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OSCILLATION CRITERIA FOR FRACTIONAL IMPULSIVE HYBRID PARTIAL DIFFERENTIAL EQUATIONS

Abstract. In this paper, we study the oscillatory behavior of the solutions of fractional-order nonlinear impulsive hybrid partial differential equations with the mixed boundary condition. By using the integral averaging method and the Riccati technique, we have obtained the oscillation criteria of all the solutions of the given system. An example is given to illustrate our main results.

Key words: *Oscillation, Fractional partial differential equation, Hybrid differential equation, impulse.*

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1. Introduction In the last few decades, there has been a lot of interest in deriving sufficient conditions for the oscillation and nonoscillation of solutions of classes of differential equations. The first investigation and publication of the oscillation theory of impulsive differential equations was in 1989 [9], and the first paper on impulsive partial differential equations was published in 1996 [2]. Its consequences were included in the book [15]. Chatzarakis et al. [3], Kalaimani et al. [13], Prakash et al. [20], Sadhasivam et al. [23, 25] and Yang et al. [30] studied the impulsive partial differential equations. The study of impulsive partial differential equations is motivated by various applications in population models [2], single species growth [6], feedback control prey-predator model [12], and various scientific models [29, 31].

Even though there is a countable number of papers on oscillatory solutions of fractional partial differential equations, refer [7, 16, 21, 22, 24], they have not dealt with the impulse effect. Many authors have investigated some of the areas of applications of fractional differential equations, like viscoelasticity, electrochemistry, signal processing and so on; in the last few decades, there have been several monographs of fractional derivatives

and integrals, see [1, 4, 14, 18, 26, 33] and the references cited therein. In the recent years, there has been much research on hybrid differential equations that refer to various aspects of quadratic perturbations of nonlinear differential equations. The reason is that hybrid differential equations include several dynamic systems as special cases. There has been a significant amount of work on the theory of hybrid differential equations that can be referred in articles [10, 27, 28, 32]. Applications with numerical solutions have been studied by several authors, see, for example, [11, 19].

In a systematic review of the literature on hybrid differential equations, it is noted that Dhage and Lakshmikantham [5] scrutinized the existence of extremal solutions and a comparison result for first order hybrid differential equation with linear perturbations of the following type:

$$\frac{d}{dt} \left(\frac{x(t)}{f(t, x(t))} \right) = g(t, x(t)), \text{ a.e. } t \in J, x(t_0) = x_0 \in \mathbb{R},$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} - \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. Ge et al. [8] analysed the existence of a mild solution for impulsive hybrid fractional differential equations:

$$\left. \begin{aligned} {}^c D_{0,t}^q \left(\frac{u(t)}{f(t, u(t))} \right) &= g(t, u(t)), \quad t \in J' := J \setminus \{t_1, t_2, \dots, t_m\} \\ u(t_k^+) &= u(t_k^-) + I_k(u(t_k^-)), \quad k = 1, 2, \dots, m \\ u(0) &= u_0, \end{aligned} \right\}$$

where ${}^c D_{0,t}^q$ is the generalized Caputo fractional derivative of order $q \in (0, 1)$ with the zero lower limit, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.

However, to the best of our knowledge, we understand that there has been no previous research made on the oscillation of nonlinear fractional impulsive hybrid partial differential equations. Hence, we believe that we are the first who initiate the oscillation criteria for fractional impulsive partial hybrid differential equations which had not been formerly studied. Motivated by the above observations, we propose the following model of

the form

$$\left. \begin{aligned} & \frac{\partial}{\partial t} \left(r(t) D_{+,t}^\alpha \left(\frac{u(x,t)}{h(t,u(x,t))} \right) \right) + q(x,t) g \left(D_{+,t}^\alpha u(x,t) \right) = \\ & = a(t) \Delta u(x,t) - f \left(t, \int_0^t (t-s)^{-\alpha} u(x,s) ds \right) + F(x,t), \\ & u(x, t_k^+) = \gamma_k(x, t_k, u(x, t_k)), \\ & u_t(x, t_k^+) = \delta_k(x, t_k, u_t(x, t_k)), \quad k = 1, 2, \dots, (x, t) \in \Omega \times \mathbb{R}_+ \equiv G. \end{aligned} \right\} \quad (1)$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, $\alpha \in (0, 1)$ is a constant, $D_{+,t}^\alpha$ is the Riemann-Liouville fractional derivative of order α of u with respect to t , and Δ is the Laplacian operator in the Euclidean n -space R^n , i. e., $\Delta u(x, t) = \sum_{r=1}^n \frac{\partial^2 u(x, t)}{\partial x_r^2}$ with the boundary conditions

$$\frac{\partial u(x, t)}{\partial \eta} + \gamma(x, t) u(x, t) = 0, \quad (x, t) \in \partial\Omega \times R_+ \quad (2)$$

and

$$\frac{\partial u(x, t)}{\partial \eta} = 0, \quad (x, t) \in \partial\Omega \times R_+, \quad (3)$$

where η is the unit exterior normal vector to $\partial\Omega$ and $\gamma(x, t)$ is a positive continuous function on $\partial\Omega \times R_+$.

