Modèles de dépendance hiérarchique pour l'évaluation des passifs et la tarification en actuariat

Thèse

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Résumé

Dans cette thèse on s'intéresse à la modélisation de la dépendance entre les risques en assurance non-vie, plus particulièrement dans le cadre des méthodes de provisionnement et en tarification. On expose le contexte actuel et les enjeux liés à la modélisation de la dépendance et l'importance d'une telle approche avec l'avènement des nouvelles normes et exigences des organismes réglementaires quant à la solvabilité des compagnies d'assurances générales.

Récemment, Shi et Frees (2011) suggère d'incorporer la dépendance entre deux lignes d'affaires à travers une copule bivariée qui capture la dépendance entre deux cellules équivalentes de deux triangles de développement. Nous proposons deux approches différentes pour généraliser ce modèle. La première est basée sur les copules archimédiennes hiérarchiques, et la deuxième sur les effets aléatoires et la famille de distributions bivariées Sarmanov.

Nous nous intéressons dans un premier temps, au Chapitre 2, à un modèle utilisant la classe des copules archimédiennes hiérarchiques, plus précisément la famille des copules partiellement imbriquées, afin d'inclure la dépendance à l'intérieur et entre deux lignes d'affaires à travers les effets calendaires. Par la suite, on considère un modèle alternatif, issu d'une autre classe de la famille des copules archimédiennes hiérarchiques, celle des copules totalement imbriquées, afin de modéliser la dépendance entre plus de deux lignes d'affaires. Une approche avec agrégation des risques basée sur un modèle formé d'une arborescence de copules bivariées y est également explorée. Une particularité importante de l'approche décrite au Chapitre 3 est que l'inférence au niveau de la dépendance se fait à travers les rangs des résidus, afin de pallier un éventuel risque de mauvaise spécification des lois marginales et de la copule régissant la dépendance.

Comme deuxième approche, on s'intéresse également à la modélisation de la dépendance à travers des effets aléatoires. Pour ce faire, on considère la famille de distributions bivariées Sarmanov qui permet une modélisation flexible à l'intérieur et entre les lignes d'affaires, à travers les effets d'années de calendrier, années d'accident et périodes de développement. Des expressions fermées de la distribution jointe, ainsi qu'une illustration empirique avec des triangles de développement sont présentées au Chapitre 4. Aussi, nous proposons un modèle avec effets aléatoires dynamiques, où l'on donne plus de poids aux années les plus récentes, et utilisons l'information de la ligne corrélée afin d'effectuer une meilleure prédiction du risque. Cette dernière approche sera étudiée au Chapitre 5, à travers une application numérique sur les nombres de réclamations, illustrant l'utilité d'un tel modèle dans le cadre de la tarification.

On conclut cette thèse par un rappel sur les contributions scientifiques de cette thèse, tout en proposant des angles d'ouvertures et des possibilités d'extension de ces travaux.

Mots clés : Dépendance hiérarchique, Triangles de développement, Méthodes de provisionnement, Copules, Maximum de vraisemblance, Estimation basée sur les rangs, Effets aléatoires, Sarmanov, Bootstrap, Tarification, Mesures de risque et allocation de capital.

Abstract

The objective of this thesis is to propose innovative hierarchical approaches to model dependence within and between risks in non-life insurance in general, and in a loss reserving context in particular.
One of the most critical problems in property/casualty insurance is to determine an appropriate reserve for incurred but unpaid losses. These provisions generally comprise most of the liabilities of a non-life insurance company. The global provisions are often determined under an assumption of independence between the lines of business. However, most risks are related to each other in practice, and this correlation needs to be taken into account.

Recently, Shi and Frees (2011) proposed to include dependence between lines of business in a pairwise manner, through a copula that captures dependence between two equivalent cells of two different runoff triangles. In this thesis, we propose to generalize this model with two different approaches. Firstly, by using hierarchical Archimedean copulas to accommodate correlation within and between lines of business, and secondly by capturing this dependence through random effects.

The first approach will be presented in chapters 2 and 3. In chapter 2, we use partially nested Archimedean copulas to capture dependence within and between two lines of business, through calendar year effects. In chapter 3, we use fully nested Archimedean copulas, to accommodate dependence between more than two lines of business. A copula-based risk aggregation model is also proposed to accommodate dependence. The inference for the dependence structure is performed with a rank-based methodology to bring more robustness to the estimation.

In chapter 4, we introduce the Sarmanov family of bivariate distributions to a loss reserving context, and show that its flexibility proves to be very useful for modeling dependence between loss triangles. This dependence is captured by random effects, through calendar years, accident years or development periods. Closed-form expressions are given, and a real life illustration is shown again. In chapter 5, we use the Sarmanov family of bivariate distributions in a dynamic framework, where the random effects are considered evolutionary and evolve

over time, to update the information and allow more weight to more recent claims. Hence, we propose an innovative way to jointly model the dependence between risks and over time with an illustration in a ratemaking context.

Finally, a brief conclusion recalls the main contributions of this thesis and provides insights into future research and possible extensions to the proposed works.

Keywords : Hierarchical dependence, Reserving, Copulas, Loss triangles, Bootstrap, Rank-based estimation, Maximum likelihood estimation, Risk capital allocation, Ratemaking, Claim counts, Random effects, Sarmanov.

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À mes parents

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Avant-propos

Cette thèse étudie des modèles de dépendance hiérarchiques pour l'évaluation des réserves et de mesures de risque, ainsi que la tarification en assurance non-vie.
Elle est constituée de six chapitres, le premier et le dernier étant une introduction et une conclusion générales. Les chapitres de 2 à 5 sont présentés sous forme de quatre articles scientifiques.

Plus spécifiquement, le Chapitre 2 est constitué d'un article co-écrit avec mon co-directeur de thèse de l'Université du Québec à Montréal, Jean-Philippe Boucher, professeur au Département de mathématiques et ma directrice de thèse à l'Université Laval, Hélène Cossette, professeure à l'École d'actuariat, s'intitulant *Modeling Dependence Between Loss Triangles with Hierarchical Archimedean Copulas* et publié dans la revue *ASTIN Bulletin*. Le modèle présenté dans cet article capture la dépendance à l'intérieur et entre les lignes d'affaires, à travers les effets calendaires à l'aide des copules archimédiennes partiellement imbriquées. Cet article a été récipiendaire du prestigieux prix *Hachemeister* de la *Casualty Actuarial Society (CAS)* pour l'année 2016. Le critère principal d'attribution de ce prix est déterminé par l'impact potentiel de la publication scientifique sur l'assurance IARD en Amérique du Nord et de sa mise en application pratique.

Le deuxième article, présenté au Chapitre 3, propose une méthode alternative à celle du chapitre précédent, avec une modélisation de la dépendance entre six lignes d'affaires à l'aide des copules archimédiennes totalement imbriquées, ainsi qu'un modèle d'agrégation par copules. Dans cet article, intitulé *Rank-Based Methods for Modeling Dependence Between Loss Triangles*, l'inférence est basée sur les rangs des résidus de façon à offrir une estimation plus robuste à la structure de dépendance. Cet article est en cours de publication à la revue *European Actuarial Journal* et co-écrit avec Marie-Pier Côté, étudiante au doctorat en statistique à l'Université McGill et son directeur de thèse Christian Genest, professeur au Département de mathématiques et de statistique de l'Université McGill.

Le quatrième chapitre est basé sur un article publié dans la revue *North American Actuarial Journal* et intitulé *Sarmanov Family of Bivariate Distributions for Multivariate Loss Reser-*

ving Analysis. Cet article est co-écrit avec mes deux directeurs de thèse Jean-Philippe Boucher et Hélène Cossette, ainsi que Julien Trufin, professeur au Département de mathématiques à l'Université Libre de Bruxelles (ULB). On s'intéresse dans ce chapitre à la modélisation de la dépendance à l'intérieur et entre les lignes d'affaires à l'aide des effets aléatoires et la famille de distributions bivariées Sarmanov. Nous démontrons la flexibilité et l'utilité de cette famille de distributions dans le contexte des réserves.

Cette même famille de distributions est utilisée dans un contexte de tarification en assurance automobile dans le chapitre 5. On expose une dépendance entre les nombres de réclamations entre différentes couvertures, basée sur des effets aléatoires dynamiques, où la dimension temporelle y est incorporée. Ce dernier travail repose sur un quatrième article, *Sarmanov Family of Multivariate Distributions for Bivariate Dynamic Claim Counts Model*, qui est également publié dans la revue *Insurance : Mathematics and Economics* et réalisé en collaboration avec mes directeurs de thèse Jean-Philippe Boucher et Hélène Cossette.

Chapitre 1

Introduction

Ce premier chapitre est une introduction générale de la thèse, où les méthodes de provisionnement en assurance générale et la modélisation de la dépendance entre les lignes d'affaires sont présentées. Les définitions et les outils qui seront utilisés pour la suite de la thèse y sont également fournis. Cette introduction se divise en trois sections : la première définit la notion de réserve en assurance générale, avec une description du contexte actuel et des méthodes de provisionnement existantes, alors que la deuxième partie traite des enjeux liés à la modélisation stochastique et de la dépendance entre les risques dans le cadre des réserves. La troisième section élabore le plan des travaux de la thèse et introduit les contributions proposées dans cette thèse.

1.1 Les provisions en assurance générale

1.1.1 Définitions et notations

En assurance, les opérations se font de manière bilatérale, de telle sorte que l'assuré se fait promettre en échange d'une prime, en cas de réalisation d'un risque, une prestation par l'assureur. Cela implique que le cycle de production dans le domaine de l'assurance est inversé, c'est-à-dire que l'assureur perçoit le prix de son produit d'assurance avant même de connaître son coût. De cette manière, ce coût de production n'est connu qu'après la prime perçue, ce qui contraint alors les assureurs à se baser sur une estimation du coût, qui doit être la plus proche possible de la réalité future, afin de rester solvable.
En effet, en raison de sa particularité et de sa position extrêmement importante et sensible dans l'économie, l'assurance est ainsi soumise à une réglementation supervisée par des organismes de réglementaires. Les assureurs doivent donc détenir suffisamment de capital pour faire face à leurs engagements envers les assurés. Un des rôles principaux de l'actuaire est par conséquent d'estimer les risques que porte une assurance et projeter les flux futurs qui y sont associés.
À ce titre, des provisions techniques doivent être calculées à chaque fin d'année, pour gérer et

couvrir les risques de l'entreprise d'assurance et lui permettre d'honorer ses engagements. Les provisions techniques, comme le définit Partrat *et al.* (2007), sont «les provisions destinées à permettre le règlement intégral des engagements pris envers les assurés et bénéficiaires de contrats. Elles sont liées à la technique même de l'assurance, et imposées par la réglementation».

Formellement, à la date t, la compagnie d'assurance est tenue de constituer une provision, aussi appelée réserve, pour les sinistres survenus avant la date t et qu'elle sera amenée à indemniser. Elle doit donc estimer le coût des sinistres survenus, et retrancher les montants déjà versés. Ceci revient à un problème de prévision. Les coûts de sinistres ne sont pas connus le jour de la survenance du sinistre, il y a tout d'abord un délai avant que le sinistre ne soit déclaré à la compagnie d'assurance par l'assuré, puis un temps (plus ou moins long) de gestion du sinistre, d'expertises, de paiements, avant de clôturer le dossier plusieurs mois, ou plusieurs années plus tard. Il est également possible parfois, dans des cas exceptionnels, que des dossiers soient réouverts après leur clôture.

De plus, le paiement des sinistres ne s'effectue pas toujours en une fois, dans l'année même de survenance. Cela est particulièrement vrai dans certaines branches de l'assurance, comme par exemple la responsabilité civile automobile dans le cas de sinistres avec dommages corporels, surtout en cas de dommage corporel grave qui nécessite de nombreuses années avant que l'état de santé se stabilise.

En effet, le règlement des sinistres s'étale au fil du temps et il devient nécessaire de constituer des réserves pour pouvoir honorer les dettes futures. Comme le montant qui sera finalement payé pour le sinistre est inconnu au départ, la somme à mettre en réserve est également inconnue et il faut l'estimer. Cet exercice est d'autant plus important sachant que les provisions représentent la plus grande partie de l'ensemble du bilan. Ainsi, l'analyse des provisions (aussi appelées provisions techniques ou des réserves) a un impact majeur sur la profitabilité d'une compagnie d'assurance.

La Figure 1.1 illustre le bilan simplifié d'une compagnie d'assurance où une grande partie de son passif est composée des provisions, souvent estimées par des méthodes statistiques.

ACTIF	PASSIF
Placements	Fonds Propres
	Provisions Techniques

Figure 1.1 – Bilan simplifié d'une compagnie d'assurance

Ces estimations peuvent être calculées par des techniques, dites techniques IBNR (*Incurred But Not Reported*), qui se basent sur l'évolution passée du coût des sinistres pour estimer son développement futur. En pratique, ces techniques sont utilisées soit pour estimer les réserves purement IBNR, soit les réserves de sinistres totales. Ceci est illustré à la Figure 1.2 à travers un diagramme de Lexis qui schématise l'évolution de la vie des sinistres. Les sinistres surviennent à la date •, sont déclarés à l'assureur à la date + et clôturés à la date × . L'exercice de provisionnement consiste à estimer à une date donnée, aussi appelée la date de coupure (ici en 2010, correspondant au trait plein vertical), le montant des paiements restant à faire pour l'ensemble des sinistres survenus (déclarés ou pas).

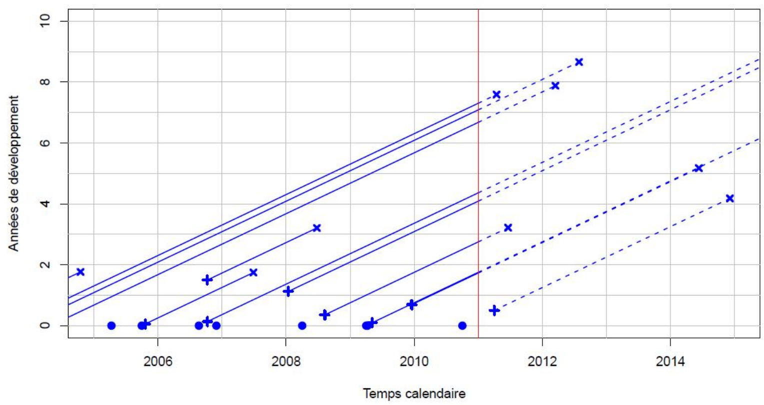

Figure 1.2 – Diagramme de Lexis de l'évolution de la vie des sinistres

Il est primordial que l'évaluation de ces provisions techniques soit faite de manière rigoureuse et scientifique, afin d'assurer la stabilité de la compagnie d'assurance. Une mauvaise évaluation peut avoir de lourdes conséquences pour l'entreprise, allant d'une simple perte à l'insolvabilité. Il est donc important de garder à l'esprit les différents enjeux que le provisionnement représente pour une société d'assurance, et ce, à plusieurs égards.

Évidemment, il y a tout d'abord un enjeu de solvabilité et de réputation pour une entreprise, dans le sens où une évaluation insuffisante des provisions techniques entraînerait la société d'assurance à ne pas pouvoir faire face à ses engagements. De plus, en supposant la présence d'événements défavorables dans le développement de la sinistralité, cette situation pourrait mener à plus ou moins long terme à la faillite de la société d'assurance.
Il y a également un enjeu fiscal pour la société lorsque dans le cas contraire, la réserve est surévaluée. En effet, un provisionnement excessif entraînerait une baisse d'impôts à payer pour la société. Ce qui explique les contrôles fiscaux sur l'estimation des réserves afin de s'assurer qu'aucun excès des provisions n'est volontairement constitué.
Enfin, une mauvaise estimation des réserves pourrait avoir comme conséquence une interprétation erronée des résultats de la compagnie. Par exemple, une surévaluation des provisions se traduira par un développement favorable des réserves, et donc par un résultat positif (bonus).

Inversement, une sous-estimation des provisions générera un développement défavorable, qui se traduira par une diminution du résultat de la compagnie (malus).

Ainsi, on voit qu'il s'agit finalement d'un problème d'optimisation sous plusieurs contraintes. Cela implique qu'en réalité, les réserves calculées seront basées sur des estimations et ne seront en pratique jamais exactement égales au montant réel du coût futur. Il est généralement préférable d'être plus prudent (ou conservateur) dans l'estimation de la réserve, c'est-à-dire une estimation à la hausse plutôt qu'à la baisse pour mieux protéger l'entreprise. Ce qui est d'autant plus vrai avec les nouvelles exigences des organismes réglementaires.

1.1.2 Méthodes de provisionnement

Nous distinguerons deux approches pour le calcul de la réserve. La première étant déterministe, ou aussi appelée méthode classique de provisionnement, alors que la deuxième approche est stochastique, où les montants futurs à payer sont considérés comme des variables aléatoires.

Techniques intuitives

Dans certaines situations où les paiements d'assurance futurs sont très stables et prévisibles, il est possible d'utiliser des méthodes simples, dites intuitives, pour l'estimation des réserves. Par exemple, en assurance automobile au Québec, comme les dommages corporels sont couverts par un régime public, il est donc plutôt simple d'estimer les dommages matériels où les fermetures de dossiers se font beaucoup plus rapidement. Nous citerons deux techniques principales:

1. Méthode des réserves enregistrées: l'idée de la méthode est d'utiliser le total de toutes les réserves individuelles de sinistres et d'ajouter un pourcentage arbitraire afin d'inclure l'évolution possible de tous les coûts des réclamations.
2. Méthode des ratios de pertes espérés: l'idée de la méthode est d'utiliser le rapport sinistres/primes prédit (attendu) de la branche d'affaire et les montants de réclamations payés à ce jour afin d'estimer la réserve. Ainsi, pour une année i fixée, nous avons :

$$\begin{aligned} \text{Pertes ultimes estimées} &= \text{Ratio de perte espéré} \times \text{Primes acquises} \\ \text{Réserve estimée} &= \text{Pertes ultimes estimées} - \text{montants payés} \end{aligned}$$

   En sommant toutes les années de couverture, nous obtenons un estimé de la réserve

$$\text{Réserve estimée totale} = \sum_i \text{Réserve estimée totale}$$

Évidemment, deux problèmes évidents de cette méthode, entre autres, seraient l'estimation du pourcentage et l'inclusion du jugement.

Triangles de développement

Il y a deux écoles de pensées pour le calcul des provisions, la première s'intéresse au provisionnement agrégé, et l'autre au provisionnement par dossier. Cette dernière est aussi appelée évaluation dossier-dossier, ou *Micro-level Reserving*, qui consiste en une anticipation du coût ultime du dossier, voir par exemple Antonio et Plat (2014), et Pigeon *et al.* (2013). La principale préoccupation de cette méthodologie est faite à un niveau granulaire, soit individuel. Cette approche est devenue d'autant plus populaire avec le nouveau domaine technologique qui a vu le jour, le *Big Data*, pour faire face à l'explosion du volume d'informations.

Dans cette thèse, on prendra la direction de la première approche, celle du provisionnement agrégé (*Macro-level Reserving*), communément appelée provisionnement avec triangles de développement.

En assurance non-vie, afin de refléter la dynamique des sinistres, une construction typique est l'utilisation de triangles de développement. Un triangle de développement consiste en un tableau à double entrée, dont seule la partie supérieure gauche est connue. Les lignes du triangle correspondent aux années d'origine des sinistres qui y seront reportés, et les colonnes aux années de développement (qui représentent la durée de règlement d'un sinistre). Il existe différents types de triangles de développement, à savoir les triangles d'encourus ou de payés qui peuvent être soit décumulés (incrémentaux) ou cumulés.

Prenons pour exemple un triangle de payés, le report des sinistres dans chaque cellule (i, j) se fait de la façon suivante :

$$C_{i,j} = \sum_{s_i \in \mathbf{S}_i} Y_j^{s_i},$$

où $\mathbf{S}_i$ est l'ensemble des sinistres déclarés à l'année i, ou d'origine i. $C_{i,j}$ correspond donc aux paiements cumulatifs des sinistres d'année d'origine i, et qui ont été effectués lors de leur année de développement j avec un paiement Y_j. Ainsi, $C_{i,j}$ est le montant cumulé jusqu'à l'année de développement j, des sinistres survenus à l'année d'accident i, pour $1 \leqslant i, j \leqslant n$.

$C_{i,j}$ peut représenter soit le montant payé, soit le coût total estimé du sinistre, appelé aussi l'encouru (paiement déjà effectué plus réserve). Les montants $C_{i,j}$ sont connus pour $i + j \leqslant n + 1$ et on cherche à estimer les valeurs des $C_{i,j}$ pour $i + j > n + 1$, et en particulier les valeurs ultimes $C_{i,n}$ pour $2 \leqslant i \leqslant n$.

Ces notations sont illustrées dans le triangle suivant :

$$C = \begin{pmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n-1} & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n-1} & \\ \vdots & \vdots & \ddots & & \\ C_{n-1,1} & C_{n-1,2} & & & \\ C_{n,n} & & & & \end{pmatrix}$$

Approche déterministe

Le modèle le plus célèbre et répandu dans le monde des méthodes de provisionnement en assurance non-vie, est certainement la méthode déterministe de Chain-Ladder. La très grande simplicité du modèle de Chain-Ladder lui vaut un engouement particulier à tel point que les modèles stochastiques tentent de correspondre au mieux à ses résultats. Certains remarquent que la charge ultime obtenue par la méthode de Chain-Ladder peut être obtenue en maximisant la vraisemblance d'un modèle de Poisson (Hachemeister et Stanard (1975), Kremer (1985) et Mack (1991)). D'autres ont associé la méthode de Chain-Ladder à un modèle linéaire généralisé (Renshaw (1989) et Renshaw et Verrall (1998)) ou encore, à la manière de Verrall (1989), ont associé un filtre de Kalman au triangle de données. Les modèles sont différents mais aboutissent souvent à des résultats proches de Chain-Ladder. Le modèle de Mack (1993a) donne exactement les mêmes résultats.
La méthode de Chain-Ladder estime les montants inconnus, $C_{i,j}$ pour $i+j>n+1$, par

$$\widehat{C}_{i,j} = C_{i,n+1-i}.\widehat{f}_{n+1-i}\cdots\widehat{f}_{j-1}\ , \quad i+j>n+1\ , \tag{1.1}$$

où

$$\widehat{f}_j = \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}}\ , \qquad 1 \leqslant j \leqslant n-1\ . \tag{1.2}$$

Il est donc supposé que $C_{i,j}$ est proportionnel à $C_{i,j-1}$ et que le coefficient de proportionnalité f_{j-1}, calculé sur base des données de sinistres du passé, ne dépend pas de l'année d'accident i.
Le montant ultime des sinistres survenus à l'année d'accident i est alors estimé par

$$\widehat{C}_{i,n} = C_{i,n+1-i}.\widehat{f}_{n+1-i}\cdots\widehat{f}_{n-1}\ , \quad 2 \leqslant i \leqslant n\ .$$

La réserve de sinistre pour l'année d'accident i (c'est-à-dire ce qui reste à payer pour les sinistres survenus en l'année i), qui est définie par $R_i = C_{i,n} - C_{i,n+1-i}$, est alors estimée par

$$\widehat{R}_i = C_{i,n+1-i}.\widehat{f}_{n+1-i}\cdots\widehat{f}_{n-1} - C_{i,n+1-i}\ , \quad 2 \leqslant i \leqslant n\ .$$

Bien que la méthode de Chain-Ladder soit populaire dans le calcul des provisions techniques en assurance de dommages et qu'elle fournisse de l'information utile sur la réserve, cette méthode reste toutefois assez limitée. En effet, la première limitation est due au fait que l'hypothèse de proportionnalité directe entre $C_{i,j}$ et $C_{i,j+1}$ est très contraignante. Cela fait que le calcul de provisions devient fortement lié au dernier montant de charge connu, ce qui n'est pas forcément le cas pour des triangles présentant des irrégularités. D'autres modèles ont été proposés pour améliorer la méthode Chain-Ladder, entre autres, la méthode London-Chain où $C_{i,j+1} = f_j C_{i,j} + a_j$. Aussi, nous pouvons citer par ailleurs les méthodes Bornhuetter-Ferguson et Cape Cod, où de l'information exogène au triangle est utilisée, notamment un avis d'expert.

En somme, ces méthodes déterministes se trouvent limitées devant la possibilité de prendre en compte une inflation non-constante, de repérer un changement de jurisprudence, de détecter des irrégularités potentielles du triangle, etc. Dans un tel contexte, nous ne pouvons détecter les irrégularités potentielles du triangle et surtout obtenir une estimation de la loi de probabilité de la provision totale, qui nous donnerait plus d'information sur la variabilité et la distribution de la réserve.
Il est connu qu'en pratique, un actuaire est plus intéressé à obtenir la distribution et toutes les valeurs possibles de la réserve, plutôt que d'avoir une seule estimation ponctuelle, qui correspondrait finalement à un seul point de la distribution. Au vu de ces limites, nous tâcherons donc d'y remédier en ayant recours aux méthodes stochastiques, ce qui nous amène à la sous-section suivante.

1.1.3 Du déterministe au stochastique

L'approche dite stochastique permet de modéliser plus justement la volatilité des réserves, de quantifier et comprendre le comportement aléatoire des réserves. Le mot stochastique est donc synonyme d'aléatoire et s'oppose par définition au déterminisme.
Dans le contexte des réserves, le premier pas vers la modélisation stochastique est fait à travers le modèle de Mack, présenté dans Mack (1993a) , Mack (1993b) et Mack (1994) et qui est une transition entre la méthode Chain-Ladder et la sphère stochastique. En effet, il s'agit de la méthode Chain-Ladder analysée dans un cadre stochastique, permettant l'estimation de la variabilité des réserves calculées. Nous aurons ainsi des expressions pour l'erreur standard sur la réserve correspondant à chaque année de survenance de sinistres examinée ainsi que sur la réserve totale. Ces mesures de variabilité nous permettront de construire des intervalles de confiance pour les réserves.
Ce modèle repose sur les trois hypothèses suivantes :

$$E[C_{i,j+1} \mid C_{i,1},...,C_{i,j}] = E[C_{i,j+1}|C_{i,j}] = f_j C_{i,j}\ , \quad 1 \leq i \leq n \quad \text{et} \quad 1 \leq j \leq n-1.$$

$$\{C_{i,1},...,C_{i,n}\}, \{C_{k,1},...,C_{k,n}\} \quad \forall i \neq k, \quad \text{sont indépendants.}$$

$$Var(C_{i,j+1} \mid C_{i,1},...,C_{i,j}) = \sigma_j^2 C_{i,j}\ , \quad 1 \leq i \leq n \quad \text{et} \quad 1 \leq j \leq n-1.$$

où,

$$\hat{\sigma}_j^2 = \frac{1}{n-j-1} \sum_{i=1}^{n-j} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j \right)^2 ;$$

$$\hat{\sigma}_{n-1}^2 = \min\left(\frac{\hat{\sigma}_{n-2}^4}{\hat{\sigma}_{n-3}^2}, \min\left(\hat{\sigma}_{n-3}^2, \hat{\sigma}_{n-2}^2\right)\right).$$

La première hypothèse signifie qu'étant donné le développement $C_{i,1},...,C_{i,j}$ des sinistres survenus à l'année i, il existe un coefficient f_j tel que l'espérance de $C_{i,j+1}$ soit égale à $C_{i,j} f_j$. Ceci est une conséquence directe de la méthode de Chain-Ladder. La deuxième hypothèse quant à elle, signifie que les années de survenance sont indépendantes entre elles. Enfin, la dernière hypothèse sert à l'obtention de la variance de $C_{i,j}$ et sous-tend implicitement la méthode de Chain-Ladder.

En somme, l'avantage majeur de cette méthode, est que non seulement elle est simple et fournit exactement les mêmes réserves que la méthode Chain-Ladder, mais elle permet aussi d'avoir un niveau d'incertitude associé à cette réserve. Un intervalle de confiance autour de la réserve peut également être obtenu, en supposant une loi sous-jacente, notamment la loi normale ou log-normale.

Ceci nous permet d'avoir une idée plus précise sur la variabilité et la volatilité de la réserve. De plus, cela permet de mieux répondre aux exigences et réglementations internes de la compagnie. Il est à noter qu'un autre avantage qu'offre le modèle de Mack, surtout d'un point de vue pratique, est la possibilité d'effectuer un test de validation des hypothèses, proposé par Denuit *et al.* (2000).

Toutefois, le modèle de Mack présente encore des lacunes, qui sont essentiellement les mêmes défauts que la méthode Chain-Ladder. En effet, la première hypothèse n'est plus vérifiée en cas de changement important dans la gestion des sinistres ou dans le taux d'inflation par exemple. Aussi, supposer une loi (log)normale pour construire un intervalle de confiance est restrictif. C'est un modèle qui a l'avantage d'être pratique et simple mais qui peut être parfois simpliste.

1.2 Réserves stochastiques

Dans cette section, nous allons commencer par introduire brièvement les modèles linéaires, par la suite on s'intéressera surtout à leur généralisation, soit les modèles linéaires généralisés, qui constituent la base des méthodes sur le provisionnement stochastique. Nous rappelons que le terme stochastique se dit de phénomènes qui, partiellement, relèvent du hasard et qui font l'objet d'une analyse statistique.

1.2.1 Modèles linéaires

On cherche à exprimer la variable aléatoire Y (variable réponse) comme une fonction linéaire des variables explicatives $(X_1, X_2, ...)$

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... \ .$$

Les paramètres de cette fonction linéaire $(\beta_0, \beta_1, \beta_2, ...)$ sont estimés par moindres carrés ordinaires, moindres carrés généralisés, etc. Il est alors possible d'estimer les valeurs futures des observations et par conséquent de compléter la partie inférieure du triangle.

Les avantages d'une telle régression, c'est qu'elle permet de calculer la distribution estimée des provisions à travers des méthodes de simulations (cf. Section 1.2.2) et de calculer les volatilités ultimes. Il y a également la possibilité d'introduire comme variable explicative des variables exogènes au triangle.

Toutefois, une telle méthodologie nécessite la vérification de nombreuses hypothèses, surtout la normalité des réponses (voir Neter *et al.* (1996)). Ainsi, il est populaire de travailler avec une généralisation des modèles linéaires, où nous avons un plus large éventail de choix quant à la distribution de la variable réponse.

1.2.2 Modèles linéaires généralisés

Comme mentionné plus tôt, les modèles linéaires généralisés (GLM) sont la base de la modélisation stochastique des réserves. La théorie des GLM est donc utilisée dans le contexte des méthodes de provisionnement. Les démonstrations et les écrits complets sur les modèles linéaires généralisés se trouvent notamment dans De Jong et Heller (2008) et McCullagh et Nelder (1989).

GLM et réserves

Les modèles stochastiques supposent que les pertes incrémentales $Y_{i,j}$, avec $Y_{i,j} = C_{i,j} - C_{i,j-1}$, sont une réalisation d'une loi de probabilité provenant de la famille exponentielle linéaire. Ainsi, l'objectif est de modéliser la moyenne $\mu_{i,j}$ de ces lois de probabilités et la variance, en utilisant la théorie des modèles linéaires généralisés.

La façon la plus élémentaire de paramétriser la moyenne $\mu_{i,j}$ de la variable aléatoire $Y_{i,j}$ est de supposer que l'effet de l'année d'accident est indépendant de l'effet de la période de développement. On suppose ainsi que chaque année d'accident aura besoin d'un paramètre distinct pour représenter une réserve finale différente.

Ensuite, dans la même logique, on suppose qu'à chaque période de développement, un pourcentage spécifique de la réserve finale sera réclamé, ainsi on aura besoin de paramètres différents selon la période de développement.

En ce qui concerne les triangles de développement, nous utilisons la modélisation suivante

$$g(\mu_{i,j}) = \gamma + \alpha_i + \beta_j,$$

où $i \in \{2, 3, ..., n\}$ représente l'année d'accident et $j \in \{2, 3, ..., n\}$, l'année de développement, alors que $g()$ est une fonction de lien qui relie la moyenne de la variable réponse (réserve) aux variables explicatives (années d'accident et périodes de développement).

On observe ainsi que la moyenne de la variable aléatoire $Y_{i,j}$ est affectée par un paramètre correspondant à la ligne du triangle de développement, et à un paramètre correspondant à la colonne de ce même triangle.

On remarque aussi γ représentant les pertes incrémentales moyennes pour la cellule de référence, qui est définie comme étant la première année d'accident et la première année de développement. Pour des fins d'identification des paramètres, on suppose $\alpha_1 = \beta_1 = 0$.

Année	1	2	$\cdots$	n-1	n
1	$g^{-1}(\gamma)$	$g^{-1}(\gamma+\beta_2)$	$\cdots$	$g^{-1}(\gamma+\beta_{n-1})$	$g^{-1}(\gamma+\beta_n)$
2	$g^{-1}(\gamma+\alpha_2)$	$g^{-1}(\gamma+\alpha_2+\beta_2)$	$\cdots$	$g^{-1}(\gamma+\alpha_2+\beta_{n-1})$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
n-1	$g^{-1}(\gamma+\alpha_{n-1})$	$g^{-1}(\gamma+\alpha_{n-1}+\beta_2)$	$\cdots$	$\cdots$	$\cdots$
n	$g^{-1}(\gamma+\alpha_n)$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 1.1 – Expression du triangle de développement à travers la fonction de lien.

Valeur incrémentale $Y_{i,j}$	X_0	c_2	c_3	c_4	$\cdots$	l_2	l_3	l_4	$\cdots$
$Y_{1,1}$	1	0	0	0	$\cdots$	0	0	0	$\cdots$
$Y_{1,2}$	1	1	0	0	$\cdots$	0	0	0	$\cdots$
$Y_{1,3}$	1	0	1	0	$\cdots$	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$Y_{2,1}$	1	0	0	0	$\cdots$	1	0	0	$\cdots$
$Y_{2,2}$	1	1	0	0	$\cdots$	1	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$Y_{4,3}$	1	0	1	0	$\cdots$	0	0	1	$\cdots$
$Y_{4,4}$	1	0	0	1	$\cdots$	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1.2 – Illustration de la forme de la matrice des régresseurs

Soit g^{-1}, la fonction réciproque de la fonction de lien, ainsi afin de pouvoir estimer nos paramètres, il est possible d'exprimer le triangle de développement à l'aide de g^{-1}, comme dans la Table 1.1.

À première vue, cette représentation peut sembler complexe, toutefois, il est possible d'exprimer le triangle des réserves d'une manière plus générale, avec une forme matricielle, à l'aide de régresseurs :

$$g(\mu_{i,j}) = \gamma X_0 + \beta' C + \alpha' L,$$

où

— α est le vecteur des α_i pour $i \in \{2, 3, ..., n\}$;

— β est le vecteur des β_j pour $j \in \{2, 3, ..., n\}$;

— C est le vecteur des c_j pour $j \in \{2, 3, ..., n\}$;

— L est le vecteur des l_i pour $i \in \{2, 3, ..., n\}$,

avec $l_k = 1$ pour $Y_{i,j}$ si $i = k$ (et 0 sinon), et $c_k = 1$ pour $Y_{i,j}$ si $j = k$ (et 0 sinon).

En utilisant ces régresseurs, il est donc possible d'exprimer les $Y_{i,j}$ dans une grande matrice tel qu'illustré à la Table 1.2.

Variance d'une réserve

Un des grands atouts des modèles stochastiques est que l'on peut obtenir le degré d'incertitude de la réserve. Nous appelons variance de la réserve, une mesure de la fluctuation de la réserve qui peut être causée par l'ensemble des types d'erreur d'assurance. On définit l'erreur comme étant la variabilité ou la fluctuation associée à un élément du modèle. Boucher (2011) indique que l'erreur d'assurance totale se décompose en quatre parties principales:

1. L'erreur du processus [*], appelée aussi parfois l'erreur statistique, est le bruit causé par la distribution choisie. Par exemple, pour un individu donné, la perte suit une loi quelconque qui aura des réalisations possiblement différentes à chaque observation. Ce type d'erreur est généralement inévitable et présent dans presque tous les modèles.
2. L'erreur d'estimation indique l'incertitude des paramètres optimaux estimés par rapport à ce que pourraient être les vrais paramètres. Par exemple, lorsqu'on utilise les estimateurs issus du principe de vraisemblance maximale, on suppose la normalité asymptotique.
3. L'erreur de sélection spécifie l'incertitude face à des variables cachées qui ne sont pas présentes dans le modèle, mais qui, en réalité ont un impact sur le modèle. Un bon exemple est lorsqu'il y a dans la réalité un effet d'année de calendrier, mais qu'on ne l'inclut pas dans le modèle.
4. L'erreur de spécification essaie d'identifier la pertinence de la loi choisie par rapport à d'autres lois. Par exemple, l'incertitude face au paramètre p du modèle de Tweedie (voir Wüthrich (2003)) sert à déterminer lequel parmi le modèle de Poisson, Gamma, ou une combinaison des deux est le plus approprié.

Finalement, il est à noter que ces quatre types d'erreurs ne constituent qu'une partie de l'incertitude d'assurance. Lorsqu'on établit une réserve, il existe d'autres risques à considérer qui ne seront pas pris en compte dans aucun des modèles présentés dans cette thèse. Par exemple, les risques financiers (rendement des actifs dans la réserve, taux d'intérêt, inflation sociale), les risques de souscription (risque de défaut, risque moral), les risques juridiques (changement dans les lois qui gouvernent l'assurance), environnementaux (catastrophes naturelles), les risques du marché (offre et demande, marketing, stratégies des compétiteurs), les risques opérationnels, etc.

De nouveaux types d'analyses émergent pour tenir compte de ces types de risques, notamment l'analyse financière dynamique [†], et la gestion du risque de l'entreprise [‡], mais ces concepts vont au-delà des objectifs de notre sujet de thèse et ne seront pas considérés.

∗. 1. Statistical Error (Process Error) 2. Estimation Error 3. Selection Error 4. Specification Error
†. Dynamic Financial Analysis
‡. Enterprise Risk Management

Erreur Quadratique Moyenne de Prédiction

Une statistique courante reliée à la variance de la réserve R_i est l'erreur quadratique moyenne de prédiction (EQMP), aussi appelée *MSEP* [§].
Cette statistique, définie dans England et Verrall (2002), est une combinaison de l'erreur du processus et de l'erreur d'estimation. Ces deux éléments, auxquels nous donnerons une définition mathématique plus bas, ont été présentés plus tôt. En bref, l'erreur d'estimation est reliée aux observations du passé, et l'erreur du processus est causée par le bruit des observations futures. Nous référons à Wüthrich et Merz (2008) pour plus de détails.

Soit $Y_{i,j}$ et sa prédiction $\widehat{Y}_{i,j}$, deux variables aléatoires représentant respectivement les pertes incrémentales (haut du triangle) et celles futures, estimées à l'aide du modèle (bas du triangle). On définit l'EQMP de la façon suivante :

$$\begin{aligned} EQMP[\widehat{Y_{i,j}}] &= E[(Y_{i,j} - \widehat{Y_{i,j}})^2] \\ &= E[((Y_{i,j} - E[Y_{i,j}]) - (\widehat{Y_{i,j}} - E[Y_{i,j}]))^2]. \end{aligned}$$

En supposant que $E[(Y_{i,j} - E[Y_{i,j}])(\widehat{Y_{i,j}} - E[\widehat{Y_{i,j}}])] = 0$, c'est-à-dire que les observations futures sont indépendantes des observations passées, on obtient:

$$\begin{aligned} EQMP[\widehat{Y_{i,j}}] &\approx E[(Y_{i,j} - E[Y_{i,j}])^2] + E[(\widehat{Y_{i,j}} - E[\widehat{Y_{i,j}}])^2] \\ &= \underbrace{Var[Y_{i,j}]}_{l'erreur\ de\ processus} + \underbrace{Var[\widehat{Y_{i,j}}]}_{l'erreur\ d'estimation}. \end{aligned}$$

Distribution prédictive

La modélisation stochastique nous permet donc de passer du meilleur estimé (*best estimate*) obtenu par les méthodes déterministes à la variabilité autour de cette réserve. Toutefois, il est maintenant connu qu'en pratique, un actuaire préfère avoir toute la distribution de la réserve, soit toutes les valeurs possibles, plutôt que d'avoir des points estimés. Il est souvent impossible d'avoir l'expression analytique explicite de la densité de la réserve totale, mais plusieurs approximations ont été proposées afin d'approcher numériquement cette distribution. L'une des méthodes les plus populaires, et qui sera utilisée dans la suite de la thèse, est la technique du Bootstrap.
Certaines applications de cette méthode ont été initialement proposées dans le contexte des réserves dans Lowe (1994), Hartl (2010) et Taylor (2000).
Dans le contexte des triangles de développement, la procédure du Bootstrap peut être résumée comme suit:

§. Mean Square Error of Prediction

1. À partir des paramètres estimés sur les données (GLM), on génère des réalisations du triangle supérieur.
2. À partir de ce nouveau triangle, on estime de nouveaux paramètres.
3. À partir de ces nouveaux paramètres, on simule des réalisations du triangle du bas (réserve simulée).
4. Nous répétons les étapes ci-dessus N fois.

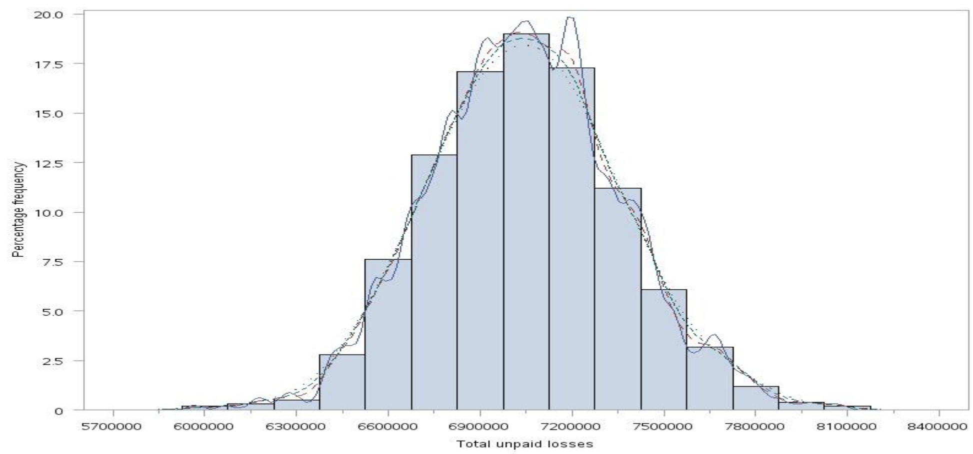

Figure 1.3 – Illustration d'une distribution prédictive d'une réserve avec une estimation par noyaux

Ainsi, nous obtenons une distribution telle que celle illustrée à la Figure 1.3. Cette approximation numérique nous donne par conséquent une information précieuse sur le comportement de la réserve, en fournissant une distribution prédictive. Ceci est particulièrement utile dans l'analyse financière dynamique (DFA) et aussi afin de satisfaire les prochaines exigences du *Office of the Superintendent of Financial Institutions* (*OSFI*). Par ailleurs, une des exigences principales qui serait à satisfaire, est la modélisation de la dépendance entre les triangles de développement. Ceci sera traité davantage dans la prochaine section.

1.2.3 Dépendance entre les lignes d'affaires

Mise en contexte et motivations

Par abus de langage, on parle souvent de corrélation au lieu de dépendance. La corrélation (souvent appelée corrélation linéaire) permet de rendre compte d'une dépendance existante entre deux variables linéairement liées. La dépendance englobe ce type de liaison, mais aussi les liaisons non linéaires.
Modéliser la dépendance entre les lignes d'affaires permet entre autres de mieux comprendre le comportement global du portefeuille, éclairer les décisions stratégiques et aider les

gestionnaires des risques dans la détermination du capital économique pour un portefeuille d'assurance.

Les risques peuvent être dépendants sous différents angles: risques dépendants au sein d'une même branche, ou encore types de risques dépendants. Ainsi, avec l'arrivée de nouvelles normes, il devient nécessaire pour une compagnie d'assurance de pouvoir estimer au mieux le montant des provisions totales de l'ensemble de ses engagements, ce qui implique de prendre en compte la dépendance entre les lignes d'affaires.

Dans la théorie classique, les risques auxquels sont soumis les sociétés d'assurance sont considérés indépendants. Toutefois, intuitivement, plusieurs exemples peuvent laisser penser que ce n'est pas le cas en pratique. Il semble alors nécessaire de s'intéresser à cette dépendance pouvant exister entre les variables aléatoires qui impactent le résultat d'une société d'assurance, et d'évaluer ses effets sur la santé financière de cette société.

En effet, un des principaux risques de dépendance auquel peut faire face une compagnie d'assurance est celui de la dépendance entre les sinistres des différentes branches. Il serait par exemple intuitif de considérer que le montant des sinistres de la branche Dommage automobile n'est pas indépendant de celui de la branche Responsabilité Civile en automobile. Toutefois, en pratique, cette évaluation se fait le plus souvent de façon indépendante, et l'un des objectifs majeurs de cette thèse est de montrer que la prise en compte de la dépendance entre les différentes branches a un impact sur l'incertitude du montant des provisions à constituer, sur les mesures de risques et sur l'allocation du capital économique.

À cet effet, nous distinguerons deux écoles de pensées différentes. La première est basée sur une approche non-paramétrique, qui se veut principalement une généralisation de la méthode de Mack, présentée plus haut, dans un contexte multivarié. La deuxième approche traite des méthodes paramétriques. Nous présentons succinctement quelques modèles existants pour chacune des deux approches.

1. Approche non-paramétrique

   Comme mentionné plus tôt, les méthodes non-paramétriques sont inspirées du modèle de Mack (1993a). La méthode de Braun (2004) par exemple, prend en compte les corrélations entre les segments en introduisant un coefficient de corrélation entre les facteurs de développement. Pröhl et Schmidt (2005) et Schmidt (2006) adoptent une approche multivariée en réalisant une étude simultanée sur l'ensemble des segments du portefeuille, et Merz et Wüthrich (2008) généralisent cette méthode de manière à pouvoir calculer l'erreur de prédiction. Des méthodes basées sur des régressions linéaires permettent également de répondre à cette problématique comme le modèle additif multivarié de Hess *et al.* (2006) et Schmidt (2006). Wüthrich et Merz (2009) ont proposé une extension permettant le calcul de l'erreur de prédiction.

2. Approche paramétrique

   L'autre volet de recherche examine les méthodes paramétriques basées sur les familles de distribution, permettant un plus large éventail de distributions et surtout, tel que soulevé à la section 1.2.2, l'obtention d'une distribution prédictive des paiements futurs, qui est considérée comme plus informative et pertinente aux actuaires que l'erreur quadratique moyenne de prédiction. De plus, et contrairement aux modèles non-paramétriques, les méthodes paramétriques permettent une allocation du capital plus éclairée entre les différentes branches d'activités.

En général, les modèles paramétriques incorporant la dépendance entre les triangles de développement, utilisent principalement les copules pour introduire la corrélation entre les lignes d'affaires.

Une copule gaussienne est souvent utilisée, notamment dans Brehm (2002) pour créer des distributions totales de sinistres impayés, tandis que Shi et Frees (2011) suggèrent d'introduire une dépendance entre les cellules équivalentes entre deux triangles de développement. Aussi, une approche flexible est suggérée par De Jong (2012) pour modéliser la dépendance entre les secteurs d'activité à travers une matrice de corrélation gaussienne.

Shi *et al.* (2012), proposent une distribution log-normale multivariée, avec une matrice de corrélation gaussienne multivariée pour tenir compte de la corrélation sous différentes formes de dépendances, tandis que Wüthrich *et al.* (2013) suggèrent une structure de corrélation flexible pour introduire la dépendance à l'intérieur et entre les sous-portefeuilles.

Le Bootstrap paramétrique est également un moyen pour introduire la dépendance entre les lignes d'affaires. Par exemple, Kirschner *et al.* (2008) utilisent un Bootstrap synchronisé au sein du portefeuille et une extension de cette méthode a également été proposée par Taylor et McGuire (2007) afin de l'utiliser avec des modèles linéaires généralisés.

L'approche paramétrique est celle que nous allons étudier dans la suite de la thèse, plus particulièrement celle utilisant les copules hiérarchiques, ainsi qu'une modélisation à l'aide des effets aléatoires communs capturés par la famille de distributions multivariées Sarmanov.

