RANKING RANGE BASED APPROACH TO MADM UNDER INCOMPLETE CONTEXT AND ITS APPLICATION IN VENTURE INVESTMENT EVALUATION

Yating LIU[1], Hengjie ZHANG[2], Yuzhu WU[3], Yucheng DONG[3*]

[1]*School of Business Administration, Xi'an University of Technology, Xi'an 710048, China*
[2]*Business School, Hohai University, Nanjing 211100, China*
[3]*Business School, State Key Laboratory of Hydraulics and Mountain River Engineering, Sichuan University, Chengdu 610065, China*

Abstract. In real-world Multiple Attribute Decision Making (MADM) problem, the attribute weights information may be unknown or partially known. Several approaches have been suggested to address this kind of incomplete MADM problem. However, these approaches depend on the determination of attribute weights, and setting different attribute weight vectors may result in different ranking positions of alternatives. To deal with this issue, this paper develops a novel MADM approach: the ranking range based MADM approach. In the novel MADM approach, the minimum and maximum ranking positions of every alternative are generated using several optimization models, and the average ranking position of every alternative is produced applying the Monte Carlo simulation method. Then, the minimum, maximum and average ranking positions of the alternative are integrated into a new ranking position of the alternative. This novel approach is capable of dealing with venture investment evaluation problems. However, in the venture investment evaluation process, decision makers will present different risk attitudes. To deal with this issue, two ranking range based MADM approaches with risk attitudes are further designed. A case study and a simulation experiment are presented to show the validity of the proposal.

Keywords: multiple attribute decision making, ranking range, incomplete attribute weights, venture investment evaluation, risk attitudes.

JEL Classification: C02, C61, D81.

Introduction

Multiple Attribute Decision Making (MADM) aims to rank the alternatives based on their evaluation information associated with multiple attributes (Fan, Ma, & Zhang, 2002; Liu, Dong, Chiclana, Cabrerizo, & Herrera-Viedma, 2017; Liu, Dong, Liang, Chiclana, & Herrera-Viedma, 2018; Wu, Cao, & Li, 2016; Xu, Wang, & Merigó, 2014; Yager, 2016), which has

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*Corresponding author. E-mail: *ycdong@scu.edu.cn*

been used widely in many areas, such as engineering, technology, economy, management and military (Butler, Morrice, & Mullarkey, 2001; Cid-López, Hornos, Carrasco-Gónzález, & Herrera-Viedma, 2018; Shevchenko, Ustinovichius, & Andruškevičius, 2008; Yoon, 1987; Singh, Gupta, & Mehra, 2017; Zaveckaite, & Ulbinaite, 2018).

To date, many MADM models have been reported in literature (Yoon & Hwang, 1981; Zeleny, 1982). Commonly used MADM approaches include: MAUT (Multiple Attribute Utility Theory) (Keeney & Raiffa, 1976); SAW (Simple Additive Weighting) (Butler et al., 2001); AHP (Analytic Hierarchy Process) (Saaty, 1981, 2013); TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) (Yoon & Hwang, 1981; Zeleny, 1982); VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje) (Opricovic & Tzeng, 2004); ELEC-TRE (ELimination and Choice Expressing REality) (Roy & Bertier, 1972); PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluation) (Mareschal, Brans, & Vincke, 1984); TODIM (an acronym in Portuguese of Interactive and Multicriteria Decision Making) (Gomes & Lima, 1992). Several comprehensive comparative analyses of different MADM models have been conducted in literature. For example, Zanakis, Solomon, Wishart, and Dublish (1998) investigated the performance of different MADM models through a simulation comparison. Karni, Sanchez, and Tummala (1990) analyzed consensus degree of the ranking results obtained by various MADM models, and the results indicated that the AHP, SAW, and ELECTRE rankings do not differ significantly. Wallenius et al. (2008) provided a famous literature review of existing MADM models. Recently, Kabak and Ervural (2017) presented a generic conceptual framework and a classification scheme for MADM models. Meanwhile, the MADM models have been successfully used in various areas, including: resource allocation, plant location and supplier selection, etc. (López, Carrillo, Chavira, & Noriega, 2017; Opricovic, 1998; Siskos & Zopounidis, 1987; Zeleny, 1982).

In ideal MADM problems, the weights of the attributes are assumed known precisely. This assumption may be unreasonable due to time pressure or limited expertise in some MADM problems. In the real-world MADM problems, the attribute weights information may be unknown or partially known, which results in incomplete MADM problem. In order to solve incomplete MADM problems, a lot of methods have been proposed. For example, Danielson, Ekenberg, and He (2014) provided an augmenting ordinal method for assessing attribute weights vector, and the obtained attribute weights are adopted in the aggregation process of the MADM. A.T. de Almeida, J.A. de Almeida, Costa, and de Almeida-Filho (2016) proposed the Flexible and Interactive Tradeoff (FITradeoff) method to obtain the attribute weights vector from the partial information on attribute weights. In addition to incomplete attribute weights, the decision making problem with incomplete assessment information has been extensively studied (Cid-López et al., 2018; Cabrerizo, Herrera-Viedma, & Pedrycz, 2013; Cabrerizo et al., 2017; Li, Rodríguez, Martínez, Dong, & Herrera, 2018). For example, Wu, Chiclana, and Herrera-Viedma (2015) presented a new trust based consensus model for managing incomplete linguistic assessment information in social network group decision making (GDM).

The existing models for incomplete MADM relied on the setting of attribute weights. A major criticism of these models is that different attribute weights may yield different results, Dong, Liu, Liang, Chiclana, and Herrera-Viedma (2018a) proposed the concept of strategic weight manipulation, in which a decision maker can strategically set the attribute weights to

obtain his/her desired ranking of alternatives. In order to overcome the shortcomings of the existing MADM models for incomplete MADM problems, we propose a ranking range based MADM approach under incomplete context. The significant characteristic of our approach is that it doesn't depend on the setting of the attribute weights. In this proposed approach, we establish a series of mixed 0-1 linear programming models to obtain the minimum and maximum ranking positions of every alternative from the multiple attribute decision matrix with incomplete attribute weights information. Then, we apply the Monte Carlo simulation method to compute the average ranking position of every alternative. Finally, the minimum, maximum and average ranking positions of every alternative are integrated into a new ranking position of the alternative. A simulation experiment is presented to show the validity of the ranking range based MADM approach.