Based on the following assumption:

(A₁) $r(t) \in C^1(R_+; R_+)$, $r'(t) \geq 0$, $a(t) \in PC(R_+; R_+)$;

(A₂) $q(x, t) \in C(\bar{G}; R_+)$ and $q(t) = \min_{x \in \bar{\Omega}} q(x, t)$;

(A₃) $g \in C(R; R)$ is convex in R_+ , $ug(u) > 0$ and $\frac{g(u)}{u} \geq \mu > 0$ for $u \neq 0$, $f(t, x(t)) \in C([t_0, \infty) \times R, R)$, there exists a function $p(t) \in C([t_0, \infty), R_+)$ such that $\frac{f(t, K)}{K} \geq p(t)$ for $K \neq 0$, $t \geq t_0$;

(A₄) $h \in C(R \times [t_0, \infty); R - \{0\})$ is convex in R_+ , there exists a function $\zeta(t) \in C([t_0, \infty), R_+)$ such that $h(t, u) \geq \xi(x)\zeta(t) \geq M\zeta(t)$, $\xi(x)$ is not identically zero on $[t_0, \infty)$ and such that $|\xi(x)| \geq M > 0$ and $D_+^\alpha h(t, y) > 0$;

- (A₅) $F \in C(\bar{G}; R)$ such that $\int_{\Omega} F(x, t) dx \leq 0$;
- (A₆) $u(x, t)$, $u_t(x, t)$ are piecewise continuous in t with discontinuities of the first kind at $t = t_k, k = 1, 2, \dots$, and left continuous at $t = t_k$, $u(x, t_k) = u(x, t_k^-)$, $u_t(x, t_k) = u_t(x, t_k^-)$, $k = 1, 2, \dots$;
- (A₇) $\gamma_k(x, t_k, u(x, t_k))$, $\delta_k(x, t_k, u_t(x, t_k)) \in PC(\bar{\Omega} \times R_+ \times R, R)$, $k = 1, 2, \dots$ and there exist positive constants b_k, b_k^*, c_k, c_k^* with $c_k \leq b_k^*$ such that

$$b_k^* \leq \frac{\gamma_k(x, t_k, u(x, t_k))}{u(x, t_k)} \leq b_k, c_k^* \leq \frac{\delta_k(x, t_k, u_t(x, t_k))}{u_t(x, t_k)} \leq c_k, k = 1, 2, \dots$$

By a solution of equation (1) and (2)–(3), we mean a nontrivial function $u(x, t) \in C^{1+\alpha}(\bar{G}, R_+)$ with

$$\int_0^t (t-s)^{-\alpha} u(x, s) ds \in C^1(\bar{G}, R_+), r(t) D_{+,t}^{\alpha} \left(\frac{u(x, t)}{h(t, u(x, t))} \right) \in C^1(\bar{G}, R_+),$$

that satisfies \bar{G} and the boundary conditions (2)–(3). A solution of equations (1) and (2)–(3) is called oscillatory if it has arbitrarily large zeros in G , and is called nonoscillatory otherwise. Equations (1) and (2)–(3) are said to be oscillatory if all their solutions are oscillatory.

This paper is organized as follows: In Section 2, we present the relevant definitions and lemmas, and in Section 3 we discuss the oscillation problem of (1) subject to the boundary conditions (2)–(3). In Section 4 we present an example to illustrate our main results.

The results in this paper extend and improve numerous findings in the earlier publications that do not include the impulsive effect. We believe that this research work would enable further researchers on the fractional impulsive partial hybrid differential equations.

2. Preliminaries. In this section, we give the fundamental definitions of fractional derivatives and integrals. There are several kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann–Liouville left-sided definition on the half-axis R_+ .

Definition 1. [14] *The Riemann–Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x, t)$ is given by*

$$(D_{+,t}^{\alpha} y)(x, t) := \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-v)^{-\alpha} y(x, v) dv \quad (4)$$

provided that the right-hand side is pointwise defined on R_+ , where Γ is the gamma function.

