Modélisation de la dépendance à l'aide des mélanges communs et applications en actuariat

Thèse

Itre Mtalai

Doctorat en actuariat
Philosophiæ doctor (Ph. D.)

Québec, Canada

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Sous la direction de:

Hélène Cossette, directrice de recherche
Étienne Marceau, codirecteur de recherche

Résumé

La modélisation de la dépendance entre les risques pour un portefeuille d'une assurance ou d'une entité financière est devenue de plus en plus importante pour la solvabilité des institutions financières et l'examen de solvabilité dynamique et l'analyse financière dynamique des compagnies d'assurance. L'hypothèse d'indépendance entre les risques est parfois réaliste et facilite l'évaluation, l'agrégation et l'allocation des risques. Cependant, dans la majorité des cas, les risques individuels sont influencés par un ou plusieurs facteurs communs, tels que l'environnement économique, les régions géographiques ou les conditions climatiques et il est donc moins réaliste, voire dangereux, de supposer l'indépendance entre les risques d'un même portefeuille. Dans la littérature, un tel cas peut être modélisé par des modèles avec mélange commun. Ces modèles ont de nombreuses applications en assurance et en finance. L'objectif de cette thèse est donc d'explorer les modèles de dépendance construits à l'aide des mélanges communs et de faire sortir, à l'aide de plusieurs applications, la dangerosité de considérer l'indépendance entre les risques au sein d'un portefeuille. En particulier, la focalisation est mise sur un modèle souvent considéré pour modéliser le montant de sinistres, notamment la loi exponentielle mélange.

Cette thèse considère les modèles de risque basés sur la loi exponentielle mélange. Le premier chapitre constitue une introduction générale aux modèles avec mélanges communs et introduit les notions qui seront utilisées dans les autres chapitres. Dans le deuxième chapitre, nous considérons un portefeuille de risques représentés par un vecteur de variables aléatoires dont la fonction de répartition conjointe est définie par une copule Archimédienne ou une copule Archimédienne imbriquée. Nous examinons le calcul de la fonction de répartition de la somme ou une variété de fonctions de ces variables aléatoires. En nous basant sur la méthodologie computationnelle présentée dans ce chapitre, nous examinons plusieurs problèmes reliés à différents modèles de risque en actuariat, tels que l'agrégation et l'allocation du capital. De plus, en utilisant une telle structure de dépendance avec des marginales spécifiques, nous obtenons des expressions explicites pour plusieurs quantités relatives au risque agrégé telles que sa fonction de masse de probabilité, sa fonction de répartition, sa TVaR, etc. L'échangeabilité des copules Archimédiennes implique que toutes les marginales sont égales. Afin de généraliser les copules Archimédiennes pour permettre les asymétries, plusieurs chercheurs utilisent une structure hiérarchique obtenue en imbriquant plusieurs copules Archimédiennes.

Toutefois, il est difficile de valider la condition d'imbrication permettant d'assurer que la structure résultante est une copule, lorsque les copules impliquées appartiennent à des familles Archimédiennes différentes. Afin de remédier à ce problème, nous présentons, au troisième chapitre, une nouvelle méthode d'imbrication basée sur la construction des lois composées multivariées exponentielles mélange. En introduisant plusieurs paramètres, un large spectre de structures de dépendance peut être couvert par cette nouvelle construction, ce qui semble être très intéressant pour des applications pratiques. Des algorithmes efficients de simulation et d'agrégation sont également présentés. En nous inspirant à la fois des chapitres 2 et 3, nous proposons et examinons en détail au quatrième chapitre une nouvelle extension au modèle collectif de risque en supposant une certaine dépendance entre la fréquence et la sévérité des sinistres. Nous considérons des modèles collectifs de risque avec différentes structures de dépendance telles que des modèles impliquant des lois mélanges d'Erlang multivariées ou, dans un cadre plus général, des modèles basés sur des copules bivariées ou multivariées. Nous utilisons également les copules Archimédiennes et Archimédiennes hiérarchiques afin de modéliser la dépendance entre les composantes de la somme aléatoire représentant le montant de sinistre global. En nous basant encore une fois sur la représentation de notre modèle sous forme d'un mélange commun, nous adaptons la méthodologie computationnelle présentée au chapitre 2 pour calculer la fonction de masse de probabilité d'une somme aléatoire incorporant une dépendance hiérarchique. Finalement, dans le cinquième chapitre, nous soulignons l'utilité des modèles avec mélange commun et nous étudions plus en détail les lois exponentielles mélange dans leurs versions univariée et multivariée et nous expliquons leur lien étroit avec les copules Archimédiennes et Archimédiennes hiérarchiques. Nous proposons également plusieurs nouvelles distributions et nous établissons leurs liens avec des distributions connues.

Abstract

Risk dependence modelling has become an increasingly important task for the solvency of financial institutions and insurance companies. The independence assumption between risks is sometimes realistic and facilitates risk assessment, aggregation and allocation. However, in most cases individual risks are influenced by at least one common factor, such as the economic environment, geographical regions or climatic conditions, and it is therefore less realistic or even dangerous to assume independence between risks. In the literature, such a case can be modelled by common mixture models. These models have many applications in insurance and finance. The aim of this thesis is to explore the dependence models constructed using common mixtures and to bring out, with the use of several applications, the riskiness of considering the independence between risks within an insurance company or a financial institution. In particular, the focus is on the exponential mixture.

Exponential mixture distributions are on the basis of this thesis. The first chapter is a general introduction to models with common mixtures and introduces the concepts that will be used in the other chapters. In the second chapter, we consider a portfolio of risks represented by a vector of random variables whose joint distribution function is defined by an Archimedean copula or a nested Archimedean copula. We examine the computation of the distribution of the sum function or a variety of functions of these random variables. Based on the computational methodology presented in this chapter, we examine risk models regarding aggregation, capital allocation and ruin problems. Moreover, by using such a dependency structure with specific marginals, we obtain explicit expressions for several aggregated risk quantities such as its probability mass function, its distribution function, and its TVaR. The exchangeability of the Archimedean copulas implies that all margins are equal. To generalize Archimedean copulas to allow asymmetries, several researchers use a hierarchical structure obtained by nesting several Archimedean copulas. However, it is difficult to validate the nesting condition when the copulas involved belong to different Archimedean families. To solve this problem, we present, in the third chapter, a new imbrication method via the construction of the multivariate compound distributions. By introducing several parameters, a large spectrum of dependency structures can be achieved by this new construction, which seems very interesting for practical applications. Efficient sampling and aggregation algorithms are also presented. Based on both Chapters 2 and 3, we propose and examine in detail, in the fourth chapter, a new extension to the collective risk model assuming a certain dependence between the frequency and the severity of the claims. We consider collective risk models with different dependence structures such as models based on multivariate mixed Erlang distributions, models involving bivariate or multivariate copulas, or in a more general setting, Archimedean and hierarchical Archimedean copulas. Once again, based on the common mixture representation, we adapt the computational methodology presented in Chapter 2 to compute the probability mass function of a random sum incorporating a hierarchical Archimedean dependency. Finally, in the last chapter, we study, in more details, the exponential mixture distributions in their univariate and multivariate versions and we explain their close relationship to Archimedean and hierarchical Archimedean copulas. We also derive several new distributions, and we establish their links with pre-existent distributions.

Keywords : Common mixture models, Exponential mixture, Bernoulli mixture, Archimedean copulas, Nested Archimedean copulas, Compounding, Marshall-Olkin, Hierarchical dependence structures.

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À ma fille qui a pu supporter
mon éloignement alors qu'elle est
encore bébé,
À ma fille qui ne cesse de
m'exprimer son amour,
À ma fille qui a donné sens et
goût à ma vie,
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Avant-propos

Cette thèse étudie les modèles de dépendance construits par des mélanges communs. Elle est composée de six chapitres dont une introduction et une conclusion générales. Les autres chapitres sont constitués de quatre articles scientifiques.

Plus spécifiquement, le Chapitre 2 est coécrit avec ma directrice de thèse Hélène Cossette de l'Université Laval, mon codirecteur de thèse Étienne Marceau de l'Université Laval et Déry Veilleux, étudiant à la maîtrise à l'Université Laval. Cet article, intitulé *Dependent risk models with Archimedean copulas : A computational strategy based on common mixture and applications*, propose une approche d'agrégation pour un portefeuille de plusieurs risques dont la dépendance est définie par une copule archimédienne ou archimédienne imbriquée. Plusieurs applications en actuariat telles que l'allocation de capital sont également considérées. Cet article a été publié en 2018 dans la revue scientifique Insurance : Mathematics and Economics.

Le Chapitre 3 repose sur un article coécrit avec Hélène Cossette, Étienne Marceau et Simon Pierre Gadoury, étudiant au baccalauréat à l'Université Laval, s'intitulant *Hierarchical Archimedean Copulas Through Multivariate Compound Distributions*. Cet article fut publié en 2017 dans la revue Insurance : Mathematics and Economics et présente une nouvelle méthode de construction de copules archimédiennes hiérarchiques basée sur les distributions composées multivariées.

Le Chapitre 4 est basé sur un article intitulé *Collective Risk Models with Hierarchical Archimedean Copulas*, soumis actuellement pour publication dans la revue scientifique Insurance : Mathematics and Economics et réalisé en collaboration avec Hélène Cossette et Étienne Marceau. Cet article propose la classe de modèles collectifs de risque incorporant une dépendance entre le nombre et les montants des sinistres.

Le Chapitre 5 est coécrit avec Hélène Cossette, Étienne Marceau et Déry Veilleux. Cet article est intitulé *A note on Univariate and Multivariate Mixed Exponential Distributions* et présentement soumis pour publication dans la revue scientifique Methodology and Computing in Applied Probability. Cet article examine en détail les distributions exponentielles mélange univariées et multivariées. La contribution principale de cet article est la proposition de nouvelles distributions univariées et multivariées et de nouvelles copules tout en faisant le lien avec des distributions connues.

Chapitre 1

Introduction

"*All models are wrong,*
though some are useful"
George E. P. Box

La théorie du risque fait référence à un ensemble de techniques permettant de modéliser et de quantifier le risque associé à un portefeuille de contrats d'assurance. Étant basée sur la statistique et la théorie des probabilités, la théorie du renouvellement, les processus stochastiques, la théorie de la ruine, l'analyse fonctionnelle et la théorie de l'optimisation, elle a comme objectif d'examiner les fluctuations des réclamations reçues par une compagnie d'assurance et de fournir les outils nécessaires d'évaluation des risques souscrits par les compagnies d'assurance. Pour plus de détails sur la théorie du risque, voir par exemple Denuit and Charpentier (2004), Denuit and Charpentier (2005), McNeil et al. (2005), Denuit et al. (2005), Panjer et al. (2008), Marceau (2013), McNeil et al. (2015).

Depuis les années 1990, la modélisation de la dépendance entre les risques est devenue un enjeu important au cœur de la théorie du risque. Elle permet de comprendre la structure de dépendance entre deux ou plusieurs variables aléatoires (v.a.) et de déterminer leur loi conjointe. Jusqu'à récemment, les actuaires avaient un ensemble assez limité d'outils pour construire la loi conjointe ou encore pour analyser, extraire et utiliser les informations intégrées dans les distributions multivariées. Les solutions les plus connues ont été de supposer l'indépendance entre les risques ou d'utiliser le coefficient de corrélation linéaire comme outil pour mesurer la dépendance entre les variables aléatoires.

Contrairement à l'indépendance qui ne peut s'exprimer que d'une seule façon, selon Joe (2014), il y a plusieurs approches pour développer des modèles multivariés incorporant la dépendance. Parmi ces modèles, on retrouve, dépendamment du contexte, les copules, les modèles par mélange, les effets aléatoires, les variables latentes, les opérateurs stochastiques ou encore les spécifications conditionnelles.

Dans cette thèse, nous nous intéressons aux modèles multivariés construits par mélange commun et nous étudions leurs liens avec les copules Archimédiennes.

Le présent chapitre est entièrement consacré à une introduction générale de la thèse. Cette introduction est divisée en quatre sections. Dans la première section, on fournit un rappel des principaux résultats sur la modélisation de la dépendance qui seront utilisés dans la thèse. Ensuite, dans la deuxième section, on présente des modèles de dépendance avec mélange commun tout en faisant le lien avec les copules. La troisième section est consacrée aux applications possibles en actuariat. Les notions sur les mesures de risque seront présentées dans cette section. Finalement, la dernière section résume les principales contributions de cette thèse.

1.1 Modélisation de la dépendance

Les relations de dépendance entre les variables aléatoires représentent un sujet fort important et très étudié en statistique et en théorie de la probabilité. Elle a fait l'objet de plusieurs travaux de recherche, dont, par exemple, le travail encyclopédique de Harry Joe (Joe (1997)), Sklar (1959), Mardia (1970), Samuel et al. (2001), Embrechts et al. (2001), Nelsen (2007), Balakrishnan (2006), Schweizer and Sklar (2011), Dall'Aglio et al. (2012), Marceau (2013) et Joe (2014).

Selon Joe (1997), il y a quatre niveaux de dépendance. Le premier type regroupe les singularités sur certaines courbes et surfaces. Ensuite, on trouve la dépendance positive et négative. Le troisième type est la dépendance échangeable ou flexible. Enfin, s'il existe un indice temporel, on retrouve la dépendance décroissante avec le décalage. L'analyse et le choix d'un modèle multivarié se basent sur les niveaux de dépendance qu'il couvre ainsi que le spectre de dépendance qu'il peut atteindre. Afin de déterminer les niveaux de dépendance d'un modèle ainsi que son adéquation à un cas spécifique, la réécriture du modèle selon une (ou plusieurs) représentation stochastique s'avère importante. En général, il y a plusieurs propriétés désirables que doit satisfaire un modèle multivarié. Parmi ces propriétés, on retrouve la représentation de la fonction de répartition conjointe sous une forme fermée, la flexibilité dans le choix des marginales, une structure de dépendance large et flexible, etc. (voir par exemple Joe (1997) pour plus de détails). Notez que certaines de ces propriétés sont plus importantes que d'autres et qu'en général, il n'est pas possible de satisfaire toutes ces propriétés désirables. Dans ce cas, il s'avère important de décider de l'importance relative des propriétés et de déterminer lesquelles abandonner. Cette décision n'est en général pas si facile et évidente surtout lors de l'utilisation des distributions multivariées. Supposons par exemple que nous voulons décrire le comportement d'un portefeuille de risques composé d'un panier de crédit et d'un stock. Si l'on suppose que les distributions marginales de ces risques proviennent de familles de distributions différentes (par exemple une loi normale et une loi inverse normale), il sera difficile de trouver une distribution multivariée qui couple ces différentes distributions univariées. De plus, bien qu'il soit assez facile d'étendre les distributions univariées au cas bivarié, tel est le cas pour la loi normale et la loi de Poisson, il est souvent plus compliqué de les généraliser à plusieurs dimensions. La distribution conjointe multivariée d'un vecteur aléatoire contient des informations complètes sur ce vecteur, donc contient également l'information sur la dépendance entre les variables aléatoires. En général, la spécification, la construction et le choix des distributions multivariées sont souvent problématiques, car il est difficile, et même impossible, de décrire la structure de dépendance indépendamment des marginales (voir e.g. Joe (1997) et Frees and Valdez (1998) pour plus de détails).

Heureusement, tous ces problèmes peuvent être résolus grâce à la construction des distributions multivariées en utilisant les copules. Provenant du mot latin « copula » signifiant « lien », une copule est une fonction de répartition multivariée qui sert à relier des fonctions de répartition univariées afin de construire une fonction de répartition multivariée. Son plus grand avantage, autre que la grande flexibilité dans le choix des distributions marginales et des paramètres, est la capacité de capturer et de décrire la structure de dépendance d'une manière distincte des marginales dans le cas de vecteur de v.a. continues. Bien que cette notion statistique constitue un sujet de recherche assez moderne et en plein essor ces dernières années, elle fut étudiée, il y a plusieurs années. On retrouve la notion de copule notamment dans les travaux de Wassily Hoeffding (voir Höffding (1940) et Hoeffding (1941)) dans lesquels il obtient de nombreux résultats pour les fonctions de distribution normalisées sur le carré $[-1/2, 1/2]$. Ensuite, on retrouve, par exemple, les travaux de Fréchet (1951) et de Féron (1956) (pour lesquels les travaux de Hoeffding étaient inconnus à cause de la Deuxième Guerre mondiale) dans lesquels la même notion a été étudiée sous le nom de distributions standardisées. Puis, en 1959, Abe Sklar introduisit pour la première fois le mot « copule » et démontra le célèbre théorème de Sklar. Dans Sklar (1959), il considère les copules dans le cas général avec le sens qu'on y connaît aujourd'hui. Depuis ce temps, ce domaine de recherche jeune et actif s'est développé assez rapidement, voir e.g. Fisher (1997), Schweizer and Sklar (1983), Joe (1997), McNeil et al. (2005), Nelsen (2007), Genest et al. (2009) et Joe (2014).

La présente section se veut une introduction à la théorie des copules et à quelques concepts connexes comme les mesures de la dépendance entre les variables aléatoires. Il est clair qu'une telle introduction ne peut en aucun cas explorer de manière exhaustive tous les concepts reliés à la notion de dépendance. Il est important toutefois de présenter quelques résultats nécessaires pour le travail qui s'en suivra.

1.1.1 Préliminaires

Dans cette section, nous présentons les définitions et les propriétés de base des concepts qui seront utilisés plus tard.

Soit $\underline{X} = (X_1, \ldots, X_n)$ un vecteur de n v.a. dont la fonction de répartition conjointe est définie par

$$F_{\underline{X}}(\underline{x}) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

Définition 1.1.1. *Une fonction $F_{\underline{X}} : \mathbb{R}^n \to [0,1]$ continue à droite est dite une fonction de répartition conjointe multivariée si et seulement si :*

1. *$F_{\underline{X}}$ est non décroissante sur $\mathbb{R}^n$;*
2. *$\lim\limits_{x_i \to -\infty} F_{\underline{X}}(\underline{x}) = 0$, pour $i \in \{1, \ldots, n\}$;*
3. *$\lim\limits_{x_1, \ldots, x_n \to \infty} F_{\underline{X}}(\underline{x}) = 1$;*
4. *pour tout $\underline{x}$, et pour tout $\underline{a} = (a_1, \ldots, a_n)$ et $\underline{b} = (b_1, \ldots, b_n)$ dans $\mathbb{R}^n$ tels que $a_i < b_i$, pour $i = 1, \ldots, n$ on a*
$$\triangle_{a_1,b_1} \ldots \triangle_{a_n,b_n} F_{\underline{X}}(\underline{x}) \geq 0,$$
où l'opérateur $\triangle$ est défini comme suit :
$$\triangle_{a_i,b_i} F_{\underline{X}}(\underline{x}) = F_{\underline{X}}(x_1, \ldots, b_i, \ldots, x_n) - F_{\underline{X}}(x_1, \ldots, a_i, \ldots, x_n).$$

La fonction de répartition marginale est obtenue à partir de la fonction de répartition conjointe comme suit :

$$F_{X_i}(x_i) = \Pr(X_i \leq x_i) = F_{\underline{X}}(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty),$$

pour $i = 1, \ldots, n$.

Enfin, la fonction de survie conjointe est définie par

$$\overline{F}_{\underline{X}}(\underline{x}) = \Pr(X_1 > x_1, \ldots, X_n > x_n).$$

Soit $\Gamma(F_{X_1}, \ldots, F_{X_n})$ la classe de Fréchet qui correspond à l'ensemble de toutes les fonctions de répartition conjointes de $\underline{X} = (X_1, \ldots, X_n)$ ayant pour marginales $F_{X_1}, \ldots, F_{X_n}$. Cette classe n'est pas vide, car elle contient le cas de la distribution multivariée de n v.a. indépendantes avec marginales $F_{X_1}, \ldots, F_{X_n}$, i.e. $F_{\underline{X}}(\underline{x}) = \prod_{i=1}^n F_{X_i}$. Un résultat très important dans la modélisation de la dépendance, revenant à Maurice Fréchet, stipule que toute fonction de répartition conjointe appartenant à la classe de Fréchet $\Gamma(F_{X_1}, \ldots, F_{X_n})$ est comprise entre deux bornes. Ces bornes sont appelées bornes (supérieure et inférieure) de Fréchet et sont données comme suit :

$$F^-(\underline{x}) = \max\left\{0, \sum_{i=1}^n F_{X_i}(x_i) - (n-1)\right\} \leq F_{\underline{X}}(\underline{x}) \leq \min\{F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\} = F^+(\underline{x}). \tag{1.1}$$

Il est important de souligner que la borne supérieure de Fréchet est une fonction de répartition multivariée appartenant à la classe $\Gamma(F_{X_1}, \ldots, F_{X_n})$ quelle que soit la dimension n, alors que, en général, ce n'est pas le cas pour la borne inférieure. La borne inférieure appartient à $\Gamma(F_{X_1}, \ldots, F_{X_n})$ seulement si $n = 2$. Voir e.g. Joe (1997) et Joe (2014) pour plus de détails à ce sujet.

La borne supérieure de Fréchet représente la dépendance positive parfaite et correspond à la comonotonicité, i.e., les composantes du vecteur $\underline{X}$ sont comonotones avec $X_i = F_{X_i}^{-1}\left(F_{X_j}(X_j)\right)$, pour tout $i, j \in \{1, \ldots, n\}$, avec $i \neq j$. De plus, pour $n = 2$, la borne inférieure de Fréchet représente la dépendance négative parfaite appelée l'antimonotonicité, i.e., X_1 et X_2 sont antimonotones avec $X_1 = F_{X_1}^{-1}(U)$ et $X_2 = F_{X_2}^{-1}(1-U)$ où $U \sim Unif(0,1)$.

1.1.2 Dépendance positive par quadrant

L'un des objectifs de la modélisation de la dépendance est la capacité de mesurer et de comparer la force des relations de dépendance entre les v.a. Soit $\underline{X} = (X_1, \ldots, X_n)$ un vecteur de n v.a. avec fonction de répartition conjointe $F_{\underline{X}} \in \Gamma(F_{X_1}, \ldots, F_{X_n})$. Il existe plusieurs façons de décider qu'un membre de la classe de Fréchet $\Gamma(F_{X_1}, \ldots, F_{X_n})$ est plus dépendant qu'un autre. La comparaison peut aussi s'établir en termes de vecteurs de v.a. Une de ces façons est l'ordre de concordance qui se base sur le concept de dépendance positive par quadrant. Dans cette section, on introduit les notions relatives à ce concept.

Pour des fins de simplification, on va considérer l'étude du cas bivarié uniquement dans cette section ($n = 2$), bien que la majorité des résultats présentés puissent facilement être généralisés au cas multivarié.

Définition 1.1.2. *On considère le couple (X_1, X_2) avec fonction de répartition $F_{X_1,X_2} \in \Gamma(F_{X_1}, F_{X_2})$. Le couple (X_1, X_2) (ou F_{X_1,X_2}) est dit dépendant positivement par quadrant (PQD) si*

$$\overline{F}_{X_1,X_2}(x_1, x_2) \geq \overline{F}_{X_1}(x_1)\overline{F}_{X_2}(x_2), \tag{1.2}$$

ou de façon équivalente

$$F_{X_1,X_2}(x_1, x_2) \geq F_{X_1}(x_1)F_{X_2}(x_2), \tag{1.3}$$

pour tout $x_1, x_2 \in \mathbb{R}$.

En d'autres mots, si le couple (X_1, X_2) est dépendant positivement par quadrant, alors les v.a. X_1 et X_2 ont plus tendance à prendre simultanément des valeurs élevées ou des petites valeurs que dans le cas d'indépendance entre X_1 et X_2. Notez qu'en inversant le symbole des inégalités dans (1.2) ou (1.3), on retrouve la notion de dépendance négative par quadrant.

La dépendance positive par quadrant sert donc à comparer un vecteur de v.a. quelconques à un vecteur de v.a. indépendantes. Ce concept peut se généraliser à la notion d'ordre de concordance, noté $\prec_c$, telle que définie dans Joe (1997). Selon cet ordre, la comparaison de relations de dépendance est faite entre des vecteurs de v.a. quelconques. En d'autres mots, en utilisant l'ordre de concordance, on peut comparer la dépendance d'une distribution multivariée par rapport à une autre.

Définition 1.1.3. *On considère les couples* (X_1, X_2) *et* (X_1', X_2') *dont les fonctions de répartition bivariées* F_{X_1,X_2} *et* $F_{X_1',X_2'}$ *appartiennent à* $\Gamma(F_1, F_2)$. *La paire* (X_1', X_2') *est dite plus concordante que* (X_1, X_2), *noté* $(X_1, X_2) \prec_c (X_1', X_2')$, *si*

$$F_{X_1,X_2}(x_1, x_2) \leq F_{X_1',X_2'}(x_1, x_2),\ \forall x_1, x_2 \in \mathbb{R},$$

ou de manière équivalente

$$\overline{F}_{X_1,X_2}(x_1, x_2) \leq \overline{F}_{X_1',X_2'}(x_1, x_2),\ \forall x_1, x_2 \in \mathbb{R}.$$

De plus, l'ordre de concordance est l'ordre à dépendance positive multivariée le plus faible. En d'autres termes, si le couple (X_1^*, X_2^*) est plus dépendant que (X_1, X_2) selon un ordre à dépendance positive quelconque, alors (X_1^*, X_2^*) est forcément plus concordant que (X_1, X_2), i.e., $(X_1, X_2) \prec_c (X_1^*, X_2^*)$.

Cette notion d'ordre de concordance a une grande importance dans ce qui suit, car elle permettra de comparer les forces de dépendance entre différentes copules et entre les copules composant une structure hiérarchique. Elle permet également d'ordonner les différentes mesures d'association et de concordance.

1.1.3 Mesures de concordance et d'association

L'outil le plus utilisé dans la littérature pour quantifier la dépendance est le coefficient de corrélation de Pearson. Cependant, ce coefficient est invariant par rapport aux transformations linéaires, n'est pas une bonne mesure pour les corrélations non linéaires et ne permet pas de mesurer adéquatement la concordance. De plus, dans le cas de v.a. continues non normales, l'utilisation d'un coefficient basé sur la linéarité n'est pas toujours adéquate. Contrairement au coefficient de corrélation de Pearson dont la valeur dépend des marginales, il existe des mesures de dépendance invariantes au changement d'échelle dont le tau de Kendall et le rhô de Spearman. Ces deux mesures valent 1 pour la borne supérieure de Fréchet et -1 pour la borne inférieure de Fréchet, ce qui rend leur utilisation très avantageuse.

Définition 1.1.4. *Soit deux paires indépendantes de v.a. continues* (X_1, X_2) *et* (X_1', X_2') *dont les fonctions de répartition conjointes appartiennent à la classe de Fréchet, i.e.,* $F_{X_1,X_2}, F_{X_1',X_2'} \in \Gamma(F_1, F_2)$. *Dans ce cas, le tau de Kendall est défini comme suit :*

$$\tau(X_1, X_2) = \Pr\left((X_1 - X_1')(X_2 - X_2') > 0\right) - \Pr\left((X_1 - X_1')(X_2 - X_2') < 0\right).$$

En d'autres mots, le tau de Kendall est la probabilité de concordance entre (X_1, X_2) et (X_1', X_2') moins la probabilité de discordance.

De la même façon, on peut définir le rhô de Spearman.

Définition 1.1.5. *Soit* (X_1, X_2) *un couple de v.a. continues avec fonction de répartition conjointe* $F_{X_1,X_2} \in \Gamma(F_{X_1}, F_{X_2})$. *Le rhô de Spearman s'écrit comme suit*

$$\rho_S(X_1, X_2) = 12 \int\int F_{X_1}(x_1) F_{X_2}(x_1) \mathrm{d}F_{X_1,X_2}(x_1, x_2) - 3.$$

En d'autres mots, le rhô de Spearman est la corrélation linéaire entre F_{X_1} et F_{X_2}.

Notez que le tau de Kendall et le rhô de Spearman prennent des valeurs entre -1 et 1 et sont invariants par rapport aux transformations strictement croissantes ce qui permettra de les écrire en termes de copules dans la prochaine section.

Il existe une relation entre le tau de Kendall et le rhô de Spearman. Connaître l'une de ces deux mesures permet de donner une idée sur l'autre. Pour un couple de v.a. (X_1, X_2), nous avons

$$\begin{cases} \frac{3\tau-1}{2} \leq \rho_S(X_1, X_2) \leq \frac{1+2\tau-\tau^2}{2}, & si\, \tau \geq 0 \\ \frac{\tau^2+2\tau-1}{2} \leq \rho_S(X_1, X_2) \leq \frac{1+3\tau}{2}, & si\, \tau \leq 0 \end{cases},$$

où $\tau = \tau(X_1, X_2)$.

Afin d'étudier la dépendance de queue d'un couple de v.a. continues, on peut utiliser les coefficients de dépendance de queue.

Définition 1.1.6. *Soit le couple de v.a. continues* (X_1, X_2) *avec fonction de répartition conjointe* $F_{X_1,X_2} \in \Gamma(F_{X_1}, F_{X_2})$. *Le coefficient de dépendance de queue supérieure est*

$$\lambda_U(X_1, X_2) = \lim_{\kappa \to 1^-} \Pr\left(X_2 > F_{X_2}^{-1}(\kappa) | X_1 > F_{X_1}^{-1}(\kappa)\right)$$

et le coefficient de dépendance de queue inférieure est

$$\lambda_L(X_1, X_2) = \lim_{\kappa \to 0^+} \Pr\left(X_2 \leq F_{X_2}^{-1}(\kappa) | X_1 \leq F_{X_1}^{-1}(\kappa)\right).$$

Les coefficients de dépendance de queue (des mesures de dépendance locales) revêtent une importance particulière en assurance, car ils permettent, notamment, de donner une idée sur la survenance simultanée de deux sinistres de montants élevés dans deux lignes d'affaires.

En utilisant l'ordre de concordance, on peut comparer ces différentes mesures de dépendance pour deux distributions différentes. Soit les couples de v.a. (X_1, X_2) et (X_1', X_2') avec $F_{X_1,X_2}, F_{X_1',X_2'} \in \Gamma(F_1, F_2)$. Si $(X_1, X_2) \prec_c (X_1', X_2')$, alors on a

$$\tau(X_1, X_2) \leq \tau(X_1', X_2'), \tag{1.4}$$
$$\rho_S(X_1, X_2) \leq \rho_S(X_1', X_2'), \tag{1.5}$$
$$\lambda_U(X_1, X_2) \leq \lambda_U(X_1', X_2'), \tag{1.6}$$
$$\lambda_L(X_1, X_2) \leq \lambda_L(X_1', X_2'). \tag{1.7}$$

1.1.4 Copules

La dernière crise financière de 2008 confirme l'importance, voire l'indispensabilité, de modéliser adéquatement les structures de dépendance entre les différentes v.a. Depuis ce temps, la théorie des copules a connu un essor important en finance (voir e.g. Mai and Scherer (2014)), en actuariat et en gestion quantitative des risques. Par exemple, on retrouve l'utilisation des copules en assurance IARD dans un contexte de tarification (voir e.g., Czado et al. (2012) et Shi (2016)) et des réserves (voir e.g., Shi and Frees (2011), Abdallah et al. (2015) et Côté et al. (2016)).

La théorie des copules est un outil mathématique très flexible permettant d'analyser et de modéliser la dépendance entre les v.a. Une copule est une fonction de répartition multivariée avec des marginales uniformes sur $[0, 1]$.

Définition 1.1.7. *Une copule multivariée $C(u_1, \ldots, u_d)$ est une fonction de répartition définie sur $[0, 1]^d$ dont les marginales sont uniformes sur $[0, 1]$. Pour que C soit une copule, elle doit satisfaire les propriétés suivantes :*

1. *$C(0, \ldots, 0) = 0$;*
2. *$C(1, \ldots, 1) = 1$;*
3. *Si toutes les composantes de C sont 1 sauf pour un u, alors la copule prend la valeur u ;*
4. *Si au moins une des composantes de C est 0, alors la copule prend la valeur 0 ;*
5. *La probabilité sur $(a_1, b_1] \times \ldots \times (a_d, b_d]$ est positive i.e.*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots+i_d} C(u_{1i_1}, \ldots, u_{di_d}) \geq 0,$$

avec $u_{j1} = a_j$ et $u_{j2} = b_j$ pour $j \in \{1, 2, \ldots, d\}$.

En 1959, Abe Sklar a présenté et démontré le résultat fondamental de la théorie des copules (voir Sklar (1959)). Connu sous le nom du théorème de Sklar, ce résultat établit un lien entre une fonction de répartition multivariée et ses marginales univariées comme suit :

Théorème 1.1.1. ***Théorème de Sklar*** *Soit $F \in \Gamma(F_1, \dots, F_d)$, où Γ correspond à la classe de toutes les fonctions de répartition conjointes ayant pour marginales $F_1, \dots, F_d$. Alors, il existe une copule C telle que pour tout $\underline{x} \in \mathbb{R}^d$*

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \tag{1.8}$$

La copule C est unique si $F_1, \dots, F_d$ sont continues. Sinon, C est uniquement déterminée sur les rangs des marginales. Inversement, si C est une copule et $F_1, \dots, F_d$ sont des marginales, alors F, telle que définie en (1.8), est une fonction de répartition multivariée avec marginales F_i, pour $i = 1, \dots, d$.

Preuve 1.1.1. *Voir e.g., Joe (1997) ou Nelsen (2007).*

Dans la littérature, on retrouve plusieurs approches pour créer des copules. Dans cette thèse, seule la méthode par inversion sera considérée. En effet, en se basant sur le théorème de Sklar, on peut construire une copule à partir d'une fonction de répartition conjointe F et l'inverse des marginales continues $F_1^{-1}, \dots, F_d^{-1}$ comme suit :

$$C(u_1, \dots, u_d) = F\left(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\right),$$

pour tout $(u_1, \dots, u_d) \in [0,1]^d$.

Inversement, à l'aide d'une copule et de n fonctions de répartition marginales, on peut construire une fonction de répartition multivariée. La copule permet donc de relier ou de "coupler" différentes fonctions de répartition marginales pour construire une fonction de répartition multivariée. La flexibilité est donc une propriété naturelle d'une telle structure de dépendance. En effet, avec une copule seulement, il est possible de construire un très grand nombre de fonctions de répartition multivariées. De plus, la propriété d'invariance par rapport aux transformations non décroissantes permet aux copules de capturer et de décrire, de façon complète, la relation de dépendance entre les v.a. La structure de dépendance est ainsi entièrement dissociée des marginales ce qui constitue l'une des raisons de la popularité de la théorie des copules.

Il est également possible de définir une copule $\hat{C}$, appelée copule de survie, qui relie les fonctions de survie marginales $\overline{F}_1, \dots, \overline{F}_d$ de $X_1, \dots, X_d$ afin de construire une fonction de survie multivariée comme suit :

$$\overline{F}(x_1, \dots, x_d) = \hat{C}\left(\overline{F}_1(x_1), \dots, \overline{F}_d(x_d)\right).$$

Il est important de noter que $\hat{C}$ est aussi une copule.

En utilisant la méthode inverse, on peut construire la copule de survie $\hat{C}$ à partir d'une fonction de survie conjointe d'une loi continue multivariée comme suit :

$$\hat{C}\left(u_1,\ldots,u_d\right)=\overline{F}\left(\overline{F}_1^{-1}\left(u_1\right),\ldots,\overline{F}_d^{-1}\left(u_d\right)\right), \tag{1.9}$$

pour tout $(u_1,\ldots,u_d)\in[0,1]^d$.

Une propriété intéressante d'une famille de copules est le spectre de dépendance qu'elle est capable de couvrir. Les valeurs que peuvent prendre les paramètres de la copule ont une influence directe sur l'intensité de la dépendance et contribuent à définir sa structure de dépendance. Il existe trois cas particuliers de copules qui sont pertinents dans la discussion du spectre de dépendance couvert par une copule. Le premier cas est la copule d'indépendance, notée Π, qui représente le cas de marginales indépendantes, i.e.,

$$\Pi\left(u_1,u_2\right)=u_1\,u_2,\;(u_1,u_2)\in[0,1]^2.$$

Les deux autres cas sont les copules borne inférieure et borne supérieure de Fréchet, notées respectivement par W et M, qui s'écrivent comme suit :

$$W(u_1,u_2)=\max\left\{0,u_1+u_2-1\right\}$$

et

$$M(u_1,u_2)=\min\left\{u_1,u_2\right\},$$

où $(u_1,u_2)\in[0,1]^2$. Ces deux copules représentent les bornes (inférieure et supérieure) de Fréchet telles que définies dans les sections précédentes. Tel qu'illustré dans la Figure 1.1, les réalisations produites par la copule borne supérieure de Fréchet sont uniformément distribuées sur la diagonale $u_1=u_2$, et celles produites par la borne inférieure sont regroupées sur la diagonale $u_1=1-u_2$.

Puisque toute copule est une fonction de répartition conjointe dont les marginales sont uniformément distribuées, l'inégalité (1.1) est vérifiée. Ainsi, pour toute copule bivariée C et pour tout $u_1,u_2\in[0,1]$, on a

$$W(u_1,u_2)\leq C(u_1,u_2)\leq M(u_1,u_2),\;(u_1,u_2)\in[0,1]^2.$$

Ce résultat est illustré dans la Figure 1.2 par une série de courbes de niveau. Une courbe de niveau ou isosphère d'altitude (en anglais "contour plot"), est une technique graphique

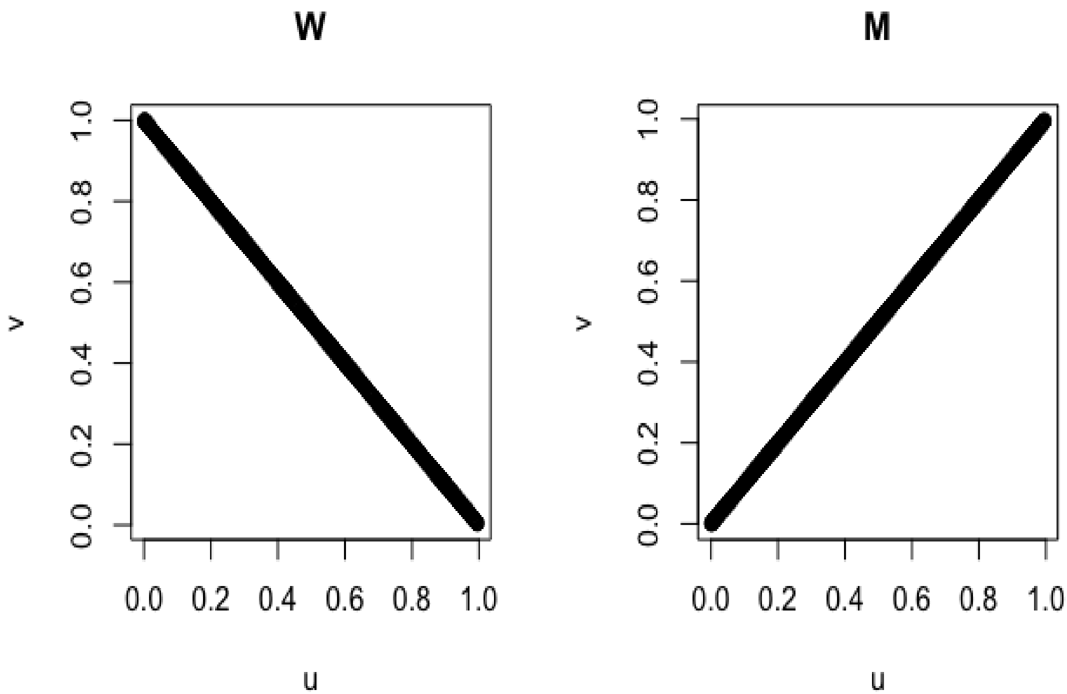


FIGURE 1.1 – Nuages de points représentant 1000 réalisations de (U_1, U_2) provenant des copules borne inférieure W (gauche) et borne supérieure de Fréchet (droite).

qui sert à représenter une surface 3-D en traçant des lignes constantes "t" sur une surface 2-D représentant le relief situé à la même altitude. En d'autres termes, pour une valeur "t" donnée, des lignes sont dessinées pour connecter les coordonnées (x, y) où cette valeur "t" se produit. Dans le cas d'une copule, étant donné une valeur pour "$t \in [0, 1]$", la courbe de niveau représente le tracé formé par $C(u_1, u_2) = t$, $\forall (u_1, u_2) \in [0, 1]^2$. On remarque que la courbe de toute copule $C(u_1, u_2)$ est comprise dans le triangle formé par les deux courbes de niveau t des copules borne supérieure et borne inférieure de Fréchet.

Notez que les copules d'indépendance et borne supérieure de Fréchet peuvent être généralisées au cas multidimensionnel.

Puisque toute copule est une fonction de répartition, il est donc possible de définir la fonction de survie qui lui est associée. Cette fonction, notée $\hat{C}$, est également une copule. Soit C une copule bivariée définie sur $[0, 1]$. La copule de survie associée à C s'écrit comme suit :

$$\hat{C}(u_1, u_2) = C(1 - u_1, 1 - u_2) + u_1 + u_2 - 1,\ \forall u_1, u_2 \in [0, 1].$$

La fonction de densité bivariée associée à la copule C est comme suit :

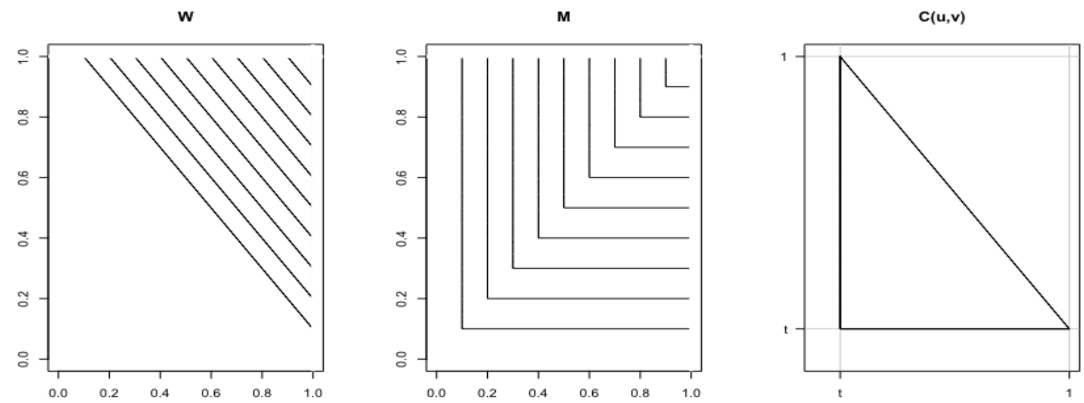


FIGURE 1.2 – Courbes de niveau des copules borne supérieure (M) et borne inférieure (W) de Fréchet.

$$c(u_1,u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1,u_2),$$

et celle associée à la copule de survie $\hat{C}$:

$$\hat{c}(u_1,u_2) = c(1-u_1, 1-u_2),$$

avec $(u_1,u_2) \in [0,1]^2$.

En ce qui concerne la notion de concordance, Joe (1997) définit l'ordre de concordance pour les copules multivariées comme suit :

Définition 1.1.8. *Soit C_1 et C_2 deux copules de dimension d. C_2 est plus concordante que C_1, noté $C_1 \prec_c C_2$, si*

$$C_1(\underline{u}) \leq C_2(\underline{u}) \text{ et } \overline{C_1}(\underline{u}) \leq \overline{C_2}(\underline{u}),$$

pour tout $\underline{u} \in [0,1]^d$.

Pour une copule bivariée, le tau de Kendall, le rhô de Spearman et les coefficients de dépendance de queue s'écrivent, respectivement, comme suit :

$$\tau(C) = 4 \int_0^1 \int_0^1 C(u_1,u_2) \mathrm{d}C(u_1,u_2) - 1,$$

$$\rho(C) = 12 \int_0^1 \int_0^1 u_1 \, u_2 \, \mathrm{d}C(u_1,u_2) - 3,$$

$$\lambda_U(C) = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u},$$

et

$$\lambda_L(C) = \lim_{u \to 0^+} \frac{C(u,u)}{u}.$$

Notez que si $C_1 \prec_c C_2$, les propriétés (1.4), (1.5), (1.6) et (1.7) sont valides.

1.1.5 Copules Archimédiennes

Plusieurs copules sont développées en deux dimensions seulement, ce qui restreint leur application. Une des familles de copules qui se généralise naturellement en plusieurs dimensions est la famille des copules Archimédiennes. En plus de sa simple procédure de construction et sa généralisation multivariée, elle a l'avantage de fournir des expressions explicites des fonctions de répartition multivariées, d'atteindre différents niveaux de dépendance de queue et d'avoir une procédure de simulation assez simple et facile à implémenter. Cette classe de copules constitue un des sujets principaux de cette thèse.

On rappelle brièvement la définition d'une copule Archimédienne.

Définition 1.1.9. *Une copule C de dimension d est dite Archimédienne si*

$$C(u_1, \ldots, u_d) = \psi\left(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)\right), \tag{1.10}$$

pour $(u_1, \ldots, u_d) \in [0,1]^d$ et où ψ est appelée le générateur de la copule C. La fonction ψ satisfait les propriétés suivantes :

1. $\psi : [0, \infty) \to [0,1]$ *;*
2. $\psi(0) = 1$ *et* $\lim_{t\to\infty} \psi(t) = 0$ *;*
3. ψ *est continue et strictement décroissante ;*
4. $\psi^{-1} : [0,1] \to [0,\infty)$ *avec* $\lim_{u\to 1} \psi^{-1}(u) = 0$ *et* $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$.

En utilisant les "Williamson d-transforms", McNeil and Nešlehová (2009) démontrent que la fonction dans (1.10) est une copule si et seulement si le générateur ψ est d-monotone, c'est-à-dire que ψ est continue sur $[0,\infty)$, $d-2$ dérivable avec

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi(t) \geq 0, \forall k \in \{0, \ldots, d-2\} \, et \, \forall t \in [0,\infty),$$

et $(-1)^{d-2} \frac{\mathrm{d}^{d-2}}{\mathrm{d}t^{d-2}}\psi(t)$ est décroissante et convexe sur $[0,\infty)$.

Dans cette thèse, nous allons considérer une classe spécifique de copules Archimédiennes dont les générateurs sont complètement monotones, i.e., $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi(t) \geq 0$, pour tout $k = 0, 1, \ldots$ et $\forall t \in [0,\infty)$. Selon le théorème de Kimberling, ce cas représente une condition nécessaire et suffisante pour générer une copule pour toute dimension $d = 2, 3, \ldots$

Théorème 1.1.2. ***Théorème de Kimberling***
La fonction C est une copule pour tout $d \geq 2$ si et seulement si ψ est complètement monotone, i.e,

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi(t) \geq 0, \ \forall k \in \{0, 1, \ldots\}, \forall t \in [0, \infty).$$

Preuve 1.1.2. *Voir e.g. Kimberling (1974) ou Hofert (2010).*

Nous allons aussi nous baser sur le théorème suivant de Bernstein afin de faire le lien entre la classe des générateurs complètement monotones, qu'on note par Ψ_∞, et la classe des transformées de Laplace-Stieltjes (TLS).
La TLS d'une v.a. strictement positive Θ avec fonction de répartition F_Θ, est définie par

$$\mathcal{L}_\Theta(t) = \int_0^\infty \mathrm{e}^{-t\,x}\,\mathrm{d}F_\Theta(x) = E\left[\mathrm{e}^{-t\Theta}\right]. \tag{1.11}$$

Théorème 1.1.3. ***Théorème de Bernstein***
Une fonction $\psi : [0, \infty) \to [0, 1]$ est complètement monotone si et seulement si ψ est la TLS d'une v.a. strictement positive Θ, i.e., $\psi = \mathcal{L}_\Theta$.

Preuve 1.1.3. *Voir e.g. Feller (1971).*

En se basant sur le Théorème 1.1.3 de Bernstein, on peut réécrire la copule C dans (1.10) en termes de la TLS d'une v.a. strictement positive Θ comme suit :

$$C\left(u_1, \ldots, u_d\right) = \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(u_1\right) + \ldots + \mathcal{L}_\Theta^{-1}\left(u_d\right)\right), \tag{1.12}$$

pour $(u_1, \ldots, u_d) \in [0, 1]^d$.

Bien que les copules Archimédiennes soient faciles à construire via la TLS d'une v.a strictement positive, il n'y a qu'un nombre limité de générateurs Archimédiens connus pour lesquels l'inverse existe et qui permettent d'obtenir une expression explicite et fermée de la copule (voir e.g., Nelsen (2007), Joe (1997) et Joe (2014)). Parmi les familles de copules Archimédiennes les plus utilisées dans la littérature et qui reviendront souvent dans cette thèse, on retrouve les copules Ali-Mikhail-Haq (AMH), Frank, Joe, Gumbel et Clayton (voir la Table 1.1). Les expressions ainsi que les différentes propriétés de ces copules seront données aux chapitres suivants.

Dans la Table 1.1, on note par $Geo(q)$ la distribution géométrique avec fonction de masse de probabilité (fmp) $p_k = q\,(1-q)^{k-1}$ pour $k = 1, 2, \ldots$, et par $Log(q)$ la distribution logarithmique avec fmp $p_k = \frac{-q^k}{k\ln(1-q)}$ pour $k = 1, 2, \ldots$ Pour la copule de Joe, $Sibuya(q)$ désigne la distribution Sibuya avec fmp $p_k = (-1)^{k-1}\binom{q}{k}$ pour $k = 1, 2, \ldots$ On note par $Gamma(\alpha, \beta)$ la distribution gamma avec paramètre de forme α et paramètre d'intensité β. Finalement, $S\left(\alpha, 1, \cos^{\frac{1}{\alpha}}\left(\frac{\pi\alpha}{2}\right), 1_{\{\alpha=1\}}; 1\right)$ désigne la loi stable avec TLS donnée par

Copule	**Paramètre**	**Générateur** $\psi = \mathcal{L}_\Theta$	Distribution de Θ
Ali-Mikhail-Haq	$\alpha \in [0,1]$	$\frac{1-\alpha}{e^t-\alpha}$	$\text{Geo}(1-\alpha)$
Frank	$\alpha \in [0,\infty)$	$-\frac{\ln\left(1-(1-e^{-\alpha})e^{-t}\right)}{\alpha}$	$\text{Log}(1-e^{-\alpha})$
Joe	$\alpha \in [1,\infty)$	$1-(1-e^{-t})^{\frac{1}{\alpha}}$	$\text{Sibuya}\big(\frac{1}{\alpha}\big)$
Gumbel	$\alpha \in [1,\infty)$	$\exp\left\{-t^{\frac{1}{\alpha}}\right\}$	$S\big(\frac{1}{\alpha}, 1, \cos^\alpha\left(\frac{\pi}{2\alpha}\right), 1_{\{\alpha=1\}}; 1\big)$
Clayton	$\alpha \in [0,\infty)$	$(1+t)^{-\frac{1}{\alpha}}$	$\text{Gamma}\big(\frac{1}{\alpha}, 1\big)$

TABLE 1.1 – Générateurs de quelques copules Archimédiennes connues.

$\mathcal{L}(t) = e^{-t^\alpha}$. Pour un exposé sur les algorithmes de simulation de ces différentes distributions, voir, e.g., Hofert (2010), Hofert (2011) et Devroye (2009).

Les copules Archimédiennes peuvent être générées en se basant sur l'algorithme de Marshall & Olkin (voir Marshall and Olkin (1988)).

Algorithme 1.1.1. ***(Marshall & Olkin)*** *Soit C une copule Archimédienne de dimension d avec générateur $\mathcal{L}_\Theta$.*

1. *Générer Θ avec LST $\mathcal{L}_\Theta$;*
2. *Générer d réalisations $R_i \sim Exp(1)$, pour $i = 1, \ldots, d$;*
3. *Retourner $\underline{U} = (U_1, \ldots, U_d)$ où $U_i = \mathcal{L}_\Theta\left(\frac{R_i}{\Theta}\right)$, pour $i = 1, \ldots, d$.*

Dans le cas où la distribution de la v.a. Θ n'est pas connue, il existe des procédures d'inversion numérique de la TLS qui permettent de générer Θ (voir e.g. Cohen (2007), Valkó and Vojta (2001) et Hofert (2010) pour un exposé sur les différentes méthodes d'inversion numériques possibles).

L'Algorithme 1.1.1 fournit une interprétation des copules Archimédiennes sous la forme de représentation stochastique suivante

$$\underline{U} = \left(\mathcal{L}_\Theta\left(\frac{R_1}{\Theta}\right), \ldots, \mathcal{L}_\Theta\left(\frac{R_d}{\Theta}\right)\right). \tag{1.13}$$

La représentation en (1.13) s'interprète comme étant une structure de dépendance, modélisant un portefeuille de d risques, induite par une v.a. Θ de telle sorte que toutes les v.a. du portefeuille sont conditionnellement indépendantes sachant la valeur de Θ.

Il existe d'autres méthodes de simulation, comme la méthode *conditionnelle*, qui peuvent être utilisées pour générer des réalisations à partir d'une copule Archimédienne. En général, ces méthodes s'avèrent, d'un point de vue computationnel, compliquées à appliquer pour un vecteur $\underline{U}$ de grande dimension d. Pour un exposé intéressant sur les différentes méthodes de simulation des copules Archimédiennes, on suggère Hofert (2010) ainsi que la bibliographie reliée à cet ouvrage.

L'étude des propriétés des copules Archimédiennes a fait l'objet de plusieurs travaux de recherche. On retrouve par exemple Joe (1997), McNeil et al. (2005), Nelsen (2007), McNeil and Nešlehová (2009) et Hofert (2010). La propriété la plus importante d'une copule Archimédienne est sa forme explicite en fonction d'un générateur. Cette propriété attrayante facilite l'étude de la copule et permet d'obtenir des expressions fermées ou semi-fermées de plusieurs de ses quantités connexes. Par exemple, pour une copule bivariée Archimédienne C, le tau de Kendall et les coefficients de dépendance de queue s'écrivent, respectivement, sous les formes suivantes :

$$\tau(C) = 1 + 4\int_0^1 \frac{\psi^{-1}(t)}{\left(\psi^{-1}(t)\right)'}\mathrm{d}t = 1 - 4\int_0^\infty t\ \left(\psi'(t)\right)^2 \mathrm{d}t,$$

$$\lambda_U(C) = 2 - \lim_{t\to 0}\frac{1-\psi(2t)}{1-\psi(t)},$$

et

$$\lambda_L(C) = \lim_{t\to\infty}\frac{\psi(2t)}{\psi(t)}.$$

(Voir e.g. Joe (1997) et Nelsen (2007)).

En ce qui concerne l'ordre de concordance défini dans la section 1.1.2, les théorèmes 4.1 et 4.7 de Joe (1997) fournissent le résultat intéressant suivant. Soit deux copules Archimédiennes de dimension d, C_1 et C_2, avec générateurs ψ_1 et ψ_2. Si la fonction $(\psi_1^{-1} \circ \psi_2)'$ est complètement monotone, alors C_2 est plus concordante que C_1, i.e. $C_1 \prec_c C_2$. Ce résultat va jouer un rôle très important dans l'étude des copules Archimédiennes hiérarchiques.

1.1.6 Copules Archimédiennes hiérarchiques

Parmi les classes de copules les plus connues, on retrouve les copules Archimédiennes et les copules elliptiques. Ces dernières sont construites à partir des distributions elliptiques multivariées connues (les lois normales et Student multivariées) en utilisant le Théorème de Sklar et elles sont radialement symétriques (les courbes de niveau et les graphiques des points forment des ellipses). Voir e.g. Embrechts et al. (2001) et Nelsen (2007) pour un exposé plus détaillé.

Les copules Archimédiennes, contrairement aux copules elliptiques, ne sont pas radialement symétriques. De plus, elles peuvent capturer différents types de dépendance de queue et s'écrivent explicitement sous des formes analytiques. Cependant, leur propriété d'échangeabilité les rend moins intéressantes pour des applications pratiques en grandes dimensions. Cette propriété implique que pour une dimension donnée d, toutes les marginales sont identiques. L'utilisation d'une telle structure de dépendance devient restrictive lors de la modélisation d'un portefeuille composé de plusieurs sous-groupes non homogènes. Il existe plusieurs alternatives proposées dans la littérature. Par exemple, on retrouve les copules *vine* introduites par Harry Joe en 1994 (voir e.g. Joe (1997) et Bedford and Cooke (2002)). Cette classe de copules hiérarchiques représente un outil très flexible de modélisation de la dépendance qui consiste à regrouper les risques deux à deux à l'aide de plusieurs copules bivariées. Cependant, le grand nombre de copules impliquées dans cette structure de dépendance nécessite l'estimation de plusieurs paramètres ce qui rend l'utilisation de cette structure moins avantageuse pour les portefeuilles volumineux. Les copules Archimédiennes imbriquées (en anglais *"nested Archimedean copulas"*) représentent une alternative très intéressante pour modéliser les asymétries. Cette structure, proposée encore une fois par Harry Joe en 1997, consiste à imbriquer des copules Archimédiennes l'une dans l'autre afin de construire une copule Archimédienne hiérarchique. Cette structure est capable de capturer différents niveaux de dépendance et de différencier entre les dépendances à l'intérieur des sous-groupes et la dépendance entre les différents sous-groupes. Elle a fait l'objet de plusieurs travaux de recherche que ce soit pour étudier ses différentes propriétés, ses algorithmes de simulation, ses applications pratiques, ou encore les conditions d'imbrication pour vérifier que la structure résultante est une copule. Par exemple, Joe (1997), McNeil (2008), Hofert (2008), Hofert (2010), Hofert (2012) et Abdallah et al. (2015) proposent des exposés sur les copules Archimédiennes hiérarchiques et leurs applications.

Afin qu'une telle structure Archimédienne hiérarchique soit une copule, des conditions d'imbrication doivent être vérifiées. Considérons un exemple simple et trivial de copule Archimédienne imbriquée de dimension 3 comme suit :

$$C(u_1, u_2, u_3) = C(u_1, C(u_2, u_3)) = \psi_0\Big(\psi_0^{-1}(u_1) + \psi_0^{-1} \circ \psi_1(\psi_1^{-1}(u_2) + \psi_1^{-1}(u_3))\Big), \quad (1.14)$$

où ψ_0 et ψ_1 sont, respectivement, le générateur de la copule mère et celui de la copule enfant.

Afin que la structure en (1.14) soit une copule, la fonction $(\psi_0^{-1} \circ \psi_1)'$ doit être complètement monotone. Cette condition représente, selon Joe (1997) et McNeil (2008)), *la condition nécessaire d'imbrication* ("*nesting condition*"). La vérification de cette condition pour plusieurs familles connues de copules Archimédiennes se trouve dans Hofert (2010). Cependant, dans le cas général, cette condition n'est pas facile à vérifier ce qui restreint l'utilisation de ces copules construites par imbrication dans les cas où des copules de familles différentes interviennent dans la structure hiérarchique.

D'autres approches ont été proposées afin de contourner ces contraintes reliées aux copules Archimédiennes imbriquées. On retrouve par exemple les copules Archimédiennes hiérarchiques basées sur les subordinateurs de Lévy proposées par Hering et al. (2010). Dans le même esprit, nous proposons au chapitre 3 de cette thèse une nouvelle classe de copules Archimédiennes hiérarchiques basée sur des lois composées multivariées. Dans ce chapitre, nous traitons plus en détail cette classe de copules et fournissons une comparaison entre les différentes méthodes hiérarchiques.

1.2 Modèles de dépendance avec mélange commun

Nous voici finalement rendus au sujet central de la thèse : les méthodes de construction de modèles de dépendance avec mélanges communs. Une des forces des copules Archimédiennes et des copules Archimédiennes hiérarchiques est leur représentation stochastique sous forme d'un mélange commun. On a pu voir cette propriété lors de l'interprétation de l'algorithme de simulation de Marshall & Olkin à la section 1.1.5 (voir Algorithme 1.1.1). Cette représentation a été utilisée par plusieurs chercheurs que ce soit, par exemple, pour fournir des algorithmes de simulation (voir e.g. Marshall and Olkin (1988), McNeil (2008) et Hofert (2011)), pour établir des formules explicites de probabilité de ruine (voir Albrecher et al. (2011)) ou encore pour déduire des expressions explicites relatives à l'agrégation des risques (voir e.g. Sarabia et al. (2017)).

Bien que le Chapitre 5 soit entièrement consacré aux modèles de dépendance avec mélange commun, nous présentons dans cette section les notions de base en rapport avec ces modèles et nous faisons le lien avec les copules Archimédiennes. Les propriétés de ces modèles et leurs différentes applications sont fournies de façon plus détaillée aux chapitres 2 et 5.

1.2.1 Lois multivariées construites par mélange commun

La méthode de construction des distributions basée sur les mélanges communs est due à Marshall and Olkin (1988) et Oakes (1989). Ces distributions construites par mélange commun, que ce soit dans un contexte univarié ou multivarié, sont très utilisées en actuariat. En particulier, les distributions exponentielles mélange ont un mode égal à zéro et un taux d'échec décroissant ce qui rend ces distributions très intéressantes pour la modélisation de la distribution des montants des sinistres ou de la distribution des durées de vie. Un exposé plus détaillé concernant la loi exponentielle mélange, ses différentes propriétés ainsi que des exemples de son application en actuariat et en théorie de la file d'attente, est fourni dans e.g., Cai (2006).

Soit Θ une v.a. dite v.a. de mélange, discrète ou continue. Soit un vecteur de v.a. $\underline{X} = (X_1, \ldots, X_d)$ où la distribution conditionnelle de $(\underline{X}|\Theta = \theta)$ est influencée par la valeur prise par la v.a. mélange Θ. Si Θ est une v.a. discrète définie sur le support $A = \{\theta_1, \theta_2, \ldots\}$, alors on a

$$F_{\underline{X}}\left(\underline{x}\right) = \sum_{\theta \in A} f_\Theta\left(\theta\right) F_{\underline{X}|\Theta=\theta}\left(\underline{x}\right),$$

où $\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Si l'on fait l'hypothèse additionnelle que les v.a. $(X_1|\Theta = \theta)$, ..., $(X_d|\Theta = \theta)$ sont conditionnellement indépendantes, alors on a

$$F_{\underline{X}}\left(\underline{x}\right) = \sum_{\theta \in A} f_\Theta\left(\theta\right) F_{X_1|\Theta=\theta}\left(\underline{x}\right) \ldots F_{X_d|\Theta=\theta}\left(\underline{x}\right).$$

Quand la v.a. mélange Θ est continue avec fonction de densité f_Θ, on écrit

$$F_{\underline{X}}\left(\underline{x}\right) = \int_{-\infty}^{\infty} F_{\underline{X}|\Theta=\theta}\left(\underline{x}\right) f_\Theta\left(\theta\right) \mathrm{d}\theta.$$

On considère que les d v.a. composant le vecteur $\underline{X}$ sont strictement positives. De plus, sachant $\Theta = \theta$, les v.a. X_i, $i = 1, \ldots, d$ sont conditionnellement indépendantes avec fonctions de survie conditionnelles suivantes

$$\overline{F}_{X_i|\Theta=\theta}\left(x_i\right) = \Pr\left(X_i > x_i|\Theta = \theta\right) = \left(\overline{F}_{Y_i}\left(x_i\right)\right)^\theta, \; x_i \in \mathbb{R}^+, \tag{1.15}$$

où Y_i $(i = 1, \ldots, n)$ sont des v.a. indépendantes.

La fonction de survie conditionnelle multivariée de $\underline{X} = (X_1, \ldots, X_d)$ est donc

$$\begin{aligned} \overline{F}_{\underline{X}|\Theta=\theta}\left(x_1, \ldots, x_d\right) &= \Pr\left(X_1 > x_1, \ldots, X_d > x_d|\Theta = \theta\right) \\ &= \left(\prod_{i=1}^d \overline{F}_{Y_i}\left(x_i\right)\right)^\theta, \; x_i \in \mathbb{R}^+. \end{aligned} \tag{1.16}$$

Les fonctions de survie univariée et multivariée inconditionnelles deviennent respectivement

$$\begin{aligned} \overline{F}_{X_i}\left(x_i\right) &= \int_{\theta \in A_\Theta} \overline{F}_{X_i|\Theta=\theta}\left(x_i\right) \mathrm{d}F_\Theta\left(\theta\right) \\ &= \int_{\theta \in A_\Theta} \left(\overline{F}_{Y_i}\left(x_i\right)\right)^\theta \mathrm{d}F_\Theta\left(\theta\right) \\ &= L_\Theta\left(-\ln\left(\overline{F}_{Y_i}\left(x_i\right)\right)\right), \; x_i \in \mathbb{R}^+, \end{aligned}$$

et

$$\begin{aligned} \overline{F}_{X_1,\ldots,X_d}\left(x_1, \ldots, x_d\right) &= \int_{\theta \in A_\Theta} \overline{F}_{X_1,\ldots,X_d|\Theta=\theta}\left(x_1, \ldots, x_d\right) \mathrm{d}F_\Theta\left(\theta\right) \\ &= \int_{\theta \in A_\Theta} \left(\overline{F}_{Y_1}\left(x_1\right) \ldots \overline{F}_{Y_d}\left(x_d\right)\right)^\theta \mathrm{d}F_\Theta\left(\theta\right) \\ &= \mathcal{L}_\Theta\left(-\ln\left(\overline{F}_{Y_1}\left(x_1\right) \ldots \overline{F}_{Y_d}\left(x_d\right)\right)\right), \; x_1, \ldots, x_d \in \mathbb{R}^+, \end{aligned}$$

où $\mathcal{L}_\Theta$ est la TLS de la v.a. Θ.

Dans tous les chapitres de cette thèse, on utilise les distributions construites par mélange commun. En particulier, on considère la loi exponentielle-mélange univariée et multivariée.

Si l'on suppose que la v.a. Y_i obéit à une loi exponentielle de moyenne 1, pour $i = 1, \ldots, d$, on obtient

$$\begin{aligned} \overline{F}_{X_i}(x_i) &= \int_{\theta \in A_\Theta} \left(\overline{F}_{Y_i}(x_i)\right)^\theta \mathrm{d}F_\Theta(\theta) \\ &= \int_{\theta \in A_\Theta} \mathrm{e}^{-\theta x_i} \mathrm{d}F_\Theta(\theta) \\ &= \mathcal{L}_\Theta(x_i),\ x_i \in \mathbb{R}^+, \end{aligned} \tag{1.17}$$

et

$$\begin{aligned} \overline{F}_{X_1,\ldots,X_d}(x_1,\ldots,x_d) &= \int_{\theta \in A_\Theta} \left(\overline{F}_{Y_1}(x_1)\ldots\overline{F}_{Y_d}(x_d)\right)^\theta \mathrm{d}F_\Theta(\theta) \\ &= \int_{\theta \in A_\Theta} \left(\mathrm{e}^{-x_1}\ldots\mathrm{e}^{-x_d}\right)^\theta \mathrm{d}F_\Theta(\theta) \\ &= \mathcal{L}_\Theta(x_1 + \ldots + x_d),\ x_1,\ldots,x_d \in \mathbb{R}^+. \end{aligned} \tag{1.18}$$

1.2.2 Lien entre les copules et les modèles avec mélange commun

Afin de faire le lien entre les copules Archimédiennes et les modèles de dépendance avec mélange commun, on construit la copule de survie associée à la fonction de survie conjointe et les fonctions de survie marginales définies respectivement en (1.18) et (1.17). En utilisant le Théorème de Sklar et la méthode par inversion en (1.9), il est ainsi possible de construire des copules de survie à partir d'une loi multivariée exponentielle mélange. En effet, en utilisant la fonction de survie multivariée telle que définie en (1.18), la copule résultante s'écrit sous la forme

$$\begin{aligned} C(u_1,\ldots,u_d) &= \overline{F}_{X_1,\ldots,X_d}\left(\overline{F}^{-1}_{X_1}(u_1) + \ldots + \overline{F}^{-1}_{X_d}(u_d)\right) \\ &= \mathcal{L}_\Theta\left(\mathcal{L}^{-1}_\Theta(u_1) + \ldots + \mathcal{L}^{-1}_\Theta(u_d)\right), \end{aligned}$$

où $\mathcal{L}^{-1}_\Theta$ est l'inverse de $\mathcal{L}_\Theta$. On retrouve ainsi l'expression d'une copule Archimédienne avec générateur $\mathcal{L}_\Theta$ telle que donnée dans (1.12).

L'introduction d'une v.a mélange afin de transformer des v.a. dépendantes en des v.a. conditionnellement indépendantes permet d'exprimer la dépendance entre les variables en tant que copule de survie Archimédienne dont le paramètre de dépendance dépend de la v.a. mélange. En d'autres mots, une copule Archimédienne, définie par un générateur correspondant à une

LST $\mathcal{L}_\Theta$, est une copule de survie construite, en utilisant le Théorème de Sklar, à partir d'une loi exponentielle mélange. Ce résultat permet de dériver et de mieux interpréter l'algorithme de simulation des copules Archimédiennes (voir Algorithme 1.1.1).

1.3 Applications en actuariat

Dans cette thèse, nous étudions les modèles de risque dépendants pour lesquels la structure de dépendance est induite par un mélange commun. L'indépendance conditionnelle cachée derrière est à la base de toute application de ces modèles de dépendance en actuariat. En effet, elle permet d'examiner l'agrégation, l'allocation de capital et les problèmes de ruine dans le cadre de différents modèles de risque. Les applications en actuariat considérées dans cette thèse diffèrent d'un chapitre à l'autre. Puisque chacun des quatre prochains chapitres contient le détail des applications qui y sont considérées ainsi que les notions connexes, nous nous contentons dans cette partie de présenter une notion très importante en actuariat et fort utilisée dans tous les chapitres, à savoir, les mesures de risque VaR et TVaR.

Les mesures de risque constituent une composante majeure dans la gestion et la quantification des risques. Elles servent à déterminer le montant, appelé capital économique, à mettre en réserve par les compagnies d'assurance et les institutions financières afin de faire face aux mauvaises surprises.

Les mesures de risque sont aussi utilisées dans le cadre de l'allocation du capital et de l'agrégation des risques. L'allocation du capital permet de décrire la procédure de répartition ou d'allocation du capital nécessaire pour l'ensemble du portefeuille à chaque risque. Le calcul de la contribution de chaque risque individuel au risque global est établi à l'aide d'une mesure de risque.

Soit un risque représenté par la v.a. X où X peut être le coût pour un contrat ou pour un ensemble de risques d'un portefeuille. Mesurer un risque représenté par la v.a. X revient à établir une relation fonctionnelle (une application) ρ entre l'espace des v.a. X et un nombre réel. Les mesures de risque scalaires permettent de comparer les risques, par exemple des investissements, en comparant leurs valeurs de risque $\rho(X)$ respectives.

Définition 1.3.1. *Soit ρ une mesure de risque. Un risque X est dit moins dangereux qu'un risque Y si $\rho(X) < \rho(Y)$.*

Il est admis qu'une mesure de risque ρ doit satisfaire quelques conditions pour être considérée comme une mesure de risque cohérente (voir e.g. Artzner et al. (1999)).

Proposition 1.3.1. *Une mesure de risque ρ est dite cohérente si elle possède les propriétés suivantes :*

1. *Homogénéité : Pour tout risque X et tout scalaire positif c,on a*

$$\rho(cX) = c\rho(X).$$

2. *Invariance par rapport à la translation : Pour tout risque X et tout scalaire positif c, on a*

$$\rho(X + c) = \rho(X) + c.$$

3. *Monotonicité : Soit deux risques X et Y, on a*

$$\Pr(X \leq Y) = 1 \Rightarrow \rho(X) \leq \rho(Y).$$

4. *Sous-additivité : Soit deux risques X et Y, on a*

$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

La Propriété 1 exprime l'invariance du risque par rapport à un changement d'unité monétaire. La Propriété 2 traduit le fait que l'ajout d'un montant certain au risque X, ne requiert aucun capital supplémentaire pour faire face à un risque sans aucune incertitude supplémentaire. Selon la Propriété 3, il faut mettre plus de capital de côté pour un risque plus dangereux. Enfin, la Propriété 4 exprime la réduction du risque par diversification.

Selon le domaine d'application, il existe d'autres propriétés désirables auxquelles peut satisfaire une mesure de risque. En actuariat, les propriétés énoncées dans la proposition ci-dessous sont jugées souhaitables.

Proposition 1.3.2. *Les propriétés désirables additionnelles pour une mesure de risque ρ sont les suivantes :*

1. *Marge de risque non excessive.*
2. *Marge de sécurité positive.*
3. *Marge de risque justifiée.*

La Propriété 1 signifie que le montant que doit mettre de côté une compagnie d'assurance pour faire face à un risque X, ne doit pas excéder le montant maximal que X peut prendre. La Propriété 2 traduit le fait que le capital minimum doit excéder les coûts espérés afin d'éviter la ruine certaine. Finalement, selon la Propriété 3, si les coûts pour un portefeuille sont constants et égalent à une constante a, il n'est pas justifié de détenir un capital différent de a. Voir e.g. Denuit et al. (2005) et Marceau (2013) pour plus de détails.

Les mesures de risque "Value-at-Risk" (VaR) et "Tail Value-at-Risk" (TVaR) sont probablement parmi les mesures de risque les plus utilisées par les compagnies d'assurance et les institutions financières.

Définition 1.3.2. *La mesure de risque VaR, de niveau de confiance κ, d'une v.a. X avec fonction de répartition F_X est définie par*

$$VaR_{\kappa}(X) = \inf \{x \in \mathbb{R} : F_X(x) \geq \kappa\},$$

où $\kappa \in (0, 1)$.

La valeur que prend $VaR_{\kappa}(X)$ correspond au montant à garder en réserve afin de permettre à l'institution financière de rencontrer ses engagements avec probabilité κ. La mesure VaR ne donne aucune information sur le comportement de la v.a. X au-delà de la valeur $VaR_{\kappa}(X)$. La VaR n'est pas une mesure cohérente, car elle ne satisfait pas la propriété de sous-additivité. De plus, cette mesure n'introduit pas une marge de risque excessive et la marge introduite par cette mesure est justifiée. Par contre, elle peut introduire une marge de risque négative.

Définition 1.3.3. *La mesure de risque TVaR, de niveau de confiance κ, d'une v.a. X avec fonction de répartition F_X est définie par*

$$\begin{aligned} TVaR_{\kappa}(X) &= \frac{1}{1-\kappa} \int_{\kappa}^{1} VaR_u(X) du \\ &= \frac{1}{1-\kappa} E\left[X \times 1_{\{X > VaR_{\kappa}(X)\}}\right] + VaR_{\kappa}(X) \left(F_X\left(VaR_{\kappa}(X)\right) - \kappa\right). \end{aligned}$$

où $\kappa \in (0, 1)$,

La mesure de risque TVaR correspond à la moyenne des VaR de niveau de confiance supérieur à κ. Contrairement à la VaR, cette mesure est cohérente et donne une indication quant à l'épaisseur de la queue de distribution de la v.a. X au-delà d'un quantile fixé (voir e.g. McNeil et al. (2005) pour plus de détails). De plus, cette mesure satisfait les trois propriétés désirables en actuariat.

1.4 Aperçu de la thèse

On fournit un aperçu global de cette thèse dont le contenu est réparti dans les chapitres 2, 3, 4 et 5.

1.4.1 Chapitre 2

L'agrégation des risques dépendants est une des préoccupations importantes des chercheurs en actuariat depuis quelques années. L'objectif de l'agrégation des risques est d'évaluer la loi de la somme des différents risques. Un excellent outil pour la modélisation de la relation de dépendance entre les risques est la théorie des copules. Depuis les années 1990, les copules revêtent une importance capitale en actuariat et en gestion quantitative des risques. Elles permettent de caractériser la dépendance entre les risques et de construire des distributions multivariées.

Dans le chapitre 2, on s'intéresse à l'identification de la distribution de la somme des risques d'un portefeuille. Soit un portefeuille de n risques représentés par les v.a. $X_1, \ldots, X_n$. Le montant total des coûts encourus pour un portefeuille est représenté par la v.a. S définie par $S = X_1 + \ldots + X_n$. La distribution conjointe du vecteur $(X_1, \ldots, X_n)$ est définie par une copule Archimédienne. On montre que cette copule peut être représentée par un mélange commun dont le facteur commun est une v.a positive pouvant être discrète (e.g. AMH, Frank et Joe) ou continue (e.g. Clayton, Gumbel).

Dans un premier temps, on considère une copule Archimédienne. En ce qui concerne la nature des marginales et du facteur commun, on suppose trois cas spécifiques. Dans le premier cas, la v.a. mélange est discrète de même que les marginales. Dans ce cas, on obtient la valeur exacte de la fonction de répartition de S. Le deuxième cas considère des marginales continues et un facteur commun discret. Dans un tel cas, on déduit des bornes sur F_S. Dans le dernier cas, la v.a. mélange est discrète et on présente deux différentes méthodes d'approximation de F_S.

Ensuite, à travers un exemple détaillé, on montre que cette approche d'agrégation peut être adaptée à des copules Archimédiennes hiérarchiques. De plus, plusieurs règles d'allocation de capital ainsi que d'autres applications sont également traitées.

En comparaison avec la méthode d'approximation basée sur la méthode de Monte Carlo, les résultats de notre approche fournissent une appréciation plus adéquate du risque global d'un portefeuille constitué de risques dépendants. Notre approche a l'avantage de calculer la valeur exacte de F_S dans certains cas, de permettre le contrôle du degré de précision dans le cas d'une approximation.

1.4.2 Chapitre 3

Afin de généraliser les copules Archimédiennes pour permettre les asymétries et remédier au problème d'échangeabilité, plusieurs chercheurs utilisent la structure d'imbrication hiérarchique. Elle consiste à imbriquer plusieurs copules Archimédiennes connues l'une dans l'autre.

Les copules Archimédiennes imbriquées sont capables de capturer différentes relations de dépendance entre et au sein de différents groupes de risques avec un nombre relativement petit de paramètres (voir par exemple Górecki et al. (2016) pour plus de détails). Pour qu'une telle structure hiérarchique soit une copule, une condition d'imbrication doit être vérifiée. Si la structure hiérarchique est construite via l'imbrication de plusieurs copules de la même famille, la vérification de la condition d'imbrication s'avère très facile. Autrement, la condition peut ne pas tenir pour n'importe quel choix de paramètres. En outre, la simulation de ces copules peut être difficile. D'autres modèles hiérarchiques multivariés utilisant une technique différente pourraient donc être introduits et étudiés pour permettre une plus grande flexibilité.