Venture investment evaluation involving multiple projects (can be seen as the alternatives in the MADM) and multiple evaluation indices (can be seen as the evaluation attribute in the MADM) can be regarded as a MADM problem. The MADM approaches have been widely used in venture investment evaluation system (Aggarwal, Kryscynski, & Singh, 2015; Barrot, 2016; Hall & Hofer, 1993; Tyebjee & Bruno, 1984). In this paper, the proposed ranking range based MADM approach can be used to help the investors select the best investment project(s). However, in a venture investment evaluation problem, the investors often present different risk attitudes. To deal with this kind of MADM, two ranking range based MADM approaches with risk attitudes are further designed. A case study is used to show the practicality of the proposal.

The rest of this paper is organized as follows. Section 1 introduces MADM problem with incomplete attribute weights information. Then, Section 2 proposes the ranking range based approach to deal with the incomplete MADM problem. Following this, Section 3 further develops two ranking range based approaches to cope with MADM problems with risk attitudes, and discusses their applications in venture investment evaluation problems. Subsequently, Section 4 provides a simulation experiment to demonstrate the validity of the proposed approach. Finally, concluding remarks and future research agenda are discussed in last Section.

1. Incomplete MADM problem

This section introduces the MADM problem with incomplete attribute weights information.

Let $X = \{x_1, ..., x_n\}$ be a finite set of alternatives, and $A = \{a_1, ..., a_m\}$ be a set of attributes. Let $w = (w_1, w_2, ..., w_m)$ be the weight vector of the attributes, where $w_j \geq 0$ denotes the weight of attribute a_j and $\sum_{j=1}^{m} w_j = 1$. Let $V = [v_{ij}]_{n \times m}$ be the multiple attribute decision matrix given by the decision maker, where v_{ij} denotes the preference value of the alternative $x_i \in X$ associated with attribute $a_j \in A$.

Generally, the resolution process of MADM problems includes the following two steps:

(1) *Normalization of the multiple attribute decision matrix*

In MADM problems, we can classify the attributes into two categories: benefit and cost attributes. The decision matrix $V = [v_{ij}]_{n \times m}$ should be normalized into a standardized decision matrix $\overline{V} = [\overline{v}_{ij}]_{n \times m}$ (Chen, Zhang, & Dong, 2015; Dong et al., 2018a), where

$$\overline{v}_{ij} = \frac{\max_i(v_{ij}) - v_{ij}}{\max_i(v_{ij}) - \min_i(v_{ij})}, \tag{1}$$

if $a_j \in A$ is a cost attribute, and

$$\overline{v}_{ij} = \frac{v_{ij} - \min_i(v_{ij})}{\max_i(v_{ij}) - \min_i(v_{ij})}, \tag{2}$$

if $a_j \in A$ is a benefit attribute.

(2) *Generating a ranking position of the alternatives*

Let $D_w(x_i)$ be the overall evaluation value to alternative x_i generated from $\overline{V} = [\overline{v}_{ij}]_{n \times m}$ and $w = (w_1, w_2, ..., w_m)^T$. Many approaches have been reported to produce $D_w(x_i)$ (Ishizaka & Nemery, 2013; Yu, Zhang, Zhong, & Sun, 2017; B. W. Zhang, Liang, & G. Q. Zhang, 2018). In particular, the WA (weighted averaging) (Chen, Zhang, & Dong, 2015; Pérez, Cabrerizo, Alonso, Dong, Chiclana, & Herrera-Viedma, 2018) and OWA (Ordered weighted averaging) operators (Dong et al., 2018a; Yager, 1988) are used widely in the MADM. Without loss of generality, this paper uses WA and OWA operators to yield $D_w(x_i)$:

$$D_w(x_i) = WA_w(\overline{v}_{i1}, \overline{v}_{i2}, ..., \overline{v}_{im}) = \sum_{j=1}^{m} w_j \overline{v}_{ij}; \tag{3}$$

$$D_w(x_i) = OWA_w(\overline{v}_{i1}, \overline{v}_{i2}, ..., \overline{v}_{im}) = \sum_{j=1}^{m} w_j \overline{v}_{i(j)}, \tag{4}$$

where $\overline{v}_{i(j)}$ is the *jth* largest value in $\{\overline{v}_{i1}, \overline{v}_{i2}, ..., \overline{v}_{im}\}$.

Based on the Eqs (3)–(4), we can obtain the ranking position of the alternative x_k as follows:

$$r_w(x_k) = |Q_{k,w}| + 1, \tag{5}$$

where $Q_{k,w} = \{x_i | D_w(x_i) > D_w(x_k), \ i = 1, 2, ..., n\}$ denotes the set of the alternatives whose evaluation value(s) is(are) greater than that of the alternative x_k, and $|Q_{k,w}|$ signifies the number of alternatives in $Q_{k,w}$ (Dong et al., 2018a).

In ideal MADM problems, the weights of the attributes are assumed known precisely. This assumption may be unreasonable due to time pressure or limited expertise in some practical MADM problems. In the real-world MADM problems, the attribute weights information may be unknown or partially known, which results in incomplete MADM problem. Usually, the known weight information on the attribute can be constructed using the following basic forms:

(i) Weak ranking: $S_1 = \{w_i \geq w_j \geq 0\} \ (i \neq j)$;

(ii) Strict ranking: $S_2 = \{w_i - w_j \geq \alpha_{ij}\} \ (\alpha_{ij} > 0; i \neq j)$;

(iii) Ranking multiples: $S_3 = \{w_i \geq \alpha_{ij} w_j\} \ (i \neq j)$;

(iv) Interval form: $S_4 = \{\gamma_i \leq w_i \leq \gamma_i + \varepsilon_i\} \ (0 \leq \gamma_i \leq \gamma_i + \varepsilon_i \leq 1)$;

(v) Ranking differences: $S_5 = \{w_i - w_j \geq w_k - w_i\} \ (i \neq j \neq k)$.