Definition 2. [14] The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : R_+ \rightarrow R$ on the half-axis R_+ is given by

$$(I_+^\alpha y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - v)^{\alpha-1} y(v) dv \quad \text{for } t > 0 \tag{5}$$

provided that the right-hand side is pointwise defined on R_+ .

Definition 3. [14] The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : R_+ \rightarrow R$ on the half-axis R_+ is given by

$$(D_+^\alpha y)(t) := \frac{d^{[\alpha]}}{dt^{[\alpha]}} \left(I_+^{[\alpha]-\alpha} y \right) (t) \quad \text{for } t > 0 \tag{6}$$

provided that the right-hand side is pointwise defined on R_+ , where $[\alpha]$ is the ceiling function of α .

Lemma 1. [14] Let $y(t)$ be a solution of (1) and

$$E(t) = \int_0^t (t - \nu)^{-\alpha} y(\nu) d\nu \quad \text{for } \alpha \in (0,1) \text{ and } t \geq 0. \tag{7}$$

Then

$$E'(t) = \Gamma(1 - \alpha) D_+^\alpha y(t). \tag{8}$$

Theorem 1. [17] Let $f(x), g(x), h(x) \geq 0$ be continuous functions on $[a, b]$ such that

$$(g(x) - g(y)) \left(\frac{f(y)}{h(y)} - \frac{f(x)}{h(x)} \right) \geq 0. \tag{9}$$

Then the inequality

$$\frac{\int_a^b f(x) dx}{\int_a^b h(x) dx} \geq \frac{\int_a^b f(x)g(x) dx}{\int_a^b h(x)g(x) dx} \tag{10}$$

holds. If (9) reverses, then (10) reverses.

Corollary 1. Let $g(x, t)$, $u(x, t)$, $f(t, u(x, t)) \geq 0$ be continuous on Ω and for any fixed $t \in R_+$ such that

$$(u(x, t) - u(y, t)) \left(\frac{g(y, t)}{f(t, u(y, t))} - \frac{g(x, t)}{f(t, u(x, t))} \right) \geq 0. \quad (11)$$

Then the inequality

$$\frac{\int_{\Omega} g(x, t) dx}{\int_{\Omega} f(t, u(x, t)) dx} \geq \frac{\int_{\Omega} g(x, t) u(x, t) dx}{\int_{\Omega} f(t, u(x, t)) u(x, t) dx} \quad (12)$$

holds. If (11) reverses, then (12) reverses.

Proof. We have to prove that

$$\int_{\Omega} g(x, t) dx \int_{\Omega} f(t, u(x, t)) u(x, t) dx \geq \int_{\Omega} f(t, u(x, t)) dx \int_{\Omega} g(x, t) u(x, t) dx,$$

which is equivalent to

$$\int_{\Omega} g(x, t) dx \int_{\Omega} f(t, u(y, t)) u(y, t) dy \geq \int_{\Omega} f(t, u(x, t)) dx \int_{\Omega} g(y, t) u(y, t) dy,$$

which implies

$$\int_{\Omega} \int_{\Omega} u(y, t) (g(x, t) f(t, u(y, t)) - g(y, t) f(t, u(x, t))) dx dy \geq 0.$$

Denote

$$I = \int_{\Omega} \int_{\Omega} u(y, t) (g(x, t) f(t, u(y, t)) - g(y, t) f(t, u(x, t))) dx dy,$$

then

$$I = \int_{\Omega} \int_{\Omega} u(x, t) (g(y, t) f(t, u(x, t)) - g(x, t) f(t, u(y, t))) dx dy.$$

Hence

$$\begin{aligned}
 2I &= \int_{\Omega} \int_{\Omega} (u(x,t) - u(y,t))(g(y,t)f(t, u(x,t)) - g(x,t)f(t, u(y,t))) dx dy = \\
 &= \int_{\Omega} \int_{\Omega} f(t, u(x,t))f(t, u(y,t))(u(x,t) - u(y,t)) \times \\
 &\quad \times \left(\frac{g(y,t)}{f(t, u(y,t))} - \frac{g(x,t)}{f(t, u(x,t))} \right) dx dy.
 \end{aligned}$$

By (11), $2I \geq 0$. \square

Remark 1. In particular, replacing $g(x,t) = 1$ and $f(t, u(x,t)) = 1/f(t, u(x,t))$ in the above Corollary 1 under the condition

$$(u(x,t) - u(y,t))(f(t, u(y,t)) - f(t, u(x,t))) \geq 0, \tag{13}$$

then

$$|\Omega| \int_{\Omega} \frac{u(x,t)}{f(t, u(x,t))} dx \geq \int_{\Omega} u(x,t) dx \int_{\Omega} \frac{1}{f(t, u(x,t))} dx \tag{14}$$

holds. If (13) reverses, then (14) reverses.