Copules et provisions

Avec leur forme de dépendance très flexible entre différentes variables aléatoires, les copules sont devenues en quelques années un outil important dans la modélisation de la structure de dépendance de deux ou plusieurs variables aléatoires.

Il existe un grand nombre de familles de copules qui conduisent à des structures de dépendance positive ou négative très variées. Certaines permettent, par exemple, de la dépendance dans les queues de distributions ce qui est particulièrement intéressant dans les domaines de l'assurance et de la finance. Pour la modélisation des réserves en assurance non-vie, le choix de cette structure est central. Par exemple, si le calcul est effectué à l'aide d'un critère de type *Value at Risk* ou *Tail Value at Risk*, il sera bien plus considérable en cas de dépendance des valeurs élevées, qu'en cas de dépendance concentrée sur les valeurs intermédiaires puisqu'il s'agit d'un quantile.

On voit ainsi que dans une telle situation, on ne peut pas se contenter d'une loi normale multivariée qui, de plus, n'a jamais été appropriée dans la modélisation des sinistres en assurance. Il est donc nécessaire d'utiliser un outil qui permet de choisir quelle est la structure de dépendance la plus adéquate possible pour les triangles de développement. C'est pourquoi, dans la suite de ce que nous allons présenter, la dépendance entre les triangles sera modélisée principalement à l'aide des copules.

Le point fort de cette approche est que nous disposons déjà d'outils de diagnostic pour valider nos méthodes et que nous disposons d'un large éventail de distributions pour mieux comprendre le lien entre les différentes lignes d'affaires.

Depuis les années 90, les applications de la théorie des copules n'ont pas cessé de se multiplier dans différents domaines, notamment en finance, par exemple Embrechts *et al.* (1999), et en actuariat, en l'occurrence Frees et Valdez (1998). Pour une revue sur les copules voir Nelsen (2006), Joe (1997) et Genest et Favre (2007).

Une copule est un outil permettant de séparer la dépendance du comportement marginal. Formellement, les copules ont été introduites par Sklar (1959) par le théorème suivant:

Théorème 1.2.1. *Soit G une fonction de répartition multidimensionnelle continue de dimension d, associée aux variables aléatoires $X_1, ..., X_d$ de fonctions de répartition continues respectives $F_1, ..., F_d$. Alors il existe une unique fonction continue $[0,1]^d \to [0,1]$ satisfaisant*

$$G(x_1, ..., x_d) = C\left\{F_1(x_1), ..., F_d(x_d)\right\}.$$

La fonction C est appelée copule. Inversement, étant donnée une copule C et des fonctions de répartition unidimensionnelles arbitraires $F_1, ..., F_d$, la fonction G construite à partir du théorème (1.2.1) est une fonction de répartition de dimension d, dont les marges sont $F_1, ..., F_d$.

Le théorème de Sklar nous permet donc de décomposer n'importe quelle fonction de répartition en deux composantes: la copule et les marges. Cette flexibilité nous permet d'étudier la structure de dépendance indépendamment des marges.

Dans cette thèse nous nous intéresserons à une classe importante des copules, soit la classe des copules archimédiennes, et plus particulièrement à la famille des copules archimédiennes hiérarchiques. Notre point de départ sera le modèle de Shi et Frees (2011).

Dans cet article, les auteurs ont considéré une dépendance bivariée entre les cellules équivalentes entre les triangles. C'est-à-dire que deux paiements qui ont eu lieu dans la même année d'accident et même période de développement seront couplés.

En considérant un portefeuille avec deux lignes d'affaires, la distribution conjointe des montants incrémentaux standardisés $(Y_{i,j}^{(1)}, Y_{i,j}^{(2)})$, peut être représentée, selon le théorème 1.2.1, par une copule unique:

$$F_{i,j}(y_{i,j}^{(1)}, y_{i,j}^{(2)}) = \Pr(Y_{i,j}^{(1)} \leq y_{i,j}^{(1)}, Y_{i,j}^{(2)} \leq y_{i,j}^{(2)}) = C(F_{i,j}^{(1)}, F_{i,j}^{(2)}; \phi) \ , \tag{1.3}$$

où $C(.;\phi)$ est la fonction de la copule, avec le vecteur de paramètres ϕ, et les fonctions de répartition des distributions marginales $F_{i,j}^{(1)}$ et $F_{i,j}^{(2)}$. Celles-ci sont définies comme suit :

$$F_{i,j}^{(\ell)} = \Pr(Y_{i,j}^{(\ell)} \leq y_{i,j}^{(\ell)}) = F^{(\ell)}(y_{i,j}^{(\ell)}; \eta_{i,j}^{(\ell)}, \gamma^{(\ell)}), \ \ell = 1, 2,$$

où

— $Y_{i,j}^{(\ell)} = \dfrac{X_{i,j}^{(\ell)}}{\omega_i^{(\ell)}}$, où $X_{i,j}^{(\ell)}$ et $\omega_i^{(\ell)}$ représentent respectivement les paiements incrémentaux et la variable d'exposition au risque du triangle ℓ.

— $\eta_{i,j}^{(\ell)}$, est la composante systématique, qui représente le paramètre de localisation, qui est souvent une fonction linéaire des variables explicatives (variables exogènes), de la forme $\eta_{i,j}^{(\ell)} = x_{i,j}^{(\ell)'} \beta_{i,j}^{(\ell)}$.

— $x_{i,j}^{(\ell)}$ est la matrice des variables indicatrices, représentant les variables explicatives, et $\beta_{i,j}^{(\ell)}$, sont les coefficients que nous devons estimer.

— $\gamma^{(\ell)}$ est le vecteur qui regroupe tous les paramètres additionnels de la distribution de $Y_{i,j}^{(\ell)}$, qui déterminent la forme et l'échelle.

Étant dans un contexte paramétrique, l'équation (1.3) peut aisément être estimée par la méthode du maximum de vraisemblance. Pour ce faire, soit $c(.)$ la fonction de densité de la copule correspondante, $C(.)$. La fonction de log-vraisemblance du modèle par paires peut alors s'écrire ainsi :

$$L = \sum_{i=0}^{I} \sum_{j=0}^{I-i} \ln \ c(F_{i,j}^{(1)}, F_{i,j}^{(2)}; \phi) + \sum_{i=0}^{I} \sum_{j=0}^{I-i} \ln(f_{i,j}^{(1)}) + \ln(f_{i,j}^{(2)}) \ , \tag{1.4}$$

où $f_{i,j}^{(\ell)}$ représente la fonction de densité de la distribution marginale $F_{i,j}^{(\ell)}$, c'est-à-dire $f_{i,j}^{(\ell)} = f^{(\ell)}(y_{i,j}^{(\ell)}; \eta_{i,j}^{(\ell)}, \gamma^{(\ell)})$, pour $\ell = 1, 2$.

Par conséquent, le modèle est estimé en utilisant les paiements observés $y_{i,j}^{(\ell)}$, pour $(i,j) \in \{(i,j) : i + j \leqslant I\}$, afin de prévoir une réserve adéquate qui couvrira les paiements futurs $y_{i,j}^{(\ell)}$, pour $(i,j) \in \{(i,j) : i + j > I\}$.

1.3 Plan des travaux de la thèse

Pour le plan élaboré de cette thèse, nous allons considérer l'article de Shi et Frees (2011) comme étant le point départ de notre analyse et nous nous intéresserons à la généralisation de leur modèle décrit à la section 1.2.3. Pour ce faire, nous allons étudier deux approches différentes, la première utilisant la famille des copules archimédiennes hiérarchiques (section

1.3.2), et l'autre utilisant des effets aléatoires et la famille de distributions bivariées Sarmanov (section 1.3.3).

1.3.1 Généralisation de Shi et Frees (2011)

La modélisation par paires incorpore la dépendance entre des cellules équivalentes de triangles de développement. C'est une méthode intuitive et simple qui a été utilisée avec succès dans la littérature.
Toutefois, cette méthode a aussi été critiquée par le fait qu'elle ne tient pas compte des effets exogènes qui ne peuvent être détectés par cette modélisation. En effet, plusieurs facteurs communs, tels que l'inflation, les taux d'intérêt, les changements de jurisprudence, ou les décisions stratégiques comme l'accélération des paiements pour tout le portefeuille, peuvent impacter simultanément toutes les lignes d'affaires, mais ne peuvent être directement capturés par une dépendance par paires.
Afin de mieux prendre en considération ces effets exogènes, une dépendance par année de calendrier, année d'accident ou période de développement pourrait être considérée. Une telle modélisation capturerait également la dépendance à l'intérieur et entre les triangles de développement.
D'autre part, un modèle utilisant une seule copule par paires entre différents triangles de développement ne serait pas approprié dans le cas où nous avons plus de deux lignes d'affaires et plusieurs secteurs d'activité différents. En effet, il ne serait pas adéquat de considérer que tous les risques sont capturés à travers le même degré de dépendance, peu importe le secteur d'activité ou la région géographique. Pour pallier cela, nous nous intéresserons à une modélisation avec dépendance hiérarchique.

Pour ce faire, nous allons considérer deux approches différentes, chacune d'elle découlera à son tour de deux articles scientifiques, selon le plan suivant:

1. Approches par copules archimédiennes
    a) Copules archimédiennes partiellement imbriquées (Chapitre 2)
    b) Copules archimédiennes totalement imbriquées (Chapitre 3)

    La famille des copules archimédiennes est largement utilisée dans différents domaines d'application. Cette popularité est notamment due au fait que l'un des plus grands avantages de l'utilisation de cette famille de copules est la possibilité de travailler avec des fonctions de densités explicites, définies en terme du générateur unidimensionnel de la copule archimédienne. Les copules elliptiques n'ont pas cette propriété importante et n'ont pas de forme explicite. Aussi, la famille des copules archimédiennes est très flexible et variée en terme de structure de dépendance, alors que les copules elliptiques supposent une dépendance symétrique, aussi forte dans les deux queues de la distribution. Voir Genest et MacKay (1986), Joe (1997) et Nelsen (2006).

La généralisation aux copules archimédiennes multidimensionnelles de dimension $d > 2$ nécessite d'abord la définition des notions de d-monotonicité et de monotonicité complète.

Définition 1 1. *Une fonction g est dite d-monotone si elle est continue, différentiable jusqu'à l'ordre $d-2$ vérifiant* $(-1)^k \frac{\partial^k g(t)}{\partial t^k} \geq 0, \ \forall k = 0, 1, ..., d-2$ *et telle que* $(-1)^{d-2} \frac{\partial^{d-2} g(t)}{\partial t^{d-2}}$ *est décroissante et convexe.*

Définition 2 1. *Une fonction g est dite complètement monotone si elle est continue, admettant des dérivées de tout ordre vérifiant* $(-1)^k \frac{\partial^k g(t)}{\partial t^k} \geq 0, \ \forall k \geq 0.$

Soit ϕ_θ un générateur de copule archimédienne de paramètre de dépendance θ et $C_{\theta,d}$ la fonction de $[0,1]^d \to [0,1]$ donnée par

$$C_{\theta,d}(u_1, u_2, \cdots, u_d) = \phi_\theta^{-1} \{\phi_\theta(u_1) + ... + \phi_\theta(u_n)\} \ , (u_1, \cdots, u_d) \in [0,1]^d. \tag{1.5}$$

McNeil et Nešlehová (2009) ont montré que la fonction $C_{\theta,d}$ est une copule si et seulement si ϕ^{-1} est d-monotone. Elle est alors appelée copule archimédienne de dimension d. La copule C_d ainsi construite est une fonction de répartition conjointe de d variables aléatoires uniformément distribuées dans $[0,1]$ et échangeables.

Dans le contexte des réserves, cette propriété d'échangeabilité peut être parfois vue comme une restriction. En effet, il est difficile de concevoir une dépendance qui soit aussi forte partout entre toutes les lignes d'affaires, et aussi à travers tous les effets exogènes qui peuvent induire une dépendance autant à l'intérieur qu'entre les lignes d'affaires. Ainsi, les copules archimédiennes hiérarchiques sont une généralisation des copules archimédiennes multidimensionnelles qui permettent de pallier ce problème et de contourner cette restriction.

2. Approches par effets aléatoires
   a) Distribution Sarmanov et modélisation multivariée des réserves (Chapitre 4)
   b) Distribution Sarmanov avec effets aléatoires dynamiques (Chapitre 5)

La famille de distributions bivariées Sarmanov a été introduite par Sarmanov (1966), et proposée en physique par Cohen (1984) sous une forme plus générale. Lee (1996) suggère une forme multivariée avec des applications en médecine. Récemment, et grâce à sa flexibilité, cette famille de distributions devient de plus en plus populaire dans les domaines appliquées. Par exemple, Schweidel *et al.* (2008) utilise cette forme de dépendance pour modéliser l'expérience client, en l'occurrence la rétention. Miravete (2009) compare les tarifs de deux compétiteurs téléphoniques en ayant recours à la famille de distributions Sarmanov. Danaher et Smith (2011) discutent des applications en marketing. Dans le domaine de l'assurance, Hernández-Bastida *et al.* (2009) et

Hernández-Bastida et Fernández-Sánchez (2012) évaluent les primes à l'aide de la famille de distributions bivariées Sarmanov.
À partir de deux fonctions de distributions $u^{(\ell)}$ et $\psi^{(\ell)}$ avec $\ell \in \{1,2\}$ tel que $\int_{-\infty}^{\infty} \psi^{(\ell)}(t)\, u^{(\ell)}(t)\, dt = 0$, la distribution jointe peut être écrite comme suit:

$$u^S\left(\theta^{(1)}, \theta^{(2)}\right) = u^{(1)}\left(\theta^{(1)}\right) u^{(2)}\left(\theta^{(2)}\right)\left(1 + \omega\psi^{(1)}\left(\theta^{(1)}\right)\psi^{(2)}\left(\theta^{(2)}\right)\right),$$

où ω est le paramètre de dépendance qui satisfait la condition suivante:

$$1 + \omega\psi^{(1)}\left(\theta^{(1)}\right)\psi^{(2)}\left(\theta^{(2)}\right) \geq 0 \ \forall\theta^{(\ell)}\ ,\ \ell \in \{1,2\}.$$

La fonction ψ est appelée fonction de mélange et peut-être obtenue de la façon suivante $\psi^{(\ell)}\left(\theta^{(\ell)}\right) = \exp\left(-\theta^{(\ell)}\right) - L_{\theta^{(\ell)}}(1)$, où $L_{\theta^{(\ell)}}$ est la transformée de Laplace de $u^{(\ell)}$, évaluée à 1. L'une des propriétés les plus importantes de cette famille de distributions est la flexibilité d'utilisation d'un large spectre de distributions pour les marginales, ainsi que la possibilité de d'obtention d'expressions explicites pour les distributions jointes. Les nombreuses caractéristiques de cette famille de distributions sont notamment présentées dans Lee (1996).

1.3.2 Approches par copules archimédiennes

Dans les chapitres 2 et 3 de cette thèse, on suggère deux approches différentes, à l'aide de la famille des copules hiérarchiques (imbriquées), afin de proposer une généralisation au modèle de Shi et Frees (2011). Il s'agit de deux sous-classes de la famille des copules imbriquées: les copules partiellement et totalement imbriquées.

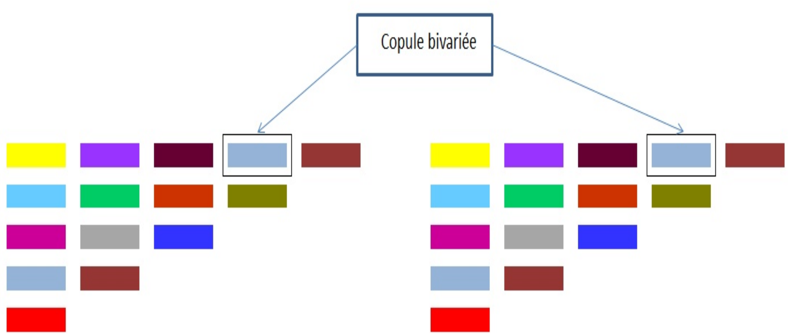

Figure 1.4 – Structure de dépendance du modèle PWD

Copules archimédiennes partiellement imbriquées

Dans le chapitre 2, on généralise le modèle de Shi et Frees (2011) en utilisant le même jeu de données pour lequel, au lieu de considérer une dépendance par paires tel que illustré sur la

Figure 1.4, nous allons capturer la dépendance (intra et inter) à travers les effets calendaires. En effet, tel que mentionné plus tôt, plusieurs facteurs exogènes peuvent avoir un impact simultané sur les triangles de développement à une année de calendrier donnée. En utilisant des copules archimédiennes hiérarchiques, aussi appelées copules imbriquées ou emboîtées, cela nous permet d'ajouter un autre niveau de dépendance et par conséquent plus de flexibilité et de complexité à notre modélisation. Nous allons donc considérer une dépendance à l'intérieur des lignes d'affaires, reliée à l'effet calendaire, et une autre dépendance entre les lignes d'affaires, qui capture également la dépendance par année de calendrier.
Pour ce faire, nous considérons un modèle hiérarchique où des copules archimédiennes ordinaires $C_{1,1}$ et $C_{1,2}$ seront utilisées au premier niveau (année de calendrier) et une copule hiérarchique $C_{2,1}$ sera considérée au deuxième niveau de hiérarchie pour capturer la dépendance entre les deux lignes d'affaires.
Cette famille de copules a été introduite dans la littérature par Joe (1997) et a été présentée plus en détails dans Savu et Trede (2010). Plus récemment, Okhrin *et al.* (2013) fournissent une méthode pour estimer les distributions multivariées définies par des copules archimédiennes hiérarchiques.
Plusieurs conditions doivent être remplies pour l'existence des copules hiérarchiques partiellement imbriquées.
Le nombre de copules doit décroître à chaque niveau h, i.e. $n_h < n_{h-1}$, tout comme le degré de dépendance, i.e. $\theta_{h+1,k'} < \theta_{h,k}$ pour tout $h = 0, ..., H$ et $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ tel que $C_{h,k} \in C_{h+1,k'}$ où $\theta_{h,k}$ est le paramètre de dépendance du générateur $\phi_{h,k}$. Dans le contexte des triangles de développement, cela signifie que les éléments appartenant à la même ligne d'affaire sont plus fortement corrélés entre eux qu'avec ceux d'une autre ligne d'affaire, ce qui est intuitivement logique. Mathématiquement, les conditions qui doivent être vérifiées par une copule archimédienne hiérarchique se résument comme suit:

1. Toutes les fonctions inverses du générateur $\phi_{h,k}^{-1}$ sont complètement monotones.
2. Toutes les fonctions composées $\phi_{h+1,k'} \circ \phi_{h,k}^{-1}$ sont convexes pour tout $h = 0, ..., H$ et $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ tel que $C_{h,k} \in C_{h+1,k'}$.

La Figure 1.5 illustre le modèle partiellement imbriqué dans le cas à 4-dimensions. Ce modèle sera présenté plus en détails au chapitre 2.

Copules archimédiennes totalement imbriquées

Dans le chapitre 3, nous utiliserons une approche alternative et différente de celle utilisée dans le chapitre 2 pour généraliser le modèle de Shi et Frees (2011) dans un contexte multivarié (plus que deux lignes d'affaires) et en utilisant un jeu de données d'une compagnie d'assurance canadienne. Pour ce faire, nous considérons la classe des copules archimédiennes totalement imbriquées.
Lorsque nous travaillons avec plus de deux lignes d'affaires, cela requiert une connaissance plus profonde et globale de tout le portefeuille pour pouvoir construire une hiérarchie appropriée à

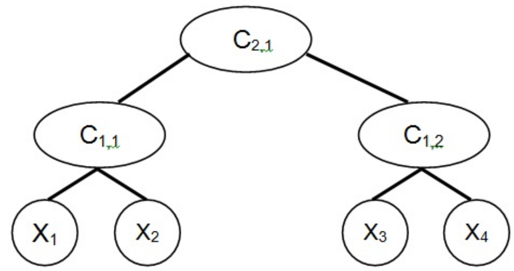

Figure 1.5 – Dépendance avec copules archimédiennes partiellement imbriquées

l'aide des copules archimédiennes partiellement imbriquées, étant donné que tous les risques doivent être regroupés dès le premier niveau de hiérarchie. Nous verrons que cette forte condition n'est pas requise pour le cas des copules archimédiennes totalement imbriquées.

Cette classe de copules est une simple généralisation de la famille des copules archimédiennes multivariées donnée par Joe (1997) et également discutée dans Embrechts *et al.* (2003), et Whelan (2004). Les notations peuvent parfois paraître complexes, mais il est assez simple de construire une telle hiérararchie.

Soit φ, le générateur d'une copule archimédienne. On définit une copule à 3 dimensions comme suit

$$C_3(u_1, u_2, u_3) = \varphi_2^{-1}(\varphi_2(u_3) + \varphi_2 \circ \varphi_1^{-1}(\varphi_1(u_2) + \varphi_1(u_1))) \tag{1.6}$$

et on définit récursivement une copule à d dimensions, nécessitant $d-1$ générateurs distincts:

$$C_d(u_1, \ldots, u_d) = \varphi_{d-1}^{-1}(\varphi_{d-1}(u_d) + \varphi_{d-1}(C_{d-1}(u_1, \ldots, u_{d-1}))).$$

Plus concrètement, on couple u_1 et u_2, puis on couple la paire (u_1, u_2) avec u_3, puis le triplet (u_1, u_2, u_3) avec u_4, et ainsi de suite. La Figure 1.6 illustre la structure en forme d'arbre pour quatre risques liés par des copules archimédiennes totalement imbriquées.

Comme les copules partiellement imbriquées, il y a quelques conditions pour l'existence des copules totalement imbriquées. Notamment, le degré de dépendance devrait diminuer en augmentant dans le niveau de hiérarchie, i.e. $\theta_{h+1} < \theta_h$, où θ_h est le paramètre de dépendance au niveau h de la hiérarchie, avec $h = 0, \ldots, d-1$. Dans le chapitre 3, nous verrons que les lignes d'affaires géographiquement proches, et du même secteur d'activité, devraient avoir un plus grand degré de dépendance qu'avec d'autres lignes d'affaires géographiquement plus éloignées.

Mathématiquement, les conditions qui doivent être vérifiées pour l'existence de copules archimédiennes totalement imbriquées sont résumées comme suit:

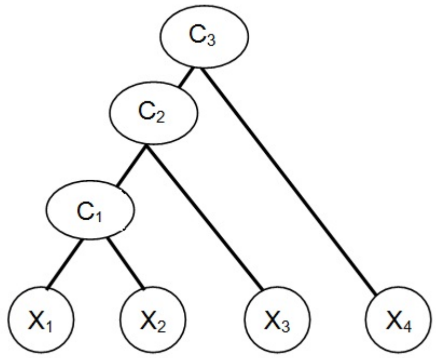

Figure 1.6 – Dépendance avec copules archimédiennes totalement imbriquées.

1. Toutes les fonctions inverses du générateur φ_h^{-1} sont complètement monotones.
2. Toutes les fonctions composées $\varphi_{h+1} \circ \varphi_h^{-1}$ sont convexes pour tout $h = 0, ..., d-1$.

Cette structure totalement imbriquée est aussi comparée à une autre approche alternative au chapitre 3. Il s'agit de l'approche par copule avec risques agrégés. On somme les risques d'un niveau à un autre, au lieu de les imbriquer à l'aide de copules hiérarchiques. Ce modèle est plus flexible car on ne se limite plus aux conditions d'existence d'une copule archimédienne, ce qui élargit le spectre pour le choix des copules et des possibilités de dépendance. Cette méthode qui remonte à Arbenz *et al.* (2012) et reprise par Côté et Genest (2015) est introduite à la littérature des réserves dans le papier scientifique présenté au chapitre 3. Dans cet article, nous introduisons également au contexte des réserves une nouvelle technique d'estimation des paramètres de dépendance. En effet, l'estimation de la dépendance entre les triangles de développement se fait indépendamment de celle des marginales, à l'aide des rangs des résidus. Tous les travaux sur la modélisation de la dépendance entre les triangles de développement qui ont été cités jusqu'à présent ont réussi à incorporer la dépendance entre les lignes d'affaires. Toutefois, la plus grande critique qu'on pourrait apporter à ces modèles est que l'inférence sur les marginales se fait conjointement et simultanément avec celle sur la dépendance, ce qui fait que les paramètres des marginales, et par conséquent les réserves, deviennent sensibles à la structure de dépendance utilisée. Ceci contredit la propriété de linéarité de l'espérance, dans le sens où la somme des espérances doit être égale à l'espérance de la somme. Cela serait contre-intuitif d'un point de vue pratique aussi, c'est-à-dire qu'un assureur ne voudrait pas adapter la réserve d'une ligne d'affaire en fonction d'une autre ligne d'affaire. Il voudrait que cet impact soit plutôt répercuté sur la distribution agrégée et sur les mesures de risque. Enfin, une autre limitation d'une inférence simultanée est qu'une mauvaise spécification des marginales pourrait influencer l'estimation de la structure de dépendance.

Au chapitre 3 de cette thèse, on propose ainsi une stratégie alternative où on estime premièrement les marginales, puis seuls les résidus des marginales sont liés à travers une structure de dépendance où l'estimation se fera à l'aide des rangs de ces résidus. Cette approche avec pseudo-vraisemblance est justifiée par le fait que l'estimation de la dépendance ne devrait pas être affectée par le comportement individuel de chaque secteur d'activité. Cette méthodologie qui remonte à Oakes (1994) et qui est apparue plus en détails dans Genest *et al.* (1995) puis Shih et Louis (1995), apporte plus de robustesse à l'estimation lorsque l'ajustement des marginales n'est pas tout à fait parfait. Cela est d'autant plus vrai lorsqu'on travaille avec des petits échantillons, ce qui est le cas dans le contexte des triangles de développement.

1.3.3 Approches par effets aléatoires

Dans les chapitres 4 et 5 de la thèse, on s'intéresse aux effets aléatoires permettant d'introduire la dépendance entre les lignes d'affaires.

Dans un contexte bayésien, on voudrait introduire des effets aléatoires qui relieraient les éléments à l'intérieur et entre les triangles de développement. La flexibilité d'un tel modèle permet de capturer des dépendances à travers les effets calendaires, années d'accidents ou encore périodes de développement.

Les méthodes bayésiennes ne sont pas nouvelles dans le domaine des réserves, voir Shi *et al.* (2012) pour une revue complète. Dans cette thèse, on utilisera les effets aléatoires pour capturer la dépendance au sein d'une même ligne d'affaire, alors que la dépendance entre les lignes d'affaires sera spécifiée par une famille de distributions que nous introduisons au contexte des réserves, appelée la famille de distributions bivariées Sarmanov.

Distribution Sarmanov et modélisation multivariée des réserves

Dans le chapitre 4 de cette thèse, nous utilisons le même jeu de données que Shi et Frees (2011) et le chapitre 2, afin de capturer la dépendance entre ces deux lignes d'affaires à l'aide d'effets aléatoires et de la famille de distributions bivariées Sarmanov.

En effet, contrairement au chapitre 2, la dépendance à l'intérieur d'une même ligne d'affaire est capturée à l'aide des effets aléatoires au lieu d'une copule archimédienne multivariée, alors que la dépendance entre les deux lignes d'affaires est capturée par la famille de distributions bivariées Sarmanov à travers ces effets aléatoires, au lieu d'une copule hiérarchique.

Pour les marginales, nous choisissons des conjuguées naturelles, conduisant ainsi à une distribution *a posteriori* qui soit une mise à jour de la distribution *a priori*. Pour ces deux triangles de développement, une loi normale sera choisie comme loi *a priori* pour le premier triangle, alors qu'une loi gamma sera considérée pour le deuxième.

En ce qui concerne le paramètre de localisation $\mu_{i,j}^{(\ell)}$, on considère la forme $\mu_{i,j}^{(1)} = \eta_{i,j}^{(1)}$ pour une loi log-normale et la forme $\mu_{i,j}^{(2)} = \dfrac{\exp(\eta_{i,j}^{(2)})}{\phi}$ pour une loi gamma, où $\eta_{i,j}^{(\ell)}$ désigne la composante systématique du modèle linéaire généralisé s'écrivant sous la forme suivante:

$$\eta_{i.j}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)},\ \ell = 1, 2,$$

où $\zeta^{(\ell)}$ est l'ordonnée à l'origine, alors que pour l'identification des paramètres, on suppose que $\alpha_1^{(\ell)} = \beta_1^{(\ell)} = 0$.

Pour la première ligne d'affaire, où une log-normale a été choisie comme distribution marginale, nous avons

$$Y_{i,j}^{(1)} \mid \Theta_t^{(1)} \ \sim\ \text{Lognormal}\left(\mu_{i,j}^{(1)}\Theta^{(1)}, \sigma^2\right)\ ,$$

$$E[Y_{i,j}^{(1)} \mid \Theta_t^{(1)}] = \exp^{\mu_{i.j}^{(1)}\Theta_t^{(1)}+\sigma^2/2}\ ,$$

$$\text{Var}[Y_{i,j}^{(1)} \mid \Theta_t^{(1)}] = \left(\exp^{\sigma^2} - 1\right)\left(\exp^{2\mu_{i.j}^{(1)}\Theta_t^{(1)}+\sigma^2}\right).$$

À l'aide du théorème de Bayes, on obtient que la distribution *a posteriori* de $[\Theta_t^{(1)} \mid \mathbf{Y_t}^{(1)}]$ est une normale de paramètres, où t désigne l'année de calendrier, d'accident ou période de développement:

$$\begin{cases} a_{post} = \frac{\sum_{k=1}^{t}\log(y_k^{(1)})\mu_k^{(1)}b^2+a\sigma^2}{\sum_{k=1}^{t}\mu_k^{(1)2}b^2+\sigma^2}; \\ b_{post}^2 = \frac{b^2\sigma^2}{\sum_{k=1}^{t}\mu_k^{(1)2}b^2+\sigma^2}. \end{cases}$$

De la même façon, nous obtenons ce qui suit lorsque la distribution gamma est considérée pour la deuxième ligne d'affaire

$$Y_{i,j}^{(2)} \mid \Theta_t^{(2)} \ \sim\ \text{Gamma}\left(\phi, \frac{\mu_{i,j}^{(2)}}{\Theta^{(2)}}\right)\ ,$$

$$E[Y_{i,j}^{(2)} \mid \Theta_t^{(2)}] = \phi\mu_{i,j}^{(2)}\frac{1}{\Theta_t^{(2)}}\ ,$$

$$\text{Var}[Y_{i,j}^{(2)} \mid \Theta_t^{(2)}] = \phi\mu_{i,j}^{(2)2}\frac{1}{\Theta_t^{(2)2}}.$$

À l'aide du théorème de Bayes, on obtient que la distribution *a posteriori* de $[\Theta_t^{(2)} \mid \mathbf{Y_t}^{(2)}]$ est une gamma de paramètres

$$\begin{cases} \alpha_{post} = t\phi + \alpha; \\ \tau_{post} = \left(\sum_{k=1}^{t}\frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau}\right)^{-1}. \end{cases}$$

Une fois que ces effets aléatoires sont incorporés au sein de chaque triangle, ils sont par la suite couplés à l'aide de la famille de distributions Sarmanov. L'utilisation de cette famille de distribution nous permet d'avoir des formes fermées de la distribution jointe, ainsi que la distribution jointe *a posteriori*, ce qui s'avérera très utile dans le calcul des moments de la réserve. Nous utilisons la loi *a posteriori* jointe lorsque la dépendance est considérée par année d'accident (ou par période de développement), car contrairement au cas par année de calendrier, cet effet aléatoire affecte les observations passées, mais aussi futures, appartenant à la même année d'accident (ou période de développement). Ceci est schématisé à la Figure 1.7.

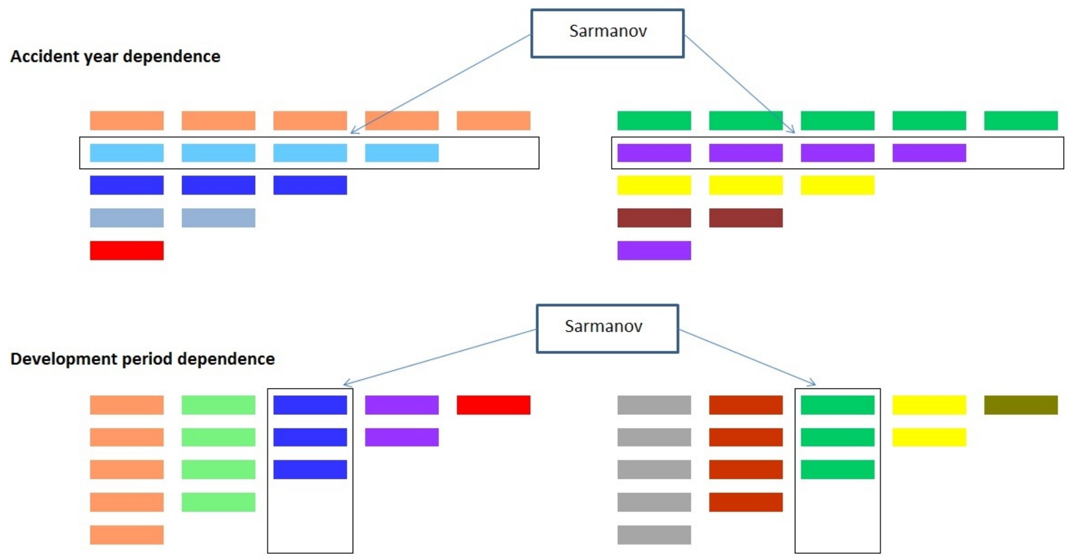

Figure 1.7 – Modeling dependence with a Sarmanov bivariate distribution

Il est intéressant de pouvoir utiliser cette flexibilité, car selon chaque situation, la dépendance entre les triangles de développement peut être causée autant par des effets d'année de calendrier, comme expliqué dans le chapitre 2, que par des effets d'années d'accidents ou de périodes de développement.
En effet, une dépendance par année d'accident pourrait provenir entre autres, d'une des situations suivantes:

— changement dans les pratiques de réserves, soit la façon dont les réserves sont allouées à l'ouverture des dossiers pour l'année d'accident en cours;
— changement législatif affectant les pertes non-survenues (pertes futures) ;
— événements majeurs ou catastrophes;
— recours collectifs reliés à un événement particulier, etc.

Les événements suivants pourraient quant à eux mener à une dépendance par période de développement:

— révision des dossiers ouverts ou inactifs depuis un certain temps;
— révision systématique des dossiers présentant un certain niveau de réserves;
— rythme des paiements qui change (initiative interne ou effet externe), etc.

Nous démontrons au chapitre 4 que la flexibilité de ce modèle nous permet de détecter la dépendance par année de calendrier, année d'accident et période de développement. Nous démontrons également que les propriétés intéressantes de cette famille de distributions s'avèrent très utiles dans le contexte des réserves. Une illustration empirique sera également présentée, où l'on voit les avantages d'une telle modélisation du point de vue du capital économique.

Distribution Sarmanov avec effets aléatoires dynamiques

Il est intuitivement intéressant et séduisant de penser que ces effets aléatoires pourraient être évolutifs dans le temps. C'est-à-dire au fur et à mesure du temps, l'information sur la loi *a priori*, et sur les effets aléatoires est mise à jour. Nous nous sommes donc intéressés à un tel modèle dans le contexte des réserves afin de voir si à travers les années de calendrier (année d'accident ou période de développement) nous avons une meilleure connaissance de la façon dont les paiements sont réglés. On voudrait aussi attribuer plus de poids aux réclamations les plus récentes car elles sont supposées être plus prédictives.

En effet, pour prédire les réclamations et paiements futurs, il a été observé et il est bien connu que les années plus récentes se sont avérées plus prédictives et informatives que les plus anciennes. Toutefois, les modèles classiques pour données longitudinales ne prennent pas cela en considération et n'allouent aucun poids à la chronologie des réclamations passées.

Des modèles avancés peuvent être utilisés pour examiner cette propriété, mais la complexité mathématique et numérique lors de l'estimation des paramètres rend la tâche difficile. De plus, lorsque nous voulons ajouter la dépendance entre les différentes couvertures et branches d'activité, la tâche est encore plus ardue. Dans l'article scientifique présenté dans le chapitre 5, nous utilisons l'information passée de la ligne corrélée, afin de mieux prédire les réclamations futures. Cette particularité du modèle est obtenue à l'aide de la famille de distributions bivariées Sarmanov.

Aussi, et contrairement au chapitre 4 où l'effet aléatoire ne dépend pas du temps, maintenant on considère que cet effet est dépendant du temps où, à chaque temps t, l'effet aléatoire est mis à jour. Ce modèle est une extension de celui décrit dans Bolancé *et al.* (2007). Soit $\Im_{\ell,T}$ la filtration des variables aléatoires $N_{\ell,1}, N_{\ell,2}, ..., N_{\ell,T}$, avec $\Im_T = (\Im_{1,T}, \Im_{2,T})$. Formellement, on écrit:

$$(\Theta_{\ell,t} \mid \Im_{\ell,t}) \sim \text{Gamma}\,(\alpha_{\ell,t}, \tau_{\ell,t}) \ , \ \ell \in \{1,2\}. \tag{1.7}$$

On suppose aussi

$$(\Theta_{\ell,t} \mid \Im_{\ell,t-1}) \sim \text{Gamma}\left(\alpha_{\ell,t|t-1}, \tau_{\ell,t|t-1}\right) \ , \ \ell \in \{1,2\} \ , \tag{1.8}$$

où

$$\begin{cases} \alpha_{\ell,t|t-1} = \nu_\ell \alpha_{\ell,t-1} \\ \tau_{\ell,t|t-1} = \nu_\ell \tau_{\ell,t-1}. \end{cases} \tag{1.9}$$

Le paramètre ν_ℓ est un coefficient de pondération plus petit ou égal à 1, et $\Im_{\ell,t}$ est l'information disponible jusqu'au temps t, pour la ligne d'affaire ℓ.
De plus, on propose un modèle croisé, où l'on utilise l'information de la ligne corrélée afin de mieux prédire le risque. Cela implique l'introduction de deux nouveaux paramètres γ_1 et γ_2 qui viennent emprunter de l'information des lignes corrélées. Mathématiquement, cela se traduit de la façon suivante:

$$\begin{cases} \alpha_{1,t|t-1} = \nu_1 \left(\alpha_{1,t-1} + \gamma_1 n_{2,t}\right) \\ \tau_{1,t|t-1} = \nu_1 \left(\tau_{1,t-1} + \gamma_1 \lambda_{2,t}\right), \end{cases} \tag{1.10}$$

et

$$\begin{cases} \alpha_{2,t|t-1} = \nu_2 \left(\alpha_{2,t-1} + \gamma_2 n_{1,t}\right) \\ \tau_{2,t|t-1} = \nu_2 \left(\tau_{2,t-1} + \gamma_2 \lambda_{1,t}\right). \end{cases} \tag{1.11}$$

En outre, afin d'obtenir une distribution jointe dynamique, une approximation de la loi *a posteriori* de la Sarmanov est proposée. Enfin, une illustration empirique est aussi présentée, où l'on observe qu'un tel modèle offre un meilleur ajustement aux données, permet des expressions explicites pour la distribution prédictive, l'espérance et la variance, et alloue des primes plus flexibles et intuitives. Toutefois, le fait qu'on travaille avec des petits échantillons, une limitation déjà mentionnée plus tôt, fait en sorte qu'il devient difficile de mettre en valeur et de faire ressortir l'intérêt de cette dynamique, ainsi que la tendance de cet effet aléatoire. Aussi, un autre défi dans le contexte des réserves, est d'établir l'ordre temporel et la chronologie des paiements.
Nous avons donc choisi d'adopter cette approche, dans un contexte différent, celui de la tarification et plus particulièrement les nombres de réclamations en assurance non-vie. En effet, cette sphère de l'actuariat est plus propice et favorable à une telle méthodologie.
Ainsi, au lieu de modéliser les montants incrémentaux de triangles de développement dans le cadre des méthodes de provisionnement, nous avons choisi de modéliser le nombre de réclamations de l'assureur.
Classiquement, la modélisation du nombre de réclamations suppose que tous les contrats et toutes les garanties sont indépendantes. Depuis quelques années, on remarque que des généralisations ont été proposées. Principalement, nous pouvons voir que des modèles de données de panel modélisent l'ensemble des contrats d'un même assuré (voir par exemple Gouriéroux et Jasiak (2004)), alors que d'autres modèles proposent une dépendance entre les couvertures (voir par exemple Boucher et Inoussa (2014)).
Récemment, des modèles plus avancés ont considéré l'aspect temporel des réclamations, afin de donner plus de poids et d'importance aux années les plus récentes, qui sont supposées être plus prédictives.
L'incorporation d'une dépendance temporelle entre les contrats d'un assuré a déjà été proposée dans la littérature. Gouriéroux et Jasiak (2004) utilisent des modèles INAR(1), Denuit et Lang (2004) proposent des modèles additifs généralisés, Bolancé *et al.* (2007) s'intéressent à

un modèle de séries temporelles dynamique pour les effets aléatoires, Pinquet *et al.* (2001) se servent des résidus d'une régression Poisson, ou plus récemment Shi et Valdez (2014) suggèrent une méthode utilisant les copules.

Toutefois, peu de modèles proposent de modéliser à la fois une dépendance temporelle dynamique, et une dépendance entre les lignes d'affaires. Le modèle que nous présentons au chapitre 5 de cette thèse propose une telle modélisation. En effet, on s'intéresse à une extension bivariée du modèle de Bolancé *et al.* (2007), où l'hétérogénéité jointe de deux couvertures est modélisée à l'aide de la distribution Sarmanov, ayant une forme dynamique.

Chapitre 2

Modeling Dependence between Loss Triangles with Hierarchical Archimedean Copulas

Résumé

L'une des tâches les plus critiques en assurance de dommages est de déterminer une réserve appropriée pour l'ensemble du portefeuille. La plupart des techniques se basent sur des segmentations en sous-portefeuilles homogènes en terme de risque où l'on additionne les provisions de chaque segment. Toutefois, une telle démarche suppose une parfaite indépendance entre les risques d'un portefeuille d'assurés, ce qui est rarement le cas en pratique. Les provisions constituent un élément majeur des états financiers d'une compagnie d'assurance et la volatilité reliée au montant des provisions totales de l'ensemble des engagements ne peut être ignorée. Pour modéliser cette dépendance, nous utilisons la classe des copules archimédiennes hiérarchiques qui généralise la famille des copules archimédiennes en introduisant une plus riche structure de corrélation. Notre modèle nous permet de capter la dépendance de façon plus réaliste et flexible en incluant une notion de niveau et de hiérarchie entre les différentes lignes d'affaires. Une illustration empirique est également présentée, où le modèle est appliqué à des données réelles.

Abstract

One of the most critical problems in property/casualty insurance is to determine an appropriate reserve for incurred but unpaid losses. These provisions generally comprise most of the liabilities of a non-life insurance company. The global provisions are often determined under an assumption of independence between the lines of business. Recently, Shi and Frees (2011) proposed to put dependence between lines of business with a copula that captures dependence between two cells of two different runoff triangles. In this paper, we propose to generalize this model in two steps. First, by using an idea proposed by Barnett and Zehnwirth (1998), we will suppose a dependence between all the observations that belong to the same calendar year for each line of business. Thereafter, we will then suppose another dependence structure that links the calendar years of different lines of business. This model is done by using hierarchical Archimedean copulas. We show that the model provides more flexibility than existing models, and offers a better, more realistic and more intuitive interpretation of the dependence between the lines of business. For illustration, the model is first applied to a dataset from a major US property-casualty insurer, and then to two lines of business from a large Canadian insurer.

2.1 Introduction

Reserves are a major component of the financial statements of a financial institution. With the advent of the new regulatory standards (e.g. Solvency II in Europe and the upcoming ORSA * guidelines in North America), insurance companies must better understand and quantify the risks associated with their activities as a whole, not just by risk classes. Thus, it is now necessary for an insurance company to not only assess a reserve for each line of business but also to better estimate the total reserves for all its insurance products. This involves taking into account dependence between lines of business. In this context, insurance companies must be particularly able to estimate the amount of provisions for the entire portfolio. For this purpose, different reserving approaches allowing dependence between lines of business must be investigated. We will focus on the parametric approach.

Parametric reserving methods have often involved copulas to model the dependence between lines of business. For example, Brehm (2002) uses a Gaussian copula to model the joint distribution of unpaid losses, while De Jong (2012) models dependence between lines of business with a Gaussian copula correlation matrix. Shi et al (2012) and Wüthrich et al (2013) have also used multivariate Gaussian copula to accommodate the correlation due to accounting years within and across runoff triangles. Bootstrapping is another popular parametric approach used to forecast the predictive distribution of unpaid losses for correlated lines of business. Kirschner et al (2008) use a synchronized bootstrap and Taylor and McGuire (2007) extend this result to a generalized linear model context.

∗. ORSA: Own Risk and Solvency Assessment

In this paper, we propose to use a parametric approach with multivariate Archimedean copulas and hierarchical Archimedean copulas. In the same vein as Frees and Shi's model, and following an idea proposed by Barnett and Zehnwirth (1998), we propose a model that allows a dependence relation between all the observations that belong to the same calendar year for each line of business using multivariate Archimedean copulas. We use another dependence structure that links the losses of calendar years of different lines of business. We show that this complex dependence structure can be constructed using hierarchial Archimedean copulas. For illustration, the model is applied to two different datasets from a major US and a large Canadian property-casualty insurers, where we conclude that the proposed model can be considered as an interesting alternative of the model proposed by Shi and Frees (2011).
In Section 2.2, we review the modeling of runoff triangles, where notations are set and copulas briefly introduced. In Section 2.3, the model of Shi and Frees (2011) is implemented (again on their dataset from a major US property-casualty insurer), but with a different parametrization. The calendar year and hierarchical dependences are explained and applied to this data and to a new pair of runoff triangles in Section 2.4. In Section 2.5, we use a parametric bootstrap to obtain the predictive distribution of unpaid losses. Section 2.6 concludes the paper.

2.2 Preliminary

2.2.1 Modeling and Reserves

Let us consider an insurance portfolio with ℓ lines of business ($\ell \in \{1, ..., L\}$). We define by $X_{i,j}^{(\ell)}$, the incremental payments of the i^{th} accident year ($i \in \{1, ..., I\}$), and the j^{th} development period ($j \in \{1, ..., J\}$). To take into account the volume of each line of business, we will work with standardized data which we denote by $Y_{i,j}^{(\ell)} = X_{i,j}^{(\ell)}/\omega_i^{(\ell)}$, where $\omega_i^{(\ell)}$ represents the exposure variable in the i^{th} accident year for the ℓ^{th} line of business. The exposure variable can be the number of policies, the number of open claims, or the earned premiums. The latter option is the one chosen in this paper.
A regression model with two independent explanatory variables, accident year and development period, will be used. Assume that $\alpha_i^{(\ell)}$ ($i \in \{1, 2, ..., I\}$) and $\beta_j^{(\ell)}$ ($j \in \{1, 2, ..., J\}$) characterize respectively the accident year effect and the development period effect. In such a context, a systematic component for the ℓ^{th} line of business can be written as:

$$\eta_{i,j}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)},\ \ell = 1, ..., L,$$

where $\zeta^{(\ell)}$ is the intercept, $I = J = n$, and for parameter identification, the constraint $\alpha_1^{(\ell)} = \beta_1^{(\ell)} = 0$ is supposed.
In our empirical illustration, we first work with the runoff triangles of cumulative paid losses exhibited in Tables 1 and 2 of Shi and Frees (2011). They correspond to paid losses of Schedule P of the National Association of Insurance Commissioners (NAIC) database. These

are data of 1997 for personal auto and commercial auto lines of business, and each triangle contains losses for accident years 1988-1997 and at most ten development years.
Shi and Frees (2011) show that a lognormal and a gamma distribution provide a good fit for the Personal Auto and the Commercial Auto line data respectively. To demonstrate the reasonable model fits for the two triangles, the authors exhibit the qq-plots of marginals for personal and commercial auto lines. We work with their conclusion and then continue with the same continuous distributions for each line of business. More specifically, we consider the form $\mu_{i,j}^{(1)} = \eta_{i,j}^{(1)}$ for a lognormal distribution with location (log-scale) parameter $\mu_{i,j}^{(1)}$ and shape parameter σ. However, for the gamma distribution, we change the parametrization and we do not use the canonical inverse link $\mu_{i,j}^{(2)} = \frac{1}{\eta_{i,j}^{(2)}\phi}$ with location (scale) parameter $\mu_{i,j}^{(2)}$ and shape parameter ϕ. Such a parametrization can lead to undesirable negative values for the lower right part of the runoff triangle, especially when one uses the bootstrap technique. To assure positive means of all the cells of the runoff triangle, we use the log link $\mu_{i,j}^{(2)} = \frac{\exp(\eta_{i,j}^{(2)})}{\phi}$, which is always positive, even for the prediction values of the runoff triangle.
With both parametrizations, the estimated total reserve is $\sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n}\widehat{y}_{i,j}^{(\ell)}\omega_i^{(\ell)}$, where $\widehat{y}_{i,j}^{(\ell)}$ is the projected unpaid loss ratio, and $\omega_i^{(\ell)}$ represents the net premiums earned in the corresponding accident year i. For the lognormal distribution, we have $\widehat{y}_{i,j}^{(1)} = \exp^{\widehat{\mu}_{i,j}^{(1)}+(\widehat{\gamma}^{(1)})^2/2}$, and for the gamma distribution, $\widehat{y}_{i,j}^{(2)} = \widehat{\mu}_{i,j}^{(2)}\widehat{\gamma}^{(2)}$, where $\widehat{\mu}_{i,j}^{(\ell)}$ and $\widehat{\gamma}^{(\ell)}$ are respectively the scale (location) and the shape parameters. Also, $\widehat{\gamma}^{(1)} = \widehat{\sigma}$ and $\widehat{\gamma}^{(2)} = \widehat{\phi}$.