In practical, the weights information may consist of above multiple basic forms. In general, we denote S as the set of incomplete attribute weights information. In particular, $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$.

In the following, we use an example to explain the basic forms of the attribute weights information.

Example 1: Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of 6 attributes.

(i) The weights information with strict ranking form may be given by:

$$S = \{(w_1, w_2, ..., w_6)^T \mid w_1 - w_5 \geq 0.17, \quad w_2 - w_3 \geq 0.5\};$$

(ii) The weights information with ranking multiples form can be given by:

$$S = \{(w_1, w_2, ..., w_6)^T \mid w_1 \geq 3w_3, \quad w_2 \geq 0.5w_6 \geq 3w_5\};$$

(iii) The weights information consists of interval and ranking differences forms may be given by:

$$S = \{(w_1, w_2, ..., w_6)^T \mid 0.1 \leq w_4 \leq 0.25, \quad w_1 - w_3 \geq w_4 - w_2\}.$$

2. Ranking range based MADM approach

In this section, we present a novel MADM approach, namely, the ranking range based MADM approach, to solve the incomplete MADM problem.

2.1. Ranking range under incomplete attribute weights

This section introduces the ranking range under incomplete attribute weights.

Let S be as above in section 2. Let $r_{w \in S}(x_h)$ be the ranking of alternative x_h generated from $V = [v_{ij}]_{n \times m}$ and $w = (w_1, w_2, ..., w_m)^T$ $(w \in S)$. Naturally, $r_{w \in S}(x_h)$ will change when setting different $w = (w_1, w_2, ..., w_m)^T$ $(w \in S)$.

Then, we give the definition of ranking ranges of the alternatives under incomplete attribute weights set S, which can be presented as follows.

Definition 1: In MADM problems, we call $R_{w \in S}(x_i) = [\underline{r}_{w \in S}(x_i), \overline{r}_{w \in S}(x_i)]$ as the ranking range of the alternative x_i under incomplete attribute weights set S, with $\underline{r}_{w \in S}(x_i) = \min_{w \in S} r_w(x_i)$ and $\overline{r}_{w \in S}(x_i) = \max_{w \in S} r_w(x_i)$ being the best and worst ranking positions of alternative x_i, respectively.

In the following, we use an example to help us understand Definition 1.

Example 2: Let $\overline{V} = [\overline{v}_{ij}]_{5 \times 4}$ be a standardized decision matrix over five alternatives $X = \{x_1, ..., x_5\}$ and four attributes $A = \{a_1, ..., a_4\}$. $\overline{V}$ is provided below:

$$\overline{V} = \begin{bmatrix} 0.89 & 1 & 0.9 & 0.76 \\ 0.74 & 0.59 & 0.8 & 0.79 \\ 0.92 & 0.45 & 0.53 & 1 \\ 1 & 0.47 & 0.91 & 0.63 \\ 0.7 & 0.89 & 1 & 0.85 \end{bmatrix}.$$

(1) When WA (i.e., Eq. (3)) is adopted as the decision model, different ranking positions of alternatives will be generated when setting different $w = (w_1, ..., w_4)^T$.

For example, (i) we have $r_w^{WA}(x_1)=1$ if we set $w=(0.09,0.75,0.03,0.13)^T$; (ii) we have $r_w^{WA}(x_1)=2$ if we set $w=(0.15,0.2,0.19,0.46)^T$; (iii) we have $r_w^{WA}(x_1)=3$ if we set $w=(0.2,0.2,0.2,0.6)^T$; (iv) we have $r_w^{WA}(x_1)=4$ if we set $w=(0.05,0.01,0.04,0.9)^T$.

In addition, $r_w^{WA}(x_1)=5$ cannot be achieved no matter what the attribute weights vector is. So, the ranking range of alternative x_1, is $R^{WA}(x_1)=[1,4]$.

Using a similar way, we can obtain the ranking ranges of alternatives $\{x_2,x_3,x_4,x_5\}$, and they are $R^{WA}(x_2)=[3,5]$, $R^{WA}(x_3)=[1,5]$, $R^{WA}(x_4)=[1,5]$, and $R^{WA}(x_5)=[1,5]$.

(2) When OWA (i.e., Eq (4)) is employed as the decision model, the ranking ranges of the five alternatives can also be generated. And they are $R^{OWA}(x_1)=[1,4]$, $R^{OWA}(x_2)=[3,5]$, $R^{OWA}(x_3)=[1,5]$, $R^{OWA}(x_4)=[1,4]$, and $R^{OWA}(x_5)=[1,4]$.

2.2. The proposed MADM approach

Based on the ranking range presented in section 3.1, here we present ranking range based MADM approach.

The proposed ranking range based MADM approach is mainly composed of three steps: (i) Generating the best ranking position ($\underline{r}_{w \in S}(x_i)$) and worst ranking position ($\overline{r}_{w \in S}(x_i)$) of every alternative by solving several mixed 0-1 linear programming models; (ii) Producing the average ranking position of every alternative ($\overline{\overline{r}}_{w \in S}(x_i)$) using Monte Carlo simulation method; (iii) Integrating the best, worst and average ranking positions of each alternative into a comprehensive ranking position of the alternative.

In the following, we give the details of the process of the ranking range based MADM approach.

(1) Generating the best and worst ranking positions of alternatives

In this step, we establish several mixed 0–1 linear programming models to compute the best and worst ranking positions of alternatives.

Let $y_i \in \{0,1\}$, M be a large enough number, and $D(x_i)$ be defined as per Eqs (3) and (4). Then, we can easily obtain the following results.

(1) $x_i \succ x_k$ if and only if $y_i=1$ under the conditions $D(x_i)>D(x_k)-(1-y_i)M$ and $D(x_i) \le D(x_k)+y_i M$.

(2) $x_i \precsim x_k$ if and only if $y_i=0$ under the conditions $D(x_i) \le D(x_k)+y_i M$ and $D(x_i)>D(x_k)-(1-y_i)M$.

Based on the above results, Theorems 1 and 2 to obtain the ranking range $R=[\underline{r}(x_k),\overline{r}(x_k)]$ of the alternative x_k under the WA and OWA operators are presented as follows.