Pour toutes ces raisons, nous proposons une nouvelle construction hiérarchique des copules Archimédiennes basée sur des distributions composées multivariées. Cette nouvelle technique d'imbrication est dérivée de la construction d'une distribution exponentielle multivariée par mélange commun et l'introduction d'un choc commun. L'absence de toute condition d'imbrication, contrairement aux copules Archimédiennes imbriquées, conduit à des avantages majeurs tels qu'une gamme flexible de combinaisons possibles dans le choix des distributions, l'existence de formules explicites pour la distribution de la somme et la facilité de calcul en grande dimension. Un équilibre entre la flexibilité et la parcimonie est visé. Après avoir présenté la technique de construction, les propriétés des copules proposées sont étudiées et des exemples illustratifs sont donnés. Une représentation sous forme de mélange commun des nouvelles copules hiérarchiques est fournie pour permettre des comparaisons avec des copules Archimédiennes imbriquées bien connues. L'agrégation des risques dans le cadre de cette structure de dépendance nouvellement proposée est également examinée.

Ce chapitre est composé de trois parties. Dans la première partie, on présente les étapes de construction de la copule multivariée en utilisant les distributions composées, suivies d'algorithmes de simulation ainsi que de notations et de propriétés pertinentes. La deuxième partie propose une représentation de la copule par mélange commun, ce qui permet de faire une comparaison avec les copules Archimédiennes hiérarchiques existantes. Finalement, la troisième partie propose une méthode d'agrégation pour un vecteur de v.a. aléatoires.

1.4.3 Chapitre 4

Le chapitre 4 propose une extension du modèle collectif de risque en supposant une structure de dépendance reliant les montants de sinistres et leur nombre. Considérer une dépendance entre les composantes de la somme aléatoire S constitue une hypothèse plus réaliste d'un point de vue pratique.

Ce chapitre est composé de deux parties. Dans la première, on présente les définitions de base et on propose quelques modèles de dépendance basés sur des copules bivariées ou multivariées. Dans la deuxième partie, on utilise les copules Archimédiennes et Archimédiennes hiérarchiques pour modéliser la dépendance entre le nombre et les montants des sinistres. Un algorithme efficient de simulation qui génère des réalisations de la somme aléatoire S est également présenté. De plus, la deuxième partie s'inspire du chapitre 2 pour proposer une méthodologie de calcul de la fonction de masse de probabilité de S.

1.4.4 Chapitre 5

En actuariat, il est assez commun de travailler avec des distributions construites par des mélanges communs que ce soit dans un contexte univarié ou multivarié. Dans ce chapitre, on considère plus spécifiquement les mélanges communs construits avec des v.a. mélange discrètes.

Le chapitre 5 est composé de quatres parties. La première est consacrée à un exemple de motivation dans lequel nous étudions l'application des copules Archimédiennes hiérarchiques pour des portefeuilles de risques dépendants de Bernoulli. Dans un contexte de portefeuille de risque de crédit, notre modèle permet de modéliser les risques dépendants de défaut et de calculer la fonction de masse de probabilité de la somme de ces risques. Cet exemple de motivation nous permet de mettre en évidence l'utilité des distributions et des copules construites avec des mélanges communs. La deuxième partie présente la loi univariée exponentielle mélange. Certaines propriétés de cette loi sont discutées et trois cas particuliers sont présentés. Dans la troisième partie, on présente la loi multivariée exponentielle mélange et on établit le lien entre cette loi et les copules Archimédiennes. Plusieurs cas particuliers sont également discutés. Finalement, une application reliée à la théorie de la ruine est présentée à la dernière section.

En résumé, l'objectif de ce chapitre est de souligner l'utilité des modèles construits avec des mélanges communs, de comprendre le lien entre les lois exponentielles-mélange univariées, multivariées et les copules Archimédiennes et de proposer de nouvelles distributions. En traitant trois cas particuliers de distributions de la v.a. latente, ce chapitre présente également une façon assez originale pour déduire les cas limites des distributions et copules proposées.

La thèse est complétée par une conclusion au chapitre 6.

1.5 Bibliographie

Abdallah, A., Boucher, J.-P., and Cossette, H. (2015). Modeling dependence between loss triangles with hierarchical archimedean copulas. *ASTIN Bulletin : The Journal of the IAA*, 45(3) :577–599.

Albrecher, H., Constantinescu, C., and Loisel, S. (2011). Explicit ruin formulas for models with dependence among risks. *Insurance : Mathematics and Economics*, 48(2) :265–270.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, 9(3) :203–228.

Balakrishnan, N. (2006). Continuous multivariate distributions. *Wiley Online Library.*

Bedford, T. and Cooke, R. M. (2002). Vines : A new graphical model for dependent random variables. *Annals of Statistics*, pages 1031–1068.

Cai, J. (2006). Mixtures of Exponential Distributions *Encyclopedia of Actuarial Science*, volume 2. Wiley.

Cohen, A. M. (2007). Numerical methods for Laplace transform inversion, volume 5. *Springer Science & Business Media.*

Côté, M.-P., Genest, C., and Abdallah, A. (2016). Rank-based methods for modeling dependence between loss triangles. *European Actuarial Journal*, 6(2) :377–408.

Czado, C., Kastenmeier, R., Brechmann, E. C., and Min, A. (2012). A mixed copula model for insurance claims and claim sizes. *Scandinavian Actuarial Journal*, 2012(4) :278–305.

Dall'Aglio, G., Kotz, S., and Salinetti, G. (2012). Advances in probability distributions with given marginals : beyond the copulas, volume 67. *Springer Science & Business Media.*

Denuit, M. and Charpentier, A. (2004). Mathématiques de l'assurance non-vie. tome 1 : principes fondamentaux de théorie du risque. *Economica*, 58.

Denuit, M. and Charpentier, A. (2005). Mathématiques de l'assurance non-vie. *Economica.*

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). Actuarial theory for dependent risks : measures, orders and models. *John Wiley & Sons.*

Devroye, L. (2009). Random variate generation for exponentially and polynomially tilted stable distributions. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 19(4) :18.

Embrechts, P., Lindskog, F., and McNeil, A. (2001). Modelling dependence with copulas. *Rapport technique, Département de mathématiques, Institut Fédéral de Technologie de Zurich, Zurich.*

Feller, W. (1971). Law of large numbers for identically distributed variables. *An introduction to probability theory and its applications*, 2 :231–234.

Féron, R. (1956). Sur les tableaux de corrélation dont les marges sont données, cas de l'espace à trois dimensions. *Publ. Inst. Statist. Univ. Paris*, 5 :3–12.

Fisher, N. I. (1997). Copulas. *Encyclopedia of statistical sciences.*

Fréchet, M. (1951). Sur les tableaux de corrélation dont les marges sont données. *Annales de l'Université de Lyon. Section A : Sciences mathématiques et astronomie 9 : 53–77.*, 14.

Frees, E. W. and Valdez, E. A. (1998). Understanding relationships using copulas. *North American actuarial journal*, 2(1) :1–25.

Genest, C., Gendron, M., and Bourdeau-Brien, M. (2009). The advent of copulas in finance. *The European Journal of Finance*, 15(7-8) :609–618.

Górecki, J., Hofert, M., and Holeňa, M. (2016). An approach to structure determination and estimation of hierarchical archimedean copulas and its application to bayesian classification. *Journal of Intelligent Information Systems*, 46(1) :21–59.

Hering, C., Hofert, M., Mai, J.-F., and Scherer, M. (2010). Constructing hierarchical archimedean copulas with Lévy subordinators. *Journal of Multivariate Analysis*, 101(6) :1428–1433.

Hoeffding, W. (1941). Masstabinvariante korrelationsmasse für diskontinuierliche verteilungen. *Archiv für mathematische Wirtschafts-und Sozialforschung*, 7 :49–70.

Hofert, M. (2008). Sampling archimedean copulas. *Computational Statistics & Data Analysis*, 52(12) :5163–5174.

Hofert, M. (2010). Sampling nested Archimedean copulas with applications to CDO pricing. *PhD thesis, Universität Ulm.*

Hofert, M. (2011). Efficiently sampling nested archimedean copulas. *Computational Statistics & Data Analysis*, 55(1) :57–70.

Hofert, M. (2012). A stochastic representation and sampling algorithm for nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 82(9) :1239–1255.

Höffding, W. (1940). Masstabinvariante korrelationstheorie. *Schriften des Mathematischen Instituts und Instituts fur Angewandte Mathematik der Universitat Berlin*, 5 :181–233.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts.* CRC Press.

Joe, H. (2014). *Dependence modeling with copulas.* CRC Press.

Kimberling, C. H. (1974). A probabilistic interpretation of complete monotonicity. *Aequationes mathematicae*, 10(2) :152–164.

Mai, J. and Scherer, M. (2014). Financial engineering with copulas explained. *Springer.*

Marceau, E. (2013). Modelisation et evaluation quantitative des risques en actuariat : Modèles sur une période. *Springer, 1st edition.*

Mardia, K. V. (1970). Families of bivariate distributions, Volume 27. *Griffin London.*

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403) :834–841.

McNeil, A. J. (2008). Sampling nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 78(6) :567–581.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). Quantitative risk management : Concepts, techniques and tools. *Princeton university press.*

McNeil, A. J., Frey, R., and Embrechts, P. (2015). Quantitative risk management : Concepts, techniques and tools. *Princeton university press.*

McNeil, A. J. and Nešlehová, J. (2009). Multivariate archimedean copulas, d-monotone functions and l1-norm symmetric distributions. *The Annals of Statistics*, pages 3059–3097.

Nelsen, R. B. (2007). An introduction to copulas. *Springer Science & Business Media.*

Oakes, D. (1989). Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, 84(406) :487–493.

Panjer, H. H., Klugman, S., and Willmot, G. (2008). Loss models : From data to decisions. *New York : John Wiley & Sons, Inc.*

Samuel, M. D. D. K., Mari, D. D., and Kotz, S. (2001). Correlation and dependence. *World Scientific.*

Sarabia, J. M., Gómez-Déniz, E., Prieto, F., and Jordá, V. (2017). Aggregation of dependent risks in mixtures of exponential distributions and extensions. *arXiv preprint arXiv :1705.00289.*

Schweizer, B. and Sklar, A. (1983). Probabilistic metric spaces. *North-Holland, New York.*

Schweizer, B. and Sklar, A. (2011). Probabilistic metric spaces. *Courier Corporation.*

Shi, P. (2016). Insurance ratemaking using a copula-based multivariate tweedie model. *Scandinavian Actuarial Journal*, 2016(3) :198–215.

Shi, P. and Frees, E. W. (2011). Dependent loss reserving using copulas. *ASTIN Bulletin : The Journal of the IAA*, 41(2) :449–486.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Université Paris 8.*

Valkó, P. and Vojta, B. (2001). The list of papers for the numerical inversion of the Laplace transforms.

Chapitre 2

Dependent risk models with Archimedean copulas : a computational strategy based on common mixtures and applications

Résumé

Dans cet article, nous considérons les portefeuilles de risques dépendants. La structure de dépendance est modélisée par une copule Archimédienne. Sur la base de la représentation par mélange des copules Archimédiennes, une approche d'agrégation des risques est présentée. Une généralisation de l'approche proposée pour les copules Archimédiennes imbriquées est également considérée. Des applications telles que le calcul des contributions individuelles au risque global selon différentes règles d'allocation sont également traitées. Plusieurs exemples numériques sont présentés.

Abstract

In this paper, we investigate dependent risk models in which the dependence structure is defined by an Archimedean copula. Using such a structure with specific marginals, we derive explicit expressions for the pdf of the aggregated risk and other related quantities. The common mixture representation of Archimedean copulas is at the basis of a computational strategy proposed to find exact or approximated values of the distribution of the sum of risks in a general setup. Such results are then used to investigate risk models in regard to aggregation, capital allocation and ruin problems. An extension to nested Archimedean copulas is also discussed.

Keywords: Archimedean Copulas; Common Mixture Representation; Aggregation Strategy; Risk Measures; Capital Allocation; Ruin Theory; Nested Archimedean Copulas

2.1 Introduction

The present paper deals with risk models incorporating dependent components whose dependence structure is induced via an Archimedean copula. Copula theory provides a flexible approach for modelling the dependency relationship between risks. A copula is a multivariate distribution function for which the marginals are standard uniformly distributed. See e.g. Joe (1997) or Nelsen (2007) for further details. One important class of copulas is the class of Archimedean copulas.

A d-dimensional copula C is said to be an *Archimedean copula* if

$$C(u_1,\ldots,u_d) = \psi\left(\psi^{-1}(u_1)+\ldots+\psi^{-1}(u_d)\right), \text{ for } (u_1,\ldots,u_d)\in[0,1]^d. \tag{2.1}$$

The continuous and strictly decreasing function ψ is called the generator of the copula, where $\psi:[0,\infty)\to[0,1]$, $\psi(0)=1$ and $\lim\limits_{t\to\infty}\psi(t)=0$. In the same manner, $\psi^{-1}:[0,1]\to[0,\infty)$, for which $\psi^{-1}(0)=\inf\{t:\psi(t)=0\}$, where ψ^{-1} is the inverse of the generator ψ. McNeil and Nešlehová (2009) show that (2.1) is a d-dimensional copula if and only if ψ is a d-monotone function. In this paper, we consider a specific class of Archimedean copulas with completely monotone generators. By using Bernstein's theorem (see e.g. Feller (1971)), it has been shown that such generators correspond to the Laplace-Stieltjes Transform (LST) of a strictly positive rv Θ with cumulative distribution function (cdf) F_Θ, where the LST of the rv Θ is given by

$$\mathcal{L}_\Theta(t) = \int_0^\infty \mathrm{e}^{-t\,\theta}\,\mathrm{d}F_\Theta(\theta) = E\left[\mathrm{e}^{-t\Theta}\right]. \tag{2.2}$$

Then, the expression in (2.1) becomes

$$C\left(u_1,\ldots,u_d\right) = \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(u_1\right)+\ldots+\mathcal{L}_\Theta^{-1}\left(u_d\right)\right). \tag{2.3}$$

The strictly positive rv Θ, which can be either discrete or continuous, corresponds to a latent mixing rv, and there is a one-to-one relation between its distribution and the expression of an Archimedean copula C. This special class of Archimedean copulas defined in (2.3) is intimately linked to common mixtures. As explained in Denuit et al. (2005) and McNeil et al. (2005), the representation of an Archimedean copula C as a common mixture allows us to identify the conditional univariate cumulative distribution functions (cdfs) of $(U_1|\Theta=\theta)$,..., $(U_d|\Theta=\theta)$, where $\underline{U}=(U_1,...,U_d)$ and $U_i\sim Unif\,(0,1)$, $i=1,2,...,d$. Using the definition in (2.2) in combination with (2.3), we have the following representation of an Archimedean copula C as a common mixture:

$$\begin{aligned} C\left(u_1,\ldots,u_d\right) &= \psi\left(\psi^{-1}\left(u_1\right)+...+\psi^{-1}\left(u_d\right)\right) \\ &= \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(u_1\right)+...+\mathcal{L}_\Theta^{-1}\left(u_d\right)\right) \\ &= \int_0^\infty \prod_{i=1}^d \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(u_i)}\mathrm{d}F_\Theta\left(\theta\right) \end{aligned}$$

which becomes

$$C(u_1, \dots, u_d) = F_{\underline{U}}(u_1, ..., u_d) = \int_0^\infty \prod_{i=1}^{d} F_{U_i|\Theta=\theta}(u_i)\, \mathrm{d}F_\Theta(\theta). \tag{2.4}$$

Then, (2.4) implies that the conditional cdf of $(U_i|\Theta = \theta)$ is $F_{U_i|\Theta=\theta}(u_i) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}(u_i)}$, for $i = 1, \dots, d$, $u_i \in [0, 1]$, $\theta > 0$, which leads to the useful representation of the copula C as a common mixture. Examples of the most popular Archimedean copulas (Θ discrete and continuous) are provided in the Appendix section (see e.g. Nelsen (2007) for an extensive list of Archimedean copulas).

The above representation provides a natural sampling algorithm (see e.g. Marshall and Olkin (1988), Hofert (2008), and references therein) useful for actuarial science and quantitative risk management purposes. Mixture representations are frequently involved in risk models. For example, in a credit-risk context, Vasicek (1987) was the first to develop the idea of the conditional independence of all defaults upon some market factor. Later, this idea was used by Li (2000), Schönbucher and Schubert (2001) and Frey and McNeil (2001) and put into a copula setup. Among the papers that exploit the conditional independence technique with Archimedean and nested Archimedean copulas in a credit risk context one finds for example Schönbucher (2002) and Hofert and Scherer (2011). Copulas in a credit risk setting is also treated with much detail in Cherubini et al. (2004). See references therein. Several researchers have also used the conditional representation of Archimedean and nested Archimedean copulas to derive their sampling algorithms see e.g. Marshall and Olkin (1988), McNeil (2008) and Hofert (2012). Archimedean risk models are also involved in the context of collective risk models with dependence. Thanks to the latter representation, Albrecher et al. (2011) were able to establish explicit formulas for the ruin probability with dependence among claim sizes and among claim inter-occurrence times modeled by Archimedean copulas. Also, a recent work of Jordanova et al. (2017), considered dependent inter-arrival times and exploited Archimedean copulas as treated in Albrecher et al. (2011).

Our objective here is to explore in more depth the mixture representation of an Archimedean copula and its advantages. More precisely, we propose a new strategy relying on (2.4), to tackle risk assessment problems such as risk aggregation for finite and random sums, capital allocation, ruin problems and so on. We show that this representation allows to avoid entirely or partially Monte-Carlo (MC) simulations. Furthermore, this methodology is accurate, exact in specific cases, very flexible, and most importantly, naturally applicable in high dimensions.

The outline of the paper is as follows. We propose in Section 2.2, a computational methodology based on the common mixture representation of Archimedean copulas in different settings. This strategy allows to derive the distribution of the aggregated risks which can be later used in different applications. Analytic expressions related to the aggregated risks are also derived for some special cases using specific marginal distributions. Section 2.3 deals with capital allocation issues involving the strategy proposed in Section 2.2. Random sums are then considered in Section 2.4. Sections 2.5 and 2.6 are devoted to the investigation of ruin problems. Finally, Section 2.7 discusses the application of the mixture-based strategy in the case of a hierarchical dependence structure based on Archimedean copulas.

2.2 Computational Methodology based on the common mixture representation of Archimedean copulas

2.2.1 Common mixture representation

Let $\underline{X} = (X_1, ..., X_d)$ be a vector of rvs with multivariate distribution defined in terms of a d-dimensional Archimedean copula C given in (2.3). The multivariate cdf $F_{\underline{X}}$ of $\underline{X}$ can be defined with the copula C and marginal univariate cdfs F_{X_1}, ..., F_{X_d} as

$$F_{\underline{X}}\left(x_1, ..., x_d\right) = C\left(F_{X_1}\left(x_1\right), ..., F_{X_d}\left(x_d\right)\right). \tag{2.5}$$

The common mixture representation of $F_{\underline{X}}$ is given by

$$F_{\underline{X}}\left(x_1, ..., x_d\right) = \int_0^\infty \prod_{i=1}^{d} F_{X_i|\Theta=\theta}\left(x_i\right) \mathrm{d}F_\Theta\left(\theta\right) = \int_0^\infty \prod_{i=1}^{d} \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(x_i)\right)} \mathrm{d}F_\Theta\left(\theta\right), \tag{2.6}$$

where

$$F_{X_i|\Theta=\theta}\left(x_i\right) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(x_i)\right)} \ (i = 1, 2, ..., d). \tag{2.7}$$

Similarly, we can define the multivariate distribution of $\underline{X}$ through its multivariate survival function with the copula C and marginal univariate survival functions $\overline{F}_{X_1}$, ..., $\overline{F}_{X_d}$, i.e.,

$$\overline{F}_{\underline{X}}\left(x_1, ..., x_d\right) = C\left(\overline{F}_{X_1}\left(x_1\right), ..., \overline{F}_{X_d}\left(x_d\right)\right). \tag{2.8}$$

Then, the common mixture representation of $\overline{F}_{\underline{X}}$ is given by

$$\overline{F}_{\underline{X}}\left(x_1, ..., x_d\right) = \int_0^\infty \prod_{i=1}^{d} \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)} \mathrm{d}F_\Theta\left(\theta\right) = \int_0^\infty \prod_{i=1}^{d} \overline{F}_{X_i|\Theta=\theta}\left(x_i\right) \mathrm{d}F_\Theta\left(\theta\right), \tag{2.9}$$

where

$$\overline{F}_{X_i|\Theta=\theta}\left(x_i\right) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)} \ (i = 1, 2, ..., d). \tag{2.10}$$

Our main objective in what follows is to find $E[\phi(S)]$, for any integrable univariate function ϕ of the rv $S = X_1 + ... + X_d$, or to find $E[\varphi(X_1, ..., X_d)]$, for any integrable d-variate function φ. The required steps to derive these quantities are as follows:

1. Find the expressions of the conditional cdfs (2.7) or the conditional survival functions (2.10) of $X_i|\Theta = \theta$.

2. Derive the conditional expectations $E[\phi(S)|\Theta=\theta]$ or $E[\varphi(X_1,...,X_d)|\Theta=\theta]$.
3. Find $E[\phi(S)]$ and $E[\varphi(X_1,...,X_d)]$ with

$$E[\phi(S)] = E_\Theta[E[\phi(S)|\Theta]] = \int_0^\infty E[\phi(S)|\Theta=\theta]\mathrm{d}F_\Theta(\theta),$$

and

$$E[\varphi(X_1,...,X_d)] = E_\Theta[E[\varphi(X_1,...,X_d)|\Theta]] = \int_0^\infty E[\varphi(X_1,...,X_d)|\Theta=\theta]\mathrm{d}F_\Theta(\theta),$$

where $(S|\Theta=\theta) = \sum_{i=1}^d (X_i|\Theta=\theta)$.

We present below three examples, with specific marginals but any Archimedean copula with generator $\mathcal{L}_\Theta$, in which the required steps, 1, 2, and 3, are easily performed. In these examples, explicit expressions for quantities related to S are obtained.

Example 2.2.1. *Let $\underline{X}=(X_1,...,X_d)$ be a vector of exchangeable Bernoulli rvs where $X_i \sim Bern(q)$ $(i=1,...,d)$ and*

$$F_{\underline{X}}(k_1,...,k_d) = C(F_{X_1}(k_1),...,F_{X_d}(k_d)),$$

for $k_1,...,k_d \in \{0,1\}$. From (2.7), we find that

$$(X_i|\Theta=\theta) \sim Bern\left(1-e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)}\right),$$

for $i=1,2,...,d$. Therefore, $(S|\Theta=\theta)$ follows a binomial distribution with

$$f_{S|\Theta=\theta}(k) = \binom{d}{k}\left(1-e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)}\right)^k e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)(d-k)} = \binom{d}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-\theta\mathcal{L}_\Theta^{-1}(1-q)(j+d-k)},$$

and we conclude

$$f_S(k) = \binom{d}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(1-q)(j+d-k)\right),$$

for $k=0,1,2,...,d$.

Example 2.2.2. *Let $\underline{X}=(X_1,...,X_d)$ be a vector of exchangeable Bernoulli rvs where $X_i \sim Bern(q)$ $(i=1,...,d)$ and*

$$\overline{F}_{\underline{X}}(k_1,...,k_d) = C\left(\overline{F}_{X_1}(k_1),...,\overline{F}_{X_d}(k_d)\right)$$

for $k_1,...,k_d \in \{0,1\}$. It follows from (2.10) that

$$(X_i|\Theta=\theta) \sim Bern\left(e^{-\theta\mathcal{L}_\Theta^{-1}(q)}\right)$$

for $i=1,2,...,d$. Since

$$f_{S|\Theta=\theta}(k) = \binom{d}{k}e^{-\theta\mathcal{L}_\Theta^{-1}(q)k}\left(1-e^{-\theta\mathcal{L}_\Theta^{-1}(q)}\right)^{n-k} = \binom{d}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j e^{-\theta\mathcal{L}_\Theta^{-1}(q)(k+j)},$$

we obtain

$$f_S(k) = \binom{d}{k} \sum_{j=0}^{d-k} \binom{d-k}{j} (-1)^j \mathcal{L}_\Theta \left(\mathcal{L}_\Theta^{-1}(q)(k+j) \right),$$

for $k = 0, 1, 2, ..., d$. This expression can be found e.g., in Cossette et al. (2002) or on page 321 in Marceau (2013).

Example 2.2.3. *Let Θ be a strictly positive rv with LST $\mathcal{L}_\Theta$. Given $\Theta = \theta$, let $(X_1|\Theta = \theta)$, ..., $(X_n|\Theta = \theta)$ be conditionally independent rvs where $(X_i|\Theta = \theta)$ follows a geometric distribution with*

$$f_{X_i|\Theta=\theta}(k_i) = \Pr(X_i = k_i|\Theta = \theta) = \mathrm{e}^{-\theta r k_i} \left(1 - e^{-r\theta}\right)$$

and

$$\overline{F}_{X_i|\Theta=\theta}(k_i) = \Pr(X_i > k_i|\Theta = \theta) = \mathrm{e}^{-\theta r(k_i+1)} \quad (i = 1, 2, ..., n)$$

for $k_i = 0, 1, 2, ...$ and $r > 0$. Then, $\underline{X} = (X_1, ..., X_n)$ follows a multivariate mixed geometric distribution where

$$\overline{F}_{\underline{X}}(k_1, ..., k_n) = \mathcal{L}_\Theta (r(k_1 + 1) + ... + r(k_n + 1))$$

with

$$\overline{F}_{X_i}(k_i) = \mathcal{L}_\Theta (r(k_i + 1)) \quad (i = 1, 2, ..., n).$$

It implies that

$$\begin{aligned} \overline{F}_{\underline{X}}(k_1, ..., k_n) &= \mathcal{L}_\Theta \left(\mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_1}(k_1)\right) + ... + \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_n}(k_n)\right)\right) \\ &= C\left(\overline{F}_{X_1}(k_1), ..., \overline{F}_{X_n}(k_n)\right) \end{aligned}$$

where C is an Archimedean copula defined with LST $\mathcal{L}_\Theta$.

Now, we are in position to derive the closed-form expression of the pmf of $S_n = \sum_{i=1}^{n} X_n$. Observe that $(S_n|\Theta = \theta)$ follows a negative binomial distribution with

$$\begin{aligned} f_{X_i|\Theta=\theta}(k) &= \Pr(X_i = k|\Theta = \theta) \\ &= \binom{k+n-1}{k} \mathrm{e}^{-\theta r k} \left(1 - e^{-r\theta}\right)^n \\ &= \binom{k+n-1}{k} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \mathrm{e}^{-\theta r(j+k)}, \end{aligned}$$

for $k = 0, 1, 2, ...$. Then, it follows that

$$f_S(k) = \binom{k+n-1}{k} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \mathcal{L}_\Theta (r(j+k)),$$

for $k = 0, 1, 2, ...$.

In the following subsection, we consider the family of multivariate mixed exponential distributions which is equivalent to choosing specific combinations of Archimedean copulas and marginals. For this class of multivariate distributions, we derive analytic expressions for the pdf of the sum of risks and other related quantities of interest.

2.2.2 Closed-form expressions for multivariate mixed exponential distributions

Let $\underline{X}$ follow a multivariate mixed exponential distribution which belongs to the class of multivariate distributions constructed by common frailty as explained in e.g. Marshall and Olkin (1988). Briefly, given $\Theta = \theta$, the conditional distribution of the rv X_i is exponential with mean $\frac{\lambda_i}{\theta}$, i.e.,

$$\overline{F}_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{\frac{-\theta x_i}{\lambda_i}}, \tag{2.11}$$

for $i = 1, ..., d$. It implies that the marginal survival function of X_i is given by

$$\overline{F}_{X_i}(x_i) = \mathcal{L}_\Theta\left(\frac{x_i}{\lambda_i}\right), \tag{2.12}$$

for $i = 1, \ldots, d$. Also, the multivariate survival function of $\underline{X}$ is given by

$$\overline{F}_{\underline{X}}(x_1, \ldots, x_d) = \mathcal{L}_\Theta\left(\frac{x_1}{\lambda_1} + \ldots + \frac{x_d}{\lambda_d}\right),$$

which implies that $\overline{F}_{\underline{X}}$ satisfies (2.8) with Archimedean copula C given in (2.3). Clearly, in this setting, the required steps 1, 2, and 3 of the methodology described in Section 2.2.1 are easily performed. The expression in (2.10) is clearly given in (2.11). We first consider the case where $0 < \lambda_1 < ... < \lambda_d$. Here, $(S|\Theta = \theta)$ follows a generalized Erlang distribution with pdf

$$f_{S|\Theta=\theta}(x) = \sum_{i=1}^{n}\left(\prod_{j=1,\, j\neq i}^{n} \frac{\lambda_i}{\lambda_i - \lambda_j}\right)\frac{\theta}{\lambda_i}\mathrm{e}^{-\frac{\theta}{\lambda_i}x}. \tag{2.13}$$

Then, using (2.13), the unconditional pdf of S is given by

$$f_S(x) = \int_0^\infty f_{S|\Theta=\theta}(x)\mathrm{d}F_\Theta(\theta) = \sum_{i=1}^{d}\frac{1}{\lambda_i}\left(\prod_{j=1,\, j\neq i}^{d}\frac{\lambda_i}{\lambda_i - \lambda_j}\right)\left((-1)\frac{\mathrm{d}\mathcal{L}_\Theta(t)}{\mathrm{d}t}\bigg|_{t=\frac{x}{\lambda_i}}\right).$$

The bivariate case ($d = 2$) is detailed on page 295 of Marceau (2013). Sarabia et al. (2017) consider the subclass of multivariate mixed exponential distributions in which $\lambda_1 = ... = \lambda_d = 1$. In such a case, $(S|\Theta = \theta)$ follows an Erlang distribution with

$$f_{S|\Theta=\theta}(x) = \frac{\theta^d x^{d-1}}{\Gamma(d)}\mathrm{e}^{-\theta x}.$$

Sarabia et al. (2017) find the following closed-form expressions for the pdf and the survival function of the rv S:

$$f_S(x) = \frac{x^{d-1}}{\Gamma(d)}\left\{(-1)^d\frac{\mathrm{d}^d}{\mathrm{d}x^d}\mathcal{L}_\Theta(x)\right\} \tag{2.14}$$

and

$$\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{k!} \left\{ (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathcal{L}_\Theta(x) \right\}, \ x \in \mathbb{R}^+.$$

From (2.14), one can deduce the following expression for the TVaR of the rv S:

$$\begin{aligned} TVaR_\kappa(S) &= \frac{E\left[S \times 1_{\{S > VaR_\kappa(S)\}}\right]}{1-\kappa} \\ &= \sum_{j=0}^{d} \frac{d \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)} \left\{ (-1)^{j-1} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \mathcal{L}_\Theta(x) \Big|_{x = VaR_\kappa(S)} \right\}, \end{aligned}$$

where $VaR_\kappa(S)$ is the solution of $F_S(x) = \kappa$, with $\kappa \in [0, 1)$. The stop-loss function of S can also be expressed in terms of $\mathcal{L}_\Theta$ as follows

$$\begin{aligned} \Pi_S(y) &= E\left[\max(S - y; 0)\right] \\ &= d \sum_{j=0}^{d} \frac{y^j}{j!} \left\{ (-1)^{j-1} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \mathcal{L}_\Theta(x) \Big|_{x=y} \right\} - y \sum_{j=0}^{d-1} \frac{y^j}{j!} \left\{ (-1)^{j} \frac{\mathrm{d}^{j}}{\mathrm{d}x^{j}} \mathcal{L}_\Theta(x) \Big|_{x=y} \right\}, \ y \in \mathbb{R}^+. \end{aligned}$$

Sarabia et al. (2017) give explicit formulas for the pdf of S for some multivariate mixed exponential distributions for which the dependence structure is an Archimedean copula with continuous common factor Θ such as the Clayton and Gumbel copulas. In the same context, Dacorogna et al. (2016) use similar explicit formulas to compute analytically risk measures and the associated diversification benefit.

Since there are several much used copulas generated from a discrete mixing rv Θ such as the Ali-Mikhail-Haq (AMH) copula, the Frank copula and the Joe copula, we provide the two following examples to complete those provided in Sarabia et al. (2017). Also, since the derivatives of the generators are known for different Archimedean copulas with discrete or continuous mixing rv (see, e.g., Hofert et al. (2012)), the results provided in the following examples follow directly from the definitions just presented above.

Example 2.2.4. *Let $\underline{X} = (X_1, \ldots, X_d)$ follow a multivariate mixed exponential-geometric distribution, i.e., $\Theta \sim Geo(q)$ with probability mass function (pmf) $f_\Theta(k) = q(1-q)^{k-1}$, for $k \in \mathbb{N}$. Clearly, the dependence structure underlying this multivariate distribution is the AMH copula with dependence parameter $\alpha = 1 - q$. Then, the following properties hold:*

1. $\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = \frac{q}{\mathrm{e}^x - (1-q)}, \ x > 0, \ i = 1, \ldots, d.$
2. $\bar{F}_{\underline{X}}(\underline{x}) = \frac{q}{\exp\left(\sum_{i=1}^d x_i\right) - (1-q)}, \ x > 0.$
3. $f_S(x) = \dfrac{x^{d-1} q}{(1-q)\,\Gamma(d)} Li_{-d}\left((1-q)\mathrm{e}^{-x}\right), \ x > 0.$
4. $\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k q}{(1-q) \times k!} Li_{-k}\left((1-q)\mathrm{e}^{-x}\right), \ x > 0.$

5. $TVaR_\kappa(S) = \sum_{j=0}^{d} \frac{dq\times(VaR_\kappa(S))^j}{j!\times(1-\kappa)(1-q)} Li_{1-j}\left((1-q)\mathrm{e}^{-VaR_\kappa(S)}\right),\ \kappa\in[0,1).$

6. $\Pi_S(y) = d\sum_{j=0}^{d} \frac{y^j\,q}{j!\,(1-q)} Li_{1-j}\left((1-q)\mathrm{e}^{-y}\right) - y\sum_{j=0}^{d-1} \frac{q\,y^j}{j!\,(1-q)} Li_{-j}\left((1-q)\mathrm{e}^{-y}\right),\ y\in\mathbb{R}^+.$

We denote by "Li" the general polylogarithm function defined as $Li_a(z) = \sum_{d=1}^{\infty} \frac{z^d}{d^a}$ *(see e.g. Lewin (1981) for more details).*

Example 2.2.5. *Let* $\underline{X} = (X_1, \ldots, X_d)$ *be an d-dimensional vector with multivariate mixed exponential-logarithmic distribution, i.e.,* $\Theta \sim Log(1-\mathrm{e}^{-\alpha})$ *with pmf* $f_\Theta(k) = \frac{\left(1-e^{-\alpha}\right)^k}{k\alpha}$, *for* $k\in\mathbb{N}$. *The corresponding dependence structure is the Frank copula with dependence parameter* α. *Then, the following properties hold:*

1. $\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = -\frac{\ln\left(1-(1-\mathrm{e}^{-\alpha})\,\mathrm{e}^{-x}\right)}{\alpha},\ x>0,\ i=1,\ldots,d.$

2. $\bar{F}_{\underline{X}}(\underline{x}) = -\frac{\ln\left(1-(1-\mathrm{e}^{-\alpha})\,\mathrm{e}^{-\sum_{i=1}^{d} x_i}\right)}{\alpha},\ x>0.$

3. $f_S(x) = \frac{x^{d-1}}{\Gamma(d)\,\alpha} Li_{1-d}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^x}\right),\ x>0.$

4. $\Pr(S>x) = \sum_{k=0}^{d-1} \frac{x^k}{\alpha\times k!} Li_{1-k}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^x}\right),\ x\in\mathbb{R}^+.$

5. $TVaR_\kappa(S) = \sum_{j=0}^{d} \frac{d\times(VaR_\kappa(S))^j}{j!\times(1-\kappa)\alpha} Li_{2-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^{VaR_\kappa(S)}}\right),\ \kappa\in[0,1).$

6. $\Pi_S(y) = d\sum_{j=0}^{d} \frac{y^j}{j!\,\alpha} Li_{2-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^y}\right) - y\sum_{j=0}^{d-1} \frac{y^j}{j!\,\alpha} Li_{1-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^y}\right),\ y\in\mathbb{R}^+.$

In this subsection, we have discussed multivariate mixed exponential distributions which, as highlighted in the examples, require specific combinations of the mixing rv Θ and the marginal distributions of X_i $(i=1,...,d)$. The dependence structure defined by the Archimedean copula is governed by Θ and hence only one choice of marginal distribution can be linked through such a dependency framework. This is very limitative since it does not allow to choose the dependence construction without any regard to the specification of the marginals. This is somewhat counter-intuitive with copulas being used as flexible tools to build dependence risk models.

In the next three subsections, we provide insight on how to benefit from the mixture representation of an Archimedean copula for any multivariate distribution defined through an Archimedean copula with a discrete mixing rv Θ and discrete marginals. Furthermore, we provide a strategy that can be used when one or both of these rvs are continuous.

2.2.3 Discrete mixing rv Θ and discrete marginals

Our strategy is to use the conditional independence assumption to identify the conditional distribution of X_i through (2.7) or (2.10). This step is usually more difficult for continuous rvs X_i than for discrete ones which is the basis of the computational strategy presented in this section. We have recourse to discretization methods in the continuous case. The conditional distributions of S given $\Theta = \theta$ are also easier to identify for discrete distributions.

The proposed strategy has many advantages. First, it is easy to implement regardless of the portfolio's dimension. Second, it yields the exact values of F_S for discrete risks X_i, $i = 1, \ldots, d$ and it gives an accurate approximation for continuous ones. The proposed strategy can also be used in the context of many actuarial risk models involving Archimedean dependence which will be discussed in Sections 2.4, 2.5 and 2.6.

Our strategy is to make use of the conditional independence representation as well as the convolution of independent rvs to derive a simple computation approach for the distribution of S and any integrable function of $\underline{X}$. Given these techniques, risk aggregation problems for a portfolio of dependent risks with a multivariate joint distribution defined with an Archimedean copula or a nested Archimedean copula, as well as other related quantities become easier to deal with.

Let $\underline{X} = (X_1, \ldots, X_d)$ be a vector of discrete rvs such that $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, ..., d)$. The univariate pmf of X_i, the univariate cdf of X_i, the multivariate pmf of $\underline{X}$, and the multivariate cdf of $\underline{X}$ are respectively denoted by $f_{X_i}(k_i h) = \Pr(X_i = k_i h)$, $F_{X_i}(k_i h) = \Pr(X_i \leq k_i h)$,

$$f_{\underline{X}}(k_1 h, ..., k_d h) = \Pr(X_1 = k_1 h, ..., X_d = k_d h),$$

and

$$F_{\underline{X}}(k_1 h, ..., k_d h) = \Pr(X_1 \leq k_1 h, ..., X_d \leq k_d h),$$

for $h > 0$ and $k_1, ..., k_n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In theory, exact values of f_S and $E[\varphi(X_1, ..., X_d)]$ can be found with

$$f_S(kh) = \sum_{k_1=0}^{k} \cdots \sum_{k_{d-1}=0}^{k-(k_1+\ldots+k_{d-2})} f_{\underline{X}}\left(k_1 h, \ldots, k_{d-1} h, \left(k - \sum_{j=1}^{d-1} k_j\right) h\right) \tag{2.15}$$

and

$$E[\varphi(X_1, ..., X_d)] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \varphi(k_1 h, ..., k_d h) f_{\underline{X}}(k_1 h, \ldots, k_d h), \tag{2.16}$$

where

$$f_{\underline{X}}(k_1 h, \ldots, k_d h) = \sum_{i_1=0,1} \cdots \sum_{i_d=0,1} (-1)^{i_1+\ldots+i_d} \times F_{\underline{X}}((k_1 - i_1) h, \ldots, (k_d - i_d) h). \tag{2.17}$$

The computation of (2.15), (2.16), and (2.17) is feasible when the dimension d of $\underline{X}$ is small (e.g. $d = 2$, 3 or 4). However, it rapidly becomes impracticable when d gets larger.

Assume that the dependence structure of $\underline{X}$ is induced via an Archimedean copula C. In such a case, the multivariate cdf of $\underline{X}$ (or its survival function) is defined with the copula C and the univariate cdfs (or the univariate survival functions) of $X_1, \ldots, X_d$, i.e, (2.5) and (2.8) become

$$F_{\underline{X}}\left(k_1 h, \ldots, k_n h\right) = C\left(F_{X_1}\left(k_1 h\right), \ldots, F_{X_d}\left(k_d h\right)\right) \tag{2.18}$$

or

$$\overline{F}_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = C\left(\overline{F}_{X_1}\left(k_1 h\right), \ldots, \overline{F}_{X_d}\left(k_d h\right)\right), \tag{2.19}$$

for $k_i \in \mathbb{N}$ and $i = 1, \ldots, d$.

In the following, let the copula C in either (2.18) or (2.19) be an Archimedean copula defined by a discrete mixing rv $\Theta \in \mathbb{N}$ such that $E[\Theta]$ is finite. Then, we can take advantage of the common mixture representation to compute the exact values of f_S and $E\left[\varphi\left(X_1, ..., X_d\right)\right]$.

Assuming discrete marginals and a discrete mixing rv Θ and using (2.6), the expression for $F_{\underline{X}}$ in (2.18) becomes

$$F_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} F_{X_i|\Theta=\theta}\left(k_i h\right) f_{\Theta}\left(\theta\right), \tag{2.20}$$

and, from (2.7), we have

$$F_{X_i|\Theta=\theta}\left(k_i h\right) = \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}\left(k_i h\right)\right)}, \tag{2.21}$$

for $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, d$, and $\theta \in \mathbb{N}$. For $i = 1, 2, ..., d$ and for each $\theta \in \mathbb{N}$, we can easily find the values of $f_{X_i|\Theta=\theta}\left(k_i h\right)$ with

$$f_{X_i|\Theta=\theta}\left(k_i h\right) = \begin{cases} \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}(k_i h)\right)} - \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(F_{X_i}((k_i-1)h)\right)} & , k_i \in \mathbb{N} \end{cases}. \tag{2.22}$$

Similarly, using (2.9), the expression for $\overline{F}_{\underline{X}}$ in (2.19) turns into

$$\overline{F}_{\underline{X}}\left(k_1 h, \ldots, k_d h\right) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} \overline{F}_{X_i|\Theta=\theta}\left(k_i h\right) f_{\Theta}\left(\theta\right), \tag{2.23}$$

and, from (2.10), we have

$$\overline{F}_{X_i|\Theta=\theta}\left(k_i h\right) = \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_i}\left(k_i h\right)\right)}, \tag{2.24}$$

for $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, d$, and $\theta \in \mathbb{N}$. It follows that

$$f_{X_i|\Theta=\theta}\left(k_i h\right) = \begin{cases} 1 - \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_i}((k_i-1)h)\right)} - \mathrm{e}^{-\theta \mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_i}(k_i h)\right)} & , k_i \in \mathbb{N} \end{cases}, \tag{2.25}$$

for $i = 1, 2, ..., d$ and for each $\theta \in \mathbb{N}$.

Then, from either (2.22) or (2.25), the expression for $f_{\underline{X}}(k_1h,\ldots,k_dh)$ is now given by

$$f_{\underline{X}}(k_1h,\ldots,k_dh)=\sum_{\theta=1}^{\infty}\prod_{i=1}^{d}f_{X_i|\Theta=\theta}(k_ih)\,f_{\Theta}(\theta)\,. \tag{2.26}$$

Let $(S|\Theta=\theta)=\sum_{i=1}^{d}(X_i|\Theta=\theta)$ be the sum of conditionally independent rvs and $f_{S|\Theta=\theta}$ be the corresponding pmf. Due to the representation of $f_{\underline{X}}$ in (2.26), the expression for the pmf of S can be written as follows

$$f_S(kh)=\sum_{\theta=1}^{\infty}f_{S|\Theta=\theta}(kh)\,f_{\Theta}(\theta)\,,\ k\in\mathbb{N}_0. \tag{2.27}$$

Since $f_{S|\Theta=\theta}$ corresponds to the convolution product of $f_{X_1|\Theta=\theta}$, ..., $f_{X_d|\Theta=\theta}$, traditional aggregation tools (e.g. FFT and DePril algorithm see e.g. Panjer et al. (2008)) from actuarial science can be applied to find values of $f_{S|\Theta=\theta}$ for each θ. It is important to note here that our strategy leads to exact values of f_S contrarily to MC simulation methods. This is also true even when n is large.

Given the representation of $f_{\underline{X}}$ in (2.26), $E[\varphi(X_1,...,X_d)]$ becomes

$$E[\varphi(X_1,...,X_d)]=\sum_{\theta=1}^{\infty}E[\varphi(X_1,...,X_d)\,|\Theta=\theta]\,f_{\Theta}(\theta)\,, \tag{2.28}$$

where

$$E[\varphi(X_1,...,X_d)\,|\Theta=\theta]=\sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty}\varphi(k_1h,...,k_dh)\prod_{i=1}^{d}f_{X_i|\Theta=\theta}(k_ih)\,. \tag{2.29}$$

Applications of (2.28) and (2.29) are provided in Section 2.3.

The procedure to compute the values of $f_S(kh)$ is summarized in the following algorithm.

Algorithm 2.2.1. ***Computation of the values of*** f_S.

1. *Let θ^* be chosen such that $F_{\Theta}(\theta^*)\leq 1-\varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon=10^{-10}$).*
2. *Fix $\theta=1$.*
3. *For $i=1,...,d$, calculate either $F_{X_i|\Theta=\theta}(k_ih)$ with (2.21) or $\overline{F}_{X_i|\Theta=\theta}(k_ih)$ with (2.24), for $k_i\in\mathbb{N}_0$.*
4. *For $i=1,...,d$, calculate $f_{X_i|\Theta=\theta}(k_ih)$ with either (2.22) ou (2.25).*
5. *Using e.g. FFT or DePril's Algorithm, compute $f_{S|\Theta=\theta}(kh)$ for $k\in\mathbb{N}_0$.*
6. *Repeat steps 3, 4, and 5 for $\theta=2,...,\theta^*$ where θ^* is chosen to be the largest integer s.t. $F_{\Theta}(\theta^*)\leq 1-\varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon=10^{-10}$).*
7. *Compute $f_S(kh)=\sum_{\theta=1}^{\theta^*}f_{S|\Theta=\theta}(kh)\,f_{\Theta}(\theta)$, for $k\in\mathbb{N}_0$.*

Some remarks must be made in regard to the proposed methodology of this section and the tail dependence of an Archimedean copula C. Let us recall that if the lower and upper-tail dependence coefficients exist for an Archimedean copula C with mixing rv Θ, then according to Joe and Hu (1996), these coefficients λ_L and λ_U can be written in terms of $\mathcal{L}_\Theta$ as follows

$$\lambda_L = \lim_{t\to\infty} \frac{\mathcal{L}_\Theta(2t)}{\mathcal{L}_\Theta(t)} = 2\lim_{t\to\infty} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}$$

and

$$\lambda_U = 2 - \lim_{t\to 0} \frac{1-\mathcal{L}_\Theta(2t)}{1-\mathcal{L}_\Theta(t)} = 2 - 2\lim_{t\to 0} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}.$$

For the special case of discrete mixing rvs Θ, the underlying Archimedean copulas cannot have lower tail dependence, i.e. $\lambda_L = 0$. This implies that our proposed methodology does not allow lower tail dependence. Also, if $E[\Theta]$ is finite, then $\lambda_U = 0$ (see Hofert (2010) page 62 for proof). This means that it is possible to find a finite θ^* for which the methodology described in Algorithm 2.2.1 leads to exact values of the pmf of the rv S. In such a case, the methodology works well for notably the AMH and Frank copulas. Note that the proposed methodology cannot be applied when $E[\Theta]$ is infinite (e.g. for Joe's copula) since Algorithm 2.2.1 suggests to truncate the distribution of Θ at θ^*. Such a truncation leads to an Archimedean copula with no upper tail dependence which violates the initial assumption $\lambda_u \neq 0$.

In the following two examples, we illustrate the accuracy of our computational methodology. More precisely, we present a first example which considers a small portfolio. This allows us to compare the values of f_S obtained with Algorithm 2.2.1 and (2.15). As expected, both results coincide. The second example is somewhat similar but illustrates the applicability of Algorithm 2.2.1 to large portfolios.

Example 2.2.6. *Let $F_{\underline{X}}$ be defined as in (2.18) with the Frank copula as given in the Appendix and $X_i \sim Bin(10, q_i)$, where $q_i = 0.1i$, for $i = 1, 2, 3, 4$. In this case, the mixing rv Θ follows a logarithmic distribution. While values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ are given in Table 2.1, Table 2.2 provides the exact values of f_S obtained with (2.15) and Algorithm 2.2.1.*

Example 2.2.7. *Let $X_i \sim Bin(10, q_i)$, with $q_i = 0.1$, for $i = 1, ..., 100$. The joint cdf $F_{\underline{X}}$ is defined as in (2.18) with the AMH copula. In Figure 2.1, we depict the exact values of F_S obtained with Algorithm 2.2.1 ($\alpha = 0.5$ and 0.9). For comparison purposes, the exact values of $F_{S^\perp}$, where $S^\perp$ is the sum of the independent rvs $X_1^\perp$, ..., $X_{100}^\perp$ and $X_i^\perp \sim X_i$ for $i = 1, 2, ..., 100$, are also provided. It is well known that the AMH copula introduces a low to moderate positive dependence relation between the rvs X_1, ..., X_{100}. However, the impact is clearly significant when the number of risks of the portfolio becomes large as illustrated in*

	$\alpha = 1$	$\alpha = 3$	$\alpha = 6$
$E[S]$	10	10	10
$Var(S)$	9.99256	15.15425	19.90096
$VaR_{0.9}(S)$	14	15	16
$VaR_{0.999}(S)$	20	21	23
$TVaR_{0.9}(S)$	15.82535	17.11038	18.04888
$TVaR_{0.999}(S)$	20.88055	22.39553	23.41423

Table 2.1 – Values of the expectation, variance, VaR and TVaR of $S = X_1 + ... + X_4$ where $F_{X_1,...,X_4}$ is defined with the Frank copula.

α	s	Exact values of $f_S(s)$ (with (2.15))	Exact values of $f_S(s)$ (with Algorithm 2.2.1)
1	0	1.992354e-05	1.992354e-05
	5	0.03925912	0.03925912
	10	0.1199567	0.1199567
3	0	0.0001013881	0.0001013881
	5	0.0619347088	0.0619347088
	10	0.0887858345	0.0887858345
6	0	0.0003887516	0.0003887516
	5	0.0691872490	0.0691872490
	10	0.0767524930	0.0767524930

Table 2.2 – Values of the pmf of $S = X_1 + ... + X_4$ where $F_{X_1,...,X_4}$ is defined with the Frank copula.

	$\alpha = 0$ (independence)	$\alpha = 0.5$	$\alpha = 0.9$
$E[S]$	100	100	100
$Var(S)$	90	1454.027	2793.690
$VaR_{0.9}(S)$	112	156	172
$VaR_{0.999}(S)$	130	225	242
$TVaR_{0.9}(S)$	116.934	176.206	192.651
$TVaR_{0.999}(S)$	133.277	233.651	250.590

Table 2.3 – Values of the expectation, variance, VaR and TVaR of $S = X_1 + ... + X_{100}$ where $F_{X_1,...,X_{100}}$ is defined with the AMH copula.

Figure 2.1. In Table 2.3, we provide the values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$ and $TVaR_\kappa(S)$. Note that the computation time increases as the dependence parameter becomes larger. Indeed, $\theta^* = F_\Theta^{-1}(1-\varepsilon;\alpha)$ increases with α. For example, if $\varepsilon = 10^{-10}$, $\theta^* = 34$ and $\theta^* = 219$ for $\alpha = 0.5$ and $\alpha = 0.9$ respectively.

Example 2.2.7 allows us to better understand the impact of the dependence between individual risks on the overall exposure evaluation. As shown in Figure 2.1, using a portfolio of 100 risks highlights this significant impact. Indeed, one cannot neglect the dependence, even moderate, for a portfolio of large dimension. Note that the proposed algorithm allowed us to make such a

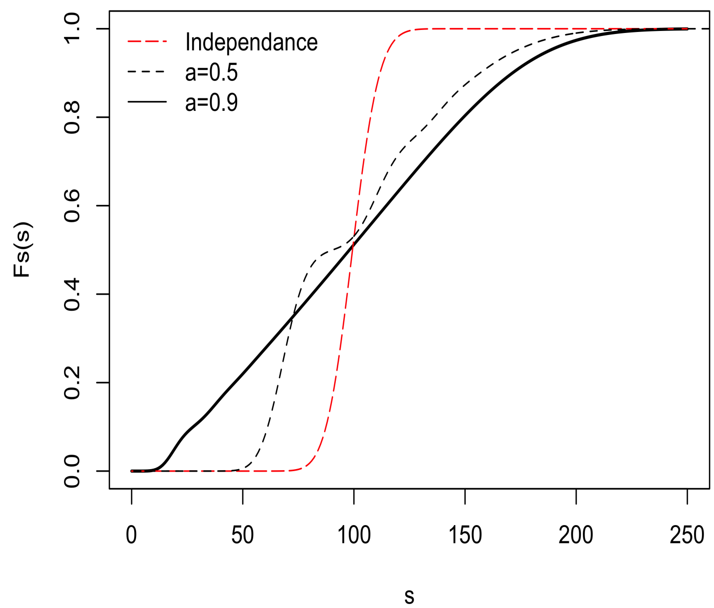


Figure 2.1 – The cdf of $S = X_1 + ... + X_{100}$ where $F_{X_1,...,X_{100}}$ is defined by the AMH copula with $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 0.9$.

conclusion, which is not possible using the convolution method in (2.15), since it only applies to small portfolios.

2.2.4 Discrete mixing rv Θ and continuous marginals

Let us assume that $\underline{X} = (X_1, ..., X_d)$ is a vector of continuous positive rvs where the multivariate cdf is defined by an Archimedean copula C as in (2.5) and the mixing rv Θ is discrete. In this case, the computation of $F_{S|\Theta=\theta}(x)$ or eventually $E[\varphi(X_1, ..., X_d)|\Theta = \theta])$ becomes more difficult. Since $(X_1|\Theta = \theta)$, ..., $(X_d|\Theta = \theta)$ are conditionally independent rvs, one can apply the tools at hand to compute an accurate (as possible) approximation of $F_{S|\Theta=\theta}(x)$ or $E[\varphi(X_1, ..., X_d)|\Theta = \theta]$ for the sum or any function of independent rvs. Note that the approach described here can be easily adapted when the multivariate survival function instead of the multivariate cdf is defined with an Archimedean copula C as in (2.8) and when the mixing rv Θ is discrete.

Inspired by Bargès et al. (2009), we propose an approximation based on discretization methods and the application of Algorithm 2.2.1. This approximation leads to numerical bounds as accurate as one may desire. Indeed, we approximate $\underline{X}$ by $\underline{\widetilde{X}} = \left(\widetilde{X}_1, ..., \widetilde{X}_d\right)$, a vector of discrete rvs with $\widetilde{X}_i \in A = \{0, 1h, 2h, \dots\}$, for $i = 1, ..., d$ and a discretization step $h > 0$. The

	$\widetilde{S}^{(u,1)}$	$\widetilde{S}^{(u,0.1)}$	$\widetilde{S}^{(l,0.1)}$	$\widetilde{S}^{(l,1)}$	$\widetilde{S}^{(m,1)}$	$\widetilde{S}^{(m,0.1)}$	MC	95%-Confidence Interval
Expectation	380.3333	398.0033	402.0033	420.3333	400	400	399.8067	[398.8303 ; 400.7831]
Variance	24667.6236	24749.6556	24749.6556	24667.6236	24796.9950	24750.9523	24817.9510	[24601.8436 ; 25036.9318]
$\text{VaR}_{0.9}$	606	624	628.4	646	627	626.4	627.3094	[624.8721 ; 629.6699]
$\text{VaR}_{0.999}$	975	993.2	997.2	1015	995	995.2	989.2484	[981.2374 ; 1000.4592]
$\text{TVaR}_{0.9}$	705.6048	723.3208	727.3191	745.6048	725.7719	725.3294	726.2473	[725.7238 ; 726.7709]
$\text{TVaR}_{0.999}$	1027.2863	1043.7218	1047.7156	1067.2863	1047.4537	1045.7465	1039.4606	[1039.1150 ; 1039.8062]

Table 2.4 – Approximated values of the expectation, variance, VaR and TVaR of $S = X_1 + ... + X_{40}$ where $F_{X_1,...,X_{40}}$ is defined with the AMH copula and continuous marginals. The discretization steps is either $h = 1$ or $h = 0.1$. For the MC method, the number of simulations is 100000.

multivariate cdf of $\underline{\widetilde{X}}$ is defined with the same copula C and with marginals $F_{\widetilde{X}_1}$, ..., $F_{\widetilde{X}_d}$ obtained with a discretization method, i.e., $F_{\underline{\widetilde{X}}}(k_1 h, ..., k_d h) = C\left(F_{\widetilde{X}_1}(k_1 h), ..., F_{\widetilde{X}_d}(k_d h)\right)$, for $k_1, ..., k_d \in \mathbb{N}_0$. In this article, we consider the upper, lower and mean preserving discretization methods (see e.g. Müller and Stoyan (2002) or Bargès et al. (2009) for details).

Example 2.2.8. *Let $X_i \sim Exp(0.1)$, for $i = 1, 2, ..., d$ ($d = 40$). The multivariate cdf of $\underline{X}$, $F_{\underline{X}}$, is defined as in (2.5) with the AMH copula ($\alpha = 0.5$). For $S = X_1 + \ldots + X_{40}$, we compute the upper and lower bounds to F_S with $h = 1$ and $h = 0.1$. Values of the expectation, variance, VaR and TVaR of the rvs $\widetilde{S}^{(u,1)}$, $\widetilde{S}^{(u,0.1)}$, $\widetilde{S}^{(l,1)}$, $\widetilde{S}^{(l,0.1)}$, $\widetilde{S}^{(m,1)}$, and $\widetilde{S}^{(m,0.1)}$ are given in Table 2.4. Note that "u", "l", and "m" in the superscripts refer respectively to the upper, lower and mean preserving discretization methods. Also, the approximated values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ ($\kappa = 0.9, 0.999$) obtained using 100000 MC simulations are provided (with 95%-level confidence intervals given in the last column). Clearly, as the discretization step h goes to 0, the difference between the bounds also tends to 0. Also, notice that the approximated values $\widetilde{VaR}^{MC}_{0.999}(S)$ and $\widetilde{TVaR}^{MC}_{0.999}(S)$ of $VaR_{0.999}(S)$ and $TVaR_{0.999}(S)$ obtained by 100000 MC simulations are outside the interval defined by the upper and the lower bounds. This is also illustrated in Figure 2.2. In the left panel, we provide the values for the cdfs of the rvs $\widetilde{S}^{(u,1)}$, $\widetilde{S}^{(u,0.1)}$, $\widetilde{S}^{(l,1)}$, and $\widetilde{S}^{(l,0.1)}$ and the approximated values of F_S, obtained with 100000 MC simulations. The close-up on the right panel of Figure 2.2 clearly shows that the approximated values of F_S obtained with MC simulations lies out of the upper and lower bounds $F_{\widetilde{S}^{(u,0.1)}}$ and $F_{\widetilde{S}^{(l,0.1)}}$. The proposed approximation has the advantage to allow us to control the precision of the approximation. On the left panel of Figure 2.2, the exact values of $F_{S^\perp}$, where $S^\perp$ is the sum of the independent rvs $X_1^\perp$, ..., $X_{100}^\perp$ and $X_i^\perp \sim X_i$ for $i = 1, 2, ..., 100$, are also depicted.*

2.2.5 Continuous mixing rv Θ

The common mixture representation in (2.6) and (2.9) leads to a two-step natural sampling procedure for $\underline{X}$. The first step is to simulate a sampled value of Θ. Then, in the second step, the sampled value of $\underline{X}$ is computed via the conditional distribution of $\underline{X}$ given the sampled value of Θ using either (2.7) or (2.10). See, e.g., Marshall and Olkin (1988) or

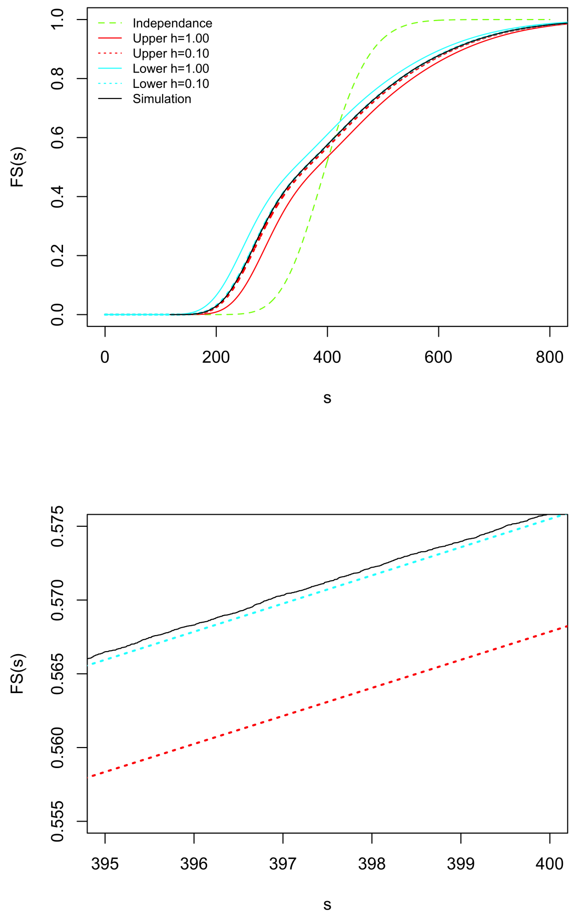


Figure 2.2 – Approximated values of the cdf of $S = X_1 + ... + X_{40}$

Hofert (2008)) for details. We examine two alternatives to this approach: one based on the simulation of the mixing rv Θ and another one on the approximation of Θ by a discrete rv.

We consider a vector of discrete (or discretized) rvs $\underline{X} = (X_1, \ldots, X_d)$ where $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, ..., d)$. Let the rv Z be the estimator of $\varphi(X_1, .., X_d)$ under a MC simulation approach, which is constructed using the two-step sampling procedure of $\underline{X}$. The standard error of the estimator Z is given by $\sqrt{Var(Z)}$. We can use a variance reduction technique, namely the conditional MC simulation method (see, e.g., Lemieux (2009) or Kroese et al. (2013) for details on this topic) to reduce this standard error. First, one generates sampled values of the rv Θ and then applies Algorithm 2.2.1 as follows. We produce m sampled values of Θ, denoted by $\Theta^{(1)}$, ..., $\Theta^{(m)}$. Then, (2.27) becomes $f_S(kh) \simeq \sum_{j=1}^{m} f_{S|\Theta=\Theta^{(j)}}(kh)\, \frac{1}{m}$, $k \in \mathbb{N}_0$, where the values of $f_{S|\Theta=\Theta^{(j)}}(kh)$ are computed using e.g. FFT or DePril's Algorithm. Then, $E[Z|\Theta]$ corresponds to the resulting conditional MC approximation of $\varphi(X_1, .., X_d)$.

Clearly, the standard error of $E[Z|\Theta]$ is given by $\sqrt{Var(E[Z|\Theta])}$. Since

$$Var(Z) = E[Var(Z|\Theta)] + Var(E[Z|\Theta]), \tag{2.30}$$

it is clear that

$$\sqrt{Var(E[Z|\Theta])} \leq \sqrt{Var(Z)}. \tag{2.31}$$

The magnitude of the difference $\sqrt{Var(Z)} - \sqrt{Var(E[Z|\Theta])}$ is analyzed in the following example.

Example 2.2.9. *Let* $\underline{X}^{(C,s,\alpha)} = \left(X_1^{(C,s,\alpha)}, ..., X_{50}^{(C,s,\alpha)}\right)$ *and* $\underline{X}^{(G,s,\alpha)} = \left(X_1^{(G,s,\alpha)}, ..., X_{50}^{(G,s,\alpha)}\right)$ *be two vectors of rvs with* $X_1^{(C,s,\alpha)} \sim X_1^{(G,s,\alpha)} \sim Bin\,(100, 0.2)$*, for* $i = 1, ..., 50$*, where their joint survival function are defined either with the Clayton copula or the Gumbel copula according to (2.8). Similarly, let* $\underline{X}^{(C,f,\alpha)} = \left(X_1^{(C,f,\alpha)}, ..., X_{50}^{(C,f,\alpha)}\right)$ *and* $\underline{X}^{(G,f,\alpha)} = \left(X_1^{(G,f,\alpha)}, ..., X_{50}^{(G,f,\alpha)}\right)$ *be two vectors of rvs with* $X_1^{(C,f,\alpha)} \sim X_1^{(G,f,\alpha)} \sim Bin\,(100, 0.2)$*, for* $i = 1, ..., 50$*, where their joint cdf are defined either with the Clayton copula or the Gumbel copula according to (2.5). For each copula, the dependence parameter* α *of the copula is fixed such that Kendall's tau is equal to 0.2, 0.5, or 0.8. The exact values of the covariances between pairs of rvs for each vectors of rvs are provided in Tables 2.5 and 2.6 for the three values of the dependence parameter. We define* $S^{(C,s,\alpha)} = \sum_{i=1}^{50} X_i^{(C,s,\alpha)}$*,* $S^{(G,s,\alpha)} = \sum_{i=1}^{50} X_i^{(G,s,\alpha)}$*,* $S^{(C,f,\alpha)} = \sum_{i=1}^{50} X_i^{(C,f,\alpha)}$*, and* $S^{(G,f,\alpha)} = \sum_{i=1}^{50} X_i^{(G,f,\alpha)}$*. Clearly,* $E[S^{(C,s,\alpha)}] = E[S^{(G,s,\alpha)}] = E[S^{(C,f,\alpha)}] = E[S^{(G,f,\alpha)}] = 1000$*, whatever the values of the dependence parameters. In Tables 2.7 and 2.8, we also give the exact values of* $Var(S^{(C,s,\alpha)})$*,* $Var(S^{(G,s,\alpha)})$*,* $Var(S^{(C,f,\alpha)})$*, and* $Var(S^{(G,f,\alpha)})$ *for the three values of their dependence parameter. In Tables 2.9 to 2.14, we provide the approximated values of* $\overline{F}_{S^{(C,s,\alpha)}}$*,* $\overline{F}_{S^{(G,s,\alpha)}}$*,* $\overline{F}_{S^{(C,f,\alpha)}}$*, and* $\overline{F}_{S^{(G,f,\alpha)}}$*. Those values are computed using both the conditional MC and the MC approaches with* $m = 100000$ *simulations. In parenthesis, we indicate the values of the standard deviation for each approximation. As expected from (2.31), we observe that the standard error of the approximation based on the conditional MC approach is lower than the corresponding one for the approximation based on the MC approach. For a given multivariate distribution, we observe that the improvement is more significant as the dependence parameter* α *decreases. The improvement is also more significant for large values of* x*. However, for a specific value of Kendall's tau, the improvement differs from one multivariate distribution to another. Notably, we observe that the improvement is the least significant for the results associated to* $\underline{X}^{(C,s,\alpha)}$ *and* $\underline{X}^{(G,f,\alpha)}$*, meaning that the improvement observed with the conditional MC approach is less significant with multivariate distributions having a non-zero right tail dependence.*

Let us now consider the case of the Clayton copula for which Θ is gamma distributed. To obtain f_S, we can approximate the Clayton copula with the shifted negative binomial copula allowing us to use Algorithm 2.2.1. This family includes the AMH copula as a

α	$Cov(X_i^{(G,s,\alpha)}, X_j^{(G,s,\alpha)})$	$Cov(X_i^{(G,f,\alpha)}, X_j^{(G,f,\alpha)})$
1.25	4.893995	5.11945
2	11.01781	11.2806
5	15.01042	15.09308

Table 2.5 – Values of $Cov(X_i^{(G,s,\alpha)}, X_j^{(G,s,\alpha)})$ and $Cov(X_i^{(G,f,\alpha)}, X_j^{(G,f,\alpha)})$, $i \neq j \in \{1, 2, ..., 50\}$.

α	$Cov(X_i^{(C,s,\alpha)}, X_j^{(C,s,\alpha)})$	$Cov(X_i^{(C,f,\alpha)}, X_j^{(C,f,\alpha)})$
0.5	5.20697	4.859871
2	11.14602	10.62849
8	14.72897	14.44488

Table 2.6 – Values of $Cov(X_i^{(C,s,\alpha)}, X_j^{(C,s,\alpha)})$ and $Cov(X_i^{(C,f,\alpha)}, X_j^{(C,f,\alpha)})$, $i \neq j \in \{1, 2, ..., 50\}$.

α	$Var(S^{(G,s,\alpha)})$	$Var(S^{(G,f,\alpha)})$
1.25	12790.29	13342.65
2	27793.64	28437.48
5	37575.52	37778.05

Table 2.7 – Values of $Var(S^{(G,s,\alpha)})$ and $Var(S^{(G,f,\alpha)})$, for $\alpha = 1.5, 2, 5$.

special case and the Clayton copula as a limit case (see Cossette et al. (2018) for more details). The idea here is to approximate the rv Θ by a discrete rv $\widetilde{\Theta}$ and apply the proposed methodology of Section 2.2.3. However, we need to be careful. We cannot blindly apply the three discretization methods used in the previous section because we aim to approximate the copula generated from Θ and not only the distribution of Θ. Thus, finding an appropriate discrete rv $\widetilde{\Theta}$ is not an easy task as we will see in the special case of the Clayton copula.

The multivariate shifted negative binomial copula is defined by

$$C_{\alpha,q_h}^{SNB}(u_1, ..., u_d) = \left(q_h \left(\prod_{i=1}^{d} \left(q_h u_i^{-\alpha} + (1 - q_h) \right) - (1 - q_h) \right)^{-1} \right)^{\frac{1}{\alpha}}. \tag{2.32}$$

The two parameters of the copula are $\alpha > 0$ and $q_h = 1 - \mathrm{e}^{-h}$, where $h > 0$ can be seen as a discretization parameter.

The underlying mixing rv, associated to (2.32) and denoted by $\Theta_{(h)}^{SNB(\alpha)}$, follows a shifted negative binomial distribution, i.e., the rv $\Theta_{(h)}^{SNB(\alpha)}$ is defined as $\Theta_{(h)}^{SNB(\alpha)} = h\left(M_{(h)}^{NB(\alpha)} + \alpha\right)$ where $M_{(h)}^{NB(\alpha)}$ follows a negative binomial distribution, i.e., $M_{(h)}^{NB(\alpha)} \sim NB\left(\frac{1}{\alpha}, q_h\right)$, with $f_{M_{(h)}^{NB(\alpha)}}(k) = \binom{\frac{1}{\alpha}+k-1}{k} (q_h)^{\frac{1}{\alpha}} (1-q_h)^k$, $k \in \mathbb{N}_0$, and $E\left[M_{(h)}^{NB(\alpha)}\right] = \frac{1-q_h}{\alpha q_h}$ with $q_h = 1 - \mathrm{e}^{-h}$,

α	$Var(S^{(C,s,\alpha)})$	$Var(S^{(C,f,\alpha)})$
0.5	13557.08	12706.68
2	28107.76	26839.79
8	36885.98	36189.97

Table 2.8 – Values of $Var(S^{(C,s,\alpha)})$ and $Var(S^{(C,f,\alpha)})$, for $\alpha = 0.5, 2, 8$.

i	x_i	CMC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$
1	1216	0.050527249 (0.198427516)	0.050140000 (0.218234866)	1157	0.050705395 (0.174562788)	0.050730000 (0.219446915)
2	1354	0.010079740 (0.090022495)	0.010160000 (0.100283972)	1201	0.010331348 (0.070988687)	0.010450000 (0.101690220)
3	1332	0.013160376 (0.102769800)	0.013100000 (0.113703647)	1243	0.001055487 (0.018413807)	0.001150000 (0.033892315)
4	1698	0.000108547 (0.009061601)	0.000090000 (0.009486453)	1273	0.000107720 (0.004440821)	0.000060000 (0.007745773)
5	1840	0.000010584 (0.002904670)	0.000020000 (0.004472114)	1296	0.000010881 (0.000807741)	0.000010000 (0.003162278)

Table 2.9 – Approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$ and $\overline{F}_{S^{(C,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 0.5$ ($\tau = 0.2$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$
1	1313	0.049968514 (0.211556800)	0.050030000 (0.218007969)	1232	0.050956074 (0.192522672)	0.050430000 (0.218831657)
2	1468	0.010020219 (0.096953570)	0.010000000 (0.099499241)	1289	0.010297759 (0.081601626)	0.010130000 (0.100137323)
3	1641	0.001008032 (0.030668361)	0.000990000 (0.031448844)	1340	0.001000416 (0.021596196)	0.000870000 (0.029483076)
4	1798	0.000100325 (0.009958315)	0.000100000 (0.009999550)	1374	0.000104887 (0.006010938)	0.000130000 (0.011401070)
5	1958	0.000010000 (0.003162278)	0.000010000 (0.003162278)	1397	0.000012494 (0.001157376)	0.000000000 (-)

Table 2.10 – Approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$ and $\overline{F}_{S^{(C,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parenthesis correspond to the standard errors.

$h > 0$. The LST of $\Theta_{(h)}^{SNB(\frac{1}{\alpha})}$ is

$$\mathcal{L}_{\Theta_{(h)}^{SNB(\alpha)}}(t) = E\left[\mathrm{e}^{-t\Theta_{(h)}^{SNB(\alpha)}}\right] = \left(\frac{\mathrm{e}^{-th} - \mathrm{e}^{-(t-1)h}}{1 - \mathrm{e}^{-(t-1)h}}\right)^{\frac{1}{\alpha}}.$$

The shifted negative binomial copula is an Archimedean copula. Indeed, (2.32) can be represented as

$$C_{\alpha,q_h}^{SNB}(u_1, ..., u_d) = \mathcal{L}_{\Theta_{(h)}^{SNB(\alpha)}}\left(\mathcal{L}^{-1}_{\Theta_{(h)}^{SNB(\alpha)}}(u_1) + ... + \mathcal{L}^{-1}_{\Theta_{(h)}^{SNB(\alpha)}}(u_d)\right).$$

When $d = 2$, $q_h = 1 - \beta$ and $\gamma = \frac{1}{\alpha}$, (2.32) becomes

$$\begin{aligned} C_{\alpha,q_h}^{SNB}(u_1, u_2) &= \left((1-\beta)\left(\prod_{i=1}^{2}\left((1-\beta)u_i^{-\frac{1}{\gamma}} + \beta\right) - \beta\right)^{-1}\right)^{\gamma} \\ &= \frac{u_1 u_2}{\left(1 - \beta\left(1 - u_1^{\frac{1}{\gamma}}\right)\left(1 - u_2^{\frac{1}{\gamma}}\right)\right)^{\gamma}}, \end{aligned}$$

which corresponds to the so-called bivariate Lomax copula in Balakrishnan (2006) , bivariate Fang-Fang-Rosen copula in Fang et al. (2000) and Genest and Rivest (2001), and bivariate BB10 copula in Joe (2014). Note that the multivariate copula provided in (2.2) of Fang et al. (2000) does not correspond to the Archimedean copula in (2.32). The copula in (2.32)

i	x_i	CMC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(C,f,\alpha)}(x_i)$
1	1337	0.050074019 (0.215637266)	0.050050000 (0.218049244)	1303	0.050640550 (0.203357022)	0.050360000 (0.218687788)
2	1489	0.010064309 (0.098773129)	0.010130000 (0.100137323)	1374	0.010921753 (0.090974062)	0.011040000 (0.104490323)
3	1649	0.001000266 (0.030807217)	0.001010000 (0.031764603)	1436	0.001052248 (0.024805081)	0.001090000 (0.032997315)
4	1799	0.000111861 (0.010159259)	0.000120000 (0.010953849)	1472	0.000103602 (0.006558965)	0.000080000 (0.008943959)
5	1960	0.000010000 (0.003162278)	0.000010000 (0.003162278)	1497	0.000010341 (0.001357239)	0.000020000 (0.004472114)

Table 2.11 – Approximated values of $\overline{F}_{S(C,s,\alpha)}$ and $\overline{F}_{S(C,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 8$ ($\tau = 0.8$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$
1	1141	0.050095336 (0.138900179)	0.050220000 (0.218399699)	1227	0.050329739 (0.208588438)	0.050500000 (0.218975408)
2	1179	0.009979292 (0.047470342)	0.009950000 (0.099252688)	1388	0.010100157 (0.095770936)	0.009950000 (0.099252688)
3	1218	0.001031343 (0.009745777)	0.001070000 (0.032693513)	1568	0.001077327 (0.031529811)	0.001040000 (0.032232418)
4	1249	0.000105875 (0.001984137)	0.000110000 (0.010487564)	1745	0.000093758 (0.009441001)	0.000090000 (0.009486453)
5	1276	0.000010431 (0.000391711)	0.000010000 (0.003162278)	1900	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 2.12 – Approximated values of $\overline{F}_{S(G,s,\alpha)}$ and $\overline{F}_{S(G,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 1.25$ ($\tau = 0.2$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,s,\alpha)}(x_i)$	x_i	CMC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$	MC approx.$\overline{F}_{S(G,f,\alpha)}(x_i)$
1	1252	0.050799403 (0.194516995)	0.051030000 (0.220060045)	1307	0.049717809 (0.211499142)	0.049840000 (0.217615367)
2	1336	0.010132678 (0.082516449)	0.010190000 (0.100430398)	1458	0.010201015 (0.097931511)	0.010140000 (0.100186230)
3	1425	0.001002746 (0.022945213)	0.001040000 (0.032232418)	1630	0.001026755 (0.031203817)	0.001040000 (0.032232418)
4	1495	0.000100023 (0.007070299)	0.000070000 (0.008366349)	1793	0.000100527 (0.009880428)	0.000100000 (0.009999550)
5	1572	0.000010349 (0.001879823)	0.000010000 (0.003162278)	1960	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 2.13 – Approximated values of $\overline{F}_{S(G,s,\alpha)}$ and $\overline{F}_{S(G,f,\alpha)}$, using conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parenthesis correspond to the standard errors.

is constructed via the approach proposed by Marshall and Olkin (1988) as for the BB10 copula.