2.2.2 Copulas

Copulas are a useful and flexible tool to model a dependence relation between runoff triangles of different lines of business. They allow a separate interpretation of the relationship (linear and non-linear) between linked random variables and their marginals. See Joe (1997) further details. We briefly recall below definitions and results that will be used later.
A multivariate copula $C(u_1, u_2, ..., u_n)$ is an application from $[0,1]^n$ to $[0,1]$, that has the same properties as a joint cumulative distribution. In other words, a copula is a function that links a multidimensional distribution to its one-dimensional margins. Let F be a n-dimensional cumulative joint function with margins $F^{(1)}$,$F^{(2)}$,...,$F^{(n)}$. Then, if the margins are all continuous, the joint distribution of n random variables $(Y^{(1)}, Y^{(2)}, ..., Y^{(n)})$, can be represented by a unique copula function:

$$F(y^{(1)}, y^{(2)}, ..., y^{(n)}) = C(F^{(1)}, F^{(2)}, ..., F^{(n)}; \theta),$$

where $F^{(i)}$, with $i \in \{1, 2, ..., n\}$, are the respective distribution functions of $Y^{(i)}$, and θ is the dependence parameter, also called the association parameter.
In this paper, we choose to use the Archimedean family of copulas, given its several interesting properties. This family of copulas offers a wide choice of copulas for which many have a closed form expression in a multivariate setting. This last property will prove to be useful in what

follows. Finally, Archimedean copulas can be constructed easily with a simple generator. Formally, we can define multivariate Archimedean copulas as

$$C(u_1, u_2, ..., u_n) = \phi^{-1}(\phi(u_1) + ... + \phi(u_n)) \ , labelch2_eq1 \tag{2.1}$$

where the function ϕ is called the generator of the copula. From (2.2), one can derive the expression for the multivariate density function of an Archimedean copula. According to McNeil and Nešlehová (2009), an Archimedean copula C admits a density c if and only if $\phi^{(n-1)}$ exists and is absolutely continuous on $(0, \infty)$. In such a case, c is given by

$$c(u_1, u_2, ..., u_n) = \phi^{(n)}(\phi^{-1}(u_1) + ... + \phi^{-1}(u_n)) \prod_{i=1}^{n} (\phi^{-1})'(u_i),$$

where functions $\phi^{(n)}$ and ϕ^{-1} correspond to the n^{th} derivative of the generator function of the copula and the inverse generator respectively. Hofert et al (2012) derive closed form expressions for the multivariate density function of a few Archimedean copulas, notably the Clayton and the Gumbel copula used in this paper.

2.3 Pairwise dependence

Dividing a portfolio into homogeneous sub-portfolios and deriving the total reserve by summing the reserve for each segment implicitly assumes independence between risks. It is generally admitted that common social or economic factors may affect several lines of business simultaneously. Allowing a possible dependence relation between the runoff triangles of different lines of business of a portfolio provides a better representation of the portfolio's behavior as a whole and hence permits to take better advantage of diversification. It is also helpful to risk managers in determining the risk capital for an insurance portfolio.

Shi and Frees (2011) propose a model that incorporates a dependence structure between two runoff triangles in a pairwise manner. More precisely, the dependence between two lines of business is based on an identical association between cells of a given accident year and development period, coming from different lines of business. This means that two paid loss ratios $Y_{i,j}^{(1)}$ and $Y_{i,j}^{(2)}$ are correlated for a given couple (i, j). This form of dependence goes back to Braun (2004). Throughout the paper, we refer to Frees and Shi's model as the pairwise dependence model (PWD).

2.3.1 Modeling

The PWD model associates two elements of the same accident year and development period, $(Y_{i,j}^{(1)}, Y_{i,j}^{(2)})$ with a bivariate copula. Mathematically, and following Sklar's theorem, the joint distribution of normalized incremental payments $(Y_{i,j}^{(1)}, Y_{i,j}^{(2)})$ will be represented by the unique copula function:

$$F_{i,j}(y_{i,j}^{(1)}, y_{i,j}^{(2)}) = \Pr(Y_{i,j}^{(1)} \leq y_{i,j}^{(1)}, Y_{i,j}^{(2)} \leq y_{i,j}^{(2)}) = C(F_{i,j}^{(1)}, F_{i,j}^{(2)}; \theta) \ , \tag{2.2}$$

where $C(.,\theta)$ denotes the copula function with parameter θ, that captures the dependence between two runoff triangles. Also, this model has the flexibility of choosing a different cumulative density function for each line of business. The log-likelihood expression can be easily derived from equation (2.2):

$$L = \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log(f_{i,j}^{(1)}) + \log(f_{i,j}^{(2)}) + \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log \ c(F_{i,j}^{(1)}, F_{i,j}^{(2)}; \theta) \ , \tag{2.3}$$

where $c(.)$ denotes the probability density function corresponding to the copula distribution function $C(.)$, $f_{i,j}^{(\ell)}$ denotes the density of marginal distribution $F_{i,j}^{(\ell)}$, for $\ell = 1, 2$. These marginals are noted as:

$$F_{i,j}^{(\ell)} = \Pr(Y_{i,j}^{(\ell)} \leq y_{i,j}^{(\ell)}) = F^{(\ell)}(y_{i,j}^{(\ell)}; \eta_{i,j}^{(\ell)}, \gamma^{(\ell)}),$$

for $i = 1, ..., I$, $j = 1, ...J$ and $\ell = 1, ..., L$. Shi and Frees (2011) choose the Gaussian and the Frank copula to model dependence, as well as the product copula that supposes independence between the cells. Their model selection is based on a likelihood-based goodness-of-fit measure, namely Akaike's Information Criterion (AIC). We will also use this criterion to select our models.

2.3.2 Empirical Illustration

We provide in Tables 2.1 and 2.2, the fit statistics and the reserves for the PWD model. Note that even if the results are close to those obtained in Shi and Frees (2011), we do not obtain the same estimates because we have changed the link function of the mean of the gamma distribution to avoid inconsistencies, as explained in Section 2.2.1.

Fit Statistics	**Independence**	**Copula** **Frank**	**Gaussian**
Dependence parameter	.	-2.7978 (1.0243)	-0.3655 (0.1190)
Log-Likelihood	346.6	350.3	350.5
AIC	-613.2	-618.5	-618.9

Table 2.1 – Fit Statistics of PWD model with Shi and Frees (2011) database

On the other hand, even if we have chosen a different parametrization, we obtain the same conclusion as their and find that the copula that leads to the smallest AIC is the Gaussian copula. This model generates a reserve of almost 7 million dollars. Interestingly, the dependence parameter obtained for the pairwise model with the Gaussian and the Frank copula is negative, meaning that the model supposes that the two lines of business are negatively correlated.

Reserves estimation	Copula Independence	Frank	Gaussian
Personal	6,464,090	6,511,363	6,423,180
Commercial	490,657	487,904	495,989
Total	6,954,747	6,999,267	6,919,169

Table 2.2 – Reserves estimation with the PWD model with Shi and Frees (2011) database

2.4 Calendar Year and Hierarchical Dependence

We propose here to further investigate the model of Shi and Frees (2011) to better capture the interactions within and between the runoff triangles of different lines of business. For that purpose, we first propose to consider a dependence construction for the different elements of a diagonal of a given runoff triangle to take into account a calendar year effect. Second, we add another level of dependence to capture the dependence between the lines of business.

2.4.1 Calendar Year Effect

We propose in this section a model that allows a dependence relation within paid claims belonging to a diagonal of a runoff triangle. This reflects a calendar year (CY) effect, more precisely the changes or inflections on paid claims in a calendar year due to jurisprudence modifications or inflationary trends for example. A CY effect can also highlight the impact of strategic decisions made in a calendar year such as an incentive to increase payments in a particular calendar year for all lines of business.

This dependence structure assumes that all cells from the same diagonal are correlated, which implies that the number of cells in the dependence structure is different for each diagonal. Indeed, the number of cells in the dependence structure varies from 1 to t for the t^{th} diagonal, with $t \in \{1, ..., n\}$, and $t = i + j - 1$. Evidently, the first cell at the top left of the runoff triangle is not linked to any other cell within the triangle.

Such a calendar year effect has already been analyzed before, for example by Barnett and Zehnwirth (1998) who added a covariate to capture the calendar year effect. The systematic component of such a model can be written as:

$$\eta_{i,j}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)} + \Upsilon_t^{(\ell)},\ \ell = 1, ..., L\ , \tag{2.4}$$

where $\zeta^{(\ell)}$ is the intercept, $\alpha_i^{(\ell)}$ ($i \in \{2, ..., I\}$) and $\beta_j^{(\ell)}$ ($j \in \{2, ..., J\}$) characterize respectively the accident year effect and the development period effect, while $\Upsilon_t^{(\ell)}$ ($t = i + j - 1$) captures the calendar year effect.

De Jong (2006) modeled the growth rates in cumulative payments in a calendar year, and Wüthrich (2010) examined the accounting year effect for a single line of business. Wüthrich and Salzmann (2012) used a multivariate Bayes Chain-Ladder model that allows the modeling of dependence along accounting years within runoff triangles. The authors showed that they are able to derive closed form solutions for the posterior distribution, the claims reserves and the corresponding prediction uncertainty. Kuang et al (2008) have also considered a canonical parametrization with three factors for a single line of business. Each factor represents time scale, in such way the inflation is taken into account. Also, they added an assumption ensuring that the forecasts do not depend on these arbitrary linear trends. They extended this assumption later by combining the canonical parametrization with a non-stationary time series forecasting model in Kuang et al (2011).

In our proposed model, instead of adding an explanatory variable for the calendar year effect, the dependence relation between the paid claims of a diagonal will be based on a multivariate Archimedean copula. More specifically, the same Archimedean copula with an identical dependence parameter is assumed for each diagonal of a runoff triangle. Hence, all random variables of the same calendar year $t = i + j - 1$ and ℓ^{th} line of business are included in the vector $\boldsymbol{Y}_{\ell t} = \{Y_{\ell i,j} : i + j - 1 = t\}$. The log-likelihood function of this model can be written as:

$$L = \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log(f_{i,j}) + \sum_{t=2}^{n} \log \ c\left(F_{t-j+1,j}, ..., F_{1,t}; \theta\right)_{j=1,...,t} , \qquad (2.5)$$

where f denotes the density of marginal distribution F, and $c(.)$ the probability density function corresponding to the copula distribution function $C(.)$.

The main advantage of the copula approach instead of adding a calendar year covariate in the mean specification, lies in the fact that the copula approach allows a more general structure of dependence between the observations of a given calendar year and allows more flexibility. Also, the use of covariates would lead to a great number of parameters to explain the calendar year effect instead of only one (dependence copula parameter). For example, for two lines of business, we would have 20 parameters instead of 2 (see equation (2.4)). This might lead to over-parametrization. Furthermore, the parameter describing a given calendar year effect, would not have any predictive power, as we cannot use it to compute the lower triangle.

2.4.2 Line of Business Dependence

A natural extension to the model behind (2.5) is to introduce a dependence structure between lines of business based on copulas, more precisely here with the Gaussian copula and hierarchical Archimedean copulas.

Another way to add dependence between lines of business is by modifying equation (2.4) and use the same calendar year covariate for the two lines of business, i.e. $\Upsilon_t = \Upsilon_t^{(1)} = \Upsilon_t^{(2)}$. The correlation induced by common calendar year effects would then be introduced through the mean specification. Also, as done in Shi et al (2012), in addition to the common calendar year covariate, a pair-wise correlation between the two runoff triangles can be added. This approach has the disadvantage however of adding a new parameter for each diagonal (Υ_t).

Multivariate Gaussian Copula

We first propose to use the Gaussian copula to capture the dependence within and between runoff triangles. The Gaussian copula which arises from the multivariate normal distribution is the most widely known copula of the elliptical family of copulas. Such a copula allows great flexibility to model dependences simply by modifying its correlation matrix.
Let us suppose, for a given calendar year t, the following set of observations

$$\mathbf{u}_t = \left(u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}, u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t}\right)_{j=1,...,t} ,$$

with multivariate Gaussian copula density:

$$c\left(\mathbf{u}_t\right) = |\mathbf{\Sigma}_t|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{\xi}_t^T\left(\mathbf{\Sigma}_t^{-1} - I\right)\boldsymbol{\xi}_t\right) ,$$

where $\boldsymbol{\xi}_t = \left(\Phi^{-1}\left(u^{(1)}_{t-j+1,j}\right), ..., \Phi^{-1}\left(u^{(1)}_{1,t}\right), \Phi^{-1}\left(u^{(2)}_{t-j+1,j}\right), ..., \Phi^{-1}\left(u^{(2)}_{1,t}\right)\right)^T_{j=1,...,t}$. The correlation matrix $\mathbf{\Sigma}_t$ for the calendar year t can be represented as a block matrix as follows, given the assumptions of the model:

$$\mathbf{\Sigma}_t = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{21} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{12} \end{pmatrix} . \tag{2.6}$$

In (2.6), the matrices $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{12}$ are correlation matrices with unit main diagonal and off-diagonal parameters $\theta_{1,1}$ and $\theta_{1,2}$ corresponding to the calendar year dependence for the first and second line of business respectively. $\mathbf{\Sigma}_{21}$ is a matrix filled with parameter $\theta_{2,1}$ representing the dependence between the two lines of business.
Numerical results obtained with the Gaussian copula are presented in the empirical illustration of section 2.4.3.

Hierarchical Archimedean Copulas

Hierarchical Archimedean copulas permit to have different levels of dependence within our framework. We use them here to add another level of dependence to the one proposed in section 2.4.1. With this second level of dependence, we capture the dependence between two different runoff triangles in a pairwise manner between corresponding diagonals, instead of between cells. Pairing diagonals instead of cells with a copula has the advantage of being applicable even in a case of missing data in one of the runoff triangles.

The hierarchical approach allows us to visualize the multi-level dependence. Indeed, this dependence structure is illustrated in Figure 2.1, where a dependence structure between cells of the same calendar year is supposed as well as a dependence structure between the two lines of business. In the next section, we will also be interested in comparing the hierarchical copula approach with the multivariate Gaussian copula approach, as the latter is often considered as a benchmark model.

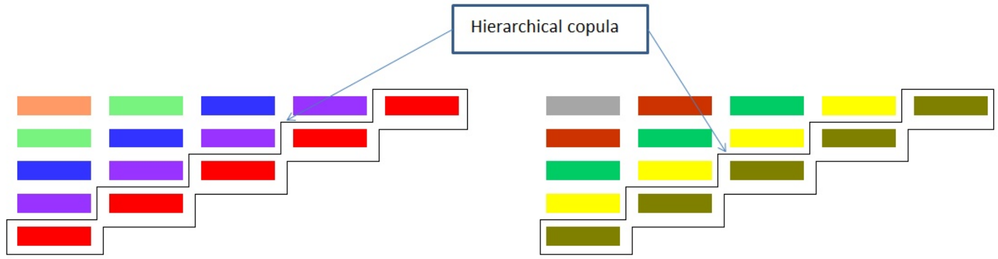

Figure 2.1 – Dependence implied by hierarchical dependence

The CY effect has not been often studied with more than one line of business. Two recent examples are De Jong (2012), where the calendar year effect was introduced through the correlation matrix and Shi et al (2012), who used random effects to accommodate the correlation due to accounting year effects within and across runoff triangles. In Shi et al (2012), they work with a Bayesian perspective, using a multivariate lognormal distribution, along with a multivariate Gaussian correlation matrix. The predictive distributions of outstanding payments are generated through Monte Carlo simulations. The calendar year effect is taken into account through an explanatory variable. A discussion of this paper is suggested in Wüthrich (2012), and where it is also explained that for the method it does not really matter whether we consider incremental or cumulative claims, as long as we have a multivariate Gaussian structure. Also, still with a Bayesian framework, Wüthrich et al (2013) used a multivariate lognormal Chain-Ladder model and derived predictors and confidence bounds in closed form. Their analytical solutions are such that they allow for any correlation structure. Their models allow a dependence between and within runoff triangles, and for any correlation structure. It has also been shown in this paper that the pair-wise dependence form is a rather weak one compared to calendar year dependence. More recently, Shi (2014) captures the dependencies introduced by various sources, including the common calendar year effects via the family of elliptical copulas, and use a parametric bootstrapping to quantify the associated reserving variability.

In this paper, to model the complex dependence structure between two runoff triangles, we introduce models based on hierarchical Archimedean copulas. The idea is to use Archimedean copulas at each level, from the lowest (calendar years) to the highest (lines of business). Hierarchical Archimedean copulas have first been mentioned in the literature by Joe (1997), and appeared in more details in Savu and Trede (2010). More recently, Okhrin et al (2013) provided a method to estimate multivariate distributions defined through hierarchical Archimedean copulas.

The main advantage of using Archimedean and hierarchical Archimedean copulas is that they can be explicitly defined in terms of a one-dimensional function called the generator of the

Archimedean copula. Elliptical copulas, used in Shi (2014), do not possess this nice property; they do not have a closed form. Archimedean copulas are also flexible and allow to model many kinds of dependencies, while Elliptical copulas, have equal lower and upper tail dependence coefficients. In high dimensions, Archimedean copulas are restricted given the exchangeability of the components. This assumption is relaxed with hierarchical Archimedean copulas.
At the lowest level, and for the calendar year t, we have $2 \times t$ standard uniformly distributed random variables $U^{(1)}_{t-j+1,j}, ..., U^{(1)}_{1,t}, U^{(2)}_{t-j+1,j}, ..., U^{(2)}_{1,t}$ where j designates the development period ($j = 1, ..., t$).
The joint distribution function is evaluated at $\mathbf{u} = (u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}, u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t}) \in [0,1]^{2t}$. Let there be H hierarchy levels indexed by h. For example, the set of elements $\mathbf{u}$ is located at level $h = 0$. At each level $h = 0, ..., H$ we have n_h distinct objects with index $k = 1, ..., n_h$.
At level $h = 1$, the $u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t}, u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t}$ are grouped into n_1 ordinary multivariate Archimedean copulas $C_{1,k}$, $k = 1, ..., n_1$ (in our case with two lines of business, we have $n_1 = 2$), of the form

$$C_{1,k}(\mathbf{u}_{1,k}) = \phi^{-1}_{1,k}\left(\sum_{\mathbf{u}_{1,k}} \phi_{1,k}(\mathbf{u}_{1,k})\right) ,$$

where $\phi_{1,k}$ denotes the generator of the copula $C_{1,k}$. Let $\mathbf{u}_{1,k}$ denote the set of elements of $u^{(k)}_{t-j+1,j}, ..., u^{(k)}_{1,t}$ belonging to the copula $C_{1,k}$ for $k = 1, ..., n_1$, which represents the elements of a given calendar year for a single line of business ℓ. At this level only, k corresponds to ℓ .In our model, we have three levels, i.e $H = 2$. At the highest level, we have a single object ($n_2 = 1$), which is the hierarchical Archimedean copula $C_{2,1}$, that aggregates the multivariate Archimedean copulas of the previous level, and can be represented as

$$C_{2,k}(\mathbf{C}_{2,k}) = \phi^{-1}_{2,k}\left(\sum_{\mathbf{C}_{2,k}} \phi_{2,k}(\mathbf{C}_{2,k})\right) ,$$

where $\phi_{2,k}$ denotes the generator of the copula $C_{2,k}$ and $\mathbf{C}_{2,k}$ represents the set of all copulas from level $h = 1$ entering copula $C_{2,k}$ for $k = 1, ..., n_2$.
Obviously, there are numerous conditions to be satisfied for the existence of a hierarchical Archimedean copula. The number of copulas must decrease at each level, i.e. $n_h < n_{h-1}$, as well as the degree of dependence, i.e. $\theta_{h+1,k'} < \theta_{h,k}$ for all $h = 0, ..., H$ and $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ such that $C_{h,k} \in C_{h+1,k'}$ where $\theta_{h,k}$ is the parameter belonging to the generator $\phi_{h,k}$. This means that for runoff triangles, elements of a same line of business can have a higher degree of dependence than elements of different lines of business. Mathematically, the conditions that have to be verified by a hierarchical Archimedean copula are summarized as follows:

1. All inverse generator functions $\phi^{-1}_{h,k}$ are completely monotone.

2. The composite $\phi_{h+1,k'} \circ \phi_{h,k}^{-1}$ are convex functions for all $h = 0, ..., H$ and $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ such that $C_{h,k} \in C_{h+1,k'}$.

In our application, we will limit the number of levels to three, and the number of lines of business to two. This means that we will have at the highest level ($h = 2$), one (hierarchical) bivariate Archimedean copula between lines of business, and for $h = 1$, two (ordinary) multivariate Archimedean copula within a runoff triangle.

As an illustration, let us consider a dependence structure between two runoff triangles for the second calendar year. The resulting hierarchical Archimedean copula has the following analytical form

$$\begin{aligned} C_{2,1}(\mathbf{u}) &= C_{2,1}(u_{2,1}^{(1)}, u_{1,2}^{(1)}, u_{2,1}^{(2)}, u_{1,2}^{(2)}) \\ &= C_{2,1}(C_{1,1}(u_{2,1}^{(1)}, u_{1,2}^{(1)}), C_{1,2}(u_{2,1}^{(2)}, u_{1,2}^{(2)})) \\ &= \phi_{2,1}^{-1}\left(\phi_{2,1} \circ \phi_{1,1}^{-1}[\phi_{1,1}(u_{2,1}^{(1)}) + \phi_{1,1}(u_{1,2}^{(1)})] + \phi_{2,1} \circ \phi_{1,2}^{-1}[\phi_{1,2}(u_{2,1}^{(2)}) + \phi_{1,2}(u_{1,2}^{(2)})]\right). \end{aligned}$$

This hierarchical Archimedean copula will be applied to each calendar year, with the dataset described in Section 2.2.1. The calendar year t takes values from 1 to 10 because the runoff triangles both have 10 diagonals, i.e. $I = J = 10$. The resulting hierarchical Archimedean copula for our model has the following general analytical form:

$$\begin{aligned} C_{2,1}(\mathbf{u}) &= C_{2,1}(u_{t-j+1,j}^{(1)}, ..., u_{1,t}^{(1)}, u_{t-j+1,j}^{(2)}, ..., u_{1,t}^{(2)}) \\ &= C_{2,1}(C_{1,1}(u_{t-j+1,j}^{(1)}, ..., u_{1,t}^{(1)}), C_{1,2}(u_{t-j+1,j}^{(2)}, ..., u_{1,t}^{(2)})) \\ &= \phi_{2,1}^{-1}\left(\phi_{2,1} \circ \phi_{1,1}^{-1}[\phi_{1,1}(u_{t-j+1,j}^{(1)}) + ... + \phi_{1,1}(u_{1,t}^{(1)})] + \phi_{2,1} \circ \phi_{1,2}^{-1}[\phi_{1,2}(u_{t-j+1,j}^{(2)}) + ... + \phi_{1,2}(u_{1,t}^{(2)})]\right). \end{aligned} \tag{2.7}$$

Finally, the log-likelihood function of the hierarchical model can be written as follows:

$$L = \sum_{\ell=1}^{2}\sum_{i=1}^{I}\sum_{j=1}^{I-i+1} \log(f_{i,j}^{(\ell)}) + \sum_{t=2}^{n} \log\left(c_{2,1}\left(F_{t,1}^{(1)}, F_{t-1,1}^{(1)}, ..., F_{1,t}^{(1)}, F_{t,1}^{(2)}, F_{t-1,1}^{(2)}, ..., F_{1,t}^{(2)}\right)\right), \tag{2.8}$$

where $c_{2,1}$ denotes the density of a hierarchical Archimedean copula, which is obtained by differentiating the copula using the chain rule. More precisely, for a given diagonal t, we have the following expression:

$$c_{2,1}(\mathbf{u}) = \frac{\partial^{2t} C_{2,1}(\mathbf{u})}{\partial u_{t-j+1,j}^{(1)} \cdots \partial u_{1,t}^{(1)} \partial u_{t-j+1,j}^{(2)} \cdots \partial u_{1,t}^{(2)}}. \tag{2.9}$$

As we have 10 diagonals, we need to derive up to 20 times. We show an example of a 4-variables case, corresponding to the second diagonal in Appendix 2.6. However, the density is computationally intensive in high dimensions when the number of observations in the diagonal increases, and a closed form expression for the maximum likelihood estimators is no longer available.

	Copula - Estimates and Standard Errors					
	Gaussian		Clayton		Gumbel	
$\theta_{1,1}$	0.6091	(0.1366)	2.2695	(0.4463)	2.7267	(0.6762)
$\theta_{1,2}$	0.7634	(0.0983)	2.9759	(0.5743)	2.7103	(0.6045)
Log-Lik.	391.5		403.9		404.3	
AIC	-699.0		-723.9		-724.6	

Table 2.3 – Fit Statistics of ICYD model with Shi and Frees (2011) database

	Copula		
Reserves estimation	**Gaussian**	**Clayton**	**Gumbel**
Personal	6,175,574	6,425,748	6,965,466
Commercial	751,725	550,179	593,945
Total	6,927,299	6,975,927	7,559,412

Table 2.4 – Reserves estimation of ICYD model with Shi and Frees (2011) database

A numerically efficient way to evaluate the log-density is presented in Hofert and Pham (2013), where an implementation of the hierarchical Clayton and Gumbel copulas is provided using the R package `copula`; see Hofert and Mächler (2011).
The simpler form of hierarchical dependence is to suppose a product copula between the two runoff triangles, meaning independence between lines of business. In this situation, the log-likelihood of the model is simply $L = L^{(1)} + L^{(2)}$, where $L^{(\ell)}$, $\ell = 1, 2$ is simply the log-likelihood obtained by (2.5). Of course, it is very easy to extend this model to more than two lines of business.

2.4.3 Empirical Illustration

Hierarchical models based on different copulas have been applied to the runoff triangles used in Section 2.2.1. For this model, the CY dependence has been modeled with four different copulas (product, Gumbel, Clayton and Gaussian). In our empirical study, we first use a model that supposes independence between lines of business, i.e. a product copula between runoff triangles. We call this model ICYD, for independence calendar year dependence. Fit statistics as long as dependence parameters of this model are shown in Table 2.3, while the estimated reserves are presented in Table 2.4. In terms of AIC, we observe that all Archimedean copula models offer a better fit than the multivariate Gaussian copula. Note that a CY dependence with a product copula within and between the two lines of business is simply a cell-by-cell modeling. The empirical results of this simple model have already been given in Section 2.3.2, for the PWD model with a product copula.
The two available copulas in the R package `copula`, which are Clayton and Gumbel, have been considered in a hierarchical model to investigate dependence between the two lines of business. The same copula is used for each level, meaning for example that if a Gumbel copula

is chosen within a runoff triangle, then it is also used between the business lines. This is due to the convexity condition on hierarchical Archimedean copulas. We call this model HCYD, for hierarchical calendar year dependence. When we apply this model to the dataset used in Section 2.2.1, the hierarchical model do not improve the independent calendar year model for the three copulas (Gaussian, Clayton and Gumbel). The values of the dependence parameters $\theta_{2,1}$ are not statistically significant, meaning that the two lines of business are uncorrelated.

To better emphasize the features of the hierarchical model, we work with two other runoff triangles that were recently used in Côté et al (2016), which come from a Canadian property-casualty insurer. The two lines of business comprise personal and commercial auto insurance. The first triangle contains paid losses of the Accident Benefits (AB) coverage from Ontario, while the second one constitutes paid losses from Bodily Injuries (BI) coverage from the Western region. The Accident Benefits (AB) coverage provides compensation, regardless of fault, if driver, passengers, or pedestrians suffer injury or death in an automobile collision. On the other hand, the Bodily Injury (BI) coverage provides compensation to the insured if he is injured or killed through the fault of a motorist who has no insurance, or by an unidentified vehicle.

Côté et al (2016) demonstrate that a gamma distribution provides a good fit for the two lines of business. We work with their conclusion and then continue with the same continuous distribution for each line of business. The cumulative paid losses and earned premiums for the two lines of business are displayed in Appendix 2.6.

We first apply the PWD model to these two lines of business, the estimation parameters and the reserves estimation are shown in Table 2.5. Whereas, the fit statistics and the reserves obtained for the independent and hierarchical calendar year models are shown in Table 2.6. To compare the degree of dependence between different copulas, we also provide the two non-linear association measures Spearman's rho ρ_S and Kandall's tau τ_K for the two copulas, see Table 2.6. We notice that the Clayton copula captures a smaller dependence than the Gumbel copula, whose association measures are slightly higher. Indeed, the Clayton family is characterized by a lower tail dependence. Also, the hierarchical calendar year model offers a better fit than the independent calendar year model as shown by the values of the log-likelihood function. This finding leads to a statistically significant dependence between the two lines of business ($\theta_{2,1}$), captured through the calendar year effects. This is also confirmed by looking at the value of the AIC, which points to the Gumbel hierarchical copula model as the one which better adjusts the data.

When we incorporate a calendar year correlation within the lines of business (level 1), the residual dependence becomes positive. Intuitively, this can be explained by the trends and common effects that are detected with the introduction of the proposed dependence structure but not with the Chain-Ladder coefficients. In a given calendar year, exogenous common factors such as inflation, interest rates, jurisprudence or strategic decisions such as the acceleration of the payments for the entire portfolio can have simultaneous impacts on all lines of

Fit Statistics	Independence	Copula Frank		Gaussian	
Dependence parameter	.	-0.6649	(0.9430)	0.0149	(0.1362)
Log-Likelihood	423.7	424.0		423.8	
AIC	-767.4	-766.0		-765.6	
Total Reserve	96,954	96,994		96,949	

Table 2.5 – Fit Statistics and Reserves of PWD model with Côté et al (2016) database

	ICYD model				HCYD model			
	Clayton		Gumbel		Clayton		Gumbel	
$\theta_{1,1}$	0.0294	(0.0708)	1.0829	(0.1292)	0.0495	(0.0608)	1.0692	(0.0515)
$\theta_{1,2}$	0.2384	(0.1881)	1.1548	(0.1315)	0.2034	(0.2259)	1.0692	(0.0496)
$\theta_{2,1}$	.		.		0.0495		1.0692	
ρ_S	.		.		0.0362		0.0948	
τ_K	.		.		0.0241		0.0648	
LogLik	424.4		425.8		426.3		427.7	
AIC	-764.8		-767.6		-766.6		-769.4	
Total Reserve	84,172		81,650		96,496		83,202	

Table 2.6 – Parameter and Reserves estimation of ICYD and HCYD models with Côté et al (2016) database

business of a given sector, such as the two lines of business considered in the present paper. These effects may as well result in trends in the development period parameters.

It is interesting to note that, unlike the slightly negative pairwise association obtained by the PWD model in Table 2.5 and also displayed for these two lines of business in Table 4 of Côté et al (2016), hierarchical models generate positive dependence between loss triangles with the same dataset.

We observe that the positive parameter $\theta_{2,1}$ is statistically significant for the Clayton and Gumbel copulas. This results highlights the fact that the choice of the dependence structure can lead to different conclusions for the dependence analysis. This was also well illustrated in Figure 4 of Shi et al (2012).

Finally, a hierarchical copula model requires a higher degree of dependence for variates linked at a lower level than those linked at a higher level. In our context, this means that the degree of dependence within lines of business should be greater than between lines of business, as illustrated in Figure 2.1. One can observe in Table 2.6 that this condition is respected with a dependence parameter $\theta_{2,1}$ lower than the dependence parameters $\theta_{1,1}$ and $\theta_{1,2}$. In this sense, this condition could also be seen as a restriction for the dependence parameter between the two lines of business. In fact, we observe that the parameters $\theta_{2,1}$ are on the boundary of their domain in Table 2.6. This actually could constitute a limitation of the hierarchical model.

2.5 Predictive distribution

In practice, actuaries are interested in knowing the uncertainty of the reserve. A parametric technique, the bootstrap, not only provides such information but most importantly lets one determine the entire predictive distribution, rarely obtained for non-Bayesian models. The predictive distribution notably allows assessment of risk capital for an insurance portfolio. Bootstrapping is also ideal from a practical point of view, because it avoids the complex theoretical calculations and can easily be implemented. Moreover, it tackles the potential model overfitting, typically encountered in loss reserving problems, due to the small sample size.

The bootstrap technique is increasingly popular in loss reserving, and allows a wide range of applications. It was first introduced in a loss reserving context with a distribution-free approach by Lowe (1994). For a multivariate loss reserving analysis, Kirschner et al (2008) used a synchronized parametric bootstrap to model dependence between correlated lines of business, and Taylor and McGuire (2007) extended this result to a generalized linear model context. Shi and Frees (2011) and more recently Shi (2014) have also performed a parametric bootstrap to incorporate the uncertainty in parameter estimates, while modeling dependence between loss triangles using copulas.

2.5.1 Parametric Bootstrap

The parametric bootstrap allows us to obtain the whole distribution of the reserves. We follow the same bootstrap algorithm of Taylor and McGuire (2007), and summarized in Shi and Frees (2011).

Copula simulation

The first step of the parametric bootstrap is to generate pseudo-responses of normalized incremental paid losses $y^{*(\ell)}_{i,j,r}$, for i, j such that $i + j \leq I$ and $\ell = 1, 2$. We know that $y^{*(\ell)}_{i,j;r} = F^{(-1)(\ell)}(u^{(\ell)}_{i,j}, \widehat{\mu}^{(\ell)}_{i,j}, \widehat{\gamma}^{(\ell)})$, with $\widehat{\mu}^{(\ell)}_{i,j}$ and $\widehat{\gamma}^{(\ell)}$ already estimated. Therefore, a technique to generate the realizations of the copula $u^{(\ell)}_{i,j}$, with $\ell = 1, 2$ should be used.

Given that the Gumbel copula generates the best fit for many models in this paper, we have decided to focus on this copula for the bootstrap. Below, the bootstrap study is performed with the datasets of Côté et al (2016).

To generate a multivariate Gumbel copula, we follow the method based on the inversion of the Laplace transform, an idea that can be traced back to Marshall and Olkin (1988).

The above cited algorithm allows us to generate the set of realizations $\mathbf{u}^{(1)}_{1,1}$ and $\mathbf{u}^{(2)}_{1,2}$ for the first level of hierarchy (CY level at $h = 1$) from the ordinary multivariate Archimedean copulas $C_{1,1}$ and $C_{1,2}$, for a given calendar year t and development period j $(j = 1, ..., t)$, with $\mathbf{u}^{(1)}_{1,1} = (u^{(1)}_{t-j+1,j}, ..., u^{(1)}_{1,t})$ and $\mathbf{u}^{(2)}_{1,2} = (u^{(2)}_{t-j+1,j}, ..., u^{(2)}_{1,t})$. To generate realizations with a

Model	Copula reserve	Bootstrap reserve	Bias	Std Error
Gumbel hierarchical model	83,202	81,574	1.95%	8,555

Table 2.7 – Bootstrap bias for the Gumbel HCYD model with Côté et al (2016) database

Gumbel copula at the highest level of the hierarchy (line of business level at $h = 2$), we used the sampling algorithm of Nested Archimedean copulas from the R package `copula`. Consequently, we have obtained the set of realizations $\mathbf{u}_{2,1}^{(1)}$ and $\mathbf{u}_{2,1}^{(2)}$ for the second level of hierarchy (business line level at $h = 2$) from the hierarchical Archimedean copula $C_{2,1}$.

Bias and MLE

The maximum likelihood estimation technique is known to be asymptotically unbiased. In practice, we work with a finite number of observations, particularly with runoff triangles. Indeed, in our empirical illustrations, only 55 observations have been used in each triangle. Consequently, regardless of the number of simulations, our estimation is done each time on limited datasets of 55 observations.
The impact of the bias on the estimation has been analyzed. Recently, the lognormal MLE bias has been studied in Johnson et al (2011), along with the gamma and Weibull distributions. Consequently, inter alia, a bias is necessarily observed in the bootstrapping procedure. In our empirical illustration, the bootstrap bias obtained for the hierarchical model is exhibited in Table 2.7.

2.5.2 Reserve indications

We show a histogram of the reserve distribution of the hierarchical model in Figure 2.2, which is important and useful for actuaries when they want to select a reserve at a desired level of conservatism.
In Table 2.7, we exhibit the bootstrap results for the Gumbel hierarchical model which mainly refers to the mean reserve and the prediction uncertainty. The latter may substantially be increased by the introduction of the accounting year dependence. In contrast, PWD models can under-estimate the variability because they implicitly assume an independence between accident years. This was also stated in Wüthrich et al (2013), where it has been shown that the CY modeling is more performant than the PWD modeling. It is worth mentioning that to compute the mean squared error of prediction, the process uncertainty must be added to this prediction error (see England and Verrall (2002)).
Note that to obtain the lower triangle in the Bootstrap procedure, we can either calculate the projected mean for each cell of the lower triangle, as shown in this paper (projected mean approach), or generating (by simulation) each cell of the lower triangle starting from the new estimates obtained for each bootstrap sample. The second approach (the simulation based approach) offers a wider range of possible reserves, and will consequently have a larger

standard error. This second approach can be particularly interesting from a capital risk standpoint where extreme loss events have to be considered. Both bootstrap approaches (projected mean approach and simulation based approach) are relevant information for property-casualty insurers.

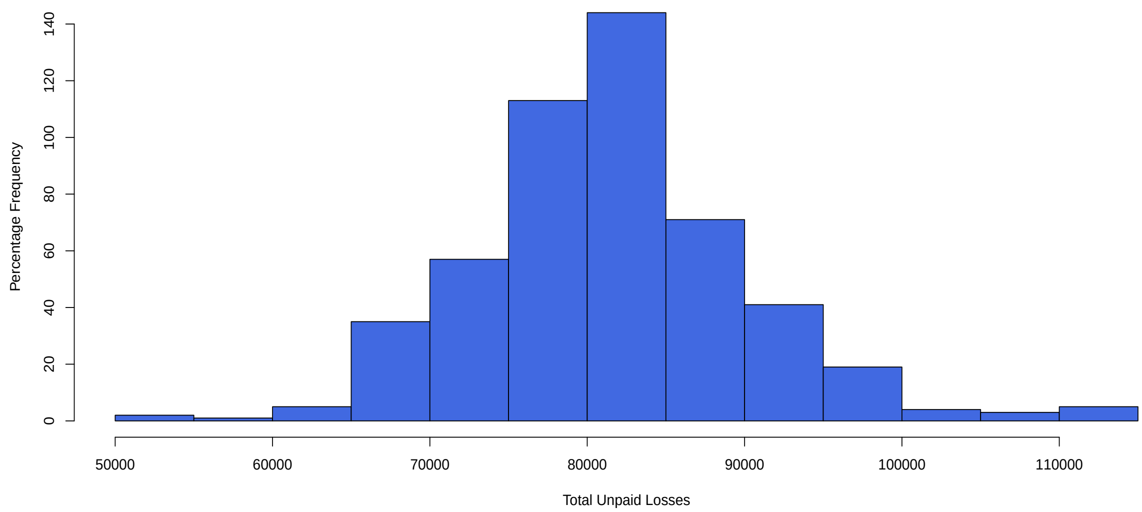

Figure 2.2 – Predictive distribution of total unpaid losses - Complete hierarchical model

2.6 Conclusion

In this paper, we have studied different approaches to model the dependence between loss triangles using multivariate copulas. If losses in different lines of business are correlated, aggregate reserves must reflect this dependence. To allow a complex dependence relation, we propose the use of new models using hierarchial Archimedean copulas. To illustrate the model, an empirical illustration was performed using the same data as the one used by Shi and Frees (2011). Based on the AIC, we show that the ICYD models provide a better fit than PWD models. Furthermore, to show the interest of HCYD models and better highlight their properties, the empirical illustration has also been performed on two other runoff triangles from a major Canadian insurance company, which also allows us to expose the proposed model to a wider range of situations. A hierarchical calendar year dependence seemed to be relevant because the hierarchical Gumbel copula model was one of the best to adjust the data.
With the proposed models, we can derive analytically the value of the reserve. However, to obtain the distribution of the reserve and to estimate the parameters, numerical evaluation is necessary. Indeed, estimation and sampling are implemented in the `R` package `copula`. Also, the total reserve estimate in the presence of dependence relies heavily on the choice of the

dependence structure and the selected copula. This is a limitation of the joint estimation of the marginal and dependence parameters. This undesirable effect will be addressed in a future work within a two-stage inference strategy; see Côté et al (2016) for more details.
These new models that use hierarchical copula theory constitute a new way to model the dependence structures of runoff triangles. Those models are promising tools to better take into account dependencies within and between business lines. Indeed, this approach can easily be generalized to more than two lines of business because hierarchical Archimedean copulas are flexible and allow more refined possible dependence constructions. Because of their flexibility, hierarchical copula models should also be considered in other areas of actuarial science.

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APPENDIX

Nested Archimedean Copula Density

To lighten the notation, let $C_\theta^{(i,j)}(u,v) = \dfrac{\partial^{i+j} C_\theta(u,v)}{\partial u^i \partial v^j}$ for $i, j \in \{0,1,2\}$.

Following the equation (2.9), the 4-dimensional density of the hierarchical Archimedean copula $C_{2,1}$ for the second diagonal ($t = 2$) will be written as follows:

$$
\begin{aligned}
c_{2,1}(u_1,u_2,u_3,u_4) &= \frac{\partial^4}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} C_{2,1}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) \\
&= \frac{\partial^3}{\partial u_1 \partial u_2 \partial u_3} C_{2,1}^{(0,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,2}^{(0,1)}(u_3,u_4) \\
&= \frac{\partial^2}{\partial u_1 \partial u_2} \big[C_{2,1}^{(0,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(0,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,2}^{(1,1)}(u_3,u_4) \big] \\
&= \frac{\partial}{\partial u_1} \big[C_{2,1}^{(1,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(1,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,1)}(u_3,u_4) \big] \\
&= C_{2,1}^{(2,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,0)}(u_1,u_2) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(1,2)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,1)}(u_1,u_2) C_{1,2}^{(1,0)}(u_3,u_4) C_{1,2}^{(0,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(2,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,0)}(u_1,u_2) C_{1,1}^{(0,1)}(u_1,u_2) C_{1,2}^{(1,1)}(u_3,u_4) \\
&\quad + C_{2,1}^{(1,1)}\left(C_{1,1}(u_1,u_2), C_{1,2}(u_3,u_4)\right) C_{1,1}^{(1,1)}(u_1,u_2) C_{1,2}^{(1,1)}(u_3,u_4)
\end{aligned}
$$

Data

Table 2.8 – Cumulative paid losses for Ontario AB.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	3043	5656	7505	8593	9403	10380	10450	10812	10856	10860	116491
2004	2070	4662	6690	8253	9286	9724	9942	10086	10121		111467
2005	2001	4825	7344	8918	9824	10274	10934	11155			107241
2006	1833	4953	7737	9524	10986	11267	11579				105687
2007	2217	5570	7898	8885	9424	10402					105923
2008	2076	5681	8577	10237	12934						111487
2009	2025	6225	9027	10945							113268
2010	2024	5888	8196								121606
2011	1311	3780									110610
2012	912										104304

Table 2.9 – Cumulative paid losses for West BI.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	2279	8683	15136	21603	27650	30428	32004	32592	33009	34140	76620
2004	2139	7077	13159	16435	20416	22598	24171	25034	25714		65691
2005	1420	4888	8762	12184	14482	15633	17089	17710			55453
2006	1510	5027	10763	15799	19269	22504	24807				54006
2007	1693	5175	8216	12263	16918	20792					55425
2008	2097	7509	10810	15673	19791						59100
2009	2094	5174	8062	12389							54438
2010	1487	4789	7448								53483
2011	1868	6196									52978
2012	2080										57879

Chapitre 3

Rank-Based Methods for Modeling Dependence Between Loss Triangles

Résumé

Un des problèmes centraux en assurance non-vie est de déterminer une structure de dépendance appropriée entre les triangles de développement. Souvent, la réserve totale estimée est affectée par le choix de cette structure de dépendance.
Dans cet article, nous pallions ce problème en utilisant une procédure d'inférence en deux étapes, où les distributions marginales sont d'abord estimées, puis un modèle de copule est estimé en utilisant des procédures basées sur les rangs.
Nous considérons deux modèles de dépendance hiérarchique : le premier utilise les copules archimédiennes imbriquées et le deuxième est basé sur une méthode d'agrégation des risques avec des copules.
Nous analysons un portefeuille d'assurance de six lignes d'affaires d'une compagnie d'assurance générale canadienne afin de supporter cette idée.

Abstract

In order to determine the risk capital for their aggregate portfolio, property and casualty insurance companies must fit a multivariate model to the loss triangle data relating to each of their lines of business. As an inadequate choice of dependence structure may have an undesirable effect on reserve estimation, a two-stage inference strategy is proposed in this paper to assist with model selection and validation. Generalized linear models are first fitted to the margins. Standardized residuals from these models are then linked through a copula selected and validated using rank-based methods. The approach is illustrated with data from six lines of business of a large Canadian insurance company for which two hierarchical dependence models are considered, i.e., a fully nested Archimedean copula structure and a copula-based risk aggregation model.

3.1 Introduction

In Canada, the Own Risk and Solvency Assessment (ORSA) guideline from the Office of the Superintendent of Financial Institutions (OSFI) requires that insurance companies set internal targets for risk capital that are tailored to their consolidated operations. In order to relate risk to capital and consider their operations as a whole, insurers are encouraged to develop internal models for the aggregation of dependent risks. Similar regulations exist in many countries worldwide.

To comply with regulatory standards, property and casualty insurance companies have to hold reserves and risk capital relating to losses that are incurred but not yet paid. For each line of business, payments relating to past claims are usually structured in a run-off triangle arranged to rows according to the accident years, and to columns according to the development periods, i.e., the years since the accident occurred. In order to determine a reserve, one must forecast the payments that these ongoing claims will induce in future years, i.e., one must extend each triangle to a rectangle by predicting the missing entries.

Several nonparametric approaches are available for developing claims in a run-off triangle, most notably the chain-ladder method. In order to account for the dependence between triangles, multivariate extensions of this technique have been proposed, e.g., by Braun (2004), Pröhl and Schmidt (2005), Schmidt (2006), Merz and Wüthrich (2008), and Zhang (2010). These techniques account for dependence in the computation of reserves and their prediction errors but they do not provide the predictive distribution needed to obtain risk measures such as Value-at-Risk (VaR) or Tail Value-at-Risk (TVaR). Their use in the determination of risk capital is therefore limited.