Remark 2. It is well known that if $f(x)$ is convex and $2f - f^2 \geq 0$ then $1/f(x)$ is convex.

3. Main Results. The following notation will be used for convenience:

$$V(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x,t) dx, \quad |\Omega| = \int_{\Omega} dx. \tag{15}$$

Throughout the paper, let us assume that $D_+^\alpha V(t) > 0$.

We begin with the following theorem.

Theorem 2. If the fractional differential inequality

$$\begin{aligned}
 &\frac{d}{dt} \left(r(t) D_+^\alpha \left(\frac{V(t)}{h(t, V(t))} \right) \right) + \mu q(t) D_+^\alpha V(t) + f(t, E(t)) \leq 0, \quad t \neq t_k, \\
 &b_k^* \leq \frac{V(t_k^+)}{V(t_k)} \leq b_k, \quad c_k^* \leq \frac{V'(t_k^+)}{V'(t_k)} \leq c_k, \quad k = 1, 2, \dots
 \end{aligned} \tag{16}$$

has no eventually positive solution, then every solution of equation (1), (2) is oscillatory in G .

Proof. Assume, for the sake of contradiction, that there is a nonoscillatory solution $u(x, t)$ of (1) and (2), which has a constant sign in the domain $\Omega \times [t_0, +\infty)$. With no loss of generality, we assume that $u(x, t) > 0$, $(x, t) \in \Omega \times [t_0, +\infty)$, $t_0 \geq 0$. Equation (1) is multiplied in both sides by $1/|\Omega|$ and integrated with respect to x over the domain Ω ; we get

$$\begin{aligned} & \frac{1}{|\Omega|} \int_{\Omega} \frac{d}{dt} \left(r(t) D_{+,t}^{\alpha} \left(\frac{u(x, t)}{h(t, u(x, t))} \right) \right) dx + \frac{1}{|\Omega|} \int_{\Omega} q(x, t) g(D_{+,t}^{\alpha} u(x, t)) dx = \\ & = \frac{1}{|\Omega|} \int_{\Omega} a(t) \Delta u(x, t) dx - \frac{1}{|\Omega|} \int_{\Omega} f \left(t, \int_0^t (t-s)^{-\alpha} u(x, s) ds \right) dx + \\ & \qquad \qquad \qquad + \frac{1}{|\Omega|} \int_{\Omega} F(x, t) dx. \quad (17) \end{aligned}$$

From Green's formula and the Robin boundary condition (2), we get

$$\int_{\Omega} \Delta u(x, t) dx = \int_{\partial\Omega} \frac{\partial u(x, t)}{\partial \eta} dS = - \int_{\partial\Omega} \gamma(x, t) u(x, t) dS \leq 0, \quad (18)$$

where dS is the surface element on $\partial\Omega$. From (A_2) , (A_3) and Jensen's inequality, we have

$$\begin{aligned} & \frac{1}{|\Omega|} \int_{\Omega} q(x, t) g(D_{+,t}^{\alpha} u(x, t)) dx \geq q(t) \frac{1}{|\Omega|} \int_{\Omega} g(D_{+,t}^{\alpha} u(x, t)) dx \geq \\ & \geq q(t) g(D_{+,t}^{\alpha} V(t)) \geq \mu q(t) D_{+,t}^{\alpha} V(t). \quad (19) \end{aligned}$$

Again applying Jensen's inequality and using Lemma 1,

$$\begin{aligned} & \frac{1}{|\Omega|} \int_{\Omega} f \left(t, \int_0^t (t-s)^{-\alpha} u(x, s) ds \right) dx \geq f \left(t, \frac{1}{|\Omega|} \int_{\Omega} \int_0^t (t-s)^{-\alpha} u(x, s) ds dx \right) \\ & = f \left(t, \int_0^t (t-s)^{-\alpha} V(s) ds \right) = f(t, E(t)). \quad (20) \end{aligned}$$

Now applying Remark 1 and 2,

$$\begin{aligned} & \frac{1}{|\Omega|} \int_{\Omega} \frac{d}{dt} \left(r(t) D_{+,t}^{\alpha} \left(\frac{u(x,t)}{h(t,u(x,t))} \right) \right) dx \geq \\ & \geq \frac{d}{dt} \left(r(t) D_{+}^{\alpha} \left(\frac{1}{|\Omega|} \int_{\Omega} \frac{u(x,t)}{h(t,u(x,t))} \right) \right) dx \geq \frac{d}{dt} \left(r(t) D_{+}^{\alpha} \left(\frac{V(t)}{h(t,V(t))} \right) \right). \end{aligned} \tag{21}$$