Clearly, when $h \to 0$, we have

$$\lim_{h\to 0} E\left[\mathrm{e}^{-t\Theta_{(h)}^{SNB(\alpha)}}\right] = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}, \tag{2.33}$$

where $\left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$ is the LST of the rv $\Theta^{Ga\left(\frac{1}{\alpha},1\right)}$ which follows a gamma distribution, i.e., $\Theta^{\left(\frac{1}{\alpha},1\right)} \sim Gamma\left(\frac{1}{\alpha},1\right)$. Then, given (2.33), $\Theta_{(h)}^{SNB\left(\frac{1}{\alpha}\right)} \overset{\mathcal{D}}{\to} \Theta^{Ga\left(\frac{1}{\alpha},1\right)}$, as the discretization parameter $h \to 0$, where "$\overset{\mathcal{D}}{\to}$" corresponds to the convergence in distribution.

As a special case, when $\alpha = 1$, the shifted negative binomial copula in (2.32) becomes the AMH copula. For a fixed $\alpha > 0$, when $h \to 0$ (i.e., $q_h \to 1$), the limit of the shifted negative binomial copula corresponds to the Clayton copula with parameter α, i.e.,

$$\lim_{h\to 0} C_{\alpha,q_h}^{SNB}(u_1,...,u_n) = \left(u_1^{-\alpha} + ... + u_n^{-\alpha} - (n-1)\right)^{-\frac{1}{\alpha}} = C_{\alpha}^{CLAY}(u_1,...,u_n).$$

In the following example, we examine the efficiency of the approximation of the Clayton copula with the shifted negative binomial copula in a risk aggregation context.

Example 2.2.10. *In this example, we compare the performance of the approximation of the Clayton copula by the shifted negative binomial copula. Let* $\underline{X}^{(C,\alpha)} = \left(X_1^{(C,\alpha)}, ..., X_4^{(C,\alpha)}\right)$ *and*

i	x_i	CMC approx. $\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	MC approx. $\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	x_i	CMC approx. $\overline{F}_{S^{(G,f,\alpha)}}(x_i)$	MC approx. $\overline{F}_{S^{(G,f,\alpha)}}(x_i)$
1	1320	0.050086643 (0.210804858)	0.050100000 (0.218152391)	1330	0.050803544 (0.217050585)	0.050620000 (0.219221569)
2	1450	0.010204031 (0.095399818)	0.010190000 (0.100430398)	1475	0.010269781 (0.099412094)	0.010220000 (0.100576601)
3	1600	0.001057080 (0.029683037)	0.001040000 (0.032232418)	1648	0.001008499 (0.031358794)	0.001020000 (0.031921306)
4	1730	0.000102036 (0.009012274)	0.000110000 (0.010487564)	1788	0.000109967 (0.010484446)	0.000110000 (0.010487564)
5	1908	0.000010152 (0.002586474)	0.000020000 (0.004472114)	1920	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 2.14 – Approximated values of $\overline{F}_{S^{(G,s,\alpha)}}$ and $\overline{F}_{S^{(G,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 5$ ($\tau = 0.8$). The values in parenthesis correspond to the standard errors.

$\underline{X}^{(SNB,\alpha,h)} = \left(X_1^{(SNB,\alpha,h)}, ..., X_4^{(SNB,\alpha,h)}\right)$ be two vectors of rvs with $X_i^{(C,\alpha)} \sim X_i^{(SNB,\alpha,h)} \sim Bin\,(10, 0.2)$, for $i = 1, \ldots, 4$, where their joint cdf are defined either with the Clayton copula (with $\alpha = 0.5, 2, 8$) or the shifted negative binomial copula (with $\alpha = 0.5, 2, 8$ and $h = 0.001, 0.0001$) according to (2.5). We define $S^{(C,\alpha)} = \sum_{i=1}^{4} X_i^{(C,\alpha)}$ and $S^{(SNB,\alpha,h)} = \sum_{i=1}^{4} X_i^{(SNB,\alpha,h)}$. Both the exact values of $\overline{F}_{S^{(C,\alpha)}}$ resulting from (2.15) and $\overline{F}_{S^{(SNB,\alpha,h)}}$ using the shifted negative binomial copula are given in Tables 2.15, 2.16 and 2.17. The results of both the conditional and the full MC simulation methods are also provided. The approximation using the shifted negative binomial copula is good and more significant as h decreases and for Kendall taus below 0.5. The stronger is the dependence relationship, the more the calculation time increases. Once again, the efficiency of the approximation of a copula with non-zero lower tail dependence by another one with $\lambda_L = 0$ is less significant when the related dependence relationship is strong.

i	x_i	$\overline{F}_{S^{(C,0.5)}}(x_i)$	$\overline{F}_{S^{(SNB,0.5,0.001)}}(x_i)$	$\overline{F}_{S^{(SNB,0.5,0.0001)}}(x_i)$	CMC approx. $\overline{F}_{S^{(C,0.5)}}(x_i)$	MC approx. $\overline{F}_{S^{(C,0.5)}}(x_i)$
1	13	0.046898732	0.046896056	0.046898466	0.046743618 (0.076386143)	0.045940000 (0.209356048)
2	14	0.022517508	0.022516026	0.022517365	0.022429261 (0.042487275)	0.022260000 (0.147528675)
3	17	0.001310321	0.001310236	0.001310341	0.001301674 (0.003570962)	0.001290000 (0.035893576)
4	19	0.000113916	0.000113946	0.000113956	0.000112915 (0.000370698)	0.000090000 (0.009486453)
5	20	0.000028520	0.000028559	0.000028562	0.000028239 (0.000099466)	0.000020000 (0.004472114)
6	26	0.000000001	0.000000044	0.000000044	0.000000001 (0.000000003)	0.000000000 (-)

Table 2.15 – Approximated values of $\overline{F}_{S^{(C,0.5)}}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 0.5$ ($\tau = 0.2$). The values in parenthesis correspond to the standard errors.

i	x_i	$\overline{F}_{S^{(C,2)}}(x_i)$	$\overline{F}_{S^{(SNB,2,0.001)}}(x_i)$	$\overline{F}_{S^{(SNB,2,0.0001)}}(x_i)$	CMC approx. $\overline{F}_{S^{(C,2)}}(x_i)$	MC approx. $\overline{F}_{S^{(C,2)}}(x_i)$
1	14	0.057715752	0.057710844	0.057715261	0.057828623 (0.134180425)	0.057780000 (0.233328130)
2	16	0.014628680	0.014627145	0.014628527	0.014636998 (0.049205147)	0.014280000 (0.118643257)
3	18	0.002313893	0.002313606	0.002313864	0.002314054 (0.011151686)	0.002190000 (0.046746398)
4	20	0.000220129	0.000220099	0.000220126	0.000220007 (0.001443237)	0.000200000 (0.014140792)
5	22	0.000012529	0.000012527	0.000012529	0.000012508 (0.000103554)	0.000000000 (-)
6	28	1.145583e-10	3.685083e-10	3.685204e-10	1.139485e-10 (0.000000001)	0.000000000 (-)

Table 2.16 – Approximated values of $\overline{F}_{S^{(C,2)}}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parenthesis correspond to the standard errors.

To summarize, if the mixing rv Θ is continuous, we can proceed either by conditional MC simulation or by the approximation of the rv Θ with a discrete one. The difference between both approximation methods depends on several elements such as the value of the chosen dependence parameter, the size of the portfolio, the number of simulations, etc. For example,

i	x_i	$\overline{F}_{S(C,8)}(x_i)$	$\overline{F}_{S(SNB,8,0.001)}(x_i)$	$\overline{F}_{S(SNB,8,0.0001)}(x_i)$	CMC approx.$\overline{F}_{S(C,8)}(x_i)$	MC approx.$\overline{F}_{S(C,8)}(x_i)$
1	16	0.042577614	0.042573043	0.042577157	0.043076480 (0.145774109)	0.042560000 (0.201863949)
2	18	0.011940884	0.011939463	0.011940742	0.012148560 (0.061046175)	0.011880000 (0.108346587)
3	20	0.001991095	0.001990824	0.001991068	0.002048498 (0.015964780)	0.002080000 (0.045559789)
4	22	0.000182527	0.000182499	0.000182525	0.000189279 (0.002184336)	0.000290000 (0.017027002)
5	24	0.000008969	0.000008967	0.000008969	0.000009308 (0.000143841)	0.000000000 (-)
6	29	3.483774e-10	3.672044e-10	3.672627e-10	3.57897e-10 (0.000000008)	0.000000000 (-)

Table 2.17 – Approximated values of $\overline{F}_{S(C,8)}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 8$ ($\tau = 0.8$). The values in parenthesis correspond to the standard errors.

the approximation of the Clayton copula with the shifted negative binomial copula is more accurate compared to the simulation method when the dependence parameter is small. In the latter case, execution times of both methods are comparable. However, in the event of a greater degree of dependence or a greater number of risks, the conditional MC simulation executes faster. Note that the aim here was to only present different approaches to solve this problem, rather than putting them through a comprehensive comparison.

2.2.6 Portfolio of exchangeable risks

Exchangeability plays an important role in the analysis of homogeneous portfolios in actuarial science and in quantitative risk management, notably in credit risk modelling (e.g. McNeil et al. (2015) and the references therein). Based on DeFinetti's Theorem (see De Finetti (1957)) and its extension in Bühlmann (1960), an infinite exchangeable sequence of rvs can be represented as a mixture over a common parameter rv of an infinite sequence of iid rvs (see e.g. Feller (1971) for more details). Let $\underline{X} = \{X_n, n \in \mathbb{N}\}$ be a sequence of n positive and exchangeable rvs. Define the sequence $\underline{W} = \{W_n, n \in \mathbb{N}\}$, where $W_n = \frac{X_1+...+X_n}{n}$, for $n \in \mathbb{N}$. As discussed in, e.g., Aldous (1985), the common parameter rv drives the asymptotic behavior of $\underline{W}$. In the context of credit risk, Frey and McNeil (2001) and Frey and McNeil (2002) have studied the asymptotic behavior of $\underline{W}$ with the distribution of $(X_1, ..., X_n)$ defined in terms of exchangeable Bernoulli mixture models, for $n = 2, 3, ...$ (see e.g. Proposition 3.2 in Frey and McNeil (2002)). Their result can be generalized to other dependence models. In the context of Section 2.2.3, we consider the joint distribution of $(X_1, ..., X_n)$ to be defined with an Archimedean copula C with either (2.18) or (2.19) for $n = 2, 3, \ldots$. Let Z be a discrete rv with $Z \in \{z_\theta, \theta \in \mathbb{N}\}$ where $z_\theta = E[X|\Theta = \theta] < \infty$ and $f_Z(z_\theta) = f_\Theta(\theta)$, $\theta \in \mathbb{N}$. Clearly, the sequence $\underline{W}$ converges in distribution to the rv Z, i.e.,

$$W_n \overset{\mathcal{D}}{\to} Z. \tag{2.34}$$

Indeed, we have

$$\mathcal{L}_{W_n}(t) = \sum_{\theta=1}^{\infty} \mathcal{L}_{W_n|\Theta=\theta}(t) f_\Theta(\theta) = \sum_{\theta=1}^{\infty} \left(\mathcal{L}_{X|\Theta=\theta}\left(\frac{t}{n}\right)\right)^n f_\Theta(\theta).$$

	$\alpha = 0.5$	$\alpha = 0.9$		$\alpha = 0.5$	$\alpha = 0.9$
$E[W_n]$	5	5	$E[Z]$	5	5
$Var(W_n)$	0.920999	1.886689	$Var(Z)$	0.879797	1.855241
$VaR_{0.9}(W_n)$	6.39	6.74	$VaR_{0.9}(Z)$	6.426813	6.702153
$VaR_{0.99}(W_n)$	7.34	7.71	$VaR_{0.99}(Z)$	7.265279	7.660453
$VaR_{0.9999}(W_n)$	8.31	8.66	$VaR_{0.9999}(Z)$	8.241060	8.579895
$TVaR_{0.9}(W_n)$	6.833850	7.191685	$TVaR_{0.9}(Z)$	6.786601	7.165780
$TVaR_{0.99}(W_n)$	7.599597	7.958826	$TVaR_{0.99}(Z)$	7.535144	7.910267
$TVaR_{0.9999}(W_n)$	8.448255	8.795515	$TVaR_{0.9999}(Z)$	8.347715	8.704634

Table 2.18 – Values of the expectation, variance, VaR and TVaR of W_n and Z where $F_{X_1,\ldots,X_n}$ is defined with the AMH copula and Poisson marginals.

Near the origin, $\mathcal{L}_{X|\Theta=\theta}(t) = 1 - z_\theta t + o(t)$ which implies that

$$\lim_{n\to\infty} \mathcal{L}_{W_n}(t) = \sum_{\theta=1}^{\infty} f_\Theta(\theta) \lim_{n\to\infty} \left(1 - z_\theta \frac{t}{n}\right)^n = \sum_{\theta=1}^{\infty} f_\Theta(\theta) \mathrm{e}^{-z_\theta t} = \mathcal{L}_Z(t),$$

leading to (2.34).

The result in (2.34) shows that for a large portfolio of exchangeable risks, the distribution of W_n can be approximated by the distribution of Z. One of the interesting features of the strategy described in Section 2.2.3 is that the exact values of z_θ ($\theta \in \mathbb{N}$) can be computed as well as its pmf. If the rv X is continuous, discretization methods presented in Section 2.2.4 are used.

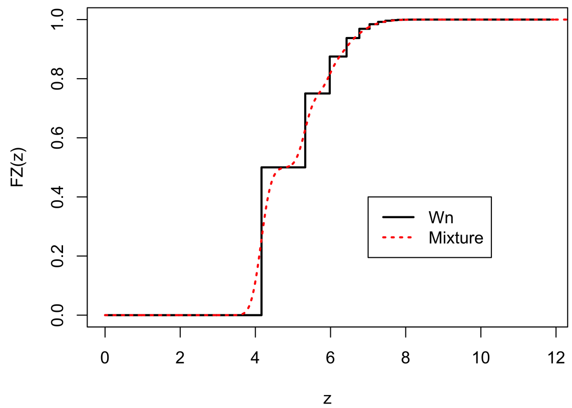
Let us look at a simple example that illustrates the convergence of W_n to Z for a portfolio of 100 exchangeable risks.

Example 2.2.11. *Let $(X_1, \ldots, X_{100})$ be a vector of 100 exchangeable rvs with $X_i \sim X \sim Pois(\lambda = 5)$, for $i = 1, \ldots, 100$. The multivariate cdf of $(X_1, \ldots, X_{100})$ is defined with an AMH copula with dependence parameter $\alpha = 0.5$ and $\alpha = 0.9$.*

As shown in Table 2.18 and illustrated in Figure 2.3, for large portfolios, the distribution of Z is a good candidate to approximate the behavior of W_n. This result is very important in terms of computation time for very large portfolios since using Z is a faster, more efficient and easier to handle tool in comparison to directly computing W_n.

2.3 Capital Allocation

Capital allocation is fundamental in actuarial science and quantitative risk management. It describes how the capital needed for the whole portfolio can be divided and allocated between risks of the portfolio. It is crucial for an insurance company or a financial institution to evaluate the overall capital charge for a portfolio of risks in order to protect itself from large rare events. The amount of capital needed for the entire portfolio is determined with a chosen



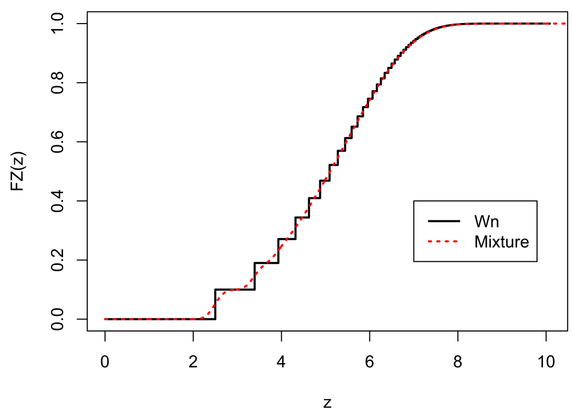


Figure 2.3 – The cdf of Z and W_n where $F_{X_1,\ldots,X_{100}}$ is defined by Poisson marginals and AMH copula with $\alpha = 0.5$ (left) and $\alpha = 0.9$ (right).

risk measure ρ. Among the desired properties for a capital allocation rule (see e.g. Furman and Zitikis (2008)), one has to be fully additive, i.e.,

$$\rho(S) = \sum_{i}^{n} C_i, \tag{2.35}$$

where $S = X_1 + \ldots + X_n$ and C_i is the contribution of the i^{th} risk to the aggregate risk of the portfolio.

In the present section, we show how to compute the values (exact or approximated) of the contributions C_i with the proposed methodology of Section 2.2. We consider Euler's capital allocation principle and the weighted risk capital allocation principle using different risk measures.

Let us assume here that we are in the context of Section 2.2.3, meaning that Θ is a discrete rv defined on $\mathbb{N}$ and that $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, ..., n)$. Also, assume that the joint cdf of $\underline{X}$ or its joint survival function is defined with the Archimedean copula C as in (2.18) or (2.19). If the rvs X_i are continuous, the procedure described in Section 2.2.4 will be used to find the contributions. For a continuous Θ, we refer to Section 2.2.5.

To apply Euler's capital allocation rule, we need to assume that the risk measure ρ is positive homogeneous. Let us define

$$L\left(\underline{\lambda}\right) = \sum_{i=1}^{n} \lambda_i X_i,$$

where $\underline{\lambda} = (\lambda_1, ..., \lambda_n)$.

For a given risk measure ρ, the contribution C_i allocated to risk i is given by

$$C_i = \rho\left(L; X_i\right) = \lambda_i \left.\frac{\partial}{\partial \lambda_i} \rho\left(L\left(\underline{\lambda}\right)\right)\right|_{\underline{\lambda}=\underline{1}}, \text{ for } i = 1, 2, ..., n,$$

where $\underline{1} = (1, ..., 1)$. We apply the Euler allocation principle with three different risk measures: covariance, VaR and TVaR (see Table 2.19 for their expressions). For more details, see e.g. Tasche (1999), McNeil et al. (2005) or Rosen et al. (2011).

The second capital allocation principle we use is the one proposed by Furman and Zitikis (2008) in which the capital amount required and the contributions are based on weighted risk measures. Furman and Zitikis (2008) propose to compute the capital amount with weighted risk measures and to use weighted allocation methods for determining the contribution of each risk i, $i = 1, 2, ..., n$. Let ω be a weight function. Then, the capital amount corresponds to ρ_ω which is defined by $\rho_\omega(S) = \frac{E[S\omega(S)]}{E[\omega(S)]}$, assuming that both expectations exist. Since

$$\frac{E\left[S\omega\left(S\right)\right]}{E\left[\omega\left(S\right)\right]} = \frac{E\left[\left(\sum_{i=1}^{n} X_i\right)\omega\left(S\right)\right]}{E\left[\omega\left(S\right)\right]} = \sum_{i=1}^{n} \frac{E\left[X_i\omega\left(S\right)\right]}{E\left[\omega\left(S\right)\right]}, \tag{2.36}$$

the share of the capital allocated to X_i is given by

$$C_i = \rho_\omega\left(X_i; S\right) = \frac{E\left[X_i\omega(S)\right]}{E\left[\omega(S)\right]},$$

for $i = 1, 2, ..., n$. Clearly, due to (2.36), property (2.35) is satisfied.

Among the several weight functions that can be used to calculate the capital allocation, we consider the following three methods: Esscher with $\omega(s) = \mathrm{e}^{\eta s}$, Kamp with $\omega(s) = 1 - \mathrm{e}^{-\eta s}$ and size-biased with $\omega(s) = s^\eta$.

Let us address the computation of C_i for the considered Euler capital allocation rules and weighted risk capital allocation rules given in Table 2.19. One encounters frequently the

Allocation rule	Contribution C_i
Euler-Covariance	$E\left[X_i\right]+\frac{Cov(X_i,S)}{Var(S)}\left\{\xi(S)-E\left[S\right]\right\}$, with ξ is is a chosen risk measure
Euler-VaR	$E\left[X_i\vert S=VaR_\kappa\left(S\right)\right]$
Euler-TVaR	$\frac{1}{1-\kappa}\left\{E\left[X_i\times 1_{S>VaR_\kappa(S)}\right]+\beta E\left[X_i 1_{S=VaR_\kappa(S)}\right]\right\}$, with $\beta=F_S\left(VaR_\kappa(S)\right)-\kappa$
Weighted-Esscher	$\frac{E\left[X_i \mathrm{e}^{\eta S}\right]}{E\left[\mathrm{e}^{\eta S}\right]}$
Weighted-Kamps	$\frac{E\left[X_i\left(1-\mathrm{e}^{-\eta S}\right)\right]}{E\left[1-\mathrm{e}^{-\eta S}\right]}$
Weighted-size-biased	$\frac{E\left[X_i S^\eta\right]}{E\left[S^\eta\right]}$

Table 2.19 – Contributions under considered allocation rules.

evaluation of the expectation $E[X_iS]$ (or a variation of it) which poses problem given that X_i and S are dependent. To circumvent this, we rewrite the product of these two rvs as

$$E\left[X_iS\right]=E_\Theta\left[E\left[X_i\left(X_i+S_{-i}\right)\vert\Theta\right]\right],$$

where $S_{-i}=X_1+\ldots+X_{i-1}+X_{i+1}+\ldots+X_n$. The conditional independence of X_i and S_{-i} given Θ simplifies the evaluation of the quantities of interest.

Note that with the VaR risk measure, it is slightly more tedious. In this case, the contribution of risk i is given by

$$C_i=E\left[X_i\vert S=VaR_\kappa\left(S\right)\right]=E\left[X_i\vert S=k_0h\right]=\frac{E\left[X_i\times 1_{\{S=k_0h\}}\right]}{\Pr\left(S=k_0h\right)},$$

assuming that $\Pr\left(S=k_0h\right)>0$. Then, using our approach, we have

$$\begin{aligned}E\left[X_i\times 1_{\{S=k_0h\}}\right] &= E_\Theta\left[E\left[X_i\times 1_{\{S=k_0h\}}\vert\Theta\right]\right]\\ &= \sum_{\theta=1}^{\infty}\left(\sum_{j=1}^{k_0} j f_{X_i\vert\Theta=\theta}(jh) f_{S_{-i}\vert\Theta=\theta}\left(\left(k_0-j\right)h\right)\right) f_\Theta\left(\theta\right)\\ &= \sum_{\theta=1}^{\infty}E\left[X_i\vert\Theta=\theta\right]\left(\sum_{j=1}^{k_0} f^*_{X_i\vert\Theta=\theta}(jh) f_{S_{-i}\vert\Theta=\theta}\left(\left(k_0-j\right)h\right)\right) f_\Theta(\theta),\end{aligned}$$

with $f^*_{X_i\vert\Theta=\theta}(jh)=\frac{jf_{X_i\vert\Theta}(j)}{E[X_i\vert\Theta=\theta]}$, for $j\in\mathbb{N}$. Finally, defining

$$g_{i\vert\Theta=\theta}(k_0h)=\sum_{j=1}^{k_0} f^*_{X_i\vert\Theta=\theta}(jh) f_{S_{-i}\vert\Theta=\theta}\left(\left(k_0-j\right)h\right)$$

as a result of a convolution product, we obtain

$$E\left[X_i\times 1_{\{S=k_0h\}}\right]=\sum_{\theta=1}^{\infty}E\left[X_i\vert\Theta=\theta\right]g_{i\vert\Theta=\theta}(k_0h)f_\Theta(\theta), \tag{2.37}$$

where the values of $g_{i\vert\Theta=\theta}\left(kh\right)$ can be easily obtained using classic aggregation methods such as FFT.

	Covariance	VaR	TVaR	$Esscher$ $(\eta=0.1)$	$Kamps$ $(\eta=10^{-6})$	$Size-biased$ $(\eta=10)$
i	$\frac{C_i}{\rho_{0.99}(S)}$	$\frac{VaR_{0.99}(X_i;S)}{VaR_{0.99}(S)}$	$\frac{TVaR_{0.99}(X_i;S)}{TVaR_{0.99}(S)}$	$\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$	$\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$	$\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$
1	11.53	28.52	31.92	14.23	10.55	14.25
2	10.75	11.88	11.65	11.52	10.27	11.58
3	10.32	9.17	8.80	10.45	10.12	10.48
4	10.04	8.15	7.75	9.85	10.01	9.87
5	9.84	7.61	7.21	9.47	9.94	9.47
6	9.69	7.28	6.88	9.21	9.89	9.20
7	9.58	7.05	6.65	9.01	9.85	8.99
8	9.49	6.89	6.49	8.86	9.81	8.84
9	9.41	6.77	6.37	8.75	9.79	8.71
10	9.35	6.68	6.28	8.65	9.77	8.61
$\rho(S)$	76.2266	96	103.0490	89.2213	57.3053	94.1052

Table 2.20 – Contributions of X_i $(i = 1, 2, ..., 10)$ in % under the 6 allocation methods. The multivariate cdf of $(X_1, \ldots, X_10)$ is defined with an AMH copula $(\alpha = 0.8)$.

Example 2.3.1. *We consider a portfolio of 10 risks, where* $X_i - 1 \sim NB\left(r_i = \frac{1+i}{2}, q_i = \frac{1}{1+\frac{8}{1+i}}\right)$*, such that* $E[X_i] = 5$*, for* $i = 1, 2, ..., 10$*. It implies* $E[S] = 50$*. The multivariate cdf of* $\underline{X}$ *is defined as in (2.18) with the AMH copula. For* $\alpha = 0$*, 0.3 and 0.8, the variance of* S *is 104.6361, 185.2821 and 343.7994 respectively. In Table 2.20, we provide the relative contributions of* X_1*, ...,* X_{10} *(i.e.,* $\frac{C_i}{\rho(S)}$*) for the methods based on Euler's capital allocation rule and the ones under the weighted risk allocation approach assuming a dependence parameter* $\alpha = 0.8$*. As shown in Table 2.20, we are able to find exact contributions based on different allocation methods. Consistent results are obtained for* $\alpha = 0$ *and* $\alpha = 0.3$*.*

2.4 Random sum of exchangeable risks

Random sums are essential in the description of many fundamental risk models in actuarial science. The so-called frequency-severity model is based on random sums. In this section, the computational methodology exposed in Section 2.2.3 is used to analyze the distribution of the aggregate claim amount rv S which is defined as the random sum of exchangeable individual claim amounts. Under this risk model, the rv S is defined by $S = \sum_{j=1}^{N} X_j$, where $\underline{X} = \{X_j, j \in \mathbb{N}\}$ forms a sequence of exchangeable rvs independent of the counting positive discrete rv N. The components of $\underline{X}$ are such that $X_j \in A = \{0, 1h, 2h, \ldots\}$ $(j \in \mathbb{N})$. The notation for the univariate pmf of X_i and the univariate cdf of X_i is given in Section 2.2. Also, we assume that the joint distribution of $(X_1, ..., X_j)$ is defined via either (2.18) or (2.19) with $d = j$.

The risk model considered in this section can be seen as an extension of the risk model described and studied in Section 2 of Albrecher et al. (2011). Indeed, they consider a risk model defined with a compound Poisson process where the vector of claim amounts follows a multivariate mixed exponential distribution as defined in Section 2.2.2. Here, the multivariate distribution of the vector of claim amounts can be defined with any Archimedean copula and any marginal distributions for the claim amounts.

To apply the computational methodology, assume that the mixing rv Θ is a strictly positive discrete rv defined on $\mathbb{N}$. Clearly, the aggregate claim amount rv $S \in A$ and the objective is to compute the values of $f_S(kh) = \Pr(S = kh)$, $k \in \mathbb{N}_0$. Given the common mixture representation of either $F_{X_1,...,X_j}$ or $\overline{F}_{X_1,...,X_j}$, we have $f_S(kh) = \sum_{\theta=1}^{\infty} f_{S|\Theta=\theta}(kh) f_\Theta(\theta)$, where $(S|\Theta = \theta) = \sum_{j=1}^{N}(X_j|\Theta = \theta)$. The usual aggregation algorithms (see e.g. Panjer et al. (2008)) such as FFT or Panjer's recursive algorithm can be used to compute the values of $f_{S|\Theta=\theta}(kh)$ for $k \in \mathbb{N}_0$ and for each $\theta = 1, 2, ..., \theta^*$ where θ^* is chosen such that $F_\Theta(\theta^*) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$). Again, note that we just need the values of $f_{X|\Theta=\theta}(kh)$, which are given in (2.22) or (2.25). Also, the LST of the rv S is given by

$$\mathcal{L}_S(t) = E\left[\mathrm{e}^{-tS}\right] = \sum_{\theta=1}^{\infty} E\left[\mathrm{e}^{-tS}|\Theta = \theta\right] f_\Theta(\theta) = \sum_{\theta=1}^{\infty} \mathcal{L}_{S|\Theta=\theta}(t) f_\Theta(\theta),$$

with $\mathcal{L}_{S|\Theta=\theta}(t) = P_N\left(\mathcal{L}_{X|\Theta=\theta}(t)\right)$ where

$$\mathcal{L}_{X|\Theta=\theta}(t) = E\left[\mathrm{e}^{-tX}|\Theta = \theta\right] = \sum_{k=0}^{\infty} \mathrm{e}^{-tkh} f_{X|\Theta=\theta}(kh)$$

and $P_N(s) = E\left[s^N\right]$ is the probability generating function of the positive discrete rv N.

Example 2.4.1. *Let $N \sim Pois(\lambda = 2)$ and $X \sim Gamma(\alpha = 2, \beta = 0.01)$ such that $E[X] = 200$. It implies that $E[S] = 400$. Also, we assume that the joint distribution of $(X_1, ..., X_j)$, $j \in \mathbb{N}$, is defined as in (2.18) with an AMH copula with dependence parameter $\alpha = 0.8$. To obtain the desired results provided in Table 2.21, we use the upper and lower discretization methods with $h = \frac{1}{20}$. Different MC simulation studies (with 10 million simulations) have been performed. We present results from one of them in the second column of Table 2.21. From one study to the next, we have observed results that may or may not fall between the upper and lower bounds given in the third and fourth columns of Table 2.21 which highlights the advantage of the proposed methodology.*

2.5 Renewal risk models with exchangeable inter-claim times

In this section, we consider a general class of continuous-time renewal risk models with exchangeable inter-claim times. This class is an extension of the class discussed in Section 3 of Albrecher et al. (2011). For an insurance portfolio, the surplus process is defined by $\underline{U} = \{U(t), t \geq 0\}$ where the surplus level at time t, $U(t)$, is given by

$$U(t) = u + ct - S(t),$$

	$\alpha=0.8$ (simul)	$\mathrm{IC}_{0.05}$	$\alpha=0.8$ (upper)	$\alpha=0.8$ (lower)	$\alpha=0$ (exact)
$E[S]$	400.0359	[399.8034 ; 400.2685]	399.950	400.050	400
$Var(S)$	140775.7887	[140652.4769 ; 140899.2633]	140912.018	140942.018	80000
$VaR_{0.9}(S)$	906.9635	[906.3475 ; 907.5886]	907.000	907.200	873.8748
$VaR_{0.99}(S)$	1644.4705	[1642.6493 ; 1646.4017]	1645.350	1645.600	1470.9808
$VaR_{0.999}(S)$	2317.5620	[2312.0092 ; 2322.2396]	2323.350	2323.650	1992.0052
$VaR_{0.9999}(S)$	2959.6858	[2942.0263 ; 2975.0607]	2967.250	2967.600	2473.3833
$TVaR_{0.9}(S)$	1230.7730	[1230.5806 ; 1230.9654]	1231.206	1231.333	1138.1220
$TVaR_{0.99}(S)$	1939.1160	[1938.9381 ; 1939.2938]	1941.278	1941.494	1699.2458
$TVaR_{0.999}(S)$	2596.5160	[2596.3469 ; 2596.6851]	2603.626	2603.935	2202.1856
$TVaR_{0.9999}(S)$	3223.8834	[3223.7239 ; 3224.0429]	3236.634	3237.032	2672.1090

Table 2.21 – Values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ where S is defined as a random sum of dependent rvs.

where $U(0)=u$ is the initial surplus and c is the premium rate. The aggregate claim amount process, denoted by $\underline{S}=\{S(t), t\geq 0\}$ with $S(t)=\sum_{j=1}^{N(t)} X_j$ is a mixed compound renewal process with exchangeable inter-claim times. The claim number process $\underline{N}=\{N(t), t\in\mathbb{R}^+\}$ is a mixed renewal process where the inter-claim times $\underline{W}=\{W_j, j\in\mathbb{N}\}$ form a sequence of exchangeable and strictly positive real-valued rvs. The time between the $(j-1)$th and the jth claim $(j=2,...)$ is defined by the rv W_j with W_1 the time of the first claim. The rvs $\{W_j, j\in\mathbb{N}\}$, are identically distributed as the canonical rv W, have pdf f_W, cdf F_W, and survival function $\overline{F}_W$.

To simplify the presentation, the joint survival function of $(W_1, W_2, ..., W_k)$ is defined with an Archimedean copula as in (2.8), i.e.,

$$\overline{F}_{W_1,W_2,...,W_k}(x_1,...,x_k)=C\left(\overline{F}_W(x_1),...,\overline{F}_W(x_k)\right), \tag{2.38}$$

for $k\in\{2,3,...\}$ and $x_1,...,x_k\geq 0$. The multivariate distribution of $(W_1,\ldots,W_k)$ can also be defined with joint cdf as in (2.5). The time of arrival of the jth claim is denoted $T_j=W_1+\cdots+W_j$.

The claim amount rvs $\underline{X}=\{X_j, j\in\mathbb{N}\}$, where X_j corresponds to the amount of the jth claim, are assumed to be a sequence of strictly positive and iid rvs with pdf f_X and cdf F_X. The sequences $\underline{W}$ and $\underline{X}$ are independent.

The time of ruin is defined by the rv $\tau_u=\inf\{t\geq 0: U(t)<0\}$ with $\tau_u=\infty$ if $U(t)\geq 0$ for all $t\geq 0$. The infinite-time ruin probability is $\zeta(u)=\Pr(\tau_u<\infty|U(0)=u)$. Throughout this section, we assume the positive security loading condition $E[cW-X]>0$ to be verified which ensures that ruin will not occur almost surely. Due to the common mixture representation, (2.38) is given by

$$\overline{F}_{W_1,...,W_k}(x_1,...,x_k)=C\left(\overline{F}_W(x_1),...,\overline{F}_W(x_k)\right)=\int_0^\infty \overline{F}_{W|\Theta=\theta}(x_1)\times...\times\overline{F}_{W|\Theta=\theta}(x_k)\,\mathrm{d}F_\Theta(\theta),$$

where

$$\overline{F}_{W|\Theta=\theta}(x) = \mathrm{e}^{-\theta\psi^{-1}\left(\overline{F}_W(x)\right)}$$

for $x \geq 0$. As mentioned in Section 3 of Albrecher et al. (2011), $\overline{F}_{W|\Theta=\theta}$ is the canonical survival function of the inter-claim time rvs for an ordinary renewal process. Let ζ_θ be the conditional ruin probability associated to the corresponding renewal process. It implies that the ruin probability ζ can be represented as a mixture, where Θ is the mixing rv, i.e.,

$$\zeta(u) = \int_0^\infty \zeta_\theta(u)\,\mathrm{d}F_\Theta(\theta). \tag{2.39}$$

The security loading condition is violated when the mixing rv Θ takes a value larger that $\theta_0 > 0$. We define θ_0 such that $\zeta_\theta(u) = 1$, for $\theta > \theta_0 > 0$. There exists a θ_0 such that $c \times E[W|\Theta = \theta] > E[X]$, for $\theta \in \{1, 2, \ldots, \theta_0\}$, and $c \times E[W|\Theta = \theta] < E[X]$, for $\theta \in \{\theta_0 + 1, \ldots\}$. Then, (2.39) becomes

$$\zeta(u) = \int_0^{\theta_0} \zeta_\theta(u)\,\mathrm{d}F_\Theta(\theta) + \overline{F}_\Theta(\theta_0), \tag{2.40}$$

for $u \geq 0$. Assuming that ζ_θ could be computed for each $\theta \in \{1, 2, \ldots, \theta_0\}$ of the ordinary renewal process associated to $\overline{F}_{W|\Theta=\theta}$, the value of $\zeta(u)$ can be computed using (2.40).

In the examples of Section 3 of Albrecher et al. (2011), the authors assume that $(W_1, W_2, ..., W_k)$ follows a multivariate mixed exponential distribution as defined in Section 2.2.2, where the LST of the mixing rv corresponds to the generator of an Archimedean copula. It means that the univariate marginal distribution of the inter-claim time is a univariate mixed exponential distribution. Indeed, the authors consider specific examples of mixed Poisson risk models.

In this section, we show that it is possible to consider any multivariate distribution for $\underline{W}$ defined with any Archimedean copula and given marginal distributions. First, let Θ be a strictly positive discrete rv defined on $\mathbb{N}$. As in Albrecher et al. (2011), we limit our analysis to exponentially distributed claim amounts with parameter β. It implies that

$$\zeta_\theta(u) = \frac{\beta - \rho_\theta}{\beta}\mathrm{e}^{-\rho_\theta u},\ u \geq 0, \tag{2.41}$$

where ρ_θ is the adjustment coefficient which is the smallest strictly positive solution to the Lundberg relation

$$E\left[\mathrm{e}^{r(X-cW)}|\Theta = \theta\right] = E\left[\mathrm{e}^{rX}\right] \times E\left[\mathrm{e}^{-rcW}|\Theta = \theta\right] = 1 \tag{2.42}$$

with $E\left[\mathrm{e}^{-rcW}|\Theta = \theta\right] = \int_0^\infty \mathrm{e}^{-rcx} f_{W|\Theta=\theta}(x)\,\mathrm{d}x$ and $f_{W|\Theta=\theta}(x) = -\frac{\mathrm{d}\overline{F}_{W|\Theta=\theta}(x)}{\mathrm{d}x}$. Given that Θ is a discrete rv and with (2.41), (2.40) becomes

$$\zeta(u) = \sum_{\theta=1}^{\theta_0} \Pr(\Theta = \theta)\frac{\beta - \rho_\theta}{\beta}\mathrm{e}^{-\rho_\theta u} + \overline{F}_\Theta(\theta_0). \tag{2.43}$$

The expression in (2.43) is illustrated in the following example.

Example 2.5.1. *Let $X \sim Exp(1)$ and $\overline{F}_{W_1,W_2,\ldots,W_k}(x_1,...,x_k)$ be defined as in (2.38) where C is an AMH copula. It means that Θ follows a geometric distribution with parameter $q = 1-\alpha$ and*

$$\mathcal{L}_\Theta(t) = \frac{q\mathrm{e}^{-t}}{1-(1-q)\,\mathrm{e}^{-t}} \text{ and } \mathcal{L}_\Theta^{-1}(u) = -\ln\left(\frac{1}{\frac{q}{t}+1-q}\right).$$

Then,

$$\overline{F}_{W|\Theta=\theta}(x) = \left(\frac{1}{q\mathrm{e}^x+1-q}\right)^\theta \text{ and } f_{W|\Theta=\theta}(x) = \frac{\theta q\mathrm{e}^x}{(q\mathrm{e}^x+1-q)^{\theta+1}}$$

for $x \geq 0$, $\theta \in \mathbb{N}$ and $0 < q < 1$. Note that $\overline{F}_{W|\Theta=1}(x) = \frac{1}{q\mathrm{e}^x+1-q}$ corresponds to the univariate survival function of the exponential distribution with tilt as defined in Marshall and Olkin (2007).

In this case, we have

$$E[W|\Theta=\theta] = \int_0^\infty \left(\frac{1}{q\,\mathrm{e}^x+1-q}\right)^\theta \mathrm{d}x = \lim_{t\to\infty}\int_0^t \left(\frac{1}{q\,\mathrm{e}^x+1-q}\right)^\theta \mathrm{d}x. \tag{2.44}$$

Let $u = q\,\mathrm{e}^x+1-q$, then (2.44) becomes

$$\int_0^\infty \left(\frac{1}{q\,\mathrm{e}^x+1-q}\right)^\theta \mathrm{d}x = \lim_{t\to\infty}\int_0^t \frac{1}{u^\theta(u-1+q)}\mathrm{d}u.$$

Using a partial fraction decomposition, we obtain

$$\int_0^\infty \left(\frac{1}{q\,\mathrm{e}^x+1-q}\right)^\theta \mathrm{d}x = -\frac{\ln(q)}{(1-q)^\theta} - \sum_{k=1}^{\theta-1}\frac{1}{(\theta-k)(1-q)^k}.$$

Also, in order to calculate the adjustment coefficient ρ_θ, we develop the expression of $E\left[\mathrm{e}^{-tW}|\Theta=\theta\right]$ as follows

$$E\left[\mathrm{e}^{-tW}|\Theta=\theta\right] = \int_0^\infty \mathrm{e}^{-tx}\frac{\theta q\mathrm{e}^x}{(q\mathrm{e}^x+1-q)^{\theta+1}}\mathrm{d}x. \tag{2.45}$$

Let $u = q\mathrm{e}^x+1-q$, then (2.45) becomes

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-tx}\frac{\theta q\mathrm{e}^x}{(q\mathrm{e}^x+1-q)^{\theta+1}}\mathrm{d}x &= \int_1^\infty \frac{\theta q^t(u+q-1)^{-t}}{u^{\theta+1}}\mathrm{d}u \\
&= \frac{\theta q^t}{(q-1)^t}\int_1^\infty \frac{1}{\left(\frac{u}{q-1}+1\right)^t u^{\theta+1}}\mathrm{d}u \\
&= \frac{\theta q^t}{(q-1)^t} \times \frac{(q-1)^t\,{}_2F_1([t,t+\theta];[t+\theta+1];1-q)}{t+\theta} \\
&= \frac{\theta q^t\,{}_2F_1([t,t+\theta];[t+\theta+1];1-q)}{t+\theta},
\end{aligned}$$

where ${}_nF_m$ denotes the generalized hypergeometric function defined as follows

$${}_nF_m([a_1,\ldots,a_n];[b_1,\ldots,b_m];z) = \sum_{k=0}^\infty \frac{(a_1)_k\ldots(a_n)_k}{(b_1)_k\ldots(b_m)_k}\frac{z^k}{k!},$$

with $(x)_k = x(x+1)\dots(x+k-1)$.

Let $\lambda = 1$*,* $\beta = 1$*, and* $c = 1.2$*. The parameter of the AMH copula is* $\alpha = 1 - q = 0.9$*. We find that* $\theta_0 = 5$ *and the expression in (2.43) for* $\zeta(u)$ *becomes*

$$\zeta(u) = \sum_{\theta=1}^{5} 0.9^{\theta-1} \times 0.1 \times (1-\rho_\theta)\mathrm{e}^{-\rho_\theta u} + 0.9^5,$$

where the values of ρ_θ *for* $\theta = 1, 2, \dots, 5$ *are given in the following table:*

θ	*1*	*2*	*3*	*4*	*5*
ρ_θ	*0.7666945*	*0.5591448*	*0.3649751*	*0.1794426*	*0.0001673811*

The values of ρ_θ *are computed by numerical optimization using (2.42) for* $\theta = 1, 2, \dots, 5$*.* □

We have defined here the multivariate distribution of $\underline{W}$ in terms of the multivariate survival function as in (2.19). Our strategy, based on common mixtures, allows us however to also do our investigation of ruin models with the multivariate cdf as in (2.18). Also, it is possible to adapt the strategy discussed in Section 2.2.5 with a continuous distribution for the mixing rv Θ. The content of this section clearly demonstrates that using our common mixture representation methodology allows to analyze a variety of risk models which opens the door to further research.

2.6 Classical discrete-time risk models with exchangeable losses

In this section, we consider a discrete-time risk model with exchangeable losses. Let $\underline{X} = \{X_k, k \in \mathbb{N}\}$ be a sequence of exchangeable rvs, where X_k is the aggregate loss for a portfolio in period $k \in \mathbb{N}$ with $X_k \sim X$, $k \in \mathbb{N}$. Let $\underline{U} = \{U_k, k \in \mathbb{N}_0\}$ be the surplus process of the portfolio, where U_k corresponds to the surplus level at period $k \in \mathbb{N}$. For $k = 0$, $U_0 = u$ corresponds to the initial amount of capital allocated to the portfolio. Then, at period $k \in \mathbb{N}$, $U_k = U_{k-1} + \pi - X_k = u - \sum_{j=1}^{k}(X_j - \pi)$. The time of ruin $\tau_u = \inf\{k \in \mathbb{N}, U_k < 0\}$, if $\underline{U}$ goes below 0 at least once, or ∞, if $\underline{U}$ never goes below 0. We define the infinite-time ruin probability by $\zeta(u) = \Pr(\tau_u < \infty)$. To prevent ruin with certainty (i.e., $\zeta(u) = 1$, $u \geq 0$), we assume that the net profit condition is satisfied, i.e., $E[X - \pi] < 0$, where π is the premium income per period with $\pi = (1+\eta)E[X]$. For simplification purposes, we assume $X \in \mathbb{N}_0$, $u \in \mathbb{N}_0$, and $\pi = 1$ with $\pi > E[X]$. Then, with these additional assumptions, the classical discrete-time risk model with exchangeable losses corresponds to an extension of the compound binomial classical risk model (see e.g. Gerber (1988), Shiu (1989), Willmot (1993), De Vylder and Marceau (1996), Dickson (1992) for details on the compound binomial risk model).

For $j = 2, 3, ...,$ the multivariate distribution of $(X_1, ..., X_j)$ is defined with an Archimedean copula C with either (2.18) or (2.19) as in Section 2.2.3. Let $\zeta_\theta(u)$ be the conditional infinite-time ruin probability given $\Theta = \theta$. There exists a θ_0 such that $E[X|\Theta = \theta] < 1$, for $\theta \in \{1, 2, ..., \theta_0\}$, and $E[X|\Theta = \theta] > 1$, for $\theta \in \{\theta_0 + 1, \theta_0 + 2, \ldots\}$. Then, when $\theta = \theta_0 + 1, \theta_0 + 2$,..., the solvency condition is not satisfied and the conditional infinite-time ruin probability $\zeta_\theta(u) = 1$, for all $u \in \mathbb{N}_0$. For $\theta = 1, 2, ..., \theta_0$, adapting expressions from Cossette et al. (2003), we have

$$\zeta_\theta(u) = \frac{\zeta_\theta(u-1) - \sum_{j=1}^{u} \zeta_\theta(u-j) \times f_{X|\Theta=\theta}(j) - \overline{F}_{X|\Theta=\theta}(u)}{f_{X|\Theta=\theta}(0)} \quad , \quad \text{for } u \in \mathbb{N}_0 \; ,$$

with initial value $\zeta_\theta(0) = \frac{E[X|\Theta=\theta] - \Pr(X>0|\Theta=\theta)}{f_{X|\Theta=\theta}(0)}$. The unconditional infinite-time ruin probability $\zeta(u)$ is given by

$$\zeta(u) = \sum_{\theta=1}^{\infty} \zeta_\theta(u) f_\Theta(\theta) = \sum_{\theta=1}^{\theta_0} \zeta_\theta(u) f_\Theta(\theta) + \overline{F}_\Theta(\theta_0), \; u \in \mathbb{N}_0. \tag{2.46}$$

Example 2.6.1. *Let X be a non-negative discrete rv with*

$$f_X(0) = (1-\delta) + \delta \times f_B(0) \quad \textit{and} \quad f_X(k) = \delta \times f_B(k), \quad k \in \mathbb{N}, \; \delta \in (0,1),$$

and $B - 1 \sim NB(r, q)$ ($r \in \mathbb{R}^+$, $q \in (0,1)$) with $E[B] = 1 + r \times \frac{1-q}{q}$. We fix the different parameters as follows: $\delta = 0.1$, $r = 2$, and $q = \frac{1}{5}$, and $E[X] = 0.9 < 1$. Finally, $F_{X_1,...,X_j}$ is defined with an AMH copula as in (2.18). The ruin probability is calculated for several values of initial capital and different values of dependence parameter α. Results are presented in Figure 2.4 from which we can see that for a small dependency parameter, the ruin probability tends to zero. Otherwise, the greater the parameter becomes, the more likely the probability of ruin tends to a value close to 0.35. Once again, we emphasize the significant impact of a low to moderate dependence relation between rvs on the overall portfolio.

In the following example we provide an analytical expression of ζ for a specific loss distribution.

Example 2.6.2. *Let X be a non-negative discrete rv with*

$$f_X(0) = (1-\delta) \quad \textit{and} \quad f_X(2) = \delta, \quad k \in \mathbb{N}, \; \delta \in (0, 0.5).$$

Assume that $F_{X_1,...,X_j}$ is defined by (2.18) with copula C. Using (2.22), we find

$$f_{X|\Theta=\theta}(0) = 1 - \delta_\theta = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}(f_X(0))}.$$

From Example 3.1 of Willmot (1993), we find

$$\zeta_\theta(u) = \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1}, \tag{2.47}$$

for $u \in \mathbb{N}_0$ and $\theta = 1, 2, ..., \theta_0$. Replacing (2.47) in (2.46), the expression $\zeta(u)$ becomes

$$\zeta(u) = \sum_{\theta=1}^{\theta_0} f_\Theta(\theta) \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1} + \overline{F}_\Theta(\theta_0), \; u \in \mathbb{N}_0.$$

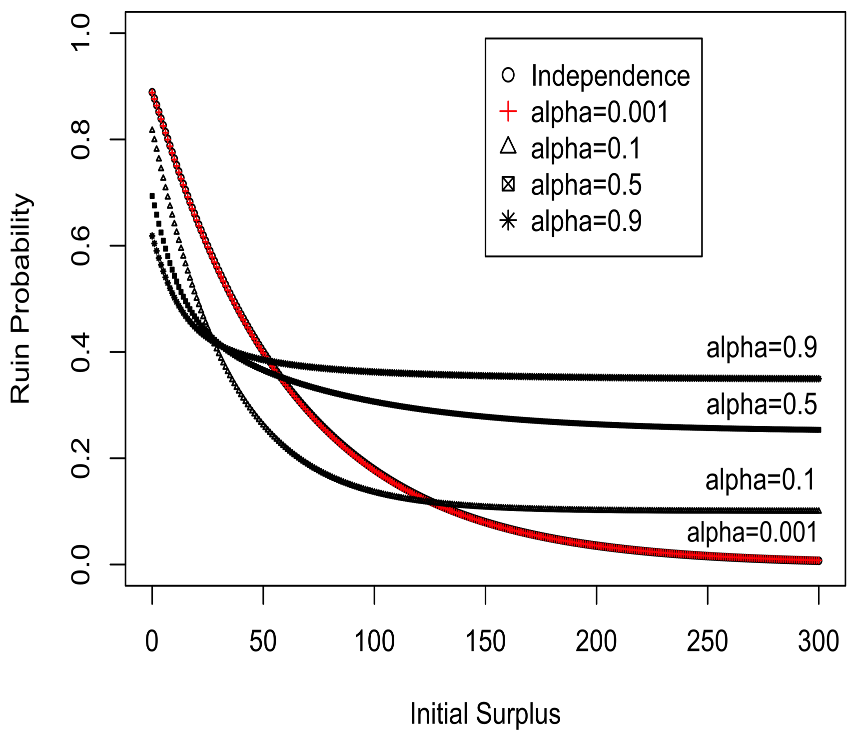


Figure 2.4 – Ultimate ruin probability for various degrees of dependence.

For illustration purposes, we fix the different parameters as follows: $\delta = 0.4$, *implying* $E[X] = 0.8 < 1$. *Finally,* $F_{X_1,\ldots,X_j}$ *is defined by (2.18) where* C *is a Frank copula with* $\alpha = 4$. *We obtain that* $\theta_0 = 9$ *and*

$$\zeta(u) = \sum_{\theta=1}^{9} \frac{(1-\mathrm{e}^{-4})^{\theta}}{4\,\theta} \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1} + \overline{F}_\Theta(9)\,,\; u \in \mathbb{N}_0,$$

where the values of δ_θ *are given in the following table:*

θ	1	2	3	4	5	6	7	8	9
δ_θ	0.92625	0.85793	0.79466	0.73604	0.68176	0.63148	0.58491	0.54177	0.50181

2.7 Partially nested Archimedean copulas

In the present section, we generalize the proposed strategy of Section 2.2 and adapt it to hierarchical structures for which at least one of the arguments is an Archimedean copula. More precisely, we consider in detail nested Archimedean copulas. Since there are several ways of nesting copulas and the computational methodology based on the mixing representation is the same for any chosen structure, we will only present a detailed example where we apply the strategy to a partially nested Archimedean copula. See Joe (1997), McNeil (2008) and Hofert (2011) for a general introduction to nested Archimedean copulas.

Let C be a one level partially nested Archimedean copula with d children, i.e., C is of the form

$$\begin{aligned} C(\underline{u}) &= C(\underline{u};\psi_0,\psi_1,...,\psi_n) \\ &= C\left(C\left(\underline{u_1};\psi_1\right),\ldots,C\left(\underline{u_d};\psi_d\right);\psi_0\right) \\ &= C\left(C\left(u_{1,1},\ldots,u_{1,n_1};\psi_1\right),\ldots,C\left(u_{d,1},\ldots,u_{d,n_d};\psi_d\right);\psi_0\right), \end{aligned} \tag{2.48}$$

with $\underline{u} = (\underline{u_1},\ldots,\underline{u_d})$, where $\underline{u_i} = (u_{i,1},\ldots,u_{i,n_i})$ for $i = 1,\ldots,d$.

An example of the partially nested Archimedean copula with arguments $\underline{u}$ is defined as follows in terms of a multivariate parent copula of dimension d and d child copulas, where the dimension of the i^{th} copula is denoted by n_i:

$$\begin{aligned} C(\underline{u}) &= C(\underline{u};\psi_0,\psi_1,...,\psi_d) \\ &= C\left(C\left(\underline{u_1};\psi_1\right),...,C\left(\underline{u_d};\psi_d\right);\psi_0\right) \\ &= C\left(C\left(u_{1,1},...,u_{1,n_1};\psi_1\right),...,C\left(u_{d,1},...,u_{d,n_d};\psi_d\right);\psi_0\right) \\ &= \psi_0\left(\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right)\right)\right)\right). \end{aligned} \tag{2.49}$$

Since ψ_0 is the LST of a strictly positive rv Θ_0, (2.49) develops into

$$\begin{aligned} C(\underline{u}) &= \int_0^{\infty} \mathrm{e}^{-\theta_0\times\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)\right)}\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \\ &= \int_0^{\infty}\prod_{i=1}^{d}\mathrm{e}^{-\theta_0\times\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)\right)}\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \end{aligned}$$

which becomes

$$\begin{aligned} C(\underline{u}) &= \int_0^{\infty}\prod_{i=1}^{d}\psi_{0,i}\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right);\theta_0\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \\ &= \int_0^{\infty}\prod_{i=1}^{d}\left(\int_0^{\infty}\prod_{j=1}^{n_i}\mathrm{e}^{-\theta_{0,i}\times\psi_i^{-1}(u_{i,j})}\mathrm{d}F_{\Theta_{0,i}}\left(\theta_{0,i}\right)\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right), \end{aligned}$$

where $\psi_{0,i}(t) = \mathrm{e}^{-\theta_0\times\psi_0^{-1}\circ\psi_i(t)}$. As mentioned notably in Hofert (2010), $\psi_0^{-1}\circ\psi_i$ must be completely monotone in order to verify the nesting condition.

Let $\underline{k} = (k_{1,1}, \ldots, k_{1,n_1}, \ldots, k_{d,1}, \ldots, k_{d,n_d})$. As in (2.18), let the multivariate distribution of $\underline{X} = (\underline{X_1}, \ldots, \underline{X_d})$ with $\underline{X_i} = (X_{i,1}, \ldots, X_{i,n_i})$, for $i = 1, \ldots, d$ be defined in terms of its joint cdf as follows

$$\begin{aligned} F_{\underline{X}}(\underline{k}h) &= C\left(F_{X_{1,1}}(k_{1,1}h), \ldots, F_{X_{1,n_1}}(k_{1,n_1}h), \ldots, F_{X_{d,1}}(k_{d,1}h), \ldots, F_{X_{d,n_d}}(k_{d,n_d}h)\right) \\ &= \int_0^\infty \prod_{i=1}^{d} \left(\int_0^\infty \prod_{j=1}^{n_i} F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h)\, \mathrm{d}F_{\Theta_{0,i}}(\theta_{0,i}) \right) \mathrm{d}F_{\Theta_0}(\theta_0), \qquad (2.50) \end{aligned}$$

where $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}) = \mathrm{e}^{-\theta_{0,i} \times \psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$ for $j = 1, 2, \ldots, n_i$ and $i = 1, \ldots, d$. Note that the multivariate distribution of $\underline{X}$ can also be defined with its joint survival function as in (2.19).

To apply our methodology in a risk aggregation context, we need to assume that we can identify the distribution of the rv $\Theta_{0,i}$ from $\psi_{0,i}$, i.e., we should be able to write $\psi_{0,i}(t) = \mathcal{L}_{\Theta_{0,i}}(t)$. Also, we assume that $\Theta_0, \Theta_{0,1}, ..., \Theta_{0,d}$ are strictly positive discrete rvs defined on $\mathbb{N}$ with pmf $f_{\Theta_{0,i}}(\theta_{0,i}) = \Pr(\Theta_{0,i} = \theta_{0,i})$ and cdf $F_{\Theta_{0,i}}(\theta_{0,i}) = \Pr(\Theta_{0,i} \leq \theta_{0,i}) = \sum_{j=1}^{\theta_{0,i}} f_{\Theta_{0,i}}(j)$ for $\theta_{0,s} \in \mathbb{N}$ and $i = 1, 2, \ldots, d$. Then, (2.50) becomes

$$F_{\underline{X}}(\underline{k_1}h, ..., \underline{k_d}h) = \sum_{\theta_0=1}^{\infty} \prod_{i=1}^{d} \left(\sum_{\theta_{0,i}=1}^{\infty} \prod_{j=1}^{n_i} F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h)\, f_{\Theta_{0,i}}(\theta_{0,i}) \right) f_{\Theta_0}(\theta_0). \quad (2.51)$$

We define $S = \sum_{i=1}^{d} \sum_{j=1}^{n_i} X_{i,j} = \sum_{i=1}^{d} S_i$ where $S_i = \sum_{j=1}^{n_i} X_{i,j}$, for $i = 1, \ldots, d$. Then, similarly to (2.27), we have

$$f_{S_i|\Theta_0=\theta_0}(kh) = \sum_{\theta_{0,i}=1}^{\infty} f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(kh)\, f_{\Theta_{0,i}}(\theta_{0,i}), \text{for } i = 1, \ldots, d,$$

and

$$f_S(kh) = \sum_{\theta_0=1}^{\infty} f_{S|\Theta_0=\theta_0}(kh)\, f_{\Theta_0}(\theta_0),$$

where

$$(S|\Theta_0 = \theta_0) = \sum_{i=1}^{d} (S_i|\Theta_0 = \theta_0),$$

and

$$(S|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i}) = \sum_{j=1}^{n_i} (X_{i,j}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i}).$$

Note that $(S_1|\Theta_0 = \theta_0)$, ..., $(S_d|\Theta_0 = \theta_0)$ are conditionally independent. Within each class $i = 1, \ldots, d$, $(X_{i,1}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$,...,$(X_{i,n_i}|\Theta_0 = \theta_0, \Theta_{0,s} = \theta_{0,i})$ are also conditionally independent. It implies that

$$f_{S|\Theta_0=\theta_0}(kh) = f_{S_1|\Theta_0=\theta_0} * \ldots * f_{S_d|\Theta_0=\theta_0}(kh), \qquad (2.52)$$

and

$$f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(kh) = f_{X_{i,1}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}} * \dots * f_{X_{i,n_i}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(kh), \tag{2.53}$$

for $k \in \mathbb{N}_0$ and $i = 1, \dots, d$ and where "$*$" denotes the convolution product. Values of (2.52) and (2.53) are computed using the same tools (e.g. DePril, FFT, etc) as the ones mentioned in Section 2.2.

Algorithm 2.7.1. *Let $\theta_0 \in \{1, 2, ..., \theta_0^*\}$.*

1. *Begin with $\theta_0 = 1$.*
2. *For each child copula C_i where $i \in \{1, \dots, d\}$, let $\theta_{0,i} \in \left\{1, 2, ..., \theta_{0,i}^*\right\}$ and proceed as follows:*
    a) *Begin with $\theta_{0,i} = 1$.*
    b) *For $j = 1, ..., n_i$, calculate $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) = \mathrm{e}^{-\theta_{0,i}\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$, for $k_{i,j} \in \mathbb{N}_0$.*
    c) *For $j = 1, \dots, n_i$, calculate*
    $$f_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) = \begin{cases} \mathrm{e}^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)} & ,k_{i,j} = 0 \\ \mathrm{e}^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)} - \mathrm{e}^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}((k_{i,j}-1)h)\right)} & ,k_{i,j} \in \mathbb{N} \end{cases}.$$
    d) *Using e.g. FFT or DePril's Algorithm, compute $f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_i h)$ for $k_i \in \mathbb{N}_0$.*
    e) *Repeat steps (2b), (2c), and (2d) for $\theta_{0,i} = 2, ..., \theta_{0,i}^*$ where $\theta_{0,i}^*$ is the largest integer such that $F_{\Theta_{0,i}}\left(\theta_{0,i}^*\right) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).*
    f) *Compute $f_{S_i|\Theta_0=\theta_0}(k_i h) = \sum_{\theta_{0,i}=1}^{\theta_{0,s}^*} f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_i h)\, f_{\Theta_{0,i}}(\theta_{0,i})$, for $k_i \in \mathbb{N}_0$.*
3. *Convolute all $f_{S_i|\Theta=\theta}$ for $i = 1, \dots, d$, to calculate $f_{S|\Theta=\theta}$.*
4. *Repeat steps (2) and (3) for $\theta_0 = 2, ..., \theta_0^*$ where θ_0^* is chosen such that $F_{\Theta_0}(\theta_0^*) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).*
5. *Compute $f_S(kh) = \sum_{\theta_0=1}^{\theta_0^*} f_{S|\Theta_0=\theta}(kh)\, f_{\Theta_0}(\theta_0)$, for $k \in \mathbb{N}_0$.*

As mentioned notably in Hofert (2010), the difficult task is to identify the distributions of $\Theta_{0,1}, \dots, \Theta_{0,d}$. For example, if we assume that $C_{\alpha_0}, C_{\alpha_1}, \dots, C_{\alpha_d}$ are AMH copulas, with respective parameters $\alpha_0, \alpha_1, \dots, \alpha_d$ (with $\alpha_0 < \min(\alpha_1, \dots, \alpha_d)$), then, it implies that $\Theta_0 \sim \mathrm{Geo}(1-\alpha_0)$ and $\Theta_{0,i} \sim$ Shifted $\mathrm{NB}(\alpha_0, \frac{1-\alpha_i}{1-\alpha_0})$, for $i = 1, \dots, d$, as shown in Hofert (2010) (see details in Appendix).

In the following example, we consider a two-dimensional partially nested Archimedean copula as defined in (2.48), where all copulas involved are AMH copulas. The example illustrates the accuracy of the proposed strategy in comparison to the MC simulation method.

Example 2.7.1. *Consider a portfolio of 80 risks* $\underline{X} = (X_{1,1}, ..., X_{1,40}, X_{2,1}, ..., X_{2,40})$ *with multivariate cdf defined as in (2.51) with* $d = 2$ *and* $n_1 = n_2 = 40$. *Assume* C_{α_0}, C_{α_1} *and* C_{α_2} *to be AMH copulas with* $\alpha_0 = 0.2$, $\alpha_1 = 0.3$, *and* $\alpha_2 = 0.4$. *Let* $X_{i,j} \sim Bin(10, q_{i,j})$ *where* $q_{i,j} = 0.05 \times i + 0.005j$, $i = 1, 2$ *and* $j = 1, 2, ..., 40$. *It implies that* $E[S] = 142$. *Relevant measures of* $S = \sum_{i=1}^{2} \sum_{j=1}^{40} X_{i,j}$ *can be obtained with Algorithm 2.7.1 or with MC simulations. Values of the expectation, the variance, the VaR and the TVaR for both methods in addition to their confidence intervals (with a confidence level of* 95%*) are given in Table 2.22. We can also calculate the exact values of Pearson's correlation coefficient between different risks. For example* $\rho_P(X_{1,1}, X_{1,2}) = 0.07548428$, $\rho_P(X_{2,1}, X_{2,2}) = 0.12173402$ *and* $\rho_P(X_{1,1}, X_{1,2}) = 0.05336731$. *Note that the simulation results (1 million simulations) are very close to the exact values obtained with the proposed approach.*

	Exact values	Simulated values	$IC_{0.05}$
$E[S]$	142	141.9887	[141.9304 ; 142.0469]
$Var(S)$	883.6003	883.3098	[880.8666 ; 885.7633]
$VaR_{0.5}(S)$	133	133	[133 ; 134]
$VaR_{0.9}(S)$	186	186	[186 ; 186]
$VaR_{0.99}(S)$	225	225	[225 ; 226]
$VaR_{0.999}(S)$	250	249	[248 ; 250]
$VaR_{0.9999}(S)$	267	267	[266 ; 268]
$TVaR_{0.5}(S)$	165.3440	165.3444	[165.3386 ; 165.3984]
$TVaR_{0.9}(S)$	204.2611	204.4585	[204.4295 ; 204.4874]
$TVaR_{0.99}(S)$	236.2996	236.3021	[236.2140 ; 236.3066]
$TVaR_{0.999}(S)$	257.3535	257.5276	[257.1387 ; 257.5444]
$TVaR_{0.9999}(S)$	273.0259	273.5895	[272.7160 ; 273.805]

Table 2.22 – Values of the expectation, variance, VaR and TVaR of $S = X_{1,1} + ... + X_{1,40} + X_{2,1} + ... + X_{2,40}$ where the joint cdf $F_{X_{1,1},...,X_{1,40},X_{2,1},...,X_{2,40}}$ is as defined in Example 2.7.1.

In the following example, we present a specific five-dimensional partially nested Archimedean copula with two nesting levels.

Example 2.7.2. *Assume a multivariate cdf of* $\underline{X} = (X_1, X_2, X_3, X_4, X_5)$ *defined with binomial marginals and the following five-dimensional partially nested Archimedean copula:*

$$C(u_1, u_2, u_3, u_4, u_5) = C_{\alpha_0}(C_{\alpha_1}(u_1, u_2), C_{\alpha_2}(u_3, C_{\alpha_3}(u_4, u_5))),$$

where C_{α_j}, $j = 0, 1, 2, 3$, *correspond to bivariate AMH copulas. Also,* $X_i \sim Bin(10, 0.05i)$, $i = 1, 2, ..., 5$, *and* $\alpha_j = 0.1j + 0.2$, $j = 0, 1, 2, 3$. *Let* $S = \sum_{i=1}^{5} X_i$ *which implies* $E[S] = 7.5$. *In Table 2.23, we provide the exact and the simulated (10 million MC simulations) values of* $f_S(k)$, $k = 0, 1, 2, ..., 50$ *in addition to their confidence intervals (with a confidence level of* 95%*). We noticed a fluctuation in the results for several distinct 10 million simulation paths, contrarily to our proposed approach.*

k	$f_S(k)$ (exact values)	$f_S(k)$ (simulated values)	$\text{IC}_{0.05}$
0	0.000808	0.000805	[0.000786 ; 0.000821]
1	0.005795	0.005785	[0.005714 ; 0.005808]
2	0.020111	0.020082	[0.019991 ; 0.020165]
3	0.045814	0.045699	[0.045598 ; 0.045856]
4	0.078337	0.078477	[0.078137 ; 0.078470]
5	0.108726	0.108730	[0.108657 ; 0.109043]
10	0.086310	0.086248	[0.086074 ; 0.086421]
15	0.006728	0.006732	[0.006667 ; 0.006768]

Table 2.23 – Values of the pmf of $S = \sum_{i=1}^{5} X_i$ where the multivariate cdf of $(X_1, X_2, X_3, X_4, X_5)$ is as defined in Example 2.7.2.

	Exact values	Simulated values	$\text{IC}_{0.05}$
$E[S]$	7.5	7.49949	[7.49771 ; 7.50128]
$Var(S)$	8.31314	8.31013	[8.30286 ; 8.31742]
$VaR_{0.5}(S)$	7	7	[7 ; 7]
$VaR_{0.9}(S)$	11	11	[11 ; 11]
$VaR_{0.99}(S)$	15	15	[15 ; 15]
$VaR_{0.999}(S)$	17	17	[17 ; 17]
$VaR_{0.9999}(S)$	19	19	[19 ; 19]
$TVaR_{0.5}(S)$	9.81112	9.81008	[9.80771 ; 9.81239]
$TVaR_{0.9}(S)$	12.85623	12.85600	[12.85160 ; 12.86020]
$TVaR_{0.99}(S)$	15.75489	15.75197	[15.744530 ; 15.759540]
$TVaR_{0.999}(S)$	17.89402	17.89070	[17.86590 ; 17.91640]
$TVaR_{0.9999}(S)$	19.72388	19.72100	[19.6540 ; 19.7920]

Table 2.24 – Values of expectation, variance, VaR and TVaR of $S = \sum_{i=1}^{5} X_i$ as defined in Example 2.7.2.

2.8 Appendix – Archimedean copulas

Archimedean copulas defined with a strictly positive discrete mixing rv Θ:

1. Ali-Mikhail-Haq (AMH) family:

   — Copula: $C_\alpha(u_1, ..., u_n) = (1-\alpha)\left(\prod_{i=1}^{n}\left((1-\alpha)u_i^{-1}+\alpha\right)-\alpha\right)^{-1}$

   — Parameter: $\alpha \in [0,1)$

   — Discrete distribution for Θ: Shifted Geometric$(1-\alpha)$

   — Pmf: $f_\Theta(k) = \alpha^{k-1}(1-\alpha)$, $k \in \mathbb{N}$

   — LST: $\mathcal{L}_\Theta(t) = \frac{1-\alpha}{e^t-\alpha}$

   — Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = \ln\left(\frac{1-\alpha}{u}+\alpha\right)$

   — Particular cases: as $\alpha \to 0$, $C_\alpha(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$

2. Frank family:

— Copula: $C_\alpha(u_1, ..., u_n) = \frac{-1}{\alpha} \ln \left(1 - \frac{\left(1-e^{-u_1\alpha}\right)\times...\times\left(1-e^{-u_n\alpha}\right)}{\left(1-e^{-\alpha}\right)^{n-1}}\right)$

— Parameter: $\alpha \in (0, \infty)$

— Discrete distribution for Θ: Logarithmic$(1 - e^{-\alpha})$

— Pmf: $f_\Theta(k) = \frac{\left(1-e^{-\alpha}\right)^k}{k\alpha}$, $k \in \mathbb{N}$

— LST: $\mathcal{L}_\Theta(t) = -\frac{1}{\alpha} \ln\left(1 - \left(1 - e^{-\alpha}\right) e^{-t}\right)$

— Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = -\ln\left(\frac{1-e^{-\alpha u}}{1-e^{-\alpha}}\right)$

— Particular cases: $C_{\alpha\to 0}(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$ and $C_{\alpha\to\infty}(u_1, ..., u_n) = C^+(u_1, ..., u_n)$

3. Shifted Negative Binomial family:

— Copula: $C_{r,q}(u_1, ..., u_n) = \left(q\left(\prod_{i=1}^{n}\left(qu_i^{\frac{-1}{r}} + 1 - q\right) - (1-q)\right)^{-1}\right)^r$

— Parameter: $r \in \mathbb{R}^+$ and $q \in (0, 1)$

— Discrete distribution for Θ: Shifted Negative Binomial(r,q)

— $\Theta = M + r$ with $M \sim NB(r, q)$

— Pmf: $f_\Theta(k) = \binom{k-1}{k-r} q^r (1-q)^{k-r}$, $k = r, r+1, \ldots$

— LST: $\mathcal{L}_\Theta(t) = \left(\frac{qe^{-t}}{1-(1-q)e-t}\right)^r$

— Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = \ln\left(qu^{-\frac{1}{r}} + (1-q)\right)$

— Particular cases: as $r \to 0$ or $r \to \infty$, $C_{r,q}(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$. Also, $C_{1,q}(u_1, ..., u_n) = C_{1-q}^{AMH}(u_1, ..., u_n)$ and when $r \to 0$, $C_{0,0}(u_1, ..., u_n) = C^+(u_1, ..., u_n)$. The most important case is when $q \to 0$, $C_{r,0}(u_1, ..., u_n) = C_{1/r}^{Clay}(u_1, ..., u_n)$

Archimedean copulas defined with a strictly positive continuous mixing rv Θ:

1. Clayton family:

— Copula: $C_\alpha(u_1, ..., u_n) = \left(u_1^{-\alpha} + ... + u_n^{-\alpha} - (n-1)\right)^{-\frac{1}{\alpha}}$

— Parameter: $\alpha \in (0, \infty)$

— Continuous distribution for Θ: Gamma($\frac{1}{\alpha}$,1)

— LST: $\mathcal{L}_\Theta(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$

— Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = u^{-\alpha} - 1$

— Particular cases: $C_{\alpha\to 0}(u_1, ..., u_n) = C^\perp(u_1, ..., u_n)$ and $C_{\alpha\to\infty}(u_1, ..., u_n) = C^+(u_1, ..., u_n)$

2. Gumbel family:

— Copula: $C_\alpha(u_1, ..., u_n) = \exp\left(-\left((-\ln(u_1))^\alpha + ... + (-\ln(u_n))^\alpha\right)^{\frac{1}{\alpha}}\right)$

— Parameter: $\alpha \in [1, \infty)$

— Continuous distribution for Θ: Positive Stable$\left(\frac{1}{\alpha}, 1, \cos^{\alpha}\left(\frac{\pi}{2\alpha}\right), 1_{\{\alpha=1\}}\right)$

— LST: $\mathcal{L}_{\Theta}(t) = \mathrm{e}^{-t^{\frac{1}{\alpha}}}$

— Inverse of LST: $\mathcal{L}_{\Theta}^{-1}(u) = (-\ln(u))^{\alpha}$

— Particular cases: $C_{\alpha\to1}(u_1, ..., u_n) = C^{\perp}(u_1, ..., u_n)$ and $C_{\alpha\to\infty}(u_1, ..., u_n) = C^{+}(u_1, ..., u_n)$

2.9 Appendix – Nested Archimedean Copula

1. **The Nested AMH family**

— $\psi_{0s}(t) = \mathcal{L}_{\Theta_{0s}}(t) = \left(\frac{1-q_s}{(1-q_0)(\mathrm{e}^t-q_s)+q0(1-q_s)}\right)^{\Theta_0}$

— Distribution of Θ_{0s}: $(\Theta_{0s}|\Theta_0 = \theta) \sim SNB\left(\theta, \frac{1-q_s}{1-q_0}\right)$ with $q_0 \leq q_s$.

— pmf of Θ_{0s}: $f_{\Theta_{0s}|\Theta_0=\theta}(k) = \binom{k-1}{k-\theta}(q^*)^r(1-q^*)^{k-r}$ with $q^* = \frac{1-q_s}{1-q_0}$ and $k \in \{\theta, \theta+1, \cdots\}$

2. **The Nested Frank family**

— $\psi_{0s}(t) = \left(\frac{\left(1-\mathrm{e}^{-\alpha_s}\right)\mathrm{e}^{-s}}{1-\mathrm{e}^{-\alpha_0}}\right)^{\Theta_0}$

— Distribution of Θ_{0s}: $(\Theta_{0s}|\Theta_0 = \theta) \sim \sum_{i=1}^{\theta} V_i$, with $P(V_i = k) = p_k$ with $\alpha_0 \leq \alpha_s$

— with $p_k = \frac{(1-\mathrm{e}^{-\alpha_s})^k}{(1-\mathrm{e}^{-\alpha_s})^{\theta_0}} \sum_{j=0}^{\infty} \binom{\theta_0}{j}\binom{j\frac{\alpha_0}{\alpha_s}}{k}(-1)^{j+k}$ for $k \in \{1, 2, \cdots\}$.

2.10 Bibliography

Albrecher, H., Constantinescu, C., and Loisel, S. (2011). Explicit ruin formulas for models with dependence among risks. *Insurance: Mathematics and Economics*, 48(2):265–270.

Aldous, D. J. (1985). Exchangeability and related topics. In *École d'Été de Probabilités de Saint-Flour XIII-1983*, pages 1–198. Springer.

Balakrishnan, N. (2006). *Continuous multivariate distributions*. Wiley Online Library.

Bargès, M., Cossette, H., and Marceau, E. (2009). Tvar-based capital allocation with copulas. *Insurance: Mathematics and Economics*, 45(3):348–361.

Bühlmann, H. (1960). *Austauschbare stochastische Variabeln und ihre Grenzwertsätze*. PhD thesis.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula methods in finance*. John Wiley & Sons.

Cossette, H., Gaillardetz, P., Marceau, É., and Rioux, J. (2002). On two dependent individual risk models. *Insurance: Mathematics and Economics*, 30(2):153–166.

Cossette, H., Landriault, D., and Marceau, É. (2003). Ruin probabilities in the compound markov binomial model. *Scandinavian Actuarial Journal*, 2003(4):301–323.

Cossette, H., Marceau, E., Mtalai, I., and Veilleux, D. (2018). A note on the univariate and multivariate mixed exponential distributions. *Submitted for publication*.

Dacorogna, M. M., Elbahtouri, L., and Kratz, M. (2016). Explicit diversification beneift for dependent risks.

De Finetti, B. (1957). Su un'impostazione alternativa della teoria collettiva del rischio. In *Transactions of the XVth international congress of Actuaries*, volume 2, pages 433–443.

De Vylder, F. and Marceau, E. (1996). Classical numerical ruin probabilities. *Scandinavian Actuarial Journal*, 1996(2):109–123.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial theory for dependent risks: measures, orders and models*. John Wiley & Sons.

Dickson, D. C. (1992). On the distribution of the surplus prior to ruin. *Insurance: Mathematics and Economics*, 11(3):191–207.

Fang, K.-T., Fang, H.-B., and Rosen, D. v. (2000). A family of bivariate distributions with non-elliptical contours. *Communications in Statistics-Theory and Methods*, 29(8):1885–1898.

Feller, W. (1971). Introduction to probability theory and its appl. ii. *New York*.

Frey, R. and McNeil, A. J. (2001). *Modelling dependent defaults*. ETH, Eidgenössische Technische Hochschule Zürich, Department of Mathematics.