Theorem 1: Let $R_{w \in S}^{WA}(x_k)=[\underline{r}_{w \in S}^{WA}(x_k),\overline{r}_{w \in S}^{WA}(x_k)]$ be the ranking range of alternative x_k under incomplete attribute weights set S, when we use the WA operator to compute the decision evaluation function as per Eq. (3). Then,

(1) The best ranking position of alternative x_k under incomplete attribute weights set S, $\underline{r}_{w \in S}^{WA}(x_k)$ can be computed via the following mixed 0-1 linear programming model (6).

$$\begin{cases} \underline{r}_{w\in S}^{WA}(x_k)=\min \sum_{i=1}^{n} y_i +1 & \text{(a)} \\ s.t. \begin{cases} \sum_{j=1}^{m} w_j \overline{v}_{ij} > \sum_{j=1}^{m} w_j \overline{v}_{kj} - (1-y_i)M, & (i=1,2,...,n) & \text{(b)} \\ \sum_{j=1}^{m} w_j \overline{v}_{ij} \le \sum_{j=1}^{m} w_j \overline{v}_{kj} + y_i M, & (i=1,2,...,n) & \text{(c)} \\ y_i =1 \ or \ 0, \ (i=1,2,...,n) & \text{(d)} \\ (w_1,w_2,...,w_m)\in S & \text{(e)} \end{cases} \end{cases} \quad \text{(6)}$$

The mixed 0–1 linear programming model (6) is denoted as P_1 in this paper.

(2) In model (6), replace the objective function (a) by

$$\overline{r}_{w\in S}^{WA}(x_k) = \max \sum_{i=1}^{n} y_i +1. \quad \text{(7)}$$

Then, the worst ranking position of alternative x_k under incomplete attribute weights set S, $\overline{r}_{w\in S}^{WA}(x_k)$, can be computed via a new mixed 0-1 linear programming model, which we called P_2 in this paper.

The proof of Theorem 1 is proved in Appendix B.

Theorem 2: Let $R_{w\in S}^{OWA}(x_k)=[\underline{r}_{w\in S}^{OWA}(x_k),\overline{r}_{w\in S}^{OWA}(x_k)]$ be the ranking range of alternative x_k under incomplete attribute weights set S, when we use the OWA operator to compute the decision evaluation function as per Eq. (4). Then,

(1) The best ranking position of alternative x_k under incomplete attribute weights set S, $\underline{r}_{w\in S}^{OWA}(x_k)$ can be computed via the following mixed 0–1 linear programming model (8).

$$\begin{cases} \underline{r}_{w\in S}^{OWA}(x_k)=\min \sum_{i=1}^{n} y_i +1 & \text{(a)} \\ s.t. \begin{cases} \sum_{j=1}^{m} w_j \overline{v}_{i(j)} > \sum_{j=1}^{m} w_j \overline{v}_{k(j)} - (1-y_i)M, & (i=1,2,...,n) & \text{(b)} \\ \sum_{j=1}^{m} w_j \overline{v}_{i(j)} \le \sum_{j=1}^{m} w_j \overline{v}_{k(j)} + y_i M, & (i=1,2,...,n) & \text{(c)} \\ y_i =1 \ or \ 0, \ (i=1,2,...,n) & \text{(d)} \\ (w_1,w_2,...,w_m)\in S & \text{(e)} \end{cases} \end{cases} \quad \text{(8)}$$

The mixed 0–1 linear programming model (8) is denoted as P_3 in this paper.

(2) In model (8), replace the objective function (a) by

$$\overline{r}_{w\in S}^{OWA}(x_k) = \max \sum_{i=1}^{n} y_i +1. \quad \text{(9)}$$

Then, the worst ranking position of alternative x_k under incomplete attribute weights set S, $\overline{r}_{w\in S}^{OWA}(x_k)$, can be computed via the mixed 0–1 linear programming model, which we called P_4 in this paper.

The proof of Theorem 2 is proved in Appendix B.

In the following, we use an example to show the application process of models $P_1 - P_4$.

Example 3: Let $\bar{V} = [\bar{v}_{ij}]_{5 \times 4}$ be as above in Example 2, and let $S = \{(w_1, w_2, w_3, w_4)^T \mid w_3 \geq w_2 \geq w_4\}$.

(1) WA based decision model

When taking $\bar{V}$ and S as the inputs of model P_1, we can obtain that: $\underline{r}_{w \in S}^{WA}(x_1) = 1$, $\underline{r}_{w \in S}^{WA}(x_2) = 3$, $\underline{r}_{w \in S}^{WA}(x_3) = 2$, $\underline{r}_{w \in S}^{WA}(x_4) = 1$ and $\underline{r}_{w \in S}^{WA}(x_5) = 1$.

Moreover, using model P_2 obtains that: $\bar{r}_{w \in S}^{WA}(x_1) = 3$, $\bar{r}_{w \in S}^{WA}(x_2) = 5$, $\bar{r}_{w \in S}^{WA}(x_3) = 5$, $\bar{r}_{w \in S}^{WA}(x_4) = 4$ and $\bar{r}_{w \in S}^{WA}(x_5) = 5$.

(2) OWA based decision model

Similarly, applying model P_3 generates that: $\underline{r}_{w \in S}^{OWA}(x_1) = 1$, $\underline{r}_{w \in S}^{OWA}(x_2) = 3$, $\underline{r}_{w \in S}^{OWA}(x_3) = 1$, $\underline{r}_{w \in S}^{OWA}(x_4) = 1$ and $\underline{r}_{w \in S}^{OWA}(x_5) = 1$.

Moreover, applying model P_4 results in: $\bar{r}_{w \in S}^{OWA}(x_1) = 4$, $\bar{r}_{w \in S}^{OWA}(x_2) = 5$, $\bar{r}_{w \in S}^{OWA}(x_3) = 5$, $\bar{r}_{w \in S}^{OWA}(x_4) = 4$ and $\bar{r}_{w \in S}^{OWA}(x_5) = 4$.

(2) Producing the average ranking positions of alternatives

In this step, we use Monte Carlo simulation to obtain the average ranking positions of alternatives.