In view of (17)–(21) and (A_4) yield

$$\frac{d}{dt} \left(r(t) D_{+}^{\alpha} \left(\frac{V(t)}{h(t,V(t))} \right) \right) + \mu q(t) D_{+}^{\alpha} V(t) + f(t, E(t)) \leq 0, \quad t \neq t_k.$$

For $t \geq t_0$, $t = t_k$, $k = 1, 2, \dots$, multiplying both sides of (1) by $\frac{1}{|\Omega|}$ and integrating with respect to x over the domain Ω , we obtain

$$b_k^* \leq \frac{u(x, t_k^+)}{u(x, t_k)} \leq b_k, \quad c_k^* \leq \frac{u_t(x, t_k^+)}{u_t(x, t_k)} \leq c_k.$$

According to (15), we get

$$b_k^* \leq \frac{V(t_k^+)}{V(t_k)} \leq b_k, \quad c_k^* \leq \frac{V'(t_k^+)}{V'(t_k)} \leq c_k, \quad k = 1, 2, \dots$$

Hence, $V(t)$ is a positive solution of impulsive differential inequality (16). \square

Theorem 3. *If there exists a positive function $\phi \in C^1([0, \infty); R_+)$ and*

$$\int_{t_0}^{+\infty} \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left(p(s)\phi(s) - \frac{1}{4M\Gamma(1-\alpha)} \frac{(\phi'(s))^2 r(s)}{\phi(s) \zeta(s)} \right) ds = +\infty, \tag{22}$$

holds, then every solution of boundary value problem (1)–(2) is oscillatory in G .

Proof. Assume that there exists a nonoscillatory solution $u(x, t)$ of (1)–(2) if the conditions of Theorem 3 hold. Suppose that $V(t) > 0$ is a solution of inequality (16) and $D_{+}^{\alpha} V(t) > 0$. By (A_3) , inequality (16) can be reduced to the following form

$$\frac{d}{dt} \left(r(t) D_{+}^{\alpha} \left(\frac{V(t)}{h(t,V(t))} \right) \right) + \mu q(t) D_{+}^{\alpha} V(t) + p(t)E(t) \leq 0, \quad t \neq t_k. \tag{23}$$

Define

$$Z(t) = \phi(t) \frac{r(t) D_+^\alpha \left(\frac{V(t)}{h(t, V(t))} \right)}{E(t)}, \quad t \geq t_1. \quad (24)$$

Then $Z(t) > 0$, since $E(t) > 0$ for $t \geq t_1$. Differentiating (24) and using (16) and Lemma 1, we have

$$\begin{aligned} Z'(t) &\leq \frac{\phi'(t)}{\phi(t)} Z(t) - \mu q(t) \phi(t) \frac{D_+^\alpha V(t)}{E(t)} - p(t) \phi(t) - Z(t) \frac{E'(t)}{E(t)} \leq \\ &\leq \frac{\phi'(t)}{\phi(t)} Z(t) - p(t) \phi(t) - Z^2(t) \frac{E'(t)}{\phi(t) r(t) D_+^\alpha \left(\frac{V(t)}{h(t, V(t))} \right)} \leq \\ &\leq \frac{\phi'(t)}{\phi(t)} Z(t) - p(t) \phi(t) - \Gamma(1 - \alpha) \frac{h(t, V(t))}{\phi(t) r(t)} Z^2(t). \end{aligned}$$

By (A_4) , we get

$$\begin{aligned} Z'(t) &\leq \frac{\phi'(t)}{\phi(t)} Z(t) - p(t) \phi(t) - M \Gamma(1 - \alpha) \frac{\zeta(t)}{\phi(t) r(t)} Z^2(t), \\ Z(t_k^+) &\leq \frac{c_k}{b_k^*} Z(t_k). \end{aligned} \quad (25)$$

Define

$$\Theta(t) = \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*} \right)^{-1} Z(t).$$

In fact, $Z(t)$ is continuous on each interval $(t_k, t_{k+1}]$ and in consideration of (25). It follows that for $t \geq t_0$,

$$\Theta(t_k^+) = \prod_{t_0 \leq t_i \leq t_k} \left(\frac{c_k}{b_k^*} \right)^{-1} Z(t_k^+) \leq \prod_{t_0 \leq t_i < t_k} \left(\frac{c_k}{b_k^*} \right)^{-1} Z(t_k) = \Theta(t_k),$$

and for all $t \geq t_0$,

$$\Theta(t_k^-) = \prod_{t_0 \leq t_i \leq t_{k-1}} \left(\frac{c_k}{b_k^*} \right)^{-1} Z(t_k^-) \leq \prod_{t_0 \leq t_i < t_k} \left(\frac{c_k}{b_k^*} \right)^{-1} Z(t_k) = \Theta(t_k),$$

which implies that $\Theta(t)$ is continuous on $[t_0, +\infty)$.