Parametric approaches leading to the distribution of unpaid losses have been considered, e.g., by Brehm (2002), Shi and Frees (2011), De Jong (2012), Shi et al (2012), Merz et al (2013) and Abdallah et al (2015). Models investigated in these articles incorporate dependence between lines of business and/or within calendar years of a line of business through Gaussian, Archimedean or Hierarchical Archimedean copulas. In these papers, the total reserve estimate in the presence of dependence is not equal to the sum of the marginal reserves estimated assuming independence. This is a by-product of the joint estimation of the marginal and dependence parameters, which relies heavily on the choice of multivariate model for the run-off triangles. An inadequate choice of dependence structure may then have a large, undesirable effect on the estimation of the reserves. This is particularly worrying given that this choice is typically based on very few data points (e.g., 55 observations for 10 accident years and 10 development periods). Tools are thus needed for assessing the dependence between run-off triangles and selecting an appropriate model.

In this paper, we address this inferential issue within the context of a multivariate extension of the pairwise model of Shi and Frees (2011), where the dependence between corresponding cells of different run-off triangles is described by a copula. We propose to use an alternative two-stage inference strategy, in which generalized linear models (GLMs) are first fitted to the margins, thereby fixing the estimates of the reserves. In the second step, standardized residuals from those models are linked through a dependence structure estimated using rank-based methods. This general approach has a long history in the copula modeling literature; see, e.g., Genest and Favre (2007) or Genest et Nešlehová (2012) for reviews. When dealing with identically distributed data, rank-based methods are well-established tools for selecting, estimating and validating copulas. To our knowledge, however, these techniques have never been applied to run-off triangles.

To illustrate the proposed approach, we consider run-off triangles for six portfolios from a large Canadian property and casualty insurance company. These data are described in Section 3.2 and appended. In Section 3.2.1, GLMs with log-normal and Gamma distributions are fitted to the individual portfolios, and the properties of these two parametric families are exploited in Section 3.2.2 to define residuals that are suitable for a dependence analysis through ranks. Two different hierarchical approaches are then explored for modeling the dependence between the lines of business.

In Section 3.3, a nested Archimedean copula model is fitted, along the same lines as Abdallah et al (2015). As this model imposes many constraints on the dependence structure and the choice of copulas, a more flexible approach considered by Arbenz et al (2012) and Côté and Genest (2015) is implemented in Section 3.4. Risk capital calculations and allocations for the two models are compared in Section 3.5, and Section 3.6 summarizes the pros and cons of these approaches. Appendix 3.6 contains density calculations for the nested Archimedean copula model, and the data (up to a multiplicative factor for confidentiality purposes) are provided in Appendix 3.6, along with parameter estimates of the marginal GLMs.

3.2 Data

The run-off triangle data considered in this paper are from a large Canadian property and casualty insurance company. They consist of the cumulative paid losses and net earned premiums for six lines of automobile and home insurance business. Tables 3.13–3.18 in Appendix 3.6 show the paid losses for accident years 2003–12 inclusively for each of the six lines of business developed over at most ten years. To preserve confidentiality, all figures were multiplied by a constant. However, this is inconsequential because in order to account for the volume of business, the analysis focuses on the paid loss ratios, i.e., the payments divided by the net earned premiums.

Table 3.1 gives a descriptive summary of each line of business (LOB). There are five run-off triangles of personal and commercial auto lines with Accident Benefits and Bodily Injury coverages from three regions (Atlantic, Ontario and the West). Atlantic Canada consists of New Brunswick, Nova Scotia, Prince Edward Island and Newfoundland/Labrador; the West comprises Manitoba, Saskatchewan, Alberta, British Columbia, Northwest Territories, Yukon, and Nunavut. Given that Québec has a public plan for this section of auto insurance, business for that province is included only in the sixth triangle, which comprises the company's country-wide Liability personal and commercial home insurance.

Bodily Injury (BI) coverage provides compensation to the insured if the latter is injured or killed through the fault of a motorist who has no insurance, or by an unidentified vehicle. The Accident Benefits (AB) coverage provides compensation, regardless of fault, if a driver, passenger, or pedestrian suffers injury or death in an automobile collision. Disability income is an insurance product that provides supplementary income when the accident results in a disability that prevents the insured from working at his/her regular employment. For this reason, AB disability income is considered separately from other AB. Finally, Liability insurance covers an insured for his/her legal liability for injuries or damage to others.

Table 3.1 – Descriptive summary of six lines of business for a Canadian insurance company.

LOB	Region	Product	Coverage
1	Atlantic	Auto	Bodily Injury
2	Ontario	Auto	Bodily Injury
3	West	Auto	Bodily Injury
4	Ontario	Auto	Accident Benefits excluding Disability Income
5	Ontario	Auto	Accident Benefits: Disability Income only
6	Country-wide	Home	Liability

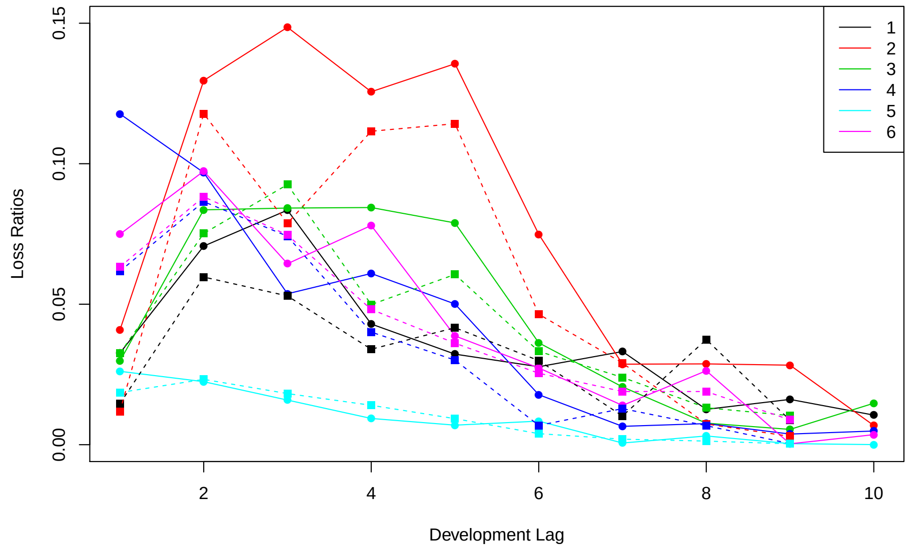

Figure 3.1 – Loss ratios for years 2003 (solid line) and 2004 (dashed line) in function of the development lag for the six lines of business.

3.2.1 Marginal GLMs for Incremental Loss Ratios

For LOB $\ell \in \{1, \ldots, 6\}$, denote by $Y_{i,j}^{(\ell)}$ the incremental payment for the ith accident year and the jth development period, where $i, j \in \{1, \ldots, 10\}$. Given that the earned premiums $p_i^{(\ell)}$ vary with accident year i and line of business ℓ, it is convenient to model the loss ratios, defined by

$$X_{i,j}^{(\ell)} = Y_{i,j}^{(\ell)} / p_i^{(\ell)}.$$

In Figure 3.1, loss ratios $X_{i,j}^{(\ell)}$ for $i = 1, 2$, $j = 1, \ldots, 11 - i$ and $\ell = 1, \ldots, 6$ are shown. It is clear from the graph that the loss ratio depends on the development lag for every portfolio. By comparing the solid and dashed lines of the same color, one can also see that the accident year has an impact. In order to capture these patterns, we consider a regression model with two explanatory variables, i.e., accident year and development period. This is in line with the classical chain-ladder approach.

For LOB $\ell \in \{1, \ldots, 6\}$, let $\kappa_i^{(\ell)}$ be the effect of accident year $i \in \{1, \ldots, 10\}$ and $\lambda_j^{(\ell)}$ be the effect of development period $j \in \{1, \ldots, 10\}$. The systematic component for the ℓth line of business can then be written as

$$\eta_{i,j}^{(\ell)} = \zeta^{(\ell)} + \kappa_i^{(\ell)} + \lambda_j^{(\ell)},$$

where $\zeta^{(\ell)}$ is the intercept, and for parameter identification, we set $\kappa_1^{(\ell)} = \lambda_1^{(\ell)} = 0$. There is no interaction term in this model, i.e., it is assumed that the effect of a given development period does not vary by accident year. While this assumption is hard to check, it is required to ensure that all parameters can be estimated from the 55 observations available.

In their analysis of dependent loss triangles using copulas, Shi and Frees (2011) use the log-normal and Gamma distributions for incremental claims. Their justification applies here as well. Following these authors, we consider the link

$$\mu_{i,j}^{(\ell)} = \eta_{i,j}^{(\ell)}$$

for a log-normal distribution with mean $\mu_{i,j}^{(\ell)}$ and standard deviation $\sigma^{(\ell)}$ on the log scale. For the Gamma distribution, however, we use the exponential link instead of the canonical inverse link in order to enforce positive means. When the Gamma distribution is selected, therefore, its scale and shape parameters are respectively denoted by $\beta_{i,j}^{(\ell)}$ and $\alpha^{(\ell)}$, and it is assumed that

$$\beta_{i,j}^{(\ell)} = \exp(\eta_{i,j}^{(\ell)})/\alpha^{(\ell)}.$$

Log-normal and Gamma distributions were fitted to all lines of business by the method of maximum likelihood. Table 3.2 shows the corresponding values of the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC). These criteria suggest the choice of the log-normal distribution for the first line of business and the Gamma distribution for all others. These choices of models are confirmed by the Kolmogorov–Smirnov goodness-of-fit test, whose p-values are also given in Table 3.2. No model is rejected at the 1% level. Q-Q plots (not shown) of standardized residuals (defined below) provide visual confirmation that the selected models are adequate, although the fit for LOB 6 is borderline.
Parameter estimates of the fitted models are given in Appendix 3.6 along with their standard errors. Using these values, one can estimate the total reserve of the portfolio by

$$\sum_{\ell=1}^{6} \sum_{i=2}^{10} \sum_{j=10-i+2}^{10} p_i^{(\ell)} \mathrm{E}(X_{i,j}^{(\ell)}),$$

where $\mathrm{E}(X_{i,j}^{(\ell)})$ is the projected unpaid loss ratio, and $p_i^{(\ell)}$ is the premiums earned in the corresponding accident year i. For $\ell = 1$, we have

$$\mathrm{E}(X_{i,j}^{(1)}) = \exp\{\hat{\mu}_{i,j}^{(1)} + (\hat{\sigma}^{(1)})^2/2\},$$

Table 3.2 – Fit statistics and goodness-of-fit test of marginals.

LOB	AIC		BIC		p-value of the
	Log-normal	Gamma	Log-normal	Gamma	Kolmogorov–Smirnov test
1	−294	−291	−254	−251	0.886
2	−266	−270	−226	−230	0.643
3	−323	−324	−283	−283	0.397
4	−272	−276	−232	−236	0.135
5	−441	−444	−401	−404	0.478
6	−259	−267	−219	−226	0.019

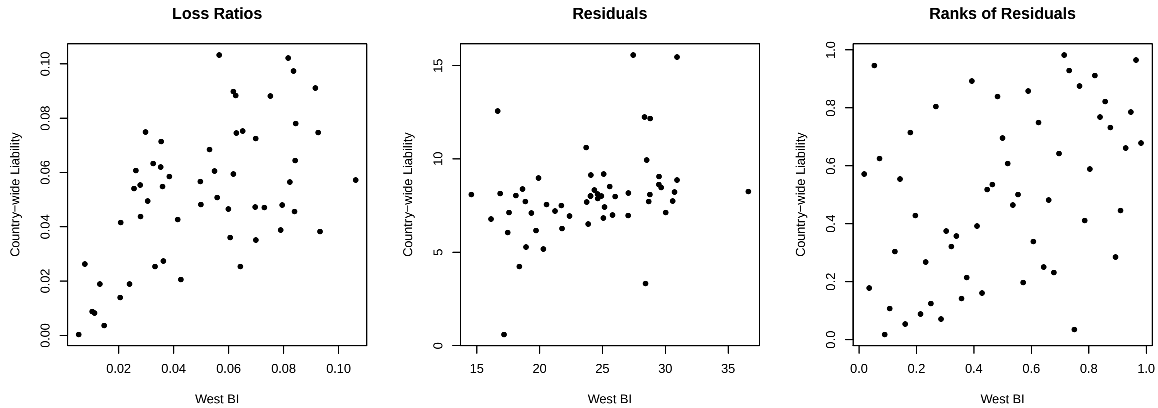

Figure 3.2 – Loss ratios (left), residuals (middle) and standardized ranks of the latter (right) for LOBs 3 and 6.

while for $\ell > 1$, $\mathrm{E}(X_{i,j}^{(\ell)}) = \hat{\beta}_{i,j}^{(\ell)} \hat{\alpha}^{(\ell)}$. The estimated reserves of the six lines of business are given at the bottom of Table 3.19 in Appendix 3.6, along with those derived from the chain-ladder method, which is the industry's benchmark. The two methods lead to similar results and total reserve estimates of \$438,088 and \$453,686, respectively.

3.2.2 Exploratory Dependence Analysis

One would expect intuitively that the AB, BI and Liability claim payments are associated, as these coverages all involve compensation for injuries or damage to the insured or to others. One may also wonder whether there exist interactions between portfolios across regions. In order to account for such dependencies between $d \geq 2$ triangles, Shi and Frees (2011) propose to link the marginal GLMs through a copula. This approach involves expressing the joint distribution of the loss ratios in the form

$$\Pr(X_{i,j}^{(1)} \leq x_{i,j}^{(1)}, \ldots, X_{i,j}^{(d)} \leq x_{i,j}^{(d)}) = C\{\Pr(X_{i,j}^{(1)} \leq x_{i,j}^{(1)}), \ldots, \Pr(X_{i,j}^{(d)} \leq x_{i,j}^{(d)})\},$$

where C is a d-variate cumulative distribution function with uniform margins on $(0, 1)$.
In order to select a copula C that appropriately reflects the dependence in the data, it is best to rely on rank-based techniques as they allow to separate the effect of the marginals from the dependence structure (Genest and Favre, 2007; Genest et Nešlehová, 2012).
To illustrate this point, consider first the graph displayed in the left panel of Figure 3.2, which shows a scatter plot of the pairs $(X_{i,j}^{(3)}, X_{i,j}^{(6)})$ with $i, j \in \{1, \ldots, 10\}$ and $j \leq i$. This graph suggests a strong, positive dependence between BI in Western Canada and country-wide Liability; in particular, the Pearson correlation is 0.56. However, the pattern of points on this graph is induced by the systematic effects of the development lags and accident years. For example, the seven points in the lower left corner of the graph all correspond to development years 7–10. As these effects are already accounted for by the marginal GLMs, this graph is uninformative (not to say misleading) for the selection of C.

To get insight into the dependence structure, it is more relevant to consider the residuals from the GLMs. For LOB 1, (standardized) residuals of the log-normal regression model can be defined, for all $i,j \in \{1,\ldots,10\}$ and $j \leq i$, as

$$\epsilon_{i,j}^{(1)} = \{\ln(X_{i,j}^{(1)}) - \hat{\mu}_{i,j}^{(1)}\}/\hat{\sigma}^{(1)},$$

while for LOB $\ell \in \{2,\ldots,6\}$, the fact that Gamma regression models were used leads to set

$$\epsilon_{i,j}^{(\ell)} = X_{i,j}^{(\ell)}/\hat{\beta}_{i,j}^{(\ell)}.$$

In this fashion, the vectors $(\epsilon_{i,j}^{(1)},\ldots,\epsilon_{i,j}^{(6)})$ with $i,j \in \{1,\ldots,10\}$ and $j \leq i$ form a pseudo-random sample from a distribution with copula C and margins approximately $\mathcal{N}(0,1)$ for $\ell = 1$ and $\mathcal{G}(\hat{\alpha}^{(\ell)},1)$, for $\ell \in \{2,\ldots,6\}$.

As an illustration, the middle panel of Figure 3.2 shows a scatter plot of the pairs $(\epsilon_{i,j}^{(3)},\epsilon_{i,j}^{(6)})$. This graph suggests a form of positive dependence (Pearson's correlation is 0.34), but the message is blurred by the effect of the Gamma marginals. As the goal is to select the copula C, which does not depend on the margins, it is preferable to plot the pairs of normalized ranks, as in the right panel of Figure 3.2. For arbitrary $i,j \in \{1,\ldots,10\}$ and $j \leq i$, the standardized rank of residual $\epsilon_{i,j}^{(\ell)}$ is defined by

$$R_{i,j}^{(\ell)} = \frac{1}{55+1}\sum_{i^*=1}^{10}\sum_{j^*=1}^{11-i^*} \mathbf{1}(\epsilon_{i^*j^*}^{(\ell)} \leq \epsilon_{i,j}^{(\ell)}),$$

where, in general, $\mathbf{1}(A)$ is the indicator function of the set A and the division by 56 rather than 55 is to ensure that all standardized ranks are strictly comprised between 0 and 1.

Let C_n be the empirical distribution function of the vectors $(R_{i,j}^{(1)},\ldots,R_{i,j}^{(d)})$, with $i,j \in \{1,\ldots,10\}$ and $j \leq i$. It can be shown, under suitable conditions on the underlying copula C, that C_n is a consistent estimator thereof. Accordingly, the vectors of standardized ranks, which form the support of C_n, are a reliable tool for copula selection, fitting and validation. In particular, all rank-based tests of bivariate or multivariate independence are based on C_n.

For example, the right panel of Figure 3.2 shows the pairs of standardized ranks associated with the residuals from the West BI and the country-wide Liability coverages. One can see from this graph that there is a residual dependence between these two portfolios. In particular, the correlation between these pairs is 0.40; this rank-based correlation is a consistent estimate of Spearman's ρ. Alternative copula-based measures of association between two variables are Kendall's τ and van der Waerden's coefficient Υ. Thus one can test the null hypothesis of bivariate independence by checking whether the empirical values of these coefficients are significantly different from 0; see, e.g., Hollander et al (2014). Table 3.3 gives estimates of ρ, τ and Υ for the pair $(\epsilon^{(3)},\epsilon^{(6)})$, along with the p-values of the corresponding tests; the null hypothesis of independence is rejected at the 1% level in all cases.

Table 3.3 – Nonparametric tests of independence.

Kendall's test		Spearman's test		Van der Waerden test	
$\hat{\tau}$	p-value	$\hat{\rho}$	p-value	$\hat{\Upsilon}$	p-value
0.29	0.0021	0.40	0.0023	18.27	0.0055

The null hypothesis of multivariate independence between the six LOBs can also be assessed globally using rank tests based on d-variate generalizations of ρ, τ or Υ. In particular, the d-variate version of Kendall's τ is given, e.g., in Genest et al (2011), by

$$\tau_{d,n} = \frac{1}{2^{d-1}-1}\left\{-1 + \frac{2^d}{n(n-1)} \sum_{(i,j)\neq(i^*,j^*)} \mathbf{1}\left(\varepsilon^{(1)}_{i^*j^*} \leq \varepsilon^{(1)}_{i,j}, \ldots, \varepsilon^{(6)}_{i^*j^*} \leq \varepsilon^{(6)}_{i,j}\right)\right\} = 0.035.$$

Under the hypothesis of multivariate independence, $\tau_{d,n}$ has mean 0, finite sample variance

$$var(\tau_{d,n}) = \frac{n(2^{2d+1} + 2^{d+1} - 4 \times 3^d) + 3^d(2^d + 6) - 2^{d+2}(2^d + 1)}{3^d(2^{d-1} - 1)^2 n(n-1)} = 1.59 \times 10^{-4},$$

and its distribution is asymptotically Gaussian. The approximate p-value of the test is 0.53%, suggesting that the residuals are dependent. The most dependent pairs of variables can be identified from Table 3.4, where all values of $\tau_{2,n}$ are displayed. Values shown in bold are those that would be significantly different from 0 at the 5% level in a single pairwise test. Although this level must be interpreted with care due to the multiple comparison issue, the two largest values in Table 3.4 are still significantly different from 0 at the global 5% level even when the very conservative Bonferroni correction is applied.

Given the presence of dependence, the challenge is then to select a copula that best reflects the association between the variables. Many parametric families of copulas are available; see, e.g., McNeil *et al.* (2015) or Nelsen (2006) for the definition and properties of the Clayton, Frank, Plackett and t copula families used subsequently. Given a class $\mathcal{C} = \{C_\theta : \theta \in \Theta\}$ of d-dimensional copulas, a rank-based estimate $\hat{\theta}$ of the dependence parameter θ can be obtained

Table 3.4 – Empirical values of Kendall's τ for all pairs in the portfolio.

	$\epsilon^{(1)}$	$\epsilon^{(2)}$	$\epsilon^{(3)}$	$\epsilon^{(4)}$	$\epsilon^{(5)}$	$\epsilon^{(6)}$
$\epsilon^{(1)}$	1.000	0.115	0.024	−0.061	0.014	0.076
$\epsilon^{(2)}$	0.115	1.000	**−0.331**	**0.244**	**0.209**	−0.090
$\epsilon^{(3)}$	0.024	**−0.331**	1.000	0.040	−0.079	**0.285**
$\epsilon^{(4)}$	−0.061	**0.244**	0.040	1.000	**0.200**	0.030
$\epsilon^{(5)}$	0.014	**0.209**	−0.079	**0.200**	1.000	0.046
$\epsilon^{(6)}$	0.076	−0.090	**0.285**	0.030	0.046	1.000

Table 3.5 – Parameter estimates and goodness-of-fit test p-value.

Copula	Parameter	Standard Deviation	p-value
Clayton	0.584	0.194	0.0804
Frank	2.804	0.836	0.7557
Plackett	3.777	1.426	0.7747
t_2	0.375	0.155	0.2323

from loss-triangle data by maximizing the pseudo log-likelihood

$$\mathcal{L}(\theta) = \sum_{i=1}^{10} \sum_{j=1}^{11-i} \ln\{c_\theta(R_{i,j}^{(1)}, \ldots, R_{i,j}^{(d)})\},$$

where c_θ is the density of C_θ. The consistency and asymptotic normality of estimators of this type was established by Genest et al (1995) under broad regularity conditions. The adequacy of the class $\mathcal{C}$ can then be tested using the Cramér–von Mises statistic defined by

$$S_n = \int_{[0,1]^d} \{C_n(u_1, \ldots, u_d) - C_{\hat{\theta}}(u_1, \ldots, u_d)\}^2 \underset{\cdot}{u}_1 \cdots \underset{\cdot}{u}_d.$$

The p-value of a test of the hypothesis $\mathcal{H}_0 : C \in \mathcal{C}$ based on the statistic S_n can be computed via a parametric bootstrap procedure described in Genest *et al.* (2009). Both the estimation and the goodness-of-fit procedures are available in the `R` package `copula`. For illustration, Table 3.5 shows the parameter estimates, standard deviation and the p-value of the goodness-of-fit test for four copula families fitted to the pairs of residuals $(\epsilon^{(3)}, \epsilon^{(6)})$ from the West BI and country-wide Liability triangles. This suggests that the Clayton copula would be a poor choice for these data; given the small sample size, however, it does not seem possible to discriminate between the other three copula families on the basis of S_n.

This model selection, fitting and validation procedure is standard and straightforward to implement in two dimensions. However, the canonical d-variate generalizations of bivariate copulas typically lack flexibility: either they are exchangeable and/or their lower-dimensional margins are all of the same type. With six lines of business, these assumptions may be too restrictive. As one can see in Figure 3.3, different pairs of residuals exhibit different types of association; this is also confirmed by the values of Kendall's τ reported earlier in Table 3.4. In particular, Ontario LOBs exhibit positive dependence, while the BI coverages for Ontario and the West are negatively associated.

The fact that many variables are positively dependent is due in part to exogenous common factors such as inflation and interest rates. Furthermore, strategic decisions can impact several portfolios, e.g., the acceleration of payments on all lines of the liability insurance sector could induce some dependence between West BI and country-wide Liability. At a more basic level, the positive association between Ontario AB and BI can be explained by the fact that the same accident will often arise in both coverages. Finally, jurisprudence can play a role. For

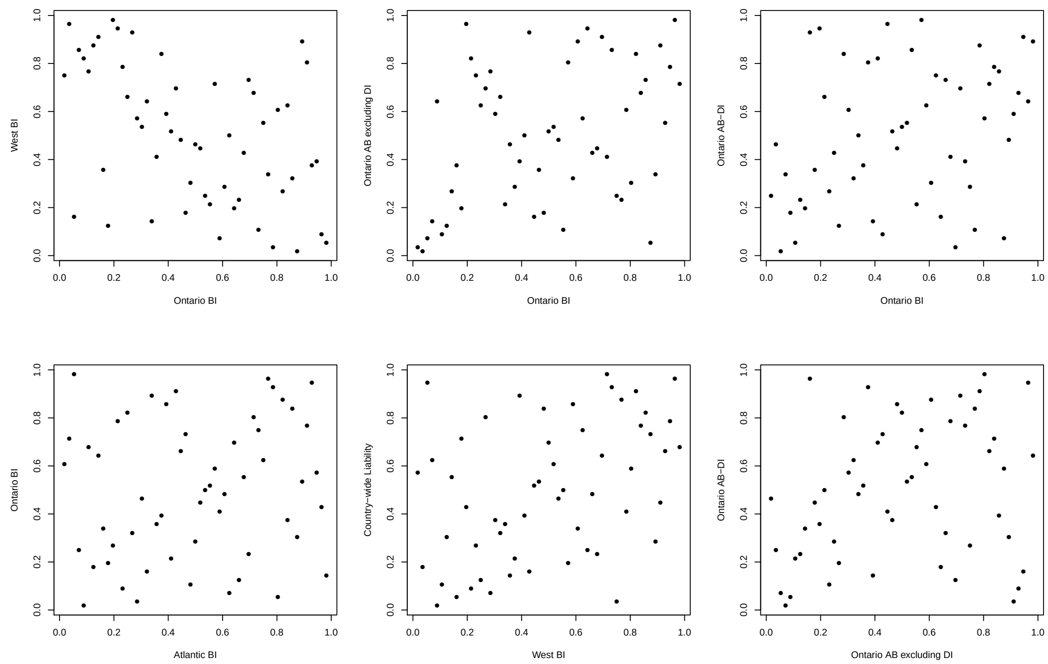

Figure 3.3 – Scatter plot of residuals between different LOBs.

example, reforms were engaged in the Atlantic region to control BI costs; this may explain why LOB 1 is seemingly independent from all other lines of business.

3.3 Nested Archimedean Copula Model

Nesting Archimedean copulas is a popular way of constructing non-exchangeable multivariate dependence models. This approach, originally proposed by Joe (1997), was further investigated, e.g., by Embrechts et al (2003), Whelan (2004) and Savu and Trede (2010). In the reserving literature, Abdallah et al (2015) exploited nested Archimedean copulas to model the dependence between two run-off triangles. In what follows, this approach is extended to higher dimensions using a specific structure called fully nested Archimedean copulas.

Following Genest and MacKay (1986) or Nelsen (2006), a bivariate copula is said to be Archimedean with generator $\varphi_1 : (0, 1] \to [0, \infty)$ if it can be expressed, for all $(u_1, u_2) \in (0, 1)^2$, in the form

$$C_1(u_1, u_2) = \varphi_1^{-1}\{\varphi_1(u_1) + \varphi_1(u_2)\},$$

where φ_1 is convex, decreasing and such that $\varphi_1(1) = 0$. More generally, a $(d + 1)$-variate copula C_d is said to be a fully nested Archimedean copula with generators $\varphi_1, \ldots, \varphi_d$ if it is defined recursively for all $(u_1, \ldots, u_{d+1}) \in (0,1)^{d+1}$, by

$$\begin{aligned}
C_2(u_1, u_2, u_3) &= \varphi_2^{-1}[\varphi_2(u_3) + \varphi_2\{C_1(u_1, u_2)\}], \\
\vdots &= \vdots \\
C_d(u_1, \ldots, u_{d+1}) &= \varphi_d^{-1}[\varphi_d(u_{d+1}) + \varphi_d\{C_{d-1}(u_1, \ldots, u_d)\}].
\end{aligned}$$

As shown by McNeil (2008), C_d is a copula when the following conditions hold:

(i) $\varphi_1^{-1}, \ldots, \varphi_d^{-1}$ are completely monotone, i.e., Laplace transforms;

(ii) $\varphi_{k+1} \circ \varphi_k^{-1}$ has completely monotone derivatives for all $k \in \{1, \ldots, d-1\}$.

This model is such that if $(U_1, \ldots, U_{d+1})$ is distributed as C_d, the copula linking variables U_j and U_k is Archimedean with generator φ_{k-1} for all $j < k$. Because of condition (ii), one must also have

$$\tau(U_k, U_\ell) \leq \tau(U_i, U_j), \quad i < j < \ell, \quad k < \ell. \tag{3.1}$$

Algorithms for generating data from C_d were given by McNeil (2008) and Hofert (2011). Hofert and Mächler (2011) also wrote the R package `nacopula` (now merged into `copula`) that can be used to simulate from fully nested Archimedean copulas in any dimension.

Figure 3.4 depicts the fully nested Archimedean structure used to model the dependence between the residuals of the six lines of business. In this structure, copula C_1 links the two components of the Ontario AB coverage. Their dependence with Ontario BI coverage is then incorporated at level 2. The West BI and the country-wide Liability coverages are then included at levels 3 and 4, respectively. Anti-ranks (i.e., the ranks of the negative residuals) had to be used at levels 3 and 4, because of the constraints imposed by (3.1) and the fact that the residuals for LOB 3 are negatively associated with LOB 2 and positively associated with LOB 6. Finally, the Atlantic BI coverage was included at the last step given its apparent lack of dependence with the other lines of business. This overall structure is in accordance with ratemaking practices, as the rating is typically performed on a territorial basis. One may thus expect the dependence between lines of business to be larger when they are from the same region than when they are not.

In what follows, it is assumed that for each $k \in \{1, \ldots, 5\}$ and all $t \in (0,1)$,

$$\varphi_k(t) = -\ln\left(\frac{e^{-t\theta_k} - 1}{e^{-\theta_k} - 1}\right)$$

for some $\theta_k \in \mathbb{R}$. In other words, the nested copulas are taken to be from the Frank family, which spans all degrees of dependence between -1 and 1, as measured by Kendall's τ. A rank-based estimate $\hat{\boldsymbol{\theta}}$ of the vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_5)$ characterizing the dependence structure is then obtained by maximizing the pseudo-likelihood function

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{i=1}^{10} \sum_{j=1}^{11-i} \ln\{c(R_{i,j}^{(4)}, R_{i,j}^{(5)}, R_{i,j}^{(2)}, 1 - R_{i,j}^{(3)}, 1 - R_{i,j}^{(6)}, R_{i,j}^{(1)}; \boldsymbol{\theta})\},$$

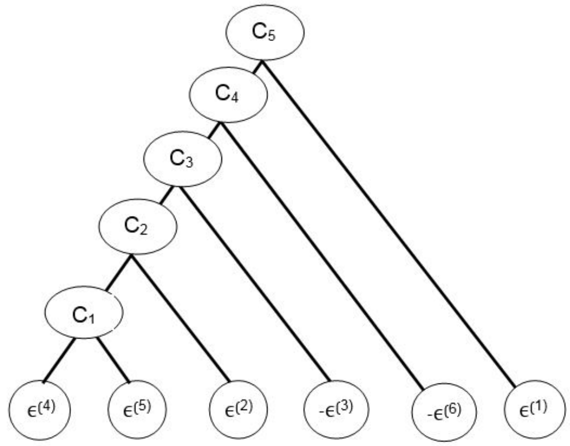

Figure 3.4 – Tree structure for the fully nested Archimedean copula model.

where c is the density of the fully nested Archimedean copula. As shown in Appendix 3.6, the evaluation of this density is straightforward but computationally intensive in high dimensions. Therefore, due to evidence that residuals for LOB 1 are independent from residuals for other LOBs, θ_5 was set equal to 0.

The maximization of the pseudo-likelihood for the model with four levels leads to the parameter estimate $\hat{\boldsymbol{\theta}} = (2.693, 2.354, 1.782, 0.867)$. However, a 95% confidence interval for θ_4 based on 1000 bootstrap replicates includes 0, which corresponds to independence in the Frank copula family. Accordingly, the dependence is significant only in the first three levels of the hierarchy. The parameters of the reduced model with $\theta_4 = \theta_5 = 0$ were estimated once again by the maximum pseudo-likelihood method. This led to $\hat{\boldsymbol{\theta}} = (2.577, 2.233, 1.776)$, whose components are all significantly different from 0.

Figure 3.5 shows the approximate distribution of $\hat{\theta}_3$ (left), $\hat{\theta}_2$ (middle), and $\hat{\theta}_1$ (right) based on 10,000 bootstrap replicates. In that figure, the dashed blue lines represent 95% confidence intervals for the parameters, none of which includes 0. There are hints in the figure that the distribution of the estimators (especially $\hat{\theta}_1$) may not be Normal. This is likely due to the constraint $\theta_3 \leq \theta_2 \leq \theta_1$. In the bottom row of Figure 3.5, one can observe that parameters on the boundary of their domain are relatively frequent: $\hat{\theta}_1 = \hat{\theta}_2$ in 14.3% of the replicates, $\hat{\theta}_3 = \hat{\theta}_2$ in 9.9% of the replicates, and $\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}_3$ in 4.8% of the replicates.

To check for model adequacy, a random sample of size 500 from the fitted model was generated. A test of the hypothesis that the underlying copula of this sample is the same as that of the original data was then carried out using the rank-based procedure of Rémillard and Scaillet (2009). The test statistic was computed with the `R` package `TwoCop` and led to an approximate p-value of 31%, suggesting that the fit is not inadequate.

As an additional informal check, random samples of size 55 were drawn from the fitted 6-dimensional copula and compared visually to the empirical copula by looking at rank plots of

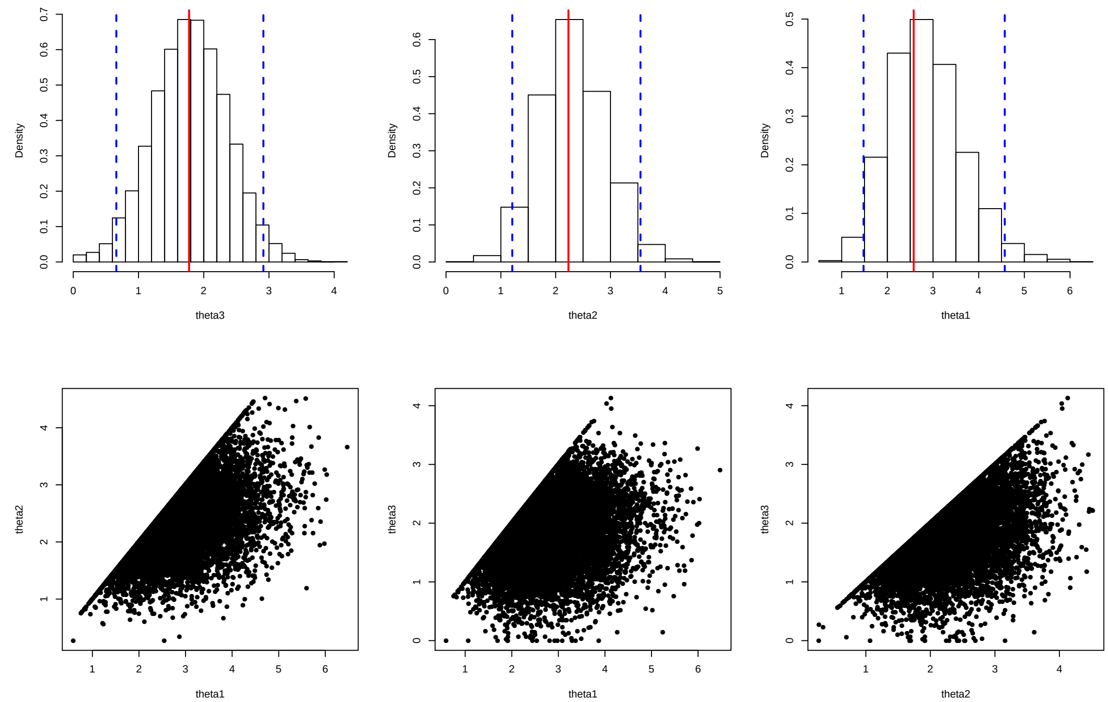

Figure 3.5 – Fully nested Archimedean copula model: histograms of bootstrap parameters with 95% confidence interval (top row) and scatter plots of bootstrap replications (bottom row).

selected pairs. Figure 3.6 shows one result from such a comparison of pairs (LOB 2, LOB ℓ) with $\ell \in \{3, 4, 5\}$ and (LOB 3, LOB 4). The rank plots derived from the residuals are in the top row, and those corresponding to the random sample are in the bottom row. The positive dependence between Ontario risks seems to be accurately captured by the model. Although the negative association between LOBs 2 and 3 is taken into account, one can see in the second column of Figure 3.6 that negative dependence is induced between LOBs 3 and 4. This is an artifact of the dependence structure, which assumes from the start that the pairs $(-3, \ell)$, with $\ell \in \{2, 4, 5\}$ have the same degree of association. Table 3.4 suggests that this is not the case. This issue could have been avoided by grouping LOBs 2 and 3 earlier in the structure, but at the expense of the overall fit of the model. A more flexible modeling approach is presented below.

3.4 Copula-Based Risk Aggregation Model

In this section, a hierarchical approach to loss triangle modeling is considered. It appears to have been originally proposed by Swiss reinsurance practitioners (Bürgi *et al.*, 2008; SCOR, 2008) but was formalized by Arbenz et al (2012). Estimation and validation procedures for this

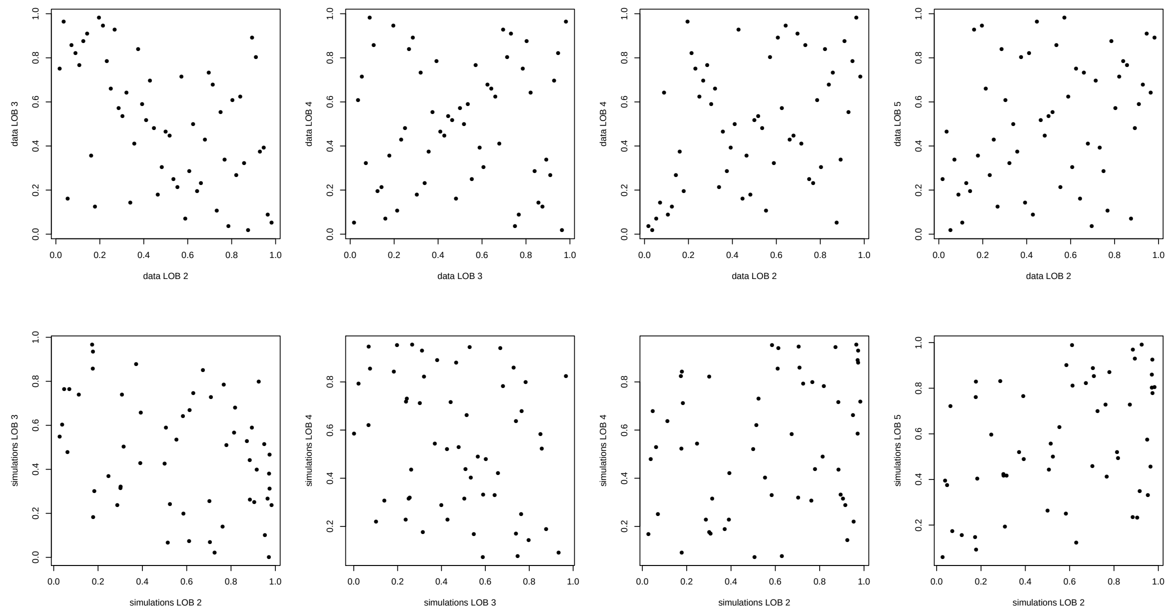

Figure 3.6 – Adequacy check for the fully nested Archimedean copula model: ranks of pairs of residuals (top row) and pairs of simulations from the model (bottom row).

class of models are described in Côté (2014) and Côté and Genest (2015), where rank-based clustering techniques are also proposed for selecting an appropriate structure.

The model is defined using a tree comprising $d-1$ nodes, each of which has two branches. An example of such a structure is shown in the left panel of Figure 3.7. At each node, a copula describes the dependence between the two components which are then summed and viewed as a single risk in higher levels of the hierarchy. For example, $C_{4,5}$ denotes the copula linking $\epsilon^{(4)}$ and $\epsilon^{(5)}$ and $S_{4,5} = \epsilon^{(4)} + \epsilon^{(5)}$, while $C_{2,\ldots,6}$ is the copula linking aggregated risks $S_{2,3,6}$ and $S_{4,5}$.

A joint distribution for the d variables is then defined in terms of $d-1$ bivariate copulas and d marginal distributions under a conditional independence assumption. This assumption, which is reasonable in the present context, states that conditional on a sum at a given node, the descendents of that node are independent of the non-descendents. For additional details, see Arbenz et al (2012) and Côté and Genest (2015).

This strategy is simple to implement, as it builds on tools already available for bivariate copula selection, inference, and validation. Furthermore, the $d-1$ copulas in the model can be chosen freely, thereby providing great flexibility in the dependence structure. Moreover, hierarchical clustering techniques can be adapted to obtain an appropriate tree structure.

As explained in Côté and Genest (2015), it is appealing to model first the risks that are the most dependent in some sense. In this paper, the distance based on Kendall's τ,

$$\Delta(\epsilon^{(\ell)}, \epsilon^{(k)}) = \sqrt{1 - \tau^2(\epsilon^{(\ell)}, \epsilon^{(k)})},$$

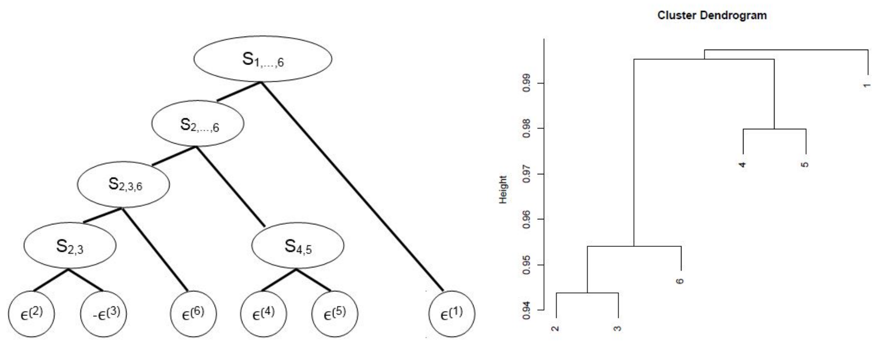

Figure 3.7 – Illustration of the tree structure and dendrogram for the copula-based aggregation model.

is maximized at each step to obtain the dendrogram displayed in the right panel of Figure 3.7. Risks 2 and 3 are grouped in the first step. Given that they are negatively associated, it was deemed preferable to work with $-\epsilon^{(3)}$ as was done in the previous section.

Before selecting appropriate copulas for each aggregation step, Kendall and van der Waerden tests of independence were performed to see if the dependence is significant. The resulting p-values are shown in Table 3.6, where one can see that independence is rejected for the first four aggregation steps, but not at the last one. This is not surprising as the preliminary analysis of the data already suggested that the Atlantic BI line of business is not related to the others. Unlike the nested Archimedean copula model, the risk aggregation model captures the existing dependence between West BI and country-wide Liability lines, and includes the latter in the dependence analysis.

Given that the independence hypothesis cannot be rejected at the last node, there are only four copulas to be fitted, namely $C_{2,3}$, $C_{2,3,6}$, $C_{4,5}$ and $C_{2,\ldots,6}$. Based on rank plots, tests of extremeness from Ben Ghorbal *et al.* (2009) and goodness-of-fit tests based on the Cramér–von Mises distance S_n, parametric families of bivariate copulas were selected and fitted by maximum pseudo-likelihood. The final choices are summarized in Table 3.7.

The model validation technique described in Côté and Genest (2015) was used. It relies on a

Table 3.6 – Results of tests of independence at each aggregation step.

Variables		τ	p-value	
			Van der Waerden test	Kendall test
$\epsilon^{(2)}$	$-\epsilon^{(3)}$	0.331	0.0004	0.0004
$S_{2,3}$	$\epsilon^{(6)}$	0.300	0.0020	0.0012
$\epsilon^{(4)}$	$\epsilon^{(5)}$	0.200	0.0541	0.0311
$S_{2,3,6}$	$S_{4,5}$	0.098	0.0406	0.2925
$S_{2,\ldots,6}$	$\epsilon^{(1)}$	0.075	0.3401	0.4204

Table 3.7 – Copula family and parameter estimates.

Step	Copula	Parameter	SD	Kendall's τ	p-value GoF test
$C_{2,3}$	Plackett	5.349	2.021	0.36	0.523
$C_{2,3,6}$	Frank	2.864	0.986	0.29	0.714
$C_{4,5}$	Clayton	0.548	0.215	0.22	0.147
$C_{2,\ldots,6}$	t_2	0.162	0.180	0.10	0.358

simulation algorithm proposed by Arbenz et al (2012) and validated by Mainik (2015). Based on a random sample of size 500 from the model, the test of Rémillard and Scaillet (2009) led to an approximate p-value of 52%. Therefore, the null hypothesis that both samples are coming from the same copula cannot be rejected. This suggests that the selected hierarchical model is appropriate, and that the conditional independence assumption is reasonable. A visual check of the latter assumption confirms this finding.

Looking at Figure 3.8, one can see that the pitfalls of the nested Archimedean copula model have been addressed: there is no negative dependence between LOBs 3 and 4, and the model induces positive dependence between LOBs 3 and 6. However, the extent of the association between Ontario AB and BI risks is not portrayed as vividly in the aggregation model as it was in the nested Archimedean copula model. Over all, the risk aggregation model provides a faithful description of the data.

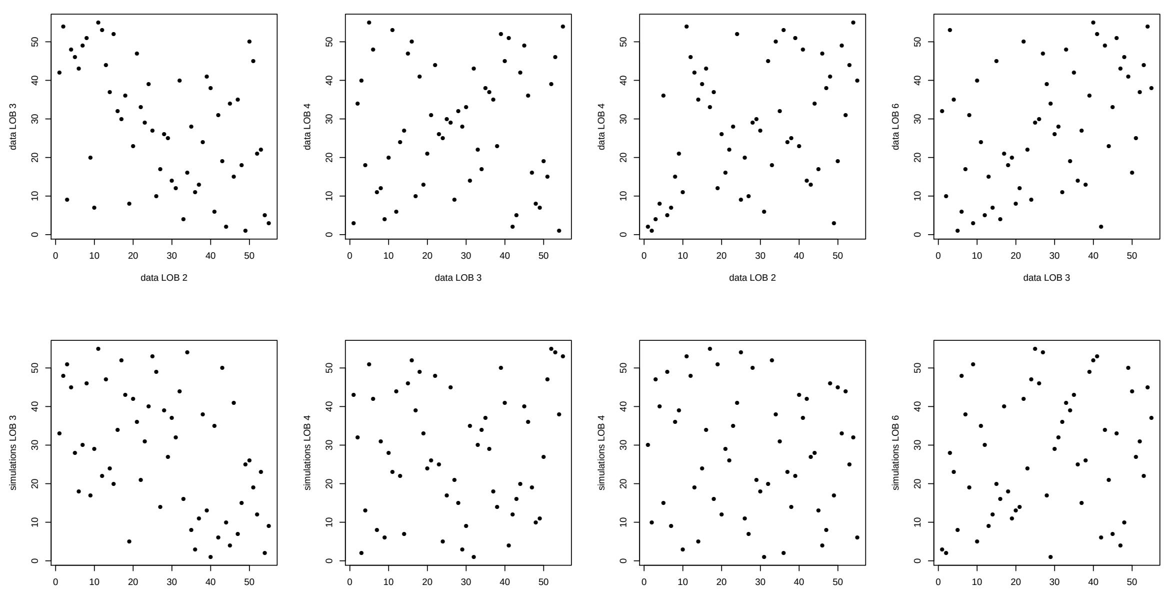

Figure 3.8 – Adequacy check for the copula-based risk aggregation model: ranks of pairs of residuals (top row) and pairs of simulations from the model (bottom row).

Note that if desired, a modification of the tree structure would make it possible to account for the dependence between LOB 2 and the pair (LOB 4, LOB 5). In that case, however, the negative dependence between LOBs 2 and 3 would be masked.