Frey, R. and McNeil, A. J. (2002). Var and expected shortfall in portfolios of dependent credit risks: conceptual and practical insights. *Journal of banking & finance*, 26(7):1317–1334.

Furman, E. and Zitikis, R. (2008). Weighted risk capital allocations. *Insurance: Mathematics and Economics*, 43(2):263–269.

Genest, C. and Rivest, L.-P. (2001). On the multivariate probability integral transformation. *Statistics & Probability Letters*, 53(4):391–399.

Gerber, H. U. (1988). Mathematical fun with the compound binomial process. *Astin Bulletin*, 18(02):161–168.

Hofert, M. (2008). Sampling archimedean copulas. *Computational Statistics & Data Analysis*, 52(12):5163–5174.

Hofert, M. (2010). *Sampling nested Archimedean copulas with applications to CDO pricing*. PhD thesis, Universität Ulm.

Hofert, M. (2011). Efficiently sampling nested archimedean copulas. *Computational Statistics & Data Analysis*, 55(1):57–70.

Hofert, M. (2012). A stochastic representation and sampling algorithm for nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 82(9):1239–1255.

Hofert, M., Mächler, M., and Mcneil, A. J. (2012). Likelihood inference for archimedean copulas in high dimensions under known margins. *Journal of Multivariate Analysis*, 110:133–150.

Hofert, M. and Scherer, M. (2011). Cdo pricing with nested archimedean copulas. *Quantitative Finance*, 11(5):775–787.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts*. CRC Press.

Joe, H. (2014). *Dependence modeling with copulas*. CRC Press.

Joe, H. and Hu, T. (1996). Multivariate distributions from mixtures of max-infinitely divisible distributions. *Journal of multivariate analysis*, 57(2):240–265.

Jordanova, P., Nefedova, Y., and Stehlík, M. (2017). Risk process approximation with mixing. *Applied Mathematical Modelling*, 41:285–298.

Kroese, D. P., Taimre, T., and Botev, Z. I. (2013). *Handbook of monte carlo methods*, volume 706. John Wiley & Sons.

Lemieux, C. (2009). *Monte Carlo and Quasi-Monte Carlo Sampling*. Springer Series in Statistics. Springer, New York.

Lewin, L. (1981). Polylogarithms and associated functions.

Li, D. X. (2000). On default correlation: A copula function approach. *The Journal of Fixed Income*, 9(4):43–54.

Marceau, E. (2013). *Modelisation et evaluation quantitative des risques en actuariat: Modèles sur une période*. Springer, 1st edition.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841.

Marshall, A. W. and Olkin, I. (2007). *Life distributions: Structure of nonparametric, semiparametric, and parametric families*. Springer Science & Business Media.

McNeil, A. J. (2008). Sampling nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 78(6):567–581.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative risk management: Concepts, techniques and tools*. Princeton university press.

McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools*. Princeton university press.

McNeil, A. J. and Nešlehová, J. (2009). Multivariate archimedean copulas, d-monotone functions and l1-norm symmetric distributions. *The Annals of Statistics*, pages 3059–3097.

Müller, A. and Stoyan, D. (2002). *Comparison methods for stochastic models and risks*, volume 389. Wiley.

Nelsen, R. B. (2007). *An introduction to copulas*. Springer Science & Business Media.

Panjer, H. H., Klugman, S., and Willmot, G. (2008). Loss models: From data to decisions.

Rosen, D., Saunders, D., Bielecki, T. R., Brigo, D., and Patras, F. (2011). Credit risk contributions. *Credit Risk Frontiers: Subprime Crisis, Pricing and Hedging, CVA, MBS, Ratings, and Liquidity*, pages 675–720.

Sarabia, J. M., Gómez-Déniz, E., Prieto, F., and Jordá, V. (2017). Aggregation of dependent risks in mixtures of exponential distributions and extensions. *arXiv preprint arXiv:1705.00289*.

Schönbucher, P. (2002). Taken to the limit: simple and not-so-simple loan loss distributions.

Schönbucher, P. J. and Schubert, D. (2001). Copula-dependent default risk in intensity models. In *Working paper, Department of Statistics, Bonn University*. Citeseer.

Shiu, E. S. (1989). The probability of eventual ruin in the compound binomial model. *Astin Bulletin*, 19(2):179–190.

Tasche, D. (1999). Risk contributions and performance measurement. *Report of the Lehrstuhl für mathematische Statistik, TU München.*

Vasicek, O. A. (1987). *Probability of loss on loan portfolio.* Wiley Online Library.

Willmot, G. E. (1993). Ruin probabilities in the compound binomial model. *Insurance: Mathematics and Economics*, 12(2):133–142.

Chapitre 3

Hierarchical Archimedean Copulas Through Multivariate Compound Distributions

Résumé

Dans cet article, on introduit une nouvelle méthode de construction de copules Archimédiennes imbriquées impliquant des distributions composées multivariées. L'absence de toute condition d'imbrication élargit les possibilités de copules imbriquées en améliorant la flexibilité à la fois dans le choix des familles et des paramètres. De plus, à l'aide de cette méthode, on construit de nouvelles copules à plusieurs paramètres, ce qui permet de couvrir une grande variété de structures de dépendance. Outre ces nouvelles copules, plusieurs autres copules archimédiennes connues peuvent être obtenues comme cas particuliers. Une représentation basée sur les mélanges communs permet d'effectuer des comparaisons avec des copules Archimédiennes imbriquées. Finalement, on examine l'agrégation des risques ainsi que la technique de simulation sous cette nouvelle méthode de construction. Des exemples sont fournis à titre d'illustration.

Abstract

In this paper, we propose a new hierarchical Archimedean copula construction based on multivariate compound distributions. This new imbrication technique is derived via the construction of a multivariate exponential mixture distribution through compounding. The absence of nesting and marginal conditions, contrarily to the nested Archimedean copulas approach, leads to major advantages, such as a flexible range of possible combinations in the choice of distributions, the existence of explicit formulas for the distribution of the sum, and computational ease in high dimensions. A balance between flexibility and parsimony is targeted. After presenting the construction technique, properties of the proposed copulas are investigated and illustrative examples are given. A detailed comparison with other construction methodologies of hierarchical Archimedean copulas is provided. Risk aggregation under this newly proposed dependence structure is also examined.

Keywords: Archimedean copulas, Mixing random variables, Compounding, Marshall-Olkin, Hierarchical structure.

3.1 Introduction

Copulas are now well known tools used for dependence modeling purposes in many research topics. A d-dimensional copula is a d-variate probability distribution function for which the marginals are uniformly distributed on $(0,1)$, with $d \geq 2$. One important class of copulas is the Archimedean copula family, popular for its simple construction procedure and multivariate generalization. A d-dimensional copula C is said to be an *Archimedean copula* if

$$C(u_1,\ldots,u_d) = \psi\left(\psi^{-1}(u_1)+\ldots+\psi^{-1}(u_d)\right), \text{ for } (u_1,\ldots,u_d) \in [0,1]^d. \tag{3.1}$$

The continuous and strictly decreasing function ψ is called the generator of the copula, where $\psi : [0,\infty) \to [0,1]$, $\psi(0) = 1$ and $\lim_{t\to\infty} \psi(t) = 0$. In the same manner, $\psi^{-1} : [0,1] \to [0,\infty)$, for which $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$, where ψ^{-1} is the inverse of the generator ψ. The set of all such functions is denoted by Ψ_∞. In fact, from Kimberling's and Bernstein's theorems, see e.g. Kimberling (1974), Feller (1971), and Hofert (2010), representation (3.1) leads to a proper copula for all $d \geq 2$ if and only if ψ is the Laplace-Stieltjes Transform (LST) of a strictly positive random variable (rv) Θ with cumulative distribution function (cdf) F_Θ, where the LST of the rv Θ is given by

$$\mathcal{L}_\Theta(t) = \int_0^\infty \mathrm{e}^{-t\,x}\,\mathrm{d}F_\Theta(x) = E\left[\mathrm{e}^{-t\Theta}\right]. \tag{3.2}$$

The use of multivariate Archimedean copulas in high dimension can be restrictive due to their exchangeability property. One can resort to another interesting class of copulas, namely vine copulas (see e.g. Bedford and Cooke (2002) and Joe (1997)). They are pair copula constructions allowing a cascade decomposition of a multivariate distribution into the product of bivariate copulas. Vine copulas force the use of $\frac{d(d-1)}{2}$ bivariate copulas which requires a high number of parameters when d, the dimension of the copula, increases.

Hierarchical Archimedean copulas provide an interesting alternative to allow asymmetries. The first approach to construct hierarchical Archimedean copulas was proposed by Joe (1997) who introduced the so-called nested Archimedean copulas in three and four dimensions. They are obtained by nesting into each other Archimedean copulas. They are able to capture different dependence relations between and within different groups of risks with a relatively small number of parameters (see e.g. Górecki et al. (2016a)). They were further studied by e.g. McNeil (2008), Hofert (2012), and Hofert (2011), in a general setting. For an Archimedean hierarchical structure to be a proper copula, a given nesting condition must be verified. For example, a 3-dimensional fully nested Archimedean copula can be written as

$$C(u_1,u_2,u_3) = C(u_1, C(u_2,u_3)) = \psi_0\Big(\psi_0^{-1}(u_1) + \psi_0^{-1} \circ \psi_1\big(\psi_1^{-1}(u_2)+\psi_1^{-1}(u_3)\big)\Big).$$

For the hierarchical structure to be a proper copula, $\psi_0^{-1} \circ \psi_1$ must have completely monotone derivatives, where ψ_0 and ψ_1 are generators of the parent and the child copulas respectively.

If generators ψ_0 and ψ_1 belong to the same family, the verification of this sufficient nesting condition can be done without much problem, see Hofert (2010) for restrictions on the parameters. However, if generators ψ_0 and ψ_1 of different Archimedean families are combined, the sufficient condition may not hold for any choice of parameters. Also, sampling from these copulas can be difficult.

To circumvent these constraints, other approaches were proposed to build hierarchical Archimedean copulas. Hering et al. (2010) suggest to build hierarchical Archimedean copulas based on Lévy subordinators. Also, Brechmann (2014) used the Kendall distribution function, the multivariate analog to the probability integral transform for univariate random variables, as a tool to construct hierarchical Archimedean copulas.

In this paper, our objective is to propose an alternative approach to build hierarchical Archimedean copulas via multivariate compound distributions. The absence of nesting and marginal conditions leads to major advantages, such as a flexible range of possible combinations in the choice of distributions, the existence of explicit formulas for the distribution of the sum, and computational ease in high dimensions. The approach given here has similarities with the one of Hering et al. (2010). Both approaches derive hierarchical Archimedean copulas from the joint survival function of a multivariate mixed exponential distribution. Also, sampling algorithms are obtained in a similar fashion. The probabilistic arguments at the basis of these two construction strategies however differ. In our opinion, the proposed approach uses simpler mathematical tools to obtain hierarchical Archimedean copulas. We provide in Section 3.4.2 a detailed comparison of both constructions. Here, random sums are key elements compared to Lévy processes in Hering et al. (2010).

The following sections are organized as follows: in Section 3.2, we provide the steps for the construction of a multivariate copula with the use of multivariate compound distributions, followed by sampling algorithms, accompanied by relevant notations and properties. Section 3.3 proposes a representation of the copula by common mixtures. In Section 3.4, we compare the proposed approach with nested Archimedean copulas and Hering et al. (2010)'s construction method and provide a brief discussion on estimation procedures. Finally, Section 3.5 is devoted to risk aggregation.

3.2 Construction of hierarchical Archimedean copulas with multivariate compound distributions

As explained in Marshall and Olkin (1988), multivariate distributions can easily be constructed through the use of exponential mixtures. In this section, we propose an alternative method based on a probabilistic argument framework using multivariate compound distributions to build hierarchical Archimedean copulas. Note that the copula obtained under the proposed approach is represented as a specific function of LSTs of rvs and their inverse. To obtain this representation, we have recourse to rvs with multivariate compound distributions. For this reason and given the context, we will refer either directly to LSTs or to the rvs associated to them. The key element of our construction is that the latter are defined with a vector of mixing rvs, denoted by $\underline{\Theta}$, which follows a multivariate compound distribution. We examine the properties of this new copula and provide simulation algorithms.

3.2.1 Multivariate compound distributions

Let $\underline{M} = (M_1, \ldots, M_d)$ be a vector of discrete strictly positive rvs with joint multivariate probability generating function (pgf) given by

$$\mathcal{P}_{\underline{M}}(t_1, \ldots, t_d) = E\left[t_1^{M_1} \ldots t_d^{M_d}\right], \tag{3.3}$$

and marginal pgfs given by

$$\mathcal{P}_{M_i}(t_i) = E\left[t_i^{M_i}\right], (i = 1, \ldots, d), \tag{3.4}$$

where $|t_1|, \ldots, |t_d| \leq 1$.

A vector of rvs $\underline{\Theta} = (\Theta_1, \ldots, \Theta_d)$ is said to follow a multivariate compound distribution if each component Θ_i can be represented as a random sum i.e. $\Theta_i = \sum_{j=1}^{M_i} B_{i,j}$, where $i = 1, \ldots, d$ for $d \geq 2$. For each i, the elements of the sequence $\underline{B_i} = \{B_{i,j}, j = 1, 2, \ldots\}$ are assumed independent and identically distributed (iid) strictly positive rvs, while the sequences are independent from each other and from the vector of rvs $\underline{M}$. By convention, for each $i = 1, \ldots, d$, $B_{i,j} \sim B_i$ with cdf F_{B_i} and LST $\mathcal{L}_{B_i}$ $(j = 1, 2, \ldots)$.

Since the rv Θ_i is defined as a random sum, its LST is given by

$$\mathcal{L}_{\Theta_i}(t) = E\left[e^{-t\Theta_i}\right] = E_{M_i}\left[E\left[e^{-t\Theta_i}|M_i\right]\right] = E_{M_i}\left[E\left[e^{-tB_i}\right]^{M_i}\right] = \mathcal{P}_{M_i}\left(\mathcal{L}_{B_i}(t)\right), \tag{3.5}$$

for $i = 1, \ldots, d$.

Similarly, the multivariate LST of the vector of mixing rvs $\underline{\Theta}$ is given by

$$\begin{aligned}
\mathcal{L}_{\underline{\Theta}}(t_1, \ldots, t_d) &= E\left[e^{-t_1\Theta_1} \ldots e^{-t_d\Theta_d}\right] \\
&= E_{\underline{M}}\left[E\left[e^{-t_1\Theta_1} \ldots e^{-t_d\Theta_d} |\underline{M}\right]\right] \\
&= E_{\underline{M}}[(E[e^{-tB_1}]^{M_1}) \ldots (E[e^{-tB_d}]^{M_d})]. \qquad (3.6)
\end{aligned}$$

With (3.6) and (3.3), the expression for the multivariate LST of $\underline{\Theta}$ is given by

$$\mathcal{L}_{\underline{\Theta}}\left(t_1, \ldots, t_d\right) = \mathcal{P}_{\underline{M}}\left(\mathcal{L}_{B_1}\left(t_1\right) \ldots \mathcal{L}_{B_d}\left(t_d\right)\right). \tag{3.7}$$

In this paper, we consider the specific case of multivariate compound distribution for $\underline{\Theta}$ assuming that $M_i = M$ $(i = 1, \ldots, d)$, where M is a strictly positive discrete rv with

$$\mathcal{P}_M(t) = E\left[t^M\right]. \tag{3.8}$$

In other words, the components of $\underline{M}$ are comonotonic and identically distributed (with $M_i \sim M$, $i = 1, \ldots, d$). Then, it means that the vector of mixing rvs $\underline{\Theta}$ follows a multivariate compound distribution and its multivariate LST in (3.7) becomes

$$\begin{aligned}
\mathcal{L}_{\underline{\Theta}}\left(t_1, \ldots, t_d\right) &= E\left[e^{-t_1\Theta_1} \ldots e^{-t_d\Theta_d}\right] \\
&= E_M\left[E\left[e^{-t_1\Theta_1} \ldots e^{-t_d\Theta_d} \,|M\right]\right] \\
&= E_M\left[E\left[e^{-t_1\Theta_1} \,|M\right] \ldots E\left[e^{-t_d\Theta_d} \,|M\right]\right] \\
&= E_M\left[\left(E\left[e^{-t_1B_1}\right] \ldots E\left[e^{-t_dB_d}\right]\right)^M\right].
\end{aligned} \tag{3.9}$$

Combining (3.9) and (3.8) leads to the following expression for the multivariate LST of $\underline{\Theta}$:

$$\mathcal{L}_{\underline{\Theta}}\left(t_1, \ldots, t_d\right) = \mathcal{P}_M\left(\mathcal{L}_{B_1}\left(t_1\right) \ldots \mathcal{L}_{B_d}\left(t_d\right)\right). \tag{3.10}$$

Also, for each $i = 1, \ldots, d$, (3.5) becomes

$$\mathcal{L}_{\Theta_i}\left(t\right) = \mathcal{P}_M\left(\mathcal{L}_{B_i}\left(t\right)\right). \tag{3.11}$$

3.2.2 Multivariate mixed exponential distributions defined with multivariate compound distributions

Let $\underline{Y} = (Y_{1,1}, \ldots, Y_{1,n_1}, \ldots, Y_{d,1}, \ldots, Y_{d,n_d})$ be a vector of $n_1 + \ldots + n_d$ rvs which can be more conveniently represented as $\underline{Y} = (\underline{Y_1}, \ldots, \underline{Y_d})$, where $\underline{Y_i} = (Y_{i,1}, \ldots, Y_{1,n_i})$ is the vector of n_i rvs for the subgroup i $(i = 1, 2, \ldots, d)$. Given $\underline{\Theta} = \underline{\theta}$, where $\underline{\theta} = (\theta_1, \ldots, \theta_d)$, it is assumed that

$$(Y_{1,1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (Y_{1,n_1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (Y_{d,1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (Y_{d,n_d} \mid \underline{\Theta} = \underline{\theta})$$

are conditionally independent. The conditional distributions of the components of $\underline{Y_i}$ are only influenced by the component Θ_i of $\underline{\Theta}$, i.e. $(Y_{i,j} \mid \underline{\Theta} = \underline{\theta})$ is identically distributed as $(Y_{i,j} \mid \Theta_i = \theta_i)$, for $i = 1, \ldots, d$ and $j = 1, \ldots, n_i$. We assume that $(Y_{i,1} \mid \Theta_i = \theta_i), \ldots, (Y_{i,n_i} \mid \Theta_i = \theta_i)$ are exponentially distributed with parameter θ_i for $i = 1, \ldots, d$. The univariate distribution of $Y_{i,j}$ is therefore a mixed exponential distribution with survival function given by

$$\overline{F}_{Y_{i,j}}\left(y_{i,j}\right) = \int_0^\infty \overline{F}_{Y_{i,j}|\Theta_i=\theta_i}(y_{i,j}) \,\mathrm{d}F_{\Theta_i}(\theta_i) = \int_0^\infty \mathrm{e}^{-y_{i,j}\,\theta_i} \,\mathrm{d}F_{\Theta_i}(\theta_i) = \mathcal{L}_{\Theta_i}\left(y_{i,j}\right), \tag{3.12}$$

for $i=1,\ldots,d$ and $j=1,\ldots,n_i$. The inverse of the survival function $\overline{F}_{Y_{i,j}}$ in (3.12) is given by

$$\overline{F}^{-1}_{Y_{i,j}}\left(u_{i,j}\right)=\mathcal{L}^{-1}_{\Theta_i}\left(u_{i,j}\right), \tag{3.13}$$

where

$$\mathcal{L}^{-1}_{\Theta_i}\left(u_{i,j}\right)=\mathcal{L}^{-1}_{B_i}\left(\mathcal{P}^{-1}_{M}\left(u_{i,j}\right)\right), \tag{3.14}$$

for $u_{i,j}\in[0,1]$, $i=1,\ldots,d$ and $j=1,\ldots,n_i$. Since

$$\mathcal{P}_M(s)=\mathcal{L}_M(-\ln(s)) \tag{3.15}$$

and

$$\mathcal{P}^{-1}_{M}\left(u\right)=\exp\left(-\mathcal{L}^{-1}_{M}\left(u\right)\right). \tag{3.16}$$

We can rewrite $\mathcal{L}^{-1}_{\Theta_i}$ in terms of $\mathcal{L}^{-1}_{M}$ as follows:

$$\mathcal{L}^{-1}_{\Theta_i}\left(u_{i,j}\right)=\mathcal{L}^{-1}_{B_i}\left(\exp(-\mathcal{L}^{-1}_{M}\left(u_{i,j}\right))\right), \tag{3.17}$$

for $u_{i,j}\in[0,1]$, $i=1,\ldots,d$ and $j=1,\ldots,n_i$.

The vector of rvs $\underline{Y_i}$ follows a multivariate mixed exponential distribution, which is defined only in terms of LST of the mixing rv Θ_i given in (3.11) and for which the multivariate survival function corresponds to

$$\begin{aligned}
\overline{F}_{\underline{Y_i}}(\underline{y_i}) &= \int_0^\infty \overline{F}_{\underline{Y_i}|\Theta_i=\theta_i}(\underline{y_i})\,\mathrm{d}F_{\Theta_i}(\theta_i) \\
&= \int_0^\infty \prod_{j=1}^{n_i} \overline{F}_{Y_{i,j}|\Theta_i=\theta_i}(y_{i,j})\,\mathrm{d}F_{\Theta_i}(\theta_i) \\
&= \int_0^\infty \prod_{j=1}^{n_i} \mathrm{e}^{-y_{i,j}\theta_i}\,\mathrm{d}F_{\Theta_i}(\theta_i) \\
&= \mathcal{L}_{\Theta_i}(y_{i,1}+\ldots+y_{i,n_i}),
\end{aligned} \tag{3.18}$$

where $\underline{y_i}=(y_{i,1},\ldots,y_{i,n_i})$, for $i=1,\ldots,d$.

Finally, the vector of rvs $\underline{Y}$ follows a multivariate mixed exponential distribution. Most importantly for us, the latter has the interesting feature of being defined by the vector of mixing rvs $\underline{\Theta}$, which follows a multivariate compound distribution. The multivariate survival function of $\underline{Y}$ is represented in terms of the LST (3.10) of $\underline{\Theta}$ i.e.

$$\overline{F}_{\underline{Y}}\left(\underline{y}\right)=\mathcal{L}_{\underline{\Theta}}\left(y_{1,1}+\ldots+y_{1,n_1},\ldots,y_{d,1}+\ldots+y_{d,n_d}\right), \tag{3.19}$$

where $\underline{y}=\left(\underline{y_1},\ldots,\underline{y_d}\right)=(y_{1,1},\ldots,y_{1,n_1},\ldots,y_{d,1},\ldots,y_{d,n_d})$.

3.2.3 Construction of hierarchical Archimedean copulas defined with multivariate compound distributions

We use the inversion method to identify the proposed hierarchical Archimedean copula C defined with a multivariate compound distribution. Indeed, applying Sklar's Theorem (see e.g. Sklar (1959) or Nelsen (2007)) and letting $y_{i,j} = \overline{F}_{Y_{i,j}}^{-1}(u_{i,j})$ in (3.13), a hierarchical Archimedean copula C defined with a multivariate compound distribution is obtained from the multivariate survival function given in (3.19) of the multivariate mixed exponential distribution as follows :

$$\begin{aligned} C(\underline{u}) &= \overline{F}_{\underline{Y}}\left(\overline{F}_{Y_{1,1}}^{-1}(u_{1,1}), \ldots, \overline{F}_{Y_{1,n_1}}^{-1}(u_{1,n_1}), \ldots, \overline{F}_{Y_{d,1}}^{-1}(u_{d,1}), \ldots, \overline{F}_{Y_{d,n_d}}^{-1}(u_{d,n_d})\right) \\ &= \mathcal{L}_{\underline{\Theta}}\left(\sum_{j=1}^{n_1} \mathcal{L}_{\Theta_1}^{-1}(u_{1,j}), \ldots, \sum_{j=1}^{n_d} \mathcal{L}_{\Theta_d}^{-1}(u_{d,j})\right), \end{aligned} \tag{3.20}$$

where $\underline{u} = (\underline{u_1}, \ldots, \underline{u_d})$ with $\underline{u_i} = (u_{i,1}, \ldots, u_{i,n_i})$ for $i = 1, \ldots, d$.

Let us examine more deeply the structure of C. The expression for the multivariate Archimedean copula in (3.20) is written in terms of the multivariate LST of the multivariate mixing random vector $\underline{\Theta}$ with the sum of the inverse of the corresponding marginal LSTs of Θ_1, ..., Θ_d evaluated at each element of a subgroup as components. Hierarchical Archimedean copulas built with our proposed approach can be seen as multivariate extensions to the classical univariate Archimedean copulas which are defined as a univariate LST evaluated at the sum of the inverse of the LST evaluated at each element of a multivariate uniform random vector as follows:

$$C(\underline{u}) = \mathcal{L}_{\Theta}\left(\sum_{i=1}^{d} \mathcal{L}_{\Theta}^{-1}(u_i)\right).$$

Given the connection with the mixing random vector $\underline{\Theta}$ and the random variables M, B_i ($i = 1, \ldots, d$), an alternative representation of the copula C in (3.20) can be established in terms of $\mathcal{L}_M$ and $\mathcal{L}_{B_i}$ ($i = 1, \ldots, d$), and $\mathcal{L}_{\Theta_i}$ ($i = 1, \ldots, d$). Indeed, letting (3.10) and (3.15) in (3.20), the hierarchical Archimedean copula C can also be written as follows:

$$C(\underline{u}) = \mathcal{L}_M\left(\sum_{i=1}^{d} -\ln\left(\mathcal{L}_{B_i}\Big(\sum_{j=1}^{n_i} \mathcal{L}_{\Theta_i}^{-1}(u_{i,j})\Big)\right)\right). \tag{3.21}$$

Finally, inserting (3.17) in (3.21), the hierarchical Archimedean copula C in (3.21) can be defined solely in terms of $\mathcal{L}_M$ and $\mathcal{L}_{B_i}$ ($i = 1, \ldots, d$) as

$$C(\underline{u}; \mathcal{L}_M, \mathcal{L}_{B_1}, \ldots, \mathcal{L}_{B_d}) = \mathcal{L}_M\left(\sum_{i=1}^{d} -\ln\left(\mathcal{L}_{B_i}\Big(\sum_{j=1}^{n_i} \mathcal{L}_{B_i}^{-1}\left(\exp(-\mathcal{L}_M^{-1}(u_{i,j}))\right)\Big)\right)\right). \tag{3.22}$$

Note that in (3.22) the terms $\mathcal{L}_M$, $\mathcal{L}_{B_1}$, ..., $\mathcal{L}_{B_d}$ are explicitly inserted as arguments of the function C. We will add these arguments only when we find relevant to insist on their implication in the copula representation.

3.2.4 Dependence structure of hierarchical Archimedean copulas defined with multivariate compound distributions

Either from (3.21) or (3.22), it is clear that the proposed hierarchical Archimedean copulas have explicit forms and offer many possibilities of dependence structures through the choices of distributions for the strictly positive discrete rv M and the strictly positive rvs $B_1, \ldots, B_d$. These copula constructions allow flexibility without requiring a large number of parameters. As stated in Brechmann (2014), "a major issue of any copula model is to find a good balance between parsimony and flexibility".

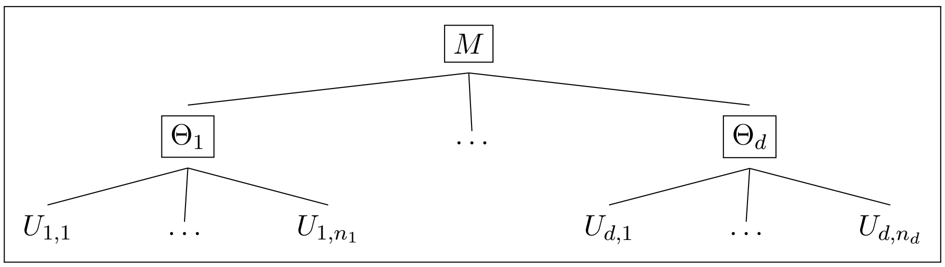


Figure 3.1 – One level tree structure

The dependence structure associated to the hierarchical Archimedean copula defined in either (3.21) or (3.22) can be illustrated with a tree representation as shown in Figure 3.1 in which $\underline{U} = (\underline{U_1}, \ldots, \underline{U_d})$ with $\underline{U_i} = (U_{i,1}, \ldots, U_{i,n_i})$ for $i = 1, \ldots, d$, is a vector of rvs with multivariate cdf C. This shows a one level tree structure composed of d subgroups joined together by their common link with the rv M. The dependence among the components of $\underline{U_i}$ is captured by Archimedean copulas generated by the mixing rvs Θ_i, for $i = 1, 2, \ldots, d$. Then, the dependence relation between the vectors $\underline{U_1}$, $\ldots$, $\underline{U_n}$ is defined through the rv M.

Let us examine the dependence structure for specific subsets of $\underline{U}$. First, the multivariate cdf of $(\underline{U_{i_1}}, \ldots, \underline{U_{i_m}})$ is given by the hierarchical Archimedean copula

$$C(\underline{u_{i_1}}, \ldots, \underline{u_{i_m}}; \mathcal{L}_M, \mathcal{L}_{B_{i_1}}, \ldots, \mathcal{L}_{B_{i_m}}), \tag{3.23}$$

for $i_1, \ldots, i_m \in \{1, 2, \ldots, d\}$ and for $m = 2, \ldots, d$. Also, if we choose at least two subgroups i_1 and i_2, the multivariate cdf of any vector formed with at least two rvs from each one of them is a hierarchical Archimedean copula of the similar form.

Now we consider the dependence structure within a subgroup i ($i = 1, 2, \ldots, d$). From (3.20), the multivariate cdf of $(U_{i,j_1}, \ldots, U_{i,j_l})$ (for $j_1, \ldots, j_l \in \{1, 2, \ldots, n_i\}$ and for $l = 2, \ldots, n_i$) is given by

$$C\left(u_{i,j_1},\ldots,u_{i,j_l};\mathcal{L}_{\Theta_i}\right)=\mathcal{L}_{\Theta_i}\left(\sum_{k=1}^{l}\mathcal{L}_{\Theta_i}^{-1}\left(u_{i,j_k}\right)\right) \tag{3.24}$$

which corresponds to an l-variate Archimedean copula with generator $\mathcal{L}_{\Theta_i}$. As a special case, the multivariate cdf of $\underline{U_i}$ is $C\left(\underline{u_i};\mathcal{L}_{\Theta_i}\right)$ for $i=1,\ldots,d$. Due to (3.11) and (3.17), copulas of the form as given in (3.24) are Archimedean copulas based on two generators $\mathcal{L}_M$ and $\mathcal{L}_{B_i}$. Such copulas allow for more flexibility by using at least two parameters (see e.g. Nelsen (2007), Section 4.5, for other examples of two parameter Archimedean copulas).

Finally, we turn our attention to the dependence structure for vectors of rvs containing at most one component from each subgroup of $\underline{U}$. For $j_1\in\{1,2,\ldots,n_1\}$, ..., $j_d\in\{1,2,\ldots,n_d\}$, the multivariate cdf of $(U_{1,j_1},\ldots,U_{d,j_d})$ is obtained from (3.20) and (3.10) and is given by

$$\begin{aligned}C\left(u_{1,j_1},\ldots,u_{d,j_d}\right)&=\mathcal{L}_{\Theta_1,\ldots,\Theta_d}\left(\mathcal{L}_{\Theta_1}^{-1}\left(u_{1,j_1}\right),\ldots,\mathcal{L}_{\Theta_d}^{-1}\left(u_{d,j_d}\right)\right)\\&=\mathcal{P}_M\left(\mathcal{L}_{B_1}\left(\mathcal{L}_{\Theta_1}^{-1}\left(u_{1,j_1}\right)\right)\ldots\mathcal{L}_{B_d}\left(\mathcal{L}_{\Theta_d}^{-1}\left(u_{d,j_d}\right)\right)\right).\end{aligned} \tag{3.25}$$

Using (3.14), (3.25) becomes

$$\begin{aligned}C\left(u_{1,j_1},\ldots,u_{d,j_d}\right)&=\mathcal{P}_M\left(\mathcal{L}_{B_1}\left(\mathcal{L}_{B_1}^{-1}\left(\mathcal{P}_M^{-1}\left(u_{1,j_1}\right)\right)\right)\ldots\mathcal{L}_{B_d}\left(\mathcal{L}_{B_d}^{-1}\left(\mathcal{P}_M^{-1}\left(u_{d,j_d}\right)\right)\right)\right)\\&=\mathcal{P}_M\left(\mathcal{P}_M^{-1}\left(u_{1,j_1}\right)\ldots\mathcal{P}_M^{-1}\left(u_{d,j_d}\right)\right)\\&=\mathcal{L}_M\left(-\ln\left(\mathcal{P}_M^{-1}\left(u_{1,j_1}\right)\right)-\ldots-\ln\left(\mathcal{P}_M^{-1}\left(u_{d,j_d}\right)\right)\right).\end{aligned} \tag{3.26}$$

Then, from (3.26) and (3.16), we conclude that the multivariate cdf of $(U_{1,j_1},\ldots,U_{d,j_d})$ is

$$C\left(u_{1,j_1},\ldots,u_{d,j_d};\mathcal{L}_M\right)=\mathcal{L}_M\left(\sum_{i=1}^{d}\mathcal{L}_M^{-1}\left(u_{i,j_i}\right)\right), \tag{3.27}$$

which corresponds to a single-generator Archimedean copula with $\mathcal{L}_M$. Given (3.27), it means that the dependence relation between the components of $(U_{1,j_1},\ldots,U_{d,j_d})$ is only defined through the rv M.

3.2.5 Dependence properties

In light of the previous results, we aim to measure and compare the strength of the dependence relations between the rvs of a given subgroup and also between rvs from different subgroups. In other words, we want to compare whether the copula generated from $\mathcal{L}_\Theta$ is more dependent than the one with a generator $\mathcal{L}_M$. To do so, we need to introduce the concept of concordance ordering as defined in Joe (1997) page 37.

Definition 3.2.1. *Let C_1 and C_2 be two d-dimensional copulas. C_2 is more* ***concordant*** *that C_1, written $C_1 \prec_c C_2$, if*

$$C_1(\underline{u}) \leq C_2(\underline{u}) \text{ and } \overline{C_1}(\underline{u}) \leq \overline{C_2}(\underline{u}),$$

for $\underline{u} \in [0,1]^d$.

A very useful consequence of the concordance ordering is its relation with dependence measures. For instance, if C_1 and C_2 are copulas with respective Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2, and $C_1 \prec_c C_2$, then $\tau_1 \leq \tau_2$, $\rho_S^{(1)} \leq \rho_S^{(2)}$, and $\lambda_1 \leq \lambda_2$. (see Joe (1997)).

Suppose now that C_1 and C_2 are Archimedean copulas with respective generators $\mathcal{L}_1$ and $\mathcal{L}_2$. Theorems 4.1 and 4.7 in Joe (1997) provide the condition that guarantees that $C_1 \prec_c C_2$ is verified. Such a condition states that $\left(\mathcal{L}_1^{-1} \circ \mathcal{L}_2\right)'$ is a completely monotone function.

We recall that our objective is to compare the copula C_Θ, generated from $\mathcal{L}_\Theta$, with the one generated from $\mathcal{L}_M$ (written C_M). Since $\mathcal{L}_\Theta$ is related to $\mathcal{L}_M$, this condition is easily verified as follows:

$$\mathcal{L}_M^{-1} \circ \mathcal{L}_\Theta = \mathcal{L}_M^{-1} \circ \mathcal{L}_M(-\ln(\mathcal{L}_B)) = -\ln(\mathcal{L}_B).$$

Therefore, if $(-\ln(\mathcal{L}_B))'$ is completely monotone, then the dependence within a subgroup is stronger than the outer dependence between subgroups, i.e. $C_M \prec_c C_\Theta$ and hence $\tau_M \leq \tau_\Theta$.

It is well known that $(-\ln(\mathcal{L}_{B_i}))'$ is completely monotone if and only if $\mathcal{L}_{B_i}^m$ is the LST of a positive rv i.e. $\mathcal{L}_{B_i}^m \in \Psi_\infty$ for all $m \in \mathbb{N}$ (see Joe (1997)). Table 3.1 provides a list of distributions, used to generate an Archimedean copula, for which it is easy to directly verify that $\mathcal{L}_{B_i}^m$ is the LST of a positive rv. We denote by $SNB(m; r, q)$ a shifted negative binomial distribution i.e. $B_{i,1} + \ldots + B_{i,m}$ admits a stochastic representation $N + m$, where $N \sim NB(r, q)$, $r \in \mathbb{R}^+$ and $q \in (0,1)$. For the logarithmic distribution, $\sum_{j=1}^m B_{i,j}$ is an $m-$fold sum of iid logarithmic rvs. If B_i follows a Sibuya distribution, $\sum_{j=1}^m B_{i,j}$ is distributed as an unknown rv N with probability mass function $p_k = \sum_{i=1}^\infty \binom{m}{i}\binom{\alpha i}{k}(-1)^{i+k}$, $k = 1, 2, \ldots$. Finally, for a stable distribution i.e. $B_i \sim S\left(\alpha, 1, \cos(\alpha\pi/2)^{\frac{1}{\alpha}}, \mathbf{1}_{\{\alpha=1\}}; 1\right)$, $\mathcal{L}_{B_i}^m$ is the LST of a positive rv exponentially tilted stable distributed $\tilde{S}\left(\alpha, 1, (\cos(\alpha\pi/2)m)^{\frac{1}{\alpha}}, m\,\mathbf{1}_{\{\alpha=1\}}, h\,\mathbf{1}_{\{\alpha\neq1\}}; 1\right)$ with $h = 0$. For sampling procedures, we refer to Hofert (2011) and Devroye (2009).

3.2.6 Examples of hierarchical Archimedean copulas

In the following two examples, we give a glimpse of the flexibility of our approach based on multivariate compound distributions in the construction of a large variety of hierarchical

Distribution of B_i	$\mathcal{L}^{\mathbf{m}}_{\mathbf{B_i}}(\mathbf{t})$	**Distribution of** $B_{i,1}+\ldots+B_{i,m}$
ShiftedGeometric(q)	$\left(\frac{qe^{-t}}{1-(1-q)e^{-t}}\right)^{m}$	$SNB(m;m,q)$
ShiftedNegativeBinomial$(r;r,q)$	$\left(\frac{qe^{-t}}{1-(1-q)e^{-t}}\right)^{m\,r}$	$SNB(rm;rm,q)$
Logarithmic(γ)	$\left(\frac{ln(1-\gamma e^{-t})}{ln(1-\gamma)}\right)^{m}$	$\sum_{i=1}^{m} B_i, B_i \sim Logarithmic(\gamma)$
Sibuya(α)	$(1-(1-e^{-t})^{\alpha})^{m}$	$N \sim p_k = \sum_{i=1}^{\infty} \binom{m}{i}\binom{\alpha i}{k}(-1)^{i+k}$
Gamma$(\alpha,1)$	$\left(\frac{1}{1+t}\right)^{m\alpha}$	Gamma$(m\alpha,1)$
Stable$\Big(\alpha,1,\cos(\alpha\pi/2)^{\frac{1}{\alpha}},\mathbf{1}_{\{\alpha=1\}};1\Big)$	$(\exp(-t^{\alpha}))^{m}$	$\tilde{S}\left(\alpha,1,(\cos(\alpha\pi/2)m)^{\frac{1}{\alpha}},m\,\mathbf{1}_{\{\alpha=1\}},0;1\right)$

Table 3.1 – Distributions for the sum of common laws

Archimedean copulas C. One must first choose the distribution for the strictly positive discrete rv M. Here, we limit ourselves to shifted-geometric and logarithmic distributions for M. Of course, other candidates would have been suitable, such as the Sibuya and shifted-negative binomial distributions (see e.g. Joe (2014)). Next, distributions for the rvs B_i are selected. Either shifted-geometric, logarithmic, or gamma distributions are picked. Among the possible combinations of distributions, we choose to depict 6 hierarchical Archimedean copulas constructed under the proposed approach. In the first example, we present 3 copulas with an underlying shifted-geometric distribution for M, while in the second example a logarithmic distribution describes the behavior of the rv M. By convention, when the distributions for the rvs B_i ($i=1,\ldots,d$) are from the same family of distributions (with different parameters), we refer to the resulting copula as "distribution for M"-"distribution for B_i" hierarchical Archimedean copula.

Example 3.2.1. *Let* $M \sim ShifGeo\,(q)$ *with* $\mathcal{L}_M\,(t) = \frac{qe^{-t}}{1-(1-q)e^{-t}}$ *and* $\mathcal{L}_M^{-1}\,(t) = -\ln\left(\frac{1}{\frac{q}{t}+1-q}\right)$. *Letting* $\alpha = 1-q$, *the multivariate cdf of* $(U_{1,j_1},\ldots,U_{d,j_d})$ *corresponds to the Ali-Mikhail-Haq (AMH) copula with generator* $\mathcal{L}_M$, *i.e.,*

$$C\left(u_{1,j_1},\ldots,u_{d,j_d};\mathcal{L}_M\right) = \frac{\alpha\prod_{i=1}^{n}u_i}{\prod_{i=1}^{n}\left(\alpha+(1-\alpha)\,u_i\right)-(1-\alpha)\prod_{i=1}^{n}u_i}.$$

1. *Geometric-Geometric hierarchical Archimedean copula:*
   — $B_i \sim ShifGeo\,(q_i)$ *with* $\mathcal{L}_{B_i}\,(t) = \frac{q_ie^{-t}}{1-(1-q_i)e^{-t}}$ *and* $\mathcal{L}_{B_i}^{-1}\,(t) = -\ln\left(\frac{1}{\frac{q_i}{t}+1-q_i}\right)$;
   — *parameters:* q, q_i, $\ldots$, $q_d \in (0,1)$;
   — *copula* C:

$$C(\underline{u}) = \frac{q\prod_{i=1}^{d} q_i \prod_{\mathrm{j=1}}^{\mathrm{n_i}}\left(\frac{u_{i,j}}{u_{i,j}-qq_i(u_{i,j}-1)}\right)}{\prod_{i=1}^{d}\left(1-(1-q_i)\prod_{\mathrm{j=1}}^{\mathrm{n_i}}\left(\frac{u_{i,j}}{u_{i,j}-qq_i(u_{i,j}-1)}\right)\right)\left(1-\frac{(1-q)\prod_{i=1}^{d}\mathrm{q_i}\prod_{\mathrm{j=1}}^{\mathrm{n_i}}\left(\frac{u_{i,j}}{u_{i,j}-qq_i\left(u_{i,j}-1\right)}\right)}{\prod_{i=1}^{d}\left(1-(1-q_i)\prod_{\mathrm{j=1}}^{\mathrm{n_i}}\left(\frac{u_{i,j}}{u_{i,j}-qq_i\left(u_{i,j}-1\right)}\right)\right)}\right)}.$$

2. *Geometric-Logarithmic hierarchical Archimedean copula :*
   - $B_i \sim Logarithmic\,(\gamma_i)$ *with* $\mathcal{L}_{B_i}(t) = \frac{\ln(1-\gamma_i e^{-t})}{\ln(1-\gamma_i)}$ *and* $\mathcal{L}_{B_i}^{-1}(t) = -\ln\left(\frac{(1-(1-\gamma_i)^t)}{\gamma_i}\right)$;
   - *parameters:* q, $\gamma_1 = (1-e^{-\alpha_1}), \ldots, \gamma_d = (1-e^{-\alpha_d}) \in (0,1)$;
   - *copula* C:

$$C(\underline{u}) = \frac{-q\prod_{i=1}^{d}\ln\left(1-(1-e^{-\alpha_i})\prod_{j=1}^{n_i}\left(\frac{1-\mathrm{e}^{-\alpha_i\left(\frac{q}{u_{i,j}}+1-q\right)^{-1}}}{(1-e^{-\alpha_i})}\right)\right)}{\prod_{i=1}^{d}\alpha_i + (1-q)\prod_{i=1}^{d}\ln\left(1-(1-e^{-\alpha_i})\prod_{j=1}^{n_i}\left(\frac{1-\mathrm{e}^{-\alpha_i\left(\frac{q}{u_{i,j}}+1-q\right)^{-1}}}{(1-e^{-\alpha_i})}\right)\right)}.$$

3. *Geometric-Gamma hierarchical Archimedean copula :*
   - $B_i \sim Gamma\,(\alpha_i, 1)$ *with* $\mathcal{L}_{B_i}(t) = \left(\frac{1}{1+t}\right)^{\alpha_i}$ *and* $\mathcal{L}_{B_i}^{-1}(t) = t^{-\frac{1}{\alpha_i}} - 1$;
   - *parameters:* $q \in (0,1)$, $\alpha_i, \ldots, \alpha_d \in \mathbb{R}^+$;
   - *copula* C:

$$C(\underline{u}) = \frac{q\prod_{i=1}^{d}\left(\sum_{j=1}^{n_i}\left(\frac{q}{u_{i,j}}+1-q\right)^{\frac{1}{\alpha_i}} - (n_i-1)\right)^{-\alpha_i}}{1-(1-q)\prod_{i=1}^{d}\left(\sum_{j=1}^{n_i}\left(\frac{q}{u_{i,j}}+1-q\right)^{\frac{1}{\alpha_i}} - (n_i-1)\right)^{-\alpha_i}}.$$

□

Example 3.2.2. *We assume that* $M \sim Logarithmic\,(\gamma)$ *with* $\mathcal{L}_M(t) = \frac{\ln(1-\gamma e^{-t})}{\ln(1-\gamma)}$ *and* $\mathcal{L}_M^{-1}(t) = -\ln\left(\frac{(1-(1-\gamma)^t)}{\gamma}\right)$. *Letting* $\alpha = \ln(1-\gamma)$ *the multivariate cdf of* $(U_{1,j_1}, \ldots, U_{d,j_d})$ *corresponds to the Frank copula with generator* $\mathcal{L}_M$, *i.e.,*

$$C(u_{1,j_1}, \ldots, u_{d,j_d}; \mathcal{L}_M) = \frac{-1}{\alpha}\ln\left(1-\frac{\prod_{i=1}^{n}(1-e^{-\alpha u_i})}{(1-e^{-\alpha})^{n-1}}\right).$$

1. *Logarithmic-Geometric hierarchical Archimedean copula:*
   - $B_i \sim ShifGeo\,(q_i)$ *with* $\mathcal{L}_{B_i}(t) = \frac{q_i e^{-t}}{1-(1-q_i)e^{-t}}$ *and* $\mathcal{L}_{B_i}^{-1}(t) = -\ln\left(\frac{1}{\frac{q_i}{t}+1-q_i}\right)$;
   - *parameters:* $\gamma = (1-e^{-\alpha})$, $q_i, \ldots, q_d \in (0,1)$;

— *copula C:*

$$C\left(\underline{u}\right)=-\frac{1}{\alpha}\ln\left(1-\frac{\left(1-e^{-\alpha}\right)\prod_{i=1}^{d}q_i\prod_{j=1}^{n_j}\left(\frac{q_i\left(1-e^{-\alpha}\right)}{1-e^{-\alpha u_{i,j}}}+1-q_i\right)^{-1}}{\prod_{i=1}^{d}\left(1-\left(1-q_i\right)\prod_{j=1}^{n_i}\left(\frac{q_i\left(1-e^{-\alpha}\right)}{1-e^{-\alpha u_{i,j}}}+1-q_i\right)^{-1}\right)}\right).$$

2. *Logarithmic-Logarithmic hierarchical Archimedean copula:*

   — $B_i \sim$ *Logarithmic* (γ_i) *with* $\mathcal{L}_{B_i}\left(t\right)=\frac{\ln(1-\gamma_i e^{-t})}{\ln(1-\gamma_i)}$ *and* $\mathcal{L}_{B_i}^{-1}\left(t\right)=-\ln\left(\frac{(1-(1-\gamma_i)^t)}{\gamma_i}\right)$*;*

   — *parameters:* $\gamma=\left(1-e^{-\alpha}\right)$, $\gamma_1=\left(1-e^{-\alpha_1}\right),\ldots,\gamma_d=\left(1-e^{-\alpha_d}\right)\in(0,1)$*;*

   — *copula C:*

$$C\left(\underline{u}\right)=\frac{-1}{\alpha}\ln\left(1-\left(1-e^{-\alpha}\right)\prod_{i=1}^{d}-\frac{1}{\alpha_i}\ln\left(1-\left(1-e^{-\alpha_i}\right)\prod_{j=1}^{n_i}\left(\frac{1-e^{-\alpha_i\left(\frac{1-e^{-\alpha\, u_{i,j}}}{1-e^{-\alpha}}\right)}}{1-e^{-\alpha_i}}\right)\right)\right).$$

3. *Logarithmic-Gamma hierarchical Archimedean copula:*

   — $B_i \sim$ *Gamma* $(\alpha_i,1)$ *with* $\mathcal{L}_{B_i}\left(t\right)=\left(\frac{1}{1+t}\right)^{\alpha_i}$ *and* $\mathcal{L}_{B_i}^{-1}\left(t\right)=t^{-\frac{1}{\alpha_i}}-1$*;*

   — *parameters:* $\gamma=\left(1-e^{-\alpha}\right)\in(0,1)$, $\alpha_1,\ \ldots,\ \alpha_d\in\mathbb{R}^+$*;*

   — *copula C:*

$$C\left(\underline{u}\right)=\frac{-1}{\alpha}\ln\left(1-\left(1-e^{-\alpha}\right)\prod_{i=1}^{d}\left(\sum_{j=1}^{n_i}\left(\frac{1-e^{-\alpha\, u_{i,j}}}{1-e^{-\alpha}}\right)^{\frac{-1}{\alpha_i}}-\left(n_i-1\right)\right)^{-\alpha_i}\right).$$

□

Obviously, hierarchical Archimedean copulas can be built assuming that the distributions of the rvs B_i belong to different families. Since we can't list all possible hierarchical Archimedean copulas that can obtained this way, we limit ourselves to providing, in the following example, an illustration of a hierarchical Archimedean copula assuming $d=2$ subgroups with different families of distributions B_1 and B_2.

Example 3.2.3. *Let* $d=2$ *and* $M\sim ShiftedGeo(q)$ *with* $\mathcal{L}_M\left(t\right)=\frac{qe^{-t}}{1-(1-q)e^{-t}}$, $B_1\sim Log(\gamma)$ *with* $\mathcal{L}_{B_1}(t)=\frac{\ln(1-\gamma e^{-t})}{\ln(1-\gamma)}$ *and finally,* $B_2\sim Gamma(\alpha,1)$ *with* $\mathcal{L}_{B_2}(t)=\left(\frac{1}{t+1}\right)^{\alpha}$. *Then, the expression of the 3-parameters copula is given by*

$$C(\underline{u_1},\underline{u_2})=\frac{q\left(\sum_{j=1}^{n_2}\left(\frac{q}{u_{2,j}}+1-q\right)^{\frac{1}{\alpha}}-1\right)^{-\alpha}\ln\left(1-\gamma\exp\left(\sum_{i=1}^{n_1}\ln\left(\frac{1-\frac{q}{u_{1,i}}+1-q\sqrt{1-\gamma}}{\gamma}\right)\right)\right)}{\ln\left(1-\gamma\right)\left(1-\frac{\left(\sum_{j=1}^{n_2}\left(\frac{q}{u_{2,j}}+1-q\right)^{\frac{1}{\alpha}}-1\right)^{-\alpha}\ln\left(1-\gamma\exp\left(\sum_{i=1}^{n_1}\ln\left(\frac{1-\frac{q}{u_{1,i}}+1-q\sqrt{1-\gamma}}{\gamma}\right)\right)\right)}{\ln(1-\gamma)(1-q)^{-1}}\right)}.$$

□

3.2.7 Sampling hierarchical Archimedean copulas

Inspired from Marshall and Olkin (1988), the sampling procedure for a hierarchical Archimedean copula C defined with a multivariate compound distribution follows from its construction. The algorithm given just below aims to simulate samples of a random vector $\underline{U} = (\underline{U_1}, \ldots, \underline{U_d})$ (with $\underline{U_i} = (U_{i,1}, \ldots, U_{i,n_i})$, for $i = 1, \ldots, d$) whose multivariate cdf is the copula C.

Algorithm 3.2.1. *Let C be a hierarchical Archimedean copula with d subgroups and root M. Define $\underline{U}$ as a vector of standard uniformly distributed rvs with cdf C.*

1. *Sample M;*
2. *For each subgroup ($i = 1, \ldots, d \geq 2$):*
   - 2.1. *Sample $B_{i,k}$ for $k = 1, \ldots, M$;*
   - 2.2. *Return $\Theta_i = \sum_{k=1}^{M} B_{i,k}$;*
   - 2.3. *Sample $R_{i,j} \sim Exp(1)$ for $j = 1, \ldots, n_i$;*
   - 2.4. *Return $U_{i,j} = \mathcal{L}_{\Theta_i}(R_{i,j}/\Theta_i)$ for $j = 1, \ldots, n_i$;*
3. *Return $\underline{U} = (U_{1,1}, \ldots, U_{1,n_1}, \ldots, U_{d,1}, \ldots, U_{d,n_d})$.*

An application of Algorithm 3.2.1 is provided in the following example.

Example 3.2.4. *Let C be the Geometric-Gamma hierarchical Archimedean copula presented in Example 3.2.1 with $d = 2$ and $n_1 = n_2 = 2$. To illustrate the pairwise dependence between components of $\underline{U} = (U_{1,1}, U_{1,2}, U_{2,1}, U_{1,2})$, we give, in Figure 3.2, a graph of sampled values of $\underline{U}$, where the parameters of the copula C are $q = 0.1$, $\alpha_1 = 0.04$ and $\alpha_2 = 0.2$. Note that τ, in Figure 3.2, corresponds to the Kendall's tau.*

It is clear that the sampled values of $(U_{1,1}, U_{1,2})$ provide scatter plots that are similar to those of the Clayton copula with a dependence subdued by the shifted geometric distribution. The same phenomenon happens for the sampled values of $(U_{2,1}, U_{2,2})$. Since the dependence structures of the four other pairs $(U_{1,1}, U_{2,1})$, $(U_{1,1}, U_{2,2})$, $(U_{1,2}, U_{2,1})$, $(U_{1,2}, U_{2,2})$ depend on M, their scatter plots are those of an AMH copula. □

3.2.8 Multi-level hierarchical Archimedean copulas

Until now, we have only considered one-level hierarchical Archimedean copulas, for which the associated typical one-level tree representation is provided in Figure 3.1. Naturally, it is possible to design multi-level hierarchical Archimedean copulas by adding ramifications such that their dependence structures can be represented as multi-level trees.

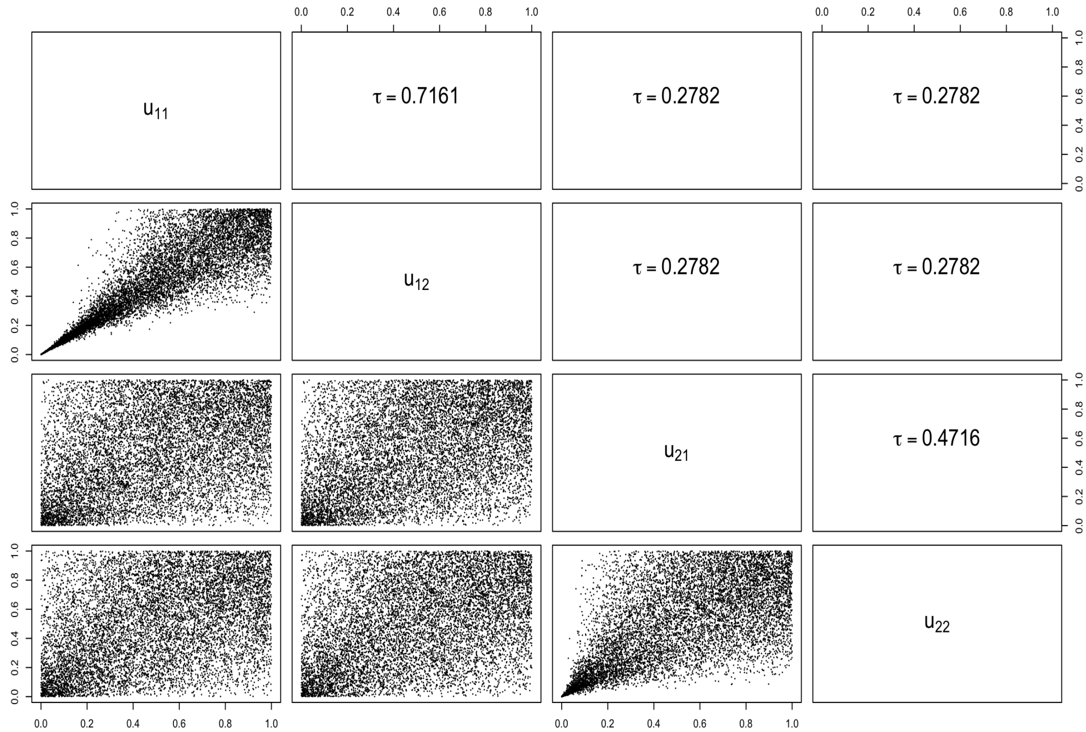


Figure 3.2 – Pairwise graph of the 10000 4-dimensional vectors of realizations sampled from the hierarchical Archimedean copula with multivariate compound distributions in Example 3.2.4

Let $\underline{U} = (\underline{U_1}, \ldots, \underline{U_d})$ be a random vector with a cdf defined by a multi-level hierarchical Archimedean copula C, such that each subgroup $\underline{U_i}$, $i = 1, \ldots, n_i$, characterized by its genetic code (the path in the hierarchy) e_i and its dimension n_i, can be written as

$$\underline{U_i} = (U_{e_i,j},\, j = 1, \ldots, n_i)\,.$$

This genetic code is represented by a vector in which the first element corresponds to the root of the tree (0 by default) and the k^{th} element represents the position of a node in the $(k-1)^{\text{th}}$ level, within its mother's direct descendants, assuming that the leaves are always at the far right. A value of 0 at the last position of the vector represents a direct link between a leaf and a mother node.

The design of the hierarchical structure comes from considering the primary distribution of the compound rvs as a random sum. Let $\underline{U_i}$ be a subgroup with genetic code e_i, and let ξ be a right truncated vector of e_i, and k the $(\dim(\xi)+1)^{th}$ element of e_i, where dim(.) is the dimension of a vector. On one hand, if $\dim(\xi) < \dim(e_i) - 1$, then the rv $M^{(\xi,k)}$ is defined as a random sum, i.e, $M^{(\xi,k)} = \sum_{j=1}^{M^{(\xi)}} N_j^{(\xi,k)}$ where $N_j^{(\xi,k)}$ is distributed as a discrete rv $N^{(\xi,k)}$ which plays the same role as the rv B_i in Section 3.2.1. On the other hand, if

$\dim(\xi) = \dim(e_i) - 1$, then the procedure is similar to the one presented in Section 3.2.1.

A typical hierarchical structure can be depicted by a tree where the root is always $M^{(0)}$ and all leaves are of type "u". A node of type "M", at a given level of the tree, can have children of all available types, meaning "M","Θ" and "u". Further, Θ nodes can only have u children (leaves), while leaves have no children at all.

To illustrate the notation, let us consider the tree construction depicted in Figure 3.3:

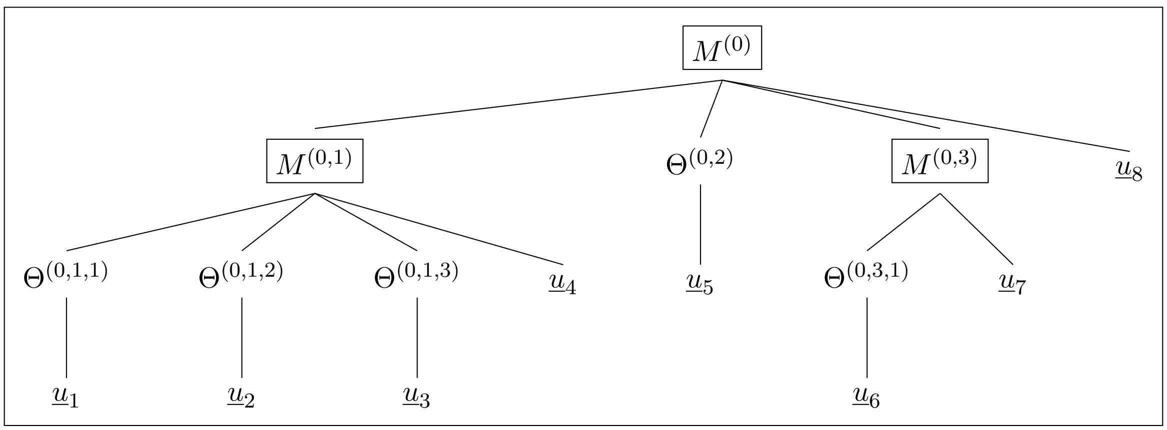


Figure 3.3 – Example of a multi-level hierarchical structure

The tree in Figure 3.3 corresponds to the dependence structure of a random vector $\underline{U} = (\underline{U_1}, \ldots, \underline{U_8})$ characterized by the following genetic codes:

$e_1 = (0, 1, 1)$	$e_2 = (0, 1, 2)$	$e_3 = (0, 1, 3)$	$e_4 = (0, 1, 0)$
$e_5 = (0, 2)$	$e_6 = (0, 3, 1)$	$e_7 = (0, 3, 0)$	$e_8 = (0, 0)$

Note that, one can deduce a tree from any given set of genetic codes. This set of codes constitutes a simple tool to summarize all the information in regard to the dependence structure depicted by the tree.

For any given multi-level hierarchical copula, it is possible to derive a tailored sampling procedure by adapting the one described in Algorithm 3.2.1. Algorithm 3.2.2 provides a sampling procedure for several hierarchical levels, assuming the three different types of children.

Algorithm 3.2.2. *Let C be a multi-level hierarchical copula with root $M^{(0)}$ and d subgroups. Define $\underline{U}$ as a vector of standard uniformly distributed rvs with cdf C. Let e_i be the genetic code of each subgroup i and let e_i' be the vector e_i without the last element.*

1. *Fix $j = 2$*
2. *For each genetic code with $\dim(e_i) = j$ do:*

2.1. *Sample* $M^{(e_i')}$

2.2. *If the last element of* e_i *is zero i.e.* $e_{i,\dim(e_i)} = 0$*:*

i. *Sample* $R \sim Exp(1)$

ii. *Set the corresponding component of* $\underline{U_i}$ *as* $\mathcal{L}_{M^{(e_i')}}\left(\frac{R}{M^{(e_i')}}\right)$

2.3. *Else:*

i. *Sample* $\Theta^{(e_i)}$

ii. *Sample* $R \sim Exp(1)$

iii. *Set the corresponding component of* $\underline{U_i}$ *as* $\mathcal{L}_{\Theta^{(e_i)}}\left(\frac{R}{\Theta^{(e_i)}}\right)$

3. *Set j=j+1*
4. *Repeat from 2*
5. *Return* $\underline{U}$

Algorithm 3.2.2 proceeds as follows: we start by simulating the root, which is always of type "M". Next, we perform a series of simulations in the following order: simulate all u-children of the root, simulate all Θ-children of the root then simulate all M-children of the root. Finally, we move the root to M and repeat the same logic over again.

In the following illustration, we provide an example of a 2-level hierarchical Archimedean copula.

Example 3.2.5. ***(Two level tree)*** *We consider a 2-level hierarchical Archimedean copula* C*, for which the associated 2-level tree representation is given in Figure 3.4. The 6-dimensional copula* C *is the multivariate cdf of the random vector* $\underline{U} = (U_{(0,2),1}, U_{(0,2),2}, U_{(0,1,1),1}, U_{(0,1,1),2}, U_{(0,1,2),1}, U_{(0,1,1),2})$*. Let the rvs* $\Theta^{(0,2)}$*,* $\Theta^{(0,1,1)}$*, and* $\Theta^{(0,1,2)}$ *be associated to the pairs* $(U_{(0,2),1}, U_{(0,2),2})$*,* $(U_{(0,1,1),1}, U_{(0,1,1),2})$ *and* $(U_{(0,1,2),1}, U_{(0,1,2),2})$*. We define* $M^{(0,1)} = \sum_{k=1}^{M^{(0)}} N_k^{(0,1)}$*,* $\Theta^{(0,2)} = \sum_{k=1}^{M^{(0)}} B_k^{(0,2)}$ *and* $\Theta^{(0,1,i)} = \sum_{k=1}^{M^{(0,1)}} B_k^{(0,1,i)}$*, for* $i = 1, 2$*.*

By first conditioning on $M^{(0,1)}$ *and then on* $M^{(0)}$*, the LST of* $\Theta^{(0,1,i)}$ *can be written as*

$$\mathcal{L}_{\Theta^{(0,1,i)}}(t) = \mathcal{P}_{M^{(0)}}\left(\mathcal{P}_{N^{(0,1)}}\left(\mathcal{L}_{B^{(0,1,i)}}(t)\right)\right).$$

Similarly, we obtain the trivariate LST of $(\Theta^{(0,2)}, \Theta^{(0,1,1)}, \Theta^{(0,1,2)})$*, which is given by*

$$\mathcal{L}_{\Theta^{(0,2)},\Theta^{(0,1,1)},\Theta^{(0,1,2)}}(t_1, t_2, t_3) = \mathcal{P}_{M^{(0)}}\left(\mathcal{L}_{B^{(0,2)}}(t_1)\ \mathcal{P}_{N^{(0,1)}}\left(\mathcal{L}_{B^{(0,1,1)}}(t_2)\ \mathcal{L}_{B^{(0,1,2)}}(t_3)\right)\right).$$

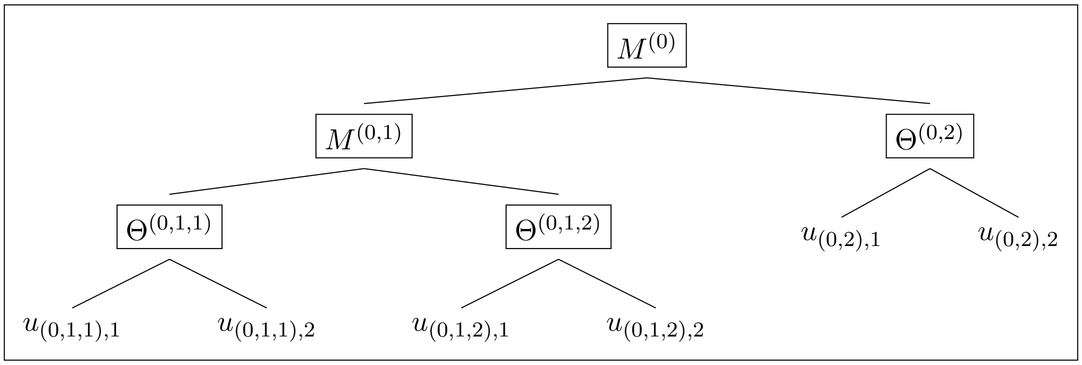


Figure 3.4 – Two level tree structure

Thus, the 6-dimensional copula C can be written as

$$C(u_{(0,2),1}, u_{(0,2),2}, u_{(0,1,1),1}, u_{(0,1,1),2}, u_{(0,1,2),1}, u_{(0,1,2),2}) = \mathcal{L}_{\Theta^{(0,2)},\Theta^{(0,1,1)},\Theta^{(0,1,2)}}\Big(\sum_{k=1}^{2}\mathcal{L}^{-1}_{\Theta^{(0,2)}}(u_{(0,2),k}), \sum_{k=1}^{2}\mathcal{L}^{-1}_{\Theta^{(0,1,1)}}(u_{(0,1,1),k}), \sum_{k=1}^{2}\mathcal{L}^{-1}_{\Theta^{(0,1,2)}}(u_{(0,1,2),k})\Big) \quad (3.28)$$

We assume that $M^{(0)} \sim ShiftedGeo(0.5)$, $N^{(0,1)} \sim ShiftedGeo(0.1)$ and $B^{(0,2)} \sim B^{(0,1,1)} \sim B^{(0,1,2)} \sim Gamma(\alpha = \frac{1}{30}, \beta = 1)$. In Figure 3.5, considering that $B^{(0,2)}$, $B^{(0,1,1)}$ and $B^{(0,1,2)}$ are iid, we clearly see the impact of $M^{(0)}$ and $M^{(0,1)}$ on their respective dependence structure. Using the numbered boxes with coordinates (x, y) in Figure 3.5, the scatter plot associated to the pair $(U_{(0,2),1}, U_{(0,1),2})$ (see box $(5, 1)$) has a shape that is the closest to the one of a Clayton copula with a dependence parameter of 30. This phenomenon is explained by the fact that the couple depends only on $M^{(0)}$ with a parameter of 0.5, which in turn has little to no impact on $\Theta^{(0,2)}$. The scatter plot for the pair $(U_{(0,1,1),1}, U_{(0,1,1),2})$ (see box $(1, 5)$), however, shows a dependence restrained by $M^{(0,1)}$, which has a much bigger impact due to its overall higher values. We can also see that boxes $\{(1, 1), (1, 2), \dots, (4, 1), (4, 2)\}$ only depend on $M^{(0)}$, which implies a very small dependence. However, boxes $\{(1, 3), (1, 4), (2, 3), (2, 4)\}$ depend on $M^{(0,1)}$, a rv compounded on $M^{(0)}$. As discussed in Section 3.2.5, the copula generated from $M^{(0,1)}$ is more concordant than the one generated from $M^{(0)}$ and hence $\tau_{M^{(0,1)}} \geq \tau_{M^{(0)}}$.

3.3 Representation of the copula as a common mixture

The functional symmetry of Archimedean copulas lead to the introduction by Joe (1997) of nested Archimedean copulas. This popular class of copulas results from nesting Archimedean copulas into each other, allowing asymmetries and multiple hierarchy levels. Such constructions are however limited by nesting conditions that must be fulfilled. In contrast, hierarchical Archimedean copulas based on multivariate compound distributions provide very flexible combinations of Archimedean copula families without such restrictions. Given their similar nature, it is of interest to push further their comparison. For that purpose, we use the

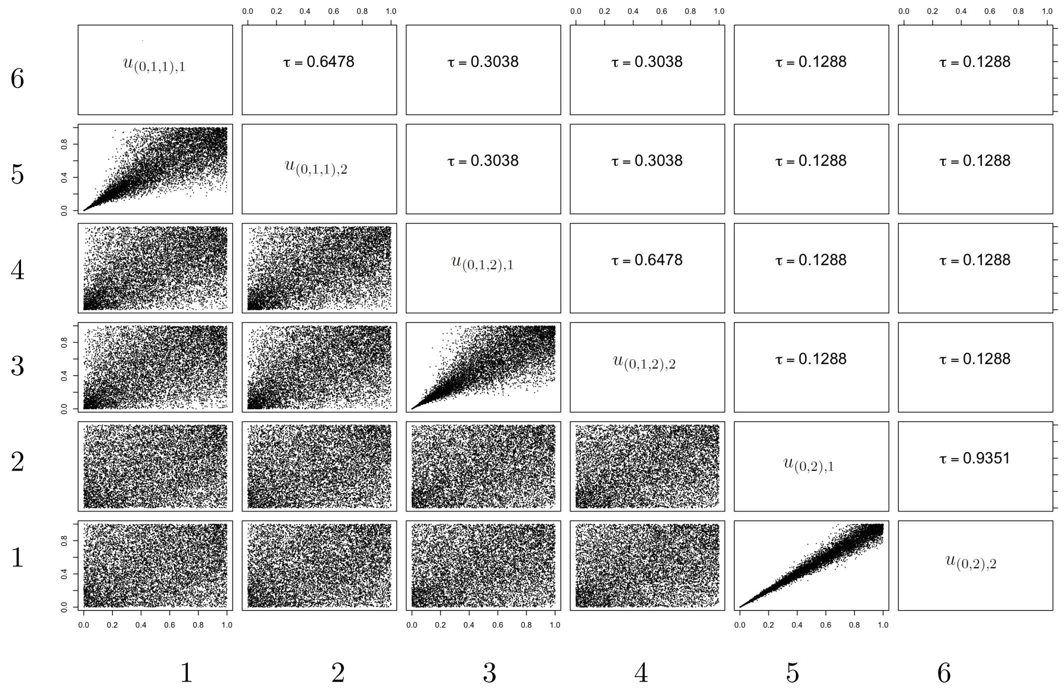


Figure 3.5 – Pairwise graph of 10000 6-dimensional vectors of realizations sampled from the hierarchical Archimedean copula with multivariate compound distributions in Example 3.2.5

common mixture representation of a copula to write the proposed hierarchical Archimedean copula. This will also prove useful for further investigation of aggregation methods.

Using (3.20) and (3.21), the common mixture representation of a hierarchical Archimedean copula based on a multivariate compound distribution is given by

$$\begin{aligned} C\left(\underline{u}\right) &= \mathcal{L}_{\Theta_1,\ldots,\Theta_d}\left(\sum_{j=1}^{n_1}\mathcal{L}_{\Theta_1}^{-1}\left(u_{1,j}\right),\ldots,\sum_{j=1}^{n_d}\mathcal{L}_{\Theta_d}^{-1}\left(u_{d,j}\right)\right) \\ &= \mathcal{P}_M\left(\mathcal{L}_{B_1}\left(\sum_{j=1}^{n_1}\mathcal{L}_{\Theta_1}^{-1}\left(u_{1,j}\right)\right)\times\ldots\times\mathcal{L}_{B_d}\left(\sum_{j=1}^{n_d}\mathcal{L}_{\Theta_d}^{-1}\left(u_{d,j}\right)\right)\right) \\ &= \sum_{m=1}^{\infty}\prod_{i=1}^{d}\mathcal{L}_{B_i}^{m}\left(\sum_{j=1}^{n_i}\mathcal{L}_{\Theta_i}^{-1}\left(u_{i,j}\right)\right)f_M\left(m\right). \end{aligned} \tag{3.29}$$

Since $\mathcal{L}_{B_i}^{m}$ is the LST of a positive rv which we denote by V_i, we can write the copula given in (3.29) as

$$C\left(\underline{u}\right) = \sum_{m=1}^{\infty}\prod_{i=1}^{d}\Big(\int_0^\infty e^{-v_i\left(\sum_{j=1}^{n_i}\mathcal{L}_{\Theta_i}^{-1}(u_{i,j})\right)}\mathrm{d}F_{V_i}(v_i)\Big)f_M(m). \tag{3.30}$$

From the common mixture representation given in (3.30), we adapt Algorithm 3.2.1 and provide below a more efficient one.

Algorithm 3.3.1. *Let C be a hierarchical Archimedean copula with d subgroups and root M_0. Define $\underline{U}$ as a vector of standard uniformly distributed rvs with cdf C.*

1. *Sample M_0;*
2. *For each subgroup ($i = 1, \ldots, d \geq 2$):*
   2.1. *Sample V_i with $\mathcal{L}_{V_i}(t) = \mathcal{L}_{B_i}^{M_0}(t)$;*
   2.2. *Sample $R_{i,j} \sim Exp(1)$ for $j = 1, \ldots, n_i$;*
   2.3. *Return $U_{i,j} = \mathcal{L}_{\Theta_i}(R_{i,j}/V_i)$ for $j = 1, \ldots, n_i$;*
3. *Return $\underline{U} = (U_{1,1}, \ldots, U_{1,n_1}, \ldots, U_{d,1}, \ldots, U_{d,n_d})$.*

In order to confirm the efficiency of Algorithm 3.3.1, let us consider a simple one level tree hierarchical Archimedean copula with two groups where $M \sim Geo(q)$, $B_1 \sim Gamma(2.5, 1)$ and $B_2 \sim Gamma(2.5, 1)$. The top graph of Figure 3.6 serves as an illustration to compare computation times (in seconds) between Algorithm 3.2.1 and Algorithm 3.3.1 for 100 000 realizations with respect to q, while the bottom one compares both algorithms with respect to the sampling size. Evidently, Algorithm 3.3.1 is far more efficient. In fact, Algorithm 3.3.1 has a very small computation time since we sample directly from the known distribution of the sum in step 2.1. of Algorithm 3.3.1.