Let S be as introduced in section 2. Let $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$ be the average ranking position of alternative x_i under the WA operator with S.

The basic idea of generating $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$ is introduced below:

We randomly generate a group of attribute weight vectors from S. For each attribute weight vector $w \in S$, we can calculate the ranking positions of alternatives using WA operator. Further, $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$ can be produced by averaging all obtained ranking positions of alternative x_i.

The detailed process to generate $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$ is described in Appendix C.

Let $\bar{\bar{r}}_{w \in S}^{OWA}(x_i)$ be the average ranking position of alternative x_i under the OWA operator with S. The method to generate $\bar{\bar{r}}_{w \in S}^{OWA}(x_i)$ is similar to that of $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$. For space limitation, we omit the process to calculate $\bar{\bar{r}}_{w \in S}^{OWA}(x_i)$.

In the following, we provide an example to show the process to generate $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$ and $\bar{\bar{r}}_{w \in S}^{OWA}(x_i)$.

Example 4: Let $\bar{V} = [\bar{v}_{ij}]_{5 \times 4}$ and $S = \{(w_1, w_2, w_3, w_4)^T \mid w_3 \geq w_2 \geq w_4\}$ be as the above in Example 3.

(1) Generating $\bar{\bar{r}}_{w \in S}^{WA}(x_i)$, $(i = 1, 2, ..., 5)$.

Step 1: Generate three attribute weights vectors.

$w_1 = (0.05, 0.25, 0.5, 0.2)^T$, $w_2 = (0.55, 0.1, 0.27, 0.08)^T$, $w_3 = (0.26, 0.25, 0.27, 0.22)^T$.

Step 2: Using Eqs (3) and (5) to generate $r_{w_k}^{WA}(x_i)$, $(i = 1, 2, 3, 4, 5; \ k = 1, 2, 3)$. They are provided below:

$\{r_{w_1}^{WA}(x_1), r_{w_1}^{WA}(x_2), r_{w_1}^{WA}(x_3), r_{w_1}^{WA}(x_4), r_{w_1}^{WA}(x_5)\} = \{2, 4, 5, 3, 1\}$;

$\{r_{w_2}^{WA}(x_1), r_{w_2}^{WA}(x_2), r_{w_2}^{WA}(x_3), r_{w_2}^{WA}(x_4), r_{w_2}^{WA}(x_5)\} = \{1, 5, 4, 2, 3\}$;

$\{r_{w_3}^{WA}(x_1), r_{w_3}^{WA}(x_2), r_{w_3}^{WA}(x_3), r_{w_3}^{WA}(x_4), r_{w_3}^{WA}(x_5)\} = \{1, 4, 5, 3, 2\}$.

Step 3: Based on $r_{w_k}^{WA}(x_i)$, we have $r_1 = \dfrac{(r_{w_1}^{WA}(x_1) + r_{w_2}^{WA}(x_1) + r_{w_3}^{WA}(x_1))}{3} = \dfrac{2+1+1}{3} = \dfrac{4}{3}$,

$r_2 = \dfrac{4+5+4}{3} = \dfrac{13}{3}$, $r_3 = \dfrac{5+4+5}{3} = \dfrac{14}{3}$, $r_4 = \dfrac{3+2+3}{3} = \dfrac{8}{3}$ and $r_5 = \dfrac{1+3+2}{3} = 2$.

Further, we obtain that $\overline{\overline{r}}_{w \in S}^{WA}(x_1) = 1$, $\overline{\overline{r}}_{w \in S}^{WA}(x_2) = 4$, $\overline{\overline{r}}_{w \in S}^{WA}(x_3) = 5$, $\overline{\overline{r}}_{w \in S}^{WA}(x_4) = 3$ and $\overline{\overline{r}}_{w \in S}^{WA}(x_5) = 2$.

(2) Meanwhile, we can obtain that $\overline{\overline{r}}_{w \in S}^{OWA}(x_1) = 1$, $\overline{\overline{r}}_{w \in S}^{OWA}(x_2) = 4$, $\overline{\overline{r}}_{w \in S}^{OWA}(x_3) = 5$ $\overline{\overline{r}}_{w \in S}^{OWA}(x_4) = 3$, and $\overline{\overline{r}}_{w \in S}^{OWA}(x_5) = 2$.

(3) Computing the comprehensive ranking positions of alternatives

In this step, we integrate the best, worst and average rankings of each alternative into a comprehensive ranking position of the alternative.

Let

$$h(x_i) = \frac{\underline{r}_{w \in S}^{WA}(x_i) + \overline{\overline{r}}_{w \in S}^{WA}(x_i) + \overline{r}_{w \in S}^{WA}(x_i)}{3}. \tag{10}$$

Obviously, the smaller $h(x_i)$ value indicates the better the alternative x_i is.

So, the comprehensive ranking position of alternative x_i under WA operator, $r_{w \in S}^{WA}(x_i)$ can be computed as follows:

$r_{w \in S}^{WA}(x_i) = g$,

if $h(x_i)$ is the gth smallest value in $\{h(x_1), h(x_2), ..., h(x_n)\}$.

Using the similar way to calculate $r_{w \in S}^{WA}(x_i)$, the comprehensive ranking position of alternative x_i under OWA operator, $r_{w \in S}^{OWA}(x_i)$ can be generated.

In the following, we give an example to show how to compute $r_{w \in S}^{WA}(x_i)$ and $r_{w \in S}^{OWA}(x_i)$.

Example 5: Let $\overline{V} = [\overline{v}_{ij}]_{5 \times 4}$, $S = \{(w_1, w_2, w_3, w_4)^T \mid w_3 \geq w_2 \geq w_4\}$, $\underline{r}_{w \in S}^{WA}(x_i)$ and $\overline{r}_{w \in S}^{WA}(x_i)$ be as the above in Example 3, and let $\overline{\overline{r}}_{w \in S}^{WA}(x_i)$ be as the above in Example 4.