3.5 Predictive Distribution and Risk Capital

The goal of loss triangle modeling is to forecast the unpaid loss by completing the triangle into a rectangle. Insurance companies are interested in the expected unpaid loss — the reserve — but also in its standard deviation, and other risk measures defined in terms of a risk tolerance $\kappa \in (0,1)$ such as the Value-at-Risk (VaR) and the Tail Value-at-Risk (TVaR). In principle, these various measures could all be computed for the nested Archimedean copula model (Model I) and the risk aggregation model (Model II), given that they both specify a distribution for the total unpaid claims. As these distributions cannot be obtained explicitly through a convolution, however, all risk measures must be estimated by simulation. To obtain one realization of the total unpaid loss, one can proceed as follows.

Simulation Procedure

1. Simulate 45 observations from the dependence model.
2. Transform these observations into loss ratios $X_{i,j}^{(\ell)}$ for each LOB $\ell \in \{1, \ldots, 6\}$, development year $j \in \{2, \ldots, 10\}$ and accident year $i \in \{12-j, \ldots, 10\}$ by using appropriate inverse probability transforms.
3. For each LOB $\ell \in \{1, \ldots, 6\}$, compute the simulated unpaid loss

$$X^{(\ell)} = \sum_{i=2}^{10} \sum_{j=12-i}^{10} p_i^{(\ell)} X_{i,j}^{(\ell)}$$

   as well as the total unpaid loss $S = X^{(1)} + \cdots + X^{(6)}$.

Consistent estimates of the risk measures can be derived easily from n independent copies of the unpaid loss $S_1, \ldots, S_n$. Let F_n be the corresponding empirical distribution function. Then

$$\widehat{\mathrm{V}aR}_\kappa(S) = \inf\{s \in \mathbb{R} | F_n(s) \geq \kappa\} = s_\kappa$$

and

$$\widehat{\mathrm{T}VaR}_\kappa(S) = \frac{1}{1-\kappa}\left[\frac{1}{n}\sum_{j=1}^{n} S_j \mathbf{1}(S_j > s_\kappa) + s_\kappa\{F_n(s_\kappa) - \kappa\}\right].$$

Table 3.8 shows risk measures for the total unpaid loss based on 500,000 simulations for Models I and II. Given the GLMs fitted to the marginal distributions, one would expect an average total unpaid loss of $438,088; the small discrepancy between this value and the approximations is due to simulation. The risk measures are all smaller for Model I than for Model II. This is slightly surprising because Model II takes into account the negative dependence between LOBs 2 and 3; intuitively, one would thus expect more risk diversification under Model II than under Model I. Nevertheless, Model II is more conservative than Model I in the sense that it does not assume that LOB 6 is independent from the other lines of business. In addition, Model II is based in part on Plackett and t_2 copulas, which exhibit tail dependence, whereas members of Frank's copula family in Model I do not.

Insurance companies also have to determine capital allocations, i.e., the share of the risk capital to be allocated to each LOB. This exercise helps to identity the most and least profitable sectors of activities in a company. Capital allocation principles have first been introduced by Tasche (1999); see Bargès *et al.* (2009) for a review. Here, TVaR-based capital allocations are used. If

$$X^{(\ell)} = \sum_{i=2}^{10} \sum_{j=12-i}^{10} p_i^{(\ell)} X_{i,j}^{(\ell)}$$

is the unpaid loss for LOB ℓ, the capital allocated to that LOB is

$$\mathrm{T}VaR_\kappa(X^{(\ell)}; S) = \frac{\mathrm{E}[X^{(\ell)}\mathbf{1}\{S > \mathrm{V}aR_\kappa(S)\}] + \beta_\kappa \, \mathrm{E}[X^{(\ell)}\mathbf{1}\{S = \mathrm{V}aR_\kappa(S)\}]}{1-\kappa},$$

where $\beta_\kappa = [F_S\{\mathrm{V}aR_\kappa(S)\} - \kappa] / \Pr\{S = \mathrm{V}aR_\kappa(S)\}$ if the denominator is strictly positive and 0 otherwise. This quantity can be estimated by

$$\widehat{\mathrm{T}VaR}_\kappa(X^{(\ell)}; S) = \frac{1}{n(1-\kappa)} \left\{ \sum_{j=1}^{n} X_j^{(\ell)} \mathbf{1}(S_j > s_\kappa) + \frac{F_n(s_\kappa) - \kappa}{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \mathbf{1}(S_k = s_\kappa)} \sum_{j=1}^{n} X_j^{(\ell)} \mathbf{1}(S_j = s_\kappa) \right\},$$

where $X_1^{(\ell)}, \ldots, X_n^{(\ell)}$ are the n realizations of $X^{(\ell)}$ corresponding to the realizations $S_1, \ldots, S_n$. In Table 3.9, TVaR-based capital allocations are shown for both models as well as for the "Silo" method, which is widespread in industry (Ajne, 1994). It is clear that the Silo method overestimates the total capital required as it implicitly assumes that risks are comonotonic,

Table 3.8 – Risk measures for 500,000 simulations.

Model	Average	St. Dev.	$\mathrm{V}aR_{95\%}$	$\mathrm{V}aR_{99\%}$	$\mathrm{T}VaR_{99\%}$
I	\$438,115	\$13,706	\$460,938	\$470,750	\$475,697
II	\$438,101	\$13,808	\$461,179	\$471,486	\$476,763

Table 3.9 – Risk capital allocation for 500,000 simulations.

	$\mathrm{T}VaR_{99\%}$ - based capital allocations						
Model	LOB 1	LOB 2	LOB 3	LOB 4	LOB 5	LOB 6	Total
Silo	\$42,510	\$157,764	\$87,141	\$90,237	\$22,027	\$118,807	\$518,485
I	\$37,006	\$151,247	\$82,578	\$74,320	\$18,639	\$111,907	\$475,697
II	\$36,891	\$147,418	\$79,719	\$81,928	\$19,285	\$111,521	\$476,763

thereby preventing any form of diversification. The results for Models I and II are similar. While the capital allocations for LOBs 4 and 5 are higher in Model II than in Model I, they are lower for LOBs 2 and 3, outlining the additional risk diversification that is possible in the presence of negative dependence.

The risk measures in Tables 3.8 and 3.9 could be used to set internal capital targets, but they do not incorporate parameter uncertainty, as the model is assumed to be correct. However, a parametric bootstrap can be used in order to quantify estimation error and to tackle potential model over-fitting; see, e.g., Taylor and McGuire (2007) or Shi and Frees (2011). For the present purpose, it was assumed that the tree structure, the copula families, and the marginal distributions are given, except for their parameter values. The following procedure was then repeated a large number of times (10,000 here) in order to obtain the approximate distribution of the unpaid loss, including parameter uncertainty.

Parametric Bootstrap Procedure

1. Simulate 55 observations from the dependence model, and transform them into observations of the loss ratios for the top triangle, i.e., all accident years $i \in \{1, \ldots, 10\}$ and development years $j \in \{1, \ldots, 11 - i\}$, using the inverse marginal distributions.
2. Fit the marginal GLMs (log-normal for LOB 1 and Gamma for LOBs 2–6).
3. Compute the residuals from the GLMs.
4. Fit the copula model to the ranks of the residuals obtained.
5. From this new model, simulate the total unpaid loss using the steps described under "Simulation Procedure". The aggregate value is the simulated total unpaid loss.

The results for the nested Archimedean copula model should be interpreted with caution, however, because the constraints on the dependence parameters in this model, and notably the fact that $\hat{\theta}_2$ is close to $\hat{\theta}_1$, may invalidate the parametric bootstrap (Andrews, 2000).
Tables 3.10 and 3.11 show risk measures and capital allocations obtained with 10,000 bootstrap simulations, while Figure 3.9 shows the predictive distribution obtained for Model I (left) and Model II (right). The risks measures in Table 3.10 are similar for both models and are much higher than those reported in Table 3.8; this highlights the importance of incorporating parameter uncertainty. Unsurprisingly, most of the increase in risk measures when including parameter uncertainty is due to the $6 \times 20 = 120$ marginal GLM parameters. Table 3.12 shows the risk measures obtained with the parametric bootstrap procedure without Step 4, i.e., the dependence parameters are fixed to their initial value estimated with the original data. The resulting risk measures are close to those found in Table 3.10, even though the uncertainty in the copula parameters is not accounted for when Step 4 is omitted.
Finally, the figures in Table 3.11 are in line with those of Table 3.9. In particular, observe that Model II allocates less capital to LOB 6 than Model I, reflecting the fact that LOB 6

Table 3.10 – Risk measures for 10,000 bootstrap simulations.

Model	Average	St. Dev.	$\mathrm{V}aR_{95\%}$	$\mathrm{V}aR_{99\%}$	$\mathrm{T}VaR_{99\%}$
I	\$443,041	\$31,291	\$496,780	\$521,293	\$539,205
II	\$442,957	\$31,038	\$496,470	\$522,417	\$535,536

Table 3.11 – Risk capital allocation for 10,000 bootstrap simulations.

	$\mathrm{T}VaR_{99\%}$ - based capital allocations						
Model	LOB 1	LOB 2	LOB 3	LOB 4	LOB 5	LOB 6	Total
Silo	\$60,740	\$189,466	\$103,465	\$111,946	\$26,637	\$157,345	\$649,599
I	\$40,519	\$167,492	\$90,228	\$75,015	\$18,565	\$147,386	\$539,205
II	\$41,919	\$158,306	\$83,978	\$88,665	\$20,858	\$141,810	\$535,536

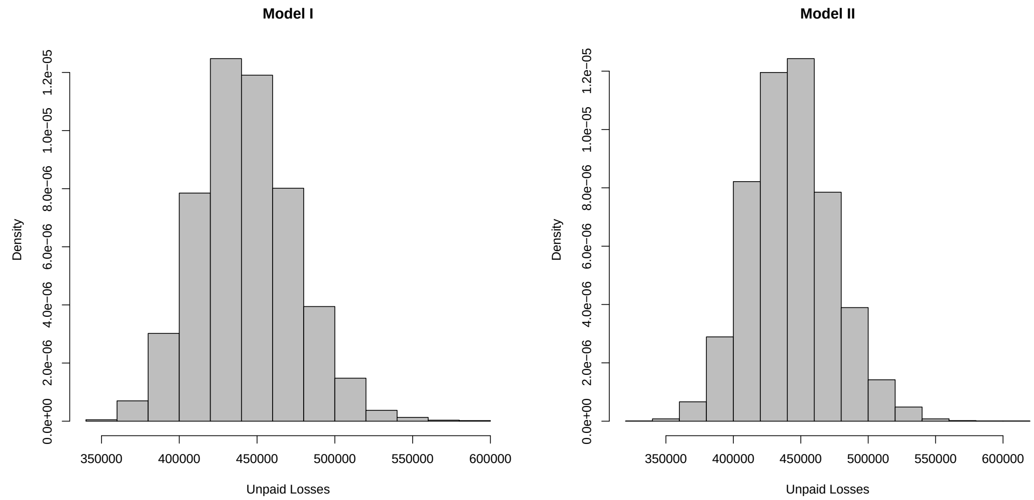

Figure 3.9 – Predictive distributions based on 10,000 bootstrap replicates.

is related to LOBs 2 and 3 in Model II. In view of these results, the insurer might consider increasing the volume of LOB 3 to take better advantage of risk diversification.

3.6 Summary and Discussion

In this paper, rank-based procedures were introduced for the selection, estimation and validation of dependence structures for run-off triangles of property and casualty insurance claim data. The approach was illustrated using data from six lines of business of a large Canadian insurance company. Two hierarchical approaches were considered for modeling the pairwise dependence between different lines of business, i.e., fully nested Archimedean copulas and a copula-based risk aggregation model.

As simple and convenient as the nested Archimedean copula model may seem, its implementation raises more issues than one might anticipate initially. The success of this approach hinges on the choice of hierarchy and Archimedean generators at each of its levels. In principle, different Archimedean generators could be used throughout the structure, but the conditions required to ensure that the construction is valid are not always easy to verify. As there is no selection technique for generators, practitioners typically assume that they are all from the

Table 3.12 – Risk measures for 10,000 bootstrap simulations including uncertainty for marginal parameters only.

Model	Average	St. Dev.	$\mathrm{V}aR_{95\%}$	$\mathrm{V}aR_{99\%}$	$\mathrm{T}VaR_{99\%}$
I	\$443,554	\$31,390	\$496,781	\$522,696	\$535,069
II	\$442,937	\$30,928	\$495,620	\$520,986	\$534,703

same parametric family φ_θ. In the latter case, conditions for the validity of the nested copula typically boil down to the constraint $\theta_1 \geq \cdots \geq \theta_d$; see, e.g., Hofert (2010).

As illustrated in the present paper, the use of the same generator throughout a fully nested Archimedean copula model has strong implications on the dependence structure. In particular, each variable is linked by the same bivariate copula to any variable appearing in a lower level of the hierarchy and, therefore, shares the same dependence characteristics with all of them in terms of symmetry, tail dependence, etc. In addition, the conditions stated in Eq. (3.1) are not only restrictive, but are also problematic for the parametric bootstrap. Indeed, when a bootstrap sample leads to unconstrained estimates $\hat{\theta}_1, \ldots, \hat{\theta}_d$ such that $\hat{\theta}_1 \geq \cdots \geq \hat{\theta}_d$ fails, one or more of the constrained parameter estimates end up being equal to 0. When this happens repeatedly, the dependence between the LOBs is underestimated. Thus, it is still unclear that this model can be used in a parametric bootstrap procedure to obtain the predictive distribution of unpaid losses, due to the optimization problem that is not standard.

Working with the risk aggregation model allows one to avoid most of these issues. The tree structure can be determined using hierarchical clustering and the copulas can be chosen freely at each aggregation step. In addition, standard tools for bivariate copula selection, estimation, and validation are available. Moreover, the application of the parametric bootstrap to this context is standard, as there are no constraints on the parameters. Overall, the model provides greater flexibility and the dependence structure can be considerably more complex than what can be achieved with the nested Archimedean approach. However, the conditional independence assumption must be satisfied (at least approximately) and formal tools for checking this assumption remain to be developed. Another minor irritant is the fact that simulation from this model relies on the Iman–Conover reordering algorithm, which is efficient but not yet included in standard software; in contrast, sampling from the fully nested Archimedean copula is easily done with the R package `copula`.

Perhaps the most significant limitation of the rank-based approach to risk aggregation modeling described here is that it can only be applied to data or residuals that are (at least approximately) identically distributed. Another requirement for this approach to make sense is that the sums that are linked by the copulas have the same number of components. This means that the risk aggregation model cannot be extended easily to include calendar year dependence, as Abdallah et al (2015) did using nested Archimedean copulas. Unfortunately, this approach is not amenable to estimation and validation procedures based on ranks, as there is then only one observation for each copula in the model.

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APPENDIX

Nested Archimedean Copula Density

The 3-dimensional fully nested Archimedean copula is defined, for all $u, v, w \in (0,1)$, by

$$C(u,v,w) = C_{\theta_2}\{w, C_{\theta_1}(u,v)\},$$

where $\theta_1 \geq \theta_2 \geq 0$. To ease notation, let $C_\theta^{(i,j)}(u,v) = \partial^{i+j} C_\theta(u,v)/\partial u^i \partial v^j$ for $i,j \in \{0,1,2\}$. The density of the nested Archimedean copula can be derived easily using the chain rule, viz.

$$\begin{aligned} c(u,v,w) &= \frac{\partial^3}{\partial u \partial v \partial w} C_{\theta_2}\{w, C_{\theta_1}(u,v)\} = \frac{\partial^2}{\partial u \partial v} C_{\theta_2}^{(1,0)}\{w, C_{\theta_1}(u,v)\} \\ &= \frac{\partial}{\partial u}\left[C_{\theta_2}^{(1,1)}\{w, C_{\theta_1}(u,v)\} C_{\theta_1}^{(0,1)}(u,v)\right] \\ &= C_{\theta_2}^{(1,2)}\{w, C_{\theta_1}(u,v)\} C_{\theta_1}^{(1,0)}(u,v) C_{\theta_1}^{(0,1)}(u,v) + C_{\theta_2}^{(1,1)}\{w, C_{\theta_1}(u,v)\} C_{\theta_1}^{(1,1)}(u,v). \end{aligned}$$

This expression is explicit, though it involves partial derivatives. In the case of the Frank family, the expressions required are the copula

$$C_\theta(u,v) = -\frac{1}{\theta} \ln\left\{1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{(e^{-\theta} - 1)}\right\},$$

its density

$$C_\theta^{(1,1)}(u,v) = c_\theta(u,v) = \frac{-\theta e^{-\theta(u+v)}(e^{-\theta} - 1)}{\{(e^{-\theta} - 1) + (e^{-\theta u} - 1)(e^{-\theta v} - 1)\}^2},$$

and the following partial derivatives:

$$\begin{aligned} C_\theta^{(1,0)}(u,v) &= \frac{\partial C_\theta(u,v)}{\partial u} = \frac{e^{-\theta u}(e^{-\theta v} - 1)}{(e^{-\theta} - 1) + (e^{-\theta u} - 1)(e^{-\theta v} - 1)} = C_\theta^{(0,1)}(v,u), \\ C_\theta^{(1,2)}(u,v) &= \frac{\partial c_\theta(u,v)}{\partial v} = \frac{-\theta^2 (e^{-\theta} - 1) e^{-\theta(u+v)}\{(e^{-\theta v} + 1)(e^{-\theta u} - 1) - (e^{-\theta} - 1)\}}{\{(e^{-\theta} - 1) + (e^{-\theta u} - 1)(e^{-\theta v} - 1)\}^3}. \end{aligned}$$

A similar procedure can be used to obtain the copula density in dimensions 4 and 5. The formulas are available from the authors upon request or can be derived through long but routine calculations facilitated by resorting to a symbolic calculator such as `Maple` or `Mathematica`.

Data and Marginals

Tables 3.13–3.18 provide the net earned premiums and the cumulative paid losses for accident years 2003–12 inclusively for each of LOBs 1–6 developed over at most ten years. To preserve confidentiality, all figures were multiplied by a constant.

Table 3.13 – Cumulative paid losses for LOB 1.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	1404	4445	8037	9885	11272	12465	13892	14433	15127	15580	43028
2004	437	2222	3805	4821	6065	6961	7266	8385	8645		29905
2005	408	2170	4369	6995	7996	9450	11104	11569			31780
2006	372	1785	4757	6368	8377	9470	10122				30381
2007	404	1965	3953	6454	7507	8142					28939
2008	355	2069	3661	5161	6121						27844
2009	1316	2955	4839	5896							25812
2010	298	2595	4582								24188
2011	402	2475									23412
2012	553										23993

Table 3.14 – Cumulative paid losses for LOB 2.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	3488	14559	27249	37979	49561	55957	58406	60862	63280	63864	85421
2004	1169	12781	20550	31547	42808	47385	50251	50978	51272		98579
2005	1478	10788	25499	34279	43057	49360	52329	52544			103062
2006	1186	11852	22913	32537	41824	48005	52542				108412
2007	1737	13881	25521	38037	43684	47755					111176
2008	1571	12153	27329	41832	51779						112050
2009	1199	17077	29876	44149							112577
2010	1263	16073	28249								113707
2011	986	10003									126442
2012	683										130484

Table 3.15 – Cumulative paid losses for LOB 3.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	2279	8683	15136	21603	27650	30428	32004	32592	33009	34140	76620
2004	2139	7077	13159	16435	20416	22598	24171	25034	25714		65691
2005	1420	4888	8762	12184	14482	15633	17089	17710			55453
2006	1510	5027	10763	15799	19269	22504	24807				54006
2007	1693	5175	8216	12263	16918	20792					55425
2008	2097	7509	10810	15673	19791						59100
2009	2094	5174	8062	12389							54438
2010	1487	4789	7448								53483
2011	1868	6196									52978
2012	2080										57879

Table 3.16 – Cumulative paid losses for LOB 4.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	13714	24996	31253	38352	44185	46258	47019	47894	48334	48902	116491
2004	6883	16525	24796	29263	32619	33383	34815	35569	35612		111467
2005	7933	22067	32801	38028	44274	44948	46507	46665			107241
2006	7052	18166	25589	31976	36092	38720	39914				105687
2007	10463	23982	31621	36039	38070	41260					105923
2008	9697	28878	41678	47135	50788						111487
2009	11387	37333	48452	55757							113268
2010	12150	32250	40677								121606
2011	5348	14357									110610
2012	4612										104304

Table 3.17 – Cumulative paid losses for LOB 5.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	3043	5656	7505	8593	9403	10380	10450	10812	10856	10860	116491
2004	2070	4662	6690	8253	9286	9724	9942	10086	10121		111467
2005	2001	4825	7344	8918	9824	10274	10934	11155			107241
2006	1833	4953	7737	9524	10986	11267	11579				105687
2007	2217	5570	7898	8885	9424	10402					105923
2008	2076	5681	8577	10237	12934						111487
2009	2025	6225	9027	10945							113268
2010	2024	5888	8196								121606
2011	1311	3780									110610
2012	912										104304

Table 3.18 – Cumulative paid losses for LOB 6.

Accident	Development Lag (in months)										
Year	12	24	36	48	60	72	84	96	108	120	Premiums
2003	4157	9558	13131	17460	19608	21124	21900	23360	23377	23575	55484
2004	4158	9956	14860	18024	20397	22068	23312	24555	25137		65705
2005	3989	10519	15877	20274	23428	26495	30974	31580			73879
2006	4012	10904	16141	19643	21954	26215	28095				91473
2007	4322	10814	16086	20186	24157	27222					87212
2008	6379	14524	19058	24108	28329						89455
2009	5291	14620	20799	25131							90341
2010	4946	12956	18007								89212
2011	5674	15026									91606
2012	5478										99982

Table 3.19 – Parameter and Reserve Estimations.

LOB ℓ		1	2	3	4	5	6
GLM		Log-normal	Gamma	Gamma	Gamma	Gamma	Gamma
$\zeta^{(\ell)}$		−4.031 (0.157)	−3.628 (0.148)	−3.501 (0.098)	−2.365 (0.173)	−4.064 (0.148)	−2.872 (0.167)
Accident Year	2	−0.226 (0.153)	−0.750 (0.151)	0.053 (0.097)	−0.413 (0.174)	−0.121 (0.151)	0.101 (0.177)
	3	0.022 (0.161)	−0.729 (0.160)	−0.156 (0.100)	−0.196 (0.183)	0.171 (0.161)	0.163 (0.177)
	4	−0.028 (0.168)	−0.651 (0.168)	0.239 (0.105)	−0.112 (0.190)	0.129 (0.168)	−0.136 (0.184)
	5	−0.112 (0.177)	−0.741 (0.174)	0.137 (0.110)	−0.095 (0.199)	0.092 (0.173)	−0.024 (0.191)
	6	−0.183 (0.189)	−0.574 (0.185)	0.120 (0.117)	−0.001 (0.210)	0.396 (0.187)	0.095 (0.203)
	7	0.170 (0.205)	−0.574 (0.200)	0.003 (0.127)	0.197 (0.227)	0.254 (0.200)	0.069 (0.219)
	8	0.032 (0.228)	−0.658 (0.220)	−0.160 (0.141)	−0.012 (0.253)	0.055 (0.222)	−0.017 (0.246)
	9	0.131 (0.268)	−1.147 (0.255)	0.169 (0.167)	−0.628 (0.295)	−0.259 (0.260)	0.131 (0.289)
	10	0.261 (0.362)	−1.625 (0.340)	0.175 (0.226)	−0.754 (0.393)	−0.676 (0.348)	−0.032 (0.390)
Dev. Lag	2	1.311 (0.154)	2.061 (0.145)	0.815 (0.096)	0.450 (0.167)	0.419 (0.149)	0.420 (0.167)
	3	1.438 (0.161)	2.065 (0.151)	0.817 (0.101)	−0.055 (0.175)	0.114 (0.155)	0.076 (0.174)
	4	1.150 (0.168)	2.018 (0.158)	0.849 (0.106)	−0.507 (0.183)	−0.358 (0.163)	−0.095 (0.182)
	5	0.874 (0.177)	1.818 (0.166)	0.717 (0.112)	−0.759 (0.193)	−0.582 (0.173)	−0.406 (0.192)
	6	0.636 (0.189)	1.297 (0.176)	0.283 (0.120)	−1.580 (0.207)	−1.154 (0.182)	−0.481 (0.206)
	7	0.392 (0.205)	0.773 (0.193)	−0.115 (0.129)	−1.899 (0.223)	−1.870 (0.201)	−0.757 (0.226)
	8	0.137 (0.228)	−0.493 (0.216)	−1.001 (0.143)	−2.670 (0.250)	−2.103 (0.219)	−1.215 (0.248)
	9	−0.291 (0.268)	−0.429 (0.255)	−1.375 (0.169)	−3.762 (0.298)	−3.849 (0.257)	−2.612 (0.304)
	10	−0.522 (0.362)	−1.358 (0.340)	−0.715 (0.226)	−2.960 (0.393)	−6.248 (0.348)	−2.764 (0.390)
sd or scale		0.326 (0.031)	10.700 (2.009)	24.046 (4.554)	8.038 (1.502)	10.078 (1.891)	8.021 (1.499)
Reserve		36,063	132,919	78,665	73,220	18,290	98,931
C-L Reserve		35,411	146,794	76,500	75,551	18,726	100,704

Chapitre 4

Sarmanov Family of Bivariate Distributions for Multivariate Loss Reserving Analysis

Résumé

Dans ce chapitre, pour modéliser la dépendance entre les triangles de développement, on propose un modèle bayésien, qui capture la dépendance à l'aide de la famille de distributions bivariées Sarmanov à travers des effets aléatoires. La flexibilité de notre modèle nous permet de détecter la dépendance par année calendrier, année d'accident et période de développement. Nous démontrons que les propriétés intéressantes de cette famille de distributions s'avèrent très utiles dans le contexte des réserves. Une illustration empirique sera également présentée avec des implications sur le capital économique.

Abstract

The correlation among multiple lines of business plays a critical role in aggregating claims and thus determining loss reserves for an insurance portfolio. We show that the Sarmanov family of bivariate distributions is a convenient choice to capture the dependencies introduced by various sources, including the common calendar year, accident year and development period effects. The density of the bivariate Sarmanov distributions with different marginals can be expressed as a linear combination of products of independent marginal densities. This pseudo-conjugate property greatly reduces the complexity of posterior computations. In a case study, we analyze an insurance portfolio of personal and commercial auto lines from a major US property-casualty insurer.

4.1 Introduction

Provisions generally represent most of the liabilities of a property/casualty insurance company. It is therefore crucial for a company to estimate its provisions well. With the advent of the new regulatory standards (e.g. Solvency II in Europe and the upcoming ORSA * guidelines in North America), it is now necessary for an insurer to be more accurate and rigorous to settle the amount of provisions for the entire portfolio. This involves taking into account the correlation between the lines of business.

To incorporate dependencies among multiple runoff triangles, the literature can be separated into two different schools of thought.

The first strand of research examines distribution-free methods, where the (conditional) mean squared prediction error can be derived to measure prediction uncertainty. For example, Braun (2004) takes into account the correlations between the segments by introducing a correlation between development factors, while Schmidt (2006) adopts a multivariate approach, by performing a simultaneous study of all segments of the portfolio.

The other approach relies on parametric methods based on distributional families, allowing predictive distribution of unpaid losses, which is believed to be more informative to actuaries in setting a reasonable reserve range than a single mean squared prediction error. We will focus on the parametric approach.

Parametric reserving methods mainly involve copulas to model dependence between lines of business. For example, Brehm (2002) uses a Gaussian copula to model the joint distribution of unpaid losses, while De Jong (2012) models dependence between lines of business with a Gaussian copula correlation matrix. Shi et al (2012) and Wüthrich et al (2013) also use multivariate Gaussian copula, to accommodate correlation due to accounting years within and across runoff triangles. Bootstrapping is another popular parametric approach used to forecast the predictive distribution of unpaid losses for correlated lines of business. Kirschner et al (2008) use a synchronized bootstrap and Taylor and McGuire (2007) extend this result

*. ORSA: Own Risk and Solvency Assessment

to a generalized linear model context. More recently, Abdallah et al (2015) use Hierarchical Archimedean copulas to accommodate correlation within and between runoff triangles.

We use random effects to accommodate correlation due to calendar year, accident year and development period effects within and across runoff triangles. Bayesian methods are not new to the loss reserving literature (see Shi et al (2012) for an excellent review). In this paper, to capture dependence between the lines of business (through random effects), we introduce the Sarmanov Family of bivariate distributions to the reserving literature. This family of bivariate distributions was first presented in Sarmanov (1966) and appeared in more detail in Lee (1996). The Sarmanov family includes Farlie–Gumbel–Morgenstern (FGM) distributions as special cases.

The applicability of Sarmanov's distribution results from its versatile structure that offers us flexibility in the choice of marginals and allows a closed form for the joint density. We aim to show the potential of this family of distributions in a loss reserving context.

In Section 4.2, we review the modeling of runoff triangles, where notations are set and random effects defined. In Section 4.3, we present the Sarmanov Family of Bivariate Distributions and introduce them to the loss reserving context in Section 4.4. We apply the model to a casualty insurance portfolio from a U.S. insurer and demonstrates the flexibility of the proposed approach in Section 4.5. Section 4.6 concludes the paper.

4.2 Modeling

4.2.1 General notations

In this paper, a dependence model within and between lines of business through calendar year, accident year and development period effects is presented. To simplify the notations, we will consider the calendar year case. The notations could be easily generalized to accident year and development period cases.

Let us consider an insurance portfolio with ℓ lines of business ($\ell \in \{1, ..., L\}$). We define by $X_{i,j}^{(\ell)}$, the incremental payments of the i^{th} accident year ($i \in \{1, ..., n\}$), and the j^{th} development period ($j \in \{1, ..., n\}$). To take into account the volume of each line of business, we work with standardized data which we denote by $Y_{i,j}^{(\ell)} = X_{i,j}^{(\ell)}/p_i^{(\ell)}$, where $p_i^{(\ell)}$ represents the exposure variable in the i^{th} accident year for the ℓ^{th} line of business. The exposure variable can be the number of policies, the number of open claims, or the earned premiums. The latter option is the one chosen in this paper. We suppose that the accident year effect is independent of the development period effect. Hence, a regression model with two independent explanatory variables, accident year and development period, is used. Assume that $\alpha_i^{(\ell)}$ ($i \in \{1, 2, ..., n\}$) and $\beta_j^{(\ell)}$ ($j \in \{1, 2, ..., n\}$) characterize the accident year effect and the development period effect respectively. In such a context, a systematic component for the ℓ^{th} line of business can be written as

$$\eta_{i.j}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)}, \ \ell = 1, ..., L, \tag{4.1}$$

where $\zeta^{(\ell)}$ is the intercept, and for parameter identification, the constraint $\alpha_1^{(\ell)} = \beta_1^{(\ell)} = 0$ is supposed. In our empirical illustration, and in the following, we work with two runoff triangles ($L = 2$) of cumulative paid losses exhibited in Tables 1 and 2 of Shi and Frees (2011). They correspond to paid losses of Schedule P of the National Association of Insurance Commissioners (NAIC) database. These are 1997 data for personal auto and commercial auto lines of business, and each triangle contains losses for accident years 1988-1997 and at most ten development years. Shi and Frees (2011) show that a lognormal distribution and a gamma distribution provide a good fit for the Personal Auto and the Commercial Auto line data respectively. To demonstrate the reasonable model fits for the two triangles, the authors exhibit the qq-plots of marginals for personal and commercial auto lines (see Figure 3 in Shi and Frees (2011)). We work with their conclusion and continue with the same continuous distributions for each line of business. More specifically, we consider the form $\mu_{i,j}^{(1)} = \eta_{i,j}^{(1)}$ for a lognormal distribution with location (log-scale) parameter $\mu_{i,j}^{(1)}$ and shape parameter σ. However, for the gamma distribution, as noted by Abdallah et al (2015) we use the exponential link instead of the canonical inverse link to ensure positive means, with $\mu_{i,j}^{(2)} = \dfrac{\exp(\eta_{i,j}^{(2)})}{\phi}$, where $\mu_{i,j}^{(2)}$ and ϕ are the scale (location) and the shape parameters respectively.

4.2.2 Random effects

The models with random effects can be interpreted as models where hidden characteristics are captured by this additional random term. Here, we want to detect the effects characterizing the loss of a given calendar year (accident year or development period) through a random variable. The latter will capture correlations within the runoff triangles for the L lines of business.

As mentioned earlier, we keep the same assumptions of Shi and Frees (2011) for the marginals, i.e. a lognormal distribution for the first line of business and a gamma distribution for the second line of business. Hence, as an associated conjugate prior, we take normal and gamma distributions, for the first and second runoff triangle respectively.

Prior distributions

Let the random variable $\Theta_t^{(\ell)}$ characterize the losses of the business line ℓ ($\ell = 1, 2$) for a given calendar year t with probability density function (pdf) denoted by $u^{(\ell)}$.

Let $\mathbf{Y}_{\mathbf{t}}^{(\ell)} = \left(Y_{t,1}^{(\ell)}, ..., Y_{1,t}^{(\ell)}\right)$ be the vector of losses for the t^{th} calendar year of the business line ℓ. This vector can also be written as $\mathbf{Y}_{\mathbf{t}}^{(\ell)} = \left(Y_1^{(\ell)}, ..., Y_j^{(\ell)}, ..., Y_t^{(\ell)}\right)$ where j indicates the j^{th} development period. Also, let $\mu_j^{(\ell)} = \mu_{t-j+1,j}^{(\ell)}$.

Let us assume that, given $\Theta_t^{(\ell)}$, the random variables $Y_1^{(\ell)}, ..., Y_t^{(\ell)}$ are conditionally independent. For $\ell = 1$, we suppose that

$$\left[Y_{i,j}^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}\right] \sim \text{Logn.}\left(\mu_{i,j}^{(1)}\theta^{(1)}, \sigma^2\right) ,$$

and

$$f_{Y_j^{(1)}|\Theta_t^{(1)}}\left(y_j^{(1)}; \theta^{(1)}\mu_j^{(1)}, \sigma^2\right) = \left(\frac{1}{y_j^{(1)}\sqrt{2\pi}\sigma}\right) \exp\left(\frac{-(\log y_j^{(1)} - \mu_j^{(1)}\theta^{(1)})^2}{2\sigma^2}\right),$$

with $E[Y_{i,j}^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}] = e^{\mu_{i.j}^{(1)}\theta^{(1)}+\sigma^2/2}$ and $\text{Var}[Y_{i,j}^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}] = \left(e^{\sigma^2} - 1\right)\left(e^{2\mu_{i.j}^{(1)}\theta^{(1)}+\sigma^2}\right)$. Also, let

$$\Theta_t^{(1)} \sim \text{Normal}\left(a, b^2\right),$$

with

$$u^{(1)}\left(\theta^{(1)}; a, b^2\right) = \frac{1}{b\sqrt{2\pi}} \exp\left(\frac{-\left(\theta^{(1)} - a\right)^2}{2b^2}\right).$$

Given these assumptions, the law of total probability leads to the following joint density function for $\mathbf{Y_t}^{(1)}$, denoted by $f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a, b^2\right)$

$$\begin{aligned}
& f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a, b^2\right) \\
= & \int_0^\infty \prod_{j=1}^t f_{Y_j^{(1)}|\Theta_t^{(1)}}\left(y_j^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}\right) u^{(1)}\left(\theta^{(1)}; a, b^2\right) d\theta^{(1)} \\
= & \prod_{j=1}^t \left(\frac{1}{y_j^{(1)}\sqrt{2\pi}\sigma}\right) \frac{\sigma}{\sqrt{\sum_{j=1}^t \mu_j^2 b^2 + \sigma^2}} \\
& \times \exp\left(-\frac{\left(\frac{1}{\sigma^2}\sum_{j=1}^t \log(y_j^{(1)})^2(b^2\sum_{j=1}^t \mu_j^2 + \sigma^2) + \frac{1}{b^2}a^2(b^2\sum_{j=1}^t \mu_j^2 + \sigma^2) - \frac{(\sum_{j=1}^t \log(y_j^{(1)})\mu_j b^2 + a\sigma^2)^2}{b^2\sigma^2}\right)}{2(b^2\sum_{j=1}^t \mu_j^2 + \sigma^2)}\right).
\end{aligned} \tag{4.2}$$

For the second line of business $\ell = 2$, we assume that

$$\left[Y_{i,j}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}\right] \sim \text{Gamma}\left(\phi, \frac{\mu_{i,j}^{(2)}}{\theta^{(2)}}\right) ,$$

and

$$f_{Y_j^{(2)}|\Theta_t^{(2)}}\left(y_j^{(2)}; \phi, \frac{\mu_j^{(2)}}{\theta^{(2)}}\right) = \frac{y_j^{(2)\phi-1}}{\Gamma(\phi)\left(\frac{\mu_j^{(2)}}{\theta^{(2)}}\right)^\phi} \exp\left(-\frac{y_j^{(2)}}{\frac{\mu_j^{(2)}}{\theta^{(2)}}}\right),$$

with $E[Y_{i,j}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}] = \phi\mu_{i,j}^{(2)}\frac{1}{\theta^{(2)}}$ and $\mathrm{Var}[Y_{i,j}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}] = \phi\mu_{i,j}^{(2)2}\frac{1}{\theta^{(2)2}}$. For the random effect $\Theta_t^{(2)}$, we suppose

$$\Theta_t^{(2)} \sim \text{Gamma}(\alpha, \tau),$$

with

$$u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right) = \frac{\theta^{(2)\alpha-1}}{\Gamma(\alpha)(\tau)^{\alpha}} \exp\left(-\frac{\theta^{(2)}}{\tau}\right).$$

The joint density function of $\mathbf{Y_t}^{(2)}$ denoted by $f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)}; \alpha, \tau\right)$ is hence given by

$$\begin{aligned} f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)}; \alpha, \tau\right) &= \int_0^{\infty} \prod_{k=1}^{t} f_{Y_k^{(2)}|\Theta_t^{(2)}}\left(y_k^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}\right) u^{(2)}\left(\theta^{(2)}; \tau, \alpha\right) d\theta^{(2)} \\ &= \prod_{k=1}^{t} \frac{y_k^{(2)\phi-1}}{\Gamma(\phi)(\mu_k^{(2)})^{\phi}} \frac{\Gamma(t\phi+\alpha)}{\Gamma(\alpha)(\tau)^{\alpha}} \frac{1}{\left(\sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau}\right)^{t\phi+\alpha}}. \end{aligned} \tag{4.3}$$

For parameter identification, we suppose that $a = 1$ and $\tau = \frac{1}{\alpha-1}$ in our empirical illustration.

Posterior distribution

Using Bayes theorem, the posterior distributions for $[\Theta_t^{(1)} = \theta^{(1)} \mid \mathbf{Y_t}^{(1)}]$ and $[\Theta_t^{(2)} = \theta^{(2)} \mid \mathbf{Y_t}^{(2)}]$ are given by

$$\begin{aligned} u^{(1)}\left(\theta^{(1)} \mid \mathbf{Y_t}^{(1)}\right) \quad &\propto f_{\mathbf{Y_t}^{(1)}|\Theta_t^{(1)}}\left(\mathbf{y_t}^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}\right) u^{(1)}\left(\theta^{(1)}; a, b^2\right) \\ &\propto u^{(1)}\left(\theta^{(1)}; a_{post}, b_{post}^2\right), \end{aligned}$$

and

$$\begin{aligned} u^{(2)}\left(\theta^{(2)} \mid \mathbf{Y_t}^{(2)}\right) \quad &\propto f_{\mathbf{Y_t}^{(2)}|\Theta_t^{(2)}}\left(\mathbf{y_t}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}\right) u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right) \\ &\propto u^{(2)}\left(\theta^{(2)}; \alpha_{post}, \tau_{post}\right). \end{aligned}$$

This shows that the posterior distributions for $[\Theta_t^{(1)} = \theta^{(1)} \mid \mathbf{Y_t}^{(1)}]$ and $[\Theta_t^{(2)} = \theta^{(2)} \mid \mathbf{Y_t}^{(2)}]$, are again Normal and gamma distributions with updated parameters

$$\begin{cases} a_{post} = \frac{\sum_{k=1}^{t} \log(y_k^{(1)})\mu_k^{(1)} b^2 + a\sigma^2}{\sum_{k=1}^{t} \mu_k^{(1)2} b^2 + \sigma^2} \text{ and } b_{post}^2 = \frac{b^2\sigma^2}{\sum_{k=1}^{t} \mu_k^{(1)2} b^2 + \sigma^2}; \\ \alpha_{post} = \alpha + t\phi \text{ and } \tau_{post} = \left(\frac{1}{\tau} + \sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}}\right)^{-1}. \end{cases}$$

These results of posterior distributions will be very helpful in the calculation of the joint Sarmanov distribution, and for the moments calculation of the total as well.

4.3 Sarmanov Family of Bivariate Distributions

Sarmanov's bivariate distribution was introduced in the literature by Sarmanov (1966), and was also proposed in physics by Cohen (1984) under a more general form. Lee (1996) suggests a multivariate version and discusses several applications in medicine. Recently, due to its flexible structure, Sarmanov's bivariate distribution gained interest in different applied studies. For example, Schweidel et al (2008) use a bivariate Sarmanov model to capture the relationship between a prospective customer's time until acquisition of a particular service and the subsequent duration for which the service is retained. Miravete (2009) presents two models based on Sarmanov distribution and uses them to compare the number of tariff plans offered by two competing cellular telephone companies. Danaher and Smith (2011) discuss applications to marketing (see also the references therein). In the insurance field, Hernández-Bastida et al (2009) and Hernández-Bastida and Fernández-Sánchez (2012) use the bivariate Sarmanov distribution for premium evaluation. Here, we want to highlight and show its usefulness in loss reserving modeling.

We suppose a dependence between the calendar years (accident years or development periods) of the two runoff triangles, i.e the elements of a given calendar year of a line of business are assumed to be correlated with the corresponding elements of the other line of business through common random effects. This will create dependence between $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$. For this purpose, we propose to use the Sarmanov Family of bivariate distributions to model the joint distribution of the random effect $\Theta_t^{(\ell)}$ with $\ell \in \{1,2\}$.

4.3.1 Definitions

Let $\psi^{(\ell)}(\theta^{(\ell)}), \ell = 1,2$ be two bounded non-constant functions such that $\int_{-\infty}^{\infty} \psi^{(\ell)}(t)\, u^{(\ell)}(t)\, dt = 0$. Let $\left(\Theta^{(1)}, \Theta^{(2)}\right)$ have a bivariate Sarmanov distribution, the joint distribution can then be expressed as

$$u^S\left(\theta^{(1)}, \theta^{(2)}\right) = u^{(1)}\left(\theta^{(1)}; a, b^2\right) u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right)\left(1 + \omega\psi^{(1)}\left(\theta^{(1)}\right)\psi^{(2)}\left(\theta^{(2)}\right)\right), \tag{4.4}$$

provided that ω is a real number that satisfies the condition

$$1 + \omega\psi^{(1)}\left(\theta^{(1)}\right)\psi^{(2)}\left(\theta^{(2)}\right) \geq 0 \text{ for all } \theta^{(\ell)}\ ,\ \ell \in \{1,2\}.$$

One of the main interesting properties of the Sarmanov is that the bivariate distribution can support a wide range of marginals, such as in this case, the normal and the gamma distributions. Different methods are proposed in Lee (1996) to construct mixing functions $\psi^{(\ell)}$ for different types of marginals. As mentioned in Lee (1996), different types of mixing functions can be used to yield different multivariate distributions with the same set of marginals. Based on Corollary 2 in Lee (1996), a mixing function can be defined as $\psi^{(\ell)}\left(\theta^{(\ell)}\right) = \exp\left(-\theta^{(\ell)}\right) - L_{(\ell)}(1)$, where $L_{(\ell)}$ is the Laplace transform of $u^{(\ell)}$, evaluated at 1. Hence, given our choice of distribution for $\Theta^{(\ell)}$, $\ell = 1,2$, we have

$$\psi^{(1)}(\theta^{(1)}) = \exp\left(-\theta^{(1)}\right) - \exp\left(-a + \frac{b^2}{2}\right)$$
$$\psi^{(2)}(\theta^{(2)}) = \exp\left(-\theta^{(2)}\right) - (1+\tau)^{-\alpha}.$$

As for the dependence parameter ω of the Sarmanov bivariate distribution, in the case of normal and gamma marginals, it is bounded as follows

$$-\frac{1}{b\exp(-a+\frac{b^2}{2})\sqrt{\alpha}\tau\,(1+\tau)^{-\alpha-1}} \leq \omega \leq \frac{1}{b\exp(-a+\frac{b^2}{2})\sqrt{\alpha}\tau\,(1+\tau)^{-\alpha-1}}.$$

The proof of this result is a direct consequence of Lee's (1996) Theorem 2.

4.3.2 Joint distribution

A critical problem when modeling dependence between runoff triangles is to obtain a joint distribution of unpaid losses. The Sarmanov distribution will be a good ally to circumvent to this problem. With normal and gamma marginals for $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ respectively, the prior joint pdf of $\left(\Theta_t^{(1)}, \Theta_t^{(2)}\right)$ is given by

$$\begin{aligned} u^S\left(\theta^{(1)}, \theta^{(2)}\right) \;=\;& u^{(1)}\left(\theta^{(1)}; a, b^2\right) u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right)\left(1 + \omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\ &+ u^{(1)}\left(\theta^{(1)}; a-b^2, b^2\right) u^{(2)}\left(\theta^{(2)}; \alpha, \frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\ &- u^{(1)}\left(\theta^{(1)}; a-b^2, b^2\right) u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\ &- u^{(1)}\left(\theta^{(1)}; a, b^2\right) u^{(2)}\left(\theta^{(2)}; \alpha, \frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}, \quad (4.5) \end{aligned}$$

which corresponds to a linear combination of the product of univariate pdfs. This last expression highlights an attractive feature of the Sarmanov family of distributions. Its simplicity and form greatly facilitate many calculations.