$$\begin{aligned} &\Theta'(t) + M\Gamma(1 - \alpha) \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right) \frac{\zeta(t)}{\phi(t)r(t)} \Theta^2(t) + \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right)^{-1} p(t)\phi(t) - \\ &\quad - \frac{\phi'(t)}{\phi(t)} \Theta(t) = \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right)^{-1} Z'(t) - \prod_{t_0 \leq t_i < t_k} \left(\frac{c_k}{b_k^*}\right)^{-1} \frac{\phi'(t)}{\phi(t)} \Theta(t) + \\ &+ \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right) \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right)^{-2} \Gamma(1 - \alpha) \frac{M\zeta(t)\Theta^2(t)}{\phi(t)r(t)} + \prod_{t_0 \leq t_i < t_k} \left(\frac{c_k}{b_k^*}\right)^{-1} p(t)\phi(t) = \\ &= \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right)^{-1} \left[Z'(t) - \frac{\phi'(t)}{\phi(t)} Z(t) + p(t)\phi(t) + M\Gamma(1 - \alpha) \frac{\zeta(t)}{\phi(t)r(t)} Z^2(t) \right] \leq 0. \end{aligned}$$

That is,

$$\begin{aligned} \Theta'(t) \leq &-M\Gamma(1 - \alpha) \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right) \frac{\zeta(t)}{\phi(t)r(t)} \Theta^2(t) - \\ &- \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right)^{-1} p(t)\phi(t) + \frac{\phi'(t)}{\phi(t)} \Theta(t). \end{aligned}$$

Then, using the inequality, $\lambda AB^{\lambda-1} - A^\lambda \leq (\lambda - 1)B^\lambda$, $\lambda \geq 1$, we have

$$\Theta'(t) \leq - \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*}\right)^{-1} \left[p(t)\phi(t) - \frac{1}{4M\Gamma(1 - \alpha)} \frac{(\phi'(t))^2 r(t)}{\phi(t) \zeta(t)} \right].$$

Integrating the last inequality from t_0 to t , which yields

$$\Theta(t) \leq \Theta(t_0) - \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*}\right)^{-1} \left[p(s)\phi(s) - \frac{1}{4M\Gamma(1 - \alpha)} \frac{(\phi'(s))^2 r(s)}{\phi(s) \zeta(s)} \right] ds.$$

Letting $t \rightarrow +\infty$, we get a contradiction to the hypothesis (22). \square

Theorem 4. Assume that there exist functions $\phi(t)$ and $\psi(t) \in C^1([0, \infty); (0, +\infty))$ in which $\phi(t)$ is nondecreasing with respect to t and the functions $H(t, s), h(t, s) \in C^1(D, R)$ in which $D = \{(t, s) | t \geq s \geq t_0 > 0\}$, such that

$$(T_1) \quad H(t, t) = 0 \text{ for } t \geq t_0 \text{ and } H(t, s) > 0 \text{ for } t > s \geq t_0;$$

$$(T_2) \quad H'_t(t, s) \geq 0, H'_s(t, s) \leq 0;$$

$$(T_3) \quad h(t, s) = -\frac{\partial H(t, s)}{\partial s} \psi(s) - H(t, s) \psi'(s) - H(t, s) \psi(s) \frac{\phi'(s)}{\phi(s)}.$$

If

$$\limsup_{t \rightarrow +\infty} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left(p(s) \phi(s) H(t, s) \psi(s) - \frac{1}{4M\Gamma(1-\alpha)} \frac{h^2(t, s) \Psi(s)}{H(t, s) \psi(s)} \right) ds = \infty, \quad (26)$$

where $\Psi(s) = \frac{\phi(s)r(s)}{\zeta(s)}$, then each solution of (1)–(2) is oscillatory in G .