3.4 Comparisons with other construction methodologies of hierarchical Archimedean Copulas

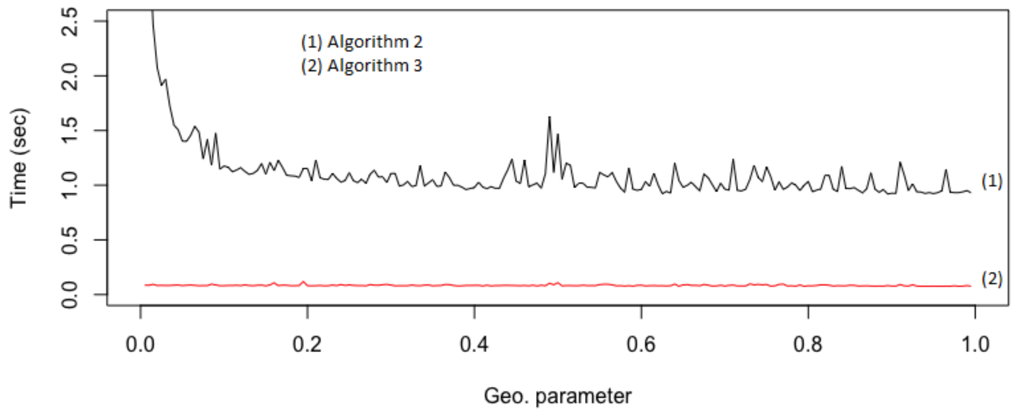
3.4.1 Links with nested Archimedean copulas

Now that hierarchical Archimedean copulas with multivariate compound distributions are written as common mixtures, we aim to compare them to nested Archimedean copulas. To do so, we consider a simple example of a one level partially nested Archimedean copula with d children copulas i.e.

$$\begin{aligned} C\left(\underline{u}\right) &= C\left(C\left(\underline{u_1};\psi_1\right),\ldots,C\left(\underline{u_d};\psi_d\right);\psi_0\right) \\ &= \psi_0\left(\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right)\right)\right)\right), \end{aligned} \tag{3.31}$$

where ψ is the generator of the Archimedean copula C. If we suppose that ψ_0 is the LST of a discrete strictly positive rv R, then, (3.31) becomes

$$\begin{aligned} C\left(\underline{u}\right) &= \sum_{r=1}^{\infty}\prod_{i=1}^{d}\mathrm{e}^{-r\,\psi_0^{-1}\circ\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)} f_R(r) \\ &= \sum_{r=1}^{\infty}\prod_{i=1}^{d}\psi_{0i}\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j});r\right) f_R(r), \end{aligned} \tag{3.32}$$



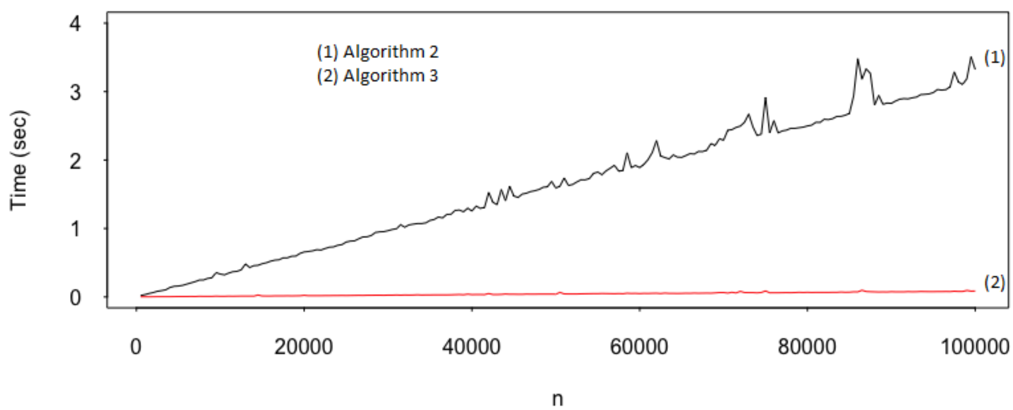


Figure 3.6 – **Top**: Computation time (sec) for Algorithm 3.2.1 and Algorithm 3.3.1 with respect to q. **Bottom**: Computation time (sec) for Algorithm 3.2.1 and Algorithm 3.3.1 with respect to the sampling size (n) with $q = 0.01$.

where $\psi_{0i}(t;r) = \exp\left(-r \times \psi_0^{-1} \circ \psi_i(t)\right)$. The representation in (3.32) is a proper copula if and only if ψ_{0i} is the LST of a positive rv, say W_i, i.e.

$$C\left(\underline{u}\right) = \sum_{r=1}^{\infty} \prod_{i=1}^{d} \left(\int_0^{\infty} e^{-w_i\left(\sum_{j=1}^{n_i} \psi_i^{-1}(u_{i,j})\right)} \mathrm{d}F_{W_i}(w_i) \right) f_R(r). \tag{3.33}$$

Remark 3.4.1. *It is important to highlight that verifying that ψ_{0i} is a LST, and hence that the sufficient nesting condition is verified, is not an easy task. Nesting together different Archimedean families has its limitations and parameter restrictions (see e.g. Hofert (2010)).*

We can clearly see that the representation in (3.30) of the hierarchical Archimedean copula with multivariate compound distributions and the one in (3.33) of the nested Archimedean copula have the same general form. In order to compare these two representations, we rewrite $\mathcal{L}_{B_i}^m(t)$ as

$$\mathcal{L}_{B_i}^m(t) = e^{-m\mathcal{L}_M^{-1} \circ \mathcal{L}_{\Theta_i}(t)} = \psi_{0i}(t; m). \tag{3.34}$$

Expressing $\mathcal{L}_{B_i}^m$ as ψ_{0i} allows us to represent the hierarchical Archimedean copula with multivariate compound distributions in (3.30) as the partially nested Archimedean copula in (3.32), where the mother copula C_0 is generated from the rv M, and the child copula C_i is generated from Θ_i. Note that C_i is an Archimedean copula derived from $\Theta_i = \sum_{j=1}^{M} B_{i,j}$, which is in general not a well known distribution and expressed in terms of the rv M. Given that $\psi_{0i} = \mathcal{L}_{B_i}^m$, we have that ψ_{0i} solely depends in our structure on the distribution of B_i unlike in the nested Archimedean copulas in which both the mother and the child copulas affect ψ_{0i}, leading to a restrictive nesting condition. Also, since $\mathcal{L}_{B_i}^m$ depends only on the parameters of B_i regardless of the distribution of the rv M, the choice of M's distribution is very flexible in comparison to the nested Archimedean copulas, where nesting different families is a recurrent problem.

The above remarks imply that the hierarchical Archimedean copula C, as defined in (3.20) can be represented as a nested Archimedean copula as follows:

$$C(\underline{u}) = C(C(\underline{u_1}; \mathcal{L}_{\Theta_1}), \ldots, C(\underline{u_d}; \mathcal{L}_{\Theta_d}); \mathcal{L}_M). \tag{3.35}$$

Since the hierarchical Archimedean copulas proposed in this paper can be written as a nested Archimedean copula, one may wonder if we can produce similar copulas from both approaches. To do so, we have to find the distribution of B_i for which (3.35) is a known nested Archimedean copula.

Let M be a positive and discrete rv and Θ_i $(i = 1, \ldots, d)$ be strictly positive rvs generating known Archimedean copulas. In this case, the distribution of the rvs B_i can be deduced from

$$\mathcal{L}_{B_i}(t) = \mathcal{P}_M^{-1}\left(\mathcal{L}_{\Theta_i}(t)\right) = \mathrm{e}^{-\mathcal{L}_M^{-1}\left(\mathcal{L}_{\Theta_i}(t)\right)}. \tag{3.36}$$

From (3.34), (3.36) becomes

$$\mathcal{L}_{B_i}^m(t) = \psi_{01}(t; m), \tag{3.37}$$

which means that

$$W_i = \sum_{k=1}^{m} B_i,$$

where W_i is the rv generating $\psi_{01}(.;m)$.

To illustrate this idea, let (3.35) be a nested AMH-AMH copula. To obtain such a copula with our approach, we must have $M \sim Geo(1-\alpha_0)$ and $\Theta_i \sim Geo(1-\alpha_i)$ which implies that $B_i \sim Geo\left(\frac{1-\alpha_i}{1-\alpha_0}\right)$, $\alpha_0 \leq \alpha_i$ $(i = 1,\ldots,d)$. Similarly, in order for (3.35) to be a nested Joe-Joe copula, our approach requires that $M \sim Sibuya(\alpha_0)$ and $\Theta_i \sim Sibuya(\alpha_i)$ which implies that $B_i \sim Sibuya\left(\frac{\alpha_i}{\alpha_0}\right)$, $\alpha_i \leq \alpha_0$ $(i = 1,\ldots,d)$.

The representation in (3.35) can be easily generalized to multi-level hierarchical structures by adapting the notation presented in Section 3.2.8. For example, (3.28) can also be written as

$$C(\underline{u_1},\underline{u_2},\underline{u_3}) = C\left(C\left(C\left(\underline{u_1};\mathcal{L}_{\Theta^{(0,1,1)}}\right), C\left(\underline{u_2};\mathcal{L}_{\Theta^{(0,1,2)}}\right);\mathcal{L}_{M^{(0,1)}}\right), C\left(\underline{u_3};\mathcal{L}_{\Theta^{(0,2)}}\right);\mathcal{L}_{M^{(0)}}\right), \tag{3.38}$$

where $\underline{u_1} = (u_{(0,1,1),1}, u_{(0,1,1),2})$, $\underline{u_2} = (u_{(0,1,2),1}, u_{(0,1,2),2})$ and $\underline{u_3} = (u_{(0,2),1}, u_{(0,2),2})$.

Notation wise, (3.38) can be simplified to

$$C(\underline{u_1},\underline{u_2},\underline{u_3}) = C^{(0)}\left(C^{(0,1)}\left(C^{(0,1,1)}\left(\underline{u_1}\right), C^{(0,1,2)}\left(\underline{u_2}\right)\right), C^{(0,2)}\left(\underline{u_3}\right)\right). \tag{3.39}$$

3.4.2 Approach based on Lévy subordinators and random time

In this section, we examine the link between the approach proposed by Hering et al. (2010) to construct hierarchical Archimedean copulas and the one described within this paper. The construction approach by Hering et al. (2010) is based on increasing Lévy processes, also called Lévy subordinators. They are notably important components for building models in financial mathematics (see e.g. Cont and Tankov (2003)).

Let $\underline{\Lambda}^{(i)} = \left\{\Lambda^{(i)}(t), t \geq 0\right\}$ be a Lévy subordinator, for $i = 1,2,\ldots,d$. For all $x,t \geq 0$, the LST of $\Lambda^{(i)}(t)$ is given by

$$\mathcal{L}_{\Lambda^{(i)}(t)}(x) = E\left[\mathrm{e}^{-x\Lambda^{(i)}(t)}\right] = \mathrm{e}^{-t\zeta_i(x)},$$

where the function ζ_i corresponds to the Laplace exponent of $\underline{\Lambda}^{(i)}$, for $i = 1,2,\ldots,d$.

Let a strictly positive rv V be a random time with LST defined by

$$\mathcal{L}_V(x) = E\left[\mathrm{e}^{-Vx}\right] = \psi_0(x),$$

for $x \geq 0$.

Table 1 of Hering et al. (2010) provides a short list of possible distributions for V, which can be either discrete or continuous. The random time rv V is used as a common factor to define the vector of rvs

$$\underline{\Lambda}(V) = \left(\Lambda^{(1)}(V), \ldots, \Lambda^{(d)}(V)\right).$$

Given V, the rvs $\Lambda^{(i)}(V)$, for $i = 1, \ldots, d$, are assumed to be conditionally independent. This implies that the multivariate LST of $\underline{\Lambda}(V)$ is obtained as follows :

$$\begin{aligned}
\mathcal{L}_{\underline{\Lambda}(V)}(x_1, \ldots, x_d) &= E\left[\mathrm{e}^{-x_1\Lambda^{(1)}(V)} \times \ldots \times \mathrm{e}^{-x_d\Lambda^{(d)}(V)}\right] \\
&= E_V\left[E\left[\mathrm{e}^{-x_1\Lambda^{(1)}(V)} \times \ldots \times \mathrm{e}^{-x_d\Lambda^{(d)}(V)}|V\right]\right] \\
&= E\left[\mathrm{e}^{-V\zeta_1(x_1)} \times \ldots \times \mathrm{e}^{-V\zeta_d(x_d)}\right] \\
&= E\left[\mathrm{e}^{-V(\zeta_1(x_1)+\ldots+\zeta_d(x_d))}\right] \\
&= \mathcal{L}_V(\zeta_1(x_1) + \ldots + \zeta_d(x_d)). \qquad (3.40)
\end{aligned}$$

Following a similar procedure to the one presented in Section 3.2.2, Hering et al. (2010) propose to construct multivariate mixed exponential distributions using $\underline{\Lambda}(V)$. Let

$$\underline{Y} = (Y_{1,1}, \ldots, Y_{1,n_1}, \ldots, Y_{d,1}, \ldots, Y_{d,n_d})$$

be a vector of $n_1 + \ldots + n_d$ rvs. Given $\underline{\Lambda}(V) = \underline{\lambda}$, it is assumed that

$$(Y_{1,1}|\underline{\Lambda}(V) = \underline{\lambda}), \ldots, (Y_{1,n_1}|\underline{\Lambda}(V) = \underline{\lambda}), \ldots, (Y_{d,1}|\underline{\Lambda}(V) = \underline{\lambda}), \ldots, (Y_{d,n_d}|\underline{\Lambda}(V) = \underline{\lambda})$$

are conditionally independent. Also, given $\underline{\Lambda}(V) = \underline{\lambda} = (\lambda_1, \ldots, \lambda_d)$, the rv $(Y_{i,j}|\underline{\Lambda}(V) = \underline{\lambda})$ is identically distributed as $\left(Y_{i,j}|\Lambda^{(i)}(V) = \lambda_i\right)$ for $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, d$. We assume that $\left(Y_{i,1}|\Lambda^{(i)}(V) = \lambda_i\right), \ldots, \left(Y_{i,n_i}|\Lambda^{(i)}(V) = \lambda_i\right)$ are exponentially distributed with parameter λ_i, for $i = 1, \ldots, d$. Then, $\underline{Y}$ follows a multivariate mixed exponential distribution with multivariate survival function given by

$$\begin{aligned}
&\overline{F}_{\underline{Y}}(y_{1,1}, \ldots, y_{1,n_1}, \ldots, y_{d,1}, \ldots, y_{d,n_d}) \\
= \; & E\left[\mathrm{e}^{-\Lambda^{(1)}(V)y_{1,1}} \times \ldots \times \mathrm{e}^{-\Lambda^{(1)}(V)y_{1,n_1}} \times \ldots \times \mathrm{e}^{-\Lambda^{(d)}(V)y_{d,1}} \times \ldots \times \mathrm{e}^{-\Lambda^{(d)}(V)y_{d,n_d}}\right]. \qquad (3.41)
\end{aligned}$$

Due to (3.40), we can write (3.41) as

$$\begin{aligned}
&\overline{F}_{\underline{Y}}(y_{1,1}, \ldots, y_{1,n_1}, \ldots, y_{d,1}, \ldots, y_{d,n_d}) \\
= \; & \mathcal{L}_{\underline{\Lambda}(V)}(y_{1,1} + \ldots + y_{1,n_1}, \ldots, y_{d,1} + \ldots + y_{d,n_d}) \\
= \; & \psi_0(\zeta_1(y_{1,1} + \ldots + y_{1,n_1}) + \ldots + \zeta_d(y_{d,1} + \ldots + y_{d,n_d})). \qquad (3.42)
\end{aligned}$$

Also, the univariate survival function of $Y_{i,j}$ is

$$\overline{F}_{Y_{i,j}}(y_{i,j}) = E\left[\mathrm{e}^{-\Lambda^{(i)}(V)y_{i,j}}\right] = \mathcal{L}_{\Lambda^{(i)}(V)}(y_{i,j}) = \mathcal{L}_V(\zeta_i(y_{i,j})) = \psi_0(\zeta_i(y_{i,j})) = \psi_0 \circ \zeta_i(y_{i,j}), \qquad (3.43)$$

for $y_{i,j} \geq 0$, $j = 1, 2, \ldots, n_i$, and $i = 1, \ldots, d$.

Then, using Sklar's Theorem, the copula associated to $\overline{F}_{\underline{Y}}$ is given by

$$C(\underline{u}) = \overline{F}_{\underline{Y}} \left(\overline{F}^{-1}_{Y_{1,1}}(u_{1,1}), \ldots, \overline{F}^{-1}_{Y_{1,n_1}}(u_{1,n_1}), \ldots, \overline{F}^{-1}_{Y_{d,1}}(u_{d,1}), \ldots, \overline{F}^{-1}_{Y_{d,n_d}}(u_{d,n_d}) \right), \quad (3.44)$$

where $\overline{F}^{-1}_{Y_{i,j}}$ is obtained from (3.43) i.e.

$$\overline{F}^{-1}_{Y_{i,j}}(u_{i,j}) = \mathcal{L}^{-1}_{\Lambda^{(i)}(V)}(u_{i,j}) = \zeta_i^{-1}\left(\psi_0^{-1}(u_{i,j})\right) = \zeta_i^{-1} \circ \psi_0^{-1}(u_{i,j}), \quad (3.45)$$

for $u_{i,j} \in [0,1]$, $j = 1, 2, \ldots, n_i$, and $i = 1, \ldots, d$. Combining (3.44) and (3.45) with (3.42), the expression for $C(\underline{u})$ becomes

$$\begin{aligned} C(\underline{u}) &= \mathcal{L}_{\underline{\Lambda}(V)}\left(\mathcal{L}^{-1}_{\Lambda^{(1)}(V)}(u_{1,1}) + \ldots + \mathcal{L}^{-1}_{\Lambda^{(1)}(V)}(u_{1,n_1}), \ldots, \mathcal{L}^{-1}_{\Lambda^{(d)}(V)}(u_{d,1}) + \ldots + \mathcal{L}^{-1}_{\Lambda^{(d)}(V)}(u_{1,n_d})\right) \\ &= \psi_0\left(\sum_{i=1}^{d} \zeta_i\left(\zeta_i^{-1} \circ \psi_0^{-1}(u_{i,1}) + \ldots + \zeta_i^{-1} \circ \psi_0^{-1}(u_{i,n_i})\right)\right). \end{aligned}$$

In Hering et al. (2010), conditions for admissible Lévy subordinators are given in Theorem 2.1 and a list of popular Lévy subordinators can be found in Table 2.

We may now compare both approaches and the resulting hierarchical Archimedean copulas. Under both approaches, a hierarchical Archimedean copula is identified from the joint survival function of a multivariate mixed exponential distribution. In both cases, it leads to natural generic sampling algorithms for hierarchical Archimedean copulas. The dependence structure of the multivariate mixed exponential distribution is defined through a vector of dependent mixing rvs,

$$\underline{\Lambda}(V) = \left(\Lambda^{(1)}(V), \ldots, \Lambda^{(d)}(V)\right)$$

in Hering et al. (2010), and through

$$\underline{\Theta} = (\Theta_1, \ldots, \Theta_d)$$

within this paper. Even if these two vectors of rvs are defined through different probabilistic arguments (Lévy subodinators with a common random time and random sums with a common counting rv), both approaches lead to identical generic structures for $\mathcal{L}_{\underline{\Lambda}(V)}$ and $\mathcal{L}_{\underline{\Theta}}$ and for $\mathcal{L}_{\Lambda^{(i)}(V)}$ and $\mathcal{L}_{\Theta_i}$. The expression for $\mathcal{L}_{\Lambda^{(i)}(V)}$ is

$$\mathcal{L}_{\Lambda^{(i)}(V)}(x) = \mathcal{L}_V(\zeta_i(x)),$$

for $i = 1, \ldots, d$. Let us define the function υ_i associated to $\mathcal{L}_{B_i}$ such that

$$\mathcal{L}_{B_i}(x) = \exp(-\upsilon_i(x)),$$

or, equivalently,

$$\upsilon_i(x) = -\ln(\mathcal{L}_{B_i}(x)),$$

for $i = 1, 2, \ldots, d$. Then, the expression for $\mathcal{L}_{\Theta_i}$ becomes

$$\mathcal{L}_{\Theta_i}(x) = \mathcal{P}_M(\mathcal{L}_{B_i}(x)) = \mathcal{L}_M(-\ln \mathcal{L}_{B_i}(x)) = \mathcal{L}_M(\upsilon_i(x)),$$

for $i = 1, \ldots, d$. Note that

$$\mathcal{L}_{\underline{\Lambda}(V)}(x_1, \ldots, x_d) = \mathcal{L}_V(\zeta_1(x_1) + \ldots + \zeta_d(x_d))$$

and

$$\mathcal{L}_{\underline{\Theta}}(x_1, \ldots, x_d) = \mathcal{L}_M(\upsilon_1(x_1) + \ldots + \upsilon_d(x_d)).$$

The rvs V and M play similar roles as a common factor. As mentioned earlier, the rv V can be either discrete or continuous. Under the approach proposed within this paper, M is a strictly positive discrete rv. If V is a strictly positive discrete rv, then $\mathcal{L}_V$ and $\mathcal{L}_M$ are identical. Also, both $\mathcal{L}_V$ and $\mathcal{L}_M$ correspond to the generator ψ_0 in the definition of a nested Archimedean copula.

Finally, both υ_i and ζ_i correspond to the function $\psi_0^{-1} \circ \psi_1$ which can be found in the definition of the nested Archimedean copula and for which it is difficult to verify the conditions of admissibility. In conclusion, the approaches proposed both in Hering et al. (2010) and within the present paper circumvent this difficulty. Note that υ_i plays the same role as ζ_i (the Laplace exponent of the Lévy subordinator $\underline{\Lambda}^{(i)}$).

The construction method provided in this paper and Hering et al. (2010)'s approach have their own advantages even if their strategies appear similar. Both approaches can lead to the same hierarchical Archimedean copula in some cases. However, others can only be obtained with one of the two methods. For all the popular Lévy subordinators provided in Table 2 of Hering et al. (2010), it is possible to define the distribution of B_i such that its Laplace exponent υ_i is equal to the Laplace exponent ζ_i of $\underline{\Lambda}^{(i)}$, $i = 1, 2, \ldots, d$. For example, if B_i follows a gamma distribution with parameters β and 1, then $\upsilon_i(x) = \beta \ln(1 + x)$ as in (ii) of Table 2 (η is a scaling factor) then the expressions for $\mathcal{L}_{\Lambda^{(i)}(V)}$ and $\mathcal{L}_{\Theta_i}$ are identical (assuming $\mathcal{L}_V = \mathcal{L}_M$). If B_i follows an inverse Gaussian with parameters $\frac{1}{\eta}$ and 1, $\upsilon_i(x) = \left(\sqrt{\frac{2x}{\eta^2} + 1} - 1\right)$ as in (iii) of Table 2 (β is a scaling factor) then it follows that $\mathcal{L}_{\Lambda^{(i)}(V)}$ and $\mathcal{L}_{\Theta_i}$ have the same expression (assuming $\mathcal{L}_V = \mathcal{L}_M$). Also, under the assumption $\mathcal{L}_V = \mathcal{L}_M$ (i.e. V is a discrete rv) and since the rv B_i can follow many distributions including the ones listed in Table 1 of Hering et al. (2010), the proposed method provides a larger range of possible hierarchical Archimedean copulas in comparison to Hering et al. (2010)'s construction method. If V is a continuous rv, the resulting hierarchical Archimedean copula obtained with Hering et al. (2010)'s approach cannot be obtained with our approach.

Finally, inspired from Hering et al. (2010), the expression of the hierarchical Archimedean copula constructed under our proposed approach given in (3.20) becomes

$$\begin{aligned} C(\underline{u}) &= \mathcal{L}_{\underline{\Theta}}\left(\mathcal{L}_{\Theta_1}^{-1}(u_{1,1}) + \ldots + \mathcal{L}_{\Theta_1}^{-1}(u_{1,n_1}), \ldots, \mathcal{L}_{\Theta_d}^{-1}(u_{d,1}) + \ldots + \mathcal{L}_{\Theta_d}^{-1}(u_{d,n_d})\right) \\ &= \mathcal{L}_M\left(\sum_{i=1}^{d} \upsilon_i \left(\upsilon_i^{-1} \circ \mathcal{L}_M^{-1}(u_{i,1}) + \ldots + \upsilon_i^{-1} \circ \mathcal{L}_M^{-1}(u_{i,n_i})\right)\right). \end{aligned}$$

3.4.3 Estimation procedure and Determination of the Tree Structure

The main objective of the present work is to propose an alternative approach to construct hierarchical Archimedean copulas. The interestingness of these new copulas also relies on the capability to estimate them. Recently, different research works have appeared in the literature on the estimation of hierarchical Archimedean copulas, more specifically on the structure determination and the parameter estimation. Okhrin et al. (2013) appears to be the first paper to address simultaneously both of these tasks through a multi-stage procedure in a bottom-up manner. Maximum-likelihood estimation is used for the parameters and the inversion of Kendall's tau is also suggested. The structure determination is investigated in several ways, with notably approaches based on a goodness-of-fit test or binary trees. In Górecki et al. (2016a) and Górecki and Holeňa (2013), the estimator for both the structure and the parameters of hierarchical Archimedean copulas is based on Kendall's tau, more precisely on the inversion of Kendall's tau estimator. They use agglomerative hierarchical clustering (with three different definitions for the distance between clusters) in which the relationship between two rvs is established through their Kendall's tau. Other research papers, such as Uyttendaele (2016) and Segers and Uyttendaele (2014) estimate the structure differently by considering either trivariate structures or supertrees.

These works mainly focus on hierarchical Archimedean copulas in which all copulas in the tree are from the same Archimedean family. An excellent paper, Górecki et al. (2016b), paves the way to the estimation of the structure and the parameters of hierarchical Archimedean copulas involving different Archimedean families. This new estimation procedure, which is in part based on goodness-of-fit tests, adopts estimation algorithms from previous papers in a way that guarantees the verification of the nesting condition. In the same context of different family generators, Zhu et al. (2016) recently proposed an approach using a three stage estimation approach based on a clustering procedure to choose the optimal hierarchical structure as an estimation of a Lévy subordinated hierarchical Archimedean copula.

Given the similarities of our construction with the one of Hering et al. (2010) discussed in Section 3.4.2, the approach suggested in Zhu et al. (2016) is evidently a good starting point for the estimation of our structure based on maximum likelihood. This should be compared with the method proposed in Górecki et al. (2016b), based on Kendall's tau estimator, which allows the use of Archimedean copulas from different copula families as the proposed construction here. Contrarily to the nested Archimedean copulas approach, we will not be confronted with the difficulties surrounding the nesting condition. However, the flexibility of our new structure leads to new challenges in regard to finding the optimal solution among all possible structures. This mainly concerns for us, firstly, the choice of the appropriate LST of M and then the choice of the LSTs of the rvs B_i $(i = 1, \ldots, d)$. This will be followed by the estimation of the parameters of the hierarchical Archimedean copulas. These questions are at the core of another ongoing research project.

3.5 Aggregation method

Risk aggregation comes into play for tasks such as the analysis of a risk portfolio and regularly capital calculations within and between risk categories. In this section, we examine the behavior of aggregated dependent risks with multivariate cdf defined through a hierarchical Archimedean copula based on multivariate compound distributions.

Let $\underline{X} = (X_{1,1}, \ldots, X_{1,n_1}, \ldots, X_{d,1}, \ldots, X_{d,n_d})$ be a random vector with the following multivariate cdf defined with an underlying hierarchical Archimedean copula as given in (3.21) and marginals $F_{X_{i,j}}$ (for $i = 1, \ldots, d$ and $j = 1, \ldots, n_i$)

$$\begin{aligned} & F_{\underline{X}}\left(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{d,1}, \ldots, x_{d,n_d}\right) \\ = \; & C\left(F_{X_{1,1}}\left(x_{1,1}\right), \ldots, F_{X_{1,n_1}}\left(x_{1,n_1}\right), \ldots, F_{X_{d,1}}\left(x_{d,1}\right), \ldots, F_{X_{d,n_d}}\left(x_{d,n_d}\right)\right) \\ = \; & \mathcal{L}_{\underline{\Theta}}\left(\sum_{i=1}^{n_1} \mathcal{L}_{\Theta_1}^{-1}\left(F_{X_{1,i}}\left(x_{1,i}\right)\right), \ldots, \sum_{j=1}^{n_d} \mathcal{L}_{\Theta_d}^{-1}\left(F_{X_{d,j}}\left(x_{d,j}\right)\right)\right). \end{aligned} \tag{3.46}$$

Since the vectors of rvs $(X_{i,1} \mid \Theta_i = \theta_i), \ldots, (X_{i,n_i} \mid \Theta_i = \theta_i)$, for $i \in 1, \ldots, d$, are conditionally independent given $\Theta_i = \theta_i$ and $(X_{1,j_1} \mid M = m), \ldots, (X_{d,j_d} \mid M = m)$, for $j_i \in 1, \ldots, n_i$, are also conditionally independent given $M = m$, $F_{\underline{X}}$ can be represented as a common mixture of conditional cdfs as

$$\begin{aligned} & F_{\underline{X}}\left(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{d,1}, \ldots, x_{d,n_d}\right) \\ = \; & \sum_{m=1}^{\infty} F_{\underline{X}|M=m}\left(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{d,1}, \ldots, x_{d,n_d}\right) f_M(m) \\ = \; & \sum_{m=1}^{\infty} \prod_{i=1}^{d}\left(\int_0^{\infty} \prod_{j=1}^{n_i} F_{X_{i,j}|M=m,\Theta_i=\theta_i}\left(x_{i,j}\right) \mathrm{d}F_{\Theta_i}(\theta_i)\right) f_M(m). \end{aligned} \tag{3.47}$$

To find the expression of the conditional cdfs $F_{X_{i,j}|M=m,\Theta_i=\theta_i}$, we use the common mixture representation given in (3.30). We obtain

$$F_{\underline{X}}\left(x_{1,1}, \ldots, x_{d,n_d}\right) = \sum_{m=1}^{\infty} \prod_{i=1}^{d}\left(\int_0^{\infty} e^{-v_i\left(\sum_{j=1}^{n_i} \mathcal{L}_{\Theta_i}^{-1}\left(F_{X_{i,j}}(x_{i,j})\right)\right)} dF_{V_i}(v_i)\right) f_M(m). \tag{3.48}$$

Given (3.47) and (3.48), we have $F_{X_{i,j}|M=m,V_i=v_i}(x_{i,j}) = e^{-v_i\left(\mathcal{L}^{-1}_{\Theta_i}\left(F_{X_{i,j}}(x_{i,j})\right)\right)}$, for $i = 1, \ldots, d \geq 2$ and $j = 1, \ldots, n_i$.

Let $S = \sum_{i=1}^{d} S_i$, where $S_i = X_{i,1} + \ldots + X_{i,n_i}$, for $i = 1, 2, \ldots, d$. To find the desired cdf of S, one may apply the approach proposed in Cossette et al. (2018). To do so however, we must assume V_i to be a strictly positive discrete rv defined on $\{1, 2, \ldots\}$, with probability mass function $f_{V_i}(v_i) = \Pr(V_i = v_i)$, $V_i = 1, 2, \ldots$ In such a case, (3.47) becomes

$$F_{\underline{X}}(x_{1,1}, \ldots, x_{d,n_d}) = \sum_{m=1}^{\infty} \prod_{i=1}^{d} \left(\sum_{v_i=1}^{\infty} \Big(\prod_{j=1}^{n_i} F_{X_{i,j}|M=m,V_i=v_i}(x_{i,j}) \Big) f_{V_i}(v_i) \right) f_M(m).$$

We provide next an example to illustrate the applicability of the aggregation procedure proposed in Cossette et al. (2018) in the context of the new construction given in this paper. Since the method of Cossette et al. (2018) leads to exact results, the following example also shows the precision of the values obtained with Algorithm 3.3.1.

Example 3.5.1. *Consider a portfolio of 80 risks* $\underline{X} = (X_{1,1}, \ldots, X_{1,40}, X_{2,1}, \ldots, X_{2,40})$ *with a multivariate cdf defined as in (3.46) with* $d = 2$ *and* $n_1 = n_2 = 40$. *We assume* $M \sim$ *Logarithmic*(0.5), $B_1 \sim$ *Geo*(0.8) *and* $B_2 \sim$ *Geo*(0.9). *Let* $X_{s,i} \sim$ *Binom*$(10, q_{s,i})$ *where* $q_{s,i} = 0.05 \times s + 0.005i$, $s = 1, 2$ *and* $i = 1, 2, \ldots, 40$. *It implies that* $E[S] = 142$. *Relevant measures of* $S = \sum_{s=1}^{2} \sum_{i=1}^{40} X_{s,i}$ *can be obtained with the approach of Cossette et al. (2018) or with MC simulations using Algorithm 3.3.1. Note that the distributions of* M *and* B_i *are not from the same families. Both risk measures,* VaR *and* $TVaR$, *are given in Table 3.2 to illustrate relevant results for both methods as a mean of comparison. Note that the approach of Cossette et al. (2018) always provides exact results for discrete marginals* $F_{X_{i,j}}$. *Moreover, the simulation results are very close to the actual results, considering 1 million simulations were done.*

3.6 Conclusion

A new hierarchical Archimedean copulas construction method involving multivariate compound distributions was presented. The absence of nesting and marginal conditions enlarges the possibilities of nested copulas by improving the flexibility in both the choice of families and parameters. Moreover, new copulas with multiple parameters were derived allowing for a large variety of dependence structures. In addition to the new parametric copulas, well-known Archimedean copulas can be obtained as special cases. Closed form expressions have been derived for the copulas. To further complement our theoretical results, efficient sampling algorithms have been developed for computational applications.

Risk Measures	Cossette et al. (2018)	Sampling (1M)
$Var(S)$	1157.4461	1156.8011
$VaR_{0.9}(S)$	193.0000	193.0000
$TVaR_{0.9}(S)$	214.4829	214.4334
$VaR_{0.99}(S)$	240.0000	240.0000
$TVaR_{0.99}(S)$	252.1244	252.2950
$VaR_{0.999}(S)$	267.0000	268.0000
$TVaR_{0.999}(S)$	276.1494	276.4880
$VaR_{0.9999}(S)$	287.0000	288.0000
$TVaR_{0.9999}(S)$	293.5822	294.8900

Table 3.2 – Values of the variance, VaR and TVaR of $S = X_{1,1} + \ldots + X_{1,40} + X_{2,1} + \ldots + X_{2,40}$ where the joint cdf $F_{X_{1,1},\ldots,X_{1,40},X_{2,1},\ldots,X_{2,40}}$ is as defined in Example 3.5.1.

3.7 Bibliography

Bedford, T. and Cooke, R. M. (2002). Vines: A new graphical model for dependent random variables. *Annals of Statistics*, pages 1031–1068.

Brechmann, E. C. (2014). Hierarchical kendall copulas: Properties and inference. *Canadian Journal of Statistics*, 42(1):78–108.

Cont, R. and Tankov, P. (2003). *Financial modelling with jump processes, Vol. 2, 2003.* Chapman & Hall/CRC: London.

Cossette, H., Marceau, E., Mtalai, I., and Veilleux, D. (2018). Dependent risk models with archimedean copulas: A computational strategy based on common mixtures and applications. *Insurance: Mathematics and Economics*, 78:53–71.

Devroye, L. (2009). Random variate generation for exponentially and polynomially tilted stable distributions. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 19(4):18.

Feller, W. (1971). Introduction to probability theory and its appl. ii. *New York*.

Górecki, J., Hofert, M., and Holeňa, M. (2016a). An approach to structure determination and estimation of hierarchical archimedean copulas and its application to bayesian classification. *Journal of Intelligent Information Systems*, 46(1):21–59.

Górecki, J., Hofert, M., and Holeňa, M. (2016b). On structure, family and parameter estimation of hierarchical archimedean copulas. *arXiv preprint arXiv:1611.09225*.

Górecki, J. and Holeňa, M. (2013). Structure determination and estimation of hierarchical archimedean copulas based on kendall correlation matrix. In *International Workshop on New Frontiers in Mining Complex Patterns*, pages 132–147. Springer.

Hering, C., Hofert, M., Mai, J.-F., and Scherer, M. (2010). Constructing hierarchical archimedean copulas with lévy subordinators. *Journal of Multivariate Analysis*, 101(6):1428–1433.

Hofert, M. (2010). *Sampling nested Archimedean copulas with applications to CDO pricing.* PhD thesis, Universität Ulm.

Hofert, M. (2011). Efficiently sampling nested archimedean copulas. *Computational Statistics & Data Analysis*, 55(1):57–70.

Hofert, M. (2012). A stochastic representation and sampling algorithm for nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 82(9):1239–1255.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts.* CRC Press.

Joe, H. (2014). *Dependence modeling with copulas.* CRC Press.

Kimberling, C. H. (1974). A probabilistic interpretation of complete monotonicity. *Aequationes mathematicae*, 10(2):152–164.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841.

McNeil, A. J. (2008). Sampling nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 78(6):567–581.

Nelsen, R. B. (2007). *An introduction to copulas.* Springer Science & Business Media.

Okhrin, O., Okhrin, Y., and Schmid, W. (2013). On the structure and estimation of hierarchical archimedean copulas. *Journal of Econometrics*, 173(2):189–204.

Segers, J. and Uyttendaele, N. (2014). Nonparametric estimation of the tree structure of a nested archimedean copula. *Computational Statistics & Data Analysis*, 72:190–204.

Sklar, M. (1959). *Fonctions de répartition à n dimensions et leurs marges.* Université Paris 8.

Uyttendaele, N. (2016). On the estimation of nested archimedean copulas: A theoretical and an experimental comparison.

Zhu, W., Wang, C.-W., and Tan, K. S. (2016). Structure and estimation of Lévy subordinated hierarchical archimedean copulas (lshac): Theory and empirical tests. *Journal of Banking & Finance*, 69:20–36.

Chapitre 4

Collective Risk Models with Hierarchical Archimedean Copulas

Résumé

La perte agrégée d'une compagnie d'assurance est toujours modélisée par une somme aléatoire. Il est très courant de supposer que le nombre de sinistres et leurs montants sont indépendants même si cela n'est pas toujours le cas en pratique. Dans ce chapitre, on considère une relation de dépendance entre la fréquence et la sévérité des sinistres. On propose dans un premier temps, différents modèles collectifs de risque avec dépendance basée sur des copules bivariées ou multivariées. Ensuite, on se sert de la famille des copules Archimédiennes et Archimédiennes hiérarchiques afin de modéliser la structure de dépendance entre les composantes de la somme aléatoire représentant le montant total des sinistres. Cette famille de copules permet d'obtenir une méthodologie computationnelle pour l'évaluation des quantités reliées aux pertes agrégées. Tout en étant très flexible, cette méthodologie est facile à mettre en œuvre et peut facilement s'adapter à des structures hiérarchiques plus complexes.

Abstract

In actuarial science, collective risk models, in which the aggregate claim amount of a portfolio is defined in terms of random sums, play a crucial role. In these models, it is common to assume that the number of claims and their amounts are independent, even if this might not always be the case. We consider collective risk models with different dependence structures. Due to the importance of such distributions in an actuarial setting, we first investigate a collective risk model with dependence involving the family of multivariate mixed Erlang distributions. Other models based on mixtures involving bivariate and multivariate copulas in a more general setting are then presented. These different structures allow to link the number of claims to each claim amount, and to quantify the aggregate claim loss. Then, we use Archimedean and hierarchical Archimedean copulas in collective risk models, to model the dependence between the claim number random variable and the claim amount random variables involved in the random sum. Such dependence structures allow us to derive a computational methodology for the assessment of the aggregate claim amount. While being very flexible, this methodology is easy to implement, and can easily fit more complicated hierarchical structures.

Keywords: Random sums; collective risk models; dependence; copulas; Archimedean copulas; hierarchical Archimedean copulas; mixed Erlang distributions

4.1 Introduction

Collective risk models are fundamental in actuarial science to model the aggregate claim amount of an insurance company. For a given portfolio of policyholders and over a fixed period of time, the aggregate claim amount random variable (rv) S is defined as a random sum, i.e.,

$$S = \sum_{i=1}^{N} X_i,$$

where $\underline{X} = \{X_i, i \in \mathbb{N}\}$ is a sequence of non-negative rvs, and N is a positive counting rv. The rv N represents the number of claims and X_i corresponds to the amount of the i^{th} claim ($i \in \mathbb{N}$).

In the classical collective risk model, the components of $\underline{X}$ are assumed to be independent of the counting rv N, and also independent and identically distributed (iid) rvs (see, e.g., Rolski et al. (1999) and Klugman et al. (2009)). However in practice, these assumptions are not always verified. For example, while analyzing a car insurance data-set, Gschlossl and Czado (2007) found that the number and the size of claims are significantly dependent. See also, e.g., Kousky and Cooke (2009) for other related examples, such as the highlighted dependency between flood damage and wind damage, using a catastrophic loss data.

While several papers proposed models that only account for the dependence between claim amounts (see e.g., Denuit et al. (2006)), few others considered an extra dependency between claim amounts and claim counts as well. For example, claim counts can be considered as predictors for the claim amounts, see e.g., Gschlossl and Czado (2007), Frees et al. (2011) and Garrido et al. (2016). In another setting, inter-claim times and claim sizes are assumed to be dependent in a compound Poisson process, see e.g., Albrecher et al. (2014), Boudreault et al. (2006), Cossette et al. (2008), and Landriault et al. (2014).

Due to their interesting properties, copulas have been widely used to model a dependence structure between rvs. In a collective risk context, such a structure has been used by several researchers to model the dependence between claim amounts and their related sizes. For example, Czado et al. (2012) and Krämer et al. (2013) propose to use families of bivariate copulas to model the dependency relationship between the number of claims and the average claim amount. In the same context, we aim in this paper to use copulas to account for the dependence between the number of claims rv N and each individual claim amount rv X_i ($i \in \mathbb{N}$). After explaining in a first setup how we can use a bivariate copula to link N and $\underline{X}$, we propose next, a general dependence structure using multivariate and hierarchical Archimedean copulas. One of the most important properties of Archimedean copulas is their great flexibility and natural generalization to any dimension. Such a property allows us to model the dependence between the rvs X_i, for $i = 1, \ldots, k$, regardless of the changing dimension k. The risk model considered in this case is an extension of the one studied in Section 4 of Cossette et al. (2018), in which $\underline{X} = \{X_i, i \in \mathbb{N}\}$ forms a sequence of exchangeable rvs independent of the counting positive discrete rv N.

It is known that quantifying the aggregate claim amount rv S using risk measures represents an important task in risk management. In the presence of mutual dependence relationships between the claim amount rvs $X_1, \ldots, X_k$ and the claim count rv N, such a task becomes more tedious. In this paper, we propose dependence models, linking N and $\underline{X}$, for which explicit formulas for quantities of interest in regard to S can be obtained. The mixture representation hidden behind Archimedean structures simplifies the computation of the cumulative distribution function (cdf) of S and other related quantities such as the risk measures VaR and TVaR of S. Similarly to Cossette et al. (2018), a computational methodology is proposed to analyze the distribution of the aggregate claim amount rv S, defined as a random sum.

The outline of the paper is as follows. Section 2 presents basic definitions and proposes motivation setups using bivariate and multivariate copulas. In Section 3, a general model using hierarchical Archimedean copulas is presented. This section presents a computational methodology for the distribution of S and related quantities using either an Archimedean or a hierarchical Archimedean copula. Increasing convex ordering inequalities on S are also derived in different settings.

4.2 Definitions, properties and dependence models

4.2.1 Basic definitions and properties

Let $\aleph = \aleph(F_N, F_X)$ be a class of collective risk models defined with the pairs $(N, \underline{X})$ where N is a counting (discrete) rv and $\underline{X} = \{X_i, i \in \mathbb{N}\}$ is a sequence of strictly positive identically distributed rvs. The cdf of the rv N is denoted by F_N, with support $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The rv X_i is distributed as the strictly positive rv X, with cdf F_X, for $i \in \mathbb{N}$. In an insurance context, the rvs N and X_i correspond, respectively, to the number of claims and the amount of the ith individual claim ($i \in \mathbb{N}$). To simplify the presentation, we assume that $F_X(0) = 0$. The joint cdf of $(N, X_1, ..., X_k)$ is denoted by $F_{N,X_1,...,X_k}$, for $k \in \mathbb{N}$. For a pair $(N, \underline{X}) \in \aleph$, we define the random sum $S = \sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i \times 1_{\{N \geq i\}}$, with $1_{\{N \geq i\}} = \begin{cases} 1, & N \geq i \\ 0, & \text{otherwise} \end{cases}$, and the convention that $\sum_{i=1}^{0} a_i = 0$. In an insurance context, the rv S corresponds to the aggregate claim amount. We also define $S_k = \sum_{i=1}^{k} X_i \times 1_{\{N \geq i\}}$, such that $S_\infty = \lim_{k \to \infty} S_k = S$, and $S_{1:k} = \sum_{i=1}^{k} X_i$, for $k \in \mathbb{N}$, with $S_{1:1} = X_1$. The general expression of the cdf of the aggregate claim amount rv S is given by

$$F_S(x) = \Pr(N=0) + \sum_{k=1}^{\infty} \Pr(N=k, S_{1:k} \leq x),\ x \geq 0, \tag{4.1}$$

and assuming that $E[N] < \infty$ and $E[X] < \infty$, the general expression for the expectation of the rv S is given by

$$E[S] = \sum_{k=1}^{\infty} \Pr(N=k) \times k \times E[X|N=k]. \tag{4.2}$$

Let $\aleph^{(\perp,\ )} \subset \aleph$ be a subset of $\aleph$, where a pair $\left(N^{\perp}, \underline{X}\right) \in \aleph^{(\perp,\ )}$ defines a collective risk model where the sequence $\underline{X}$ and $N^{\perp}$ are independent. Evidently, we have $N^{\perp} \sim N$. For $(N^{\perp}, \underline{X}) \in \aleph^{(\perp,)}$, we define the aggregate claim amount rv by $S^{(\perp,\ )} = \sum_{i=1}^{N^{\perp}} X_i$. Examples of collective risk models defined by $\left(N^{\perp}, \underline{X}\right) \in \aleph^{(\perp,\ )}$ can be found in, e.g., Section 4 of Cossette et al. (2018) and Section 2.4 of Willmot and Woo (2015). Also, the class of risk models defined in Section 3 of Albrecher et al. (2014) over a fixed time interval $(0,t]$ belong to the subset $\aleph^{(\perp,\ )}$. For a pair $\left(N^{\perp}, \underline{X}\right) \in \aleph^{(\perp,\ )}$, (4.1) becomes

$$F_{S^{(\perp,\ )}}(x) = \Pr(N=0) + \sum_{k=1}^{\infty} \Pr(N=k) \times \Pr(S_{1:k} \leq x), x \geq 0, \tag{4.3}$$

and the expectation of the aggregate claim amount rv in (4.2) becomes

$$E[S^{(\perp,\ )}] = E[N^{\perp}] \times E[X] = E[N] \times E[X], \tag{4.4}$$

which means that the expected aggregate claim amount corresponds to the product of the expected claim count with the expected claim amount.

The pair $(N^{\perp}, \underline{X}^{+}) \in \aleph^{(\perp,\ )}$ defines a collective risk model with a sequence of comonotonic individual claim amounts, independent of the claim number $N^{\perp}$. Let $U \sim Unif(0,1)$ be independent of $N^{\perp}$. This means that we have $X_1 \sim F_X^{-1}(U), ..., X_k \sim F_X^{-1}(U)$, for $k \in \mathbb{N}$. The corresponding aggregate claim amount rv is defined by

$$S^{(\perp,+)} = \sum_{i=1}^{N^{\perp}} X_i^{+} = N^{\perp} \times X^{+}. \tag{4.5}$$

Clearly, for the pair $(N^{\perp}, \underline{X}^{+}) \in \aleph^{(+,)}$, we have

$$F_{S^{(\perp,+)}}(x) = \Pr(N=0) + \sum_{k=1}^{\infty} \Pr(N=k) \times \Pr\left(X \leq \frac{x}{k}\right), x \geq 0. \tag{4.6}$$

If we consider the special case $(N^{+}, \underline{X}^{+}) \in \aleph$ where $N^{+}, X_1^{+}, \ldots, X_k^{+}$ are all comonotonic rvs (with $N^{+} \sim N$ and $X_i^{+} \sim X$, for $i = 1, \ldots, k$), this means that there exists a standard uniformly distributed rv $U \sim Unif(0,1)$, such that the representation

$$N^{+} \sim F_N^{-1}(U), X_1^{+} \sim F_X^{-1}(U), , \dots, X_k^{+} \sim F_X^{-1}(U), \tag{4.7}$$

holds for $k \in \mathbb{N}$, where F_N^{-1} and F_X^{-1} are the generalized inverse of the cdfs F_N and F_X respectively. Recall that the generalized inverse of the cdf of a given rv Y with cdf F_Y is defined as $F_Y^{-1}(\kappa) = \inf\{x \in \mathbb{R}, F_Y(x) \geq \kappa\}$. The corresponding aggregate claim amount rv is here denoted by $S^{(+,+)} = \sum_{i=1}^{N^+} X_i^{+}$, and clearly, due to the representation in (4.7), it can be shown that

$$S^{(+,+)} = N^{+} \times X^{+}. \tag{4.8}$$

For the proof of (4.8) and for additional properties on the collective risk model defined by the pair $(N^{+}, \underline{X}^{+})$, see Liu and Wang (2017). Using the representation in (4.8), Liu and Wang (2017) show that

$$E\left[S^{(+,+)}\right] = E\left[N^{+} \times X^{+}\right]. \tag{4.9}$$

Note that even though the definitions of the aggregate claim amount in (4.8) and (4.5) appear similar, the one in (4.8) is the product of two comonotonic rvs, while in (4.5), the aggregate claim amount rv is the product of two independent rvs.

Another interesting element of $\aleph^{(\perp,\ )} \subset \aleph$ is the classical collective risk model where a pair $\left(N^{\perp}, \underline{X}^{\perp}\right) \in \aleph$ defines a collective risk model where $\underline{X}^{\perp} = \left\{X_i^{\perp}, i \in \mathbb{N}\right\}$ forms a sequence of iid rvs (with $X_i^{\perp} \sim X$) also independent of the rv $N^{\perp}$ (with $N^{\perp} \sim N$). We denote the corresponding aggregate claim amount rv by $S^{(\perp,\perp)} = \sum_{i=1}^{N^{\perp}} X_i^{\perp}$. The properties of $S^{(\perp,\perp)}$ in the classical risk model have been thoroughly investigated in the actuarial literature (see, e.g., Rolski et al. (1999) and Klugman et al. (2009)). It is well known that the cdf of $S^{(\perp,\perp)}$ is as given in (4.3), and its expectation is given by

$$E[S^{(\perp,\perp)}] = E[N^{\perp}] \times E[X^{\perp}] = E[N] \times E[X], \tag{4.10}$$

which corresponds to (4.4).

To sum up, the relation in (4.4) remains unchanged for a collective risk model assuming independence between the claim number rv and a sequence of (independent or dependent) claim amounts. However, as we have seen in (4.9) from the general expression in (4.2), the relation in (4.4) is no longer valid for a collective risk model defined by a pair $(N, \underline{X}) \in \aleph \backslash \aleph^{(\perp,\ )}$, i.e., for a collective risk model with a given dependence structure between the claim number rv and the sequence of claim amount rvs. In this case, one may observe either

$$E[S] < E[N] \times E[X] \text{ or } E[S] > E[N] \times E[X]. \tag{4.11}$$

The popular risk measures Value-at-Risk (VaR) and Tail-Value-at-Risk (TVaR) are used namely to determine the capital of the insurance portfolio. The VaR at the confidence level $\kappa \in (0,1)$ of the rv S is defined as

$$VaR_\kappa(S) = F_S^{-1}(\kappa), \tag{4.12}$$

where $F_S^{-1}(\kappa) = \inf\{x \in \mathbb{R}, F_S(x) \geq \kappa\}$. Assuming $E[S] < \infty$, the TVaR at the confidence level $\kappa \in (0,1)$ of the aggregate claim rv S is given by

$$TVaR_\kappa(S) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(S)\, du. \tag{4.13}$$

See, e.g., McNeil et al. (2015) for the properties and applications of these two risk measures. These quantities are crucial for actuaries and risk managers. Generally, numerical optimization methods need to be used to perform the inversion in (4.12). In this paper, we present collective risk models with dependence and we examine the computation of F_S, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$. Note that, given the representation of $S^{(+,+)}$ in (4.8), Liu and Wang (2017) show that

$$VaR_\kappa(S^{(+,+)}) = VaR_\kappa(N^+) \times VaR_\kappa(X^+),$$

for $\kappa \in (0,1)$.

4.2.2 Stochastic orders

Stochastic orders are used to compare risks according to how risky and dangerous they are. They have many applications in, e.g., actuarial science, applied probability, reliability, and economics. See Müller and Stoyan (2002), Denuit et al. (2006), and Shaked and Shanthikumar (2007) for a review on stochastic orders.

Definition 4.2.1. *Let X and X^* be two rvs with finite expectations. Then, X is said to be smaller than X^* according to the convex order (increasing convex order), denoted $X \preceq_{cx} X^*$ ($X \preceq_{icx} X^*$), if $E[\phi(X)] \leq E[\phi(X^*)]$ for all (increasing) convex function ϕ, when the expectations exist.*

The convex and increasing convex orders are variability orders. Note that, if $X \preceq_{icx} X^*$ and $E[X] = E[X^*]$, then $X \preceq_{cx} X^*$. Proposition 3.4.8 of Denuit et al. (2006) provides an important result about the increasing convex order and the TVaR for applications in actuarial science and quantitative risk management:

$$X \preceq_{icx} X^* \text{ if and only if } TVaR_\kappa(X) \leq TVaR_\kappa(X^*),\ \forall\, \kappa \in (0,1). \tag{4.14}$$

In order to compare two random vectors, we use the supermodular dependence order.

Definition 4.2.2. *Let $f : \mathbb{R}^k \to \mathbb{R}$. The function f is supermodular if the following inequality is true:*

$$\begin{aligned} & f(x_1, \ldots, x_i + \epsilon, \ldots, x_j + \delta, \ldots, x_k) - f(x_1, \ldots, x_i + \epsilon, \ldots, x_j, \ldots, x_k) \\ \geq \ & f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_k) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k), \end{aligned}$$

$\forall (x_1, \ldots, x_k) \in \mathbb{R}^k$, $\forall \epsilon, \delta > 0$, *and* $1 \leq i \leq j \leq k$.

Definition 4.2.3. *Let $(X_1, \ldots, X_k)$ and $(X_1^*, \ldots, X_k^*)$ be two random vectors such that, for $i = 1, \ldots, k$, X_i and X_i^* have the same marginal distribution. Then, $(X_1^*, \ldots, X_k^*)$ is greater than $(X_1, \ldots, X_k)$ according to the supermodular order, denoted $(X_1, \ldots, X_k) \preceq_{sm} (X_1^*, \ldots, X_k^*)$, if $E[f(X_1, \ldots, X_k)] \leq E[f(X_1^*, \ldots, X_k^*)]$, for any supermodular function f, when the expectations exist.*

We know that if $(X_1, \ldots, X_k) \preceq_{sm} (X_1^*, \ldots, X_k^*)$ then, $S_{1:k} \preceq_{cx} S_{1:k}^*$, for $k \in \mathbb{N}$ (see e.g., Bäuerle and Müller (2006) for the proof of such a result). For more properties and details on the convex, increasing convex, and supermodular orders, see e.g., Müller and Stoyan (2002), Denuit et al. (2006), and Shaked and Shanthikumar (2007).

In the context of collective risk models with dependence, we have the following result.

Proposition 4.2.1. *Let the pairs $(N, \underline{X}) \in \aleph$ and $(N^*, \underline{X}^*) \in \aleph$ such that*

$$(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*), \forall k \in \mathbb{N}.$$

Then,

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*. \tag{4.15}$$

By (4.14), we conclude $TVaR_\kappa(S) \leq TVaR_\kappa(S^*)$, for $\kappa \in (0, 1)$.

Proof. We have

$$S_k = \sum_{i=1}^{k} X_i 1_{\{N \geq i\}} = \phi(N, X_1, ..., X_k),$$

where

$$\phi(x_0, x_1, ..., x_k) = \sum_{i=1}^{k} x_i 1_{\{x_0 \geq i\}} \tag{4.16}$$

is a supermodular function.

Since $(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*)$, for all $k \in \mathbb{N}$, and given that the function ϕ defined in (4.16) is supermodular, it implies that

$$S_k \preceq_{icx} S_k^*, \tag{4.17}$$

for all $k \in \mathbb{N}$ (see Theorem 9.A.16 on page 399 of Shaked and Shanthikumar (2007) for more details). Letting $k \to \infty$ in (4.17), we obtain the desired result. □

Since $(N, X_1, \ldots, X_k) \preceq_{sm} (N^{\perp}, X_1^{\perp}, \ldots, X_k^{\perp})$ for any pair $(N, \underline{X}) \in \aleph$, we obtain, from Proposition 4.2.1, the following corollary.

Corollary 4.2.1. *Consider a collective risk model defined by a pair* $(N, \underline{X}) \in \aleph$. *Then,* $S \preceq_{icx} S^{(+,+)}$. *Also, by (4.14),*

$$TVaR_{\kappa}(S) \leq TVaR_{\kappa}\left(S^{(+,+)}\right), \forall \kappa \in (0,1). \tag{4.18}$$

According to (4.18), it is clear that $E[S^{(+,+)}] \geq E[S]$, for any collective risk model defined by a pair $(N, \underline{X}) \in \aleph$. Also, $E[S^{(+,+)}] \geq E[S^{(\perp,)}] = E[N] \times E[X]$. Note that Liu and Wang (2017) used another approach to obtain the result (4.18) in Corollary 4.2.1.

In the next section, we consider our first collective risk model with dependence between N and $\underline{X}$. More precisely, inspired from Section 2.4 of Willmot and Woo (2015), we discuss a collective risk model based on a multivariate mixed Erlang distribution. Univariate and multivariate mixed Erlang distributions are very flexible and can be applied in various contexts in actuarial science (see, e.g., Willmot and Lin (2011), Lee and Lin (2012), Cossette et al. (2015), and Willmot and Woo (2015) for details).

4.2.3 Collective risk models with mixed Erlang distributions

We first recall a few definitions related to mixed Erlang distributions. Let the rv X follow a univariate mixed Erlang distribution with

$$F_X(x) = \sum_{j=1}^{\infty} \gamma_J(j) H(x; j, \beta), \text{ for } \beta > 0 \text{ and } x \geq 0, \tag{4.19}$$

where γ_J is the probability mass function (pmf) of a strictly positive rv J, with probability generating function (pgf) $\mathcal{P}_J(s) = \sum_{j=1}^{\infty} \gamma_j s^j$ ($|s| \leq 1$), and $H(x; j, \beta)$ corresponds to the cdf of an Erlang distribution (parameters $j \in N$ and $\beta > 0$), with

$$H(x; j, \beta) = 1 - \mathrm{e}^{-\beta x} \sum_{l=0}^{j-1} \frac{(\beta x)^l}{l!}, \text{ for } x \geq 0.$$

Given that a mixed Erlang distribution may be represented as a random sum of exponential rvs, we have $X = \sum_{j=1}^{J} C_j$, where $C_j \sim C$ is an exponential rv with mean $\frac{1}{\beta}$ and Laplace-Stieltjes Transform (LST) $\mathcal{L}_C(t) = \frac{\beta}{\beta+t}$, for $j \in \mathbb{N}$. The LST of X is hence given by

$$\mathcal{L}_X(t) = \mathcal{P}_J(\mathcal{L}_C(t)) = \mathcal{P}_J\left(\frac{\beta}{\beta+t}\right), \text{ for } t > 0.$$

Note that a mixed Erlang distribution is also called "mixture of Erlangs" distribution or "hyper-Erlang distribution", see, e.g., Fang (2001) and Lee and Lin (2010). See also, e.g., Lee and Lin (2010) and Willmot and Lin (2011) for details on univariate mixed Erlang distributions with applications in actuarial science.

Let $(X_1, ..., X_k)$ be a vector of rvs which follows a multivariate mixed Erlang distribution with

$$F_{X_1,...,X_k}(x_1, ..., x_k) = \sum_{j_1=1}^{\infty} ... \sum_{j_k=1}^{\infty} \gamma_{J_1,...,J_k}(j_1, ..., j_k) H(x_1; j_1, \beta) ... H(x_k; j_k, \beta),$$

for $\beta > 0$, $x_1 \geq 0$, ..., $x_k \geq 0$, $k \in \mathbb{N}$. Also, $\gamma_{J_1,...,J_k}(j_1, ..., j_k) \in [0, 1]$ corresponds to the joint pmf of a vector of discrete rvs $(J_1, ..., J_k)$,

$$\gamma_{J_1,...,J_k}(j_1, ..., j_k) = \Pr(J_1 = j_1, ..., J_k = j_k),$$

for $j_i \in \mathbb{N}$, $i = 1, 2, ..., k$, and with $\sum_{j_1=1}^{\infty} ... \sum_{j_k=1}^{\infty} \gamma_{J_1,...,J_k}(j_1, ..., j_k) = 1$. See, e.g., Lee and Lin (2012) and Willmot and Woo (2015) for details on the class of multivariate mixed Erlang distributions.

In Section 2.4 of Willmot and Woo (2015), the authors study the properties of a class of risk models which belongs to $\aleph^{(\perp,\ )} \subset \aleph$. In this section, we examine an extension of that class, relaxing the assumption of independence between the claim number rv N and the sequence of claim amounts $\underline{X}$. Let the vector $(N, X_1, ..., X_k)$ follow a multivariate mixed distribution with

$$E\left[1_{\{N=j_0, X_1 \leq x_1, ..., X_k \leq x_k\}}\right] = \sum_{j_1=1}^{\infty} ... \sum_{j_k=1}^{\infty} \gamma_{N,J_1,...,J_k}(j_0, j_1, ..., j_k) H(x_1; j_1, \beta) ... H(x_k; j_k, \beta),$$

for $\beta > 0$, $x_1 \geq 0$, ..., $x_k \geq 0$, $k \in \mathbb{N}$. For $k \in \mathbb{N}$, the joint pmf of $(N, J_1, ..., J_k)$ corresponds to $\gamma_{N,J_1,...,J_k}(j_0, j_1, ..., j_k)$ ($j_0 \in \mathbb{N}_0$ and $j_1, \ldots, j_k \in \mathbb{N}$) with

$$\sum_{j_0=0}^{\infty} \sum_{j_1=1}^{\infty} ... \sum_{j_k=1}^{\infty} \gamma_{N,J_1,...,J_k}(j_0, j_1, ..., j_k) = 1.$$

It implies that the joint cdf of $(N, X_1, ..., X_k)$ is

$$F_{N,X_1,...,X_k}(k, x_1, ..., x_k) = \sum_{j_0=0}^{k} \sum_{j_1=1}^{\infty} ... \sum_{j_k=1}^{\infty} \gamma_{N,J_1,...,J_k}(j_0, j_1, ..., j_k) H(x_1; j_1, \beta) ... H(x_k; j_k, \beta), \tag{4.20}$$

for $\beta > 0$, $x_1 \geq 0$, ..., $x_k \geq 0$, $k \in \mathbb{N}$, and $k \in \mathbb{N}$.

Let L be a non-negative discrete rv with pmf

$$\Pr(L = l) = \zeta_l = \begin{cases} \gamma_N(0) & , l = 0 \\ \sum_{i=1}^{l} \Pr(N = i, J_1 + ... + J_i = l), & l \in \mathbb{N} \end{cases}. \tag{4.21}$$

$$= \sum_{i=1}^{l} \sum_{j_1+...+j_i=l} \gamma_{N,J_1,...,J_{l-i+1}}(i, j_1, \ldots, j_i), \tag{4.22}$$

and pgf $\mathcal{P}_L(s) = \sum_{l=0}^{\infty} \zeta_l s^l$, $|s| \leq 1$. Then, the rv S follows a mixed Erlang distribution with

$$F_S(x) = \zeta_0 + \sum_{l=1}^{\infty} \zeta_l H(x; l, \beta), \ x \geq 0, \tag{4.23}$$

and

$$\mathcal{L}_S(t) = \mathcal{P}_L\left(\frac{\beta}{\beta + t}\right), \text{ for } t > \beta.$$

For example, if $N \in \{0, 1, 2, 3\}$, then we have the following weights in the univariate Erlang mixture for S

$$\zeta_l = \begin{cases} \gamma_N(0) & , l = 0 \\ \gamma_{N,J_1}(1,1) & , l = 1 \\ \gamma_{N,J_1}(1,2) + \gamma_{N,J_1,J_2}(2,1,1) & , l = 2 \\ \gamma_{N,J_1}(1,l) + \sum_{j=1}^{l} \gamma_{N,J_1,J_2}(2,j,l-j) & \\ + \sum_{j=1}^{l} \sum_{i=1}^{l-j} \gamma_{N,J_1,J_2,J_3}(3,j,i,l-i-j) & , l \geq 3 \end{cases} . \tag{4.24}$$

Different dependence structures can be used to link N and J_i, for $i \in \mathbb{N}$. For example, hierarchical Archimedean copulas are used in Example 4.3.1 in Section 4.3 with a multivariate mixed Erlang distribution which is in line with the aim of this paper that is to use copulas to account for dependence between claim frequency and amounts of claims. With this in mind, we investigate in the following section the use of bivariate copulas in a new fashion that has not been previously done in previous works. We will then pursue with multivariate copulas and hierarchical Archimedean copulas in respectively Section 4.2.5 and Section 4.3.

4.2.4 Collective risk models with mixing

In this section, we discuss the use of bivariate copulas to account for the dependence between the number of claims N and every claim amount X_i, for $i \in \mathbb{N}$. The idea here is to consider a dependence structure for $\underline{X}$ to be induced via a strictly positive discrete common rv $\mathcal{V}$. This differs from what has been proposed by Czado et al. (2012) and Krämer et al. (2013) who used this tool, as mentioned in the introduction, to model the dependence between the claim counts and the average claim amount.

Consider a class of risk models $(N, \underline{X}) \in \aleph$, where the canonical rv X ($X_i \sim X$, for $i \in \mathbb{N}$) follows a mixed distribution with discrete mixing rv $\mathcal{V}$. Let $\mathcal{V}$ be a strictly positive discrete mixing rv with pmf $\gamma_{\mathcal{V}}$. Then, we can construct the mixed distribution of X either by using the conditional cdf of $(X|\mathcal{V} = \nu)$, for $\nu \in \mathbb{N}$, or by using its conditional survival function, i.e.,

$$F_X(x) = \sum_{k=1}^{\infty} \gamma_{\mathcal{V}}(\nu) F_{X|\mathcal{V}=\nu}(x), \ x \geq 0, \tag{4.25}$$

or

$$\overline{F}_X(x) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu)\, \overline{F}_{X|\mathcal{V}=\nu}(x),\ x \geq 0. \tag{4.26}$$

In order to link N and $\underline{X}$ using a bivariate copula, we can simply consider an existing dependence between the claim counts N and the mixing rv $\mathcal{V}$. Such a dependence structure can be easily modelled using a bivariate copula. Then, let $(N, \mathcal{V})$ be a pair of discrete rvs with bivariate pmf denoted by

$$\gamma_{N,\mathcal{V}}(k,\nu) = \Pr(N = k, \mathcal{V} = \nu),\ \text{for } k \in \mathbb{N}_0 \text{ and } \nu \in \mathbb{N}.$$

If the distribution of X is defined using (4.25), then the joint cdf of $(X_1, \ldots, X_k)$ is given by

$$F_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) \prod_{i=1}^{k} F_{X_i|\mathcal{V}=\nu}(x_i),\ x_i \geq 0. \tag{4.27}$$

Also, the joint cdf of $(N, X_1, ..., X_k)$ is given by

$$F_{N,X_1,\ldots,X_k}(k, x_1, ..., x_k) = \sum_{l=0}^{k} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(l,\nu) \prod_{i=1}^{l} F_{X_i|\mathcal{V}=\nu}(x_i),\ k \in \mathbb{N}, x_1, ..., x_k \geq 0. \tag{4.28}$$

If the distribution of X is defined using (4.26), then the joint survival function of $(X_1, \ldots, X_k)$ is given by

$$\overline{F}_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) \prod_{i=1}^{k} \overline{F}_{X_i|\mathcal{V}=\nu}(x_i),\ x_i \geq 0. \tag{4.29}$$

Also, the joint survival function of $(N, X_1, ..., X_k)$ is given by

$$\overline{F}_{N,X_1,\ldots,X_k}(k, x_1, ..., x_k) = \sum_{l=k+1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(l,\nu) \prod_{i=1}^{l} \overline{F}_{X_i|\mathcal{V}=\nu}(x_i),\ k \in \mathbb{N}, x_1, ..., x_k \geq 0. \tag{4.30}$$

Then, in both cases, i.e., with (4.28) or (4.30), (4.1) becomes

$$F_S(x) = \Pr(N = 0) + \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(k,\nu)\, F_{S_{1:k}|\mathcal{V}=\nu}(x),\ x \geq 0. \tag{4.31}$$

Let us consider specifically the class of risk models defined with pairs $(N, \underline{X}) \in \aleph$, where X follows a univariate mixed Erlang distribution. The cdf of X given in (4.25) becomes

$$F_X(x) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu)\, H(x; \nu, \beta),\ x \geq 0, \tag{4.32}$$

and the joint cdf of $(X_1, ..., X_k)$ given in (4.27) becomes

$$F_{X_1,\ldots,X_k}(x_1, ..., x_k) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu)\, H(x_1; \nu, \beta) \ldots H(x_k; \nu, \beta),\ x_1, ..., x_k \geq 0, \tag{4.33}$$

with $\beta > 0$. Note that in this case, the mixing occurs over the shape parameter. Also, if $J_1, \ldots, J_k$ in Section 4.2.3 are comonotonic rvs with $J_i = F_{\mathcal{V}}^{-1}(U)$, where $U \sim Unif(0,1)$, then

$$\gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) = \begin{cases} \gamma_{\mathcal{V}}(j), & j_1 = j_2 = \ldots = j_k = j \\ 0, & \text{otherwise} \end{cases}.$$

The joint cdf of $(N, X_1, ..., X_k)$ in (4.28) and (4.20) becomes

$$F_{N,X_1,...,X_k}(k, x_1, ..., x_k) = \sum_{l=0}^{k} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(l,\nu) H(x_1;\nu,\beta) ... H(x_k;\nu,\beta), \; k \in \mathbb{N}, x_1, ..., x_k \geq 0, \tag{4.34}$$

which leads to the following cdf of S

$$F_S(x) = \Pr(N = 0) + \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(k,\nu) H(x; k\nu, \beta), \; x \geq 0. \tag{4.35}$$

As in Willmot and Woo (2007), (4.35) reduces to (4.23) where

$$\zeta_l = \begin{cases} \gamma_N(0) & , \quad l = 0 \\ \sum_{k=1}^{l} \gamma_{N,\mathcal{V}}\left(k, \frac{l}{k}\right) \times 1_{\left\{\frac{l}{k} = \left[\frac{l}{k}\right]\right\}} & , \quad l \in \mathbb{N} \end{cases},$$

where the function $t \to [t]$ represents the integer function.

Also, the expression for the expectation of the aggregate claim amount is

$$E[S] = \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(k,\nu) \frac{k\nu}{\beta} = \sum_{l=1}^{\infty} \zeta_l \frac{l}{\beta} \neq E[N] \times E[X]. \tag{4.36}$$

If the rvs $N^{\perp} \sim N$ and $\mathcal{V}^{\perp} \sim \mathcal{V}$ are independent, i.e., the corresponding collective risk model belongs to the subset $\aleph^{(\perp,)}$, it implies that

$$\gamma_{N,\mathcal{V}}(k,\nu) = \gamma_N(k) \times \gamma_{\mathcal{V}}(\nu), \tag{4.37}$$

leading, as expected, to

$$E\left[S^{(+,)}\right] = \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_N(k) \times \gamma_{\mathcal{V}}(\nu) \frac{k\nu}{\beta} = E[N] \times E[X].$$

We provide below a numerical illustration in which we have recourse to Frank and Gumbel bivariate copulas for the joint distribution of $(N, X_1, \ldots, X_k)$.

Example 4.2.1. *Let the rv N follow a Poisson distribution with mean 5, and the rv $\mathcal{V}$ follow a geometric distribution with $\gamma_{\mathcal{V}}(\nu) = (1-\theta)\theta^{\nu-1}$, $\nu \in \mathbb{N}$, and $\beta = \frac{\lambda}{1-\theta}$, with $\lambda = 1$. These*

	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 4$	$(N^{\perp}, X^{\perp})$	(N^{+}, X^{+})
$E[S]$	5	5.633093	6.027384	5	7.092215
$Var(S)$	18.33333	39.87369	51.14767	10	108.9786
$VaR_{0.9}(S)$	10.16318	12.34318	14.04169	9.275575	18.42068
$VaR_{0.99}(S)$	20.98362	31.23657	34.76929	14.404382	50.65687
$VaR_{0.999}(S)$	33.43638	56.54057	60.93267	18.850045	89.80081
$TVaR_{0.9}(S)$	14.81275	20.30066	22.85213	11.54844	31.64512
$TVaR_{0.99}(S)$	26.34496	42.04653	45.96633	16.35276	67.39474
$TVaR_{0.999}(S)$	39.43228	69.55788	74.20058	20.64058	111.7606

Table 4.1 – Collective risk model defined with the Gumbel copula with different values of the dependence parameter α, and the Moran-Downton's multivariate exponential distribution with dependence parameter $\theta = \frac{1}{3}$.

assumptions imply that $X \sim Exp(\lambda)$ and, using (4.33), that $(X_1, ..., X_k)$ follows a Moran-Downton's multivariate exponential distribution with dependence parameter θ for $k = 2, 3, ...$ (see, e.g., details in Kotz et al. (2004) and Cossette et al. (2015)). Also, we define

$$F_{N,\mathcal{V}}(k, \nu) = C(F_N(k), F_{\mathcal{V}}(\nu)), \text{ for } k \in \mathbb{N}_0, \ \nu \in \mathbb{N},$$

where C is a copula with dependence parameter α. In Tables 4.1 and 4.2, we assume that C is a Gumbel copula with $\alpha = 1, 1.5,$ and 4, and we consider $\theta = \frac{1}{3}$ and $\theta = \frac{2}{3}$, respectively. When $\alpha = 1$, N and $\underline{X}$ are independent. The corresponding collective risk model belongs to $\aleph^{(\perp,)}$ and the aggregate claim amount rv is denoted by $S^{(\perp,)}$. In Table 4.3, $\theta = \frac{1}{3}$ and C is a Frank copula with $\alpha = -10, -5, 5,$ and 10. In all these three tables, we provide the exact values of F_S, $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$, for $\kappa = 0.9, 0.99, 0.999$. For comparison purposes, the same quantities are also provided for $S^{(+,+)}$ and $S^{(\perp,\perp)}$. From Tables 4.1,4.2, and 4.3, we can see that the expectation, the variance and the TVaR increase when the dependence parameter of the copula linking N to $\mathcal{V}$ increases. These results are also confirmed in Figures 4.1 and 4.2, where all the curves (F_S) intersect multiple times. Also, for a fixed α, the expectation, the variance and the TVaR increase with the dependence parameter θ of the Moran-Downton's multivariate distribution. Moreover, for Tables 4.1 and 4.2, $E[S]$ is the same for both $S^{(\perp,)}$ and $S^{(\perp,\perp)}$, and the values of the TVaR always fall between $TVaR_\kappa(S^{(\perp,\perp)})$ and $TVaR_\kappa(S^{(+,+)})$. This is not the case however for Table 4.3 since the Frank copula admits negative parameters, and hence, negative dependence between N and $\mathcal{V}$.

Using the second representation based on the survival distribution as given in (4.26) and (4.30), we consider here a context in which the mixing is done over the scale parameter contrarily to the previous Erlang mixture resulting from shape mixing. More specifically, we consider the class of risk models defined with pairs $(N, \underline{X}) \in \aleph$, where X follows a mixed exponential distribution.

	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 4$	$(N^{\perp}, X^{\perp})$	(N^{+}, X^{+})
$E[S]$	5	5.951677	6.580654	5	7.092215
$Var(S)$	26.66667	62.00988	79.54353	10	108.9786
$VaR_{0.9}(S)$	11.19305	14.06950	16.27849	9.275575	18.42068
$VaR_{0.99}(S)$	24.88079	38.43050	42.98413	14.404382	50.65687
$VaR_{0.999}(S)$	40.89441	71.56438	77.20073	18.850045	89.80081
$TVaR_{0.9}(S)$	17.05438	24.30166	27.58036	11.54844	31.64512
$TVaR_{0.99}(S)$	31.76864	52.57158	57.63292	16.35276	67.39474
$TVaR_{0.999}(S)$	48.66817	88.72821	94.78002	20.64058	111.7606

Table 4.2 – Collective risk model defined with the Gumbel copula with different values of the dependence parameter α, and the Moran-Downton's multivariate exponential distribution with dependence parameter $\theta = \frac{2}{3}$.

	$\alpha = -10$	$\alpha = -5$	$\alpha = 5$	$\alpha = 10$	$(N^{\perp}, X^{\perp})$	(N^{+}, X^{+})
$E[S]$	4.233481	4.420326	5.615181	5.840004	5	7.092215
$Var(S)$	5.389221	7.851772	32.34743	38.88801	10	108.9786
$VaR_{0.9}(S)$	7.261149	7.88958	12.89829	13.79695	9.275575	18.42068
$VaR_{0.99}(S)$	11.28135	13.69451	26.95522	29.43549	14.404382	50.65687
$VaR_{0.999}(S)$	15.88196	21.41967	41.53259	44.99325	18.850045	89.80081
$TVaR_{0.9}(S)$	9.038549	10.39373	18.99590	20.62353	11.54844	31.64512
$TVaR_{0.99}(S)$	13.256628	16.987532	33.27113	36.19493	16.35276	67.39474
$TVaR_{0.999}(S)$	18.462177	25.60353	48.21852	51.98976	20.64058	111.7606

Table 4.3 – Collective risk model defined with the Frank copula with different values of the dependence parameter α, and the Moran-Downton's multivariate exponential distribution with dependence parameter $\theta = \frac{1}{3}$.

In this case, given $\mathcal{V} = \nu$, the conditional distribution of the rv $(X|\mathcal{V} = \nu)$ is exponential with mean $\frac{\beta}{\nu}$, i.e., $\overline{F}_{X|\mathcal{V}=\nu}(x_i) = \mathrm{e}^{\frac{-\nu x}{\beta}}$. Note that a mixed exponential distribution can also be constructed using a continuous mixing rv. Here, (4.26) and (4.29) become

$$\overline{F}_X(x) = \mathcal{L}_{\mathcal{V}}\left(\frac{x}{\beta}\right), \tag{4.38}$$

and

$$\overline{F}_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = \mathcal{L}_{\mathcal{V}}\left(\frac{x_1}{\beta} + \ldots + \frac{x_k}{\beta}\right), \tag{4.39}$$

where $\mathcal{L}_{\mathcal{V}}$ denotes the LST of the rv $\mathcal{V}$. The joint survival function in (4.39) is the one of a multivariate mixed exponential distribution which belongs to the class of multivariate distributions constructed by common frailty, as explained in e.g., Marshall and Olkin (1988).

Also, the joint survival function of $(N, X_1, \ldots, X_k)$ given in (4.30) becomes

$$\overline{F}_{N,X_1,\ldots,X_k}(k, x_1, ..., x_k) = \sum_{l=k+1}^{\infty}\sum_{\nu=1}^{\infty}\gamma_{N,\mathcal{V}}(l,\nu)\prod_{i=1}^{l}\mathrm{e}^{\frac{-\nu x_i}{\beta}},\ k \in \mathbb{N}, x_1, ..., x_k \geq 0.$$

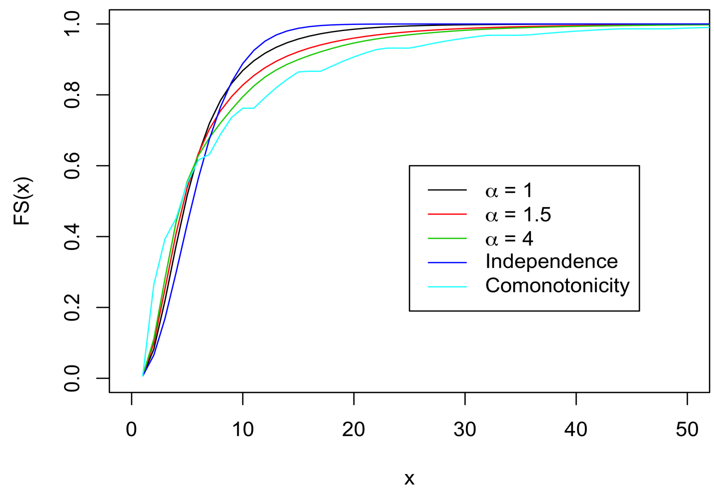


Figure 4.1 – Cdf of S for different dependence structures as defined in Example 4.2.1: Gumbel copula C, for different values of α, comonotonicity, and independence.

