Bounded-Degree Polyhedronization of Point Sets

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Abstract

In 1994 Grünbaum [2] showed, given a point set S in $\mathbb{R}^3$, that it is always possible to construct a polyhedron whose vertices are exactly S. Such a polyhedron is called a *polyhedronization* of S. Agarwal et al. [1] extended this work in 2008 by showing that a polyhedronization always exists that is decomposable into a union of tetrahedra (*tetrahedralizable*). In the same work they introduced the notion of a *serpentine* polyhedronization for which the dual of its tetrahedralization is a chain. In this work we present an algorithm for constructing a serpentine polyhedronization that has vertices with bounded degree of 7, answering an open question by Agarwal et al. [1].

1 Introduction

It is well-known that any set S of points in the plane (not all of which are collinear) admits a *polygonalization*, that is, there is a simple polygon whose vertex set is exactly S. Similarly, a point set $S \subset \mathbb{R}^3$ admits a *polyhedronization* if there exists a simple polyhedron that has exactly S as its vertices. In 1994 Grünbaum proved that every point set in $\mathbb{R}^3$ admits a polyhedronization. Unfortunately, the polyhedronizations generated by Grünbaum's method can be impossible to tetrahedralize. This is because they may contain *Schönhardt polyhedra*, a class of non-tetrahedralizable polyhedra [3].

In 2008, Agarwal, Hurtado, Toussaint, and Trias described a variety of methods for producing polyhedronizations with various properties [1]. One of these methods, called hinge polyhedronization, produces *serpentine polyhedronizations*, meaning they are composed of tetrahedra whose dual (a graph where each tetrahedron is a node and each edge connects a pair of nodes whose primal entities are tetrahedra sharing a face) is a chain. Serpentine polyhedronizations produced by the hinge polyhedronization method are guaranteed to have two vertices with edges to every other vertex in the set. As a result, two vertices in these constructions have degree $n-1$, where n is the number of points in the set. A natural question, and one posed by Agarwal et al., is whether it is always possible to create serpentine polyhedronizations with bounded degree.

In this work we describe an algorithm for constructing serpentine polyhedronizations that have $O(1)$ degree. The constant bound of the produced polyhedronizations is 7, which we show is nearly optimal for all point sets with greater than 12 vertices. Such bounded-degree serpentine polyhedronizations are useful in applications of modeling and graphics where low local complexity is desirable for engineering and computational efficiency.

2 Setting

Let the point set P in $\mathbb{R}^3$ be in general position in the sense that it contains no four coplanar points. The convex hull of P, written $\mathcal{CH}(P)$, is the intersection of all half-spaces containing P. The boundary of each face of $\mathcal{CH}(P)$ is a polygon with coplanar vertices. Since P contains no four coplanar points, each of the faces of $\mathcal{CH}(P)$ is triangular. The three vertices composing a face of $\mathcal{CH}(P)$ we call a *face triplet*.

We will make reference to points and faces that *see* each other. We say that a pair of points p, q can see each other if the segment pq does not intersect a portion of any polyhedron present. A face f is the planar region bounded by a triangle formed by three points. A point p can see a face f if p can see every point in f (strong visibility). Similarly, a point p can see a segment s if p can see every point on s.

3 Algorithm

In this section we present a high-level overview of the algorithm. Begin with a point set $S \subset \mathbb{R}^3$. Select a face triplet of $\mathcal{CH}(S)$ arbitrarily. Call this face triplet T_0. Let $S_0 = S \setminus T_0$. Assign the labels u_0, v_0, w_0 to the

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vertices of T_0 and connect the three vertices to form a triangle.

Next we search for a face triplet T_1 of $\mathcal{CH}(S_0)$ that we can attach to the triangle T_0 via a polyhedron *tunnel* (see Figure 1). The tunnel has the face triplet T_0 at one end, the face triplet T_1 at the other end and is disjoint with the interior of $\mathcal{CH}(S_0)$. The tunnel needs to be tetrahedralizable and the vertices u_0, v_0 have degree 5 and 4, and w_0 has degree 3. Moreover, the vertices of the face triplet T_1 that we will call u_1, v_1, w_1 should have degree 3, 4 and 5, respectively. Note that the constructed tunnel must *meet* the degree requirements for the vertices of T_0 while it *determines* the vertex naming assignments for the vertices of T_1.