(1) Based on Eq (10), we have

$h(x_1) = \dfrac{\underline{r}_{w \in S}^{WA}(x_1) + \overline{\overline{r}}_{w \in S}^{WA}(x_1) + \overline{r}_{w \in S}^{WA}(x_1)}{3} = \dfrac{1 + \frac{4}{3} + 3}{3} = \dfrac{16}{9}$, $h(x_2) = \dfrac{3 + \frac{13}{3} + 5}{3} = \dfrac{37}{9}$,

$h(x_3) = \dfrac{2 + \frac{14}{3} + 5}{3} = \dfrac{35}{9}$, $h(x_4) = \dfrac{1 + \frac{8}{3} + 4}{3} = \dfrac{23}{9}$ and $h(x_5) = \dfrac{1 + 2 + 5}{3} = \dfrac{8}{3}$.

Clearly, we have $r_{w \in S}^{WA}(x_1) = 1$, $r_{w \in S}^{WA}(x_2) = 5$, $r_{w \in S}^{WA}(x_3) = 4$, $r_{w \in S}^{WA}(x_4) = 2$ and $r_{w \in S}^{WA}(x_5) = 3$.

(2) Meanwhile, we can obtain that: $h(x_1) = \dfrac{\underline{r}_{w \in S}^{OWA}(x_1) + \overline{\overline{r}}_{w \in S}^{OWA}(x_1) + \overline{r}_{w \in S}^{OWA}(x_1)}{3} = \dfrac{1+1+4}{3} = 2$,

$h(x_2) = \dfrac{3+4+5}{3} = 4$, $h(x_3) = \dfrac{1 + \frac{14}{3} + 5}{3} = \dfrac{32}{9}$, $h(x_4) = \dfrac{1 + \frac{10}{3} + 4}{3} = \dfrac{25}{9}$ and

$h(x_5) = \dfrac{1+2+4}{3} = \dfrac{7}{3}$.

Clearly, we have $r_{w \in S}^{OWA}(x_1) = 1$, $r_{w \in S}^{OWA}(x_2) = 5$, $r_{w \in S}^{OWA}(x_3) = 4$, $r_{w \in S}^{OWA}(x_4) = 3$ and $r_{w \in S}^{OWA}(x_5) = 2$.

3. The application of the ranking range based approach in venture investment evaluation with risk attitudes

In this section, we introduce the venture investment evaluation problem, and construct two ranking range based MADM approaches with risk attitudes to select the best project(s). Meanwhile, the validity of the developed MADM model is demonstrated using a case study.

3.1. Venture investment evaluation

Venture investment is an integrative product of technological innovation and capital market, and which plays an important role in the economic development of a country (Fried & Hisrich, 1994; Tyebjee & Bruno, 1984). Venture investment evaluation method is often adopted to help the venture capitalists to rank or select the best venture project(s) according to the performances of the projects associated with multiple criteria (or attributes). A lot of methods have been reported in literature for venture investment evaluation problems (Barrot, 2016; Nanda & Rhodes-Kropf, 2016; Townsend, 2015). For example, Barrot (2016) studied whether and how the contractual horizon of venture capital funds affects investors' investments in innovative firms. Nanda and Rhodes-Kropf (2016) provided a new investment model that reveals the reasons for the correlation between some places, times, as well as industries and a greater degree of experimentation by investors. Townsend (2015) examined how the investors' companies were affected when other companies with the same investors received negative shocks. Additional methods for venture investment evaluation problems can be found in (Aggarwal et al., 2015; Fried & Hisrich, 1994; Hall & Hofer, 1993; Tyebjee & Bruno, 1984).

In general, the evaluation criteria in the venture investment evaluation can be classified into two categories: risk and income criteria. Here, we use $C = \{c_1, c_2, ..., c_{m_1}\}$ and $D = \{d_1, d_2, ..., d_{m_2}\}$ to signify the risk and income criteria, respectively. Let $w^C = (w_1^C, w_2^C, ..., w_{m_1}^C)$ be the weight vector of the risk criteria, where $w_j^C \geq 0$ denotes the weight of attribute c_j and $\sum_{j=1}^{m_1} w_j^C = 1$. Let $w^D = (w_1^D, w_2^D, ..., w_{m_2}^D)$ be the weight vector of the income criteria, where $w_j^D \geq 0$ denotes the weight of attribute d_j and $\sum_{j=1}^{m_2} w_j^D = 1$. And, we denote S^C and S^D as the incomplete attribute weights set of risk and income criteria, respectively.

Let $X = \{x_1, x_2, ..., x_n\}$ be n investment projects. Let $U = [u_{ij}]_{n \times m_1}$ be risk evaluation matrix, where u_{ij} represents the risk evaluation value of project x_i associated with c_j, $(j = 1, 2, ..., m_1)$. Let $Z = [z_{ij}]_{n \times m_2}$ be income evaluation matrix, where z_{ij} indicates the income evaluation value of alternative x_i with respect to d_j, $(j = 1, 2, ..., m_2)$.

In the venture investment valuation, the investor wants to obtain a best investment project according to $U = [u_{ij}]_{n \times m_1}$ and $Z = [z_{ij}]_{n \times m_2}$. However, different investors have different risk attitudes in the selection of best investment project(s). To deal with this issue, two approaches are developed in the following section.

3.2. Ranking range based MADM approaches with risk attitudes

In this section, two ranking range based MADM approaches with risk attitudes are developed to help investor select the best investment project(s).

Let $U = [u_{ij}]_{n \times m_1}$ be as the above. Let $R^{risk} = (r_1^{risk}, r_2^{risk}, ..., r_n^{risk})^T$ be the risk ranking position vectors over projects $X = \{x_1, x_2, ..., x_n\}$ derived from U, where r_i^{risk} denotes the risk ranking position of project x_i. Using models p_2 or p_4, we can obtain $\overline{r}_{w \in S}^{WA}(x_i)$ or $\overline{r}_{w \in S}^{OWA}(x_i)$ $(i = 1, 2, ..., n)$ from U. We assume that the investor ranks the projects based on $\overline{r}_{w \in S}^{WA}(x_i)$ or $\overline{r}_{w \in S}^{OWA}(x_i)$ $(i = 1, 2, ..., n)$, due to the investor wants to avoid risk to decrease loss from risk. Specifically, $R^{risk} = (r_1^{risk}, r_2^{risk}, ..., r_n^{risk})^T$ is computed by:

$$r_i^{risk} = g$$

if $\overline{r}_{w \in S}^{WA}(x_i)$ is the gth smallest value in $\{\overline{r}_{w \in S}^{WA}(x_1), ..., \overline{r}_{w \in S}^{WA}(x_n)\}$ (or $\overline{r}_{w \in S}^{OWA}(x_i)$ is the gth smallest value in $\{\overline{r}_{w \in S}^{OWA}(x_1), ..., \overline{r}_{w \in S}^{OWA}(x_n)\}$).