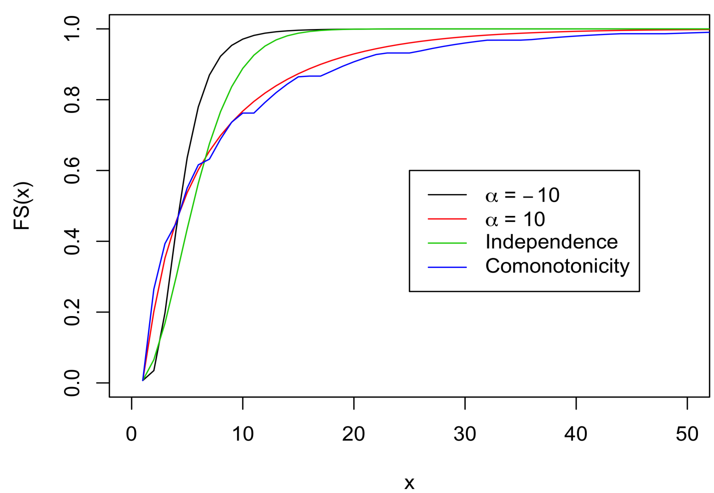


Figure 4.2 – Cdf of S for different dependence structures as defined in Example 4.2.1: Frank copula C, for different values of α, comonotonicity, and independence.

Since $(S_{1:k}|\mathcal{V}=\nu)$ follows an Erlang distribution, then (4.31) is given by

$$F_S(x) = \gamma_N(0) + \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(k,\nu)\, H\left(x, k, \frac{\beta}{\nu}\right),\ x \geq 0. \tag{4.40}$$

In particular, for $\left(N^{\perp}, \underline{X}\right) \in \aleph$, (4.40) becomes

$$F_{S^{(\perp,\ )}}(x) = \gamma_N(0) + \sum_{k=1}^{\infty} \gamma_N(k) \times \left(1 - \sum_{j=0}^{k-1} \frac{x^j}{j!\beta} \left\{ (-1)^j \frac{\mathrm{d^j} \mathcal{L}_{\mathcal{V}}(t)}{\mathrm{d}t^j} \bigg|_{t=\frac{x}{\beta}} \right\} \right), \; x \geq 0.$$

For more details, see e.g., Cossette et al. (2018).

4.2.5 Collective risk models with multivariate copulas

Another strategy to model the dependence between the number of claims and claim amounts is to consider multivariate copulas. Since we aim to propose dependence models for which the quantification of the aggregate claim amount is feasible, we consider, for the remainder of the paper, multivariate copulas that naturally extend to any dimension k and for which computational formulas for the cdf of S are obtained.

We consider a collective risk model defined by a pair $(N, \underline{X}) \in \aleph$ where the multivariate cdf of $(N, X_1, \ldots, X_k)$ (or its multivariate survival function) is defined with a copula C and univariate cdfs $F_N, F_{X_1}, \ldots, F_{X_k}$ (or the univariate survival functions $\overline{F}_N, \overline{F}_{X_1}, \ldots, \overline{F}_{X_k}$), i.e., either

$$F_{N,X_1,\ldots,X_k}(k, x_1, \ldots, x_k) = C\left(F_N(k), F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right), \tag{4.41}$$

or

$$\overline{F}_{N,X_1,\ldots,X_k}(k, x_1, \ldots, x_k) = C\left(\overline{F}_N(k), \overline{F}_{X_1}(x_1), \ldots, \overline{F}_{X_k}(x_k)\right), \tag{4.42}$$

where $(x_1, \ldots, x_k) \in [0, \infty)^k$ and $k \in \mathbb{N}$.

A well known and flexible multivariate copula that can be generalized to any dimension $k \in \mathbb{N}$ is the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copula. The EFGM copula is generated as a perturbation of the independence copula allowing for more flexibility in the dependence structure. See e.g., Durante and Sempi (2010) and references therein for more details.

Let $(U_0, U_1, ..., U_k)$ be a vector of uniformly distributed rvs for which the dependence structure is induced via the following $(k+1)$-dimensional copula C

$$\begin{aligned} C\left(u_0, u_1, ..., u_k\right) &= u_0 u_1, ... u_k \times \left(1 + \alpha \sum_{i=1}^{k} \left(1 - u_0\right)\left(1 - u_i\right)\right) \\ &= u_0 u_1 ... u_k + \alpha u_0 u_1 ... u_k \times \sum_{i=1}^{k} \left(1 - u_0\right)\left(1 - u_i\right), \end{aligned} \tag{4.43}$$

where the dependence parameter $\alpha \in [-1, 1]$ and for $u_0, \ldots, u_k \in [0, 1]$. The copula C defined in (4.43) is a special case of the generalized EFGM copula family presented in Durante and Sempi (2010). Such a copula allows a dependence structure for the $k+1$ rvs $U_0, U_1, ..., U_k$, in which there is a dependence relationship between U_0 and each U_i, for $i = 1, 2, \ldots, k$, meaning, there is only a dependence between U_0 and U_1, U_0 and U_2, etc. Note that the independence copula is obtained when $u_0 = 1$.

Let $(N, \underline{X}) \in \aleph$ such that $N = F_N^{-1}(U_0)$, and the continuous rv $X_i = F_{X_i}^{-1}(U_i)$, for $i = 1, 2, \ldots, k$. In this case, the multivariate cdf of $(N, X_1, \ldots, X_k)$ is given by (4.41) with the copula in (4.43). The joint probability function of $(N, X_1, \ldots, X_k)$ is given by

$$\begin{aligned}
& f_{N,X_1,\ldots,X_k}(k, x_1, \ldots, x_k) \\
= & \frac{\partial^k}{\partial x_1 ... \partial x_k} C\left(F_N(k), F_{X_1}(x_1), ..., F_{X_k}(x_k)\right) \\
& - \frac{\partial^k}{\partial x_1 ... \partial x_k} C\left(F_N(k-1), F_{X_1}(x_1), ..., F_{X_k}(x_k)\right) \\
= & F_N(k) \prod_{i=1}^{k} f_{X_i}(x_i) + \alpha F_N(k)(1 - F_N(k)) \prod_{i=1}^{k} f_{X_i}(x_i) \sum_{i=1}^{k} (1 - 2F_{X_i}(x_i)) \\
& - F_N(k-1) \prod_{i=1}^{k} f_{X_i}(x_i) - \alpha F_N(k-1)(1 - F_N(k-1)) \prod_{i=1}^{k} f_{X_i}(x_i) \sum_{i=1}^{k} (1 - 2F_{X_i}(x_i)),
\end{aligned}$$

where $(x_1, \ldots, x_k) \in [0, \infty)^k$, for $k \in \mathbb{N}$.

If we further assume that $X_i \sim Exp(\beta)$, $i = 1, 2, ..., k$, for $k \in \mathbb{N}$, we can obtain the following explicit formula for the cdf of the aggregate loss S

$$\begin{aligned}
F_S(x) &= \Pr(N = 0) + \sum_{k=1}^{\infty} \Pr(N = k, S_k \leq x) \\
&= \Pr(N = 0) + \sum_{k=1}^{\infty} \Pr(N = k) H(x; k, \beta) + \alpha \sum_{k=1}^{\infty} k\, F_N(k) \overline{F}_N(k) \left\{ \sum_{j=0}^{\infty} p_j H(x; k+j, 2\beta) - H(x; k, \beta) \right\} \\
&\quad - \alpha \sum_{k=1}^{\infty} k\, F_N(k-1) \overline{F}_N(k-1) \left\{ \sum_{j=0}^{\infty} p_j H(x; k+j, 2\beta) - H(x; k, \beta) \right\}, \qquad (4.44)
\end{aligned}$$

where $p_j = 0.5^k\, \xi_j$, for $j \in \mathbb{N}_0$, with $\xi_0 = 1$, $\xi_j = \frac{k}{j} \sum_{i=1}^{j} 0.5^i\, \xi_{j-i}$, for $j \in \mathbb{N}$. As expected, when $\alpha = 0$ (4.44) become the cdf of $S^{(\perp,\perp)}$.

Another family of multivariate copulas that can be used to account for the dependence between the number of claims and the individual claim amounts are the well known hierarchical Archimedean copulas. Due to their great flexibility, simple construction procedure, multivariate generalization, and their ability to capture different tail dependencies, much attention has been devoted to Archimedean copulas and their different hierarchical extensions in the last few years. These copulas are very good candidates to model the dependence structure within $(N, \underline{X}) \in \aleph$. We treat in details this class of copulas and discuss a computation methodology for the aggregate claim amount S under such a dependence construction.

4.3 Collective risk models with hierarchical Archimedean copulas

4.3.1 Definitions and basic relations

In this section, we examine the class of collective risk models defined with either an Archimedean or a hierarchical Archimedean copula. Archimedean copulas are good candidates to model such a dependence structure. However, the inherent exchangeability in Archimedean copulas implies that the dependence between the number of claims N and their amounts X_i for $i \in \mathbb{N}$ is the same as the dependence between the components of $\underline{X}$. In practice, this exchangeability property is a very strong assumption. A more realistic dependence structure could be a hierarchical one. For example, we can consider a one level hierarchical Archimedean copula allowing to have different dependency relationships between N and the components of $\underline{X}$ and also between all the components of $\underline{X}$. Such a dependence structure can be illustrated with a tree representation as shown in Figure 4.3. In this section, we consider nested Archimedean copulas (introduced by Joe (1997)) and hierarchical Archimedean copulas through compounding proposed in Cossette et al. (2017) to model such a dependence structure.

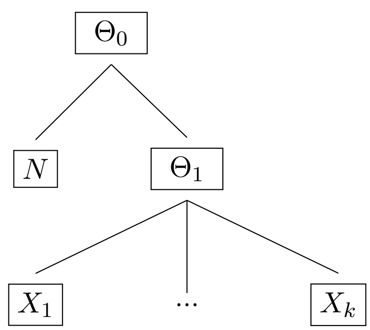


Figure 4.3 – One level hierarchical tree structure.

A copula C is said to be a nested Archimedean copula if at least one of its arguments is an Archimedean copula. For instance, a one level $(k+1)$-dimensional nested Archimedean copula, that can fit the dependence structure depicted in Figure 4.3, is given by

$$C(u_0, u_1, \ldots, u_k) = C(u_0, C(u_1, \ldots, u_k)) = \psi_0 \left(\psi_0^{-1}(u_0) + \psi_0^{-1} \circ \psi_1 \left(\sum_{i=1}^{k} \psi_1^{-1}(u_i) \right) \right), \tag{4.45}$$

where ψ_0 and ψ_1 are respectively the generators of the outer copula (also named the mother copula) and the inner copula (or the child copula), and $k \in \mathbb{N}$. Only nested Archimedean copulas of the form (4.45) for which ψ_0, ψ_1, and $\psi_{0,1}(t;\theta) = \exp\left\{-\theta\, \psi_0^{-1} \circ \psi_1(t)\right\}$ are LSTs of positive rvs Θ_0, Θ_1, and $\Theta_{0,1}$ respectively will be considered. This insures that the copula is properly defined (see e.g., Joe (2014) for details).

To bypass the constraints related to the nesting condition of nested Archimedean copulas, Cossette et al. (2017), among others, proposed a hierarchical copula obtained from a multivariate mixed exponential distribution and using multivariate compound distributions. To fit the dependence model presented in this paper, these hierarchical Archimedean copulas are given by

$$C\left(u_0, u_1, \ldots, u_k\right) = \mathcal{L}_M\left(\mathcal{L}_M^{-1}\left(u_0\right) - \ln\left(\mathcal{L}_{B_i}\left(\sum_{i=1}^{d} \mathcal{L}_{\Theta}^{-1}\left(u_i\right)\right)\right)\right), \tag{4.46}$$

where M is a positive discrete rv, the elements of the sequence $\underline{B} = \{B_i, i = 1, 2, \ldots\}$ are assumed to be iid and strictly positive rvs, independent of the rv M, and $\Theta = \sum_{i=1}^{M} B_i$.

In the example just below, we consider a collective risk model with mixed Erlang distributions as presented in Section 4.2.3. Let $(N, X_1, ..., X_k)$ follow a mixed multivariate distribution where $N \in \{0, 1, 2, 3\}$ and X_i follows a univariate mixed Erlang distribution, for $i = 1, 2, 3$. We consider the joint distribution of $(N, J_1, ..., J_k)$ (for $k \in \mathbb{N}$) to be defined as in (4.41) with a hierarchical Archimedean copula as in (4.45) or in (4.46), with dependence parameters α_0 and α_1.

Note that the pmf of S is as given in (4.24), and the joint pmf of $(N, J_1, \ldots, J_k)$ can be derived from its joint cdf as follows

$$\gamma_{N,J_1,\ldots,J_k}\left(j_0, j_1, \ldots, j_k\right) = \sum_{i_0=0,1} \cdots \sum_{i_k=0,1} (-1)^{i_1+\ldots+i_k} F_{N,\underline{X}}\left(\left(j_0 - i_0\right), \ldots, \left(j_k - i_k\right)\right), \tag{4.47}$$

for $j_1 \in \{0, 1, 2, 3\}$ and $j_i \in \mathbb{N}$, for $i = 1, \ldots, k$.

Remark 4.3.1. *Note that a similar procedure can be used if $\underline{X}$ is a sequence of discrete rvs. In this case, (4.24) and (4.47) are still valid ($J_1, \ldots, J_k$ are simply replaced by $X_1, \ldots, X_k$).*

We provide below a numerical illustration in which we consider a collective risk model with mixed Erlang distributions within the context just discussed.

Example 4.3.1. *Let $N \sim Binomial(3, 0.1)$ and X_i, follow a univariate mixed Erlang distribution, for $i = 1, 2, 3$ with cdf as given in (4.19), where*

$$\gamma_J(j) = \begin{cases} 0.2281946, & j = 2 \\ 0.5429590, & j = 9 \\ 0.2288464, & j = 14 \\ 0, & \textit{otherwise} \end{cases},$$

(α_0, α_1)	$(0.7, 5)$	$(2.3, 5)$	$(2.3, 25)$	$(\perp, \perp)$	$(+, +)$
$E[S]$	1.391406	1.565219	1.565219	1.307986	2.364319
$Var(S)$	7.99	9.444512	9.481301	7.103238	21.18255
$VaR_{0.9}(S)$	5.753878	6.30505	6.30371	5.513185	7.706855
$VaR_{0.99}(S)$	11.759088	12.78519	12.86657	10.934496	21.54156
$VaR_{0.999}(S)$	17.542042	18.47185	18.70519	15.981132	26.32088
$TVaR_{0.9}(S)$	8.361956	9.013769	9.022227	7.921505	12.22021
$TVaR_{0.99}(S)$	14.372150	15.395022	15.574824	13.152781	25.05844
$TVaR_{0.999}(S)$	19.624744	20.582120	20.882731	17.852269	42.23223

Table 4.4 – Values of the cdf, expectation, variance, VaR and TVaR of S as defined in Example 4.3.1.

and $\beta = 1.960312$. *Such a distribution is an approximation of a mixture of two gamma distributions obtained in Lee and Lin (2010) (see Table 15). The joint distribution of* $(N, J_1, \ldots, J_k)$ *is assumed to be defined with a hierarchical Archimedean copula through compounding as given in (4.46), with* $M \sim Logarithmic(q = 1 - \mathrm{e}^{-\alpha_0})$ *and* $B_i \sim B \sim Gamma(\lambda = 1/\alpha_1, 1)$. *Note that the rv S follows a mixed Erlang distribution. We consider the values* 0.5 *and* 0.9 *for* q, *and the values* 0.04 *and* 0.2 *for* λ, *which results in the following dependence parameters of the copula* $\alpha_0 \simeq 0.7, 2.3$, *and* $\alpha_1 = 25, 5$, *respectively.*

Values of F_s, $E[S]$, $Var(S)$, $VaR_\kappa(S)$, *and* $TVaR_\kappa(S)$ *are given in Table 4.4. For comparison purposes, the same quantities are also provided for the case of comonotonic case* $S^{(+,+)}$ *and for the independence case* $S^{(\perp,\perp)}$. *From this example, we can see that as the dependence parameter* α *(*α *being the parameter of either the outer or the inner copula) increases, the expectation, the variance and the TVaR increase as well. Also, if the outer parameter is fixed, the expectation* $E[S]$ *does not change when the inner dependence parameter changes, which is to be expected. Moreover, we can see that the values of the TVaR always fall between the TVaR in the independence case and the ones in the comonotonic case. In general, for collective risk models, the comonotonic case is considered as the worst case dependence scenario.*

Another approach can be used to model the dependence structure of $(N, \underline{X})$. Let C be a $(k+1)$-dimensional Archimedean or hierarchical Archimedean copula and $(U_0, U_1, \ldots, U_k)$ a vector of uniformly distributed rvs such that

$$F_{U_0,U_1,\ldots,U_k}(u_0, u_1, \ldots, u_k) = C(u_0, u_1, \ldots, u_k),$$

where $k \in \mathbb{N}$. We also suppose that $N = F_N^{-1}(1 - U_0)$, and $X_i = F_{X_i}^{-1}(U_i)$, for $i = 1, \ldots, k$.

(α_0, α_1)	$(0.7, 5)$	$(2.3, 5)$	$(2.3, 25)$	$(\perp, \perp)$
$E[S]$	1.223656	1.041449	1.041449	1.307986
$Var(S)$	6.620315	5.164109	5.211133	7.103238
$VaR_{0.9}(S)$	5.187718	4.544783	4.528977	5.513185
$VaR_{0.99}(S)$	10.806303	9.748571	9.803962	10.934496
$VaR_{0.999}(S)$	16.390610	14.708676	15.087800	15.981132
$TVaR_{0.9}(S)$	7.676748	6.850087	6.876406	7.921505
$TVaR_{0.99}(S)$	13.252042	11.901964	12.056931	13.152781
$TVaR_{0.999}(S)$	18.444212	16.666773	17.193214	17.852269

Table 4.5 – Values of the cdf, expectation, variance, VaR and TVaR of S as defined in Example 4.3.2.

Then, the joint cdf of $(N, X_1, \ldots, X_k)$ can be written as

$$\begin{aligned}
& F_{N,X_1,\ldots,X_k}(k, x_1, \ldots, x_k) \\
= \; & \Pr\left(N \leq k, X_1 \leq x_1, \ldots, X_k \leq x_k\right) \\
= \; & \Pr\left(F_N^{-1}(1-U_0) \leq k, F_{X_1}^{-1}(U_1) \leq x_1, \ldots, F_{X_k}^{-1}(U_k) \leq x_k\right) \\
= \; & \Pr\left(U_0 > 1 - F_N(k), U_1 \leq F_{X_1}(x_1), \ldots, U_k \leq F_{X_k}(x_k)\right) \\
= \; & \Pr\left(U_1 \leq F_{X_1}(x_1), \ldots, U_k \leq F_{X_k}(x_k)\right) \\
& - \Pr\left(U_0 \leq 1 - F_N(k), U_1 \leq F_{X_1}(x_1), \ldots, U_k \leq F_{X_k}(x_k)\right) \\
= \; & C\left(F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right) - C\left(1 - F_N(k), F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right). \qquad (4.48)
\end{aligned}$$

Example 4.3.2. *For this numerical illustration, we use the marginal distributions given in Example 4.3.1 and we suppose that C is as given in Example 4.3.1 with the same dependence parameters. From Table 4.5, we can see that the expectation of the aggregate loss S under this dependence structure is smaller than the one in the classical risk model, i.e.,* $E[S] \leq E[N] \times E[X] = 1.307986$*. Also, in this case, and contrarily to the result of Example 4.3.1, we can see that when the outer dependence parameter α_0 increase, the expectation, the variance, and TVaR decrease. Recall that the parameter α_0 captures the degree of dependence between N and the components of $\underline{X}$. Finally, for a fixed α_0, we observe that the expectation, the variance and the TVaR increase as the inner dependence parameter α_1 becomes larger, since the parameter only affects the strength of dependence among the claim amounts.*

Since the computation of (4.47) becomes more cumbersome, and even impossible for dimensions larger than 5, we need to find a calculation method that is more suitable in large dimensions.

4.3.2 Sampling Algorithm

In this section, we propose an efficient algorithm to generate samples of S. Inspired by the sampling algorithms of both nested Archimedean copulas and hierarchical copulas through compounding (see e.g., Marshall and Olkin (1988), Hofert (2008), and Cossette et al. (2017)), we propose a general sampling algorithm that generates samples of a random sum S incorporating a dependence relationship between the number and amounts of claims using hierarchical Archimedean copulas.

Let Θ_0 and Θ_1 be a two strictly positive rvs. Given $\Theta_0 = \theta_0$, we denote by $\Theta_{0,1}$ the conditional rv $(\Theta_1|\Theta_0 = \theta_0)$. Note that Θ_0 and $\Theta_{0,1}$ represent the mixing rvs such that given $\Theta_0 = \theta_0$ and $\Theta_{0,1} = \theta_{0,1}$, $(X_1|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1}), \ldots, (X_k|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1})$ are conditionally iid and independent of $(N|\Theta_0 = \theta_0)$. Note that if C is a nested Archimedean copula as in (4.45), Θ_0 and Θ_1 represent the mixing rvs related to the outer and inner copulas respectively, and $\Theta_{0,1}$ is such that $\mathcal{L}_{\Theta_{0,1}}(t;\theta) = \exp\left\{-\theta\mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}(t)\right\}$. As for the case where C is defined as in (4.46), the rv Θ_0 plays the same role as the rv M, $\Theta_1 = \sum_{j=1}^{M} B_j$, and, given $\Theta_0 = \theta_0$, $\Theta_{0,1} = \sum_{j=1}^{\theta_0} B_j$, for $\theta_0 \in \mathbb{N}$.

Algorithm 4.3.1. *Let C be a one level hierarchical Archimedean copula with generators $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$ allowing a dependence structure as depicted in Figure 4.3.*

1. *Sample Θ_0;*
2. *Sample $R \sim Exp(1)$;*
3. *Calculate $U = \mathcal{L}_{\Theta_0}\left(\frac{R}{\Theta_0}\right)$;*
4. *Return $N = F_N^{-1}(U)$;*
5. *If $N = 0$ return $S = 0$; else*
   - 5.1. *Sample $\Theta_{0,1}$;*
   - 5.2. *Sample $R_i \sim Exp(1)$ for $i = 1, \ldots, N$;*
   - 5.3. *Calculate $U_i = \mathcal{L}_{\Theta_1}\left(\frac{R_i}{\Theta_{0,1}}\right)$ for $i = 1, \ldots, N$;*
   - 5.4. *Calculate $X_i = F_X^{-1}(U_i)$ for $i = 1, \ldots, N$;*
   - 5.5. *Return $S = \sum_{k=1}^{N} X_i$;*
6. *Return S.*

In the following example, we provide an application of Algorithm 4.3.1.

Example 4.3.3. *Let $N \sim Poisson(2)$ and $X_i \sim Pareto(3, 100)$, for $i = 1, 2, \ldots$. We consider collective risk model defined with the pair $(N, \underline{X})$, where the multivariate cdf is defined with the same copula and dependence parameters as in Example 4.3.1. Approximated values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$, using 10 million simulations, are given in Table 4.6. As we can see, when one or both dependence parameters increase, $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ also increase. Moreover, the expectation of S does not change when the outer parameter is fixed. From Figure 4.4 and Figure 4.5, we can also see that the three curves, representing the cdf of S for different values of the dependence parameters, intersect multiple times.*

	$\alpha_0 = 0.7,\ \alpha_1 = 5$	$\alpha_0 = 2.3,\ \alpha_1 = 5$	$\alpha_0 = 2.3,\ \alpha_1 = 25$
$E[S]$	108.042	125.441	125.440
$Var(S)$	34426.712	44913.801	52347.687
$VaR_{0.9}(S)$	293.243	348.577	347.406
$VaR_{0.99}(S)$	852.977	966.297	1070.826
$VaR_{0.999}(S)$	1664.452	1837.663	2048.952
$TVaR_{0.9}(S)$	534.521	616.365	658.370
$TVaR_{0.99}(S)$	1211.572	1351.950	1500.564
$TVaR_{0.999}(S)$	2301.543	2540.498	2748.067

Table 4.6 – Values of the expectation, variance, VaR and TVaR of S as defined in Example 4.3.3.

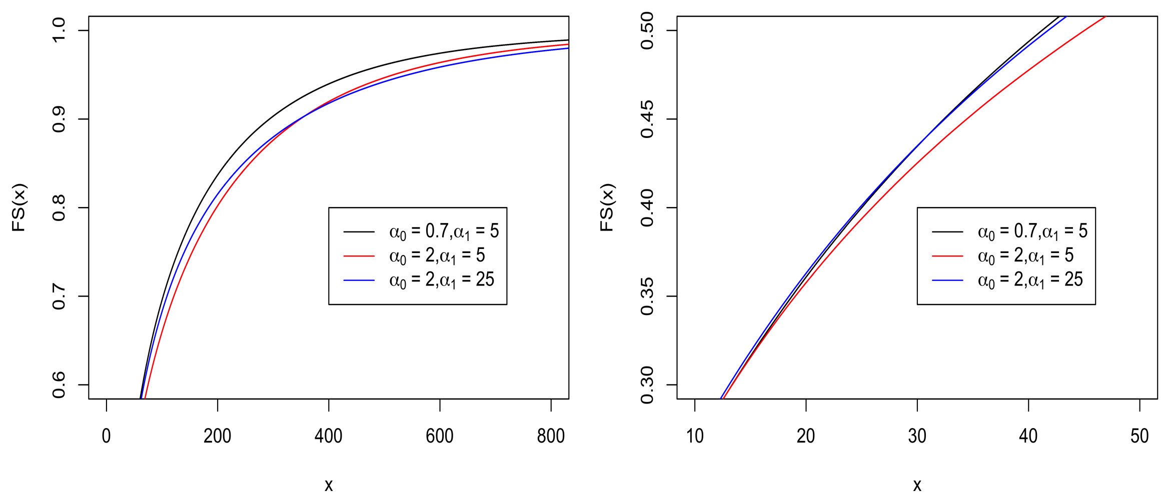


Figure 4.4 – Cdf of S, for different values of α_0 and α_1, as defined in Example 4.3.3.

As discussed within Example 4.3.1, a computational methodology is needed to calculate the cdf of S in high dimensions. The sampling algorithm just presented can be used to derive approximated values of the cdf of S via Monte Carlo simulations. This approach is efficient and very practical especially in the case of continuous mixing rvs and/or continuous marginals. Based on Cossette et al. (2018), we derive, in the following section, another computational methodology allowing to compute exact values of the cdf of S for discrete rvs X_i, $i = 1, 2, \ldots$, even in high dimensions.

4.3.3 Computational Methodology

In this section, we adapt the computational methodology presented in Cossette et al. (2018) to derive an algorithm to compute the cdf of a random sum S, incorporating a dependence

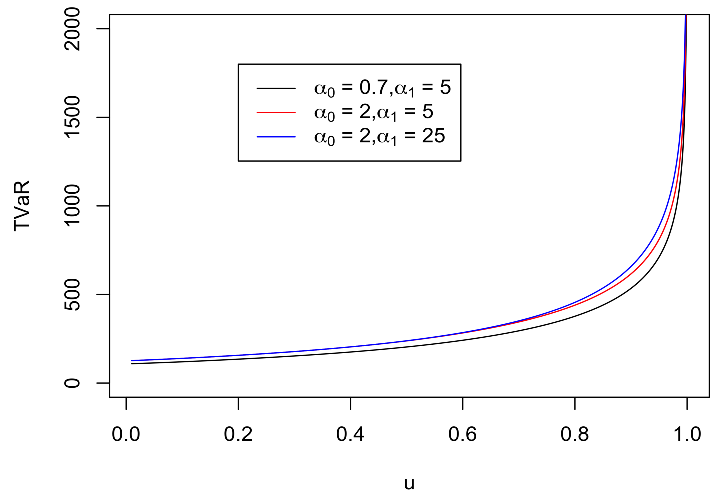


Figure 4.5 – $TVaR_u(S)$ for different values of α_0 and α_1, as defined in Example 4.3.3.

relationship between the claim number rv N and the claim amount rvs $X_1, X_2, \ldots$. Here, we only discuss the case of discrete mixing rvs and discrete marginals, whilst the case of continuous marginals or continuous mixing rvs can be treated in a similar fashion as in Cossette et al. (2018) where discretization methods are used to derive upper and lower bounds for the exact value of the cdf of S.

A simple hierarchical structure

The collective risk model with dependence presented in this paper is an extension of the one treated in Section 4 of Cossette et al. (2018). We adapt here its proposed computational strategy to our context. The idea behind it is to use the conditional independence assumption to identify the conditional distribution of N and X_i. While this computational strategy works naturally for discrete rvs X_i, discretization methods can be used to approximate continuous rvs X_i, $i = 1, 2, \ldots$ (see details in Cossette et al. (2018); see also e.g. Müller and Stoyan (2002) and Bargès et al. (2009) for a review of different discretization methods).

In this section, we assume discrete marginals for X_i with $X_i \in \{0, 1h, 2h, \ldots\}$, $i = 1, 2, \ldots$, and discrete mixing rvs Θ_0 and $\Theta_{0,1}$ with respective LSTs $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$, respective pmfs $f_{\Theta_0}(\theta_0) = \Pr(\Theta_0 = \theta_0)$ and $f_{\Theta_{0,1}}(\theta_{0,1}) = \Pr(\Theta_{0,1} = \theta_{0,1})$, and respective cdfs $F_{\Theta_0}(\theta_0) = \Pr(\Theta_0 \leq \theta_0) = \sum_{j=1}^{\theta_0} f_{\Theta_0}(j)$ and $F_{\Theta_{0,1}}(\theta_{0,1}) = \Pr(\Theta_{0,1} \leq \theta_{0,1}) = \sum_{j=1}^{\theta_{0,1}} f_{\Theta_{0,1}}(j)$, for $\theta_0, \theta_{0,1} \in \mathbb{N}$.

The multivariate cdf of $(N, X_1, \ldots, X_k)$ becomes

$$\begin{aligned} & F_{N,X_1,\ldots,X_k}(k, m_1 h, \ldots, m_k h) \\ = & \sum_{\theta_0=1}^{\infty} F_{N|\Theta_0=\theta_0}(k) \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} F_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}(m_i h) f_{\Theta_{0,1}}(\theta_{0,1}) f_{\Theta_0}(\theta_0) \\ = & \sum_{\theta_0=1}^{\infty} \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}(F_N(k))} \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(m_i h)\right)} f_{\Theta_{0,1}}(\theta_{0,1}) f_{\Theta_0}(\theta_0), \end{aligned} \tag{4.49}$$

where $m_1, \ldots, m_k \in \mathbb{N}_0$, $F_{N|\Theta_0=\theta_0}(k) = \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}(F_N(k))}$, $\forall k \in \mathbb{N}_0$, and

$$F_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}(m_i h) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(m_i h)\right)}, \tag{4.50}$$

for $m_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, k$, and $\theta_{0,1} \in \mathbb{N}$.

Let N_{θ_0} and $X_{i,\theta_0,\theta_{0,1}}$ denote, respectively, the conditional rvs $(N|\Theta_0 = \theta_0)$ and $(X_i|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1})$, for $i = 1, 2, \ldots$ The pmf and pgf of N_{θ_0} are respectively given by

$$\Pr(N_{\theta_0} = k) = \begin{cases} \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}(F_N(0))} & , k = 0 \\ \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}(F_N(k))} - \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}(F_N(k-1))} & , k \in \mathbb{N} \end{cases}, \tag{4.51}$$

and

$$\mathcal{P}_{N_{\theta_0}}(t) = E\left[t^{N_{\theta_0}}\right] = \sum_{k=0}^{\infty} t^k \Pr(N_{\theta_0} = k). \tag{4.52}$$

For $i = 1, 2, ...$ and for each $\theta_{0,1} \in \mathbb{N}$, we can find the values of $f_{X_{i,\theta_0,\theta_{0,1}}}(m_i h)$ with

$$f_{X_{i,\theta_0,\theta_{0,1}}}(m_i h) = \begin{cases} \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(0)\right)} & , m_i = 0 \\ \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(m_i h)\right)} - \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}((m_i-1)h)\right)} & , m_i \in \mathbb{N} \end{cases}. \tag{4.53}$$

Similar results are obtained when the dependence structure of $(N, X_1, \ldots, X_k)$ is induced via a copula C and survival functions as in (4.42). The survival function of $(N, X_1, \ldots, X_k)$ is given by

$$\begin{aligned} & \overline{F}_{N,X_1,\ldots,X_k}(k, m_1 h, \ldots, m_k h) \\ = & \sum_{\theta_0=1}^{\infty} \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}\left(\overline{F}_N(k)\right)} \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}(m_i h)\right)} f_{\Theta_{0,1}}(\theta_{0,1}) f_{\Theta_0}(\theta_0), \end{aligned} \tag{4.54}$$

where $k, m_1, \ldots, m_k \in \mathbb{N}_0$. In this case, the pmf of N_{θ_0} is

$$\Pr(N_{\theta_0} = k) = \begin{cases} 1 - \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}\left(\overline{F}_N(0)\right)} & , k = 0 \\ \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}\left(\overline{F}_N(k-1)\right)} - \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}\left(\overline{F}_N(k)\right)} & , k \in \mathbb{N} \end{cases}. \tag{4.55}$$

As for $X_{i,\theta_0,\theta_{0,1}}$, we have

$$\overline{F}_{X_{i,\theta_0,\theta_{0,1}}}(m_i h) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}(m_i h)\right)}, \tag{4.56}$$

for $m_i \in \mathbb{N}_0$, $i = 1, 2, \ldots$, and $\theta \in \mathbb{N}$.

The expression for $f_{N,X_1,\ldots,X_k}(k, m_1 h, \ldots, m_k h)$ in this case is consequently given by

$$f_{N,X_1,\ldots,X_k}(k, m_1 h, \ldots, m_k h) = \sum_{\theta_0=1}^{\infty} \Pr\left(N_{\theta_0} = k\right) \left\{ \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} f_{X_{i,\theta_0,\theta_{0,1}}}(m_i h) f_{\Theta_{0,1}}(\theta_{0,1}) \right\} f_{\Theta_0}(\theta_0). \tag{4.57}$$

Let $S_{\theta_0,\theta_{0,1}} = \sum_{i=1}^{N_{\theta_0}} X_{i,\theta_0,\theta_{0,1}}$ be the sum of conditionally independent rvs and $f_{S_{\theta_0,\theta_{0,1}}}$ be its corresponding pmf. Let $\mathcal{L}_{X_{\theta_0,\theta_{0,1}}}$ be the LST of $X_{\theta_0,\theta_{0,1}}$, where $X_{i,\theta_0,\theta_{0,1}} \sim X_{\theta_0,\theta_{0,1}}$, for $i \in \mathbb{N}$. Then, the LST of $S_{\theta_0,\theta_{0,1}}$ is given by

$$\mathcal{L}_{S_{\theta_0,\theta_{0,1}}}(t) = \mathcal{P}_{N_{\theta_0}}\left(\mathcal{L}_{X_{\theta_0,\theta_{0,1}}}(t)\right). \tag{4.58}$$

Using (4.58) and FFT, it is easy to compute the exact values of $f_{S_{\theta_0,\theta_{0,1}}}$ for each θ_0 and $\theta_{0,1}$. Finally, due to the representation of $f_{N,X_1,\ldots,X_k}$ in (4.57), the unconditional pmf of S can be computed using

$$f_S(mh) = \sum_{\theta_0=1}^{\infty} \sum_{\theta_{0,1}=1}^{\infty} f_{S_{\theta_0,\theta_{0,1}}}(mh) f_{\Theta_{0,1}}(\theta_{0,1}) f_{\Theta_0}(\theta_0), \; m \in \mathbb{N}_0. \tag{4.59}$$

The computational methodology used to find the exact values of f_S is summarized in the following algorithm.

Algorithm 4.3.2. ***Computation of the values of*** f_S

1. *Fix* $\theta_0 = 1$*;*
2. *Fix* $\theta_{0,1} = 1$*;*
3. *Calculate either* $F_{X_{i,\theta_0,\theta_{0,1}}}(m_i h)$ *with (4.50) or* $\overline{F}_{X_{i,\theta_0,\theta_{0,1}}}(m_i h)$ *with (4.56), for* $m_i \in \mathbb{N}_0$*;*
4. *Calculate* $f_{X_{i,\theta_0,\theta_{0,1}}}(m_i h)$*, for* $m_i \in \mathbb{N}_0$*;*
5. *Use FFT to return the vector* $\underline{\tilde{f}}_{X_{i,\theta_0,\theta_{0,1}}}$*, where* $\underline{\tilde{f}}$ *denotes the vector of values of the characteristic function, also known as the Fourier transform (see e.g., Klugman et al. (2009));*
6. *For* $k = 0, 1, \ldots,$ *calculate* $\Pr\left(N_{\theta_0} = k\right)$ *using either (4.51) or (4.55);*
7. *Use the pgf of* N_{θ_0} *given in (4.52) to calculate* $\underline{\tilde{f}}_{S_{\theta_0,\theta_{0,1}}} = \mathcal{P}_{N_{\theta_0}}\left(\underline{\tilde{f}}_{X_{\theta_0,\theta_{0,1}}}\right)$*;*
8. *Use FFT (inverse) to compute* $f_{S_{\theta_0,\theta_{0,1}}}(mh)$ *for* $m \in \mathbb{N}_0$*;*

	Exact values (Alg.4.3.2)	1M MC simulations (Alg.4.3.1)	Comonotonicity
$E[S]$	4.039336	4.040922	5.25118
$Var(S)$	10.482232	10.450744	35.58824
$VaR_{0.9}(S)$	8.000000	8.000000	12.00000
$VaR_{0.99}(S)$	14.000000	14.000000	30.00000
$VaR_{0.9999}(S)$	23.000000	23.000000	63.00000
$TVaR_{0.9}(S)$	10.809860	10.793440	18.79353
$TVaR_{0.99}(S)$	15.835839	15.781300	34.38692
$TVaR_{0.9999}(S)$	24.651145	24.630000	68.10100

Table 4.7 – Values of the expectation, variance, VaR and TVaR of S as defined in Example 4.3.4.

9. *Repeat steps 3-8 for* $\theta_{0,1} = 2, ..., \theta^*_{0,1}$ *where* $\theta^*_{0,1}$ *is chosen such that* $F_{\Theta_{0,1}}\left(\theta^*_{0,1}\right) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*);*
10. *Compute* $f_{S|\Theta_0=\theta_0}(mh) = \sum_{\theta_{0,1}=1}^{\theta^*_{0,1}} f_{S_{\theta_0,\theta_{0,1}}}(mh) f_{\Theta_{0,1}}(\theta_{0,1})$*, for* $m \in \mathbb{N}_0$*;*
11. *Repeat steps 2-10 for* $\theta_0 = 2, ..., \theta^*_0$ *where* θ^*_0 *is chosen such that* $F_{\Theta_0}(\theta^*_0) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.,* $\varepsilon = 10^{-10}$*);*
12. *Compute* $f_S(mh) = \sum_{\theta_0=1}^{\theta^*_0} f_{S|\Theta_0=\theta_0}(mh) f_{\Theta_0}(\theta_0)$*, for* $m \in \mathbb{N}_0$*.*

Example 4.3.4. *Let* $N \sim Poisson(2)$ *and* $X_i - 1 \sim Binomial(10, 0.1)$*, for* $i = 1, 2, \ldots$ *Also, assume that the joint distribution of* $(N, X_1, \ldots, X_k)$ *is defined as in (4.41) with a nested Ali-Mikhail-Haq (AMH) copula with parameters* $\alpha_0 = 0.1$ *and* $\alpha_1 = 0.2$*, i.e.,* $\Theta_0 \sim Geometric(1-\alpha_0)$*,* $\Theta_1 \sim Geometric(1-\alpha_1)$*, and* $\Theta_{0,1} \sim Shifted\,Negative\,Binomial\left(\alpha_0, \frac{1-\alpha_1}{1-\alpha_0}\right)$*. Using Algorithm 4.3.2, we compute the values of* f_S *allowing to derive the exact values of* $E[S]$*,* $Var(S)$*,* $VaR_\kappa(S)$*, and* $TVaR_\kappa(S)$*, provided in Table 4.7. The same quantities are also calculated using 1000000 Monte Carlo simulations (using Algorithm 4.3.1), and also for comonotonic rvs* N^+ *and* X_i^+*, for* $i = 1, 2, \ldots$*. As expected, the highest values for the TVaR of* S *are obtained for the perfect positive dependence case, which is in line with Corollary 4.2.1. Also, we can see that the results for both Algorithms 4.3.2 and 4.3.1 are close, with comparable computation times.*

Remark 4.3.2. *If* C *is an Archimedean copula with mixing rv* Θ*, meaning that* $N, X_1, X_2, \ldots$ *are conditionally independent given* $\Theta = \theta$*, computation procedure is nearly the same. One has only to replace* $\Theta_{0,1}$ *and* Θ_0 *by* Θ *and Algorithm 4.3.2 becomes the following:*

Algorithm 4.3.3. ***Computation of the values of*** f_S

1. *Fix* $\theta = 1$*;*
2. *Calculate either* $F_{X|\Theta=\theta}(m_i h) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(F_X(m_i h))}$ *or* $\overline{F}_{X|\Theta=\theta}(m_i h) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}\left(\overline{F}_X(m_i h)\right)}$*, for* $m_i \in \mathbb{N}_0$*;*
3. *Deduce* $f_{X|\Theta=\theta}(m_i h)$ *from step 2, for* $m_i \in \mathbb{N}_0$*;*

4. *Use FFT to return the vector* $\underline{\tilde{f}}_{X|\Theta=\theta}$*;*
5. *For* $k = 0, 1, 2, \ldots$*, calculate* $\Pr(N_\theta = k)$ *using either (4.51) or (4.55);*
6. *Use the pgf of* N_θ *given in (4.52) to calculate* $\underline{\tilde{f}}_{S|\Theta=\theta} = \mathcal{P}_{N_\theta}\left(\underline{\tilde{f}}_{X|\Theta=\theta}\right)$*;*
7. *Use FFT (inverse) to compute* $f_{S|\Theta=\theta}(mh)$*, for* $m \in \mathbb{N}_0$*;*
8. *Repeat steps 2-7 for* $\theta = 2, ..., \theta^*$ *where* θ^* *is chosen such that* $F_\Theta(\theta^*) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*);*
9. *Compute* $f_S(mh) = \sum_{\theta=1}^{\theta^*} f_{S|\Theta=\theta}(mh) f_\Theta(\theta)$*, for* $m \in \mathbb{N}_0$*.*

Until now, we have only considered one level hierarchical dependence structures. We can generalize the proposed model to a multi-level hierarchical structure by considering the dependence structure linking the rvs X_i, for $i = 1, 2, \ldots$, to be a hierarchical one. Using the mixture representation as in Section 4.3.3, Algorithm 4.3.2 can be easily modified to fit a multi-level hierarchical Archimedean copula (see Section 7 of Cossette et al. (2018) for details).

4.3.4 Impact of dependence on the aggregate claim amount

Observations made in Examples 4.3.1, 4.3.2, 4.3.3, and 4.3.4 concerning the impact of dependence parameters on the expectation and the TVaR drove us to further investigate the dependence relationship connecting the components of $(N, X_1, \ldots, X_k)$. In Corollary 4.2.1, we compare our proposed dependence structure for $(N, X_1, \ldots, X_k)$ with the case of perfect positive dependence. What happens if we slightly increase the dependence? Can we compare two random sums with the same dependence structure but different dependence parameters? To address these questions, we resort to the supermodular dependence order as defined in Definition 4.2.3. More precisely, in order to use the result of Proposition 4.2.1, we need to identify the criteria for establishing the supermodular order for the considered dependence structures.

The following two properties aim to compare, according to the supermodular order, two random vectors $(N, Y_1, \ldots, Y_k)$ and $(N^*, Y_1^*, \ldots, Y_k^*)$ for which the multivariate distribution is defined with either an Archimedean copula or a hierarchical Archimedean copula.

Corollary 4.3.1. *Let the multivariate distributions of* $(N, Y_1, \ldots, Y_k)$ *and* $(N^*, Y_1^*, \ldots, Y_k^*)$ *be defined using either (4.41) or (4.42), where* C *is an Archimedean copula with dependence parameter* α *and* α^* *respectively. If* $\alpha \leq \alpha^*$*, then,*

$$C_\alpha \preceq_{sm} C_{\alpha^*},$$

$$(N, Y_1, \ldots, Y_k) \preceq_{sm} (N^*, Y_1^*, \ldots, Y_k^*).$$

Proof. To show $C_\alpha \preceq_{sm} C_{\alpha^*}$ see Wei and Hu (2002). Using the property of closure under all increasing (or decreasing) transforms of the supermodular order (see, e.g. Theorem 9.A.9.(a) of Shaked and Shanthikumar (2007)), we can conclude that $(N, Y_1, \ldots, Y_k) \preceq_{sm} (N^*, Y_1^*, \ldots, Y_k^*)$, for all $k \in \mathbb{N}$. □

Corollary 4.3.2. *Let the multivariate distributions of $(N, Y_1, \ldots, Y_k)$ and $(N^*, Y_1^*, \ldots, Y_k^*)$ be defined using either (4.41) or (4.42), where C is a one level hierarchical Archimedean copula as illustrated in Figure 4.3, with parameters α_0, α_1 and α_0^*, α_1^* respectively.*

1. *If $\alpha_0 \le \alpha_0^*$ and $\alpha_1 = \alpha_1^*$, then $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1}$;*
2. *If $\alpha_0 = \alpha_0^*$ and $\alpha_1 \le \alpha_1^*$, then $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0,\alpha_1^*}$;*
3. *If $\alpha_0 \le \alpha_0^*$ and $\alpha_1 \le \alpha_1^*$, then $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1^*}$.*

Therefore, for all three cases, we have

$$(N, Y_1, \ldots, Y_k) \preceq_{sm} (N^*, Y_1^*, \ldots, Y_k^*)\,, \forall k \in \mathbb{N}.$$

Proof. The proofs for 4.3.2.1 and 4.3.2.2 can be found in Wei and Hu (2002) for nested Archimedean copulas. Given the links between nested Archimedean copulas and hierarchical Archimedean copulas through compounding (see Cossette et al. (2017) for details), this result also holds here for the one-level hierarchical Archimedean copula defined in (4.46). If $\alpha_0 \le \alpha_0^*$ and $\alpha_1 \le \alpha_1^*$, then, using 4.3.2.1 and 4.3.2.2, we have $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1^*}$. Once again the property of closure under all increasing (or decreasing) transforms of the supermodular order can be used to show that $(N, Y_1, \ldots, Y_k) \preceq_{sm} (N^*, Y_1^*, \ldots, Y_k^*)$, for all $k \in \mathbb{N}$. □

Note that the observations made in Example 4.3.3 can now be explained using both Proposition 4.2.1 and Corollary 4.3.2. Also, conclusions in regard to Example 4.3.4 hold for other chosen inner or outer parameter values.

Corollary 4.3.3. *Let $(J_1, \ldots, J_k)$ and $(J_1^*, \ldots, J_k^*)$ be two random vectors, for $k \in \mathbb{N}$. Define $Y_i = \sum_{j=1}^{J_i} B_{i,j}$, where $B_{i,1}, \ldots, B_{i,J_i}$ are iid and independent of J_i, for $i = 1, \ldots, k$. Define also $Y_i^* = \sum_{j=1}^{J_i^*} B_{i,j}^*$, where $B_{i,1}^*, \ldots, B_{i,J_i}^*$ are iid and independent of J_i^*, for $i = 1, \ldots, k$. If the vectors $(N, J_1, \ldots, J_k)$ and $(N^*, J_1^*, \ldots, J_k^*)$ are such that*

$$(N, J_1, \ldots, J_k) \preceq_{sm} (N^*, J_1^*, \ldots, J_k^*)\,, \forall k \in \mathbb{N}.$$

Then,

$$(N, Y_1, \ldots, Y_k) \preceq_{sm} (N^*, Y_1^*, \ldots, Y_k^*)\,, \forall k \in \mathbb{N}.$$

Proof. We have $N = \sum_{i=1}^{N} 1$, then, using Proposition 2 of Denuit et al. (2002), we obtain the desired result. □

If we consider two collective risk models $(N, \underline{X}) \in \aleph$ and $(N^*, \underline{X}^*) \in \aleph$ defined with mixed Erlang distributions as presented in Section 4.2.3, then, using Proposition 4.2.1 and Corollary 4.3.3, we have: if $(N, J_1, \ldots, J_k) \preceq_{sm} (N^*, J_1^*, \ldots, J_k^*)$, $\forall k \in \mathbb{N}$, then $S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*$ and hence $TVaR_\kappa(S) \leq TVaR_\kappa(S^*)$, for $\kappa \in (0,1)$. Such a result justifies the observations made in Examples 4.2.1 and 4.3.1.

Remark 4.3.3. *Note that if the dependence structures of both $(N, X_1, \ldots, X_k)$ and $(N^*, X_1^*, \ldots, X_k^*)$ are modelled with the same hierarchical Archimedean copula with identical outer dependence parameter but different inner dependence parameters, then the expectations of S and S^* coincide. In this case, the result given in (4.15) of Proposition 4.3.2 is given in terms of convex order instead of increasing convex order, i.e., $S \preceq_{cx} S^*$.*

Remark 4.3.4. *Note that if the joint cdf of $(N, X_1, \ldots, X_k)$ is as given in (4.48), then, using the supermodular order, we can show that $E[S] \leq E[N] \times E[X]$. We know that*

$$\left(U_0^{\perp}, U_1^{\perp}, \ldots, U_k^{\perp}\right) \preceq_{sm} (U_0, U_1, \ldots, U_k).$$

Since $f : x \to F_N^{-1}(1-x)$ and $g : x \to -F_X^{-1}(x)$ are decreasing functions, using the closure under all decreasing transforms, we have

$$\left(F_N^{-1}\left(1-U_0^{\perp}\right), -F_{X_1}^{-1}\left(U_1^{\perp}\right), \ldots, -F_{X_k}^{-1}\left(U_k^{\perp}\right)\right) \preceq_{sm} \left(F_N^{-1}(1-U_0), -F_{X_1}^{-1}(U_1), \ldots, -F_{X_k}^{-1}(U_k)\right),$$

i.e.,

$$\left(N^{\perp}, -X_1^{\perp}, \ldots, -X_k^{\perp}\right) \preceq_{sm} (N, -X_1, \ldots, -X_k).$$

Then, $-S^{(\perp,\perp)} \preceq_{icx} -S$, which means that $E[N] \times E[-X] \leq E[-S]$, and hence, $E[S] \leq E[N] \times E[X]$.

4.4 Bibliography

Albrecher, H., Boxma, O. J., and Ivanovs, J. (2014). On simple ruin expressions in dependent Sparre Andersen risk models. *Journal of Applied Probability*, 51(1):293–296.

Bargès, M., Cossette, H., and Marceau, E. (2009). TVaR-based capital allocation with copulas. *Insurance: Mathematics and Economics*, 45(3):348–361.

Bäuerle, N. and Müller, A. (2006). Stochastic orders and risk measures: consistency and bounds. *Insurance: Mathematics and Economics*, 38(1):132–148.

Boudreault, M., Cossette, H., Landriault, D., and Marceau, E. (2006). On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, 2006(5):265–285.

Cossette, H., Gadoury, S.-P., Marceau, É., and Mtalai, I. (2017). Hierarchical archimedean copulas through multivariate compound distributions. *Insurance: Mathematics and Economics*, 76:1–13.

Cossette, H., Marceau, E., and Marri, F. (2008). On the compound Poisson risk model with dependence based on a generalized farlie–gumbel–morgenstern copula. *Insurance: Mathematics and Economics*, 43(3):444–455.

Cossette, H., Marceau, E., Mtalai, I., and Veilleux, D. (2018). Dependent risk models with archimedean copulas: A computational strategy based on common mixtures and applications. *Insurance: Mathematics and Economics*, 78:53–71.

Cossette, H., Marceau, E., and Perreault, S. (2015). On two families of bivariate distributions with exponential marginals: Aggregation and capital allocation. *Insurance: Mathematics and Economics*, 64:214–224.

Czado, C., Kastenmeier, R., Brechmann, E. C., and Min, A. (2012). A mixed copula model for insurance claims and claim sizes. *Scandinavian Actuarial Journal*, 2012(4):278–305.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2006). *Actuarial theory for dependent risks: measures, orders and models.* John Wiley & Sons.

Denuit, M., Genest, C., and Marceau, É. (2002). Criteria for the stochastic ordering of random sums, with actuarial applications. *Scandinavian Actuarial Journal*, 2002(1):3–16.

Durante, F. and Sempi, C. (2010). Copula theory: an introduction. In *Copula theory and its applications*, pages 3–31. Springer.

Fang, Y. (2001). Hyper-Erlang distribution model and its application in wireless mobile networks. *Wireless Networks*, 7(3):211–219.

Frees, E. W., Gao, J., and Rosenberg, M. A. (2011). Predicting the frequency and amount of health care expenditures. *North American Actuarial Journal*, 15(3):377–392.

Garrido, J., Genest, C., and Schulz, J. (2016). Generalized linear models for dependent frequency and severity of insurance claims. *Insurance: Mathematics and Economics*, 70:205–215.

Gschlössl, S. and Czado, C. (2007). Spatial modelling of claim frequency and claim size in non-life insurance. *Scandinavian Actuarial Journal*, 2007(3):202–225.

Hofert, M. (2008). Sampling archimedean copulas. *Computational Statistics & Data Analysis*, 52(12):5163–5174.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts.* CRC Press.

Joe, H. (2014). *Dependence modeling with copulas.* CRC Press.

Klugman, S. A., Panjer, H. H., Willmot, G. E., and Venter, G. (2009). Loss models: From data to decisions . *Annals of Actuarial Science*, 4(2):343.

Kotz, S., Balakrishnan, N., and Johnson, N. L. (2004). *Continuous multivariate distributions, Volume 1: Models and applications*, volume 1. John wiley & sons.

Kousky, C. and Cooke, R. M. (2009). The unholy trinity: fat tails, tail dependence, and micro-correlations. *The unholy trinity: fat tails, tail dependence, and micro-correlations*, 2009(36):1–36.

Krämer, N., Brechmann, E. C., Silvestrini, D., and Czado, C. (2013). Total loss estimation using copula-based regression models. *Insurance: Mathematics and Economics*, 53(3):829–839.

Landriault, D., Lee, W. Y., Willmot, G. E., and Woo, J.-K. (2014). A note on deficit analysis in dependency models involving coxian claim amounts. *Scandinavian Actuarial Journal*, 2014(5):405–423.

Lee, S. C. and Lin, X. S. (2010). Modeling and evaluating insurance losses via mixtures of erlang distributions. *North American Actuarial Journal*, 14(1):107–130.

Lee, S. C. and Lin, X. S. (2012). Modeling dependent risks with multivariate erlang mixtures. *ASTIN Bulletin: The Journal of the IAA*, 42(1):153–180.

Liu, H. and Wang, R. (2017). Collective risk models with dependence uncertainty. *ASTIN Bulletin: The Journal of the IAA*, 47(2):361–389.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841.

McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools.* Princeton university press.

Müller, A. and Stoyan, D. (2002). *Comparison methods for stochastic models and risks*, volume 389. Wiley.

Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. L. (1999). *Stochastic processes for insurance and finance.* John Wiley & Sons.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders.* Springer Science & Business Media.

Wei, G. and Hu, T. (2002). Supermodular dependence ordering on a class of multivariate copulas. *Statistics & probability letters*, 57(4):375–385.

Willmot, G. E. and Lin, X. S. (2011). Risk modelling with the mixed erlang distribution. *Applied Stochastic Models in Business and Industry*, 27(1):2–16.

Willmot, G. E. and Woo, J.-K. (2007). On the class of erlang mixtures with risk theoretic applications. *North American Actuarial Journal*, 11(2):99–115.

Willmot, G. E. and Woo, J.-K. (2015). On some properties of a class of multivariate erlang mixtures with insurance applications. *ASTIN Bulletin: The Journal of the IAA*, 45(1):151–173.

Chapitre 5

A note on the univariate and multivariate mixed Exponential Distributions

Résumé

En science actuarielle, il est assez courant d'utiliser des distributions construites avec des mélanges communs, que ce soit dans un contexte univarié ou multivarié, dans la modélisation des montants de sinistres. Dans cet article, des mélanges basés sur des variables aléatoires discrètes sont considérés. Trois cas particuliers sont traités en détail et de nouvelles distributions univariées et multivariées sont obtenues. Le lien avec les copules Archimédiennes est également établi.

Abstract

In actuarial science, it is common to work with distributions built with common mixtures, either in a univariate or a multivariate context. In this paper, mixtures based on discrete random variables (rv) are considered. Three particular cases are treated in detail and, thus, new univariate and multivariate distributions are obtained. The link with Archimedean copulas is also established.

Keywords: Common mixture representation, Negative binomial distribution, Discrete stable distribution, Linnik distribution, Archimedean copulas.

5.1 Introduction

Mixed distributions are frequently used in actuarial risk modeling. Distributions obtained through mixtures allow greater flexibility in the modeling of loss amounts. Often, the use of mixtures leads to heavy tailed distributions such as the Pareto distribution which arises as a mixed exponential-gamma distribution. More specifically, mixed exponential distributions have a mode of zero and a coefficient of variation greater than one which makes them good candidates for claim amount modeling. Mixed exponential distributions are also frequently used as life-time distributions in the context of failure data analysis. Several mixed exponential distributions were proposed in the literature in such a context, see, e.g., Adamidis and Loukas (1998), Adamidis et al. (2005), Kuş (2007), Tahmasbi and Rezaei (2008), Chahkandi and Ganjali (2009), Barreto-Souza and Bakouch (2013), Hajebi et al. (2013), Gui et al. (2014) and Yilmaz et al. (2016). Mixed exponential distributions have a decreasing failure rate (DFR) which makes them very useful in reliability theory (see, e.g., Proschan (1963) for a proof of the DFR property of mixtures of distributions with a constant hazard rate).

This mixing technique is not only used in a univariate setting. It can also serve to construct multivariate distributions. Marshall and Olkin (1988) and Oakes (1989) were among the first ones to use such a construction technique to obtain new multivariate distributions. In such multivariate models, the dependence between risks is induced via a common random variable (rv) (or several common rvs) representing common economic, geographical or climate conditions, or any other common external factor. Also, this mixing technique can also be used to define a multivariate dependence structure as an Archimedean copula (see, e.g., Marshall and Olkin (1988)).

As previously stated, several research works have studied mixed exponential distributions in the univariate and multivariate cases. The present paper highlights the usefulness of such distributions and lays the story of the mixing technique behind them. It also explains the underlying link between all these works. In addition, a comprehensive study of three special cases of mixing distributions is considered. Throughout the paper, the negative binomial distribution, the discrete stable distribution, and the discrete Linnik distribution are considered as special cases of the distribution of the mixing rv.

The outline of the paper is as follows. In Section 5.2, we illustrate through a detailed example in a credit risk portfolio context, the utility of common mixture models which highlights the importance of developing new distributions based on common mixtures. Other motivation examples are discussed in Section 5.3, 5.4, and 5.5. Section 5.3 presents the univariate mixed exponential distribution, discusses some of its interesting properties and studies three special cases. A generalization of this distribution to the multivariate case is treated in Section 5.4.

The link with Archimedean copulas is also established in this section. Finally, Section 5.5 is devoted to the investigation of ruin problems in portfolios with exchangeable inter-claim times.

5.2 A motivation example

Before carrying out a detailed study on different univariate and multivariate distributions based on common mixtures, we present an application in credit risk which illustrates the usefulness of such dependence models.

To comply with internal risk management policies and other regulations, financial institutions need to model and evaluate risk within their credit portfolios. Credit risk is now considered to be one of the most important risk facing companies, banks and financial institutions, as it includes all default risks associated with the incapacity of a borrower to repay a loan or a bond. In most cases, credit risks are influenced by one or more common factors, such as bankruptcy rules, economic environment or geographical regions. For this reason, it is not realistic to assume the independence among risks. In this section we discuss a default model based on hierarchical Archimedean copulas allowing to capture hierarchical dependence structures among risks of a credit portfolio. In the same context, hierarchical Archimedean copulas have been used to model the dependence structure of the default times in a credit risk portfolio (see, e.g., Whelan et al. (2004), Hofert and Scherer (2011), and Hering et al. (2010)). In this paper, we consider the dependence between the rvs representing the occurrences of defaults and we are interested in modelling the distribution of the global number of defaults.

Let us consider a one period credit risk default model, which is divided into d groups. Typically, risks belonging to the same group are exchangeable and are affected by the same factors such as consumer trends and macroeconomic factors. Also, we suppose that all groups are affected by certain factors at the same time, such as geographical regions and global economic shocks.

Let $\underline{X} = (\underline{X}_1, ..., \underline{X}_d)$, where $\underline{X}_i = (X_{i,1}, ..., X_{i,n_i})$ for $i = 1, \ldots, d$, be a vector of dependent Bernoulli rvs, where $X_{i,j} \sim Bernoulli\,(q_i)$, $j = 1, 2, ..., n_i$ and $i = 1, 2, ..., d$. Note that $X_{i,j} = 1$ means default of the j^{th} risk of the i^{th} group. We aim to examine the random behaviour of the number of defaults for the entire portfolio $S = \sum_{i=1}^{d} \sum_{j=1}^{n_i} X_{i,j} = \sum_{i=1}^{d} S_i$, where $S_i = \sum_{j=1}^{n_i} X_{i,j}$, for $i = 1, \ldots, d$. We suppose that the dependence structure inherent in $\underline{X}$ is modelled with a one level hierarchical Archimedean copula C of the form as depicted in Figure 5.1.

Let $\underline{\Theta} = (\Theta_0, \Theta_1, \ldots, \Theta_d)$ be a vector of strictly positive rvs representing all factors affecting

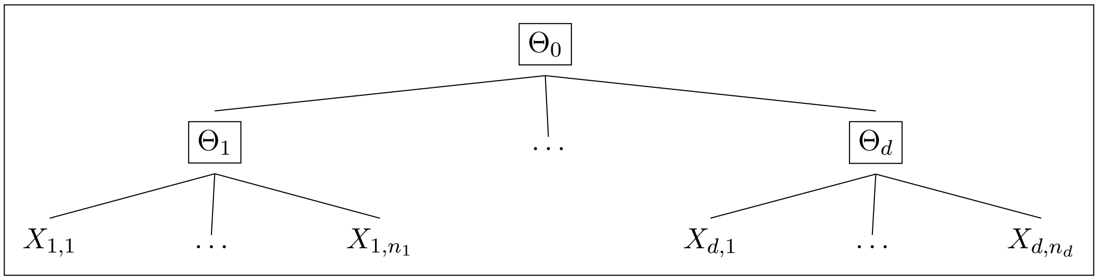


Figure 5.1 – One level hierarchical tree structure.

the vector of risks $\underline{X}$. Given $\Theta = \theta$, where $\underline{\theta} = (\theta_0, \theta_1, \ldots, \theta_d)$, it is assumed that

$$(X_{1,1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (X_{1,n_1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (X_{d,1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (X_{d,n_d} \mid \underline{\Theta} = \underline{\theta})$$

are conditionally independent. Given $\Theta_0 = \theta_0$, we denote by $\Theta_{0,i}$ the conditional rv $(\Theta_i | \Theta_0 = \theta_0)$, for $i = 1, \ldots, d$. The conditional distributions of the components of $\underline{X}_i$ are only influenced by the components Θ_0 and Θ_i of $\underline{\Theta}$, i.e., $(X_{i,j} \mid \underline{\Theta} = \underline{\theta})$ is identically distributed as $(X_{i,j} \mid \Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$, for $i = 1, \ldots, d$ and $j = 1, \ldots, n_i$. In this case, since C is a hierarchical Archimedean copula, the cumulative distribution function (cdf) of $(X_{i,j} \mid \Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$ is given by

$$F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}\right) = \mathrm{e}^{-\theta_{0,i} \times \mathcal{L}^{-1}_{\Theta_i}\left(F_{X_{i,j}}(k_{i,j})\right)}, \tag{5.1}$$

where $\mathcal{L}_{\Theta_i}$ denotes the Laplace-Stieltjes Transform (LST) of the positive rv Θ_i, $k_{i,j} \in \{0, 1\}$, $j = 1, 2, \ldots, n_i$ and $i = 1, \ldots, d$.

Let $\underline{k} = (k_{1,1}, \ldots, k_{1,n_1}, \ldots, k_{d,1}, \ldots, k_{d,n_d}) \in \{0, 1\}^{n_1+\ldots+n_d}$. The multivariate distribution of $\underline{X}$ is then defined in terms of its joint cdf as follows

$$\begin{aligned} F_{\underline{X}}\left(\underline{k}\right) &= C\left(F_{X_{1,1}}\left(k_{1,1}\right), \ldots, F_{X_{1,n_1}}\left(k_{1,n_1}\right), \ldots, F_{X_{d,1}}\left(k_{d,1}\right), \ldots, F_{X_{d,n_d}}\left(k_{d,n_d}\right)\right) \\ &= \int_0^\infty \prod_{i=1}^{d} \left(\int_0^\infty \prod_{j=1}^{n_i} F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}\right) \mathrm{d}F_{\Theta_{0,i}}\left(\theta_{0,i}\right) \right) \mathrm{d}F_{\Theta_0}\left(\theta_0\right), \end{aligned} \tag{5.2}$$

where $k_{i,j} \in \{0, 1\}$ and $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}$ is as given in (5.1) for $j = 1, 2, \ldots, n_i$ and $i = 1, \ldots, d$.

The multivariate distribution of $\underline{X}$ can also be defined with its survival function as follows

$$\overline{F}_{\underline{X}}\left(\underline{k}\right) = C\left(\overline{F}_{X_{1,1}}\left(k_{1,1}\right), \ldots, \overline{F}_{X_{1,n_1}}\left(k_{1,n_1}\right), \ldots, \overline{F}_{X_{d,1}}\left(k_{d,1}\right), \ldots, \overline{F}_{X_{d,n_d}}\left(k_{d,n_d}\right)\right).$$

Since the procedure is the same, we simply consider the model given in (5.2). In this case, the conditional probability mass function (pmf) of $(X_{i,j}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$ is as follows

$$\Pr\left(X_{i,j} = k|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i}\right) = \begin{cases} \mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)} & ,k=0 \\ 1-\mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)} & ,k=1 \end{cases}.$$

and its conditional probability generating function (pgf) can be given by

$$\mathcal{P}_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(t\right) = \mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)} + \left(1-\mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)}\right)t,$$

for $j = 1, 2, \ldots, n_i$ and $i = 1, \ldots, d$.

Since, in the same subgroup i ($i = 1, \ldots, d$), the rvs $(X_{i,j}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$ are independent and identically distributed (iid), for $j = 1, 2, \ldots, n_i$, then, the conditional pgf of $(S_i|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$ can be written as

$$\begin{aligned}\mathcal{P}_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(t\right) &= \left(\mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)} + \left(1-\mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)}\right)t\right)^{n_i} \\ &= \left(\mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)}\left(1-t\right)+t\right)^{n_i} \\ &= \sum_{l=0}^{n_i}\binom{n_i}{l}\mathrm{e}^{-\theta_{0,i}\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)\times l}\left(1-t\right)^l\left(t\right)^{n_i-l}.\end{aligned}$$

Then, the conditional pgf of $(S_i|\Theta_0 = \theta_0)$ is given by

$$\begin{aligned}\mathcal{P}_{S_i|\Theta_0=\theta_0}\left(t\right) &= \int_0^\infty\sum_{l=0}^{n_i}\binom{n_i}{l}\mathrm{e}^{-\theta_{0,i}\times l\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)}\left(1-t\right)^l\left(t\right)^{n_i-l}dF_{\Theta_{0,i}}\left(\theta_{0,i}\right) \\ &= \sum_{l=0}^{n_i}\binom{n_i}{l}\left(1-t\right)^l\left(t\right)^{n_i-l}\int_0^\infty\mathrm{e}^{-\theta_{0,i}\times l\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)}dF_{\Theta_{0,i}}\left(\theta_{0,i}\right) \\ &= \sum_{l=0}^{n_i}\binom{n_i}{l}\left(1-t\right)^l\left(t\right)^{n_i-l}\mathcal{L}_{\Theta_{0,i}}\left(l\times\mathcal{L}_{\Theta_i}^{-1}\left(1-q_i\right);\theta_0\right). \qquad (5.3)\end{aligned}$$

Since $\mathcal{L}_{\Theta_{0,i}}(t;\theta_0) = \mathrm{e}^{-\theta_0\times\mathcal{L}_{\Theta_0}^{-1}\circ\mathcal{L}_{\Theta_i}(t)}$, the conditional pgf given in (5.3) becomes

$$\begin{aligned}\mathcal{P}_{S_i|\Theta_0=\theta_0}\left(t\right) &= \sum_{l=0}^{n_i}\binom{n_i}{l}\left(1-t\right)^l\left(t\right)^{n_i-l}\mathcal{L}_{\Theta_{0,i}}\left(l\times\mathcal{L}_{\Theta_i}^{-1}\left(1-q_i\right);\theta_0\right) \\ &= \sum_{l=0}^{n_i}\binom{n_i}{l}\left(1-t\right)^l\left(t\right)^{n_i-l}\mathrm{e}^{-\theta_0\times\mathcal{L}_{\Theta_0}^{-1}\circ\mathcal{L}_{\Theta_i}\left(l\times\mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)}.\end{aligned}$$

The unconditional pgf of S_i can be written as

$$\begin{aligned}
\mathcal{P}_{S_i}(t) &= \int_0^\infty \mathcal{P}_{S_i|\Theta_0=\theta_0}(t)\, \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \int_0^\infty \sum_{l=0}^{n_i} \binom{n_i}{l} (1-t)^l (t)^{n_i-l} \mathrm{e}^{-\theta_0 \times \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)} \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \sum_{l=0}^{n_i} \binom{n_i}{l} (1-t)^l (t)^{n_i-l} \int_0^\infty \mathrm{e}^{-\theta_0 \times \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)} \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \sum_{l=0}^{n_i} \binom{n_i}{l} (1-t)^l (t)^{n_i-l} \mathcal{L}_{\Theta_0}\left(\mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)\right) \\
&= \sum_{l=0}^{n_i} \binom{n_i}{l} (1-t)^l (t)^{n_i-l} \mathcal{L}_{\Theta_i}\left(l \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right).
\end{aligned}$$

As for the conditionnal pgf of $(S|\Theta_0 = \theta_0)$, we have

$$\begin{aligned}
\mathcal{P}_{S|\Theta_0=\theta_0}(t) &= \Pi_{i=1}^d \mathcal{P}_{S_i|\Theta_0=\theta_0}(t) \\
&= \Pi_{i=1}^d \left(\sum_{l_i=0}^{n_i} \binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i} \mathcal{L}_{\Theta_{0,i}}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)\right) \\
&= \Pi_{i=1}^d \left(\sum_{l_i=0}^{n_i} \binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i} \mathrm{e}^{-\theta_0 \times \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_0}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)}\right) \\
&= \sum_{l_1=0}^{n_1} \cdots \sum_{l_d=0}^{n_d} \Pi_{i=1}^d \left(\binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i} \mathrm{e}^{-\theta_0 \times \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)}\right) \qquad (5.4)
\end{aligned}$$

Then, using (5.4), the unconditional pgf of the rv S can be obtained as follows

$$\begin{aligned}
\mathcal{P}_S(t) &= \int_0^\infty \mathcal{P}_{S|\Theta_0=\theta_0}(t)\, \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \int_0^\infty \sum_{l_1=0}^{n_1} \cdots \sum_{l_d=0}^{n_d} \Pi_{i=1}^d \left(\binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i} \mathrm{e}^{-\theta_0 \times \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)}\right) \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \int_0^\infty \sum_{l_1=0}^{n_1} \ldots \sum_{l_d=0}^{n_d} \Pi_{i=1}^d \left(\binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i}\right) \mathrm{e}^{-\theta_0 \times \sum_{i=1}^d \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)} \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \sum_{l_1=0}^{n_1} \ldots \sum_{l_d=0}^{n_d} \Pi_{i=1}^d \left(\binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i}\right) \int_0^\infty \mathrm{e}^{-\theta_0 \times \sum_{i=1}^d \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)} \mathrm{d}F_{\Theta_0}(\theta_0) \\
&= \sum_{l_1=0}^{n_1} \ldots \sum_{l_d=0}^{n_d} \Pi_{i=1}^d \left(\binom{n_i}{l_i} (1-t)^{l_i} (t)^{n_i-l_i}\right) \mathcal{L}_{\Theta_0}\left(\sum_{i=1}^d \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)\right). \qquad (5.5)
\end{aligned}$$

From (5.5) follows the characteristic function of S given by

$$\phi_S(t) = \sum_{l_1=0}^{n_1} \ldots \sum_{l_d=0}^{n_d} \Pi_{i=1}^d \left(\binom{n_i}{l_i} \left(1-\mathrm{e}^{\mathrm{i}t}\right)^{l_i} \left(\mathrm{e}^{\mathrm{i}t}\right)^{n_i-l_i}\right) \mathcal{L}_{\Theta_0}\left(\sum_{i=1}^d \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}\left(l_i \times \mathcal{L}_{\Theta_i}^{-1}(1-q_i)\right)\right). \qquad (5.6)$$

Finally, using FFT with (5.6), it is easy to compute the exact values of $f_S(k)$, for $k = 0, 1, 2, ..., n_S$ where $n_S = n_1 + ... + n_d$. Note that the expression for the characteristic function of S given in (5.6) depends only on Θ_0 and Θ_i, for $i = 1, \ldots, d$, which facilitates the calculation of f_S for several known hierarchical Archimedean copulas.

Moreover, note that the use of a one level hierarchical Archimedean copula to model the dependence structure between all default risks of a credit portfolio allows us to derive explicit expressions for the covariances between and within subgroups as follows:

$$\begin{aligned}
Cov\left(X_{i,j}, X_{i,j'}\right) &= \Pr\left(X_{i,j} = 1, X_{i,j'} = 1\right) - q_i^2 \\
&= 1 - 2\left(1 - q_i\right) + \Pr\left(X_{i,j} = 0, X_{i,j'} = 0\right) - q_i^2 \\
&= 2q_i - 1 + \mathcal{L}_{\Theta_i}\left(2\mathcal{L}_{\Theta_i}^{-1}\left(1 - q_i\right)\right) - q_i^2,
\end{aligned}$$

and

$$\begin{aligned}
Cov\left(X_{i,j}, X_{i*,j^*}\right) &= \Pr\left(X_{i,j} = 1, X_{i^*,j^*} = 1\right) - q_i \times q_{i^*} \\
&= 1 - \left(1 - q_i\right) - \left(1 - q_{i^*}\right) + \Pr\left(X_{i,j} = 0, X_{i^*,j^*} = 0\right) - q_i \times q_{i^*} \\
&= q_i + q_{i^*} - q_i \times q_{i^*} - 1 + \mathcal{L}_{\Theta_0}\left(\mathcal{L}_{\Theta_0}^{-1}(1 - q_i) + \mathcal{L}_{\Theta_0}^{-1}(1 - q_{i^*})\right),
\end{aligned}$$

where $j, j' \in \{1, \ldots, n_i\}$, $j^* \in \{1, \ldots, n_{i^*}\}$, and $i, i^* \in \{1, \ldots, d\}$, such that $j \neq j'$, $j \neq j^*$, and $i \neq i^*$.

In order to illustrate the results above, we consider the following numerical examples in the context of two known constructions of hierarchical Archimedean copulas, namely nested Archimedean copulas (see, e.g., Joe (1997), McNeil (2008) and Hofert (2011) for a general introduction) and hierarchical Archimedean copulas obtained through compounding proposed by Cossette et al. (2017). Briefly, a nested Archimedean copula is an Archimedean copula for which at least one of the arguments is an Archimedean copula. In the context of our proposed dependence model, Θ_0 and Θ_i are the mixing rvs related to, respectively, the outer copula $C(., \mathcal{L}_{\Theta_0})$ and the inner copula $C(., \mathcal{L}_{\Theta_i})$, for $i = 1, \ldots, d$. In this case, given $\Theta_0 = \theta_0$, the conditional rv $\Theta_{0,i}$ is such that $\mathcal{L}_{\Theta_{0,i}}(t) = \exp\left\{-\theta_0 \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_i}(t)\right\}$, for $i = 1, \ldots, d$. As for the hierarchical Archimedean copulas obtained through compounding, this construction is, as indicated by its name, based on a multivariate compound distribution modelling the dependence between $\Theta_1, \ldots, \Theta_d$, such that $\Theta_i = \sum_{j=1}^{\Theta_0} B_{i,j}$, where Θ_0 is a discrete and positive rv and $B_{i,j}$ is a positive rv, for $j = 1, \ldots, n_i$ and $i = 1, \ldots, d$ (for more details, see Cossette et al. (2017)). Using such a hierarchical copula and given $\Theta_0 = \theta_0$, we have $\Theta_{0,i} = \sum_{j=1}^{\theta_0} B_{i,j}$ for $\theta_0 \in \mathbb{N}$, $j = 1, \ldots, n_i$, and $i = 1, \ldots, d$.

