- 0.02
EX <- 40

v.ruine <- numeric(0)
v.ruine[1] <- (q*EX-q)/(1-q)

v.ruine[1] #=prob de ruine si u=0

u <- 2000
i <- 2
for(i in 1:u)
    {
        ff1 <- c(v.ruine[1:i],0)
        ff2 <- c(0,dgeom(0:(i-1),1/40))
        v.ruine[i+1] <- (v.ruine[i]-q*sum(ff1[i:1]*ff2[2:(i+1)])-q*
                         pgeom(i-1,1/40,lower=FALSE))/(1-q)
    }

v.ruine[200]
v.ruine[1000]
v.ruine[2000]     #Tend vers 0 lorsque u tend vers infini

plot(x=0:1999,v.ruine[1:2000],ylim=c(0,1),xlab="u",ylab="Prob. de ruine"
     ,main="Prob. de ruine pour le cas independant")
```

A.6.3 Dépendance entre les montants de sinistre

```
Fxi.theta <- function(ki,theta,alpha,i)
    {
        exp(-theta*psi.inverse(1-alpha,pgeom(ki-1,qi(i))))
    }
```

```
prob.ruine <- function(u,alpha,ll,qocc,lx)
#u=reserve ini(max, on calcule de 0 ), alpha= dep. param, ll=theta de 1 l,
    #qocc=Prob. occurence, lx=somme les geom de 1 x
    {
        ruin.theta <- rep(0,ll)
        ex.theta <- rep(0,ll)
        p.ruin <- matrix(0,ll,u+1)
        m.vfx <-matrix(0,ll,lx)
        for (theta in 1:ll)
            {
                m.vfx[theta,] <- sapply(1:lx,fxi.theta,theta,alpha,1)
                                                                #1 car geo i.d.
                ex.theta[theta] <- sum((1:lx)*m.vfx[theta,])
                p.ruin[theta,1] <- qocc*(ex.theta[theta]-1)/(1-qocc) #psi(0)
            }
        ruine.certaine <- which(ex.theta*qocc>1)
        if(u!=0)
            {
                p.ruin[,2] <- (p.ruin[,1]-qocc*(p.ruin[,(1:1)]*m.vfx[,(1:1)])
                               -qocc*rowSums(m.vfx[,(2):lx]))/(1-qocc)
                if(u>1)
                    {
                    for (i in 2:u)
                    {
                    p.ruin[,i+1] <- (p.ruin[,i]-qocc*rowSums(p.ruin[,(i:1)]*
                                                            m.vfx[,(1:i)])
                                     -qocc*rowSums(m.vfx[,(i+1):lx]))/(1-qocc)
                    }
                    }
                else
                    {
                    }
            }
        else
            {
            }
        vfmp <- sapply(1:ll,fmp.theta,alpha)
        m.vfmp <- matrix(rep(vfmp,u+1),ll,u+1)
        p.ruin[ruine.certaine,] <- rep(1,u+1) #Si qocc*EX.theta >1, ruine cert.
```

```
    result <- colSums(p.ruin*m.vfmp)
    return(c("Prob de ruine u=0"=result[1],"Prob de ruine u=" =c(result[-1])))
}
```

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