The joint distribution $f_{\mathbf{Y}_\mathbf{t}^{(1)}, \mathbf{Y}_\mathbf{t}^{(2)}}\left(\mathbf{y}_\mathbf{t}^{(1)}, \mathbf{y}_\mathbf{t}^{(2)}\right)$ in the case of the Sarmanov family of bivariate distributions with normal and gamma marginals is expressed by

$$\begin{aligned} & f_{\mathbf{Y}_\mathbf{t}^{(1)}, \mathbf{Y}_\mathbf{t}^{(2)}}\left(\mathbf{y}_\mathbf{t}^{(1)}, \mathbf{y}_\mathbf{t}^{(2)}\right) \\ &= \int_0^{+\infty}\int_{-\infty}^{\infty}\prod_{k=1}^{t} f_{Y_k^{(1)}|\Theta_t^{(1)}}\left(y_k^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}\right) f_{Y_k^{(2)}|\Theta_t^{(2)}}\left(y_k^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}\right) u^S\left(\theta^{(1)}, \theta^{(2)}\right) d\theta^{(1)} d\theta^{(2)}. \end{aligned}$$

Following (4.2), (4.3) and (4.5), we obtain a closed-form expression for the density function of $\left(\mathbf{Y}_\mathbf{t}^{(1)}, \mathbf{Y}_\mathbf{t}^{(2)}\right)$, namely

$$
\begin{aligned}
f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right) &= f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)};a,b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)};\alpha,\tau\right)\left(1+\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\
&\quad + f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)};a-b^2,b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
&\quad - f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)};a-b^2,b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)};\alpha,\tau\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
&\quad - f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)};a,b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}
\end{aligned}
\tag{4.6}
$$

4.3.3 Posterior Sarmanov distribution

The posterior distribution can be used for the calculation of the moments of the total reserve. The posterior bivariate joint density function of the couple $\left(\Theta_t^{(1)},\Theta_t^{(2)}\right)$ conditioned on $\left(\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}\right)$ is given by

$$
\begin{aligned}
& u^S(\theta^{(1)},\theta^{(2)} \mid \mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}) \\
&= \frac{f(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)} \mid \theta^{(1)},\theta^{(2)})u^S(\theta^{(1)},\theta^{(2)})}{f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}(y_1^{(1)},...,y_t^{(1)},y_1^{(2)},...,y_t^{(2)})} \\
&= C_1u^{(1)}\left(\theta^{(1)};a_{post},b^2_{post}\right)u^{(2)}\left(\theta^{(2)};\alpha_{post},\tau_{post}\right) + C_2u^{(1)}\left(\theta^{(1)};a'_{post},b^2_{post}\right)u^{(2)}\left(\theta^{(2)};\alpha_{post},\tau'_{post}\right) \\
& -C_3u^{(1)}\left(\theta^{(1)};a'_{post},b^2_{post}\right)u^{(2)}\left(\theta^{(2)};\alpha_{post},\tau_{post}\right) - C_4u^{(1)}\left(\theta^{(1)};a_{post},b^2_{post}\right)u^{(2)}\left(\theta^{(2)};\alpha_{post},\tau'_{post}\right),
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
a_{post} &= \frac{\sum_{k=1}^t \log(y_k^{(1)})\mu_k^{(1)}b^2 + a\sigma^2}{\sum_{k=1}^t \mu_k^{(1)2}b^2+\sigma^2} \\
a'_{post} &= \frac{\sum_{k=1}^t \log(y_k^{(1)})\mu_k^{(1)}b^2 + \left(a-b^2\right)\sigma^2}{\sum_{k=1}^t \mu_k^{(1)2}b^2+\sigma^2} \\
b^2_{post} &= \left(\frac{\sum_{k=1}^t \mu_k^{(1)2}}{\sigma^2}+\frac{1}{b^2}\right)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_{post} &= t\phi+\alpha \\
\tau_{post} &= \left(\sum_{k=1}^t \frac{y_k^{(2)}}{\mu_k^{(2)}}+\frac{1}{\tau}\right)^{-1} \\
\tau'_{post} &= \left(\sum_{k=1}^t \frac{y_k^{(2)}}{\mu_k^{(2)}}+\frac{1}{\tau}+1\right)^{-1},
\end{aligned}
$$

with

$$
\begin{aligned}
C_1 &= \frac{1}{f_{\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)}\right)} f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\tau\right)\left(1+\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right)\\
C_2 &= \frac{1}{f_{\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)}\right)} f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a-b^2,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\\
C_3 &= \frac{1}{f_{\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)}\right)} f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a-b^2,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\tau\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\\
C_4 &= \frac{1}{f_{\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)}\right)} f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}.
\end{aligned}
$$

This last expression shows that the posterior bivariate density function of $\left(\Theta^{(1)},\Theta^{(2)}\right)$, is again a linear combination of the product of univariate normal and gamma pdfs. The posterior density is hence a pseudo-conjugate to the prior density in the sense that the posterior density is a linear combination of products of densities from the univariate natural exponential family of distributions (normal and gamma in our case). It would be interesting to investigate the link between the posterior Sarmanov distribution and the linear credibility theory, where the Bayesian premium is considered linear.

4.4 Claims reserving

4.4.1 Calendar year dependence

To accommodate correlation, most multivariate loss reserving methods focus on a pairwise association between corresponding cells in multiple runoff triangles. Recently, Shi and Frees (2011) successfully incorporated dependence between two lines of business with a pairwise association. However, such a practice usually relies on an independence assumption across accident years and ignores the calendar year effects that could affect all open claims simultaneously and induce dependencies among loss triangles. In fact, most dependencies among loss triangles could arguably be driven by certain calendar year effects and exogenous common factors such as inflation, interest rates, jurisprudence or strategic decisions such as the acceleration of the payments for the entire portfolio can have simultaneous impacts on all lines of business of a given sector, which could be the case here for the two lines of business considered in the present paper.

Such a calendar year effect has already been analyzed, for example by Barnett and Zehnwirth (1998) who add a covariate to capture the calendar year effect. De Jong (2006) models the growth rates in cumulative payments in a calendar year, and Wüthrich (2010) examines the accounting year effect for a single line of business. Wüthrich and Salzmann (2012) use a multivariate Bayes Chain-Ladder model that allows modeling of dependence along accounting years within runoff triangles. The authors derive closed form solutions for the posterior

distribution, claims reserves and corresponding prediction uncertainty. Kuang et al (2008) also consider a canonical parametrization with three factors for a single line of business.
In our proposed model, instead of adding an explanatory variable for the calendar year effect, the dependence relation between the paid claims of a diagonal will be based on a random effect. More specifically, the same random variable $\Theta_t^{(\ell)}$ is assumed for each diagonal of a runoff triangle. The likelihood function of this model can be easily derived from (4.2) and (4.3).

4.4.2 Line of Business dependence

Motivations

In the same view of Abdallah et al (2015), we propose a model that allows a dependence relation between all the observations that belong to the same calendar year for each line of business using random effects instead of multivariate Archimedean copulas. Additionally, we use another dependence structure that links the losses of calendar years of different lines of business with a Sarmanov family of bivariate distributions instead of hierarchical copula. With this second level of dependence, we capture the dependence between two different runoff triangles in a pairwise manner between corresponding diagonals, instead of between cells. Hence, instead of pairing cells with a copula as in Shi and Frees (2011), we will pair diagonals through random effects using the Sarmanov family of bivariate distributions.
The calendar year effect has rarely been studied with more than one line of business. Two recent examples are De Jong (2012), where the calendar year effect is introduced through the correlation matrix and Shi et al (2012), who use random effects to accommodate the correlation due to accounting year effects within and across runoff triangles. Shi et al (2012) work with a Bayesian perspective, using a multivariate lognormal distribution, along with a multivariate Gaussian correlation matrix. The predictive distributions of outstanding payments are generated through Monte Carlo simulations. The calendar year effect is taken into account through an explanatory variable. Again with a Bayesian framework, Wüthrich et al (2013) use a multivariate lognormal Chain-Ladder model and derive predictors and confidence bounds in closed form. Their analytical solutions are such that they allow for any correlation structure. Their models permit dependence between and within runoff triangles, along with any correlation structure. It has also been shown in this paper that the pairwise dependence form is rather weak compared with calendar year dependence. More recently, Shi (2014) captures the dependencies introduced by various sources, including the common calendar year effects via the family of elliptical copulas, and uses parametric bootstrapping to quantify the associated reserving variability.
In this paper, to model the complex dependence structure between two runoff triangles, we introduce models based on the Sarmanov family of bivariate distributions. The idea is to use random effects to capture dependence within lines of business, and then join the two random

effects through a Sarmanov distribution to capture dependence between lines of business. Empirical results are shown in the next section. Finally, the log-likelihood function of this model can be obtained from (4.6).

Mean and Variance

To compute the resulting reserve for this model, the estimated total unpaid losses for $i+j > n+1$, can be expressed as follows

$$E\left[R_{tot}\right] = E\left[R^{(1)} + R^{(2)}\right] = E\left[\sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} Y_{i,j}^{(\ell)}\right] = \sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} E[Y_{i,j}^{(\ell)}],$$

where

$$E[Y_{i,j}^{(1)}] = E[E[Y_{i,j}^{(1)} \mid \Theta_t^{(1)}]] = E[e^{\mu_{i,j}^{(1)}\Theta_t^{(1)} + \sigma^2/2}] = e^{a\mu_{i,j}^{(1)} + \frac{1}{2}b^2\mu_{i,j}^{(1)2} + \sigma^2/2}$$

and

$$E[Y_{i,j}^{(2)}] = E[E[Y_{i,j}^{(2)} \mid \Theta_t^{(2)}]] = E\left[\phi\mu_{i,j}^{(2)}\frac{1}{\Theta_t^{(2)}}\right] = \phi\mu_{i,j}^{(2)}\frac{1}{\tau(\alpha-1)},$$

with $t = i+j-1$.

Consequently, the total unpaid losses can be written as

$$E\left[R_{tot}\right] = \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} e^{a\mu_{i,j}^{(1)} + \frac{1}{2}b^2\mu_{i,j}^{(1)2} + \sigma^2/2} + \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} \frac{p_i^{(2)}\phi\mu_{i,j}^{(2)}}{\tau(\alpha-1)}. \tag{4.8}$$

When we model dependence between loss triangles, the global variance can be very informative. Knowing that the two runoff triangles are correlated, it is interesting to observe how the two random effects $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ change together, i.e whether the two variables tend to show similar (positive dependence) or opposite behavior (negative dependence). Note that when $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ are assumed unrelated (independent case), we will have $\text{Cov}(R^{(1)}, R^{(2)}) = 0$.

The total claims reserve variance can be written as

$$\begin{aligned}\text{Var}(R_{tot}) &= \text{Var}\left(R^{(1)} + R^{(2)}\right) = \text{Var}(R^{(1)}) + \text{Var}(R^{(2)}) + 2\text{Cov}(R^{(1)}, R^{(2)}) \\ &= \sum_{\ell=1}^{2}\text{Var}(R^{(\ell)}) + 2\text{Cov}(R^{(1)}, R^{(2)}).\end{aligned}$$

Using the conditional independence of $Y_{i,j}^{(\ell)}$ given $\Theta_t^{(\ell)} = \theta^{(\ell)}$ $(t = i+j-1))$, we have

$$\begin{aligned}\text{Var}(R^{(1)}) &= E\left[\text{Var}\left(\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right)\right] + \text{Var}\left[E\left(\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right)\right] \\ &= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)2}\left(E\left[\text{Var}\left(Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right)\right] + \text{Var}\left(E\left[Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right]\right)\right) \\ &= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)2}\left(e^{2a\mu_{i,j}^{(1)} + 2b^2\mu_{i,j}^{(1)2} + 2\sigma^2} - e^{2a\mu_{i,j}^{(1)} + b^2\mu_{i,j}^{(1)2} + \sigma^2}\right)\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(R^{(2)}) &= E\left[\mathrm{Var}\left(\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)} Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right)\right] + \mathrm{Var}\left(E\left[\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)} Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right]\right) \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)2}\left(E\left[\mathrm{Var}\left(Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right)\right] + \mathrm{Var}\left(E\left[Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right]\right)\right) \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} \phi(p_i^{(2)}\mu_{i,j}^{(2)})^2\left(\frac{\alpha+\phi-1}{\tau^2(\alpha-1)^2(\alpha-2)}\right).
\end{aligned}$$

For the covariance calculation, we have

$$\mathrm{Cov}(R^{(1)}, R^{(2)}) = \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} p_i^{(2)}\left(E\left[Y_{i,j}^{(1)} Y_{i,j}^{(2)}\right] - E\left[Y_{i,j}^{(1)}\right] E\left[Y_{i,j}^{(2)}\right]\right).$$

From (4.8) we have

$$E\left[Y_{i,j}^{(1)}\right] E\left[Y_{i,j}^{(2)}\right] = \left(e^{a\mu_{i,j}^{(1)} + \frac{1}{2}b^2\mu_{i,j}^{(1)2} + \sigma^2/2}\right)\left(\frac{\phi\mu_{i,j}^{(2)}}{\tau(\alpha-1)}\right),$$

and given (4.5), we obtain

$$\begin{aligned}
E[Y_{i,j}^{(1)} Y_{i,j}^{(2)}] &= E\left[E\left(Y_{i.j}^{(1)} Y_{i.j}^{(2)} \mid \Theta_t^{(1)}, \Theta_t^{(2)}\right)\right] \\
&= e^{\sigma^2/2}\phi\mu_{i,j}^{(2)} E[e^{\mu_{i,j}^{(1)}\Theta_t^{(1)}}\frac{1}{\Theta_t^{(2)}}],
\end{aligned}$$

with

$$E[e^{\mu_{i,j}^{(1)}\Theta_t^{(1)}}\frac{1}{\Theta_t^{(2)}}] = \int_0^\infty \int_{-\infty}^\infty e^{\mu_{i,j}^{(1)}\theta^{(1)}}\frac{1}{\theta^{(2)}} u^S(\theta^{(1)}, \theta^{(2)}) d\theta^{(1)} d\theta^{(2)}.$$

Consequently, the total variance of unpaid losses is expressed as follows

$$
\begin{aligned}
\mathrm{Var}(R_{tot}) &= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)2}\left(e^{2a\mu_{i,j}^{(1)}+2b^2\mu_{i,j}^{(1)2}+2\sigma^2} - e^{2a\mu_{i,j}^{(1)}+b^2\mu_{i,j}^{(1)2}+\sigma^2}\right)\\
&+ \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} \phi(p_i^{(2)}\mu_{i,j}^{(2)})^2\left(\frac{\alpha+\phi-1}{\tau^2(\alpha-1)^2(\alpha-2)}\right)\\
&+ 2\sum_{i=2}^{n}\sum_{j=n-i+2}^{n}\Bigg\{p_i^{(1)}p_i^{(2)}e^{\sigma^2/2}\phi\mu_{i,j}^{(2)}\times\\
&\Bigg[e^{a\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1}{\tau(\alpha-1)}\left(1+\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right)\\
&+e^{\left(a-b^2\right)\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1+\tau}{\tau(\alpha-1)}\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\\
&-e^{\left(a-b^2\right)\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1}{\tau(\alpha-1)}\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\\
&-e^{a\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1+\tau}{\tau(\alpha-1)}\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\Bigg]\\
&-\left(p_i^{(1)}e^{a\mu_{i,j}^{(1)}+\frac{1}{2}b^2\mu_{i,j}^{(1)2}+\sigma^2/2}\right)\left(\frac{p_i^{(2)}\phi\mu_{i,j}^{(2)}}{\tau(\alpha-1)}\right)\Bigg\}. \qquad (4.9)
\end{aligned}
$$

4.4.3 Accident year and development period dependence

We consider here a dependence structure captured through accident year and development period effects. In fact, some exogenous factors could result in an accident year trend. Change in reserving practices for example, in the way case reserves are settled at the opening of the claim, for current accident year claims. Further, a court judgment, a change in legislation affecting future losses, major events and disasters can all result in an accident year trend as well. The development period trend could result from the same exogenous factors cited for the calendar year case, but also from management decisions. For example, a revision of inactive claims or a changing pace of payments (internal or external initiative) are widespread practices in the industry that might affect several lines of business simultaneously.

Credibility loss reserving

As discussed earlier in this paper, the flexibility of the Sarmanov family of bivariate distribution allows us to easily change the dependence structure. Hence, as in extension and in addition to the calendar year approach, we will consider here two other approaches in which the random effect characterizes the loss of a given accident year or development period. Such modeling is well illustrated in Figure 4.1. In fact, we can see that a given accident year or development period effect will also impact the observations in the lower triangle belonging to the same accident year or development period. This is a great advantage when working with random effects rather than copulas, where the predictive power for the lower triangle might be limited.

Henceforth, we consider a situation where an insurer has access to claims experience and has the potential to improve prediction of outstanding liabilities by incorporating past information. The link here with linear credibility is pretty straightforward.

The accident year and development period effect has rarely been studied in the literature. It is interesting to note that depending on the dependence structure we use, we could get different conclusions from the analysis of dependence between the two business lines. This was also well illustrated in Figure 4 of Shi et al (2012). This will be discussed in greater detail in the next section, where an empirical illustration is presented.

The idea here is that future payments will be updated through past experience. In fact, the random effect characterizing the loss of a given accident year or development period affects payments in the lower triangle as well. More importantly, it would be interesting here to see how these random effects impact the two runoff triangles simultaneously.

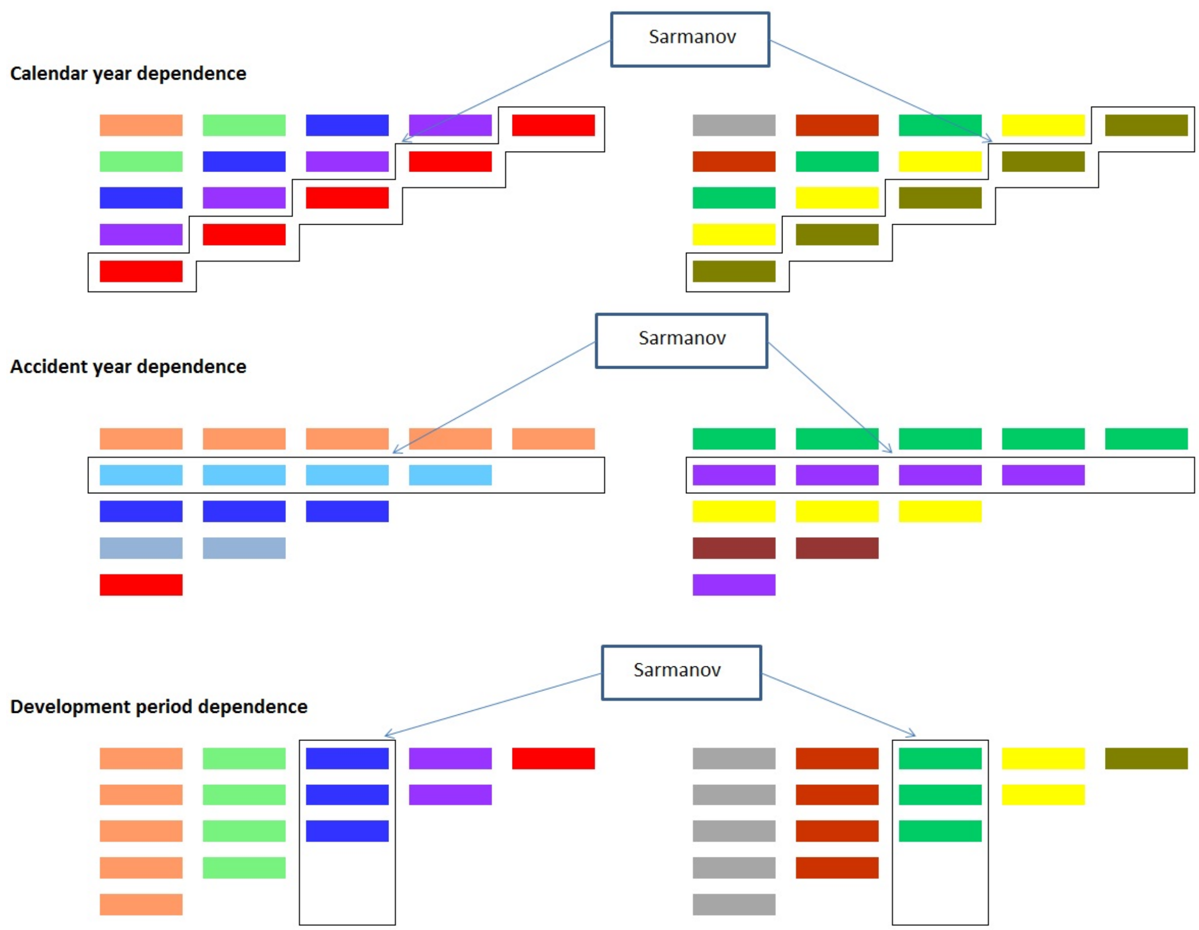

Figure 4.1 – Modeling dependence with a Sarmanov bivariate distribution

Expected claim reserve

For the accident year or development period approach, unlike the calendar year case, the projections in the lower part of the triangle will be now impacted by the values of the upper part, because they are, henceforth, linked by the random effect $\Theta_t^{(\ell)}$.
Let $\Theta_t = \left(\Theta_t^{(1)}, \Theta_t^{(2)}\right)$ and $\Im_t = \left(\mathbf{Y_t^{(1)}}, \mathbf{Y_t^{(2)}}\right)$ where $\Theta_t^{(\ell)}$, $\ell = 1, 2$, characterizes the loss of a given accident year ($t = i$) or development period ($t = j$).
Given the conditional independence of $Y_{i,j}^{(\ell)}$ given $\Theta_t^{(\ell)} = \theta^{(\ell)}$, the total estimated projected paid loss ratio is given by

$$\begin{aligned} E\left[R_{tot} \mid \Im_t\right] &= E\left[R^{(1)} + R^{(2)} \mid \Im_t\right] \\ &= E\left[\sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} Y_{i.j}^{(\ell)} \mid \Im_t\right] \\ &= \sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} E\left[Y_{i.j}^{(\ell)} \mid \Im_t\right], \end{aligned}$$

where $E[Y_{i,j}^{(\ell)} \mid \Im_t] = E[E[Y_{i,j}^{(\ell)} \mid \Theta_t, \Im_t] \mid \Im_t] = E[E[Y_{i,j}^{(\ell)} \mid \Theta_t^{(\ell)}] \mid \Im_t]$ with $u^{(1)}\left(\theta^{(1)} \mid \Im_t\right) = \int_0^{+\infty} u^S(\theta^{(1)}, \theta^{(2)} \mid \Im_t) d\theta^{(2)}$ and $u^{(2)}\left(\theta^{(2)} \mid \Im_t\right) = \int_{-\infty}^{+\infty} u^S(\theta^{(1)}, \theta^{(2)} \mid \Im_t) d\theta^{(1)}$.
Hence, from (4.7), we have

$$u^{(1)}\left(\theta^{(1)} \mid \Im_t\right) = \left(C_1 - C_4\right) u^{(1)}\left(\theta^{(1)}; a_{post}, b_{post}^2\right) + \left(C_2 - C_3\right) u^{(1)}\left(\theta^{(1)}; a_{post}^{'}, b_{post}^2\right),$$

which leads to

$$\begin{aligned} E[Y_{i,j}^{(1)} \mid \Im_t] &= e^{\sigma^2/2} \int_{-\infty}^{+\infty} e^{\mu_{i,j}^{(1)}\theta^{(1)}} u^{(1)}\left(\theta^{(1)} \mid \Im_t\right) d\theta^{(1)} \\ &= \left(C_1 - C_4\right) e^{a_{post}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2} + \left(C_2 - C_3\right) e^{a_{post}^{'}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2}. \end{aligned}$$

Similarly, we obtain for the second line of business

$$u^{(2)}\left(\theta^{(2)} \mid \Im_t\right) = \left(C_1 - C_3\right) u^{(2)}\left(\theta^{(2)}; \alpha_{post}, \tau_{post}\right) + \left(C_2 - C_4\right) u^{(2)}\left(\theta^{(2)}; \alpha_{post}, \tau_{post}^{'}\right),$$

and hence

$$\begin{aligned} E[Y_{i,j}^{(2)} \mid \Im_t] &= \phi\mu_{i,j}^{(2)} E[\frac{1}{\Theta_t^{(2)}} \mid \Im_t] \\ &= \phi\mu_{i,j}^{(2)} \int_0^{+\infty} \frac{1}{\theta^{(2)}} u^{(2)}\left(\theta^{(2)} \mid \Im_t\right) d\theta^{(2)} \\ &= \phi\mu_{i,j}^{(2)} \left(\frac{C_1 - C_3}{\tau_{post}(\alpha_{post} - 1)} + \frac{C_2 - C_4}{\tau_{post}^{'}(\alpha_{post} - 1)}\right). \end{aligned}$$

Consequently, the total unpaid losses in this case can be written as

$$\begin{aligned}
E\left[R_{tot} \mid \Im_t\right] &= \sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} E[Y_{i.j}^{(\ell)} \mid \Im_t] \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} \left(\left(C_1 - C_4\right) e^{a_{post}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2} + \left(C_2 - C_3\right) e^{a'_{post}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2} \right) \\
&\quad + \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)} \phi \mu_{i,j}^{(2)} \left(\frac{C_1 - C_3}{\tau_{post}(\alpha_{post} - 1)} + \frac{C_2 - C_4}{\tau'_{post}(\alpha_{post} - 1)} \right),
\end{aligned} \tag{4.10}$$

with parameters a_{post}, a'_{post}, b^2_{post}, α_{post}, τ_{post}, τ'_{post} and $C_i, i \in \{1,2,3,4\}$ as given in (4.7). The expression of the claims reserve variance is more cumbersome for this approach but can be handily derived given that the posterior density of the Sarmanov family is a pseudo-conjugate prior.

4.5 Empirical illustration

4.5.1 Model calibration

We implement the three models proposed in the previous sections with the runoff triangles described in section 4.2.1. We want to compare the fit of our models with that obtained in Shi and Frees (2011), where pairwise dependence (PWD) between cells is supposed through a copula. The Gaussian copula was selected for this model based on Akaike's Information Criterion (AIC). In our empirical study, we first use a model that supposes independence between lines of business, with dependence within runoff triangles captured through random effects. This model is described in section 4.4.1. Fit statistics are shown in Table 4.1. In terms of the AIC, we observe that the three models offer a better fit than the PWD model, which is a promising result for what follows. Now, we suppose pairwise dependence between random effects that affect a given calendar year, accident year or development period. This dependence between runoff triangles is captured with the Sarmanov family of bivariate distributions. The fit statistics and the reserves obtained for this model are shown in Tables 4.2 and 4.3 respectively.

The reserve estimations, for the calendar year approach are based on (4.8), with the systematic component described in (4.1). As for the accident year and development period approach, the calculation is performed following (4.10). However, the accident year (development period) parameter is missing in the mean specification for accident year (development period) approach. Hence, we borrow the information from the calendar year trend to complete the projection of the lower triangle. We note that a gamma curve, also known as a Hoerl's curve, could also have been investigated for this case.

Fit Statistics	Dependence			
	PWD	**Dev. period**	**Calendar year**	**Accident year**
Log-Likelihood	350.5	376.4	396.4	402.3
AIC	-618.9	-669.0	-708.9	-720.8
BIC	-508.3	-656.2	-696.2	-708.1

Table 4.1 – Fit Statistics of PWD model vs Independent lines of business with random effects

Fit Statistics	Dependence		
	Dev. period	**Calendar year**	**Accident year**
Dependence parameter	628.76 (194.20)	-387.10 (746.77)	12083 (22300)
Log-Likelihood	381.1	396.6	403.1
AIC	-676.2	-707.2	-718.8
BIC	-663.1	-694.2	-705.8

Table 4.2 – Fit Statistics of Sarmanov model

Reserves estimation	Dependence			
	PWD	**Dev. period**	**Calendar year**	**Accident year**
Personal	6,423,180	6,547,988	6,476,093	6,616,171
Commercial	495,989	504,928	551,478	438,716
Total	6,919,169	7,052,916	7,027,571	7,054,888

Table 4.3 – Reserve estimation with different models

We observe that the model with accident year dependence offers the best fit of all the models. Indeed, according to the fit statistics, the data seem to favour the model emphasising accident year effects. However, the model with development period dependence seems to favor dependence between lines of business. Given that the three models nest the independence case as a special case, we can perform a likelihood ratio test to examine the model fit. Compared with the independent case, the accident year model gives a χ^2 statistics of 0.2, the calendar year model gives a χ^2 statistics of 0.4, whereas the development period model gives a χ^2 statistics of 9.4. Henceforth, the dependence is rejected over the independence model for the calendar year and accident year cases, because ω is not statistically significant, meanwhile a dependence model is preferred for the development period case. A Wald test (see Boucher et al (2007) for a detailed discussion on one-sided statistic tests) based on the estimated values of ω and its standard errors (see Table 4.2) leads to the same conclusions drawn from the likelihood ratio test. Interestingly, the model has a better fit when incorporating dependence between the two lines of business only for the development period approach. This is also confirmed by the results of the AIC.

4.5.2 Predictive distribution

In practice, actuaries are interested in knowing the uncertainty of the reserve. A modern parametric technique, the bootstrap, not only gives such information but most importantly provides the entire predictive distribution of aggregated reserves for the portfolio. The predictive distribution notably allows assessment of risk capital for an insurance portfolio. Bootstrapping is also ideal from a practical point of view, because it avoids complex theoretical calculations and can easily be implemented. Moreover, it tackles potential model overfitting, typically encountered in loss reserving problems due to the small sample size. Henceforth, we implement a parametric bootstrap analysis to quantify predictive uncertainty.

The bootstrap technique is increasingly popular in loss reserving, and allows a wide range of applications. It was first introduced in a loss reserving context with a distribution-free approach by Lowe (1994). For a multivariate loss reserving analysis, Kirschner et al (2008) used a synchronized parametric bootstrap to model dependence between correlated lines of business, and Taylor and McGuire (2007) extended this result to a generalized linear model context. Shi and Frees (2011), and more recently Shi (2014), have also performed a parametric bootstrap to quantify the uncertainty in parameter estimates, while modeling dependence between loss triangles using copulas.

Sarmanov simulation

The parametric bootstrap allows us to obtain the whole distribution of the reserves. We follow the same bootstrap algorithm as Taylor and McGuire (2007), also summarized in Shi and Frees (2011).

The first step of the parametric bootstrap is to generate pseudo-responses of normalized incremental paid losses $y_{i,j}^{*(\ell)}$, for i, j such that $i + j - 1 \leq n$ and $\ell = 1, 2$.

For the first line of business, we generate a realization $y_{i,j}^{*(1)}$ of a lognormal distribution with location (log-scale) parameter $\widehat{\mu}_{i,j}^{(1)}\Theta^{(1)}$ and shape parameter $\widehat{\sigma}$. As for the second line of business, $y_{i,j}^{*(2)}$ is a generated realization of a gamma distribution with location (scale) parameter $\frac{\widehat{\mu}_{i,j}^{(2)}}{\Theta^{(2)}}$ and shape parameter $\widehat{\phi}$.

Therefore, a technique to generate realizations of the couple $\left(\theta^{(1)}, \theta^{(2)}\right)$ from a Sarmanov family of bivariate distributions should be used.

Given that the calendar year dependence is the most widely used for its intuitive and practical purposes, we focus solely on this approach.

To generate a bivariate Sarmanov distribution we follow the method based on the conditional simulation. Thus, for a given calendar year t, the algorithm for a Sarmanov bivariate distribution between the lines of business is as follows

1. Generate a realization $\theta^{(1)}$, from the random variable $\Theta_t^{(1)} \sim Normal\left(\widehat{a}, \widehat{b}^2\right)$.
2. Generate a realization from the conditional cumulative distribution of the random variable $\left(\Theta_t^{(2)} \mid \Theta_t^{(1)} = \theta^{(1)}\right)$.

	Estimated reserve	Bootstrap reserve	Estimation error	Process error
CY Sarmanov	7,027,571	7,047,931	312,331	153,413

Table 4.4 – Bootstrap results for the calendar year Sarmanov model

3. Get a realization $\theta^{(2)}$ from the previous stage.

Consequently, we have obtained realizations of the couple $\left(\theta^{(1)}, \theta^{(2)}\right)$ from a Sarmanov family of bivariate distributions.

MSEP

A common statistic to measure the total variance uncertainty of the portfolio R_{tot}, is the mean squared error of prediction (MSEP).

The MSEP is a combination of process error and estimation error. Estimation error is linked to past observations and process error is due to the variation of future observations. The definition can be expressed as follows

$$\begin{aligned} MSEP[\widehat{R_{tot}}] &= E[(R_{tot} - \widehat{R_{tot}})^2] \\ &= E[((R_{tot} - E[R_{tot}]) - (\widehat{R_{tot}} - E[R_{tot}]))^2]. \end{aligned}$$

Assuming $E[(R_{tot} - E[R_{tot}])(\widehat{R_{tot}} - E[\widehat{R_{tot}}])] = 0$, i.e. future observations are independent of past observations, we get

$$\begin{aligned} MSEP[\widehat{R_{tot}}] &\approx E[(R_{tot} - E[R_{tot}])^2] + E[(\widehat{R_{tot}} - E[\widehat{R_{tot}}])^2] \\ &= \underbrace{Var[R_{tot}]}_{\text{Process error}^2} + \underbrace{Var[\widehat{R_{tot}}]}_{\text{Estimation error}^2} . \end{aligned}$$

The main advantage of using the Sarmanov family of bivariate distributions lies in the fact that we are able to derive a closed-form expression for the process error of the whole portfolio (see (4.9)), which is not straightforward to obtain analytically with a copula model. We quantify the estimation error with the parametric bootstrap. In our empirical illustration, the obtained bootstrap results are exhibited in Table 4.4.

Also, because we can obtain the estimation error and process error for a Sarmanov model, it would be interesting to compare them with their analytic equivalent from Mack's model, which has long been considered as a benchmark model. This comparison is shown in Table 4.5. We note that the two methods provide results in the same order of magnitude.

Risk capital analysis

In addition to the bootstrap results for the calendar year dependence model with a Sarmanov family of bivariate distributions exhibited in Table 4.4, we provide a histogram of the re-

Model	Reserve	$\sqrt{MSEP}$
Sarmanov	7,027,571	347,947
Mack	6.925.951	334,929

Table 4.5 – Comparison between Sarmanov model and Mack model

serve distribution, with the corresponding percentiles in Figure 4.2. The latter information is important and useful for actuaries when they want to select a reserve at a desired level of conservatism. We also superimposed kernel density estimates on the histogram of Figure 4.2 in Figure 4.3 with several choices for the bandwidth parameter to determine the smoothness and closeness of the fit. Smoothing the data distribution with a kernel density estimate can be more effective than using a histogram to identify features that might be obscured by the choice of histogram bins.

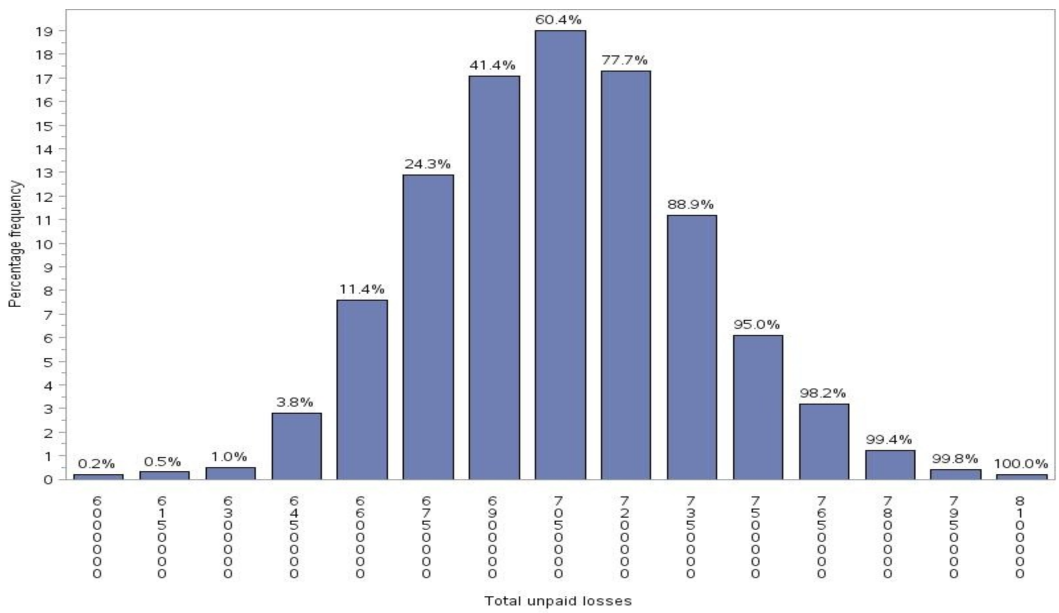

Figure 4.2 – Percentiles of total unpaid losses (in millions) - Sarmanov calendar year model

The predictive distribution of unpaid losses is very helpful to obtain reserve ranges, but it is also useful from a risk capital standpoint. Risk capital is the amount that property/casualty insurers set aside as a buffer against potential losses from extreme and adverse events.
We want to show here the impact of assuming a dependence structure based on the Sarmanov family of bivariate distribution on the risk capital calculation instead of summing up the risk capital for each subportoflio. In fact, the most common approach in practice, called the "Silo" method, is to divide the portfolio into several subportfolios and to evaluate the risk capital for each silo and then add them up for the portfolio. The main criticism to this method is that it

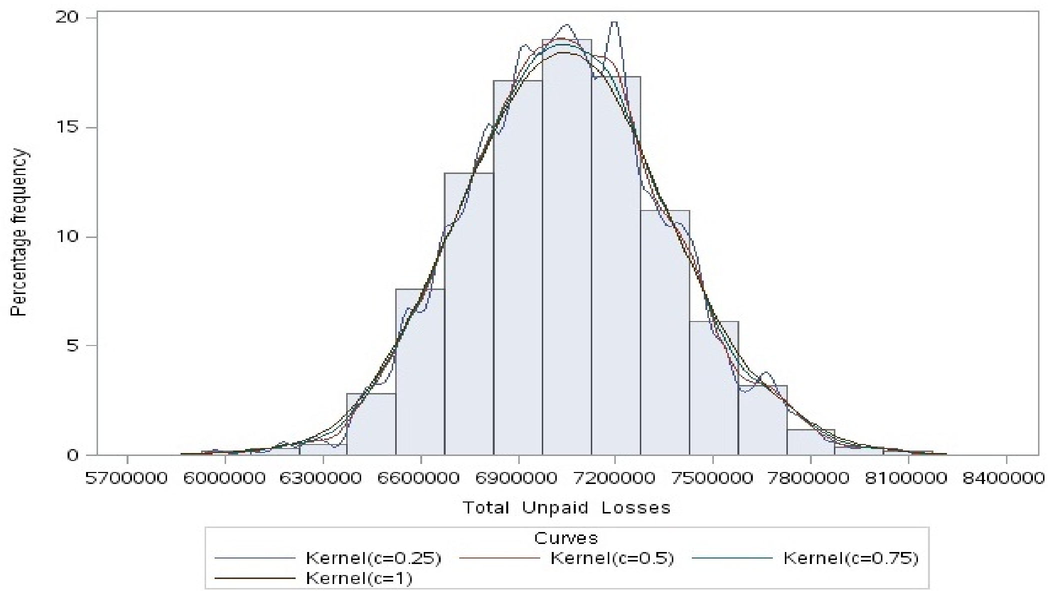

Figure 4.3 – Total unpaid losses distribution with density Kernel estimation (in millions) - Sarmanov calendar model

implicitly assumes a perfect positive linear relationship among subportfolios, which does not allow any form of diversification. We aim to show, following the parametric bootstrap, that one can take advantage of this diversification between the two lines of business, allowing risk capital analysts to be less conservative.

Mathematically, the risk capital is the difference between the risk measure and the expected unpaid losses of the portfolio, which are $7,047,931$. For the risk measure, we consider the tail value-at-risk (TVaR) that has been widely used by actuaries. This measure is more informative than the value at risk (VaR) in the distribution tail, and the subadditivity of VaR is not guaranteed in general.

To examine the role of dependencies we calculate the risk measure for each sub-portfolio (i.e. the personal auto line and the commercial auto line), and then use the simple sum as the risk measure for the entire portfolio. This is the result reported under the silo method. The silo method gives the largest estimates of risk measures because it does not account for any diversification effect in the portfolio. We provide the results for the case where no random effects are considered within lines of business (Silo - independent), and the case where random effects within lines of business are assumed (Silo - random effects). Both cases assume independence between lines of business and are compared with the case that treats the two lines of business as related through the Sarmanov bivariate distribution. We show in Table 4.6 that the gain in terms of risk capital is important when we capture the association between the two triangles, and this difference is even greater in the distribution tail where most adversed situations are encountered for the two lines of business. This result indicates

Risk measure	TVaR (80%)	TVaR (85%)	TVaR (90%)	TVaR (95%)	TVaR (99%)
Silo - random effects	7,671,066	7,755,618	7,862,446	8,041,361	8,441,168
Silo - independent	7,582,963	7,656,635	7,760,671	7,922,635	8,259,798
Sarmanov	7,491,092	7,542,301	7,609,383	7,720,910	7,910,013
Risk capital					
Silo - random effects	623,135	707,686	814,515	993,429	1,393,237
Silo - independent	535,032	608,703	712,739	874,704	1,211,866
Sarmanov	443,160	494,369	561,451	672,979	862,082
Gain					
vs independent	17.17%	18.78%	21.23%	23.06%	28.86%
vs random effect	28.88%	30.14%	31.07%	32.26%	38.12%

Table 4.6 – Risk capital estimation with different scenarios

that the silo method leads to more conservative risk capital, while the Sarmanov model leads to more aggressive risk capital.

4.6 Conclusion

In this paper, we have studied different approaches to model dependence between loss triangles. If losses in different lines of business are correlated, aggregate reserves must reflect this dependence. To allow a flexible dependence relation, we propose the use of the Sarmanov family of bivariate distributions. To illustrate the model, an empirical illustration was performed using the same data as that used by Shi and Frees (2011). Based on the AIC and on the BIC, we show that our models provide a better fit than the PWD model does.
With the proposed model, we can derive analytically the expression of total the reserve and the total process variance with a calendar year, accident year and development period dependence model, thanks to the pseudo-conjugate properties. Also, we use a parametric bootstrap to derive a predictive distribution and incorporate parameter uncertainty in our analysis.
By coupling various sources of dependencies with a Sarmanov bivariate distribution through random effects, we propose a new approach to model dependence structures between runoff triangles. This model is a promising tool to better take into account dependencies within and between business lines. Indeed, this approach can easily be generalized to more than two lines of business because it is possible to extend the Sarmanov's family of distributions to the multivariate case. As an extension, one can also consider these random effects dynamic or evolutionary, i.e. that they evolve over time and are updated through past experience. We leave the detailed discussion of this complicated case to the future study.

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Chapitre 5

Sarmanov Family of Multivariate Distributions for Bivariate Dynamic Claim Counts Model

Résumé

Pour prévoir les futures réclamations, il est connu que les plus récentes réclamations sont considérées plus prédictives que les plus anciennes.
Cependant, les modèles classiques de données de panel pour les nombres de sinistres tels que la distribution binomiale négative multivariée ne permettent pas d'allouer des poids aux réclamations passées.
Des modèles plus avancés peuvent être utilisés pour examiner cette propriété, mais ont souvent besoin de procédures numériques très complexes pour estimer les paramètres. Lorsque nous voulons ajouter une dépendance entre les différents types de réclamation, la tâche devient d'autant plus difficile.
Dans cet article, nous proposons un modèle dynamique bivarié pour les nombres de sinistres, où l'expérience des réclamations passées d'un type de réclamation donné est utilisée pour mieux prédire l'autre type de réclamation corrélé. Cette nouvelle distribution dynamique bivariée pour les nombres de réclamations est basée sur des effets aléatoires provenant de la famille de distributions multivariées Sarmanov.
Pour obtenir une distribution dynamique appropriée, une approximation de la distribution *a posteriori* des effets aléatoires est proposée. Le modèle qui en résulte peut être considéré comme une extension du modèle de l'hétérogénéité dynamique décrit dans Bolancé et al (2007).
Nous appliquons ce modèle à deux échantillons de données à partir d'une compagnie d'assurance canadienne, où nous montrons que le modèle proposé est l'un des meilleurs modèles

pour ajuster les données. Nous montrons aussi que qu'une telle modélisation permet plus de flexibilité dans le calcul des primes prédictives, puisque que des expressions fermées sont obtenues pour la distribution prédictive, les moments et les moments prédictifs.

Abstract

To predict future claims, it is well-known that the most recent claims are more predictive than older ones. However, classic panel data models for claim counts, such as the multivariate negative binomial distribution, do not put any time weight on past claims. More complex models can be used to consider this property, but often need numerical procedures to estimate parameters. When we want to add a dependence between different claim count types, the task would be even more difficult to handle. In this paper, we propose a bivariate dynamic model for claim counts, where past claims experience of a given claim type is used to better predict the other type of claims. This new bivariate dynamic distribution for claim counts is based on random effects that come from the Sarmanov family of multivariate distributions. To obtain a proper dynamic distribution based on this kind of bivariate priors, an approximation of the posterior distribution of the random effects is proposed. The resulting model can be seen as an extension of the dynamic heterogeneity model described in Bolancé et al (2007). We apply this model to two samples of data from a major Canadian insurance company, where we show that the proposed model is one of the best models to adjust the data. We also show that the proposed model allows more flexibility in computing predictive premiums because closed-form expressions can be easily derived for the predictive distribution, the moments and the predictive moments.

5.1 Introduction

One of the most critical problems in property and casualty insurance is to determine future numbers of claims and cost of claims. A related task is the calculation of the premium, i.e ratemaking. Parametric modeling of these random variables allows to identify the risk level through explanatory variables and clarifies the behavior of insureds. In this paper, we will focus only on the frequency part of this task, i.e. the modeling of the number of claims.

Risk classification techniques for claim counts have been the topic of many papers in the actuarial literature. For example, Denuit et al (2007) provide an exhaustive overview of count data models for insurance claims. In recent years, dependence between all the contracts of the same insured has been supposed in actuarial models, leading to what is called panel data modeling. Panel data modeling allows the premiums to depend on past claims experience, where the classic credibility theory can be used. In this paper, panel data models for claim counts are generalized in two ways: 1) by allowing different claim types to be modeled simultaneously, and 2) by allowing a time weight for past claims, because we expect that the most recent claims are more predictive than the oldest ones. To our knowledge, the proposed model is the first parametric model with continuous random effects to achieve these generalizations.

In actuarial sciences, the modeling of two different types of claims has already been studied. For example, Pinquet (1998) uses Poisson residuals to create dependence between at-fault and not-at-fault claims in automobile insurance, while Boucher and Inoussa (2014) use Bonus-Malus Systems with specific penalty rules allowing different claim types to affect the premium. Frees and Valdez (2008) also model various type of claims by decomposing all possibilities of claim types that may occur for a single accident. Generalizations to time-dependent heterogeneous models have also often been studied in the actuarial literature. To obtain a dynamic approach with continuous random effects, a parametric model would normally need T-dimensional integrals to express the joint distribution of all claims of a single insured (Xu et al (2007)). Consequently, complex numerical procedures that are not suited for panel data framework are sometimes needed (see for example Jung and Liesenfeld (2001)). Other approaches have been proposed to put a dynamic effect into count models: evolutionary credibility models in Gerber and Jones (1975), Jewell (1975), Poisson residuals in Pinquet et al (2001), or more recently copulas with the jittering method in Shi and Valdez (2014).

In this paper, to obtain this generalization of panel data models, the bivariate claim count distribution will be based on two conditional Poisson distributions with two gamma random effects distributions. Dependence will be supposed between the random effects, based on the Sarmanov family of multivariate distributions. This family of multivariate distributions has nice properties. Indeed, we show that this family of distributions offers flexibility in the choice of marginals and allows a closed-form expression for the joint density function. Additionally, we show that the posterior density of the bivariate random effects has approximately the same form as its prior. In particular, we show that the proposed model allows closed-form expressions for the predictive distribution, and a closed-form expression for the predictive premium, which can be an important insight for the insurer. Note that even if the illustrated model is used based on Poisson-gamma combinations, the proposed model can be easily used to generalize models with different conditional distributions or different random effects distributions.

In Section 5.2, we review the modeling of claim counts, where notations are set and random effects defined. In Section 5.3, we define the Sarmanov family of multivariate distribution. A multivariate extension of the dynamic model based on Harvey and Fernandes (1989) is presented in Section 5.4. The proposed model can be seen as an extension of the dynamic heterogeneity model described in Bolancé et al (2007). To be able to use such a dynamic approach, an approximation of the *a posteriori* Sarmanov distribution of the random effects is proposed, where it is supposed that this *a posteriori* distribution has the same form as the *a priori* distribution. Using data from a sample of a major Canadian insurance company, two numerical illustrations are performed in Section 5.5, where different claim types are used. Predictive premiums as well as the predictive variance are also computed and compared for various models. Section 5.6 concludes the paper.

5.2 Claim Count Modeling

5.2.1 General notations

We are interested in modeling the number of claims $N_{i,\ell,t}$, for the ith policyholder ($i = 1, ..., n$) of an insurance portfolio, of a given type of claim ℓ ($\ell = 1, 2$), at time t ($t = 1, ..., T$). To simplify the notations, subscript i will be removed for the remainder of the paper. To construct our model, we will suppose a conditional Poisson distribution of mean $\lambda_{\ell,t}\theta_\ell$, i.e.

$$(N_{\ell,t} \mid \Theta_\ell = \theta_\ell) \;\sim\; \text{Poisson}\left(\lambda_{\ell,t}\theta_\ell\right),$$

where $\lambda_{\ell,t} = \exp(\beta' x_{\ell,t})$ and $x_{\ell,t}$ represents the vector of all the pertinent covariates for claim type ℓ during year t. Classically, most of the ratemaking techniques rely on generalized linear models (GLM) (see McCullagh and Nelder (1989)) to estimate the regression parameters.

For each claim type, hidden characteristics are usually captured by an additional random term that affects all the contracts of the same insured. Each random effect is denoted by the random variable $\Theta_\ell, \ell = 1, 2$. Even if each claim type shares common hidden characteristics, we will first suppose that Θ_1 and Θ_2 are independent. This assumption will be relaxed later.