Example 5.2.1. *Consider a portfolio of 20 default risks* $\underline{X} = (X_{1,1}, \ldots, X_{1,10}, X_{2,1} \ldots, X_{2,10})$ *with multivariate cdf as defined in (5.2), with* $d = 2$ *and* $n_1 = n_2 = 10$. *Assume* C *to be a* *nested Clayton-Clayton copula, i.e.,* $\Theta_0 \sim Gamma(1/\alpha_0, 1)$*,* $\Theta_1 \sim Gamma(1/\alpha_1, 1)$*, and* $\Theta_2 \sim Gamma(1/\alpha_2, 1)$*, where* $\alpha_0 = 0.2839531$*,* $\alpha_1 = 1.710445$*, and* $\alpha_2 = 3.160818$ *are the dependence parameters of the copula* C*. Let* $X_{i,j} \sim Bernoulli(q_i)$*, for* $j = 1, \ldots, 10$ *and* $i = 1, 2$*. We fix* $q_1 = 0.05$ *and* $q_2 = 0.1$*, such that* $E[S] = 1.5$ *and* $Var(S) = 3.9232$*. Values of the pmf of* S *computed with FFT are provided in Table 5.1. The values of the pmf of* $S^{\perp}$ *in the case of independence between all risks* $\underline{X}^{\perp}$ *are also given in Table 5.1. We can clearly see that the impact of the dependence induced via the copula* C *is significant even for a small portfolio of 20 risks and two subclasses.*

k	$\Pr(S = k)$	$\Pr(S^{\perp} = k)$	k	$\Pr(S = k)$	$\Pr(S^{\perp} = k)$
0	0.432588	0.208767	11	0.001052	0.000000
1	0.214996	0.341840	12	0.000413	0.000000
2	0.128632	0.264091	13	0.000153	0.000000
3	0.081100	0.127975	14	0.000052	0.000000
4	0.052542	0.043621	15	0.000016	0.000000
5	0.034491	0.011116	16	0.000004	0.000000
6	0.022619	0.002197	17	0.000001	0.000000
7	0.014566	0.000345	18	0.000000	0.000000
8	0.009003	0.000044	19	0.000000	0.000000
9	0.005166	0.000005	20	0.000000	0.000000
10	0.002604	0.000000			

Table 5.1 – Values of the pmf of S as defined in Example 5.2.1.

In the following example, we consider the hierarchical dependence structure of $\underline{X}$ to be defined with a hierarchical Archimedean copula through compounding. We opt for the same assumptions concerning the hierarchical structure, i.e., $d = 2$, $n_1 = n_2 = 10$ and the marginal distributions as in Example 5.2.1 and we fix the dependence parameters in order to obtain the same expectation and variance for S.

Example 5.2.2. *Assume now* C *to be a hierarchical Archimedean copula through compounding. We consider two cases: in the first case,* $\Theta_0 \sim Geometric(1 - \alpha_0 = 0.7)$*,* $B_1 \sim Gamma(1/2, 1)$*, and* $B_2 \sim Gamma(1/4, 1)$*. In the second case,* $\Theta_0 \sim Geometric(1 - \alpha_0 = 0.7)$*,* $B_1 \sim Stable(1/\alpha_1, 1, \cos^{\alpha_1}(\pi/2\alpha_1), 1_{\{\alpha_1=1\}}; 1)$*, and* $B_2 \sim Stable(1/\alpha_2, 1, \cos^{\alpha_2}(\pi/2\alpha_2), 1_{\{\alpha_2=1\}}; 1)$*, where* $\alpha_1 = 1.052107$ *and* $\alpha_2 = 1.2158$*. We denote by* $S^{(1)}$ *and* $S^{(2)}$*, the rv* S *in the first case and the second case respectively, such that* $E[S^{(1)}] = E[S^{(2)}] = 1.5$ *and* $Var(S^{(1)}) = Var(S^{(2)}) = 3.9232$*. Values of the pmf of* $S^{(1)}$ *and* $S^{(2)}$ *computed with FFT are provided in Table 5.2. We can see that the distribution of* S *depends significantly on the nature of the hierarchical Archimedean copula, which makes the choice of the copula an important task in the analysis of portfolios of credit risks. We can also see that the results of* $S^{(2)}$ *in Table 5.2 and those of* S *in Example 5.2.1 are close since they are both based on gamma mixing rvs.*

k	$\Pr(S^{(1)} = k)$	$\Pr(S^{(2)} = k)$	k	$\Pr(S^{(1)} = k)$	$\Pr(S^{(2)} = k)$
0	0.355022	0.430440	11	0.004014	0.001103
1	0.302044	0.217998	12	0.001364	0.000459
2	0.159786	0.128819	13	0.000450	0.000180
3	0.076385	0.080640	14	0.000168	0.000065
4	0.038248	0.052183	15	0.000078	0.000021
5	0.020971	0.034324	16	0.000043	0.000006
6	0.012699	0.022557	17	0.000028	0.000002
7	0.008492	0.014527	18	0.000019	0.000000
8	0.006279	0.008956	19	0.000014	0.000000
9	0.005271	0.005123	20	0.000016	0.000000
10	0.008609	0.002597			

Table 5.2 – Values of the pmf of $S^{(1)}$ and $S^{(2)}$ as defined in Example 5.2.2.

Throughout Examples 5.2.1 and 5.2.2, we illustrate the usefulness of dependence models based on the conditional independence representation, also known as the common mixture representation, in a credit risk modeling context. Other interesting applications in actuarial science and quantitative risk management, relying on such dependence structures, can also motivate the use of such dependence models. See, e.g., Mai and Scherer (2014), Abdallah et al. (2015), Zhu et al. (2016), Cossette et al. (2018b), and Cossette et al. (2018a). Therefore, developing new distributions and copulas based on the common mixture representation can be of great use in different research fields.

5.3 Univariate mixed exponential distributions

Let Θ be a strictly positive mixing rv with cdf F_Θ (i.e., $F_\Theta(0) = 0$) and LST $\mathcal{L}_\Theta$, where

$$\mathcal{L}_\Theta(t) = \int \mathrm{e}^{-\theta t} dF_\Theta(\theta) = E[\mathrm{e}^{-\Theta t}]. \tag{5.7}$$

A rv X follows a mixed exponential distribution if, given $\Theta = \theta$, the conditional distribution of $(X|\Theta = \theta)$ is exponential with mean $\frac{1}{\theta}$ and its survival function is $\bar{F}_{X|\Theta=\theta}(x) = \mathrm{e}^{-\theta x}$ ($x \in \mathbb{R}^+$). Then, from (5.7), it implies that the survival function of the unconditional rv X is given by

$$\overline{F}_X(x) = \int \mathrm{e}^{-\theta x} dF_\Theta(\theta) = \mathcal{L}_\Theta(x). \tag{5.8}$$

A brief review of applications of mixed exponential distributions in actuarial science and queueing theory is provided in, e.g., Cai (2006) and references therein. Also, mixed exponential distributions can be notably used to fit long tail distributions such as the Pareto distribution (see, e.g., Feldmann and Whitt (1997)).

In the following proposition, we list different general properties of a univariate mixed exponential distribution.

Proposition 5.3.1. *Let X a positive rv with univariate mixed exponential distribution and survival function as given in (5.8). Properties of such a distribution can be written in terms of the mixing rv Θ as follows (assuming that the expectations exist):*

1. *LST* $\mathcal{L}_X(t) = E_\Theta\left[\frac{\Theta}{\Theta+t}\right]$, $t > 0$;
2. *probability density function* $f_X(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_\Theta(x)$, $x \in \mathbb{R}^+$;
3. *failure rate* $h_X(x) = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_\Theta(x)}{\mathcal{L}_\Theta(x)}$, $x \in \mathbb{R}^+$;
4. *moments* $E[X^n] = \Gamma(n+1) \times E\left[\Theta^{-n}\right]$, *if it exists;*
5. *stop-loss function* $\pi_d(X) = E\left[\max(X-d;0)\right] = E_\Theta\left[\Theta^{-1}\mathrm{e}^{-d\,\Theta}\right]$, *if* $E[X] < \infty$;
6. $VaR_\kappa(X) = F_X^{-1}(\kappa) = \inf\{x \in \mathbb{R}, F_X(x) \geq \kappa\} = \mathcal{L}_\Theta^{-1}(1-\kappa)$, $\kappa \in (0,1)$;
7. $TVaR_\kappa(X) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(X)\,du = \frac{E_\Theta\left[\left(\frac{1}{\Theta}+VaR_\kappa(X)\right)\mathrm{e}^{-VaR_\kappa(X)\,\Theta}\right]}{1-\kappa}$, $\kappa \in (0,1)$, *if* $E[X] < \infty$.

Proof. All properties are obtained by simply conditioning with respect to Θ and using the fact that $(X|\Theta=\theta) \sim Exp(\theta)$. For example, the expression for the TVaR in [7] can be obtained as follows:

$$\begin{aligned} TVaR_\kappa(X) &= \frac{E\left[X \times 1_{\{X > VaR_\kappa(X)\}}\right]}{1-\kappa} \\ &= \frac{E\left[E\left[X \times 1_{\{X > VaR_\kappa(X)\}}|\Theta\right]\right]}{1-\kappa}. \end{aligned}$$

Since $(X|\Theta=\theta) \sim Exp(\theta)$, then

$$TVaR_\kappa(X) = \frac{E_\Theta\left[\left(\Theta^{-1} + VaR_\kappa(X)\right)\mathrm{e}^{-VaR_\kappa(X)\,\Theta}\right]}{1-\kappa},$$

if the expectation exists. We proceed similarly to obtain expressions for the other properties. Note that these results are not all new and have been found implicitly while studying the properties of some specific mixed exponential distributions. □

Several known continuous distributions arise as mixed exponential distributions. For example, if the mixing rv is gamma distributed, then, the resulting mixture is a Pareto distribution or a Lomax distribution (also called Pareto of type II). Another well-known result is the Weibull distribution which arises from an mixed exponential-Stable distribution.

Several other distributions can also be constructed via the exponential mixture method by taking different discrete mixing distributions. For example, Adamidis and Loukas (1998) considered a geometric mixing distribution, whereas Chahkandi and Ganjali (2009) proposed exponential-power series distributions which include binomial, Poisson (Kuş (2007)) and Logarithmic (Tahmasbi and Rezaei (2008)) mixing distributions.

Later, Hajebi et al. (2013) studied an exponential-negative binomial distribution using another construction method. Let Θ be a strictly positive discrete rv. Then, a mixed exponential distribution can be represented as the distribution of the random minimum $X = \min(Y_1, \ldots, Y_\Theta)$ where $Y_1, \ldots, Y_\Theta$ is a random sample from an exponential distribution and Θ a discrete rv independent of $(Y_1, Y_2, \ldots)$. This construction method provides an alternative approach to the mixing representation of $\bar{F}_X(x) = \Pr(X > x)$ provided in (5.8), i.e.,

$$\begin{aligned}
\Pr(X > x) &= \Pr\left(\min\left(Y_1, \ldots, Y_\Theta\right) > x\right) \\
&= \sum_k \Pr\left(\min\left(Y_1, \ldots, Y_k\right) > x\right) \Pr(\Theta = k) \\
&= \sum_k \Pr(Y_1 > x) \ldots \Pr(Y_k > x) \Pr(\Theta = k) \\
&= \sum_k (\mathrm{e}^{-x})^k \Pr(\Theta = k) \\
&= \mathcal{L}_\Theta(x).
\end{aligned}$$

5.3.1 Univariate mixed Exponential - Negative Binomial Distribution

Let Θ be a discrete rv following a negative binomial distribution (i.e., $\Theta \sim NB(r, q)$) with pmf

$$\Pr(\Theta = k) = \binom{k-1}{r-1} q^r \, (1-q)^{k-r}, \quad \forall\, k \in \{r, r+1, \ldots\}, \tag{5.9}$$

and LST given by

$$\mathcal{L}_\Theta(t) = \left(\frac{q}{\mathrm{e}^t - (1-q)}\right)^r. \tag{5.10}$$

Combining (5.8) and (5.10) leads to the following unconditional survival function of X

$$\overline{F}_X(x) = \left(\frac{q}{\mathrm{e}^x - (1-q)}\right)^r,$$

and probability density function (pdf) of X

$$f_X(x) = \frac{r\, q^r\, \mathrm{e}^{-rx}}{\left(1 - (1-q)\, \mathrm{e}^{-x}\right)^{r+1}}, \; x \in \mathbb{R}^+.$$

Note that this formula can be found in Hajebi et al. (2013) with scale parameter $\beta = 1$. Also, since the geometric distribution is a special case of the negative binomial distribution with $r = 1$, the mixed exponential-geometric distribution introduced by Adamidis and Loukas (1998) is a special case of the mixed exponential-negative binomial distribution presented here. A multivariate extension of such a distribution is presented in Section 5.4.1.

While some properties of the mixed exponential-negative binomial distribution are already given in Hajebi et al. (2013) such as the expression of the hazard rate, the moments, order statistics, extreme values and parameter estimators, other interesting properties have not been studied yet. The aim here is to consider a more comprehensive study of the mixed exponential-negative binomial distribution including a detailed survey of limit cases and to discuss some of its characteristics in relation with actuarial science and quantitative risk management.

We first briefly recall the representation of the Pareto distribution as a mixed exponential-gamma distribution. Indeed, let the mixing rv $\Theta^{Ga}_{(\alpha)}$ follow a gamma distribution, i.e., $\Theta^{Ga}_{(\alpha)} \sim Gamma(\frac{1}{\alpha}, 1)$ with LST

$$\mathcal{L}_{\Theta^{Ga}_{(\alpha)}}(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}. \tag{5.11}$$

Then, combining (5.8) and (5.11) leads to

$$\bar{F}_{X^{Pa}_{(\alpha)}}(x) = \left(\frac{1}{1+x}\right)^{\frac{1}{\alpha}}, \, x \in \mathbb{R}^{+}, \tag{5.12}$$

i.e., the rv $X^{Pa}_{(\alpha)}$ follows a Pareto distribution with shape parameter $\frac{1}{\alpha}$ and scale parameter 1.

In order to discuss the limit cases of the univariate mixed exponential-negative binomial distribution, a new parametrization is needed. Let M be a discrete rv which follows a negative binomial distribution with parameters r ($r \in \mathbb{R}^{+}$) and $q = 1 - \mathrm{e}^{-h}$ ($h \in \mathbb{R}^{+}$), pmf as defined in (5.9) and LST given by

$$\mathcal{L}_{M}(t) = \left(\frac{1 - \mathrm{e}^{-h}}{\mathrm{e}^{t} - \mathrm{e}^{-h}}\right)^{r}. \tag{5.13}$$

Let the mixing rv $\Theta_{(h,r)}$ be defined as $\Theta_{(h,r)} = h \times M$. Then, it is said that $\Theta_{(h,r)}$ follows a negative binomial distribution with, using (5.13), LST given by

$$\mathcal{L}_{\Theta_{(h,r)}}(t) = E[\mathrm{e}^{-t\,h \times M}] = \left(\frac{\mathrm{e}^{-th} - \mathrm{e}^{-(t+1)h}}{1 - \mathrm{e}^{-(t+1)h}}\right)^{r} = \left(\frac{1 - \mathrm{e}^{-h}}{\mathrm{e}^{t\,h} - \mathrm{e}^{-h}}\right)^{r}. \tag{5.14}$$

Proposition 5.3.2. *Let $\Theta^{Ga}_{(r)} \sim Gamma\left(\frac{1}{r}, 1\right)$ and $\Theta_{(h,r)} \sim NB\left(r, 1 - \mathrm{e}^{-h}\right)$ with LST given in (5.11) and (5.14) respectively. Then,*

$$\Theta_{(h,r)} \overset{\mathcal{D}}{\to} \Theta^{Ga}_{(r)},$$

as $h \to 0$.

Proof. Clearly, the LST of the discrete rv $\Theta_{(h,r)}$ in (5.14) tends to the LST of the continuous rv $\Theta_{(r)}^{Ga}$ in (5.11) as $h \to 0$, i.e.,

$$\lim_{h \to 0} \left(\frac{\mathrm{e}^{-th} - \mathrm{e}^{-(t+1)h}}{1 - \mathrm{e}^{-(t+1)h}} \right)^r = \left(\frac{1}{1+t} \right)^r . \tag{5.15}$$

By Lévy's continuity theorem, (5.15) implies that $\Theta_{(h,r)}$ converges in distribution to $\Theta_{(r)}^{Ga}$. □

In Figures 5.2 and 5.3, the convergence of the negative binomial distribution to the gamma distribution (as $h \to 0$) is clearly illustrated. The negative binomial distribution can therefore be seen as a discrete version of the continuous gamma distribution.

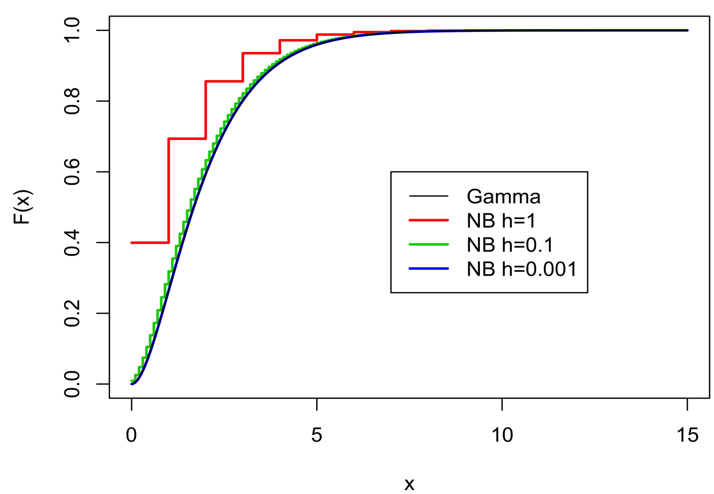


Figure 5.2 – Illustration of the convergence of the cdf of the negative binomial distribution to the cdf of the gamma distribution with $r = 2$.

Let the rv $X_{(h,r)}$ follow a univariate mixed exponential-negative binomial distribution, i.e $(X_{(h,r)} | \Theta_{(h,r)} = \theta) \sim Exp(\theta)$ where $\Theta_{(h,r)} \sim NB(r, 1 - \mathrm{e}^{-h})$. Then, combining (5.8) and (5.14), the unconditional survival function of $X_{(h,r)}$ is given by

$$\overline{F}_{X_{(h,r)}}(x) = \mathcal{L}_{\Theta_{(h,r)}}(x) = \left(\frac{1 - \mathrm{e}^{-h}}{\mathrm{e}^{xh} - \mathrm{e}^{-h}} \right)^r . \tag{5.16}$$

Moments and other risk related quantities of the mixed exponential-negative binomial distribution are defined in terms of generalized hypergeometric functions which allow to effectively perform numerical calculations. It requires the use of the rising factorial, also called Pochammer symbol, defined as $(q)_j = q \times (q+1) \times \ldots \times (q+j-1)$, for $j > 0$ and $(q)_0 = 1$.

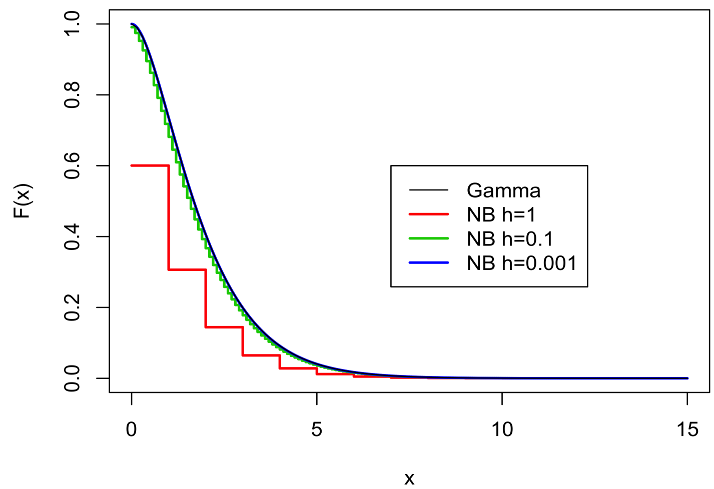


Figure 5.3 – Illustration of the convergence of the cdf of the negative binomial distribution to the cdf of the gamma distribution with $r = 2$.

Definition 5.3.1. *For $x \in \mathbb{R}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $b_j \in \mathbb{R}^+$, $j = 1, \ldots, m$, the generalized hypergeometric function, denoted by ${}_nF_m$, is defined as*

$$ {}_nF_m\left([a_1, \ldots, a_n], [b_1, \ldots, b_m], x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \times \ldots \times (a_n)_k}{(b_1)_k \times \ldots \times (b_m)_k} \frac{x^k}{k!}, \tag{5.17} $$

where $n \leq m + 1$. Special care must be taken when $b_j \in \mathbb{Z}^-$ for $j = 1, \ldots, m$. For more details see, e.g., Abramowitz and Stegun (1964).

Proposition 5.3.3. *Let $X_{(h,r)}$ follow a univariate mixed exponential-negative binomial distribution with survival function as given in (5.16). Then, the following properties hold:*

1. $X_{(h,r)} \xrightarrow{\mathcal{D}} X_{(r)}^{Pa}$, *as $h \to 0$, where* $X_{(r)}^{Pa} \sim$ *Pareto* $\left(\frac{1}{r}, 1\right)$;
2. $\mathcal{L}_{X_{(h,r)}}(t) = \frac{1}{hr+t}\left(hr\left(1 - \mathrm{e}^{-h}\right)^r {}_2F_1\left(\left[1 + r, \frac{hr+t}{h}\right], \left[\frac{h+hr+t}{h}\right], \mathrm{e}^{-h}\right)\right)$, $t > -hr$;
3. $f_{X_{(h,r)}}(x) = rh\mathrm{e}^{xh}\frac{\left(1-\mathrm{e}^{-h}\right)^r}{\left(\mathrm{e}^{xh}-\mathrm{e}^{-h}\right)^{r+1}}$, $x>0$;
4. *Failure rate* $h_{X_{(h,r)}}(x) = \frac{f_{X_{(h,r)}}(x)}{\bar{F}_{X_{(h,r)}}(x)} = \frac{rh\mathrm{e}^{xh}}{\mathrm{e}^{xh}-\mathrm{e}^{-h}}$, $x > 0$;
5. $E[X_{(h,r)}^n] = \frac{n!}{h^n r^n} \times \left(1 - \mathrm{e}^{-h}\right)^r \times {}_{n+1}F_n\left([r, \ldots, r], [1 + r, \ldots, 1 + r], \mathrm{e}^{-h}\right)$;
6. $\pi_d\left(X_{(h,r)}\right) = E\left[\max(X_{(h,r)} - d; 0)\right] = \frac{\left(1-\mathrm{e}^{-h}\right)^r {}_2F_1\left([r,r],[r+1],\mathrm{e}^{-h(d+1)}\right)}{hr\mathrm{e}^{hrd}}$, $d > 0$;
7. $VaR_\kappa(X_{(h,r)}) = \inf\{x > 0 : F_X(x) \geq \kappa\} = \frac{1}{h}\ln\left(\frac{1-\mathrm{e}^{-h}}{(1-\kappa)^{\frac{1}{r}}} + \mathrm{e}^{-h}\right)$, $\kappa \in (0, 1)$;
8. $TVaR_\kappa(X_{(h,r)}) = \frac{\left(\frac{1}{hr}+\xi\right)\left(1-\mathrm{e}^{-h}\right)^r}{\mathrm{e}^{hr\xi}(1-\kappa)} \times {}_3F_2\left(\left[r, r, \frac{1+h(1+r)\xi}{h\xi}\right], \left[1 + r, \frac{1+hr\xi}{h\xi}\right], \mathrm{e}^{-h(\xi+1)}\right)$, *where* $\xi = VaR_\kappa(X_{(h,r)})$.

Proof. For Property 1, clearly, $\lim_{h \to 0} \overline{F}_{X_{(h,r)}}(x) = \left(\frac{1}{x+1}\right)^r$, which corresponds to the survival function provided in (5.12) of the Pareto distribution (with $r = \frac{1}{\alpha}$). It implies that $X_{(h,r)} \xrightarrow{\mathcal{D}} X_{(r)}^{Pa}$ as $h \to 0$.
For Properties 3, 4, 5, and 7, see Hajebi et al. (2013) for similar results. Finally, the expressions given in Properties 2, 6, and 8 are obtained directly with Proposition 5.3.1. □

Remark 5.3.1. *Figures 5.4 and 5.5 show that the exponential-negative binomial mixture indeed converges to the Pareto distribution as h gets smaller.*

For a fixed $r > 0$ and for any $h_1 > h_2 > 0$, note that $\overline{F}_{X_{(h_1,r)}}(x) \leq \overline{F}_{X_{(h_2,r)}}(x) \leq \overline{F}_{X_{(r)}^{Pa}}(x)$ for all $x > 0$. It implies that $X_{(h_1,r)} \preccurlyeq_{sd} X_{(h_2,r)} \preccurlyeq_{sd} X_{(r)}^{Pa}$, where "$\preccurlyeq_{sd}$" denotes the usual stochastic dominance order (see, e.g., Müller and Stoyan (2002), Denuit et al. (2005), Shaked and Shanthikumar (2007) for details on the usual stochastic dominance order and its properties). Consequently,

$$VaR_\kappa(X_{(h_1,r)}) \leq VaR_\kappa(X_{(h_2,r)}) \leq VaR_\kappa(X_{(r)}^{Pa}) \tag{5.18}$$

for all $\kappa \in (0, 1)$.

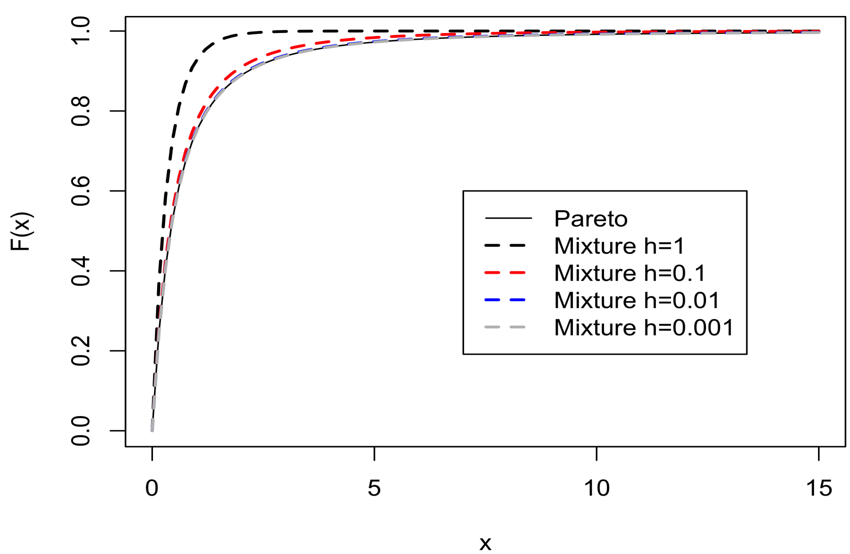


Figure 5.4 – Illustration of the convergence of the mixed exponential-negative binomial distribution to the Pareto distribution with $r = 2$.

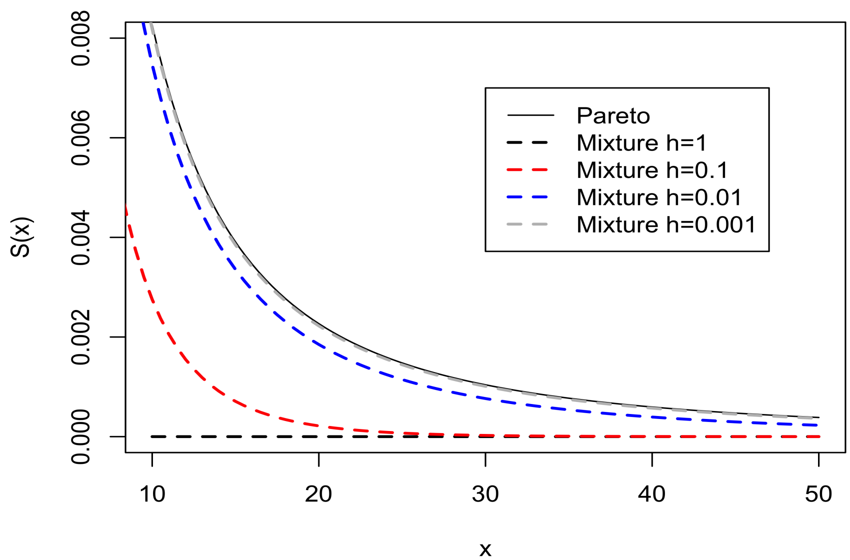


Figure 5.5 – Illustration of the convergence of the mixed exponential-negative binomial distribution to the Pareto distribution with $r = 2$.

Remark 5.3.2. *As expected, for a mixed exponential distribution (see, e.g., Cai (2006)), the failure rate function for the mixed exponential-negative binomial distribution given in Proposition 5.3.3 is decreasing in x which is very useful in reliability theory and in modeling lifetime data (see, e.g., Barlow et al. (1963) for properties of distributions with a monotone hazard rate).*

5.3.2 Univariate mixed Exponential - Discrete Stable distribution

The discrete stable distribution was introduced by Steutel and Van Harn (1979). The properties of this discrete distribution are discussed notably in Devroye (1993), Christoph and Schreiber (1998) and Rémillard et al. (2000). Also, Devroye (1993) established a sampling algorithm to generate observations from the discrete stable distribution.

Let Θ be a rv with a discrete stable distribution with parameters $\alpha \in (0, 1]$ and $\lambda > 0$, i.e., $\Theta \sim Stable(\lambda; \alpha)$. Its pgf P_N is given by

$$P_\Theta(t) = \exp\left\{-\lambda (1 - t)^\alpha\right\}, \ |t| \leq 1. \tag{5.19}$$

Clearly, if $\alpha = 1$, then (5.19) becomes the pgf of a Poisson distribution with parameter λ, i.e., $\Theta \sim Poisson(\lambda)$. For this reason, we can consider the discrete stable distribution as an extension of the Poisson distribution. As explained in Christoph and Schreiber (1998), the following explicit expression for the pmf of the discrete stable distribution is obtained from (5.19)

$$\Pr(\Theta = k) = (-1)^k \sum_{j=0}^{\infty} \binom{\alpha\, j}{k} \frac{(-1)^j \lambda^j}{j!}, \tag{5.20}$$

for $k = 0, 1, \ldots$ Note that the sum in (5.20) is absolutely convergent.

The rv $\Theta \sim \mathit{Stable}(\lambda, \alpha)$ can be represented as a random sum as follows

$$\Theta \overset{D}{=} Z_1 + \ldots + Z_M, \tag{5.21}$$

where $M \sim \mathit{Poisson}(\lambda)$ and $\{Z_k, k = 1, 2, \ldots\}$ is a sequence of iid rvs where $Z_k \sim Z \sim \mathit{Sibuya}(\alpha)$, for $k = 1, 2, \ldots$, with pgf $\mathcal{P}_Z(t) = 1 - (1-t)^\alpha$. The rvs Z_k, $k = 1, 2, ...$, are also independent of the rv M. This representation can be used to recursively compute the exact values of the pmf of the discrete stable distribution.

In order to construct the univariate mixed exponential - discrete stable distribution, let Θ' be a truncated discrete stable rv with LST given by

$$\mathcal{L}_{\Theta'}(t) = \frac{\exp\{-\lambda\, \left(1 - \mathrm{e}^{-t}\right)^{\alpha}\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}. \tag{5.22}$$

In this case, combining (5.8) and (5.22), the unconditional survival function of a rv X with an exponential-discrete stable distribution is given by

$$\overline{F}_X(x) = \frac{\exp\{-\lambda\,(1 - \mathrm{e}^{-x})^{\alpha}\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}, \tag{5.23}$$

where $x \in \mathbb{R}^+$.

As for the negative binomial distribution presented in the previous section, we use a transformation of the rv Θ to explore the limit cases of the mixed exponential-discrete stable distribution. More precisely, let us define the rv $\Theta_{(h,\alpha)} = h \times \Theta'$. Then, using (5.22), the LST of $\Theta_{(h,\alpha)}$ is given by

$$\mathcal{L}_{\Theta_{(h,\alpha)}}(t) \;=\; \frac{\exp\{-\lambda\, \left(1 - \mathrm{e}^{-t\,h}\right)^{\alpha}\} - \mathrm{e}^{-\lambda}}{1 - \mathrm{e}^{-\lambda}}. \tag{5.24}$$

Proposition 5.3.4. *Let $\Theta^{St}_{(\alpha)}$ follow a continuous positive stable distribution with LST $\mathcal{L}_{\Theta^{St}_{(\alpha)}}(t) = \mathrm{e}^{-t^{\alpha}}$ and $\Theta_{(h,\alpha)}$ be a discrete rv following a discrete stable distribution with LST given in (5.24). Then,*

$$\Theta_{(h,\alpha)} \overset{\mathcal{D}}{\to} \Theta^{St}_{(\alpha)},$$

as $h \to 0$.

Proof. Using the Taylor series expansion of the exponential function, (5.24) becomes

$$\begin{aligned}\mathcal{L}_{\Theta_{(h,\alpha)}}(t) &= \frac{\exp\{-\lambda\left(1-\mathrm{e}^{-t\,h}\right)^{\alpha}\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}\\ &= \frac{\exp\left\{-\lambda\left(1-\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(t\,h)^n}{n!}\right)\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}\\ &= \frac{\exp\left\{-\lambda\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(t\,h)^n}{n!}\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}. \qquad (5.25)\end{aligned}$$

Letting $h=\frac{1}{\lambda^{\frac{1}{\alpha}}}$, (5.25) becomes

$$\mathcal{L}_{\Theta_{(h,\alpha)}}(t) = \frac{\exp\left\{-t^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(t\,h)^n}{(n+1)!}\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}. \qquad (5.26)$$

Clearly, as $h\to 0$ (or equivalently $\lambda\to+\infty$), the LST of the mixing rv $\Theta_{(h,\alpha)}$ given in (5.26) tends to the LST of a continuous rv $\Theta_{(\alpha)}^{St}$ with a stable distribution with parameter α with LST $\mathcal{L}_{\Theta_{(\alpha)}^{St}}(t)=\mathrm{e}^{-t^{\alpha}}$, i.e.,

$$\lim_{h\to 0}\frac{\exp\left\{-t^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(t\,h)^n}{n!}\right)^{\alpha}\right\}-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}=\mathrm{e}^{-t^{\alpha}}. \qquad (5.27)$$

Therefore, from (5.27), we can deduce the convergence in distribution of $\Theta_{(h,\alpha)}$ to $\Theta_{(h)}^{St}$ by using Lévy's continuity theorem. □

Proposition 5.3.5. *Let the rv $X_{(\alpha)}$ follow a mixed exponential-discrete stable distribution with mixing rv $\Theta_{(\alpha)}$ and LST as in (5.24). Then, the following properties hold:*

1. $X_{(h,\alpha)}\xrightarrow{\mathcal{D}}X^{W}$, *as $h\to 0$, where $X_{(\alpha)}^{W}\sim Weibull(\alpha,1)$;*
2. $f_{X_{(h,\alpha)}}(x)=\dfrac{\lambda\left(1-\mathrm{e}^{-xh}\right)^{\alpha-1}\alpha h\,\mathrm{e}^{-xh-\lambda\left(1-\mathrm{e}^{-xh}\right)^{\alpha}}}{1-\mathrm{e}^{-\lambda}}$, $x\geq 0$;
3. $h_{X_{(h,\alpha)}}(x)=\dfrac{\lambda\left(1-\mathrm{e}^{-xh}\right)^{\alpha-1}\alpha h\,\mathrm{e}^{-xh-\lambda\left(1-\mathrm{e}^{-xh}\right)^{\alpha}}}{\mathrm{e}^{-\lambda(1-\mathrm{e}^{-hx})^{\alpha}}-\mathrm{e}^{-\lambda}}$, $x\geq 0$;
4. $VaR_{\kappa}(X)=-\frac{1}{h}\ln\left(1-\left(\frac{\lambda-\ln\left((1-\kappa)(\mathrm{e}^{\lambda}-1)+1\right)}{\lambda}\right)^{\frac{1}{\alpha}}\right)$, $\kappa\in(0,1)$.

Proof. For property 1, combining (5.8) and (5.26) leads to the unconditional survival function of $X_{(h,\alpha)}$ written as

$$\overline{F}_{X_{(h,\alpha)}}(x) = \frac{\exp\left\{-x^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n (x\,h)^n}{n!}\right)^{\alpha}\right\} - \mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}},\ x \geq 0.$$

Clearly, as $h \to 0$, the survival function of $X_{(h,\alpha)}$ tends to the survival function of a continuous rv $X^W_{(\alpha)}$ with a Weibull distribution with scale parameter 1 and shape parameter α, i.e.,

$$\lim_{h\to 0}\overline{F}_{X_{(h,\alpha)}}(x) = \mathrm{e}^{-x^{\alpha}},\ x \geq 0.$$

For properties 2, 3, and 4, the expressions are obtained directly from their definitions. □

Note that other properties of the mixed exponential-discrete stable distribution such as expectation, variance, moments and TVaR can only be obtained numerically.

Remark 5.3.3. *The convergence of an exponential-discrete stable distribution to a Weibull distribution was expected since a Weibull distribution arises from a mixed exponential-stable distribution. In Figures 5.6 and 5.7, an illustration of the convergence of the exponential-discrete stable distribution to the Weibull distribution is provided.*

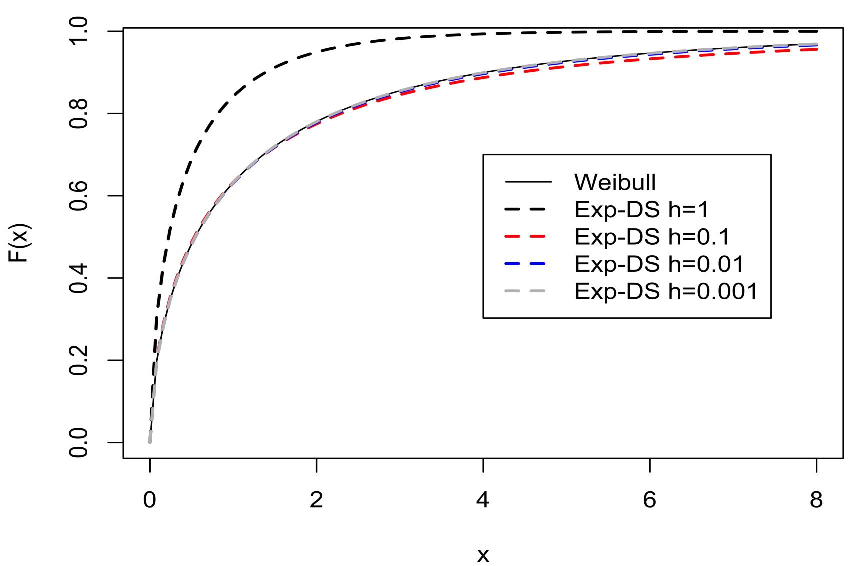


Figure 5.6 – Illustration of the convergence of the mixed exponential-discrete stable distribution to the weibull distribution with $\alpha = 0.9$.

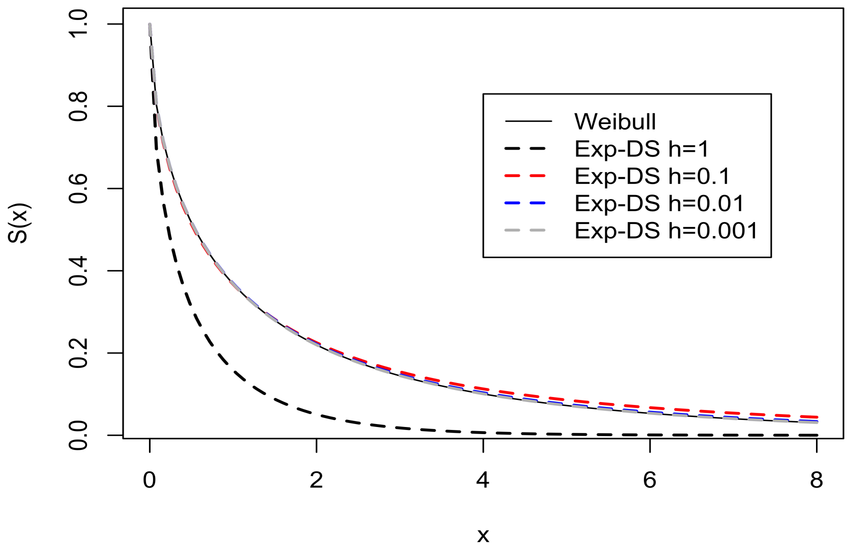


Figure 5.7 – Illustration of the convergence of the mixed exponential-discrete stable distribution to the weibull distribution with $\alpha = 0.9$.

5.3.3 Univariate mixed Exponential - Discrete Linnik distribution

The discrete Linnik distribution was proposed by Devroye (1993) as a specific mixed Poisson distribution. This distribution is a discrete version of the Linnik distribution, originally introduced by Ju. V. Linnik in 1963 (see Linnik Yu (1963)), with characteristic function $\left(1+|t|^{\delta}\right)^{\beta}$. For more details about the continuous Linnik and its generalizations see, e.g., Arnold (1973), Devroye (1990), Anderson (1992), Anderson and Arnold (1993), Kozubowski (1998) and Pakes (1998).

Let M be a discrete rv with positive discrete Linnik distribution proposed by Devroye (1993) with support $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Its pgf can be written as

$$\mathcal{P}_M(t) = \left(1 + \beta\,(1-t)^{\alpha}\right)^{-\gamma}, \ |t| \leq 1, \tag{5.28}$$

where $\alpha \in (0, 1]$, $\gamma > 0$ and $\beta > 0$.

Devroye (1993) showed that a discrete Linnik rv, with parameters α, γ and β and with pgf as in (5.28), is a conditional Poisson rv with parameter $G^{\frac{1}{\alpha}} \times S_\alpha$, where $G \sim Gamma\left(\gamma, \frac{1}{\beta}\right)$ and $S_\alpha \sim Stable(\alpha)$ with LST $\mathcal{L}_{S_\alpha}(t) = \mathrm{e}^{-t^{\alpha}}$, $t > 0$, meaning that

$$M \sim Poisson\left(G^{\frac{1}{\alpha}} \times S_\alpha\right).$$

Note that, thanks to this mixture representation of the discrete Linnik distribution, a simulation algorithm can be derived to generate observations from the discrete Linnik distribution. Also, S_α can be generated using Kanter's method (see, e.g., Kanter et al. (1975) and Devroye (1993)).

Inspired from the results of Pakes (1998) obtained in the continuous case of a generalized Linnik distribution, Bouzar (2002) proposed an interesting generalization of the discrete Linnik distribution. He showed that such a distribution is derived from the discrete Stable distribution and proved that any discrete Linnik distribution is a mixture of negative binomial distributions.

A discrete Linnik distribution can also be represented as a compound distribution. Let $M \sim Linnik(\alpha, \beta)$. We can represent M as follows

$$M = \begin{cases} \sum_{k=1}^{N} Z_k, & N > 0 \\ 0, & N = 0 \end{cases},$$

where the primary distribution follows a negative binomial distribution, i.e, $N \sim NB(\gamma, \beta)$ with pgf $\mathcal{P}_N(t) = (1-\beta(t-1))^{-\gamma}$ and where the rvs Z_k, $k = 1, 2, \ldots$, have a Sibuya distribution with parameter α and pgf $\mathcal{P}_{Z_k}(t) = 1 - (1-t)^\alpha$. See, e.g., Cossette et al. (2001) for more details. Note that the parametrization of the negative binomial distribution here is different from the one used in Section 5.3.1.

To construct the univariate mixed exponential-discrete Linnik distribution, a transformation of the rv M with pgf in (5.28) is made here. Let the mixing rv Θ be defined as follows

$$\Theta = Z_1 + Z_2 + \ldots + Z_{N^*},$$

where $N^* = N + \gamma$ with $N \sim NB(\gamma, \beta)$ and $Z_k \sim Sibuya(\alpha)$, for $k = 1, 2, \ldots$, as defined just below.

Then, the LST of Θ can be written as

$$\mathcal{L}_\Theta(t) = \left(\frac{1-(1-\mathrm{e}^{-t})^\alpha}{1+\beta\,(1-\mathrm{e}^{-t})^\alpha}\right)^\gamma, \; t > 0. \tag{5.29}$$

Note that, if $\gamma = 1$, then Θ is said to follow a discrete Mittag-Leffler distribution (see, e.g., Pillai and Jayakumar (1995)).

Combining (5.8) and (5.29), the unconditional survival function of a rv X with an exponential-discrete Linnik distribution is given by

$$\overline{F}_X(x) = \left(\frac{1-(1-\mathrm{e}^{-x})^{\alpha}}{1+\beta\,(1-\mathrm{e}^{-x})^{\alpha}} \right)^{\gamma},\ x>0. \tag{5.30}$$

Proposition 5.3.6. *Let the rv X follow a mixed exponential-discrete Linnik distribution with univariate survival function as in (5.30). Then, the following properties hold:*

1. $f_X(x) = \dfrac{\gamma\,\alpha\,(1+\beta)\left((1-\mathrm{e}^{-x})^{2\,\alpha}\,\beta + (1-\mathrm{e}^{-x})^{\alpha}\right)}{(1-(1-\mathrm{e}^{-x})^{\alpha})\,(1+\beta\,(1-\mathrm{e}^{-x})^{\alpha})^2\,(\mathrm{e}^{x}-1)} \left(\dfrac{1-(1-\mathrm{e}^{-x})^{\alpha}}{1+\beta\,(1-\mathrm{e}^{-x})^{\alpha}} \right)^{\gamma},\ x>0.$
2. *Failure rate* $h_X(x) = \dfrac{\gamma\,\alpha\,(1+\beta)\left((1-\mathrm{e}^{-x})^{2\,\alpha}\,\beta + (1-\mathrm{e}^{-x})^{\alpha}\right)}{(1-(1-\mathrm{e}^{-x})^{\alpha})\,(1+\beta\,(1-\mathrm{e}^{-x})^{\alpha})^2\,(\mathrm{e}^{x}-1)},\ x>0.$
3. $VaR_{\kappa}(X) = -\ln\left(1-\left(\dfrac{1-s^{\frac{1}{\gamma}}}{1+s^{\frac{1}{\gamma}}\beta}\right)^{\frac{1}{\alpha}}\right).$

Proof. The expressions follow directly from their definitions. □

Remark 5.3.4. *As expected, the failure rate of a mixed exponential-discrete Linnik distribution is decreasing in x.*

Note that the expectation, the moments in addition to other properties given in Proposition 5.3.1, can only be obtained numerically.

In order to study limit cases, we need to define a family of distributions introduced by Joe (2014).

Definition 5.3.2. *Let Z be a positive rv with LST $\mathcal{L}_Z$ and let Θ be a positive rv such that, given $Z=z$, the conditional LST of Θ is written as*

$$\mathcal{L}_{\Theta|Z=z}(t) = \left(\mathcal{L}_W(t)\right)^{z},\ z>0 \text{ and } t>0,$$

where W is a positive rv with LST $\mathcal{L}_W$.

Then, the unconditional distribution of Θ is referred to as "the distribution of Z"-stopped-"the distribution of W" with LST

$$\mathcal{L}_{\Theta}(t) = E\left[\left(\mathcal{L}_W(t)\right)^{Z}\right] = \mathcal{L}_Z\left(-\ln\left(\mathcal{L}_W(t)\right)\right).$$

For example, if $Z \sim Gamma(\gamma, 1)$ and $W \sim Sibuya(\alpha)$, then Θ follows a gamma-stopped-Sibuya distribution with parameters α and γ and LST given by

$$\mathcal{L}_{\Theta}(t) = \left(1 - \ln\left(1 - \left(1 - \mathrm{e}^{-t}\right)^{\alpha}\right)\right)^{-\gamma}.$$

Also, if $Z \sim BN(\gamma, \beta)$ and $W \sim Sibuya(\alpha)$, then Θ follows a negative binomial-stopped-Sibuya distribution with parameters α, β and γ and LST given by

$$\mathcal{L}_{\Theta}(t) = \left(\frac{1 - (1 - \mathrm{e}^{-t})^{\alpha}}{1 + \beta\,(1 - \mathrm{e}^{-t})^{\alpha}}\right)^{\gamma},$$

which is exactly the form of the LST of a discrete Linnik distribution given in (5.29). Therefore, the discrete Linnik distribution coincides with the negative binomial-stopped-Sibuya distribution. Since there is a link between negative binomial distributions and gamma distributions, we use the same representation of the negative binomial distribution as in Section 5.3.1 to verify if such a link still holds when these two distributions are involved in distributions of the form "the distribution of Z"-stopped-"the distribution of W". To do so, let $\beta = \frac{1-q}{q}$, where $q = 1 - \mathrm{e}^{-h}$. Then, the mixing rv $\Theta_{(h,\alpha,\gamma)}$ can be represented as follows

$$\Theta_{(h,\alpha,\gamma)} = Z_1 + Z_2 + \ldots + Z_{N^*}, \tag{5.31}$$

where $N^* = h \times (N + \gamma)$ with $N \sim NB(\gamma, \frac{1}{\mathrm{e}^h - 1})$. Since $\Theta_{(h,\alpha,\gamma)}$ follows a negative binomial-stopped-Sibuya distribution with parameters α, β and γ, its LST is given by

$$\mathcal{L}_{\Theta_{(h,\alpha,\gamma)}}(t) = E\left[(\mathcal{L}_Z(t))^{N^*}\right] = \mathcal{L}_{N^*}\left(-\ln\left(\mathcal{L}_Z(t)\right)\right), \tag{5.32}$$

where $Z \sim Sibuya(\alpha)$.

Proposition 5.3.7. *Let $X_{(h,\alpha,\gamma)}$ follow a univariate mixed exponential-discrete Linnik distribution with mixing rv $\Theta_{(h,\alpha,\gamma)}$ as defined in (5.31) and (5.32). Then,*

$$\Theta_{(h,\alpha,\gamma)} \xrightarrow{\mathcal{D}} \Theta^{GsS},$$

where Θ^{GsS} follows a gamma-stopped-Sibuya distribution with parameters α and γ. Also, when $h \to 0$

$$X_{(h,\alpha,\gamma)} \xrightarrow{\mathcal{D}} X^{GsS},$$

where X^{GsS} follows a mixed exponential distribution for which the mixing rv Θ^{GsS} follows a gamma-stopped-Sibuya distribution with parameters α and γ.

Proof. We know that $N^* \xrightarrow{\mathcal{D}} Y$, where $Y \sim Gamma(\gamma, 1)$ when $h \to 0$, and $-\ln\left(\mathcal{L}_Z(t)\right) > 0$, $\forall t > 0$. Then,

$$
\begin{aligned}
\lim_{h\to 0}\mathcal{L}_{\Theta_{(h,\alpha,\gamma)}}(t) &= \lim_{h\to 0}\mathcal{L}_{N^*}\left(-\ln\left(\mathcal{L}_Z(t)\right)\right) \\
&= \mathcal{L}_Y\left(-\ln\left(\mathcal{L}_Z(t)\right)\right).
\end{aligned}
$$

We conclude that, $\Theta_{(h,\alpha,\gamma)} \xrightarrow{\mathcal{D}} \Theta^{GsS}$ and then $X_{(h,\alpha,\gamma)} \xrightarrow{\mathcal{D}} X^{GsS}$. □

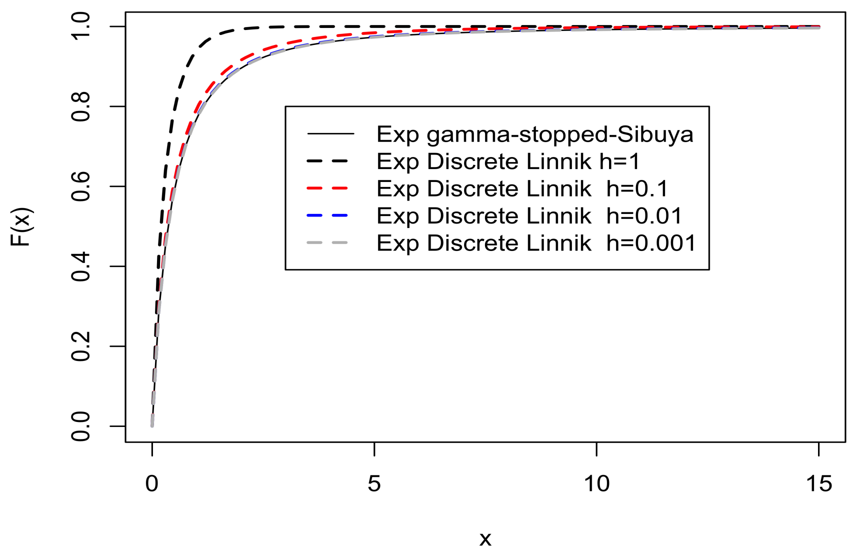


Figure 5.8 – Illustration of the convergence of the mixed exponential-discrete Linnik distribution to the Gamma-stopped-Sibuya distribution with $\alpha = 0.9$ and $\gamma = 2$.

Another interesting limit case is discussed in the following Proposition.

Proposition 5.3.8. *Let $\Theta_{h,\alpha,\gamma} = h \times \Theta$ where Θ follows a discrete Linnik distribution with LST as given in (5.29) and let Θ^{CL} be a positive rv with continuous positive Linnik distribution (with parameters $\alpha \in (0, 1]$, $\lambda > 0$ and $\gamma > 0$) and LST given by*

$$
\mathcal{L}_{\Theta^{CL}}(t) = \left(1 + \lambda t^{\alpha}\right)^{-\gamma},\ t > 0. \tag{5.33}
$$

Then, when $h \to 0$, i.e., $\beta \to \infty$, we have

$$
\Theta_{h,\alpha,\gamma} \xrightarrow{\mathcal{D}} \Theta^{CL}.
$$

Let $X_{h,\alpha,\gamma}$ and X^{CL} two positive rvs following mixed exponential distributions with mixing rv $\Theta_{h,\alpha,\gamma}$ and Θ^{CL} respectively. Then, when $h \to 0$, we have

$$
X_{h,\alpha,\gamma} \xrightarrow{\mathcal{D}} X^{CL}.
$$

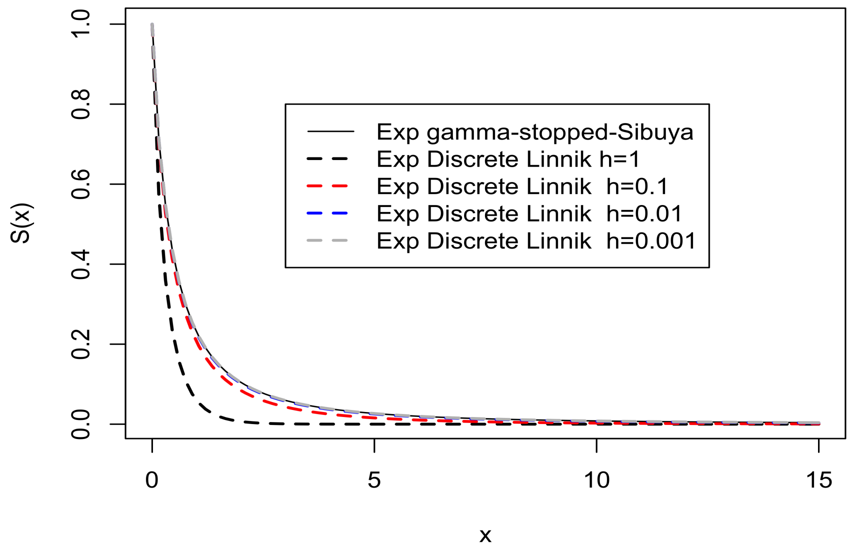


Figure 5.9 – Illustration of the convergence of the mixed exponential-discrete Linnik distribution to the Gamma-stopped-Sibuya distribution with $\alpha = 0.9$ and $\gamma = 2$.

Proof. We have

$$\begin{aligned}
\mathcal{L}_{\Theta_{h,\alpha,\gamma}}(t) &= \left(\frac{1-(1-\mathrm{e}^{-th})^{\alpha}}{1-\beta(1-\mathrm{e}^{-th})^{\alpha}}\right)^{\gamma} \\
&= \left(1-(1-\mathrm{e}^{-th})^{\alpha}\right)^{\gamma} \times \left(1-\beta\left(1-\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}(ht)^n\right)\right)^{\alpha}\right)^{-\gamma} \\
&= \left(1-(1-\mathrm{e}^{-th})^{\alpha}\right)^{\gamma} \times \left(1-\beta\, h^{\alpha}\, t^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}(ht)^n\right)^{\alpha}\right)^{-\gamma}.
\end{aligned}$$

By letting $h = \left(\frac{\lambda}{\beta}\right)^{\frac{1}{\alpha}}$, we obtain

$$\mathcal{L}_{\Theta_{h,\alpha,\gamma}}(t) = \left(1-(1-\mathrm{e}^{-th})^{\alpha}\right)^{\gamma} \times \left(1-\lambda\, t^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}(ht)^n\right)^{\alpha}\right)^{-\gamma}.$$

Then,

$$\lim_{h\to 0}\mathcal{L}_{\Theta_{h,\alpha,\gamma}}(t) = (1+\lambda t^{\alpha})^{-\gamma} = \mathcal{L}_{\Theta^{CL}}(t).$$

Therefore, we conclude that $\Theta_{h,\alpha,\gamma} \xrightarrow{\mathcal{D}} \Theta^{CL}$. □

Remark 5.3.5. *Note that Proposition 5.3.8 still holds when $\gamma = 1$. In this case, $\Theta_{h,\alpha,\gamma}$ and Θ^{CL} follow, respectively, discrete and continuous Mittag-leffler distributions.*

5.3.4 Other univariate mixed exponential distributions

Other univariate mixed exponential distributions defined with discrete rvs Θ could have been examined. However, we prefer to briefly present, in the Appendix, two of them: the univariate mixed Exponential - Sibuya and Exponential - Logarithmic distributions.

5.3.5 Classical renewal risk model with mixed exponential interclaim times

We present in this section another motivation for the class of univariate mixed exponential distributions. More precisely, we consider the classical Sparre-Andersen renewal risk model in which inter-claim times are assumed to follow a mixed exponential distribution. For an insurance portfolio, the surplus process is defined by $\underline{U} = \{U(t), t \geq 0\}$ where the surplus level at time t, $U(t)$, is given by

$$U(t) = u + ct - S(t),$$

where $U(0) = u$ is the initial surplus and c is the premium rate. The aggregate claim amount process, denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ is a compound renewal process with iid inter-claim times. The claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a renewal process where the inter-claim times $\underline{W} = \{W_j, j \in \mathbb{N}\}$ form a sequence of iid and strictly positive real-valued rvs. The time between the $(j-1)$th and the jth claim ($j = 2, ...$) is defined by the rv W_j with W_1 the time of the first claim. The rvs $\{W_j, j \in \mathbb{N}\}$, are identically distributed as the canonical rv W, have pdf f_W, cdf F_W, and survival function $\overline{F}_W$. The univariate rv W follows a univariate mixed exponential distribution.

The time of arrival of the jth claim is denoted $T_j = W_1 + \cdots + W_j$. The claim amount rvs $\underline{X} = \{X_j, j \in \mathbb{N}\}$, where X_j corresponds to the amount of the jth claim, are assumed to be a sequence of strictly positive and iid rvs with pdf f_X and cdf F_X. The sequences $\underline{W}$ and $\underline{X}$ are independent.

The time of ruin is defined by the rv $\tau_u = \inf\{t \geq 0 : U(t) < 0\}$ with $\tau_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The infinite-time ruin probability is $\zeta(u) = \Pr(\tau_u < \infty | U(0) = u)$. Throughout this section, we assume the positive security loading condition $E[cW - X] > 0$ to be verified which ensures that ruin will not occur almost surely.

We limit our analysis to exponentially distributed claim amounts with parameter β, which implies (see, e.g., Asmussen and Albrecher (2010))

$$\zeta(u) = \frac{\beta - \rho}{\beta} \mathrm{e}^{-\rho u},\ u \geq 0, \tag{5.34}$$

where ρ is the adjustment coefficient which is the smallest strictly positive solution to the Lundberg relation

$$E\left[\mathrm{e}^{r(X-cW)}\right] = E\left[\mathrm{e}^{rX}\right] \times E\left[\mathrm{e}^{-rcW}\right] = 1. \tag{5.35}$$

Let us look at a numerical example in which we compare the adjustment coefficients obtained for different choices of mixed exponential distributions for the inter-claim times

Example 5.3.1. *Let W follows a univariate mixed exponential distribution with expectation 1 and let $X \sim Exp(1)$. We consider a scale parameter β_s such that $W = \beta_s Y$ where Y follows a univariate mixed exponential distribution with expectation $1/\beta_s$ and with $\overline{F}_W$ as given in (5.8). Let the mixing rv Θ follow one of the distributions considered in this paper (i.e., negative binomial, gamma, Sibuya, Discrete stable, stable, Linnik, and logarithmic distributions), and let $c = 1.25$. Table 5.3 lists the values of the adjustment coefficient ρ for different mixing distributions. As we can see, the smaller the h becomes, the closer become the adjustment coefficients related to both the exponential negative binomial distribution and the Pareto distribution (exponential gamma distribution). Such a result is expected since the exponential-negative binomial distribution converges to the Pareto distribution. Moreover, the values of ρ differ from one distribution to another.*

Θ's distribution	Parameters	ρ
Negative binomial	$r = 2,\ h = 0.01,\ \beta_s = 1.04335$	0.09051169
	$r = 2,\ h = 0.001,\ \beta_s = 1.006456$	0.08110895
	$r = 2,\ h = 0.0001,\ \beta_s = 1.000872$	0.07934719
Gamma	$\alpha = 2,\ \beta_s = 1$	0.07903681
Sibuya	$\alpha = 0.5,\ \beta_s = 1.629446$	0.4909645
Logarithmic	$\alpha = 0.1,\ \beta_s = 0.97527775$	0.1570996
Discrete stable	$\alpha = 0.5,\ h = 0.01,\ \beta_s = 0.4653859$	0.0858715
	$\alpha = 0.5,\ h = 0.001,\ \beta_s = 0.4969561$	0.09494302
	$\alpha = 0.5,\ h = 0.0001,\ \beta_s = 0.4996998$	0.09565799
Discrete Linnik	$\alpha = 0.5,\ \gamma = 2\ h = 0.01,\ \beta_s = 1.647551$	0.1164167
	$\alpha = 0.5,\ \gamma = 2\ h = 0.001,\ \beta_s = 1.612658$	0.1163437
	$\alpha = 0.5,\ \gamma = 2\ h = 0.0001,\ \beta_s = 1.60928$	0.1163387

Table 5.3 – Adjustment coefficients for different distributions.

5.4 From multivariate mixed exponential distributions to Archimedean copulas

A multivariate mixed exponential distribution can be constructed with the elegant procedure introduced by Oakes (1989) and Marshall and Olkin (1988). Let $\underline{X} = (X_1, \ldots, X_n)$ be an n-dimensional random vector. The dependence between the rvs $X_1, \ldots, X_n$ is introduced via the common mixing strictly positive rv Θ which can be discrete or continuous. Given

$\Theta = \theta$, the conditional rvs $(X_1|\Theta = \theta), \ldots, (X_n|\Theta = \theta)$ are conditionally independent with $(X_i|\Theta = \theta) \sim Exp(\theta)$, for $i = 1, \ldots, n$. It implies that

$$\bar{F}_{\underline{X}|\Theta=\theta}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathrm{e}^{-\theta\, x_i} = \exp\left\{-\theta(x_1 + \ldots + x_n)\right\}. \tag{5.36}$$

Then, using (5.7) with (5.36), the joint survival function of $\underline{X}$ is given by

$$\bar{F}_{\underline{X}}(x_1, \ldots, x_n) = \mathcal{L}_{\Theta}\left(x_1 + \ldots + x_n\right). \tag{5.37}$$

Multivariate mixed exponential distributions have been frequently used in actuarial science, quantitative risk management and reliability theory, see, e.g., Sarabia et al. (2017), Dacorogna et al. (2016), and Albrecher et al. (2011).

A prominent problem in actuarial science and quantitative risk management is to determine the share C of the total capital to be allocated to each risk. Different capital allocation methods were proposed in the literature. We provide in Proposition 5.4.1 explicit formulas in terms of Θ for different capital allocation methods proposed in Furman and Zitikis (2008).

Proposition 5.4.1 lists several general properties of a multivariate mixed exponential distribution. The conditional independence underlying this multivariate dependence model allows to better study the relationship between risks and to calculate different expectations of the form $E\left[g(X_i, X_j)\right]$, for $i, j \in \{1, \ldots, n\}$, where g is a bivariate function. It also allows to derive different capital allocation formulas without knowing the distribution of the global risk $S = X_1 + \ldots + X_n$ (see properties 6 - 8).

Proposition 5.4.1. *Consider $\underline{X}$ to be an n-dimensional vector with multivariate mixed exponential distribution and joint survival function as given in (5.37). Let $S = X_1 + \ldots + X_n$ be the aggregated risk. Then, properties of such a distribution can be written in terms of the mixing rv Θ as follows (provided that the expectations exist):*

1. *Joint density function* $f_{\underline{X}}(x_1, \cdots, x_m) = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}x_1 \ldots \mathrm{d}x_m} \mathcal{L}_{\Theta}(x)$;
2. *Joint moments* $E[X_1^{n_1} \ldots X_m^{n_m}] = \prod_{i=1}^{m} \Gamma(n_i + 1) \times E\left[\Theta^{-(n_1 + \ldots + n_m)}\right]$;
3. *Covariance* $Cov\left(X_1, X_2\right) = E\left[\Theta^{-2}\right] - E^2\left[\Theta^{-1}\right]$;
4. *Pearson's correlation coefficient* $\rho\left(X_1, X_2\right) = \frac{E\left[\Theta^{-2}\right] - E^2\left[\Theta^{-1}\right]}{2\, E\left[\Theta^{-2}\right] - E^2\left[\Theta^{-1}\right]}$;
5. *Kendall's tau* $\tau\left(X_1, X_2\right) = 1 + 4\int_0^1 \frac{\mathcal{L}_{\Theta}^{-1}(t)}{\left(\mathcal{L}_{\Theta}^{-1}(t)\right)'}\mathrm{d}t = 1 - 4\int_0^{\infty} t \times \left(\mathcal{L}'_{\Theta}(t)\right)^2 \mathrm{d}t$;
6. *MCov capital allocation* $C_{\omega}(X_i, S) = \frac{n+1}{n} \times \frac{E\left[\Theta^{-2}\right]}{E\left[\Theta^{-1}\right]}$, $i \in \{1, 2, \ldots, n\}$;
7. *Esscher's capital allocation* $C_{\omega}(X_i, S) = \frac{E\left[\Theta(\Theta - t)^{-2}\right] E\left[\left(\frac{\Theta}{\Theta - t}\right)^{n-1}\right]}{E\left[\left(\frac{\Theta}{\Theta - t}\right)^{n}\right]}$, $i \in \{1, 2, \ldots, n\}$;

8. *Kamps's capital allocation* $C_\omega(X_i, S) = \frac{E[\Theta^{-1}] - E[\Theta(\Theta+t)^{-2}] \times E\left[\left(\frac{\Theta}{\Theta+t}\right)^{n-1}\right]}{1 - E\left[\left(\frac{\Theta}{\Theta+t}\right)^{n}\right]}, \quad i \in \{1, 2, \ldots, n\}$.

Proof. Properties 1 - 5 are known results (see, e.g., Sarabia et al. (2017)). Expressions of capital allocations given in properties 6 - 8 are obtained by first conditioning with respect to Θ and then using the fact that $X|\Theta = \theta \sim Exp(\theta)$. We use also the fact that $S|\Theta = \theta \sim Gamma(n, \theta)$. For example, we have

For MCov capital allocation method, we have

$$\begin{aligned}
C_\omega(X_i, S) &= E[X_i] + \frac{Cov(X_i, S)}{E[S]} \\
&= E\left[\Theta^{-1}\right] + \frac{E\left[X_i \times S\right] - n \times E\left[\Theta^{-1}\right]^2}{n \times E\left[\Theta^{-1}\right]} \\
&= E\left[\Theta^{-1}\right] + \frac{E\left[X_i \times (S_{-i} + X_i)\right] - n \times E\left[\Theta^{-1}\right]^2}{n \times E\left[\Theta^{-1}\right]},
\end{aligned}$$

where $i = 1, \ldots, n$ and $S_{-i} = X_1 + \ldots + X_{i-1} + X_{i+1} + \ldots + X_n$. Since $S_{-i}|\Theta = \theta \sim Gamma(n-1, \theta)$, we have

$$\begin{aligned}
E\left[X_i \times (S_{-i} + X_i)\right] &= E\left[E\left[X_i \times (S_{-i} + X_i)|\Theta\right]\right] \\
&= E\left[E\left[X_i \times S_{-i}|\Theta\right]\right] + E\left[E\left[X_i^2|\Theta\right]\right] \\
&= E\left[E\left[X_i|\Theta\right] \times E\left[S_{-i}|\Theta\right]\right] + E\left[E\left[X_i^2|\Theta\right]\right] \\
&= E\left[\Theta^{-1} \times (n-1)\,\Theta^{-1}\right] + 2\,E\left[\Theta^{-2}\right] \\
&= (n+1)\,E\left[\Theta^{-2}\right].
\end{aligned}$$

Then, the MCov capital allocation can be written as

$$C_\omega(X_i, S) = \frac{n+1}{n} \times \frac{E\left[\Theta^{-2}\right]}{E\left[\Theta^{-1}\right]}.$$

For Esscher's capital allocation, we have

$$\begin{aligned}
C_\omega(X_i, S) &= \frac{E\left[X_i\, \mathrm{e}^{t\,S}\right]}{E\left[\mathrm{e}^{t\,S}\right]} \\
&= \frac{E\left[E\left[X_i\, \mathrm{e}^{t\,S}|\Theta\right]\right]}{E\left[E\left[\mathrm{e}^{t\,S}|\Theta\right]\right]} \\
&= \frac{E\left[E\left[X_i\, \mathrm{e}^{t\,(S_{-i}+X_i)}|\Theta\right]\right]}{E\left[\mathcal{L}_S(-t)\right]} \\
&= \frac{E\left[E\left[X_i\, \mathrm{e}^{t\,X_i}|\Theta\right] \times E\left[\mathrm{e}^{t\,S_{-i}}|\Theta\right]\right]}{E\left[\mathcal{L}_S(-t)\right]} \\
&= \frac{E\left[E\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t\,X_i}\right] \times \mathcal{L}_{S_{-i}}(-t)\right]}{E\left[\mathcal{L}_S(-t)\right]} \\
&= \frac{E\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Theta}{\Theta-t}\right)\right] \times E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n-1}\right]}{E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n}\right]} \\
&= \frac{E\left[\Theta\,\left(\Theta-t\right)^{-2}\right] \times E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n-1}\right]}{E\left[\left(\frac{\Theta}{\Theta-t}\right)^{n}\right]},
\end{aligned}$$

for $i \in \{1, 2, \ldots, n\}$. The procedure is exactly the same for property 8. □

The multivariate model defined in (5.36) and (5.37) can also be constructed with completely monotone marginals joined by a dependence structure defined with an Archimedean copula with generator $\mathcal{L}_\Theta$. The equivalence of these two models is shown notably in Marshall and Olkin (1988), Oakes (1989) and Albrecher et al. (2011). In other words, an Archimedean copula $\overline{C}$ with generator $\mathcal{L}_\Theta$ can be constructed, using Sklar's theorem, from a multivariate mixed exponential distribution as defined in Section 5.4 and marginals with univariate mixed exponential distribution described in Section 5.3, i.e.,

$$\overline{F}^{ME}_{\underline{X}}(x_1, \ldots, x_n) = \overline{C}\left(\overline{F}^{ME}_{X_1}(x_1), \ldots, \overline{F}^{ME}_{X_n}(x_n)\right)$$

and

$$\overline{C}\left(u_1, ..., u_n\right) = \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(u_1\right) + ... + \mathcal{L}_\Theta^{-1}\left(u_n\right)\right), \tag{5.38}$$

where $\overline{F}^{ME}_{\underline{X}}$ and $\overline{F}^{ME}_{X_i}$, $i = 1, \ldots, n$, are as given in (5.37) and (5.8) respectively.

This method of construction, based on Marshall and Olkin (1988), provides the natural sampling algorithm given below based on the idea that, given Θ, all rvs X_i, $i = 1, \ldots, n$, are iid and exponentially distributed with mean 1. Such an Algorithm will be used in the examples of the following section.

Algorithm 5.4.1. *Let C be a d-dimensional Archimedean copula with generator $\mathcal{L}_\Theta$.*

1. *Generate Θ with LST $\mathcal{L}_\Theta$;*
2. *Generate $R_i \sim Exp(1)$ for $i = 1, \ldots, d$;*
3. *Produce $U_i = \mathcal{L}_\Theta\left(\frac{R_i}{\Theta}\right)$, for $i = 1, \ldots, d$;*
4. *Return $\underline{U} = (U_1, \ldots, U_d)$.*

Since in a multivariate model incorporating dependence using a copula, the dependence relationship between the rvs of this model is fully described by the copula, it would be interesting to see how this dependence relationship behaves as a result of a variation of the parameter of dependence. For this purpose, we need to introduce the concept of concordance ordering as defined in Joe (1997) page 37.

Definition 5.4.1. *Let C_1 and C_2 be two d-dimensional copulas with respective Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. C_2 is more* ***concordant*** *that C_1, written $C_1 \prec_c C_2$, if*

$$C_1\left(\underline{u}\right) \leq C_2\left(\underline{u}\right) \text{ and } \overline{C_1}\left(\underline{u}\right) \leq \overline{C_2}\left(\underline{u}\right),$$

for $\underline{u} \in [0,1]^d$. As a result of such a concordance ordering, $\tau_1 \leq \tau_2$, $\rho_S^{(1)} \leq \rho_S^{(2)}$, and $\lambda_1 \leq \lambda_2$. (see Joe (1997))

If C_1 and C_2 are Archimedean copulas with respective generators $\mathcal{L}_1$ and $\mathcal{L}_2$. Theorems 4.1 and 4.7 in Joe (1997) prove that the condition $\left(\mathcal{L}_1^{-1} \circ \mathcal{L}_2\right)'$ is a completely monotone function guarantees that $C_1 \prec_c C_2$ is verified.

For an Archimedean copula with mixing rv Θ such that $F_\Theta(0) = 0$ and generator $\mathcal{L}_\Theta$, if the lower and upper tail-dependence coefficients exist, then, according to Joe and Hu (1996), λ_L and λ_U can be written in terms of $\mathcal{L}_\Theta$ as follows

$$\lambda_L = \lim_{t\to\infty} \frac{\mathcal{L}_\Theta(2t)}{\mathcal{L}_\Theta(t)} = 2 \lim_{t\to\infty} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}$$

and

$$\lambda_U = 2 - \lim_{t\to 0} \frac{1 - \mathcal{L}_\Theta(2t)}{1 - \mathcal{L}_\Theta(t)} = 2 - 2\lim_{t\to 0} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}.$$

For the special case of discrete mixing rvs, the related Archimedean copulas cannot have lower tail dependence, i.e., $\lambda_L = 0$. Also, if $E[\Theta]$ is finite, then $\lambda_U = 0$ (see Hofert (2010) page 62 for proof).

5.4.1 Multivariate mixed Exponential - Negative Binomial Distribution

Once again, to investigate the limit cases of the multivariate exponential-negative binomial distribution, we will use the representation and the parametrization of the negative binomial distribution provided in Section 5.3.1. Consequently, let the mixing rv Θ follow a negative binomial distribution with LST given in (5.14). Consider $\underline{X} = (X_1, \ldots, X_n)$ a vector of n rvs for which the dependence is introduced via the mixing rv Θ. In this case, combining (5.14) and (5.37), the joint survival function of $\underline{X}$ can be written as

$$\Pr\left(X_1 > x_1, \ldots, X_n > x_n\right) = \left(\frac{1 - \mathrm{e}^{-h}}{\mathrm{e}^{\sum_{i=1}^{n} x_i\, h} - \mathrm{e}^{-h}}\right)^r,\ x_i \in \mathbb{R}^+,\ i = 1, \ldots, n. \tag{5.39}$$

Proposition 5.4.2. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential negative binomial distribution with joint survival function given in (5.39). Then, the following properties hold:*

1. *$\underline{X} \xrightarrow{\mathcal{D}} \underline{Y}$, as $h \to 0$, where $\underline{Y}$ follows a multivariate Pareto of the second kind;*
2. *Joint moments $E[X_1^{m_1} \ldots X_n^{m_n}] = \frac{\left(1-\mathrm{e}^{-h}\right)^r}{h^m r^m}\ {}_{m+1}F_m\Big([r, \ldots, r], [1 + r, \ldots, 1 + r], \mathrm{e}^{-h}\Big) \times \prod_{i=1}^{n}\Gamma(m_i + 1)$, where $m = m_1 + \ldots + m_n$;*
3. *$Cov\left(X_1, X_2\right) = \dfrac{-\left(1-\mathrm{e}^{-h}\right)^{2\,r}\left({}_2F_1([r, r],\, [r+1],\, \mathrm{e}^{-h})\right)^2 + \left(1-\mathrm{e}^{-h}\right)^r\, {}_3F_2([r, r, r],\, [r+1, r+1],\, \mathrm{e}^{-h})}{h^2\, r^2}$;*
4. *Pearson's correlation coefficient $\rho_P\left(X_1, X_2\right) = \frac{\left(1-\mathrm{e}^{-h}\right)^r\left({}_2F_1(r,r;\, r+1;\, \mathrm{e}^{-h})\right)^2 - {}_3F_2(r,r,r;\, r+1,r+1;\, \mathrm{e}^{-h})}{\left(1-\mathrm{e}^{-h}\right)^r\left({}_2F_1(r,r;\, r+1;\, \mathrm{e}^{-h})\right)^2 - 2\, {}_3F_2(r,r,r;\, r+1,r+1;\, \mathrm{e}^{-h})}$.*

Proof. For property 1, clearly, the joint survival function of $\underline{X}$ given in (5.39) tends to the joint survival function of a Pareto of the second kind distribution, i.e.,

$$\lim_{h \to 0}\left(\frac{1 - \mathrm{e}^{-h}}{\mathrm{e}^{\sum_{i=1}^{n} x_i\, h} - \mathrm{e}^{-h}}\right)^r = \left(1 + \sum_{i=1}^{n} x_i\right)^{-r}$$

where $x_i \in \mathbb{R}^+$, $i = 1, \ldots, n$. See, e.g., Arnold (1983) and Arnold (2015) for more details concerning the Pareto of the second kind distribution.