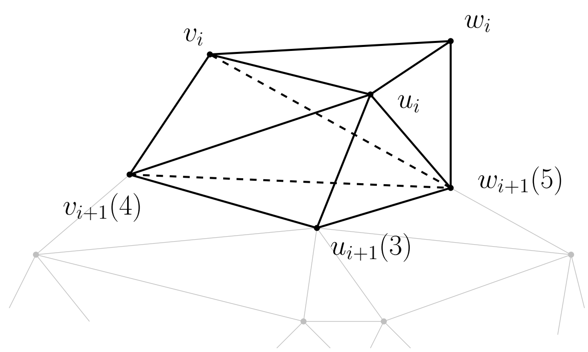

Figure 1: Constructing a tunnel between T_i, T_{i+1}. The vertices u_i and v_i have degree 5 and 4, while w_i has degree 3. The other end of the tunnel, T_{i+1}, has three vertices that will be labeled $u_{i+1}, v_{i+1}, w_{i+1}$ with degree 3, 4 and 5 (shown in parentheses), respectively.

After finding a face triplet T_1 that meets these requirements, the process is repeated for T_1 and S_1, T_2 and S_2 where $S_i = S_{i-1} \backslash T_i$, until S_i contains fewer than three points. At this point a degenerate tunnel is built out of the remaining points and the algorithm stops. In the next two sections we prove that such a construction is always possible, producing a valid serpentine polyhedronization with bounded vertex-degree 7.

4 Tunnel Construction

Here we prove that given T_i it is always possible to find a face triplet T_{i+1} such that a three-tetrahedra tunnel $(\Delta_1\Delta_2\Delta_3)$ can be constructed between them.

Let L_1 denote the line through u_iv_i. Call H_1 the plane containing T_i (and thus L_1). Note that the plane supporting T_i does not intersect $\mathcal{CH}(S_i)$ because T_i is a face of $\mathcal{CH}(S_{i-1})$. Rotate H_1 about L_1 in the direction that maintains separation of w_i and $\mathcal{CH}(S_i)$ until $\mathcal{CH}(S_i)$ is intersected. This intersection will be at a vertex, an edge, or a face. Let v_{cone} be a vertex of the intersection and H_2 the plane through L_1 and v_{cone}. Let R_1 be the swept-out region between H_1 and H_2.

Now let L_2 denote the line parallel to L_1 through v_{cone}. Rotate H_2 about L_2, starting at u_iv_i, in the direction that maintains the separation of u_iv_i and $\mathcal{CH}(S_i)$ until $\mathcal{CH}(S_i)$ is intersected. The intersection is either an edge or a face. If it is an edge, call this edge e. If it is a face, select an edge e of this face that has v_{cone} as an endpoint. Let H_3 be the plane containing L_2 and e, and let R_2 be the swept-out region between H_1 and H_2. Refer to Figure 2.

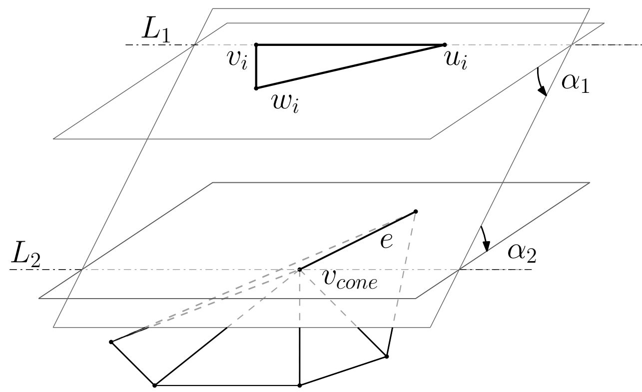

Figure 2: A visualization of the arrangement created by T_i. The angles α_1, α_2 denote the swept angular regions forming R_1 and R_2, respectively.

Lemma 1 *The segment u_iv_i can see edge e.*

Proof. Recall the plane supporting T_i does not intersect $\mathcal{CH}(S_i)$, so w_i cannot interfere with visibility. Now consider a segment connecting a point on u_iv_i and a point on e. This segment is contained in R_2, which is empty. Thus, neither w_i nor $\mathcal{CH}(S_i)$ can block visibility between u_iv_i and e. □

Connect the endpoints of e to u_i and v_i with four edges to form the middle tetrahedron Δ_2.