We assume that each Θ_ℓ is gamma distributed with shape parameter α_ℓ and scale parameter τ_ℓ. Both parameters α_ℓ and τ_ℓ are first considered stationary. Hence, we have

$$\Theta_\ell \;\sim\; \text{Gamma}\left(\alpha_\ell, \tau_\ell\right),$$

with probability density function (pdf) denoted by

$$h(\theta_\ell; \alpha_\ell, \tau_\ell) = \frac{\tau_\ell^{\alpha_\ell}}{\Gamma(\alpha_\ell)} \theta_\ell^{\alpha_\ell - 1} \exp(-\tau_\ell\theta_\ell).$$

Let us denote by $\mathbf{N}_\ell = (N_{\ell,1}, ..., N_{\ell,T})$ the vector of the number of claims, and $f_{N_{\ell,t}}(n_{\ell,t} \mid \Theta_\ell = \theta_\ell)$ the discrete conditional probability mass function of $(N_{\ell,t} \mid \Theta_\ell = \theta_\ell)$. Consequently, the joint probability mass function (pmf) of $\mathbf{N}_\ell$, denoted by $f_{\mathbf{N}_\ell}(\boldsymbol{n}_\ell; \alpha_\ell, \tau_\ell)$, is given by

$$
\begin{aligned}
f_{\mathbf{N}_\ell}(\boldsymbol{n}_\ell;\alpha_\ell,\tau_\ell) &= \Pr(\mathbf{N}_\ell=\mathbf{n}_\ell)\\
&= \int_0^\infty f_{\mathbf{N}_\ell}(\mathbf{n}_\ell|\Theta_\ell=\theta_\ell)h(\theta_\ell;\tau_\ell,\alpha_\ell)d\theta_\ell\\
&= \left(\prod_{t=1}^T \frac{\lambda_{\ell,t}^{n_{\ell,t}}}{n_{\ell,t}!}\right)\frac{\Gamma(n_{\ell,\bullet}+\alpha_\ell)}{\Gamma(\alpha_\ell)}\left(\frac{\tau_\ell}{\lambda_{\ell,\bullet}+\tau_\ell}\right)^{\alpha_\ell}(\lambda_{\ell,\bullet}+\tau_\ell)^{-n_{\ell,\bullet}}, \qquad (5.1)
\end{aligned}
$$

which corresponds to the joint pdf of a multivariate negative binomial random vector (MVNB), with $n_{\ell,\bullet}=\sum_{t=1}^T n_{\ell,t}$ and $\lambda_{\ell,\bullet}=\sum_{t=1}^T \lambda_{\ell,t}$. See Boucher et al. (2008) for details. In the stationary case, for parameter identification, we suppose that $\alpha_\ell=\tau_\ell$. In this case, the marginal moments of $N_{\ell,t}$ are given by

$$
E[N_{\ell,t}]=\lambda_{\ell,t}\frac{\alpha_\ell}{\tau_\ell}=\lambda_{\ell,t} \quad\text{and}\quad \text{Var}(N_{\ell,t})=\lambda_{\ell,t}\frac{\alpha_\ell}{\tau_\ell}+\lambda_{\ell,t}^2\frac{\alpha_\ell}{\tau_\ell^2}=\lambda_{\ell,t}+\frac{\lambda_{\ell,t}^2}{\alpha_\ell}.
$$

For ratemaking purposes, $E[N_{\ell,t}]$ is often called the *a priori* premium because it is the premium charged to new insureds, or insureds without claims experience.

5.2.2 Predictive Distribution

The random effect term models the heterogeneity of the model and incorporates the hidden characteristics. Consequently, it is reasonable to believe that these hidden characteristics are partly revealed by the number of claims reported by the policyholders. Indeed, at each insured period, the random effects can be updated given the past claim experience, revealing some insured-specific information. Henceforth, insightful information can be retrieved from the claim experience.

The Poisson and gamma distributions are natural conjugates, thus the *a posteriori* distribution is again a gamma distribution with updated parameters $\alpha_\ell^*=\alpha_\ell+\sum_{t=1}^T n_{\ell,t}$ and $\tau_\ell^*=\tau_\ell+\sum_{t=1}^T \lambda_{\ell,t}$ (see Boucher et al (2008) for details). Thus, the predictive mean can be expressed as

$$
E[N_{\ell,T+1}\mid N_{\ell,1},...,.N_{\ell,T}] = \lambda_{\ell,T+1}\frac{\alpha_\ell^*}{\tau_\ell^*}=\lambda_{\ell,T+1}\frac{\alpha_\ell+\sum_{t=1}^T n_{\ell,t}}{\alpha_\ell+\sum_{t=1}^T \lambda_{\ell,t}}. \qquad (5.2)
$$

In a ratemaking context, this is often called the predictive premium. We clearly observe how past experience is incorporated in the computation of the predictive mean. Indeed, we can see that all past claims have equal weight in the predictive premium calculation, meaning that an old claim increases the premium as much as a newer claim does. A more intuitive model would suppose that both Θ_ℓ, $\ell=1,2$, can evolve over time, resulting in a model where the most recent claims are more predictive than the oldest ones.

5.3 Sarmanov Family of Bivariate Distributions

Sarmanov's bivariate distribution was introduced in the literature by Sarmanov (1966), and was also proposed in physics by Cohen (1984) under a more general form. Lee (1996) suggests a multivariate version and discusses several applications in medicine. Recently, due to its flexible structure, Sarmanov's bivariate distribution gained interest in different applied studies. For example, Schweidel et al (2008) use a bivariate Sarmanov model to capture the relationship between a prospective customer's time until acquisition of a particular service and the subsequent duration for which the service is retained. Miravete (2009) presents two models based on the Sarmanov distribution and uses them to compare the number of tariff plans offered by two competing cellular telephone companies. Danaher and Smith (2011) discuss applications to marketing (see also the references therein). In the insurance field, Hernández-Bastida and Fernández-Sánchez (2012) use the bivariate Sarmanov distribution for premium evaluation, more recently Abdallah et al (2015) use this family of distributions to show its suitability in a loss reserving context. In this paper, we use the Sarmanov distribution to accommodate correlation of unknown characteristics of a driver that might impact all types of claims simultaneously.

5.3.1 Definitions

Let the random couple $\mathbf{\Theta} = (\Theta_1, \Theta_2)$ have a bivariate Sarmanov distribution, with gamma marginals

$$u^S(\theta_1, \theta_2) = h(\theta_1; \alpha_1, \tau_1)\, h(\theta_2; \alpha_2, \tau_2)\,(1 + \omega\phi_1(\theta_1)\,\phi_2(\theta_2)), \tag{5.3}$$

where ϕ_ℓ, $\ell = 1, 2$ are two bounded non-constant functions such that $\int_{-\infty}^{\infty} \phi_\ell(t)\, u_\ell(t)\, dt = 0$ and ω is a real number that satisfies the condition

$$1 + \omega\phi_1(\theta_1)\,\phi_2(\theta_2) \geq 0 \text{ for all } \theta_\ell\ ,\ i \in \{1, 2\}.$$

One of the main interesting properties of the Sarmanov distribution is that the multivariate distribution can support a wide range of marginals, such as the gamma distribution. Different methods are proposed in Lee (1996) to construct mixing functions ϕ_ℓ for different types of marginals. As mentioned in Lee (1996), different types of mixing functions can be used to yield different multivariate distributions with the same set of marginals. Based on Corollary 2 in Lee (1996), a mixing function can be defined as $\phi_\ell(\theta_\ell) = \exp(-\theta_\ell) - L_\ell$, where L_ℓ is the Laplace transform of the marginal distribution evaluated at 1. Hence, given our choice of distribution for Θ_ℓ, $\ell = 1, 2$, we have

$$\phi_\ell(\theta_\ell) = \exp(-\theta_\ell) - \left(\frac{\tau_\ell}{1 + \tau_\ell}\right)^{\alpha_\ell}.$$

As for the dependence parameter ω of the bivariate Sarmanov distribution, in the case of gamma marginals, it is bounded as follows $B_{\inf} < \omega < B_{\sup}$ with

$$B_{\inf} = \frac{-1}{\max\left\{\left(\frac{\tau_1}{1+\tau_1}\right)^{\alpha_1}\left(\frac{\tau_2}{1+\tau_2}\right)^{\alpha_2}, \left(1-\left(\frac{\tau_1}{1+\tau_1}\right)^{\alpha_1}\right)\left(1-\left(\frac{\tau_2}{1+\tau_2}\right)^{\alpha_2}\right)\right\}}$$

$$B_{\sup} = \frac{1}{\max\left\{\left(\frac{\tau_1}{1+\tau_1}\right)^{\alpha_1}\left(1-\left(\frac{\tau_2}{1+\tau_2}\right)^{\alpha_2}\right), \left(\frac{\tau_2}{1+\tau_2}\right)^{\alpha_2}\left(1-\left(\frac{\tau_1}{1+\tau_1}\right)^{\alpha_1}\right)\right\}}.$$

This result is given in Corollary 2 of Lee (1996). Consequently, for gamma marginals, and using the notations of the previous section, the prior joint pdf of (Θ_1, Θ_2) is given by

$$\begin{aligned} u^S(\theta_1, \theta_2) &= (1+\vartheta)\, h(\theta_1; \alpha_1, \tau_1) h(\theta_2; \alpha_2, \tau_2) + \vartheta h(\theta_1; \alpha_1, \tau_1+1) h(\theta_2; \alpha_2, \tau_2+1) \\ &\quad -\vartheta h(\theta_1; \alpha_1, \tau_1) h(\theta_2; \alpha_2, \tau_2+1) - \vartheta h(\theta_1; \alpha_1, \tau_1+1) h(\theta_2; \alpha_2, \tau_2), \end{aligned} \tag{5.4}$$

where $\vartheta = \omega \left(\frac{\tau_1}{1+\tau_1}\right)^{\alpha_1}\left(\frac{\tau_2}{1+\tau_2}\right)^{\alpha_2}$. This last expression corresponds to a linear combination of the product of univariate (gamma) pdf's and highlights the attractive features of the Sarmanov family of distributions.

5.3.2 Bivariate Count Distributions

A critical problem when modeling dependence between claim counts is to obtain a closed-form expression for the joint distribution. The Sarmanov distribution will be a good ally to circumvent this problem. Let us denote by $f_{\mathbf{N}_1,\mathbf{N}_2}$ the discrete joint probability mass function of $(\mathbf{N}_1, \mathbf{N}_2)$, i.e. $f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1, \mathbf{n}_2) = \Pr(\mathbf{N}_1 = \mathbf{n}_1, \mathbf{N}_2 = \mathbf{n}_2)$ which can be expressed as

$$\begin{aligned} f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1, \mathbf{n}_2) &= \int_0^\infty \int_0^\infty f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1, \mathbf{n}_2 | \Theta_1 = \theta_1, \Theta_2 = \theta_2) u^S(\theta_1, \theta_2) d\theta_1 d\theta_2 \\ &= (1+\vartheta)\, f_{\mathbf{N}_1}(\boldsymbol{n}_1; \alpha_1, \tau_1) f_{\mathbf{N}_2}(\boldsymbol{n}_2; \alpha_2, \tau_2) + \vartheta f_{\mathbf{N}_1}(\boldsymbol{n}_1; \alpha_1, \tau_1+1) f_{\mathbf{N}_2}(\boldsymbol{N}_2; \alpha_2, \tau_2+1) \\ &\quad -\vartheta f_{\mathbf{N}_1}(\boldsymbol{N}_1; \alpha_1, \tau_1) f_{\mathbf{N}_2}(\boldsymbol{N}_2; \alpha_2, \tau_2+1) - \vartheta f_{\mathbf{N}_1}(\boldsymbol{N}_1; \alpha_1, \tau_1+1) f_{\mathbf{N}_2}(\boldsymbol{N}_2; \alpha_2, \tau_2). \end{aligned} \tag{5.5}$$

Note that we obtain a linear combination of products of MVNB distributions. The simplicity and form of the model greatly facilitate many calculations, such as the moments of the distribution. For the *a priori* mean, we obtain the following result:

$$E\left[N_{1,t} + N_{2,t}\right] = \lambda_{1,t}\frac{\alpha_1}{\tau_1} + \lambda_{2,t}\frac{\alpha_2}{\tau_2}, \tag{5.6}$$

Note that the mean of the sum is the same as the one obtained for the sum of two MVNB distributions. However, a covariance term is added to the sum of the variance, as shown in the following result:

$$\mathrm{Var}(N_{1,t}+N_{2,t}) = \lambda_{1,t}\frac{\alpha_1}{\tau_1}+\lambda_{1,t}^2\frac{\alpha_1}{\tau_1^2}+\lambda_{2,t}\frac{\alpha_2}{\tau_2}+\lambda_{2,t}^2\frac{\alpha_2}{\tau_2^2}$$
$$+2\lambda_{1,t}\lambda_{2,t}\left(\vartheta\frac{\alpha_1}{\tau_1}\frac{\alpha_2}{\tau_2}+\vartheta\frac{\alpha_1}{\tau_1+1}\frac{\alpha_2}{\tau_2+1}-\vartheta\frac{\alpha_1}{\tau_1+1}\frac{\alpha_2}{\tau_2}-\vartheta\frac{\alpha_1}{\tau_1}\frac{\alpha_2}{\tau_2+1}\right). \tag{5.7}$$

Similarly to what Purcaru and Denuit (2002) did for univariate claim count models, it would be interesting to analyze the dependence induced by this kind of model. Recently, Bolancé et al (2014) show that the Sarmanov family of distributions has upper tail dependence equal to zero when the marginal distributions have tail Gumbel type, as Gamma distribution. However, this impact is mitigated in our context and findings, because the marginals of the Sarmanov family of multivariate distributions represent the heterogeneity components in our model. The interpretation of such a finding in our context would mean that the probability that the couple (Θ_1, Θ_2) has two extreme heterogeneity components tends to zero.

Predictive Joint Distribution

As done with the univariate analysis of Section 5.2.2, the posterior distribution is also useful as it reveals insured-specific information. Because of the bivariate structure of the random effects, past claims experience of a given claim type can be used to update the random effects distribution of the other type of claims. The posterior bivariate joint density function of the couple (Θ_1, Θ_2) conditioned on $(\mathbf{N}_1, \mathbf{N}_2)$ is given by

$$u^S(\theta_1,\theta_2 \mid \mathbf{n}_1,\mathbf{n}_2) = \psi_1 h(\theta_1;\alpha_1^*,\tau_1^*)h(\theta_2;\alpha_2^*,\tau_2^*)+\psi_2 h(\theta_1;\alpha_1^*,\tau_1^*+1)h(\theta_2;\alpha_2^*,\tau_2^*+1)$$
$$-\psi_3 h(\theta_1;\alpha_1^*,\tau_1^*)h(\theta_2;\alpha_2^*,\tau_2^*+1)-\psi_4 h(\theta_1;\alpha_1^*,\tau_1^*+1)h(\theta_2;\alpha_2^*,\tau_2^*), \tag{5.8}$$

where $\alpha_\ell^* = \alpha_\ell + n_{\ell,\bullet}$, $\tau_\ell^* = \tau_\ell + \lambda_{\ell,\bullet}$, $\ell = 1,2$, and

$$\psi_1 = \frac{1}{f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1,\mathbf{n}_2)}(1+\vartheta)\, f_{\mathbf{N}_1}(\boldsymbol{n}_1;\alpha_1,\tau_1)f_{\mathbf{N}_2}(\boldsymbol{n}_2;\alpha_2,\tau_2)$$
$$\psi_2 = \frac{1}{f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1,\mathbf{n}_2)}\vartheta f_{\mathbf{N}_1}(\boldsymbol{n}_1;\alpha_1,\tau_1+1)f_{\mathbf{N}_2}(\boldsymbol{n}_2;\alpha_2,\tau_2+1)$$
$$\psi_3 = \frac{1}{f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1,\mathbf{n}_2)}\vartheta f_{\mathbf{N}_1}(\boldsymbol{n}_1;\alpha_1,\tau_1)f_{\mathbf{N}_2}(\boldsymbol{n}_2;\alpha_2,\tau_2+1)$$
$$\psi_4 = \frac{1}{f_{\mathbf{N}_1,\mathbf{N}_2}(\mathbf{n}_1,\mathbf{n}_2)}\vartheta f_{\mathbf{N}_1}(\boldsymbol{n}_1;\alpha_1,\tau_1+1)f_{\mathbf{N}_2}(\boldsymbol{n}_2;\alpha_2,\tau_2).$$

This last expression shows that the posterior bivariate density function of (Θ_1, Θ_2), is again a linear combination of the product of univariate gamma pdfs. The posterior density is hence called a pseudo-conjugate to the prior density (Lee (1996)) in the sense that the posterior density is a linear combination of products of densities from the univariate natural exponential family of distributions (gamma in our case).

The joint predictive distribution of $\mathbf{N}_1$ and $\mathbf{N}_2$ at time $T+1$, given all the past observations up to time T can also be computed. This will enable us to evaluate notably the expected annual claim frequency conditionally on past experience. The Sarmanov distribution allows us to obtain a closed-form expression for this joint prediction. Indeed, using (5.1) and (5.8), we get

$$
\begin{aligned}
& f_{N_{1,T+1},N_{2,T+1}|\mathbf{N}_1,\mathbf{N}_2}\left(n_{1,T+1}, n_{2,T+1}\right) = \\
& \quad \psi_1 f_{N_{1,T+1}|\mathbf{N}_1}\left(n_{1,T+1}; \alpha_1 + n_{1,\bullet}, \tau_1 + \lambda_{1,\bullet}\right) f_{N_{2,T+1}|\mathbf{N}_2}\left(n_{2,T+1}; \alpha_2 + n_{2,\bullet}, \tau_2 + \lambda_{2,\bullet}\right) \\
& \quad +\psi_2 f_{N_{1,T+1}|\mathbf{N}_1}\left(n_{1,T+1}; \alpha_1 + n_{1,\bullet}, \tau_1 + 1 + \lambda_{1,\bullet}\right) f_{N_{2,T+1}|\mathbf{N}_2}\left(n_{2,t+1}; \alpha_2 + n_{2,\bullet}, \tau_2 + 1 + \lambda_{2,\bullet}\right) \\
& \quad -\psi_3 f_{N_{1,T+1}|\mathbf{N}_1}\left(n_{1,T+1}; \alpha_1 + n_{1,\bullet}, \tau_1 + \lambda_{1,\bullet}\right) f_{N_{2,T+1}|\mathbf{N}_2}\left(n_{2,T+1}; \alpha_2 + n_{2,\bullet}, \tau_2 + 1 + \lambda_{2,\bullet}\right) \\
& \quad -\psi_4 f_{N_{1,T+1}|\mathbf{N}_1}\left(n_{1,T+1}; \alpha_1 + n_{1,\bullet}, \tau_1 + 1 + \lambda_{1,\bullet}\right) f_{N_{2,T+1}|\mathbf{N}_2}\left(n_{2,T+1}; \alpha_2 + n_{2,\bullet}, \tau_2 + \lambda_{2,\bullet}\right),
\end{aligned}
\tag{5.9}
$$

with ψ_j, $j = 1, 2, 3, 4$ as given in (5.8) and $n_{\ell,\bullet}$ and $\lambda_{\ell,\bullet}$ as given in (5.1), where we suppose

$$
\begin{aligned}
f_{N_{\ell,T+1}|\mathbf{N}_\ell}\left(n_{\ell,t+1}; \alpha_*, \tau_*\right) &= \left(\frac{\lambda_{\ell,t+1}^{n_{\ell,t+1}}}{n_{\ell,t+1}!}\right) \frac{\Gamma\left(n_{\ell,t+1} + \alpha_*\right)}{\Gamma(\alpha_*)} \frac{\tau_*^{\alpha_*}}{\left(\lambda_{\ell,t+1} + \tau_*\right)^{n_{\ell,t+1}+\alpha_*}} \\
&= \frac{\Gamma(\alpha_* + n_{\ell,t+1})}{\Gamma(\alpha_*)\Gamma(n_{\ell,t+1}+1)} \left(\frac{\tau_*}{\lambda_{\ell,t+1} + \tau_*}\right)^{\alpha_*} \left(\frac{\lambda_{\ell,t+1}}{\lambda_{\ell,t+1} + \tau_*}\right)^{n_{\ell,t+1}},
\end{aligned}
\tag{5.10}
$$

which corresponds to a negative binomial distribution with parameters α_* and $\dfrac{\tau_*}{\lambda_{\ell,t+1} + \tau_*}$.

One of the main advantages of using the Sarmanov family of bivariate distributions is the possibility to derive closed-form expressions for the mean and variance of the total future number of claims. Let $\Im_{\ell,T}$ be the history of claim counts of type ℓ up to time T. Mathematically, $\Im_{\ell,T}$ is the sigma algebra generated by the random variables $N_{\ell,1}, N_{\ell,2}, ..., N_{\ell,T}$, with $\Im_T = (\Im_{1,T}, \Im_{2,T})$ and $N_{T+1}^{\text{tot}} = N_{1,T+1} + N_{2,T+1}$. It can be shown that the total expected annual claim frequency for year $T+1$ is

$$
\begin{aligned}
E\left[N_{T+1}^{\text{tot}} \mid \Im_T\right] &= \lambda_{1,T+1}\left(\left(\psi_1 - \psi_4\right)\frac{\alpha_1^*}{\tau_1^*} + \left(\psi_2 - \psi_3\right)\frac{\alpha_1^*}{\tau_1^* + 1}\right) \\
&+ \lambda_{2,T+1}\left(\left(\psi_1 - \psi_3\right)\frac{\alpha_2^*}{\tau_2^*} + \left(\psi_2 - \psi_4\right)\frac{\alpha_2^*}{\tau_2^* + 1}\right),
\end{aligned}
\tag{5.11}
$$

with $\alpha_\ell^* = \alpha_\ell + n_{\ell,\bullet}$, $\tau_\ell^* = \tau_\ell + \lambda_{\ell,\bullet}$, $\ell = 1, 2$. The total variance of the sum can be expressed as

$$
\begin{aligned}
\mathrm{Var}(N^{\mathrm{tot}}_{T+1} \mid \Im_T) &= \lambda_{1,T+1}\left((\psi_1-\psi_4)\frac{\alpha_1^*}{\tau_1^*}+(\psi_2-\psi_3)\frac{\alpha_1^*}{\tau_1^*+1}\right)\\
&+ \lambda_{1,T+1}^2\left((\psi_1-\psi_4)\frac{\alpha_1^*}{\tau_1^{*2}}+(\psi_2-\psi_3)\frac{\alpha_1^*}{(\tau_1^*+1)^2}\right)\\
&+ \lambda_{2,T+1}\left((\psi_1-\psi_3)\frac{\alpha_2^*}{\tau_2^*}+(\psi_2-\psi_4)\frac{\alpha_2^*}{\tau_2^*+1}\right)\\
&+ \lambda_{2,T+1}^2\left((\psi_1-\psi_3)\frac{\alpha_2^*}{\tau_2^{*2}}+(\psi_2-\psi_4)\frac{\alpha_2^*}{(\tau_2^*+1)^2}\right)\\
&+ 2\lambda_{1,T+1}\lambda_{2,T+1}\left(\psi_1\frac{\alpha_1^*}{\tau_1^*}\frac{\alpha_2^*}{\tau_2^*}+\psi_2\frac{\alpha_1^*}{\tau_1^*+1}\frac{\alpha_2^*}{\tau_2^*+1}-\psi_3\frac{\alpha_1^*}{\tau_1^*+1}\frac{\alpha_2^*}{\tau_2^*}-\psi_4\frac{\alpha_1^*}{\tau_1^*}\frac{\alpha_2^*}{\tau_2^*+1}\right)\\
&- 2\left(\lambda_{1,T+1}\left((\psi_1-\psi_4)\frac{\alpha_1^*}{\tau_1^*}+(\psi_2-\psi_3)\frac{\alpha_1^*}{\tau_1^*+1}\right)\right)\left(\lambda_{2,t+1}\left((\psi_1-\psi_3)\frac{\alpha_2^*}{\tau_2^*}+(\psi_2-\psi_4)\frac{\alpha_2^*}{\tau_2^*+1}\right)\right),
\end{aligned}
\tag{5.12}
$$

with $\alpha_\ell^* = \alpha_\ell + n_{\ell,\bullet}$, $\tau_\ell^* = \tau_\ell + \lambda_{\ell,\bullet}$, $\ell = 1, 2$.

It is worth mentioning that, as expected, the model borrows past information from one series to predict future claim counts of the other series. Indeed, the terms ψ_j for $j = 1, 2, 3, 4$ depend on $N_{\ell,1}, N_{\ell,2}, ..., N_{\ell,T}$, $\ell = 1, 2$, and are used to compute the expected value of each type of claim. The dependence parameter ω intervenes in the predictive mean and variance computations through the same terms ψ_j. Simplified and developed expressions of the terms ψ_j's are presented later in the paper in another context.

5.4 Multivariate Dynamic Random Effects

5.4.1 Model and motivations

Models where the random effects Θ_ℓ evolve over time would normally need T-dimensional integrals to express the joint distribution of all claims of a single insured. Consequently, complex numerical procedures or approximated inference methods are sometimes needed (see for example Jung and Liesenfeld (2001) or Xu et al (2007)). Other approaches have been proposed to put a dynamic effect into count models: evolutionary credibility models in Gerber and Jones (1975), Jewell (1975), Poisson residuals in Pinquet et al (2001), or more recently copulas with the jittering method in Shi and Valdez (2014).

In our paper, we propose an extension of Bolancé et al (2007), which is based on the idea of Harvey and Fernandes (1989). We note this model as the H-F model, referring directly to Harvey-Fernandes. Their model supposes that the risk characteristics are captured through a dynamic effect $\Theta_{\ell,t}$, which is considered evolutionary and time-dependent, i.e. that its distribution evolves over time and is updated through past experience. Formally, the classic Poisson-gamma model described in Section 5.2.1 is generalized and allows the underlying risk parameter to vary in successive periods, with the following dynamic:

$$(\Theta_{\ell,t} \mid \Im_{\ell,t}) \;\sim\; \text{Gamma}\left(\alpha_{\ell,t}, \tau_{\ell,t}\right) \;,\; \ell \in \{1,2\}. \tag{5.13}$$

It is also supposed that

$$(\Theta_{\ell,t} \mid \Im_{\ell,t-1}) \;\sim\; \text{Gamma}\left(\alpha_{\ell,t|t-1}, \tau_{\ell,t|t-1}\right) \;,\; \ell \in \{1,2\}, \tag{5.14}$$

where

$$\begin{cases} \alpha_{\ell,t|t-1} = \nu_\ell \alpha_{\ell,t-1} \\ \tau_{\ell,t|t-1} = \nu_\ell \tau_{\ell,t-1}. \end{cases} \tag{5.15}$$

The parameter ν_ℓ is a weighting parameter less than or equal to 1. The initial conditions of the dynamic model, i.e. the distribution of $\Theta_{\ell,1}$, is supposed $\text{Gamma}\left(\alpha_{\ell,0}, \tau_{\ell,0}\right)$, with $\alpha_{\ell,0} = \tau_{\ell,0}$. This means that the premium for the first year equals $\lambda_{\ell,1}$, because $E\left[\Theta_{\ell,1}\right] = 1$.

Using Bayes' theorem, the posterior distribution for $(\Theta_{\ell,t} \mid \Im_{\ell,t})$ is again a gamma distribution with updated parameters

$$\begin{cases} \alpha_{\ell,t} = \nu_\ell \alpha_{\ell,t-1} + n_{\ell,t} \\ \tau_{\ell,t} = \nu_\ell \tau_{\ell,t-1} + \lambda_{\ell,t}. \end{cases}$$

By induction, the above parameters can be expressed recursively as follows:

$$\begin{cases} \alpha_{\ell,t} = \left(\nu_\ell\right)^t \alpha_{\ell,0} + \sum_{k=0}^{t-1} \left(\nu_\ell\right)^k n_{\ell,t-k} \\ \tau_{\ell,t} = \left(\nu_\ell\right)^t \alpha_{\ell,0} + \sum_{k=0}^{t-1} \left(\nu_\ell\right)^k \lambda_{\ell,t-k}. \end{cases} \tag{5.16}$$

Given the past experience, the resulting joint distribution of $\mathbf{N}_\ell = N_{\ell,1}, ..., N_{\ell,T}$ can then be expressed as

$$f_{\mathbf{N}_\ell}(\mathbf{n}_\ell; \alpha_{\ell,t}, \tau_{\ell,t}) = \left(\prod_{t=1}^{T} \frac{\lambda_{\ell,t}^{n_{\ell,t}}}{n_{\ell,t}!} \right) \frac{\Gamma(n_{\ell,\bullet} + \alpha_{\ell,t|t-1})}{\Gamma(\alpha_{\ell,t|t-1})} \left(\frac{\tau_{\ell,t|t-1}}{\lambda_{\ell,\bullet} + \tau_{\ell,t|t-1}} \right)^{\alpha_{\ell,t|t-1}} (\lambda_{\ell,\bullet} + \tau_{\ell,t|t-1})^{-n_{\ell,\bullet}}, \tag{5.17}$$

where $\alpha_{\ell,t|t-1}$ and $\tau_{\ell,t|t-1}$ are as given in (5.15).

We observe that the multivariate joint distribution is similar to equation (5.1), but the random effects parameters are now time-dependent. Note that unlike the stationary model, where the sum of claim counts was a sufficient statistic, the dynamic model keeps the time period of each claim. The *a priori* moments of the H-F model, are given by

$$E[N_{\ell,t}] = \lambda_{\ell,t} \;\text{ and }\; \text{Var}(N_{\ell,t}) = \lambda_{\ell,t} + \frac{\lambda_{\ell,t}^2}{\alpha_{\ell,0}},$$

meanwhile the predictive moments can be expressed as

$$E[N_{\ell,T+1} \mid N_{\ell,1}, ..., .N_{\ell,T}] = \lambda_{\ell,T+1} \frac{\alpha_{\ell,T}}{\tau_{\ell,T}} \;\text{ and }\; \text{Var}(N_{\ell,T+1}) = \lambda_{\ell,T+1} \frac{\alpha_{\ell,T}}{\tau_{\ell,T}} + \lambda_{\ell,T+1}^2 \frac{\alpha_{\ell,T}}{\tau_{\ell,T}^2},$$

where $\alpha_{\ell,T}$ and $\tau_{\ell,T}$ are obtained following equation (5.16) with $\ell = 1, 2$. We observe that the *a priori* and predictive moments of this model have the same expressions as in the MVNB model, with time-dependent underlying parameters.

5.4.2 Sarmanov distribution and dynamic heterogeneity

One of the main advantages of the Sarmanov family of bivariate distributions is its pseudo-conjugate property for the posterior distribution (see equation (5.8)). However, this property might not be sufficient to directly suppose a dynamic structure for the Sarmanov distribution. Indeed, to be able to assume a dynamic approach with the Sarmanov distribution, like the one proposed for the Poisson-gamma model (or MVNB) in equation (5.16), the bivariate *a posteriori* distribution of the random effects needs to be a conjugate to the prior, where updated parameters α_ℓ^*, τ_ℓ^*, $\ell = 1, 2$ can be modified easily using a structure similar to the equations above. To obtain such a structure, the *a posteriori* distribution of the correlated random effects needs to be, once again, a member of the family of Sarmanov multivariate distribution. As just specified, the Sarmanov family of bivariate distributions does not possess this conjugate property, but its pseudo-conjugate property might enable us to construct an interesting alternative.

The posterior distribution of random effects obtained in (5.8) is a weighted sum of posterior gamma distributions, with $\alpha_\ell^* = \alpha_\ell + n_{\ell,\bullet}$, $\tau_\ell^* = \tau_\ell + \lambda_{\ell,\bullet}$, $\ell = 1, 2$. The difference between what would have been called a conjugate distribution and the pseudo-conjugate comes from $\psi_1, \psi_2, \psi_3, \psi_4$ that are not expressed solely in terms of α_ℓ^* and τ_ℓ^*, but also α_ℓ and τ_ℓ. We propose to modify the posterior distribution to obtain a distribution that is only a function of α_ℓ^* and τ_ℓ^*. This modification will be the first step to obtain a dynamic bivariate count distribution. The proposed posterior Sarmanov distribution, now referred to as Approximated Sarmanov, is then expressed as:

$$
\begin{aligned}
u^{S^*}(\theta_1, \theta_2 \mid \mathbf{n}_1, \mathbf{n}_2) \quad = \quad & (1 + \vartheta^*)\, h(\theta_1; \alpha_1^*, \tau_1^*) h(\theta_2; \alpha_2^*, \tau_2^*) + \vartheta^* h(\theta_1; \alpha_1^*, \tau_1^* + 1) h(\theta_2; \alpha_2^*, \tau_2^* + 1) \\
& - \vartheta^* h(\theta_1; \alpha_1^*, \tau_1^*) h(\theta_2; \alpha_2^*, \tau_2^* + 1) - \vartheta^* h(\theta_1; \alpha_1^*, \tau_1^* + 1) h(\theta_2; \alpha_2^*, \tau_2^*),
\end{aligned}
\tag{5.18}
$$

where the distribution has the same form as the *a priori* distribution of (Θ_1, Θ_2) with updated parameters $\vartheta^* = \omega \left(\frac{\tau_1^*}{1+\tau_1^*}\right)^{\alpha_1^*} \left(\frac{\tau_2^*}{1+\tau_2^*}\right)^{\alpha_2^*}$, where $\alpha_\ell^* = \alpha_\ell + n_{\ell,\bullet}$, $\tau_\ell^* = \tau_\ell + \lambda_{\ell,\bullet}$, for $\ell = 1, 2$.

Quality of the Approximation

The Approximated Sarmanov distribution for random effects has the desired properties to be generalized into a dynamic approach. However, before adding the dynamic structure, we need to quantify the approximation of the *a posteriori* Sarmanov distribution. The difference between the Sarmanov and Approximated Sarmanov can be expressed as:

$$
\begin{aligned}
& u^{S^*}(\theta_1,\theta_2 \mid \mathbf{n}_1,\mathbf{n}_2) - u^{S}(\theta_1,\theta_2 \mid \mathbf{n}_1,\mathbf{n}_2) = \\
& \quad \delta_1 h(\theta_1;\alpha_1^*,\tau_1^*)h(\theta_2;\alpha_2^*,\tau_2^*) + \delta_2 h(\theta_1;\alpha_1^*,\tau_1^*+1)h(\theta_2;\alpha_2^*,\tau_2^*+1) \\
& \quad -\delta_3 h(\theta_1;\alpha_1^*,\tau_1^*)h(\theta_2;\alpha_2^*,\tau_2^*+1) - \delta_4 h(\theta_1;\alpha_1^*,\tau_1^*+1)h(\theta_2;\alpha_2^*,\tau_2^*).
\end{aligned} \tag{5.19}
$$

Each term δ_j, $j = 1,2,3,4$ can be simplified as:

$$
\begin{aligned}
\delta_1 &= \left[1+\omega\left(\frac{\tau_1^*}{1+\tau_1^*}\right)^{\alpha_1^*}\left(\frac{\tau_2^*}{1+\tau_2^*}\right)^{\alpha_2^*}\right] - \psi_1 \\
&= \vartheta^*\left(1-\frac{L_1/L_1^*+L_2/L_2^*-1}{1+\vartheta+\vartheta^*(1-L_1/L_1^*-L_2/L_2^*)}\right) \\
\delta_2 &= \omega\left(\frac{\tau_1^*}{1+\tau_1^*}\right)^{\alpha_1^*}\left(\frac{\tau_2^*}{1+\tau_2^*}\right)^{\alpha_2^*} - \psi_2 \\
&= \vartheta^*\left(1-\frac{1}{1+\vartheta+\vartheta^*\left(1-L_1/L_1^*-L_2/L_2^*\right)}\right) \\
\delta_3 &= \omega\left(\frac{\tau_1^*}{1+\tau_1^*}\right)^{\alpha_1^*}\left(\frac{\tau_2^*}{1+\tau_2^*}\right)^{\alpha_2^*} - \psi_3 \\
&= \vartheta^*\left(1-\frac{L_2/L_2^*}{1+\vartheta+\vartheta^*\left(1-L_1/L_1^*-L_2/L_2^*\right)}\right) \\
\delta_4 &= \omega\left(\frac{\tau_1^*}{1+\tau_1^*}\right)^{\alpha_1^*}\left(\frac{\tau_2^*}{1+\tau_2^*}\right)^{\alpha_2^*} - \psi_4. \\
&= \vartheta^*\left(1-\frac{L_1/L_1^*}{1+\vartheta+\vartheta^*\left(1-L_2/L_2^*-L_1/L_1^*\right)}\right),
\end{aligned}
$$

where $\vartheta^* = \omega\left(\frac{\tau_1^*}{1+\tau_1^*}\right)^{\alpha_1^*}\left(\frac{\tau_2^*}{1+\tau_2^*}\right)^{\alpha_2^*}$, $L_\ell = \left(\frac{\tau_\ell}{1+\tau_\ell}\right)^{\alpha_\ell}$, and $L_\ell^* = \left(\frac{\tau_\ell^*}{1+\tau_\ell^*}\right)^{\alpha_\ell^*}$, for $\ell = 1,2$. It is worth-mentioning that the differences δ_j's are complementary and offsetting one another, resulting in a sum of differences equal to zero. This condition comes from the fact that (5.19) is a difference between two proper distributions.

We analyzed the approximation for different values of the parameters. The Approximated Sarmanov is identical to the Sarmanov distribution when $\omega = 0$, which is the particular case of independent random effects. We also observed that the values of δ_j, for $j = 1,...,4$ are proportional to ω. The difference caused by the approximation also depends on the parameters α_ℓ^*, τ_ℓ^*, and thus depends on the time of each claim and also on sums of past claims, i.e. $n_{\ell,\bullet}$ for $\ell = 1,2$. In Figure 5.1, for a specific choice of parameters, as a function of $n_{\ell,\bullet}$ for $\ell = 1,2$, we illustrate the values of the δ_j's for a large observation period $T = 100$.

We see that the approximation is less accurate in the cases where $n_{1,\bullet}$ and $n_{2,\bullet}$ tend to behave inversely. This represents unusual situations because claims of different types are assumed to be positively correlated, which means that an insured with a high value of $n_{1,\bullet}$ should also normally have a high number of past claims of type 2. When the numbers of past claims are similar for each type, the approximation seems to be more accurate and reasonable.

We also analyze the model for smaller time periods because it is more realistic for insurance data. Indeed, insurance datasets are usually constructed with $T < 10$ (see Boucher et al (2008) for example). We illustrate the values of the δ_j's for $T = 5$ in Figure 5.2. Our analysis shows that the differences expressed by the δ_j's are closer to zero (note that the scales are different than those of Figure 5.1). Henceforth, one observes differences between the Approximated Sarmanov and the original Sarmanov models, but the highest differences occur for unusual situations. However, empirically, most insureds are located in the area $n_{j,t} = \{0, 1\}$, for $j = 1, 2$ where the differences are much smaller. Thus, for small values of T, the Approximated Sarmanov distribution is close to the original Sarmanov, but it is important to understand that they are not the same model.

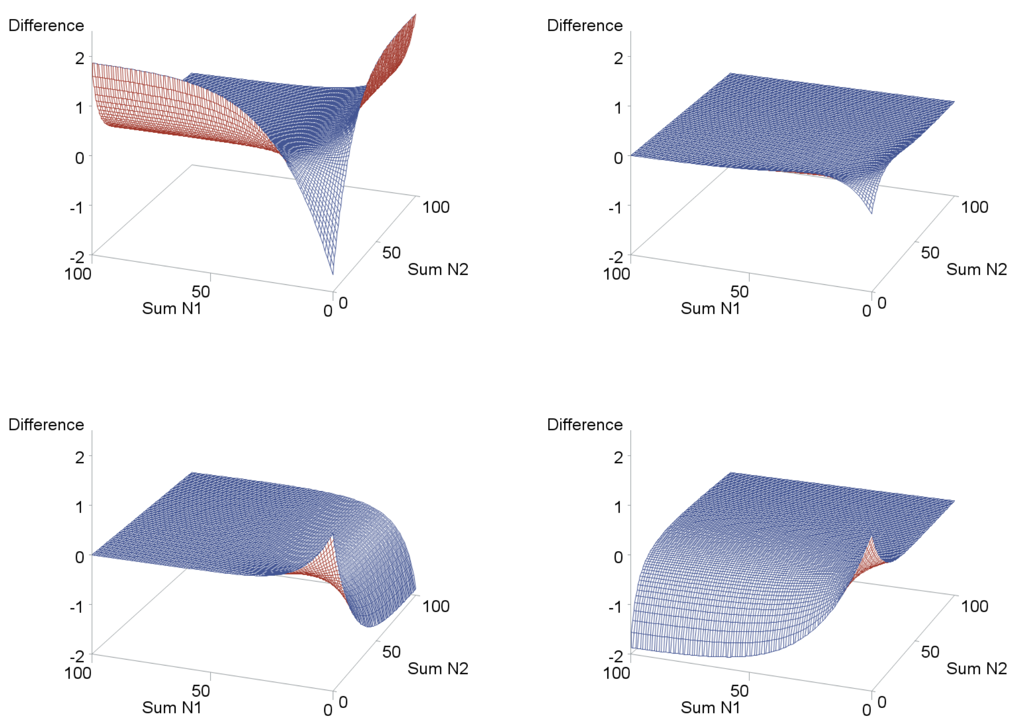

Figure 5.1 – Graphs of δ_1, δ_2, δ_3 and δ_4 respectively, for a time series of $T = 100$ periods, with $\lambda_1 = \lambda_2 = 0.15$, $\alpha_1 = \alpha_2 = 0.7$ and $\omega = 2$

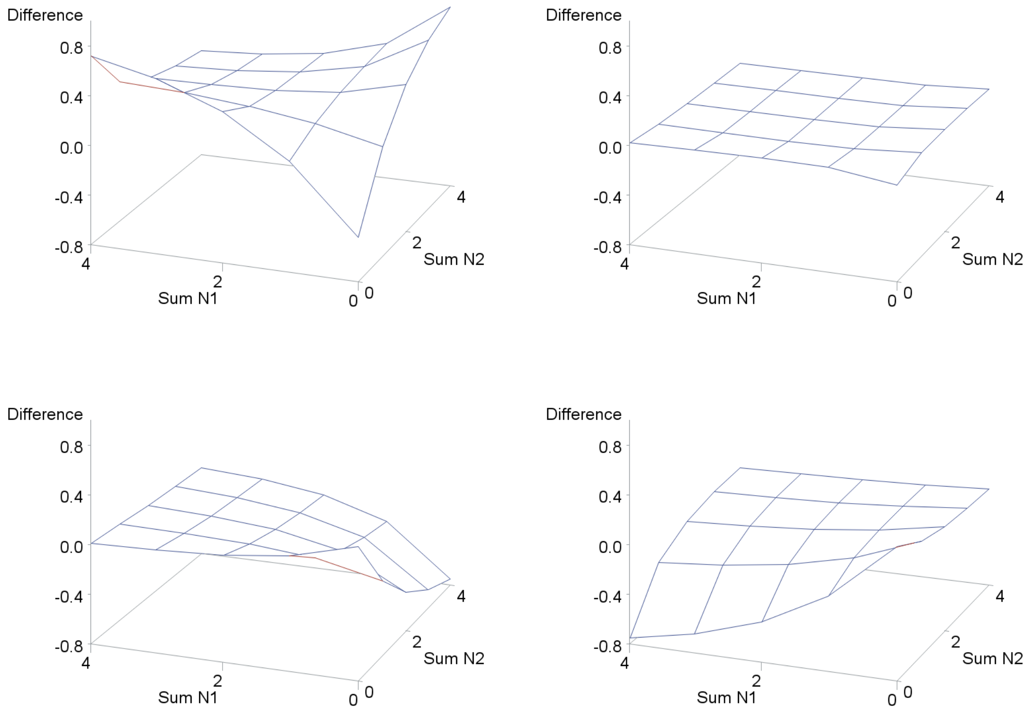

Figure 5.2 – Graphs of δ_1, δ_2, δ_3 and δ_4 respectively, for a time series of $T = 5$ periods, with $\lambda_1 = \lambda_2 = 0.15$, $\alpha_1 = \alpha_2 = 0.7$ and $\omega = 2$

Dynamic Sarmanov

The closed-form expressions for the moments of the Approximated Sarmanov do not change from those obtained in (5.7). It can be shown that the Approximated Sarmanov model generates a predictive annual claim frequency given by

$$
\begin{aligned}
E\left[N_{T+1}^{\text{tot}} \mid \Im_T\right] &= \left((1+\vartheta^* - \vartheta^*)\,\frac{\alpha_1^*}{\tau_1^*} + (\vartheta^* - \vartheta^*)\,\frac{\alpha_1^*}{\tau_1^*+1}\right) + \lambda_{2,T+1}\left((1+\vartheta^* - \vartheta^*)\,\frac{\alpha_2^*}{\tau_2^*} + (\vartheta^* - \vartheta^*)\,\frac{\alpha_2^*}{\tau_2^*+1}\right), \\
&= \lambda_{1,T+1}\frac{\alpha_1^*}{\tau_1^*} + \lambda_{2,t+1}\frac{\alpha_2^*}{\tau_2^*} \\
&= \lambda_{1,T+1}\frac{\alpha_1 + \sum_{t=1}^{T} n_{1,t}}{\tau_1 + \sum_{t=1}^{T}\lambda_{1,t}} + \lambda_{2,T+1}\frac{\alpha_2 + \sum_{t=1}^{T} n_{2,t}}{\tau_2 + \sum_{t=1}^{T}\lambda_{2,t}},
\end{aligned}
\tag{5.20}
$$

where α_ℓ^* and τ_ℓ^*, $\ell = 1, 2$, are given by (5.1). The predictive variance is expressed as follows

$$
\begin{aligned}
\text{Var}(N_{T+1}^{\text{tot}} \mid \Im_T) &= \lambda_{1,T+1}\frac{\alpha_1^*}{\tau_1^*} + \lambda_{1,T+1}^2\frac{\alpha_1^*}{\tau_1^{*2}} + \lambda_{2,T+1}\frac{\alpha_2^*}{\tau_2^*} + \lambda_{2,T+1}^2\frac{\alpha_2^*}{\tau_2^{*2}} \\
&+ 2\lambda_{1,T+1}\lambda_{2,T+1}\vartheta^*\left(\frac{\alpha_1^*}{\tau_1^*}\frac{\alpha_2^*}{\tau_2^*} + \frac{\alpha_1^*}{\tau_1^*+1}\frac{\alpha_2^*}{\tau_2^*+1} - \frac{\alpha_1^*}{\tau_1^*+1}\frac{\alpha_2^*}{\tau_2^*} - \frac{\alpha_1^*}{\tau_1^*}\frac{\alpha_2^*}{\tau_2^*+1}\right),
\end{aligned}
\tag{5.21}
$$

where $\alpha_\ell^* = \alpha_\ell + n_{\ell,\bullet}$, $\tau_\ell^* = \tau_\ell + \lambda_{\ell,\bullet}$, $\ell = 1, 2$.

We observe that we obtain closed-from expressions for the predictive mean and variance, which is convenient for premium calculation. However, we can see that the model does not use the parameter ω in the calculation of the predictive mean. Moreover, the premium for insurance coverage ℓ only uses α_ℓ^* and τ_ℓ^*, which are based on information of the claim type ℓ only. Consequently, for a specific type of claim, the Approximated Sarmanov model cannot borrow information from the other types of claims in predictive modeling. This is contradictory to the objective of our model because we expect that past claims experience of a given claim type should be used to better predict future claims of another correlated claim type.