Let $Z = [z_{ij}]_{n \times m_2}$ be as the above. Let $R^{income} = (r_1^{income}, r_2^{income}, ..., r_n^{income})^T$ be the income ranking position vectors over projects $X = \{x_1, x_2, ..., x_n\}$ derived from Z, where r_i^{income} denotes the income ranking position of project x_i. Here, we consider that r_i^{income} $(i = 1, 2, ..., n)$ is the comprehensive raking position of x_i using the approach presented from section 3.2.

By taking the risk attitudes of investor in the venture investment evaluation process, two methods are designed.

(1) Method I

In this approach, we consider that the investor wants to select the best project(s) from those projects with acceptable risk level. Let $\overline{r^{risk}}$ be a parameter that used to judge whether the risk level of a project is acceptable or not. In particular, if $r_i^{risk} \leq \overline{r^{risk}}$, then the risk level of project x_i is acceptable; otherwise, x_i is unacceptable. Following this idea, we propose the following model to help the investor select a best project:

$$\begin{cases} Min_i \ r_i^{income} \\ s.t. \ r_i^{risk} \leq \overline{r^{risk}}, \ i = 1, 2, ..., n. \end{cases} \quad (11)$$

In model (11), the constraint guarantees that the risk level(s) of the project(s) is acceptable. The objective function finds out the best project from the projects with acceptable risk level.

Example 5: Let $\{x_1, ..., x_4\}$ be a set of four investment projects. Suppose that $R^{risk} = (r_1^{risk}, r_2^{risk}, r_3^{risk}, r_4^{risk})^T = (2, 3, 4, 1)^T$ and $R^{income} = (r_1^{income}, r_2^{income}, r_3^{income}, r_4^{income})^T = (4, 1, 2, 3)^T$.

When setting $\overline{r^{risk}} = 2$, we find that the risk levels of projects x_1 and x_4 are acceptable, due to $r_1^{risk} \leq \overline{r^{risk}}$ and $r_4^{risk} \leq \overline{r^{risk}}$. For projects x_1 and x_4, the best one is x_4 due to its income level is better.

(2) Method II

In this approach, we consider that the investor wants to select the best project(s) from those projects with acceptable income level. Let $\overline{r^{income}}$ be a parameter that used to judge whether the income level of a project is acceptable or not. In particular, if $r_i^{income} \leq \overline{r^{income}}$, then the income level of project x_i is acceptable; otherwise, x_i is unacceptable. Following this idea, we propose the following model to help the investor select a best project:

$$\begin{cases} Min_i \ r_i^{risk} \\ s.t. \ r_i^{income} \geq \underline{r^{income}}, \ \ i = 1,2,...,n. \end{cases} \tag{12}$$

In model (12), the constraint guarantees that the income level(s) of the project(s) is acceptable. The objective function finds out the best project(s) from the projects with acceptable income level.

Example 6: Let $\{x_1,...,x_4\}$, $R^{risk} = (2,3,4,1)^T$ and $R^{income} = (4,1,2,3)^T$ be as above in Example 5.

When setting $\overline{r^{income}} = 2$, we find that the income level of projects x_2 and x_3 are acceptable, due to $r_2^{income} \leq \overline{r^{income}}$ and $r_3^{income} \leq \overline{r^{income}}$. For projects x_2 and x_3, the best one is x_2 due to its risk level is better.

3.3. Case study

In this section, we use a case study of the venture investment evaluation with risk attitudes to show the application process of the proposed approach.

An investor wants to select the best investment project(s) from the following four investment projects: advanced equipment manufacturing (x_1), new energy (x_2), new materials (x_3) and new-energy vehicles (x_4). The risk and income criteria to evaluate the four projects are provided in Tables 1 and 2, respectively.

Table 1. The risk criteria

Management risk	c_1 Commitment of the management team
	c_2 Division of labor and cooperation between management team
	c_3 The ability of managers
Technical risk	c_4 Technical maturity
	c_5 The reliability of technology
	c_6 Venture capitalists' understanding of technology
Production risk	c_7 The supply capability of raw material
	c_8 Production capacity
Market risk	c_9 Market maturity
	c_{10} Familiarity with the marke
	c_{11} Potential competition
Environment risk	c_{12} Government policy
	c_{13} Legal environment
	c_{14} Exit environment

Table 2. The income criteria

Enterprise management level	d_1 Enterprises' strategic planning
	d_2 The experience of managers
	d_3 Management philosophy
Market attractiveness	d_4 Market size
	d_5 Market demand
	d_6 Market potential
	d_7 Competitive advantage in the market
Product differentiation	d_8 The innovative of products and technologies
	d_9 Level of technology
	d_{10} Margins
	d_{11} Intellectual property regime
Financial evaluation	d_{12} Liquidity
	d_{13} Expected return
	d_{14} When break even

Moreover, risk evaluation matrix $U = [u]_{4 \times 14}$ and income evaluation matrix $Z = [z]_{4 \times 14}$ are provided in Tables 3 and 4, respectively.