Lemma 2 *Vertex w_i can see face $u_iv_iv_{cone}$ of Δ_2.*

Proof. The swept-out region R_1 does not contain any portion of $\mathcal{CH}(S_i)$ or Δ_2. Furthermore, every segment connecting w_i to a point on the face $u_iv_iv_{cone}$ is contained in R_1. Thus, w_i can see the face $u_iv_iv_{cone}$. □

Connect w_i to v_{cone} (it is already connected to u_i and v_i) to form a tetrahedron Δ_3.

Lemma 3 *A face f incident to e is seen by u_i or v_i.*

Proof. First consider Δ_3. The plane H_2 separates Δ_3 from $\mathcal{CH}(S_i)$ and Δ_2. So Δ_3 cannot obscure visibility between a vertex of Δ_2 and either face of $\mathcal{CH}(S_i)$ incident to e. Now refer to Figure 3. Consider rotating each face f of $\mathcal{CH}(S_i)$ incident to e away from $\mathcal{CH}(S_i)$

until a face of Δ_2 is intersected. These rotations are disjoint and both occur around the line containing e. So both cannot be greater than 180°. Let f be a face that rotates less than 180°. The face f is seen by the vertices of the face of Δ_2 it intersects, including either u_i or v_i. Call the vertex u_i or v_i intersected q. □

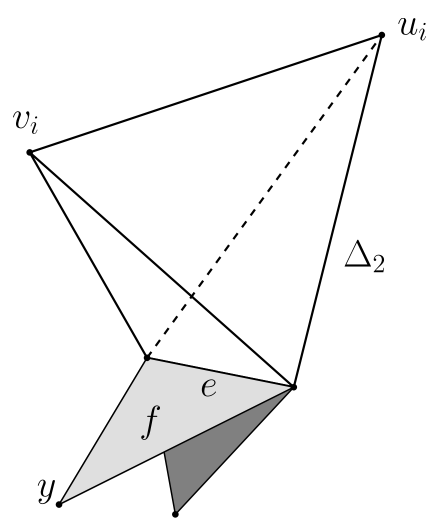

Figure 3: The scenario described in Lemma 3. Either u_i or v_i must see a face of $\mathcal{CH}(S_i)$ incident to e. In this case, v_i sees f. So $q = v_i$.

Connect q to y, the third vertex of this face (q is already connected to the other two vertices of f, the endpoints of e) to form tetrahedron Δ_1.

Theorem 4 *The tetrahedra $\Delta_1, \Delta_2, \Delta_3$ form a three-tetrahedron tunnel in which u_i, v_i have degree 5 and 4, and w_i has degree 3.*

Proof. See Figure 4. The vertices u_i, v_i, w_i each have two edges connecting them to the other two vertices of T_i. Vertex w_i is also connected to v_{cone}, so it has degree 3. Vertices u_i and v_i are also connected to the endpoints of e. Vertex q, which is either u_i or v_i, is also connected to y. Thus, one vertex from $\{u_i, v_i\}$ has degree 5, while the other has degree 4. □

Once the tunnel between T_0 and T_1 is constructed, repeat the process to build a tunnel from T_1 to T_2, etc. When T_i is reached such that S_i contains fewer than three points, construct a four- or five-vertex polyhedron. In the next section we prove that this construction produces a valid polyhedronization that is serpentine and has optimal bounded degree.

5 Polyhedronization Properties

In this section we prove that the union of the constructed tunnels is a serpentine polyhedronization with bounded-degree 7 and that this bound is nearly optimal.

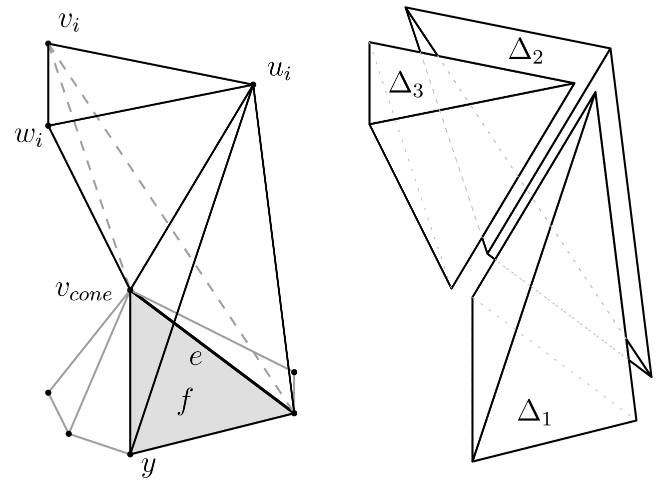

Figure 4: A complete tunnel and the three tetrahedra $\Delta_1, \Delta_2, \Delta_3$ composing it.