Properties 2, 3, and 4 are directly obtained from their definitions. □

Let us now consider a special case where $r = 1$. Then, the resulting multivariate distribution related to the survival function in (5.39) is a multivariate extension of the mixed exponential-geometric distribution proposed by Adamidis and Loukas (1998). In this case, the mixing rv Θ follows a geometric distribution with parameter q and pmf given by

$$\Pr(\Theta = k) = q \times (1-q)^{k-1}, \;\; k \in \mathbb{N}.$$

Since the limit case, when $h \to 0$, of such a distribution is already discussed in Proposition 5.4.2, there is no need for the representation using $q = 1 - \mathrm{e}^{-h}$. Then, the joint survival function in (5.39) becomes

$$\bar{F}_{\underline{X}}(\underline{x}) = \frac{q}{\exp\left(\sum_{i=1}^{m} x_i\right) - (1-q)}, \tag{5.40}$$

where $\underline{X}$ now follows a multivariate mixed exponential-geometric distribution with parameter q.

Since it is easier to work with a geometric distribution than a negative binomial one, more explicit formulas of different properties of the resulting multivariate mixed distribution can be found as shown in the following Proposition.

Proposition 5.4.3. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential geometric distribution with joint survival function given in (5.40). Then, the following properties hold:*

1. *Joint density function* $f_{\underline{X}}(\underline{x}) = \sum_{k=0}^{m-1} (-1)^{2m-k} (m-k)!\, S_2(m, m-k) \dfrac{\left(\mathrm{e}^{x_1+\ldots+x_m}\right)^{m-k}}{\left(\mathrm{e}^{x_1+\ldots+x_m} - (1-q)\right)^{m+1-k}}$;
2. *Joint moments* $E[X_1^{n_1} \ldots X_m^{n_m}] = \prod_{i=1}^{m} \Gamma(n_i + 1) \times \dfrac{q\, Li_d(1-q)}{1-q}$, *where* $d = n_1 + \ldots + n_m$;
3. *Covariance* $Cov\,(X_1, X_2) = \dfrac{q\, Li_2(1-q)}{1-q} - \dfrac{q^2\,(\ln{(q)})^2}{(1-q)^2}$;
4. *Pearson's correlation coefficient* $\rho\,(X_1, X_2) = \dfrac{\frac{q\, Li_2(1-q)}{1-q} - \frac{q^2 (\ln(q))^2}{(1-q)^2}}{2\,\frac{q\, Li_2(1-q)}{1-q} - \frac{q^2 (\ln(q))^2}{(1-q)^2}}$;
5. *Kendall's tau* $\tau\,(X_1, X_2) = \dfrac{3(1-q)^2 - 2(1-q) - 2q^2 \ln(q)}{3\,(1-q)^2}$,

where "Li" denotes the general polylogarithm function defined as $Li_a(z) = \sum_{d=1}^{\infty} \frac{z^d}{d^a}$. Note that the polylogarithm function is already built in R, Maple, Matlab.

Proof. For property 5, see, e.g., Hofert (2010). For all other properties, the expressions follow directly from their definitions. □

Now, we use the link between Archimedean copulas and multivariate mixed exponential distributions to construct a multivariate Archimedean copula using a mixing rv with negative binomial distribution. We also show that such a copula can be seen as an extension of the well-known Clayton copula.

Proposition 5.4.4. *Let $\Theta_{(h,r)}$ be a discrete rv with negative binomial distribution with LST as given in (5.14). The associated multivariate Archimedean copula with generator $\mathcal{L}_{\Theta_{(h,r)}}$, called the multivariate negative binomial copula, is given by*

$$C_{r,q}(u_1, \cdots, u_n) = \left(q \left(\prod_{i=1}^{n} \left(q u_i^{\frac{-1}{r}} + (1-q) \right) - (1-q) \right)^{-1} \right)^{r}, \tag{5.41}$$

where $q = 1 - \mathrm{e}^{-h}$.

Proof. The result follows by combining (5.14) and (5.38). □

Remark 5.4.1. *The negative binomial copula has some interesting limit cases. When $r = 1$, we have $C_{1,q}(\underline{u}) = C^{AMH}_{1-q}(\underline{u})$. If $r = 0$ or $r \to \infty$, we get the independence copula. The comonotonic copula is obtained when $r \to 0$ and $q = 0$. Finally, the most important limit case is the one when h tends to zero, i.e., $q \to 0$, which leads to the Clayton copula. The negative binomial copula can be seen as an approximation of the Clayton copula with dependence parameter $\alpha = \frac{1}{r}$.*

Proposition 5.4.5. *Let C be an n-dimensional negative binomial copula as defined in (5.41). Then, if $q > 0$, C cannot capture lower and upper tail-dependence, i.e.,*

$$\lambda_L = \lambda_U = 0.$$

If $q \to 0$, then $\lambda_L = 2^{-r}$ and $\lambda_U = 0$

Proof. When $q > 0$, we have that Θ is a strictly positive and discrete rv, therefore $\lambda_L = 0$. Also, $E[\Theta] = r + \frac{r(1-q)}{q} < \infty$, then $\lambda_U = 0$. When $q \to 0$, $\lambda_L = 2^{-r}$ and $\lambda_U = 0$ since the resulting copula is the Clayton copula with parameter $\frac{1}{r}$. □

We give in Figure 5.10 scatterplots of random points simulated, using Algorithm 5.4.1, from a negative binomial copula with $r = \frac{1}{3}$ and different values of h and a Clayton copula with $\alpha = 3$. One sees that the dependence introduced by a negative binomial copula is indeed really similar to the dependence structure of a Clayton copula.

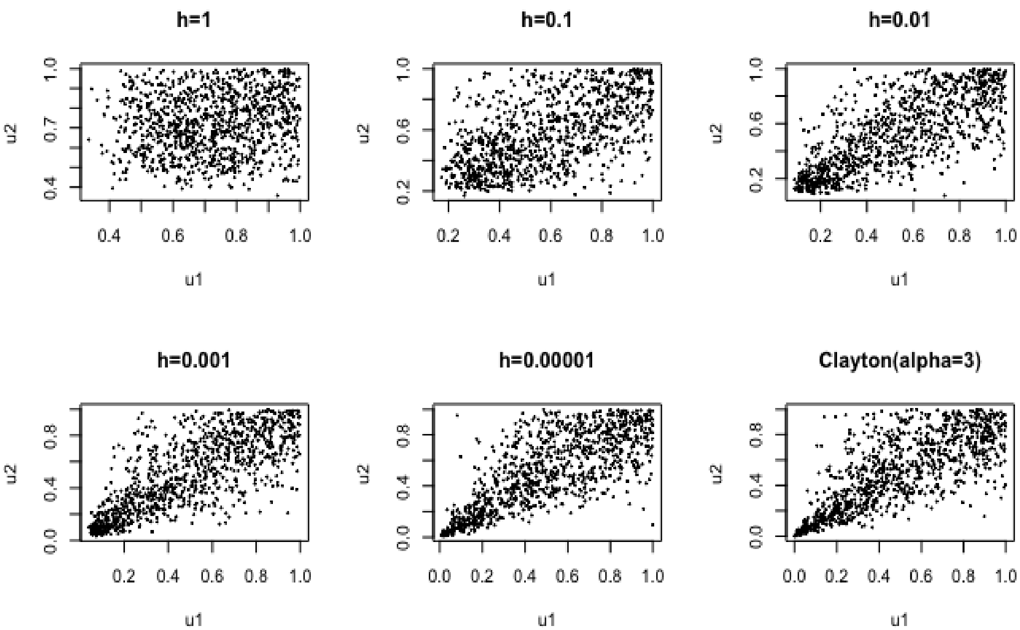


Figure 5.10 – Illustration of the approximation of a Clayton copula by an negative binomial Copula.

The bivariate negative binomial copula can be written as

$$C_{r,q}\left(u_1,u_2\right) \quad = \quad \frac{u_1 u_2}{\left(1-(1-q)\left(1-u_1^{\frac{1}{r}}\right)\left(1-u_2^{\frac{1}{r}}\right)\right)^r}.$$

This bivariate copula was first introduced by Fang et al. (2000) and was given different names such as the Fang-Fang copula as in Fang et al. (2000), the bivariate Lomax copula as in Balakrishnan and Lai (2009) and the BB10 copula as in Joe (1997). Also, Balakrishnan and Lai (2009) shows that such a copula can also be constructed via the bivariate Lomax distribution defined by its survival function $\bar{F}(x,y) = (1+\alpha_1 x+\alpha_2 y+\alpha_{1,2}xy)^{-\eta}$. Such a copula was studied by several researchers. For example, Genest and Rivest (2001) and Balakrishnan and Lai (2009) developed interesting characteristics of this copula and Jovanovic (2011) showed that it provided a better fit than the Gaussian, Clayton or Frank copulas in a context of stock market uncertainty.

Spearman's correlation coefficient and Kendall's tau between U_1 and U_2 are respectively given by

$$\rho_S(U_1, U_2) = 3 \times \Big({}_3F_2(\,[1,1,r]\,,[1+2r,1+2r]\,,1-q) - 1\Big)$$

and

$$\tau(U_1, U_2) = \frac{2r(1-q)}{(2r+1)^2} \times {}_2F_1(\,[1,1]\,,[2+2r]\,,1-q). \tag{5.42}$$

See Fang et al. (2000) for proof and more details.

Proposition 5.4.6. *Let C be a negative binomial copula with Kendall's tau τ as given in (5.42). Then*

$$0 \leq \tau \leq \tau_{Cl},$$

where $\tau_{Cl} = \frac{1}{2r+1}$ is the Kendall's tau of a Clayton copula with parameter $\frac{1}{r}$.

Proof. The construction of the negative binomial copula from a multivariate mixed exponential distribution implies that $\tau \geq 0$. To demonstrate the second inequality, let $g : [0,1] \to \mathbb{R}$, such that $g(x) = \frac{2rx}{(2r+1)^2} \times {}_2F_1(\,[1,1]\,,[2+2r]\,,x)$. We can easily show that g is a decreasing function, i.e.,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2rx}{(2r+1)^2} \times {}_2F_1(\,[1,1]\,,[2+2r]\,,x)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2rx}{(2r+1)^2} \times \sum_{k=0}^{\infty} \frac{1}{(2+2r)_k}\frac{x^k}{k!}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2r}{(2r+1)^2} \times \sum_{k=0}^{\infty} \frac{1}{(2+2r)_k}\frac{x^{k+1}}{k!}\right) \\
&= \frac{2r}{(2r+1)^2} \times \sum_{k=0}^{\infty} \frac{k+1}{(2+2r)_k}\frac{x^k}{k!} > 0.
\end{aligned}$$

Then,

$$0 \leq \tau \leq \tau_{Cl},$$

where

$$\begin{aligned}
\tau_{Cl} &= \lim_{q \to 0} \frac{2r(1-q)}{(2r+1)^2} \times {}_2F_1(\,[1,1]\,,[2+2r]\,,1-q) \\
&= \frac{2r}{(2r+1)^2} \times {}_2F_1(\,[1,1]\,,[2+2r]\,,1) \\
&= \frac{2r}{(2r+1)^2} \times \frac{2\,r+1}{2\,r} \\
&= \frac{1}{2\,r+1}.
\end{aligned}$$

As expected, when q tends to zero (or equivalently $h \to 0$), the Kendall's tau related to the negative binomial copula tends to the Kendall's tau of a Clayton copula with dependence parameter $\alpha = \frac{1}{r}$. $\square$

In order to analyze the dependence strength of such a copula, we recourse to concordance ordering as defined in Definition 5.4.1.

Proposition 5.4.7. *Let C_{r,q_1} and C_{r,q_2} be two d-dimensional negative binomial copulas with the same first parameter r and respective second parameters q_1, q_2, Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. Then*

$$C_{r,q_1} \prec_c C_{r,q_2} \text{ if } q_2 \leq q_1.$$

In this case, we have $\tau_1 \leq \tau_2$, $\rho_S^{(1)} \leq \rho_S^{(2)}$, and $\lambda_1 \leq \lambda_2$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the generators of C_{r,q_1} and C_{r,q_2} respectively. As discussed before, in order to show that $C_{r,q_1} \prec_c C_{r,q_2}$ we only have to show that $\left(\mathcal{L}_1^{-1} \circ \mathcal{L}_2\right)'$ is a completely monotone function. Then, for $x \in (0, \infty)$, we have

$$\begin{aligned}
\mathcal{L}_1^{-1} \circ \mathcal{L}_2 &= \ln\left(\frac{q_1 \mathrm{e}^x - q_1 + q_2}{q_2}\right) \\
&= -\ln\left(\frac{\frac{q_2}{q_1}}{\mathrm{e}^x - \left(1 - \frac{q_2}{q_1}\right)}\right) \\
&= -\ln\left(\mathcal{L}_N(x)\right),
\end{aligned}$$

where $N \sim Geo\left(\frac{q_2}{q_1}\right)$ if $q_2 \leq q_1$. It is well known that $(-\ln(\mathcal{L}_N))'$ is completely monotone if and only if $\mathcal{L}_N^m$ is the LST of a positive rv for all $m \in [0, \infty)$ (see Joe (1997)). Since $\mathcal{L}_N^m$, given by

$$\mathcal{L}_N^m(x) = \left(\frac{\frac{q_2}{q_1}}{\mathrm{e}^x - \left(1 - \frac{q_2}{q_1}\right)}\right)^m,$$

is clearly the LST of a negative binomial distribution with parameters m and $\frac{q_2}{q_1}$, we conclude that $C_{r,q_1} \prec_c C_{r,q_2}$ where $q_2 \leq q_1$. $\square$

Remark 5.4.2. *Proposition 5.4.7 also implies that the Clayton copula is more concordant that the negative binomial copula, i.e., $C_{r,q} \prec_c C^{Cl}_{\frac{1}{r}}$. The result of Proposition 5.4.6 is then, an implication of such an ordering.*

Remark 5.4.3. *Note that Fang et al. (2000) also proposed a multivariate extension of the Lomax copula which is given by*

$$C(u_1, \cdots, u_n) = \left(\prod_{i=1}^{n} u_i\right) \times \left(1 - (1-q)\prod_{i=1}^{n}\left(1 - u_i^{\frac{1}{r}}\right)\right)^{-r}.$$

This version of the multivariate Lomax copula is not Archimedean and does not correspond to the multivariate negative binomial copula we presented in Proposition 5.4.4.

5.4.2 Multivariate mixed Exponential - Discrete Stable distribution

Consider the strictly positive mixing rv Θ with $\Theta \sim Stable(\lambda, \alpha)$ and LST as in (5.22). Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n rvs such that, given $\Theta = \theta$, X_i, $i = 1, \ldots, n$ are conditionally independent and distributed as $Exp(\theta)$. Therefore, combining (5.22) and (5.37), the joint survival function of $\underline{X}$ is given by

$$\Pr\left(X_1 > x_1, \ldots, X_n > x_n\right) = \frac{\mathrm{e}^{-\lambda\left(1-\mathrm{e}^{-\sum_{i=1}^{n} x_i}\right)^{\alpha}} - \mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}, \; x_i \in \mathbb{R}^+, \; i = 1, \ldots, n. \tag{5.43}$$

Note that, as in the univariate case, properties related to the multivariate mixed exponential-discrete stable distribution with joint survival function given in (5.43) can only be obtained numerically.

Once again, in order to investigate limit cases, we will opt for the parametrization given in Section 5.3.4. In this case, combining (5.26) and (5.37), the joint survival function of $\underline{X}$ becomes

$$\Pr\left(X_1 > x_1, \ldots, X_n > x_n\right) = \frac{\exp\left\{-\left(\sum_{i=1}^{n} x_i\right)^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n\left(h\sum_{i=1}^{n} x_i\right)^n}{n!}\right)^{\alpha}\right\} - \mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}, \tag{5.44}$$

where $x_i \in \mathbb{R}^+$ for $i = 1, \ldots, n$.

Proposition 5.4.8. *Let $\underline{X} = (X_1, \ldots, X_n)$ follow an n-dimensional multivariate mixed exponential-Discrete stable distribution with joint survival function given in (5.44). Then,*

$$\underline{X} \xrightarrow{\mathcal{D}} \underline{Y},$$

as $h \to 0$, where $\underline{Y}$ follows a multivariate Weibull distribution proposed by Hougaard (1986).

Proof. Clearly, when $(h \to 0)$, (5.44) becomes

$$\lim_{h\to 0}\frac{\exp\left\{-\left(\sum_{i=1}^{n}x_i\right)^{\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(h\sum_{i=1}^{n}x_i)^n}{n!}\right)^{\alpha}-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}\right\}}{1-\mathrm{e}^{-\left(\frac{1}{h}\right)^{\alpha}}}=\exp\left\{-\left(\sum_{i=1}^{n}x_i\right)^{\alpha}\right\},$$

which is the joint survival function of a special case of the multivariate Weibull distribution proposed by Hougaard (1986) and defined by its survival function $\overline{F}(x_1,\ldots,x_n) = \exp\left\{\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\ldots+\left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}}\right\}^{\alpha}$. Here $\lambda_i = 1$ and $\gamma_i = \alpha$ for all $i = 1,\ldots,n$. See, e.g., Rinne (2008) and Lee and Wen (2006) for more details concerning this multivariate distribution. □

As for the negative binomial copula presented in the previous section, we will make use of the link between Archimedean copulas and exponential mixture models to construct a new multivariate copula with a discrete stable mixing distribution.

Proposition 5.4.9. *Let $\Theta_{(h,\alpha)}$ be a discrete rv with discrete stable distribution with LST as given in (5.24). The associated multivariate Archimedean copula with generator $\mathcal{L}_{\Theta_{(h,r)}}$, called the multivariate discrete stable copula, is given by*

$$C_{\lambda,\alpha}(u_1,\ldots,u_n)=\frac{\exp\left\{-\lambda\left(1-\prod_{i=1}^{n}\left(1-\left(\frac{\lambda-\ln\left(\left(\mathrm{e}^{\lambda}-1\right)u_i+1\right)}{\lambda}\right)^{\frac{1}{\alpha}}\right)\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}},\tag{5.45}$$

where $\lambda = h^{-\alpha}$.

Proof. The result is obtained by combining (5.24) and (5.38). □

The bivariate discrete stable copula is given by

$$C_{(h,\alpha)}(u_1,u_2) = \frac{\exp\left\{-\left(x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}}-(\lambda^{-1}\,x\,y)^{\frac{1}{\alpha}}\right)^{\alpha}\right\}-\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}},\tag{5.46}$$

where $x=\lambda-\ln\left((\mathrm{e}^{\lambda}-1)u_1+1\right)$ and $y=\lambda-\ln\left((\mathrm{e}^{\lambda}-1)u_2+1\right)$.

The discrete stable copula has some interesting limit cases. When $h\to 0$, i.e., $\lambda\to+\infty$, the copula associated to the mixed exponential-discrete stable distribution can be seen as an approximation of the Gumbel copula with dependence parameter $\frac{1}{\alpha}$, i.e.,

$$\begin{aligned}\lim_{\lambda\to\infty}C_{(h,\alpha)}(u_1,u_2) &= \exp\left\{-\left((-\ln(u_1))^{\frac{1}{\alpha}}+(-\ln(u_2))^{\frac{1}{\alpha}}\right)^{\alpha}\right\}\\ &= C^{Gu}(u_1,u_2),\end{aligned}$$

since,

$$\begin{aligned}\lim_{\lambda\to+\infty}\Big(\lambda-\ln\big((\mathrm{e}^{\lambda}-1)u_i+1\big)\Big) &= \lim_{\lambda\to+\infty}\ln\left(\frac{\mathrm{e}^{\lambda}}{(\mathrm{e}^{\lambda}-1)u_i+1}\right)\\ &= \lim_{\lambda\to+\infty}\ln\left(\frac{\frac{\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda}-1}}{u_i+\frac{1}{\mathrm{e}^{\lambda}-1}}\right)\\ &= -\ln(u_i),\end{aligned}$$

for $i = 1, 2$.

When $\lambda \to +\infty$ and $\alpha = 0$, we obtain the comonotonic copula and when $\lambda \to +\infty$ and $\alpha = 1$, we get the independence copula.

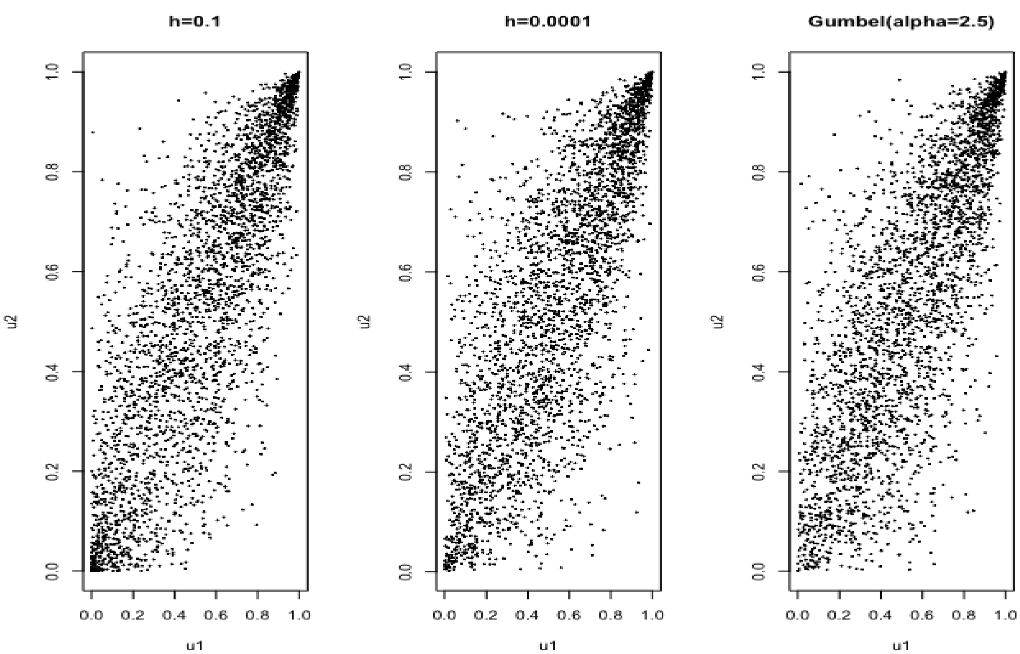


Figure 5.11 – Illustration of the approximation of a Gumbel copula by a discrete stable Copula with $\alpha = 0.4$.

Remark 5.4.4. *In order to generate realizations from the discrete stable copula, we can use Algorithm 5.4.1 where Θ can be sampled using the representation of a discrete stable distribution as a random sum as given in (5.21) where M follows a truncated Poisson distribution. Figure 5.11 represents scaterplots of 3000 realizations generated, using Algorithm 5.4.1, from the discrete stable copula with parameter $\alpha = 0.4$ and different values of h. We can see the convergence of the discrete stable copula to the Gumbel copula when h tends to zero.*

Proposition 5.4.10. *Let C be an n-dimensional discrete stable copula as defined in (5.45). Then, the lower and upper tail-dependence coefficients are given by*

$$\lambda_L = 0 \text{ and } \lambda_U = 2 - 2^{\alpha}.$$

Proof. $\lambda_L = 0$ since Θ is a strictly positive and discrete rv. Concerning the upper tail-dependence coefficient, we have

$$\begin{aligned}
\lambda_U &= 2 - 2\lim_{t\to 0}\frac{\mathcal{L}'_{\Theta}(2t)}{\mathcal{L}'_{\Theta}(t)} \\
&= 2 - 2\lim_{t\to 0}\frac{(1-\mathrm{e}^{-2t})^{\alpha-1}\mathrm{e}^{-t-\lambda(1-\mathrm{e}^{-2t})^{\alpha}+\lambda(1-\mathrm{e}^{-t})^{\alpha}}}{(1-\mathrm{e}^{-t})^{\alpha-1}} \\
&= 2 - 2\lim_{t\to 0}\frac{\left(1-\mathrm{e}^{-t}\right)^{1-\alpha}}{\left(1-\mathrm{e}^{-2t}\right)^{1-\alpha}} \\
&= 2 - 2\times 2^{\alpha-1} = 2 - 2^{\alpha}.
\end{aligned}$$

This result still holds when $\lambda \to \infty$, i.e., $\lambda_L = 0$ and $\lambda_U = 2 - 2^{\alpha}$, since the resulting copula is the Gumbel copula with parameter $\frac{1}{\alpha}$. □

Once again, we will use concordance ordering as defined in Definition 5.4.1 to analyze the dependence strength of the discrete stable copula.

Proposition 5.4.11. *Let C_{λ,α_1} and C_{λ,α_2} be two d-dimensional discrete stable copulas with the same first parameter λ and respective second parameters α_1, α_2, Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. Then*

$$C_{\lambda,\alpha_1} \prec_c C_{\lambda,\alpha_2} \text{ if } \alpha_2 \leq \alpha_1.$$

In this case, we have $\tau_1 \leq \tau_2$, $\rho_S^{(1)} \leq \rho_S^{(2)}$, and $\lambda_1 \leq \lambda_2$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the generators of C_{λ,α_1} and C_{λ,α_2} respectively. In order for $C_{\lambda,\alpha_1} \prec_c C_{\lambda,\alpha_2}$ to be true, we have to show that $\left(\mathcal{L}_1^{-1}\circ\mathcal{L}_2\right)'$ is a completely monotone function. Then, for $x \in (0,\infty)$, we have

$$\begin{aligned}
\mathcal{L}_1^{-1}\circ\mathcal{L}_2(t) &= -\ln\left(1-\left(1-\mathrm{e}^{-t}\right)^{\frac{\alpha_2}{\alpha_1}}\right) \qquad (5.47)\\
&= -\ln\left(\mathcal{L}_N(t)\right),
\end{aligned}$$

where $N \sim Sibuya\left(\frac{\alpha_2}{\alpha_1}\right)$ if $\alpha_2 \leq \alpha_1$. It is well known that $(-\ln(\mathcal{L}_N))'$ is completely monotone if and only if $\mathcal{L}_N^m$ is the LST of a positive rv for all $m \in [0, \infty)$ (see Joe (1997)). We have

$$
\begin{aligned}
\mathcal{L}_N^m(t) &= \Big(1-(1-e^{-t})^{\alpha}\Big)^m \\
&= \sum_{k=0}^{+\infty}\binom{m}{k}(-1)^k(1-e^{-t})^{\alpha k} \\
&= \sum_{k=0}^{+\infty}\binom{m}{k}(-1)^k\sum_{j=0}^{+\infty}\binom{\alpha k}{j}(-1)^j e^{-jt} \\
&= \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\binom{\alpha k}{j}\binom{m}{k}(-1)^{k+j}e^{-jt} \\
&= \sum_{j=0}^{+\infty}p_j e^{-jt},
\end{aligned}
$$

with $\alpha = \frac{\alpha_2}{\alpha_1}$, $p_j = \sum_{k=1}^{\infty}\binom{m}{k}\binom{\alpha k}{j}(-1)^{k+j}$, $p_j \geq 0$, $j \in \{0, 1, \ldots, \}$, and $\sum_{j=0}^{\infty} p_j = 1$ (see, e.g., Hofert (2010) page 95 for proof). Then, $\mathcal{L}_N^m$ is the LST of a positive rv for all $m \in [0, \infty)$. We conclude that $C_{\lambda,\alpha_1} \prec_c C_{\lambda,\alpha_2}$ where $\alpha_2 \leq \alpha_1$. □

5.4.3 Multivariate mixed Exponential - Discrete Linnik distribution

Consider the positive and discrete mixing rv $\Theta \sim Linnik(\alpha, \beta, \gamma)$ with LST as in (5.29). Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n rvs such that, given $\Theta = \theta$, X_i, $i = 1, \ldots, n$ are conditionally independent and distributed as $Exp(\theta)$. Then, combining (5.29) and (5.37), the joint survival function of $\underline{X}$ is given by

$$
\Pr(X_1 > x_1, \ldots, X_n > x_n) = \left(\frac{1-\left(1-\mathrm{e}^{-\sum_{i=1}^n x_i}\right)^{\alpha}}{1+\beta\left(1-\mathrm{e}^{-\sum_{i=1}^n x_i}\right)^{\alpha}}\right)^{\gamma},\ x_i > 0,\ i = 1, \ldots, n. \tag{5.48}
$$

Similarly to the univariate case, properties related to this multivariate mixed distribution can only be obtained numerically.

When $\gamma = 1$, we obtain the multivariate mixed exponential-discrete Mittag-Leffler distribution.

We will construct a new multivariate Archimedean copula using a mixed exponential-discrete Linnik distribution.

Proposition 5.4.12. *Let Θ be a rv with a discrete Linnik distribution with LST as given in (5.29). The associated multivariate Archimedean copula with generator $\mathcal{L}_\Theta$, called the* *multivariate discrete Linnik copula, is given by*

$$C_{\alpha,\beta,\gamma}\left(u_1,\ldots,u_n\right)=\left(\frac{1-\left(1-\prod_{i=1}^{n}x_i\right)^{\alpha}}{1-\beta\left(1-\prod_{i=1}^{n}x_i\right)^{\alpha}}\right)^{\gamma}, \tag{5.49}$$

where $x_i=\left(1-\left(\frac{1-u_i^{\frac{1}{\gamma}}}{1+\beta u_i^{\frac{1}{\gamma}}}\right)^{\frac{1}{\alpha}}\right)$.

Proof. The result is obtained by combining (5.29) and (5.38). □

Proposition 5.4.13. *Let C be an n-dimensional discrete Linnik copula as defined in (5.49). Then, the lower and upper tail-dependence coefficients are given by*

$$\lambda_L=0 \textit{ and } \lambda_U=2-2^{\alpha}.$$

Proof. Since the mixing rv underlying the Archimedean copula C is a strictly positive discrete rv, then $\lambda_L=0$. For the upper tail dependence coefficient, we have

$$\begin{aligned}
\lambda_U &= 2-2\lim_{t\to 0}\frac{\mathcal{L}'_{\Theta}(2t)}{\mathcal{L}'_{\Theta}(t)}\\
&= 2-2\lim_{t\to 0}\frac{\left(1-\mathrm{e}^{-2x}\right)^{\alpha}\mathrm{e}^{-2x}\left(1+\beta\left(1-\mathrm{e}^{-x}\right)^{\alpha}\right)\left(-1+\mathrm{e}^{-x}\right)\left(-1+\left(1-\mathrm{e}^{-x}\right)^{\alpha}\right)}{\left(1+\beta\left(1-\mathrm{e}^{-2x}\right)^{\alpha}\right)\left(-1+\mathrm{e}^{-2x}\right)\left(-1+\left(1-\mathrm{e}^{-2x}\right)^{\alpha}\right)\left(1-\mathrm{e}^{-x}\right)^{\alpha}\mathrm{e}^{-x}}\frac{\left(\frac{1-\left(1-\mathrm{e}^{-2x}\right)^{\alpha}}{1+\beta\left(1-\mathrm{e}^{-2x}\right)^{\alpha}}\right)^{\gamma}}{\left(\left(\frac{1-\left(1-\mathrm{e}^{-x}\right)^{\alpha}}{1+\beta\left(1-\mathrm{e}^{-x}\right)^{\alpha}}\right)^{\gamma}\right)}\\
&= 2-2\lim_{t\to 0}\frac{\left(1-\mathrm{e}^{-x}\right)^{1-\alpha}}{\left(1-\mathrm{e}^{-2x}\right)^{1-\alpha}}\\
&= 2-2\lim_{t\to 0}\left(\frac{\frac{1-\mathrm{e}^{-2x}}{1+\mathrm{e}^{-x}}}{1-\mathrm{e}^{-2x}}\right)^{1-\alpha}\\
&= 2-2\times 2^{\alpha}-1=2-2^{\alpha}.
\end{aligned}$$

□

Proposition 5.4.14. *Let C_{λ,α_1} and C_{λ,α_2} be two d-dimensional discrete Linnik copulas with the same last parameters β and γ and respective first parameters α_1, α_2, Kendall taus τ_1, τ_2, Spearman rhos $\rho_S^{(1)}$, $\rho_S^{(2)}$, tail dependence parameters λ_1, λ_2. Then*

$$C_{\alpha_1,\beta,\gamma}\prec_c C_{\alpha_2,\beta,\gamma} \textit{ if } \alpha_2\leq\alpha_1.$$

In this case, we have $\tau_1\leq\tau_2$, $\rho_S^{(1)}\leq\rho_S^{(2)}$, and $\lambda_1\leq\lambda_2$.

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the generators of $C_{\alpha_1,\beta,\gamma}$ and $C_{\alpha_2,\beta,\gamma}$ respectively. In order for $C_{\alpha_1,\beta,\gamma} \prec_c C_{\alpha_1,\beta,\gamma}$ to be true, we have to show that $\left(\mathcal{L}_1^{-1} \circ \mathcal{L}_2\right)'$ is a completely monotone function. Then, for $t \in (0, \infty)$, we have

$$\mathcal{L}_1^{-1} \circ \mathcal{L}_2(t) \quad = \quad -\ln\left(1 - \left(1 - \mathrm{e}^{-t}\right)^{\frac{\alpha_2}{\alpha_1}}\right),$$

which is of the same form as (5.47). Therefore, $C_{\alpha_1,\beta,\gamma} \prec_c C_{\alpha_1,\beta,\gamma}$ if $\alpha_2 \leq \alpha_1$. □

Joe's copula with parameter $\frac{1}{\alpha}$ is obtained if $\beta \to 0$ and $\gamma \to 1$. Depending on the chosen parameterization, several particular cases can be found. For example, using the result of Proposition 5.3.7, we can obtain, as a limit case, the Archimedean copula based on the gamma-stopped-Sibuya LST (see Joe (2014) page 216 for more details). Using Proposition 5.3.8, copulas based on the Mittag-leffler LST can be derived as limit cases of the discrete Linnik copula (see, e.g., BB1 copula in Joe (2014) page 190 for more details).

5.4.4 Compound Poisson risk models with exchangeable claim amounts

We consider a variant of the risk model described in Section 5.3.5 in which we assume that the univariate rv W follows a univariate mixed exponential distribution. This means that $\underline{N}$ is here a compound Poisson process. We also assume that $(X_1, X_2, ..., X_k)$ follows a multivariate mixed exponential distribution as defined in Section 5.4, for $k \in \{2, 3, ...\}$ and $x_1, ..., x_k \geq 0$. The joint survival function of $(W_1, W_2, ..., W_k)$ is defined as in (5.36) and (5.37), for $k \in \{2, 3, ...\}$. See Section 2 of Albrecher et al. (2011) for details on this class of risk models. In this context, let us define as before a strictly positive rv Θ such that, given $\Theta = \theta$, the X_j's ($j \geq 1$) are exponentially distributed with parameter θ and conditionally independent with joint conditional distribution (5.36). Let ζ_θ be the conditional ruin probability associated to the corresponding classical compound Poisson risk model which assumes independent and exponentially distributed claim amounts. Then, ζ_θ can be written as

$$\zeta_\theta(u) = \min\left\{\frac{\lambda}{\theta\, c} \exp\left\{-\left(\theta - \frac{\lambda}{c}\right) u\right\}; 1\right\},$$

for $u \geq 0$. The unconditional ruin probability can therefore be represented as a mixture with mixing rv Θ as follows

$$\zeta(u) = \int_0^\infty \zeta_\theta(u) \mathrm{d}F_\Theta(\theta). \tag{5.50}$$

The net profit condition is violated when the mixing rv Θ takes a value smaller that $\theta_0 > 0$. We define θ_0 such that $\zeta_\theta(u) = 1$ for all $u \geq 0$ ($\theta_0 = \frac{\lambda}{c}$). Then (5.50) becomes

$$\zeta(u) = F_\Theta(\theta_0) + \int_{\theta_0}^\infty \zeta_\theta(u) \mathrm{d}F_\Theta(\theta). \tag{5.51}$$

In particular, we have

$$\lim_{u \to \infty} \zeta(u) = F_{\Theta}(\theta_0). \tag{5.52}$$

Note that in Section 2 of Albrecher et al. (2011) examples of explicit formulas for the ruin probability as in (5.50) are given. In all the three provided examples, they have considered the mixing rv Θ to be continuous (gamma, Pareto and a continuous distribution with a Pareto-type tail). In the following examples, we propose explicit ruin formulas based on a discrete mixing rv Θ. In this case, the ruin probability in (5.51) becomes

$$\zeta(u) = F_{\Theta}(\theta_0) + \sum_{\theta=\theta_0+1}^{\infty} \zeta_{\theta}(u) \Pr(\Theta = \theta). \tag{5.53}$$

Example 5.4.1. *(Multivariate mixed Exponential - Negative Binomial distribution)*
Let $(X_1, X_2, ..., X_k)$ ($k \in \{2, 3, ...\}$) follow a k-dimensional multivariate mixed Exponential - Negative Binomial distribution as described in Section 5.4.1 with joint survival function as given in (5.39). The corresponding model in this case is the negative binomial copula proposed in Section 5.4.1 with specific univariate mixed Exponential - Negative Binomial distribution marginals as defined in Section 5.3.1. From (5.53), we obtain

$$\zeta(u) = \begin{cases} \xi + \frac{\theta_0}{\theta_0+1} \mathrm{e}^{-u} q^r p^{\theta_0+1-r} \binom{\theta_0}{r-1} {}_3F_2\left([1, \theta_0+1, \theta_0+1]; [\theta_0+2, \theta_0+2-r]; p\,\mathrm{e}^{-u}\right), & u > 0 \\ \xi + \frac{\theta_0}{\theta_0+1} q^r p^{\theta_0+1-r} \binom{\theta_0}{r-1} {}_3F_2\left([1, \theta_0+1, \theta_0+1]; [\theta_0+2, \theta_0+2-r]; p\right), & u = 0 \end{cases},$$

where $p = 1 - q$ and $\xi = F_{\Theta}(\theta_0) = 1 - \binom{\theta_0}{r-1} q^r (1-q)^{\theta_0+1-r} {}_2F_1\left([1, \theta_0+1]; [2+\theta_0-r]; 1-q\right)$. Also,

$$\lim_{u \to \infty} \zeta(u) = 1 - \binom{\theta_0}{r-1} q^r p^{\theta_0+1-r} {}_2F_1\left([1, \theta_0+1]; [2+\theta_0-r]; p\right),$$

with $\theta_0 = \frac{\lambda}{c}$.

□

Example 5.4.2. *(Multivariate mixed Exponential - Geometric distribution)*

Let $(X_1, X_2, ..., X_k)$ ($k \in \{2, 3, ...\}$) follow a k-dimensional multivariate mixed Exponential - Geometric distribution as described in Section 5.4.1 (see Proposition 5.4.3) with joint survival function as given in (5.40). In this case, the corresponding dependence structure is the AMH copula (with parameter $\alpha = 1 - q$) and the marginal distributions, given by

$$\overline{F}_{X_i}(x) = \mathcal{L}_{\Theta}(x) = \frac{q}{\mathrm{e}^x - (1-q)}, \; i = 1, \ldots, n,$$

correspond to univariate mixed Exponential-Geometric distributions (see Section 5.3.1 and Adamidis and Loukas (1998) for more details).

From (5.53), it follows that, for this model, the ruin probability is given by

$$\zeta(u) = \begin{cases} 1-(1-q)^{\theta_0} + q\,\mathrm{e}^{-u}\,\theta_0\,(1-q)^{\theta_0}\,\Phi\left((1-q)\mathrm{e}^{-u},1,\theta_0+1\right), & u>0 \\ 1-(1-q)^{\theta_0} + q\,\theta_0\,(1-q)^{\theta_0}\,\Phi\left((1-q),1,\theta_0+1\right), & u=0 \end{cases},$$

where $\Phi(z,a,v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^a}$ *denotes the Lerch Phi function (also called the Lerch transcendent function).*

We can easily verify (also from (5.52)) that

$$\lim_{u\to\infty} \zeta(u) = 1-(1-q)^{\theta_0},$$

where $\theta_0 = \frac{\lambda}{c}$.

□

Example 5.4.3. *(Multivariate mixed Exponential - Sibuya distribution)*

Let $(X_1, X_2, ..., X_k)$ $(k \in \{2,3,...\})$ *follow a k-dimensional multivariate mixed Exponential - Sibuya distribution as described in Section 5.6.3 with joint survival function as given in (5.63). The corresponding model in this case is the Joe copula (with parameter* $\frac{1}{\alpha}$*) with specific mixed Exponential - Sibuya distribution marginals as defined in Section 5.6.1. Then, from (5.53), we obtain*

$$\zeta(u) = \begin{cases} 1-\frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha} + \frac{\theta_0(-1)^{\theta_0}\binom{\alpha}{\theta_0+1}\mathrm{e}^{-u}{}_3F_2\left([1,1+\theta_0,1+\theta_0-\alpha];[2+\theta_0,2+\theta_0];\mathrm{e}^{-u}\right)}{\theta_0+1}, & u>0 \\ 1-\frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha} + \frac{\theta_0(-1)^{\theta_0}\binom{\alpha}{\theta_0+1}{}_3F_2([1,1+\theta_0,1+\theta_0-\alpha];[2+\theta_0,2+\theta_0];1)}{\theta_0+1}, & u=0 \end{cases}.$$

Also,

$$\lim_{u\to\infty} \zeta(u) = 1-\frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha},$$

where $\theta_0 = \frac{\lambda}{c}$.

□

Example 5.4.4. *(Multivariate mixed Exponential - Logarithmic distribution)*

Let $(X_1, X_2, ..., X_k)$ $(k \in \{2, 3, ...\})$ *follow a* k*-dimensional multivariate mixed Exponential - Logarithmic distribution as described in Section 5.6.4 with joint survival function as given in (5.65) (see Proposition 5.6.5 for more details concerning such a distribution). As discussed before, the dependence structure underlying this multivariate distribution corresponds to the Frank copula with parameter* $\alpha = -\ln(1-p)$*. In this case, the marginal distribution of* X_i*,* $i = 1, \ldots, n$*, given by*

$$\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = \frac{\ln\left(1 - p\,\mathrm{e}^{-x}\right)}{\ln(1-p)},\ i = 1, \ldots, n,$$

corresponds to the univariate mixed Exponential - Logarithmic distribution proposed by Tahmasbi and Rezaei (2008).

From (5.53), we obtain

$$\zeta(u) = \begin{cases} 1 + \frac{B(p,\theta_0+1,0)}{\ln(1-p)} - \frac{\theta_0\,\mathrm{e}^{-u}\,p^{\theta_0+1}}{\ln(1-p)}\,\Phi\left(p\,\mathrm{e}^{-u}, 2, \theta_0+1\right), & u > 0 \\ 1 + \frac{B(p,\theta_0+1,0)}{\ln(1-p)} - \frac{\theta_0\,p^{\theta_0+1}}{\ln(1-p)}\,\Phi\left(p, 2, \theta_0+1\right), & u = 0 \end{cases},$$

where B *is the incomplete beta function. Clearly,*

$$\lim_{u\to\infty} \zeta(u) = 1 + \frac{B(p, \theta_0 + 1, 0)}{\ln(1-p)},$$

with $\theta_0 = \frac{\lambda}{c}$*.*

□

5.5 Renewal risk models with exchangeable inter-claim times

In this section, we consider the class of renewal risk models with exchangeable inter-claim times, which can be seen as extensions to the classical renewal risk model described in Section 5.3.5. Albrecher et al. (2011) and Cossette et al. (2018b) also discussed the continuous-time renewal risk model with exchangeable inter-claim times. In this class of renewal risk models, the claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is a mixed renewal process where the inter-claim times $\underline{W} = \{W_j, j \in \mathbb{N}\}$ form a sequence of exchangeable and strictly positive real-valued rvs.

In the present class of risk models, $(W_1, W_2, ..., W_k)$ follows a multivariate mixed exponential distribution as defined in Section 5.4, for $k \in \{2, 3, ...\}$ and $x_1, ..., x_k \geq 0$. The joint survival function of $(W_1, W_2, ..., W_k)$ is defined as in (5.37), for $k \in \{2, 3, ...\}$. Due to this assumption on the inter-claim times, the claim number process $\underline{N}$ can be also seen as a mixed Poisson process, which have found many applications in actuarial science (see, e.g., Grandell (1997) and Albrecher et al. (2017)).

From Cossette et al. (2018b), the expression for the ruin probability $\zeta(u)$ is given by

$$\zeta(u) = \overline{F}_{\Theta}(\theta_0) + \int_0^{\theta_0} \zeta_{\theta}(u) \mathrm{d}F_{\Theta}(\theta), \tag{5.54}$$

where $u \geq 0$ and $\theta_0 = \frac{c}{E[X_i]}$. The security loading is violated when $\theta > \theta_0$. In particular, we have

$$\lim_{u \to \infty} \zeta(u) = \overline{F}_{\Theta}(\theta_0). \tag{5.55}$$

When the mixing rv Θ follows a discrete distribution as in Section 5.3, the expression for the ruin probability in (5.54) becomes

$$\zeta(u) = \overline{F}_{\Theta}(\theta_0) + \sum_{\theta=1}^{\theta_0} \zeta_{\theta}(u) \Pr(\Theta = \theta). \tag{5.56}$$

As in Section 5.3.5, we suppose that the claim amount is exponentially distributed with parameter β. Then, $\theta_0 = c\beta$ and the conditional ruin probability given in (5.56) becomes

$$\zeta_{\theta}(u) = \min\left\{ \frac{\theta}{\beta c} \exp\left\{ -\left(\beta - \frac{\theta}{c}\right) u \right\}; 1 \right\}, \; u \geq 0. \tag{5.57}$$

Example 5.5.1. *We consider the following multivariate mixed exponential distributions for* $(W_1, W_2, ..., W_k)$, *for* $k \in \{2, 3, ...\}$.

(a) Multivariate mixed Exponential - Negative Binomial distribution:

$$\psi(u) = \begin{cases} \xi + \frac{r\, q^r \, \mathrm{e}^{-(\theta_0+1)\frac{u}{c}}}{\theta_0 \left(\mathrm{e}^{-\frac{u}{c}} - 1 + q\right)^{r+1}} - \frac{q^r(\theta_0+1)(1-q)^{n-r+1}\binom{\theta_0}{r-1}\mathrm{e}^{\frac{u}{c}}\, {}_2F_1\left([1,\theta_0+2];[2+\theta_0-r];\mathrm{e}^{\frac{u}{c}}(1-q)\right)}{\theta_0}, & u > 0 \\ \xi + \frac{r}{q\,\theta_0} - \frac{q^r(\theta_0+1)(1-q)^{n-r+1}\binom{\theta_0}{r-1}{}_2F_1([1,\theta_0+2];[2+\theta_0-r];1-q)}{\theta_0}, & u = 0 \end{cases},$$

where $\xi = \overline{F}_{\Theta}(\theta_0) = \binom{\theta_0}{r-1} q^r (1-q)^{\theta_0+1-r} {}_2F_1\left([1, \theta_0+1]; [2+\theta_0-r]; 1-q\right)$. *Also,*

$$\lim_{u \to \infty} \psi(u) = \binom{\theta_0}{r-1} q^r (1-q)^{\theta_0+1-r} {}_2F_1\left([1, \theta_0+1]; [2+\theta_0-r]; 1-q\right).$$

(b) Multivariate mixed Exponential - Geometric distribution (i.e., $\Theta \sim$ *Geometric*(q)*):*

$$\psi(u) = \begin{cases} (1-q)^{\theta_0} - \frac{\left(\left(\theta_0(q-1)\mathrm{e}^{\frac{u}{c}} + \theta_0 + 1\right)(1-q)^{\theta_0}\mathrm{e}^{\frac{(\theta_0+1)u}{c}} - \mathrm{e}^{\frac{u}{c}}\right) q \mathrm{e}^{-\frac{\theta_0 u}{c}}}{\theta_0\left(1+(q-1)\mathrm{e}^{\frac{u}{c}}\right)^2}, & u > 0 \\ (1-q)^{\theta_0} - \frac{\left((\theta_0(q-1)+\theta_0+1)(1-q)^{\theta_0} - 1\right) q}{\theta_0(1+(q-1))^2}, & u = 0 \end{cases},$$

and

$$\lim_{u\to\infty}\psi(u) = (1-q)^{\theta_0}.$$

(c) *Multivariate mixed Exponential - Sibuya distribution:*

$$\psi(u) = \begin{cases} \xi + \frac{\alpha}{\theta_0}\left(1-\mathrm{e}^{\frac{u}{c}}\right)^{\alpha-1}\mathrm{e}^{-\frac{u(\theta_0-1)}{c}} - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}\mathrm{e}^{\frac{u}{c}}\,{}_2F_1\left([1,\theta_0+1-\alpha];[\theta_0+1];\mathrm{e}^{\frac{u}{c}}\right)}{\theta_0}, & u>0 \\ \xi - \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}{}_2F_1([1,\theta_0+1-\alpha];[\theta_0+1];1)}{\theta_0}, & u=0 \end{cases},$$

where $\xi = \overline{F}_\Theta(\theta_0) = \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha}$. *In particular, we have*

$$\lim_{u\to\infty}\psi(u) = \frac{(-1)^{\theta_0}(\theta_0+1)\binom{\alpha}{\theta_0+1}}{\alpha}.$$

(d) *Multivariate mixed Exponential - Logarithmic distribution:*

$$\psi(u) = \begin{cases} -\frac{B(p,\theta_0+1,0)}{\ln(1-p)} + \frac{p\,\mathrm{e}^{\frac{u(1-\theta_0)}{c}} - p^{\theta_0+1}\,\mathrm{e}^{\frac{u}{c}}}{\theta_0\ln(1-p)\left(p\mathrm{e}^{\frac{u}{c}}-1\right)}, & u>0 \\ -\frac{B(p,\theta_0+1,0)}{\ln(1-p)} + \frac{p-p^{\theta_0+1}}{\theta_0\ln(1-p)(p-1)}, & u=0 \end{cases},$$

and

$$\lim_{u\to\infty}\psi(u) = -\frac{B(p,\theta_0+1,0)}{\ln(1-p)}.$$

□

Example 5.5.2. *Let* $X \sim Exp(1)$ *and* $(W_1,\ldots,W_k)$, *for* $k=1,2,\ldots$, *follow a multivariate mixed exponential distribution with* $E[W_k]=1$, *for* $k=1,2,\ldots$. *We consider three cases for the mixing distribution, namely, the negative binomial distribution, the Sibuya distribution, and the logarithmic distribution. Let the parameters be the same as in Example 5.3.1 and* $c=1.25$. *Figures 5.12, 5.13, and 5.14 illustrate the significant impact of the dependence relationship between inter-claim times on the overall portfolio.*

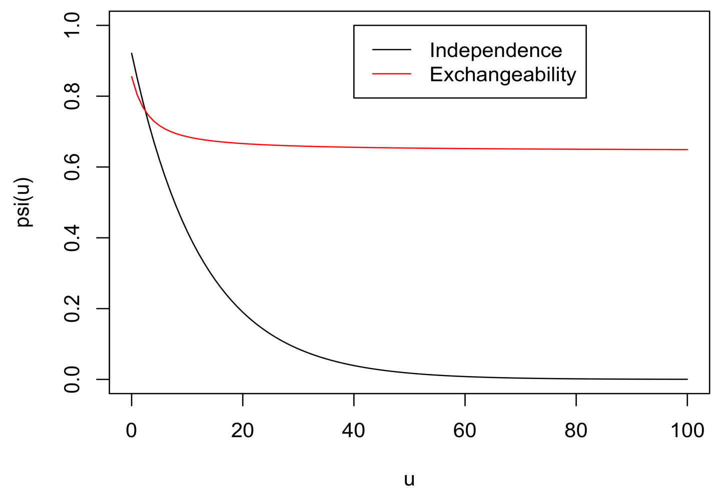


Figure 5.12 – Ruin probability for an exp-negative binomial distribution with $r = 2$ and $h = 0.0001$

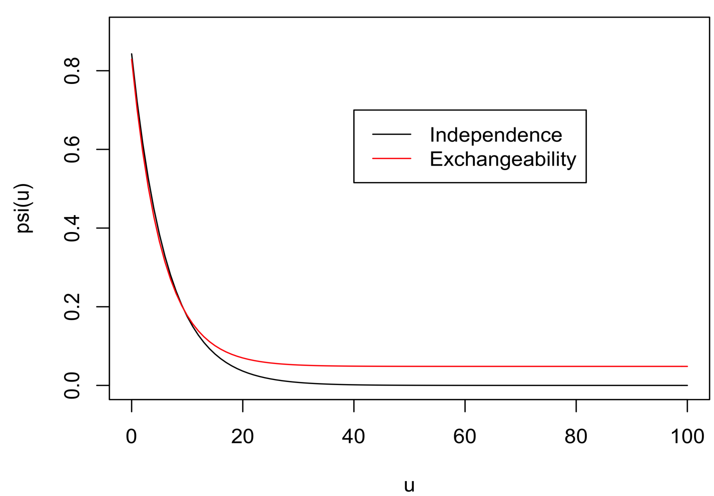


Figure 5.13 – Ruin probability for an exp-logarithmic distribution with $\alpha = 0.1$

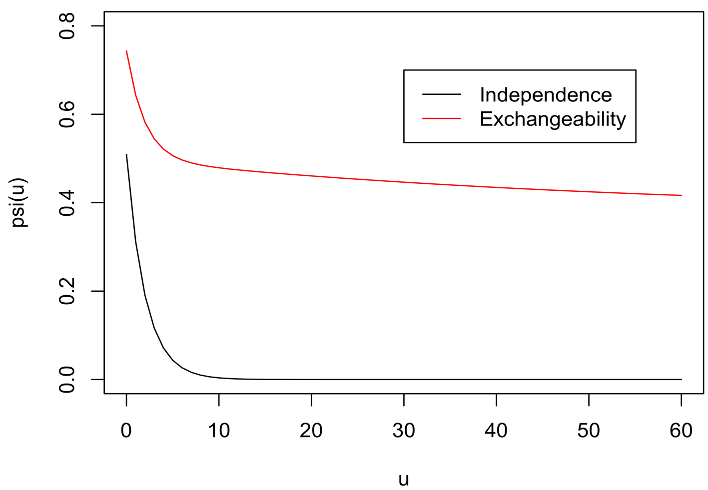


Figure 5.14 – Ruin probability for an exp-Sibuya distribution with $\alpha = 0.5$

5.6 Appendix

5.6.1 Univariate mixed Exponential - Sibuya distribution

The Sibuya distribution was first introduced by Masaaki Sibuya in Sibuya (1979) and later studied by several reserchers (see, e.g., Devroye (1993), Christoph and Schreiber (1998) and Christoph and Schreiber (2000)). This discrete distribution has some interesting properties such as self-decomposability and infinite divisibility and can be generalized to a distribution with two or three parameters, often called scaled Sibuya or generalized Sibuya distribution (see, e.g., Christoph and Schreiber (2000) and Huillet (2016)). Also, Devroye (1993) derived a distributional representation of the Sibuya distribution useful for sampling observations from such a distribution.

Let Θ be a discrete rv with a Sibuya distribution with parameter $\alpha \in (0,1]$, i.e., $\Theta \sim Sibuya(\alpha)$, and pgf given by

$$P_\Theta(t) = 1 - (1-t)^\alpha. \tag{5.58}$$

The pmf and LST of Θ are respectively given by

$$\Pr(\Theta = k) = (-1)^{k-1}\binom{\alpha}{k},\ k \in \mathbb{N},$$

and

$$\mathcal{L}_\Theta(t) = 1 - \left(1 - \mathrm{e}^{-t}\right)^\alpha. \tag{5.59}$$

Let X be a positive rv with univariate mixed exponential-Sibuya distribution derived from a discrete mixing rv Θ with LST as in (5.59). Combining (5.8) and (5.59), the univariate survival function of X can be written as

$$\bar{F}_X(x) = 1 - \left(1 - \mathrm{e}^{-x}\right)^\alpha. \tag{5.60}$$

Proposition 5.6.1. *Let the rv X follow a mixed exponential-Sibuya distribution with univariate survival function as defined in (5.60). Then, the following properties hold:*

1. $\mathcal{L}_X(t) = \dfrac{\alpha\Gamma(1+t)\Gamma(\alpha)}{\Gamma(t+1+\alpha)},\ t > 0$;
2. $f_X(x) = \alpha\left(1 - \mathrm{e}^{-x}\right)^{\alpha-1}\mathrm{e}^{-x},\ x \in \mathbb{R}^+$;
3. $h_X(x) = \dfrac{\alpha\mathrm{e}^{-x}}{\left(1 - (1-\mathrm{e}^{-x})^\alpha\right)}\left(1 - \mathrm{e}^{-x}\right)^{\alpha-1},\ x \in \mathbb{R}^+$;
4. $E[X] = \alpha\, {}_3F_2\left(1, 1, 1-\alpha;\ 2, 2;\ 1\right)$;
5. $E[X^n] = \alpha\Gamma(n+1)\, {}_{n+2}F_{n+1}\left(1, \ldots, 1, 1-\alpha;\ 2, \ldots, 2;\ 1\right),\ n = 1, 2, \ldots$;
6. $VaR_\kappa(X) = -\ln\left(1 - \kappa^{\frac{1}{\alpha}}\right),\ \kappa \in (0,1)$;

7. $TVaR_\kappa(X) = \frac{\alpha \ln(\kappa_\alpha)\kappa_\alpha - \kappa_\alpha}{(\kappa-1)} {}_4F_3\left(1,1,1-\alpha, \frac{2\ln(\kappa_\alpha)-1}{\ln(\kappa_\alpha)}; 2,2, \frac{\ln(\kappa_\alpha)-1}{\ln(\kappa_\alpha)}; \kappa_\alpha\right)$, *where* $\kappa_\alpha = 1-\kappa^{\frac{1}{\alpha}}$, $\kappa \in (0,1)$.

Proof. All expressions follow directly from Proposition 5.3.1. □

5.6.2 Univariate mixed Exponential - logarithmic distribution

Let Θ be a discrete rv with a logarithmic distribution with parameter $p \in (0,1]$, i.e., $\Theta \sim Log(p)$, and pmf

$$\Pr(\Theta = k) = \frac{-p^k}{k\,\ln(1-p)},\ k \in \mathbb{N}.$$

Note that we can also consider another parametrization using $\alpha = -\ln(1-p)$.

The LST of Θ is given by

$$\mathcal{L}_\Theta(t) = \frac{\ln\left(1-p\mathrm{e}^{-t}\right)}{\ln(1-p)},\ x > 0. \tag{5.61}$$

The resulting univariate mixed distribution, first introduced by Tahmasbi and Rezaei (2008), is called the mixed exponential-logarithmic distribution. Let X be a positive rv with univariate mixed exponential-logarithmic distribution derived from a discrete mixing rv Θ with LST as in (5.61). Combining (5.8) and (5.61), the univariate survival function of X can be written as

$$\bar{F}_X(x) = \frac{\ln\left(1-p\mathrm{e}^{-x}\right)}{\ln(1-p)}. \tag{5.62}$$

Proposition 5.6.2. *Let the rv X follow a mixed exponential-Sibuya distribution with univariate survival function as defined in (5.62). Then, the following properties hold:*

1. $\mathcal{L}_X(t) = \frac{-(\Phi(p,1,t)-1/t)}{\ln(1-p)}\ t > 0$;
2. $f_X(x) = \frac{-p\mathrm{e}^{-x}}{\ln(1-p)(1-p\mathrm{e}^{-x})}$, $x \in \mathbb{R}^+$;
3. $h_X(x) = \frac{-p\mathrm{e}^{-x}}{(1-p\mathrm{e}^{-x})\ln(1-p\mathrm{e}^{-x})}$, $x \in \mathbb{R}^+$;
4. $E[X] = \frac{-Li_2(p)}{ln(1-p)}$;
5. $E[X^n] = \frac{-\Gamma(n+1)Li_{n+1}(p)}{ln(1-p)}$, $n = 1, 2, \ldots$;
6. $VaR_\kappa(X) = -\ln\left(\frac{p-1+(1-p)^\kappa}{p}\right) + \kappa\ln(1-p)$, $\kappa \in (0,1)$;

7. $TVaR_\kappa(X) = \dfrac{p\,\mathrm{e}^{-d}(d+1)\,{}_4F_3(1,1,1,(2*d+1)/d;2,2,(d+1)/d;p\mathrm{e}^{-d})}{\ln(1-p)(\kappa-1)}$, *where* $d = VaR_\kappa(X)$ *and* $\kappa \in (0,1)$.

Proof. All expressions follow directly from Proposition 5.3.1. □

5.6.3 Multivariate mixed Exponential - Sibuya distribution

Consider Θ to be a positive distrete rv with a Sibuya distribution, i.e., $\Theta \sim \mathit{Sibuya}(\alpha)$, and LST as in (5.29). Let $\underline{X} = (X_1, \dots, X_n)$ follow a multivariate mixed exponential distribution with mixing rv Θ. Then, combining (5.29) and (5.37), the joint survival function of $\underline{X}$ can be written as

$$\bar{F}_{\underline{X}}(\underline{x}) = 1 - \left(1 - \mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)^{\frac{1}{\alpha}}. \tag{5.63}$$

Proposition 5.6.3. *Let* $\underline{X} = (X_1, \dots, X_n)$ *follow an* n*-dimensional multivariate mixed exponential-Sibuya distribution with joint survival function given in (5.63). Then, the following properties hold:*

1. $f_{\underline{X}}(\underline{x}) = \sum_{k=1}^{m} \left(1 - \mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)^{\frac{1}{\alpha}} \left(\dfrac{\mathrm{e}^{-\sum_{i=1}^{m} x_i}}{\alpha\left(1 - \mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)}\right)^{k} S_2(m,k) \sum_{j=1}^{k} (-1)^{k-1} S_1(k, k+1-j)\,\alpha^{j-1}$.
2. *Joint moments* $E[X_1^{n_1} \dots X_m^{n_m}] = \prod_{i=1}^{m} \dfrac{\Gamma(n_i+1)}{\alpha}\, {}_{d+2}F_{d+1}\left(1, \dots, 1, \dfrac{-1+\alpha}{\alpha};\, 2, \dots, 2;\, 1\right)$ *with* $d = n_1 + \dots, n_m$.
3. $Cov\,(X_1, X_2) = \dfrac{1}{\alpha^2}\left({}_4F_3\left(1,1,1,\dfrac{-1+\alpha}{\alpha};\, 2,2,2;\, 1\right)\alpha - \left({}_3F_2\left(1,1,\dfrac{-1+\alpha}{\alpha};\, 2,2;\, 1\right)\right)^2\right)$.
4. $\rho_P\,(X_1, X_2) = \dfrac{{}_4F_3(1,1,1,\frac{\alpha-1}{\alpha};\, 2,2,2;\, 1)\alpha - \left({}_3F_2(1,1,\frac{\alpha-1}{\alpha};\, 2,2;\, 1)\right)^2}{2\,{}_4F_3(1,1,1,\frac{\alpha-1}{\alpha};\, 2,2,2;\, 1)\alpha - \left({}_3F_2(1,1,\frac{\alpha-1}{\alpha};\, 2,2;\, 1)\right)^2}$.
5. *Kendall's tau* $\tau\,(X_1, X_2) = 1 - 4\sum_{k=1}^{\infty}\left(k\,(\alpha k+2)(\alpha(k-1)+2)\right)^{-1}$.

where S_1 *and* S_2 *denote the Stirling number of the first and the second kinds respectively.*

Proof. For property 5, see, e.g., Hofert (2010). For all other properties, the expessions are derived directly from their definitions. □

Using the link between multivariate mixed exponential-Sibuya distribution and Archimedean copulas, the resulting copula is the well known Joe copula.

Proposition 5.6.4. *Let Θ be a discrete rv with Sibuya distribution with LST as given in (5.29). The associated multivariate Archimedean copula with generator $\mathcal{L}_\Theta$, called the Joe copula with parameter $\frac{1}{\alpha}$, is given by*

$$C_{\frac{1}{\alpha}}(u_1,\ldots,u_n) = 1 - \left(1 - \prod_{i=1}^{n}\left(1-(1-u_i)^{\frac{1}{\alpha}}\right)\right)^{\alpha}, \tag{5.64}$$

for $u_i \in [0,1]$, $\forall i \in \{1,\ldots,n\}$, and $\alpha \in [0,1]$.

Proof. See Joe (2014) for proof and more details concerning such a copula. □

5.6.4 Multivariate mixed Exponential - Logarithmic distribution

In this section, we consider the mixing rv Θ to follow a logarithmic ditribution with parameter p and pmf

$$\Pr(\Theta = k) = \frac{-p^k}{k\ \ln(1-p)},\ k \in \mathbb{N}.$$

Note that we can also consider another parametrization using $\alpha = -\ln(1-p)$.

The resulting univariate mixed distribution, first introduced by Tahmasbi and Rezaei (2008), is called the mixed exponential-logarithmic distribution. In this section, we propose a multivariate extension of such a distribution. Let $\underline{X} = (X_1,\ldots,X_n)$ be an n-dimensional random vector. The dependence between the rvs $X_1,\ldots,X_n$ is introduced via the mixing rv Θ as described in the begining of Section 5.4. Then, from (5.37), the joint survival function of $\underline{X}$ can be written as

$$\bar{F}_{\underline{X}}(\underline{x}) = -\frac{\ln\left(1 - p\mathrm{e}^{-\sum_{i=1}^{m} x_i}\right)}{-\ln(1-p)}. \tag{5.65}$$

Proposition 5.6.5. *Let $\underline{X} = (X_1,\ldots,X_n)$ follow an n-dimensional multivariate mixed exponential logarithmic distribution with joint survival function given in (5.65). Then, the following properties hold:*

1. *Joint density function* $f_{\underline{X}}(x_1,\cdots,x_m) = \sum_{k=1}^{m}\frac{1}{k}\left(\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}i^m\right)\frac{-p^k\mathrm{e}^{-k\sum_{i=1}^{m}x_i}}{\ln(1-p)\left(1-p\mathrm{e}^{-\sum_{i=1}^{m}x_i}\right)^k}.$
2. *Joint moments* $E[X_1^{n_1}\ldots X_m^{n_m}] = \prod_{i=1}^{m}\Gamma(n_i+1)\times\frac{Li_d(p)}{-\ln(1-p)}$, *where* $d = n_1+\ldots+n_m+1$.
3. *Covariance* $Cov(X_1,X_2) = \frac{-Li_3(p)\ln(1-p) - (Li_2(p))^2}{\ln(1-p)^2}.$

4. *Pearson's correlation coefficient* $\rho(X_1, X_2) = \dfrac{Li_3(p)\ln(1-p)(Li_2(p))^2}{2\,Li_3(p)\ln(1-p)(Li_2(p))^2}$.
5. *Kendall's tau* $\tau = 1+4\frac{D_1(-\ln(1-p))-1}{-\ln(1-p)}$, $D_1(\alpha) = \frac{1}{\alpha}\int_0^{\alpha} t\,(\mathrm{t}-1)^{-1}\mathrm{d}t$*: "The Debye function of order one".*

Proof. For property 5, see, e.g., Hofert (2010). For all other properties, the expressions are derived directly from their definitions. □

Using the link between multivariate mixed exponential-logarithmic distribution and Archimedean copulas, the resulting copula is the well known Frank copula.

Proposition 5.6.6. *Let* Θ *be a discrete rv with logarithmic distribution with parameter* p*. The associated multivariate Archimedean copula with generator* $\mathcal{L}_\Theta$*, called the Frank copula with parameter* $\delta = -\ln(1-p)$*, is given by*

$$C_{\frac{1}{\alpha}}(u_1, \ldots, u_n) = -\delta^{-1}\ln\left(1 - \frac{\prod_{i=1}^{n}\left(1-\mathrm{e}^{-\delta u_i}\right)}{\left(1-\mathrm{e}^{-\delta}\right)^{n-1}}\right), \tag{5.66}$$

for $u_i \in [0,1]$*,* $\forall i \in \{1, \ldots, n\}$*.*

Proof. See, e.g., Joe (2014) for proof and more details concerning such a copula. □

5.7 Bibliography

Abdallah, A., Boucher, J.-P., and Cossette, H. (2015). Modeling dependence between loss triangles with hierarchical archimedean copulas. *ASTIN Bulletin: The Journal of the IAA*, 45(3):577–599.

Abramowitz, M. and Stegun, I. A. (1964). *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation.

Adamidis, K., Dimitrakopoulou, T., and Loukas, S. (2005). On an extension of the exponential-geometric distribution. *Statistics & probability letters*, 73(3):259–269.

Adamidis, K. and Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics & Probability Letters*, 39(1):35–42.

Albrecher, H., Beirlant, J., and Teugels, J. L. (2017). *Reinsurance: Actuarial and Statistical Aspect.* Wiley.

Albrecher, H., Constantinescu, C., and Loisel, S. (2011). Explicit ruin formulas for models with dependence among risks. *Insurance: Mathematics and Economics*, 48(2):265–270.

Anderson, D. N. (1992). A multivariate linnik distribution. *Statistics & probability letters*, 14(4):333–336.

Anderson, D. N. and Arnold, B. C. (1993). Linnik distributions and processes. *Journal of applied probability*, 30(2):330–340.

Arnold, B. C. (1973). Some characterizations of the exponential distribution by geometric compounding. *SIAM Journal on Applied Mathematics*, 24(2):242–244.

Arnold, B. C. (1983). *Pareto distribution*. International Cooperative Publishing House, Fairland, MD.

Arnold, B. C. (2015). *Pareto distribution*. Wiley Online Library.

Asmussen, S. and Albrecher, H. (2010). *Ruin probabilities*. World Scientific Publishing Co Pte Ltd.

Balakrishnan, N. and Lai, C.-D. (2009). *Continuous bivariate distributions*. Springer Science & Business Media.

Barlow, R. E., Marshall, A. W., and Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. *The Annals of Mathematical Statistics*, pages 375–389.

Barreto-Souza, W. and Bakouch, H. S. (2013). A new lifetime model with decreasing failure rate. *Statistics*, 47(2):465–476.

Bouzar, N. (2002). Mixture representations for discrete linnik laws. *Statistica neerlandica*, 56(3):295–300.

Cai, J. (2006). *Mixtures of Exponential Distributions in Encyclopedia of Actuarial Science*, volume 2. Wiley.

Chahkandi, M. and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics & Data Analysis*, 53(12):4433–4440.

Christoph, G. and Schreiber, K. (1998). Discrete stable random variables. *Statistics & Probability Letters*, 37(3):243–247.

Christoph, G. and Schreiber, K. (2000). Scaled sibuya distribution and discrete self-decomposability. *Statistics & probability letters*, 48(2):181–187.

Cossette, H., Gadoury, S.-P., Marceau, É., and Mtalai, I. (2017). Hierarchical archimedean copulas through multivariate compound distributions. *Insurance: Mathematics and Economics*.

Cossette, H., Landriault, D., and Marceau, E. (2001). A general recursive formula for the discrete stable and linnik distributions and a family of extensions. *Corpus Université Laval. Available at : http://hdl.handle.net/20.500.11794/14921.*

Cossette, H., Marceau, E., and Mtalai, I. (2018a). Collective risk models with hierarchical archimedean copulas. *Submitted for publication.*

Cossette, H., Marceau, E., Mtalai, I., and Veilleux, D. (2018b). Dependent risk models with archimedean copulas: A computational strategy based on common mixtures and applications. *Insurance: Mathematics and Economics*, 78:53–71.

Dacorogna, M. M., Elbahtouri, L., and Kratz, M. (2016). Explicit diversification beneift for dependent risks.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial theory for dependent risks: measures, orders and models.* John Wiley & Sons.

Devroye, L. (1990). A note on linnik's distribution. *Statistics & probability letters*, 9(4):305–306.

Devroye, L. (1993). A triptych of discrete distributions related to the stable law. *Statistics & Probability Letters*, 18(5):349–351.

Fang, K.-T., Fang, H.-B., and Rosen, D. v. (2000). A family of bivariate distributions with non-elliptical contours. *Communications in Statistics-Theory and Methods*, 29(8):1885–1898.

Feldmann, A. and Whitt, W. (1997). Fitting mixtures of exponentials to long-tail distributions to analyze network performance models. In *INFOCOM'97. Sixteenth Annual Joint Conference of the IEEE Computer and Communications Societies. Driving the Information Revolution., Proceedings IEEE*, volume 3, pages 1096–1104. IEEE.

Furman, E. and Zitikis, R. (2008). Weighted risk capital allocations. *Insurance: Mathematics and Economics*, 43(2):263–269.

Genest, C. and Rivest, L.-P. (2001). On the multivariate probability integral transformation. *Statistics & Probability Letters*, 53(4):391–399.

Grandell, J. (1997). *Mixed poisson processes*, volume 77. CRC Press.

Gui, W., Zhang, S., and Lu, X. (2014). The lindley-poisson distribution in lifetime analysis and its properties. *Hacet J Math Stat*, 43(6):1063–1077.

Hajebi, M., Rezaei, S., and Nadarajah, S. (2013). An exponential-negative binomial distribution. *REVSTAT Stat J*, 11(2):191–210.

Hering, C., Hofert, M., Mai, J.-F., and Scherer, M. (2010). Constructing hierarchical archimedean copulas with lévy subordinators. *Journal of Multivariate Analysis*, 101(6):1428–1433.

Hofert, M. (2010). *Sampling nested Archimedean copulas with applications to CDO pricing*. PhD thesis, Universität Ulm.

Hofert, M. (2011). Efficiently sampling nested archimedean copulas. *Computational Statistics & Data Analysis*, 55(1):57–70.

Hofert, M. and Scherer, M. (2011). Cdo pricing with nested archimedean copulas. *Quantitative Finance*, 11(5):775–787.

Hougaard, P. (1986). A class of multivanate failure time distributions. *Biometrika*, 73(3):671–678.

Huillet, T. E. (2016). On mittag-leffler distributions and related stochastic processes. *Journal of Computational and Applied Mathematics*, 296:181–211.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts*. CRC Press.

Joe, H. (2014). *Dependence modeling with copulas*. CRC Press.

Joe, H. and Hu, T. (1996). Multivariate distributions from mixtures of max-infinitely divisible distributions. *Journal of multivariate analysis*, 57(2):240–265.

Jovanovic, M. (2011). Does monetary policy affect stock market uncertainty?

Kanter, M. et al. (1975). Stable densities under change of scale and total variation inequalities. *The Annals of Probability*, 3(4):697–707.

Kozubowski, T. J. (1998). Mixture representation of linnik distribution revisited. *Statistics & probability letters*, 38(2):157–160.

Kuş, C. (2007). A new lifetime distribution. *Computational Statistics & Data Analysis*, 51(9):4497–4509.

Lee, C. K. and Wen, M.-J. (2006). A multivariate weibull disitribution. *arXiv preprint math/0609585*.

Linnik Yu, V. (1963). Linear forms and statistical criteria. *I II Selected Translations Math Stat Probab*, 3:1–90.

Mai, J. and Scherer, M. (2014). *Financial engineering with copulas explained*. Springer.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841.

McNeil, A. J. (2008). Sampling nested archimedean copulas. *Journal of Statistical Computation and Simulation*, 78(6):567–581.

Müller, A. and Stoyan, D. (2002). *Comparison methods for stochastic models and risks*, volume 389. Wiley.

Oakes, D. (1989). Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, 84(406):487–493.

Pakes, A. G. (1998). Mixture representations for symmetric generalized linnik laws. *Statistics & probability letters*, 37(3):213–221.

Pillai, R. and Jayakumar, K. (1995). Discrete mittag-leffler distributions. *Statistics & probability letters*, 23(3):271–274.

Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, 5(3):375–383.

Rémillard, B., Theodorescu, R., et al. (2000). Inference based on the empirical probability generating function for mixtures of poisson distributions. *Statistics and Decisions-International Journal for Stochastic Methods and Models*, 18(4):349–366.

Rinne, H. (2008). *The Weibull distribution: a handbook*. CRC Press.

Sarabia, J. M., Gómez-Déniz, E., Prieto, F., and Jordá, V. (2017). Aggregation of dependent risks in mixtures of exponential distributions and extensions. *arXiv preprint arXiv:1705.00289*.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders*. Springer Science & Business Media.

Sibuya, M. (1979). Generalized hypergeometric, digamma and trigamma distributions. *Annals of the Institute of Statistical Mathematics*, 31(1):373–390.

Steutel, F. and Van Harn, K. (1979). Discrete analogues of self-decomposability and stability. *The Annals of Probability*, pages 893–899.

Tahmasbi, R. and Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics & Data Analysis*, 52(8):3889–3901.

Whelan, N. et al. (2004). Sampling from archimedean copulas. *Quantitative finance*, 4(3):339–352.

Yilmaz, M., Hameldarbandi, M., and Kemaloglu, S. A. (2016). Exponential-modified discrete lindley distribution. *SpringerPlus*, 5(1):1660.

Zhu, W., Wang, C.-W., and Tan, K. S. (2016). Structure and estimation of lévy subordinated hierarchical archimedean copulas (lshac): Theory and empirical tests. *Journal of Banking & Finance*, 69:20–36.

Conclusion

Dans cette thèse, nous nous sommes intéressés à la modélisation de la dépendance à l'aide des modèles construits par des mélanges communs. Plus précisément, nous avons essentiellement utilisé le modèle exponentiel mélange et son lien avec les copules Archimédiennes et Archimédiennes hiérarchiques.

Dans le chapitre 2, nous utilisons la représentation des copules archimédiennes sous forme de mélange commun pour obtenir une méthodologie computationnelle d'agrégation pour un portefeuille pour lequel la dépendance entre ses risques est modélisée par une copule Archimédienne ou Archimédienne imbriquée. Cette représentation nous a aussi permis d'établir l'allocation du capital global selon différentes méthodes. Le problème d'asymétrie des copules Archimédiennes, hypothèse qui n'est pas trop réaliste d'un point de vue pratique, a poussé les chercheurs à proposer des structures hiérarchiques obtenues en imbriquant des copules Archimédiennes l'une dans l'autre. Cependant, la condition d'imbrication s'avère difficile à vérifier en général. Afin de résoudre ce problème, nous avons proposé au chapitre 3, une nouvelle méthode d'imbrication se basant sur des distributions composées multivariées. Cette méthode permet d'élargir les possibilités de structures de dépendance et a le grand avantage de s'appliquer sans aucune condition d'imbrication. Le chapitre 4 étudie les modèles collectifs de risques incorporant une dépendance entre le nombre et les montants des sinistres. Cette hypothèse de dépendance est plus réaliste d'un point de vue pratique. Le dernier chapitre traite en détail les distributions exponentielles mélange et propose plusieurs nouvelles distributions univariées et multivariées.

Plusieurs applications en science actuarielle ont été considérées dans les quatre articles constituant cette thèse. Ces différentes applications montrent l'utilité et le potentiel d'utiliser les modèles proposés dans cette thèse. Ainsi, ces articles ouvrent des perspectives de recherche en matière d'application des modèles proposés dans plusieurs branches d'actuariat et de finance.