One way to correct this gap in the model is to consider a dynamic model that extends the one proposed for the H-F model. Instead of adding only a weighting parameter ν_ℓ in the model, we also introduce other parameters γ_1 and γ_2 to borrow information from the claim types 2 and 1, respectively. We intuitively believe that this modification allows us to better predict future claims of each type. Formally, the parameters of such a model can be expressed as follows:

$$\begin{cases} \alpha_{1,t|t-1} = \nu_1 \left(\alpha_{1,t-1} + \gamma_1 n_{2,t}\right) \\ \tau_{1,t|t-1} = \nu_1 \left(\tau_{1,t-1} + \gamma_1 \lambda_{2,t}\right), \end{cases} \tag{5.22}$$

and

$$\begin{cases} \alpha_{2,t|t-1} = \nu_2 \left(\alpha_{2,t-1} + \gamma_2 n_{1,t}\right) \\ \tau_{2,t|t-1} = \nu_2 \left(\tau_{2,t-1} + \gamma_2 \lambda_{1,t}\right), \end{cases} \tag{5.23}$$

where ν_ℓ is again a weighting parameter less than 1. It can be shown by induction that the following general recursive relation holds:

$$\begin{cases} \alpha_{1,t} = \left(\nu_1\right)^t \alpha_{1,0} + \sum_{k=0}^{t-1} \Big((\nu_1)^k n_{1,t-k} + (\nu_1)^{k+1} \gamma_1 n_{2,t-k} \Big) \\ \tau_{1,t} = \left(\nu_1\right)^t \alpha_{1,0} + \sum_{k=0}^{t-1} \Big((\nu_1)^k \lambda_{1,t-k} + (\nu_1)^{k+1} \gamma_1 \lambda_{2,t-k} \Big), \end{cases} \tag{5.24}$$

and

$$\begin{cases} \alpha_{2,t} = \left(\nu_2\right)^t \alpha_{2,0} + \sum_{k=0}^{t-1} \Big((\nu_2)^k n_{2,t-k} + (\nu_2)^{k+1} \gamma_2 n_{1,t-k} \Big) \\ \tau_{2,t} = \left(\nu_2\right)^t \alpha_{2,0} + \sum_{k=0}^{t-1} \Big((\nu_2)^k \lambda_{2,t-k} + (\nu_2)^{k+1} \gamma_2 \lambda_{1,t-k} \Big). \end{cases} \tag{5.25}$$

With these proposed parameters, the *a posteriori* distribution and the predictive analysis incorporate information on the correlated type of claims and borrow insightful past experience of a given claim type to better predict the correlated claim type. As a generalization of the H-F model, the resulting model also puts time weight on the correlated claim. This is reflected by (5.24) and (5.25).

The proposed modification allows us to construct a dynamic structure for the bivariate count model with Sarmanov random effects. The *a posteriori* distribution of $(\Theta_{1,t}, \Theta_{2,t})$ for the Dynamic Sarmanov model is assumed to have the same form as the *a priori* joint pdf of $(\Theta_{1,t}, \Theta_{2,t})$, and can be expressed as

$$\begin{aligned}
u^S\left(\theta_{1,t}, \theta_{2,t} \mid \Im_t\right) \quad = \quad & \left(1+\omega\left(\frac{\tau_{1,t}}{1+\tau_{1,t}}\right)^{\alpha_{1,t}}\left(\frac{\tau_{2,t}}{1+\tau_{2,t}}\right)^{\alpha_{2,t}}\right) h(\theta_{1,t}; \alpha_{1,t}, \tau_{1,t}) h(\theta_{2,t}; \alpha_{2,t}, \tau_{2,t}) \\
& +\omega\left(\frac{\tau_{1,t}}{1+\tau_{1,t}}\right)^{\alpha_{1,t}}\left(\frac{\tau_{2,t}}{1+\tau_{2,t}}\right)^{\alpha_{2,t}} h(\theta_{1,t}; \alpha_{1,t}, \tau_{1,t}+1) h(\theta_{2,t}; \alpha_{2,t}, \tau_{2,t}+1) \\
& -\omega\left(\frac{\tau_{1,t}}{1+\tau_{1,t}}\right)^{\alpha_{1,t}}\left(\frac{\tau_{2,t}}{1+\tau_{2,t}}\right)^{\alpha_{2,t}} h(\theta_{1,t}; \alpha_{1,t}, \tau_{1,t}) h(\theta_{2,t}; \alpha_{2,t}, \tau_{2,t}+1) \\
& -\omega\left(\frac{\tau_{1,t}}{1+\tau_{1,t}}\right)^{\alpha_{1,t}}\left(\frac{\tau_{2,t}}{1+\tau_{2,t}}\right)^{\alpha_{2,t}} h(\theta_{1,t}; \alpha_{1,t}, \tau_{1,t}+1) h(\theta_{2,t}; \alpha_{2,t}, \tau_{2,t}), \quad (5.26)
\end{aligned}$$

where $\alpha_{\ell,t}$ and $\tau_{\ell,t}$ are non-stationary parameters given by (5.24) and (5.25).

Note that when $\omega = 0$, the model can be seen as a bivariate version of the H-F model, noted Bivariate H-F. Hence, given known past experience, the joint distribution of $(\mathbf{N}_1, \mathbf{N}_2)$ for the Dynamic Sarmanov model has the following closed-form expression:

$$\begin{aligned}
f_{\mathbf{N}_1,\mathbf{N}_2}(\boldsymbol{n}_1, \boldsymbol{n}_2) \quad = \quad & (1+\vartheta_t) f_{\mathbf{N}_1}(\boldsymbol{n}_1; \alpha_{1,t}, \tau_{1,t}) f_{\mathbf{N}_2}(\boldsymbol{n}_2; \alpha_{2,t}, \tau_{2,t}) \\
& +\vartheta_t f_{\mathbf{N}_1}(\boldsymbol{n}_1; \alpha_{1,t}, \tau_{1,t}+1) f_{\mathbf{N}_2}(\boldsymbol{n}_2; \alpha_{2,t}, \tau_{2,t}+1) \\
& -\vartheta_t f_{\mathbf{N}_1}(\boldsymbol{n}_1; \alpha_{1,t}, \tau_{1,t}) f_{\mathbf{N}_2}(\boldsymbol{n}_2; \alpha_{2,t}, \tau_{2,t}+1) \\
& -\vartheta_t f_{\mathbf{N}_1}(\boldsymbol{n}_1; \alpha_{1,t}, \tau_{1,t}+1) f_{\mathbf{N}_2}(\boldsymbol{n}_2; \alpha_{2,t}, \tau_{2,t}),
\end{aligned} \quad (5.27)$$

with $\alpha_{\ell,t}$ and $\tau_{\ell,t}$ given by (5.24) and (5.25) for $\ell = 1, 2$, where $\vartheta_t = \omega\left(\frac{\tau_{1,t}}{1+\tau_{1,t}}\right)^{\alpha_{1,t}}\left(\frac{\tau_{2,t}}{1+\tau_{2,t}}\right)^{\alpha_{2,t}}$. Note that the dependence parameter ω does not depend on time.

The moments of the model can be expressed in closed-form, and do not change from those obtained in equation (5.7). For the predictive premium and variance of the Dynamic Sarmanov model, it can be reduced to the following

$$\begin{aligned}
E\left[N_{T+1}^{\text{tot}} \mid \Im_T\right] \quad = \quad & \lambda_{1,T+1}\frac{\alpha_{1,T}}{\tau_{1,T}} + \lambda_{2,T+1}\frac{\alpha_{2,T}}{\tau_{2,T}} \\
= \quad & \lambda_{1,T+1}\frac{(\nu_1)^T \alpha_{1,0} + \sum_{k=0}^{T-1}\left(\nu_1^k n_{1,T-k} + \nu_1^{k+1}\gamma_1 n_{2,T-k}\right)}{(\nu_1)^T \alpha_{1,0} + \sum_{k=0}^{T-1}\left(\nu_1^k \lambda_{1,T-k} + \nu_1^{k+1}\gamma_1 \lambda_{2,T-k}\right)} \\
& +\lambda_{2,T+1}\frac{(\nu_2)^T \alpha_{2,0} + \sum_{k=0}^{T-1}\left(\nu_2^k n_{2,T-k} + \nu_2^{k+1}\gamma_2 n_{1,T-k}\right)}{(\nu_2)^T \alpha_{2,0} + \sum_{k=0}^{T-1}\left(\nu_2^k \lambda_{2,T-k} + \nu_2^{k+1}\gamma_2 \lambda_{1,T-k}\right)} \quad (5.28)
\end{aligned}$$

and

$$\begin{aligned}\text{Var}(N_{T+1}^{\text{tot}} \mid \Im_T) &= \lambda_{1,T+1}\frac{\alpha_{1,T}}{\tau_{1,T}} + \lambda_{1,T+1}^2\frac{\alpha_{1,T}}{\tau_{1,T}^2} + \lambda_{2,T+1}\frac{\alpha_{2,T}}{\tau_{2,T}} + \lambda_{2,T+1}^2\frac{\alpha_{2,T}}{\tau_{2,T}^2} \\ &+ 2\lambda_{1,T+1}\lambda_{2,T+1}\vartheta_T\left(\frac{\alpha_{1,T}}{\tau_{1,T}}\frac{\alpha_{2,T}}{\tau_{2,T}} + \frac{\alpha_{1,T}}{\tau_{1,T}+1}\frac{\alpha_{2,T}}{\tau_{2,T}+1} - \frac{\alpha_{1,T}}{\tau_{1,T}+1}\frac{\alpha_{2,T}}{\tau_{2,T}} - \frac{\alpha_{1,T}}{\tau_{1,T}}\frac{\alpha_{2,T}}{\tau_{2,T}+1}\right),\end{aligned} \tag{5.29}$$

where $\vartheta_T = \omega\left(\frac{\tau_{1,T}}{1+\tau_{1,T}}\right)^{\alpha_{1,T}}\left(\frac{\tau_{2,T}}{1+\tau_{2,T}}\right)^{\alpha_{2,T}}$, $\alpha_{\ell,T}$ and $\tau_{\ell,T}$ are again obtained from equations (5.24) and (5.25), with $\ell = 1, 2$.

We observe that the mean of a given claim type uses the past information of the correlated type of claims, through the crossed parameters γ_1 and γ_2. This link between the claim types does not directly depend on the dependence parameter ω, as it was the case for the stationary Sarmanov model. However, the ω intervenes in the calculation of the predictive variance of the Dynamic Sarmanov model, which can be a crucial additional information for various premium principles.

5.5 Empirical Illustration

5.5.1 Data used

We implement all the models presented in this paper with a sample of insurance data that comes from a major Canadian insurance company. Only private used cars have been considered in this sample. We consider 11 exogenous variables, shown in Table 5.5.1. For every policy we have the initial information at the beginning of the period to describe the profile of the driver. The unbalanced panel data contain information from 2003 to 2008. The sample contains 79,755 insurance contracts, which come from 26,251 policyholders.

The empirical illustration is performed on two pairs of claims types: collision vs comprehensive (noted pair COL/COM) and at-fault vs non-at-fault collision claims (noted pair AF/NAF). We decided to work with two different empirical illustrations to better describe the behavior of our models. We thus expose the models to a wider possibility of situations, which allows us to better analyze their performance and better highlight their properties.

Comprehensive coverage protects damage to the car that results from covered perils not related to a collision. Namely, a scenario that could cause damage to the car that has nothing to do with striking another vehicle. In many cases, this can include theft, vandalism, fire, natural disasters like a hurricane or a tornado, falling objects, etc. Thus, one would expect that if the accident is really a pure comprehensive accident, it should not give any indication of the competence of the driver or better predict future collisions. However, dependence may come from unobservable risk characteristics. In fact, some insureds tend to claim more than

Variable	Description
X1	equals 1 if the insured is between 16 and 25 years old
X2	equals 1 if the insured is between 26 and 60 years old
X3	equals 1 if the vehicle is 0 years old
X4	equals 1 if the vehicle is 1-3 years old
X5	equals 1 if the vehicle is 4-5 years old
X6	equals 1 if the insured owns a home
X7	equals 1 if there is only one driver
X8	equals 1 if there are two drivers
X9	equals 1 if the insured is single
X10	equals 1 if the insured is divorced
X11	equals 1 if the insured has no minor convictions

Table 5.1 – Binary variables summarizing the information available about each policyholder.

others, regardless of the type of claim. This might be explained by a social context as well, in the sense that an insured who lives in a riskier area could be exposed to both types of claims. Moreover, this dependence might also be caused by several factors, such as the driving competence of a driver (collision with a vehicle and collision with an object are often positively correlated), but this might also be explained by the behavior of the insured. Hence, the use of a model that allows dependence between coverages is justified.

In the second illustration, the collision coverage is separated into at-fault and non-at-fault claims. If the non-at-fault claims were really defined as pure bad luck, meaning that they have nothing to do with the behavior of the insured, then it would be irrational to believe that non-at-fault claims would be correlated with at-fault claims. However, in Canada, non-at-fault claims correspond to specific type of accidents, more related to the car's location in the accident. This is well known in Canada, and even if insurers cannot increase the premium for non-at-fault claims, insurers must sometimes find original ways to penalize drivers with non-at-fault claims (see Boucher and Inoussa (2014)). Consequently, for possibly the same reasons cited above for collision and comprehensive coverages that might lead to dependence, it seems logical to believe that dependence can exist between these two types of collision claims as well.

5.5.2 Model Calibration

Tables 5.2 and 5.3 exhibit the fit statistics along with the estimated parameters for the MVNB distribution, compared to the most popular count distributions, i.e. the Poisson and Negative Binomial type-2 (NB2) distributions. When the null hypothesis is on the boundary of the parameter space, a correction must be done to the likelihood ratio test, namely one-sided statistic tests (see Boucher et al (2007) for more details). Consequently, a modified likelihood ratio test has been used to check if the Poisson is rejected against the NB2 or against the

	Poisson				NB2				MVNB			
	COL		COM		COL		COM		COL		COM	
	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error
β_0	-2.5517	0.0847	-4.1896	0.1641	-2.5474	0.0913	-4.1889	0.1604	-2.5739	0.0904	-4.1912	0.1709
β_{X1}	0.6349	0.0717	0.4553	0.1388	0.6419	0.0769	0.4562	0.1409	0.6416	0.0774	0.4644	0.1468
β_{X2}	0.2231	0.0470	0.3962	0.0854	0.2229	0.0501	0.3962	0.0858	0.2261	0.0508	0.3990	0.0877
β_{X3}	0.3369	0.0413	0.3927	0.0695	0.3383	0.0436	0.3924	0.0711	0.3207	0.0434	0.3885	0.0719
β_{X4}	0.3298	0.0382	0.3944	0.0643	0.3314	0.0402	0.3948	0.0650	0.3228	0.0400	0.3924	0.0660
β_{X5}	0.2245	0.0408	0.0919	0.0721	0.2250	0.0427	0.0923	0.0738	0.2149	0.0423	0.0907	0.0749
β_{X6}	0.0904	0.0310	0.1392	0.0528	0.0917	0.0331	0.1395	0.0542	0.0929	0.0328	0.1400	0.0541
β_{X7}	-0.4855	0.0564	-0.0731	0.1191	-0.4905	0.0607	-0.0736	0.1198	-0.4780	0.0615	-0.0775	0.1232
β_{X8}	-0.3888	0.0577	-0.0781	0.1195	-0.3917	0.0623	-0.0781	0.1252	-0.3891	0.0629	-0.0830	0.1258
β_{X9}	0.1901	0.0375	0.0940	0.0643	0.1927	0.0393	0.0943	0.0647	0.1932	0.0401	0.0942	0.0663
β_{X10}	0.2097	0.0566	0.2080	0.0929	0.2134	0.0595	0.2085	0.0947	0.2031	0.0611	0.2126	0.0955
β_{X11}	-0.2349	0.0490	-0.1445	0.0884	-0.2356	0.0525	-0.1448	0.0928	-0.2054	0.0529	-0.1377	0.0886
α_ℓ					1.3170	0.1331	0.6615	0.3875	0.6855	0.0651	0.6769	0.1722
LogLik	-25,635.6				-25,542.26				-25,532.79			
AIC	51,319.20				51,136.52				51,117.58			

Table 5.2 – Parameter estimation - Stationary models for the pair COL/COM

	Poisson				NB2				MVNB			
	AF		NAF		AF		NAF		AF		NAF	
	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error
β_0	-3.3021	0.1258	-3.1918	0.1162	-3.3037	0.1282	-3.1893	0.1173	-3.3132	0.1374	-3.1967	0.1200
β_{X1}	0.7298	0.1031	0.5407	0.1003	0.7310	0.1045	0.5409	0.1016	0.7360	0.1103	0.5412	0.1040
β_{X2}	0.1586	0.0705	0.2745	0.0636	0.1586	0.0707	0.2738	0.0648	0.1606	0.0758	0.2747	0.0668
β_{X3}	0.3745	0.0630	0.3085	0.0549	0.3752	0.0633	0.3090	0.0555	0.3696	0.0648	0.3035	0.0560
β_{X4}	0.3960	0.0582	0.2786	0.0513	0.3963	0.0580	0.2792	0.0515	0.3949	0.0594	0.2760	0.0519
β_{X5}	0.3097	0.0608	0.1571	0.0547	0.3104	0.0611	0.1576	0.0554	0.3067	0.0628	0.1544	0.0554
β_{X6}	0.1230	0.0468	0.0648	0.0420	0.1230	0.0473	0.0642	0.0422	0.1242	0.0482	0.0654	0.0428
β_{X7}	-0.5720	0.0811	-0.4153	0.0754	-0.5717	0.0845	-0.4141	0.0766	-0.5726	0.0869	-0.4115	0.0798
β_{X8}	-0.4568	0.0833	-0.3329	0.0769	-0.4567	0.0857	-0.3325	0.0786	-0.4587	0.0880	-0.3322	0.0815
β_{X9}	0.2097	0.0558	0.1749	0.0497	0.2094	0.0567	0.1745	0.0505	0.2130	0.0584	0.1742	0.0511
β_{X10}	0.1536	0.0885	0.2501	0.0737	0.1536	0.0888	0.2502	0.0746	0.1545	0.0896	0.2461	0.0759
β_{X11}	-0.2442	0.0726	-0.2273	0.0659	-0.2433	0.0730	-0.2298	0.0663	-0.2295	0.0759	-0.2208	0.0694
α_ℓ					0.1112	0.1620	0.2950	0.1678	0.4943	0.1330	0.3478	0.0939
LogLik	-20,919.09				-20,917.05				-20,899.84			
AIC	41,886.18				41,886.10				41,851.68			

Table 5.3 – Parameter estimation - Stationary models for the pair AF/NAF

MVNB for both datasets. On the other hand, because NB2 and MVNB are non-nested models, the Akaike Information Criterion (AIC) has to be preferred to compare the models. In our case, despite the fact that the Poisson cannot always be rejected against the NB2 for all four coverages, we observe that the MVNB model is preferred over Poisson and NB2 for all coverages. This conclusion is interesting because unlike the Poisson or the NB2 distributions, the MVNB distribution allows temporal dependence between all contracts of the insured. This means that our intuition that a premium should somewhat depend on past claim experience is confirmed.

We also fit the Sarmanov and the Approximated Sarmanov models for comparison purposes, to validate the quality of the approximation supposed in the construction of the Approximated Sarmanov. Results are shown in Tables 5.4 and 5.5. We calculated the loglikelihood on the Approximated Sarmanov by using the estimated parameters of the Sarmanov model.

	Sarmanov				**Approximated Sarmanov**			
	COL		COM		COL		COM	
	Estimates	std error	Estimates	std error	Estimates	std error	Estimates	std error
β_0	-2.5739	0.0923	-4.1927	0.1655	-2.5736	0.0885	-4.1905	0.1686
β_{X1}	0.6423	0.0773	0.4645	0.1395	0.6415	0.0770	0.4646	0.1434
β_{X2}	0.2263	0.0512	0.3989	0.0889	0.2260	0.0502	0.3988	0.0872
β_{X3}	0.3203	0.0432	0.3864	0.0713	0.3207	0.0436	0.3884	0.0705
β_{X4}	0.3225	0.0398	0.3912	0.0648	0.3228	0.0404	0.3923	0.0651
β_{X5}	0.2148	0.0422	0.0896	0.0743	0.2149	0.0421	0.0906	0.0722
β_{X6}	0.0930	0.0328	0.1402	0.0538	0.0929	0.0328	0.1401	0.0538
β_{X7}	-0.4785	0.0615	-0.0768	0.1104	-0.4781	0.0595	-0.0780	0.1132
β_{X8}	-0.3897	0.0628	-0.0832	0.1149	-0.3892	0.0609	-0.0834	0.1139
β_{X9}	0.1933	0.0393	0.0943	0.0644	0.1932	0.0397	0.0942	0.0656
β_{X10}	0.2037	0.0609	0.2121	0.0966	0.2031	0.0612	0.2129	0.0959
β_{X11}	-0.2048	0.0519	-0.1349	0.0870	-0.2055	0.0529	-0.1378	0.0897
α_ℓ	0.6853	0.0648	0.6776	0.1728	0.6853	0.0647	0.6766	0.1626
ω	2.0703				2.0489			
LogLik	-25,531.98				-25,532.55			
AIC	51,117.96				51,119.10			

Table 5.4 – Parameter estimation - Sarmanov Approximation for the pair COL/COM

	Sarmanov				**Approximated Sarmanov**			
	AF		NAF		AF		NAF	
	Estimates	std error	Estimates	std error	Estimates	std error	Estimates	std error
β_0	-3.3217	0.1662	-3.2043	0.1457	-3.3188	0.1280	-3.2001	0.1191
β_{X1}	0.7385	0.1263	0.5423	0.1118	0.7391	0.1067	0.5414	0.1041
β_{X2}	0.1613	0.0814	0.2754	0.0716	0.1616	0.0718	0.2748	0.0671
β_{X3}	0.3642	0.0642	0.2962	0.0566	0.3669	0.0647	0.2992	0.0563
β_{X4}	0.3927	0.0590	0.2724	0.0524	0.3941	0.0592	0.2738	0.0523
β_{X5}	0.3032	0.0617	0.1504	0.0559	0.3049	0.0624	0.1521	0.0558
β_{X6}	0.1252	0.0487	0.0662	0.0428	0.1249	0.0483	0.0660	0.0426
β_{X7}	-0.5709	0.0903	-0.4073	0.0897	-0.5728	0.0843	-0.4088	0.0786
β_{X8}	-0.4594	0.0932	-0.3316	0.0887	-0.4596	0.0868	-0.3319	0.0810
β_{X9}	0.2146	0.0578	0.1742	0.0514	0.2148	0.0575	0.1737	0.0520
β_{X10}	0.1528	0.0925	0.2425	0.0770	0.1551	0.0908	0.2431	0.0769
β_{X11}	-0.2175	0.0917	-0.2102	0.0722	-0.2220	0.0755	-0.2154	0.0672
α_ℓ	0.8176	0.3940	0.7122	0.3092	0.7696	0.1301	0.6539	0.1111
ω	4.4886				5.6594			
LogLik	-20,878.76				-20,884.45			
AIC	41,811.52				41,822.90			

Table 5.5 – Parameter estimation - Sarmanov Approximation for the pair AF/NAF

We obtain close loglikelihood values ($-25,531.98$ vs. $-25,532.55$) for the pair COL/COM, meaning that models are close. Additionally, expressed with 2 decimals, we observe that the optimized loglikelihood of the Approximated Sarmanov ($-25,532.55$) is approximately the same as the one calculated with the MLE parameters of the Sarmanov distribution. For the pair AF/NAF, the loglikelihood obtained by Approximated Sarmanov by using the MLE of the Sarmanov distribution is equal to $-20,884.63$, while the maximum loglikelihood obtained by the Sarmanov is equal to $-20,878.76$, a slightly higher difference. The maximum loglikelihood obtained with the Approximated Sarmanov is also a little bit different, at $-20,884.45$. To summarize, the Approximated Sarmanov is not similar to the Sarmanov model, but the approximation seems to be reasonable.

Tables 5.6 and 5.7 show the estimated parameters for the dynamic models presented earlier in the paper: H-F, Bivariate H-F and Dynamic Sarmanov. We observe that the estimated parameters (the intercept and the eleven covariates from Table 5.5.1) are approximately the

	Harvey-Fernandes				Bivariate Harvey-Fernandes				Dynamic Sarmanov			
	COL		COM		COL		COM		COL		COM	
	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error
β_0	-2.5749	0.0921	-4.1856	0.1701	-2.5749	0.0915	-4.2021	0.1687	-2.5739	0.1009	-4.1879	0.1771
β_{X1}	0.6466	0.0773	0.4664	0.1400	0.6465	0.0777	0.4706	0.1486	0.6482	0.0811	0.4681	0.1413
β_{X2}	0.2286	0.0514	0.4007	0.0868	0.2286	0.0509	0.4028	0.0901	0.2294	0.0577	0.4006	0.0878
β_{X3}	0.3219	0.0433	0.3854	0.0721	0.3218	0.0433	0.3824	0.0714	0.3209	0.0434	0.3792	0.0711
β_{X4}	0.3236	0.0400	0.3895	0.0658	0.3236	0.0401	0.3860	0.0662	0.3226	0.0407	0.3864	0.0657
β_{X5}	0.2179	0.0423	0.0881	0.0751	0.2179	0.0424	0.0854	0.0745	0.2174	0.0426	0.0866	0.0758
β_{X6}	0.0907	0.0330	0.1400	0.0543	0.0907	0.0329	0.1453	0.0543	0.0911	0.0333	0.1398	0.0551
β_{X7}	-0.4775	0.0621	-0.0811	0.1138	-0.4775	0.0616	-0.0725	0.1115	-0.4798	0.0626	-0.0789	0.1255
β_{X8}	-0.3892	0.0637	-0.0854	0.1197	-0.3892	0.0622	-0.0799	0.1094	-0.3908	0.0640	-0.0847	0.1268
β_{X9}	0.1932	0.0397	0.0947	0.0648	0.1932	0.0401	0.0918	0.0666	0.1941	0.0398	0.0947	0.0679
β_{X10}	0.2022	0.0613	0.2122	0.0960	0.2022	0.0608	0.2094	0.0964	0.2034	0.0610	0.2115	0.0963
β_{X11}	-0.2067	0.0524	-0.1392	0.0905	-0.2067	0.0521	-0.1396	0.0887	-0.2058	0.0521	-0.1336	0.0935
$\alpha_{\ell,0}$	0.5021	0.0689	0.4472	0.1807	0.5021	0.0692	0.4273	0.1776	0.5027	0.0709	0.4482	0.1781
ν_ℓ	0.7057	0.0539	0.6421	0.1330	0.7057	0.0546	0.6083	0.1155	0.7062	0.0549	0.6240	0.1549
γ_ℓ					0	0.1438	0.2996	0.1611	0	0.2106	0.4162	0.2771
ω									0.7417			
LogLik	-25,521.67				-25,519.06				-25,517.31			
AIC	51,099.34				51,098.12				51,096.62			

Table 5.6 – Parameter estimation - Dynamic Models for the pair COL/COM

	Harvey-Fernandes				Bivariate Harvey-Fernandes				Dynamic Sarmanov			
	AF		NAF		AF		NAF		AF		NAF	
	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error	Estim	error
β_0	-3.3129	0.1281	-3.1975	0.1184	-3,312953	0.1352	-3.2068	0.1237	-3.3223	0.1289	-3.2097	0.1200
β_{X1}	0.7385	0.1066	0.5413	0.1032	0.7380	0.1067	0.5408	0.1049	0.7420	0.1080	0.5470	0.1046
β_{X2}	0.1618	0.0711	0.2747	0.0651	0.1615	0.0730	0.2779	0.0685	0.1642	0.0743	0.2782	0.0680
β_{X3}	0.3697	0.0638	0.3037	0.0556	0.3660	0.0645	0.2983	0.0564	0.3633	0.0647	0.2962	0.0565
β_{X4}	0.3953	0.0587	0.2760	0.0520	0.3925	0.0593	0.2747	0.0526	0.3925	0.0591	0.2726	0.0525
β_{X5}	0.3072	0.0615	0.1550	0.0554	0.3060	0.0627	0.1514	0.0559	0.3041	0.0627	0.1527	0.0562
β_{X6}	0.1236	0.0478	0.0651	0.0428	0.1238	0.0477	0.0639	0.0428	0.1234	0.0479	0.0647	0.0428
β_{X7}	-0.5739	0.0854	-0.4104	0.0782	-0.5714	0.0892	-0.4072	0.0818	-0.5705	0.0870	-0.4050	0.0818
β_{X8}	-0.4592	0.0883	-0.3318	0.0806	-0.4582	0.0915	-0.3314	0.0828	-0.4595	0.0887	-0.3314	0.0830
β_{X9}	0.2128	0.0579	0.1742	0.0505	0.2115	0.0573	0.1762	0.0513	0.2136	0.0580	0.1754	0.0519
β_{X10}	0.1554	0.0903	0.2451	0.0753	0.1545	0.0905	0.2438	0.0767	0.1508	0.0913	0.2407	0.0771
β_{X11}	-0.2300	0.0738	-0.2207	0.0670	-0.2287	0.0783	-0.2077	0.0701	-0.2182	0.0756	-0.2074	0.0675
$\alpha_{\ell,0}$	0.3975	0.1413	0.2901	0.0998	0.3725	0.1472	0.2962	0.1826	0.5453	0.1589	0.4755	0.1598
ν_ℓ	0.7827	0.1600	0.8090	0.1616	0.7848	0.1613	0.8329	0.1794	0.6939	0.0832	0.6742	0.1264
γ_ℓ					0.7697	0.4015	0.9659	1.1541	0.4296	0.2008	0.4864	0.2118
ω									9.6347			
LogLik	-20,898.77				-20,890.34				-20,870.36			
AIC	41,853.54				41,840.68				41,802.72			

Table 5.7 – Parameter estimation - Dynamic Models for the pair AF/NAF

same for all (stationary and dynamic) models, which is a condition that shows consistency between models (see for example Gourieroux et al (1984)).

Specification Tests

All the models presented in this paper that generalize the MVNB model are somewhat related given certain linked parameter restrictions. We illustrated the situation in Figure 5.3, where links between nested models are shown. Note that model $DS1$ refers to an intermediary model in the scheme.

This illustration is used to test all the linked models via a likelihood ratio test to check which model to retain. We perform likelihood ratio tests between all nested models used

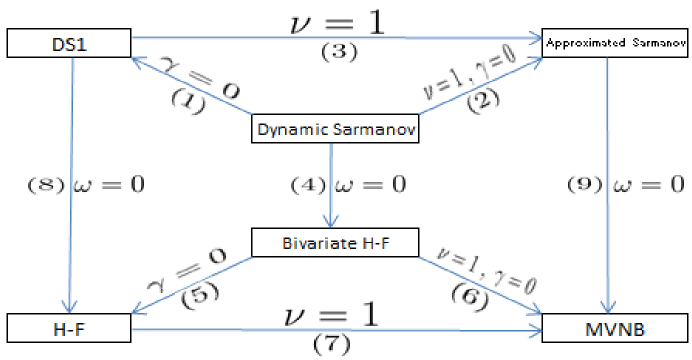

Figure 5.3 – Links between the models

in the empirical illustration. As mentioned, corrections to the likelihood ratio test are used when the null hypothesis is on the boundary of the parameter space. Because we have to test many parameters simultaneously, we use the result of Self and Liang (1987), namely that the distribution of the likelihood ratio statistic under the null hypothesis is:

$$F(x) = \sum_{d_f=0}^{k} p_{d_f} F(x; d_f),$$

where $F(x; d_f)$ is the cumulative distribution function of a Chi-Square distribution with d_f degrees of freedom, and p_{d_f} represents the probability of success of a binomial distribution with parameters $n = k$ and $p = 0.5$. The parameter k is the difference of parameters between the null and the alternative hypothesis. For example, for a likelihood ratio test where 4 parameters are tested simultaneously on the boundary of their parameter space, the null distribution can be defined as:

$$\frac{1}{16}0 + \frac{4}{16}\chi^2(1) + \frac{6}{16}\chi^2(2) + \frac{4}{16}\chi^2(3) + \frac{1}{16}\chi^2(4).$$

We summarize the main results of these tests in Table 5.8.

For the pair COM/COL, we can observe that all forms of dependence between comprehensive and collision claims are rejected. Indeed, compared with the independence case, all alternative models supposing dependence between the types of claims are rejected. However, following Table 5.7, we see that in a dynamic setting, collision claims might provide insight into comprehensive claims prediction. In fact, we observe that γ_2 is significant, meaning that collision loss experience could enhance comprehensive claim prediction. We also see that $\widehat{\gamma}_1 = 0$, meaning that no information is brought from the comprehensive claims to collision claims prediction. This is also confirmed by test (5), where a Bivariate H-F is not rejected against an H-F

$*$. Corrected likelihood ratio test: the null hypothesis is on the boundary of the parameter space.

Test	Alternative model	Null model	DF	**COL/COM** Test statistic	p-value	AF/NAF Test statistic	p-value
(2)*	Dynamic Sarma.	Approx. Sarma.	4	30.48	< 0.001	28.18	< 0.001
(4)	Dynamic Sarma.	Biv. H-F	1	3.50	0.061	39.96	< 0.001
(5)*	Biv. H-F	H-F	2	5.22	0.003	16.86	< 0.001
(6)*	Biv. H-F	MVNB	4	27.46	< 0.001	19.00	< 0.001
(7)*	H-F	MVNB	2	22.24	< 0.001	2.14	0.097
(9)	Approx. Sarma.	MVNB	1	0.48	0.488	30.78	< 0.001

Table 5.8 – Specification Tests

model, meaning that borrowing information from the correlated type of claim could improve the prediction. However, tests indicate that all stationary models are rejected against dynamic models. This shows that the data favor a model allowing greater weight to the most recent claims. Finally, when we compare the p-values between tests (4) and (9), we observe that the dependence parameter ω is becoming much more significant in a dynamic context.

For the pair AF/NAF, we observe, overall, very significant dependence between AF and NAF claims. When comparing stationary and dynamic models, we see that the tests reject stationary models in favor of dynamic models, meanwhile the MVNB is not rejected over the H-F model. Interestingly, this means that for the pair AF/NAF, a dynamic model is preferred when the information of the correlated type of claim is incorporated. Thus, the prediction is improved when additional information of the other type of claims is added to the model, which justifies and supports the intuition of adding the parameter γ_ℓ. Finally, note that by introducing a dependence parameter ω in the model, the values of $\hat{\gamma}_\ell$ changed considerably. Indeed, while $\hat{\gamma}_{AF}$ was equal to 0.7697 for the Bivariate H-F model, it goes down to 0.4296 for the Dynamic Sarmanov model. This means that NAF loss experience has a greater impact on the AF premium with a Bivariate H-F model than with a Dynamic Sarmanov model. We think that this can be explained by the flexibility induced by the ω parameter in the Dynamic Sarmanov model, where this extra parameter can be used to model the variance independently from the mean.

5.5.3 Premium Comparison

Each of the models presented in this paper has different properties, and generates different *a priori* and predictive premiums. Beside comparisons of the fit of the model to empirical data, it is useful to compare the premiums. For illustration purposes, we consider three different profiles classified as good, average and bad drivers, given their risk characteristics. The selected profiles are described in Table 5.9 and their respective *a priori* premiums are given in Tables 5.10 and 5.11. These tables show that the values exhibit small differences for the six most useful models presented in this paper. We observe the same trend for variance, with a

Profile Number	Type of Profile	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
1	Good	0	1	1	0	0	0	1	0	0	0	1
2	Average	0	1	0	1	0	0	0	1	0	0	0
3	Bad	1	0	0	0	0	0	0	1	0	0	0

Table 5.9 – Profiles Analyzed

	Good Profile			Average Profile			Bad Profile		
	Mean		Variance	Mean		Variance	Mean		Variance
Models	COL	COM	Total	COL	COM	Total	COL	COM	Total
MVNB	0.0665	0.0268	0.1008	0.0894	0.0307	0.1332	0.0981	0.0222	0.1351
Sarmanov	0.0665	0.0268	0.1015	0.0894	0.0306	0.1342	0.0981	0.0221	0.1358
Approx. Sarma	0.0665	0.0268	0.1015	0.0895	0.0307	0.1344	0.0981	0.0222	0.1359
H-F	0.0666	0.0268	0.1039	0.0896	0.0308	0.1386	0.0985	0.0223	0.1412
Bivariate H-F	0.0666	0.0265	0.1037	0.0896	0.0304	0.1382	0.0985	0.0221	0.1411
Dynamic Sarma	0.0666	0.0268	0.1042	0.0896	0.0306	0.1388	0.0986	0.0223	0.1418

Table 5.10 – *A priori* premiums for the pair COL/COM

	Good Profile			Average Profile			Bad Profile		
	Mean		Variance	Mean		Variance	Mean		Variance
Models	AF	NAF	Total	AF	NAF	Total	AF	NAF	Total
MVNB	0.0277	0.0387	0.0723	0.0401	0.0509	0.1017	0.0480	0.0504	0.1104
Sarmanov	0.0277	0.0388	0.0704	0.0397	0.0504	0.0972	0.0477	0.0501	0.1060
Approx. Sarma	0.0277	0.0388	0.0709	0.0398	0.0506	0.0986	0.0479	0.0503	0.1077
H-F	0.0277	0.0388	0.0736	0.0402	0.0509	0.1040	0.0481	0.0504	0.1131
Bivariate H-F	0.0277	0.0389	0.0739	0.0401	0.0505	0.1035	0.0482	0.0499	0.1127
Dynamic Sarma	0.0278	0.0389	0.0743	0.0398	0.0503	0.1039	0.0478	0.0501	0.1140

Table 5.11 – *A priori* premiums for the pair AF/NAF

slight increase for the dynamic models compared with the stationary ones. These results are not surprising because all models have the same form of expected values and, as specified in the previous section, all estimates of βs are similar.

We expect more differences for the predictive premiums, because some models are dynamic, others depend only on past claims and still other models also depend on the claim experience of the other type of claims. For illustration purposes, we use the pair AF/NAF only. We have kept the estimated parameters of the *a priori* analysis and projected a loss experience of 10 years for a medium-risk profile. Although other situations can easily be illustrated, because closed-form formulas have been found to compute the predictive premiums for each model, we focus here on five specific situations. Table 5.12 provides a detailed description of these loss experience situations. The first loss experience describes a claim-free situation. The second loss experience illustrates the situation of an insured with old claims for both coverages. The third experience is a situation with recent claims for both types of claims. Finally, the fourth

	Exp. #1		Exp. #2		Exp. #3		Exp. #4		Exp. #5	
Year	AF	NAF	AF	NAF	AF	NAF	AF	NAF	AF	NAF
1	0	0	1	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	1	0	0
3	0	0	1	1	0	0	0	1	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	1	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	0	1
7	0	0	0	0	1	1	0	0	0	1
8	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	1	0	0	0	1
10	0	0	0	0	1	0	0	0	0	0

Table 5.12 – Various 10-Year Loss Experiences

and fifth loss experiences correspond to claim-free situation for AF coverage, while old claims (Exp. #4) or recent claims (Exp. #5) are considered for NAF coverage.

The computed predictive premiums are presented in Tables 5.13 and 5.14 . For the claim-free situation of loss experience #1, the predicted premiums of the dynamic models are much lower than for the stationary models. We also observe the same trend for the situation where insured claimed three times in the first five years, but showed a neat progression in the most recent years. This is expected given that dynamic models have an extra parameter ν that allows us to weight past claims. For example, to compute next year's premium using a dynamic model with ν approximately equal to 70%, we would assign a claim that happens in the previous year a weight of 100%, a claim 5 years old a weight of 24%, and a claim 10 years old a weight of only 4% on the predictive premium. Meanwhile, for static models, each claim weights 100% in the calculation of the premium regardless of the occurrence time. This highlights an interesting feature of the dynamic models, where an insurer using a dynamic model in its ratemaking system would reward the positive evolution of its insured's claim experience. The insured would therefore be encouraged to improve his profile in the future even if he had more claims in the past. On the other hand, the dynamic models compensate the low premiums of those first two situations by offering higher premiums for insureds with recent claims. We also observe the same trend for predictive variance, with a more significant difference between stationary and dynamic models.

The dependence between claim types can be studied in a similar way by analyzing predictive premiums of the two types of claims simultaneously. We know that models that suppose independence between claim types do not consider the claim experience of the other type of claims in the computation of the predictive premiums. The AF premiums calculated by the MVNB and the H-F models illustrate this situation. Indeed, the AF premium does not depend on the NAF loss experience, because the AF premium is the same for loss experiences

	Exp. #1			Exp. #2			Exp. #3		
	Mean		Variance	Mean		Variance	Mean		Variance
Models	AF	NAF	Total	AF	NAF	Total	AF	NAF	Total
MVNB	0.0221	0.0206	0.0450	0.1565	0.1988	0.3742	0.1565	0.1988	0.3742
Sarma	0.0230	0.0261	0.0513	0.1308	0.1616	0.3038	0.1308	0.1616	0.3038
Approx. Sarma.	0.0262	0.0285	0.0571	0.1284	0.1594	0.2970	0.1284	0.1594	0.2970
H-F	0.0067	0.0065	0.0159	0.1222	0.1496	0.3243	0.4203	0.3404	0.9082
Bivariate H-F	0.0039	0.0051	0.0102	0.1213	0.1808	0.3366	0.3776	0.4105	0.8785
Dynamic Sarma.	0.0029	0.0023	0.0063	0.0795	0.0831	0.2173	0.4656	0.4463	1.0825

Table 5.13 – Predictive Premiums for the pair AF/NAF (1)

	Exp. #4			Exp. #5		
	Mean		Variance	Mean		Variance
Models	AF	NAF	Total	AF	NAF	Total
MVNB	0.0221	0.1988	0.2338	0.0221	0.1988	0.2338
Sarma	0.0439	0.1501	0.2026	0.0439	0.1501	0.2026
Approx. Sarma.	0.0262	0.1594	0.1914	0.0262	0.1594	0.1914
H-F	0.0067	0.1496	0.1860	0.0067	0.3404	0.4128
Bivariate H-F	0.0495	0.1044	0.1714	0.1241	0.2114	0.3736
Dynamic Sarma.	0.0213	0.0640	0.1097	0.0822	0.3010	0.4672

Table 5.14 – Predictive Premiums for the pair AF/NAF (2)

#1, #4 and #5. In contrast, the AF premium of the Sarmanov model, which allows for dependence between claim types, shows that the loss experience of the NAF coverage has an impact. Indeed, the AF premium is different between loss experiences #1 and #4. However, the premium is the same for loss experiences #4 and #5, because the Sarmanov model is static. It is interesting to see that the AF premiums of the Approximated Sarmanov model do not behave the same way as in the Sarmanov model. Indeed, we cannot observe differences between AF premiums for loss experiences #1, #4 and #5. As explained in Section 5.4.2, this comes from the construction of the Approximated Sarmanov model.

Finally, it is interesting to analyze the premiums of the Bivariate H-F and the Dynamic Sarmanov models. Both models allow past claims experience of NAF coverage to affect the AF premium. We see clear differences between the premiums of loss experiences #1, #4 and #5. Another striking observation is the difference between the computed premiums of each model. This can be explained straightforwardly by looking at the estimated parameters $\hat{\gamma}_{AF}$ of Table 5.7, which we analyzed earlier. Lastly, we observe a difference in variances between the Dynamic Sarmanov and Bivariate Harvey-Fernandes model, due to the addition of the dependence parameter ω as discussed above.

5.6 Concluding Remarks

Panel data models for claims count are used to model the potential dependence between the number of claims of contracts of the same insured. A generalization into bivariate panel data models can illustrate dependence between coverages. A dynamic approach allows the most recent claims to be more predictive than oldest ones in the prediction. In this paper, we proposed a new model that captures all these features of the panel data models for claims count.

The Sarmanov family of multivariate distribution has been used to model the joint density of the random effects. We show that the form of the posterior density of this family of distributions is almost the same form as that of the prior density. To be able to use the dynamic approach proposed by Bolancé et al (2007), an approximation of the posterior density has been made. We showed that the approximation is reasonnable, but not identical to the Sarmanov model. The Approximated Sarmanov model allowed us to construct a Dynamic Sarmanov model that possesses nice properties: closed-form expressions of the predictive distribution and closed-form expressions of the predictive premium.

We implemented the model with a sample of insurance data that comes from a major Canadian insurance company. The empirical illustration has been performed on two pairs of claim types: collision vs comprehensive and at-fault vs non-at-fault collision claims, which allows us to expose the proposed model to a wider range of situations. For each pair of coverage, a dynamic structure seemed to be relevant, the Dynamic Sarmanov model was one of the best models to adjust the data.

The Dynamic Sarmanov has been applied to a Poisson-gamma structure, but other combinations are easily possible (Poisson-Inverse Gaussian, NB2-Beta, etc.), as long as a conjugate property can be found. Also, the proposed approach can easily be generalized to more than two lines of business. Indeed, it is possible to extend the Sarmanov family of distributions to the multivariate case. Based on our data, a triplet of claim types using comprehensive, at-fault and non-at-fault claims could be interesting for future research. The trivariate Sarmanov joint density would be expressed as:

$$\begin{aligned} u^S(\theta_1, \theta_2, \theta_3) &= h(\theta_1, \alpha_1, \tau_1)\, h(\theta_2, \alpha_2, \tau_2)\, h(\theta_3, \alpha_3, \tau_3) \\ &\quad \times (1 + \omega_{12}\phi_1\phi_2 + \omega_{13}\phi_1\phi_3 + \omega_{23}\phi_2\phi_3 + \omega_{123}\phi_1\phi_2\phi_3). \end{aligned}$$

The correlation structure is expensive in terms of parameters: the model supposes four parameters to model dependence. It would be interesting to understand how each parameter affects dependence between claim types. Moreover, a multivariate model could also be performed to incorporate claim severity analysis.

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Conclusion

Dans cette thèse, nous nous sommes intéressés à des approches novatrices pour la modélisation de la dépendance en assurance non-vie. Nous avons proposé différentes approches pour capturer la dépendance entre les triangles de développement dans le contexte des réserves, mais aussi entre les nombres de réclamations pour des données de comptage.
Le point de départ a été le modèle de Shi et Frees (2011), où la dépendance entre les lignes d'affaires est capturée par paires. Une généralisation est proposée dans cette thèse sous deux angles différents.

Dans un premier temps, nous nous sommes penchés sur la théorie des copules hiérarchiques afin de mieux capturer la dépendance à l'intérieur et entre les lignes d'affaires. Le chapitre 2 a donc introduit le concept de copules hiérarchiques archimédiennes dans le contexte des réserves, afin de modéliser la dépendance à travers des effets calendaires, pour mieux prendre en considération les effets communs dus à l'inflation, à des changements de jurisprudence, ou encore à des décisions stratégiques. Dans le chapitre 3, nous avons utilisé une approche hiérarchique alternative et avons utilisé deux modèles différents afin de modéliser la dépendance. Également, une contribution importante de ce chapitre a été l'introduction de la notion d'estimation basée sur les rangs au contexte des réserves afin de préserver la linéarité de l'espérance et d'offrir une estimation plus robuste face au risque de mauvaise spécification des marges ou de la structure de dépendance.

Le deuxième angle d'étude est celui des effets aléatoires et l'utilisation de la famille de distributions bivariées Sarmanov. Le chapitre 4 a introduit cette famille de distributions aux méthodes de provisionnement et a démontré l'intérêt et l'utilité de cette famille de distributions dans le contexte des réserves. Aussi, nous présentons des formes explicites de la distribution jointe et capturons la dépendance entre les lignes d'affaires à travers les effets d'années de calendrier, d'années d'accident et périodes de développement. Cette idée de combiner le contexte bayésien et la dépendance a donc été appliquée dans le cas des triangles de développement. On suppose que chaque année de calendrier (année d'accident ou période de développement) est caractérisée par un paramètre caractéristique de la sinistralité de l'année, mais inobservable, où les montants des sinistres déjà réglés permettront d'obtenir un indicateur de ce paramètre. Une telle méthodologie est connexe à la tarification et le système bonus/malus. Nous avons donc proposé au chapitre 5 un nouveau modèle qui utilise l'information de la ligne corrélée pour mieux prédire les futures réclamations, tout en prenant en considération l'effet temporel. En effet, nous avons utilisé un modèle bivarié avec des effets aléatoires dynamiques à des données de comptage dans un cadre de tarification en assurance IARD.

Cela a démontré l'intérêt et le potentiel d'exporter de tels modèles à un contexte autre que les réserves, en l'occurrence celui de la tarification en assurance non-vie. Ainsi, ces travaux constituent des avenues de recherche très intéressantes et pourraient être appliqués et utilisés dans d'autres sphères de l'actuariat et de la finance.

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