Proof. We prove that the equation (26) has no eventually positive solutions if the conditions of Theorem 4 hold. As in the proof of Theorem 3, we obtain

$$\begin{aligned} \Theta'(t) \leq & -M\Gamma(1-\alpha) \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*} \right) \frac{\zeta(t)}{\phi(t)r(t)} \Theta^2(t) - \\ & - \prod_{t_0 \leq t_k < t} \left(\frac{c_k}{b_k^*} \right)^{-1} p(t) \phi(t) + \frac{\phi'(t)}{\phi(t)} \Theta(t). \end{aligned}$$

Multiplying the above inequality by $H(t, s)\psi(s)$ for $t \geq s \geq T$ and integrating from T to t , we have

$$\begin{aligned} \int_T^t H(t, s) \psi(s) \Theta'(s) ds & \leq - \int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} H(t, s) \psi(s) p(s) \phi(s) ds - \\ & - M\Gamma(1-\alpha) \int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right) H(t, s) \psi(s) \frac{\zeta(s)}{\phi(s)r(s)} \Theta^2(s) ds + \\ & + \int_T^t H(t, s) \psi(s) \frac{\phi'(s)}{\phi(s)} \Theta(s) ds. \end{aligned}$$

Thus

$$\int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} H(t, s) \psi(s) p(s) \phi(s) ds \leq H(t, T) \psi(T) \Theta(T) -$$

$$\begin{aligned}
 & - \int_T^t \left[- \frac{\partial H(t, s)}{\partial s} \psi(s) - H(t, s) \psi'(s) - H(t, s) \psi(s) \frac{\phi'(s)}{\phi(s)} \right] ds - \\
 & - M\Gamma(1 - \alpha) \int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right) H(t, s) \psi(s) \frac{\zeta(s)}{\phi(s)r(s)} \Theta^2(s) ds,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & \int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left[H(t, s) \psi(s) p(s) \phi(s) ds - \right. \\
 & \left. - \frac{1}{4M\Gamma(1 - \alpha)} \frac{h^2(t, s) \phi(s) r(s)}{\zeta(s) H(t, s) \psi(s)} \right] ds \leq H(t, T) \psi(T) \Theta(T), \tag{27}
 \end{aligned}$$

from (27), for $t \geq T \geq t_0$, we have

$$\begin{aligned}
 & \frac{1}{H(t, t_0)} \int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left[H(t, s) \psi(s) p(s) \phi(s) - \right. \\
 & \left. - \frac{1}{4M\Gamma(1 - \alpha)} \frac{h^2(t, s) \phi(s) r(s)}{\zeta(s) H(t, s) \psi(s)} \right] ds = \\
 & = \frac{1}{H(t, t_0)} \left[\int_{t_0}^T + \int_T^t \right] \left\{ \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left[H(t, s) \psi(s) p(s) \phi(s) - \right. \right. \\
 & \left. \left. - \frac{1}{4M\Gamma(1 - \alpha)} \frac{h^2(t, s) \phi(s) r(s)}{\zeta(s) H(t, s) \psi(s)} \right] \right\} ds \leq \\
 & \leq \int_{t_0}^T \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \psi(s) p(s) \phi(s) ds + \psi(T) \Theta(T).
 \end{aligned}$$

Letting $t \rightarrow +\infty$, we have

$$\begin{aligned}
 & \limsup_{t \rightarrow +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left[H(t, s) \psi(s) p(s) \phi(s) - \right. \\
 & \left. - \frac{1}{4M\Gamma(1 - \alpha)} \frac{h^2(t, s) \Psi(s)}{H(t, s) \psi(s)} \right] ds \leq
 \end{aligned}$$

$$\leq \int_{t_0}^T \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \psi(s)p(s)\phi(s)ds + \psi(T)\Theta(T) < \infty,$$

which leads to a contradiction with (26). \square

Remark 3. In Theorem 4, by choosing $\phi(t) = \psi(t) = 1$, we have the following result.

Corollary 1. Assume that the conditions of Theorem 4 hold, and

$$\limsup_{t \rightarrow +\infty} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left(p(s)H(t, s) - \frac{1}{4M\Gamma(1 - \alpha)} \frac{h^2(t, s)\Psi(s)}{H(t, s)\zeta(s)} \right) ds = \infty, \tag{28}$$

where $h(t, s) = -\frac{\partial H(t, s)}{\partial s}$, then each solution of (1), (2) is oscillatory.

Remark 4. All the above theorems and corollaries are true for the equation (1) with the boundary condition (3).

Remark 5. The results obtained in this paper can be extended for the more general differential equations having a damped term with a delay of the form

$$\begin{aligned} & \frac{\partial}{\partial t} \left(r(t)D_{+,t}^\alpha \left(\frac{u(x, t)}{f(t, u(x, t))} \right) \right) + q(x, t)g \left(D_{+,t}^\alpha \left(\frac{u(x, t)}{f(t, u(x, t))} \right) \right) + \\ & + r(t)u(x, \sigma(t)) + h \left(\int_0^t (t - \xi)^{-\alpha} u(x, \xi) d\xi \right) = a(t)\Delta u(x, t) + F(x, t), \end{aligned}$$

$$u(x, t_k^+) = \gamma_k(x, t_k, u(x, t_k)),$$

$$u_t(x, t_k^+) = \delta_k(x, t_k, u_t(x, t_k)), \quad k = 1, 2, \dots, \quad (x, t) \in \Omega \times R_+$$