Lemma 5 *Tunnel interiors are disjoint.*

Proof. Consider the two tunnels between T_i, T_{i+1} and T_j, T_{j+1} for $j \neq i$. Without loss of generality, let $j > i$. All of the vertices of the tunnel between T_i, T_{i+1} are on the boundary or exterior of $\mathcal{CH}(S_i)$. Additionally, all of the vertices of the tunnel between T_j and T_{j+1} are on boundary or interior of $\mathcal{CH}(S_i)$. Therefore, the two tunnels may only intersect on the boundary of $\mathcal{CH}(S_i)$. Hence, their interiors are disjoint. □

Theorem 6 *The resulting polyhedronization of S is a serpentine polyhedron.*

Proof. Each tunnel is constructed of three tetrahedra that form a chain from T_i to T_{i+1} in the order $\Delta_3, \Delta_2, \Delta_1$. The tunnel between face triplets T_i and T_{i+1} shares T_i (resp., T_{i+1}) with the previous (resp., next) tunnels, except, of course, for $i = 0$ in which case there is no previous tunnel and the tetrahedron with face T_0 is the first element of the dual chain. For the last tunnel, T_k, either a degenerate tunnel is formed with the remaining one, or two points or the last tetrahedron of T_k is the end of the chain. In the degenerate case, a face of T_k shares a face with the final degenerate tunnel. The final degenerate tunnel must be tetrahedralizable and have a dual chain since it is a polyhedron with four or five vertices. Therefore, in both cases the dual of the polyhedronization is a chain. □

Lemma 7 *Every vertex in the polyhedronization of S has degree at most 7.*

Proof. First consider the face triplets that are not first or last. Each vertex is part of some triangle T_i and has two edges connecting it to the other vertices of T_i.

For a vertex u_i, one additional edge is connected to u_i in the tunnel between T_{i-1} and T_i, and at most three additional edges are connected to u_i in the tunnel between T_i and T_{i+1} (this occurs when $u_i = q$). So u_i has degree at most $1+2+3=6$. For a vertex v_i, two additional edges are connected to v_i in the tunnel between T_{i-1} and T_i, and at most three additional edges are connected to v_i in the tunnel between T_i and T_{i+1} (this occurs when $v_i = q$). So v_i has degree at most $2+2+3=7$. For a vertex w_i, three additional edges are connected to w_i in the tunnel between T_{i-1} and T_i, and one additional edge is connected to w_i in the tunnel between T_i and T_{i+1}. So w_i has degree at most $3+2+1=6$.

Now consider the vertices involved in the final four- or five-vertex polyhedron (called D). Let T_k be the last non-degenerate face triplet. There exists a polyhedronization of D such that w_k has only 1 additional edge in D (excluding the edges to v_k, u_k). Using this polyhedronization gives w_k degree at most $3+2+1=6$ when combined with edges from the tunnel between T_{k-1} and T_k. All other vertices have at most 2 additional edges in the polyhedronization (since there are at most two vertices in the degenerate face triplet) and gain at most 2 vertices from the tunnel between T_k and T_{k-1}.. So each of these vertices has degree at most $4+2=6$.

In conclusion, the maximum degree of any vertex in the polyhedronization is 7. □

Lemma 8 *No polyhedronization of an arbitrary number of points in $\mathbb{R}^3$ can obtain a bounded degree of less than 6.*

Proof. By Euler's formula, every polyhedron in general position with $|S|$ vertices has $3|S|-6$ edges. Hence, the average degree of a vertex is $\frac{2(3|S|-6)}{|S|} = 6 - \frac{12}{|S|}$. Therefore, for $|S| > 12$, some vertex must have degree at least 6. □

The algorithm described produces a nearly optimal bounded-degree polyhedronization. Indeed, Lemma 7 proved that the construction produces a polyhedronization with bounded-degree 7, while by Lemma 8, every polyhedronization of an arbitrary number of points must have some vertex with degree at least 6. So the construction has vertices with degree at most one greater than the minimum possible degree.

6 Conclusion

In this paper we show that any point set in 3-space admits a polyhedronization with vertex degree at most 7, while 6 is a simple lower bound. Future work includes showing that either 6 or 7 is the true bound in the worst case. Furthermore, we believe that our technique can be generalized to higher dimensions.

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