4. Examples. In this section, we give examples to illustrate our main results.

Example 1. Consider the fractional nonlinear partial hybrid differen-

tial equations

$$\left. \begin{aligned} & \frac{\partial}{\partial t} \left(\sqrt{2} D_{+,t}^{\frac{1}{2}} \left(\frac{u(x,t)}{e^{-t}} \right) \right) + D_{+,t}^{\frac{1}{2}} u(x,t) = e^t \Delta u(x,t) - \\ & - \frac{1}{\sqrt{2\pi}(\cos tC(x) + \sin tS(x))} \int_0^t (t-s)^{-\frac{1}{2}} u(x,s) ds + F(x,t), \quad t \neq t_k, \\ & u(x, t_k^+) = \frac{k}{k+1} (x, t_k, u(x, t_k)), \\ & u_t(x, t_k^+) = u_t(x, t_k, u_t(x, t_k)), \quad k = 1, 2, \dots, \end{aligned} \right\} \quad (29)$$

for $(x, t) \in (0, \pi) \times [0, +\infty) = G$, with the boundary condition

$$\frac{\partial}{\partial x} u(0, t) = \frac{\partial}{\partial x} u(\pi, t) = 0. \quad (30)$$

Here $\alpha = \frac{1}{2}$, $r(t) = \sqrt{2}$, $a(t) = e^t$, $h(t,u) = e^{-t}$, $\zeta(t) = e^{-t}$, $q(t) = 1$, $b_k = b_k^* = 1$, $c_k = c_k^* = \frac{k+1}{k}$, $p(t) = 1/(\sqrt{2\pi}(\cos tC(x) + \sin tS(x)))$ where $C(x)$ and $S(x)$ are the Fresnel integrals, namely

$$C(x) = \int_0^x \cos \left(\frac{1}{2} \pi t^2 \right) dt, \quad S(x) = \int_0^x \sin \left(\frac{1}{2} \pi t^2 \right) dt$$

with $|C(x)| \leq \pi$, $|S(x)| \leq \pi$, $g(u) = u$, $\mu = 1$, $M = 1$ and

$$F(x, t) = \cos x \widehat{C}_t \left(\frac{1}{2}, -1, 1 \right) + \sqrt{\frac{t}{\pi}} - 1 - \cos x \sin t,$$

where

$$\widehat{C}_t \left(\frac{1}{2}, -1, 1 \right) = \frac{e^{-t}}{\Gamma(\frac{1}{2})} \int_0^t s^{-\frac{1}{2}} e^{-s} \cos(t-s) ds.$$

It is easy to see that $D^{\frac{1}{2}} e^{-t} = e^{-t}(\sqrt{\frac{\pi}{t}} - 1) > 0$ if $t < \pi$.

If we take $\phi(t) = 1$ then $\phi'(t) = 0$, since $t_0 = 1$, $t_k = 2^k$. Consider

$$\int_{t_0}^{+\infty} \prod_{t_0 \leq t_k < s} \left(\frac{c_k}{b_k^*} \right)^{-1} \left(p(s)\phi(s) - \frac{1}{4M\Gamma(1-\alpha)} \frac{(\phi'(s))^2 r(s)}{\phi(s) \zeta(s)} \right) ds =$$

$$\begin{aligned}
&= \lim_{t \rightarrow +\infty} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \frac{k}{k+1} \frac{1}{\sqrt{2\pi}(\cos sC(x) + \sin sS(x))} ds \geq \\
&\geq (2\pi)^{\frac{3}{2}} \int_1^{+\infty} \prod_{1 < t_k < s} \frac{k}{k+1} ds = \\
&= (2\pi)^{\frac{3}{2}} \left(\int_1^{t_1} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \int_{t_1^+}^{t_2} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \dots \right) = \\
&= (2\pi)^{\frac{3}{2}} \left(1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \dots \right) = (2\pi)^{\frac{3}{2}} \sum_{n=0}^{+\infty} \frac{2^n}{n+1} = +\infty.
\end{aligned}$$

Thus, all the conditions of Theorem 3 are satisfied. Hence, every solution of (29) and (30) is oscillatory. In fact, $u(x, t) = \cos x e^{-t} \cos t$ is one such solution of (29)–(30).

5. Conclusion. In this article, we have identified some new sufficient conditions for all solutions of fractional impulsive partial differential equations to be oscillatory, which has a scope beyond the available results in the existing literature. Required example has also been incorporated in the paper for the confirmation of the results.

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