Table 3. The risk evaluation matrix $U = [u]_{4 \times 14}$

x_i	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
x_1	0.74	0.54	0.14	0.94	0.26	0.1	0.34	0.9	0.86	0.54	0.78	0.74	0.3	0.06
x_2	0.58	0.02	0.58	0.26	0.86	0.48	0.54	0.06	0.7	0.3	0.14	0.94	0.74	0.54
x_3	0.26	0.34	0.06	0.58	0.58	0.14	0.66	0.94	0.18	0.06	0.86	0.5	0.98	0.98
x_4	0.98	0.14	0.74	0.34	0.66	0.26	0.38	0.52	0.86	0.58	0.7	0.34	0.18	0.22

Table 4. The income evaluation matrix $Z = [z]_{4 \times 14}$

x_i	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}
x_1	0.66	0.22	0.86	0.18	0.62	0.98	0.9	0.1	0.34	0.86	0.98	0.66	0.7	0.66
x_2	0.58	0.58	0.14	0.86	0.26	0.7	0.86	0.26	0.74	0.7	0.34	0.7	0.5	0.94
x_3	0.3	0.02	0.1	0.66	0.06	0.9	0.58	0.86	0.9	0.42	0.58	0.46	0.74	0.62
x_4	0.14	0.7	0.34	0.42	0.46	0.34	0.06	0.46	0.18	0.98	0.42	0.9	0.06	0.1

In this example, we set $S^C = \{(w_1^C, w_2^C, ..., w_{14}^C)^T \mid w_1^C \geq w_2^C ... \geq w_{11}^C, \ w_4^C - w_9^C \geq w_6^C - w_{13}^C, \ w_2^C / w_5^C \leq w_7^C / w_{11}^C\}$ and $S^D = \{(w_1^D, w_2^D, ..., w_{14}^D)^T \mid w_1^D \geq w_2^D \geq, ..., \geq w_{14}^D\}$. In the following, we use the proposed approach to help the investor select a best project. In particular, OWA operator is examined in detail.

(1) First, we transform $U = [u]_{4 \times 14}$ and $Z = [z]_{4 \times 14}$ into $\bar{U} = [\bar{u}]_{4 \times 14}$ and $\bar{Z} = [\bar{z}]_{4 \times 14}$ using Eq. (1) and Eq. (2), respectively. $\bar{U} = [\bar{u}]_{4 \times 14}$ and $\bar{Z} = [\bar{z}]_{4 \times 14}$ are listed in Tables 5 and 6, respectively.

Table 5. The normalized evaluation matrix $\overline{U} = [\overline{u}]_{4 \times 14}$

x_i	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
x_1	0.33	0	0.88	0	1	1	1	0.05	0	0.08	0.11	0.33	0.85	1
x_2	0.56	1	0.24	1	0	0	0.38	1	0.24	0.54	1	0	0.3	0.48
x_3	1	0.38	1	0.53	0.47	0.89	0	0	1	1	0	0.73	0	0
x_4	0	0.77	0	0.88	0.33	0.58	0.88	0.48	0	0	0.22	1	1	0.83

Table 6. The normalized evaluation matrix $\overline{Z} = [\overline{z}]_{4 \times 14}$

x_i	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}
x_1	1	0.30	1	0	1	1	1	0	0.22	0.79	1	0.45	0.94	0.67
x_2	0.85	0.82	0.05	1	0.36	0.562	0.95	0.21	0.78	0.5	0	0.55	0.65	1
x_3	0.31	0	0	0.71	0	0.88	0.62	1	1	0	0.38	0	1	0.62
x_4	0	1	0.32	0.35	0.71	0	0	0.47	0	1	0.13	1	0	0

(2) Using the method presented in section 4.2, we can obtain $R^{risk} = (2,2,1,2)^T$ from $\overline{U} = [\overline{u}]_{4 \times 14}$. Meanwhile, $R^{income} = (1,4,2,3)^T$ is generated from $\overline{Z} = [\overline{z}]_{4 \times 14}$ using the method proposed in section 4.2.

(3) Using methods I and II to help the investor select the best project(s).

   (i) Method I

   When setting $\overline{r^{risk}} = 1$, only the risk level of project x_3 is acceptable, it is obviously that x_3 is the best project.

   When setting $\overline{r^{risk}} = 2$, the risk levels for all projects are acceptable, due to $r_i^{risk} \leq \overline{r^{risk}}$ $(i = 1,2,3,4)$. For all projects $\{x_1, ..., x_4\}$, the best one is x_1 due to its income level is the best.

   (ii) Method II

   When setting $\overline{r^{income}} = 2$, then income levels of projects x_1 and x_3 are acceptable, due to $r_1^{income} \leq \overline{r^{income}}$ and $r_3^{income} \leq \overline{r^{income}}$. For projects x_1 and x_3, the best one is x_3 due to its risk level is better.

4. Simulation analysis

In this section, we design a simulation experiment to discuss the performance of the proposed model under different aggregation operators.

   Let *S* be as above in section 2. Traditional MADM models assumed that the attribute weights are known precisely. So, when using these models to deal with the incomplete MADM problem, a specific approach is often adopted to generate the complete attribute weights vector from *S*. As a result, the ranking positions of alternatives generated from the traditional MADM models rely on the setting of attribute weights vector.

   To date, a lot of MADM models have been reported (Wu & Liu, 2013; Wu et al., 2016; Wu & Xu, 2016). In particular, the MADM models with the WA and OWA operators can be de-

scribed as Eqs (3) and (4), respectively. Based on Eqs (3) or (4), the ranking positions of the alternatives can be generated. It is noted that the attribute weight vector $w = (w_1, w_2, ..., w_m)^T$ needs to be obtained from the incomplete attribute weight information when using Eqs (3) and (4).

Here, we compare the performance of the TM (denotes the traditional MADM model) and NM (denotes the novel MADM model presented in this paper) under WA and OWA operators based on the following criterion. The distance between the ranking positions of alternative x_i generated using WA and OWA operators, which can be calculated by

$$D_i = |r_i^{WA} - r_i^{OWA}|. \tag{13}$$

The main idea of the simulation experiment is that we randomly generate a multiple attribute decision matrix $V = [v_{ij}]_{n \times m}$. Then, the TM with WA and OWA operators are adopted to yield the ranking positions of alternatives from V, respectively. In particular, the weight vectors used in the WA and OWA operators are generated from S. Meanwhile, the ranking positions of alternatives under NM with WA and OWA operators can also be yielded from V, respectively. Further, based on Eq. (13), the distances between the ranking positions of alternative x_i generated using WA and OWA operators under TM and NM can be calculated, respectively. For simplification, the distances of alternative x_i under TM and NM are denoted as TMD_i and NMD_i, respectively.
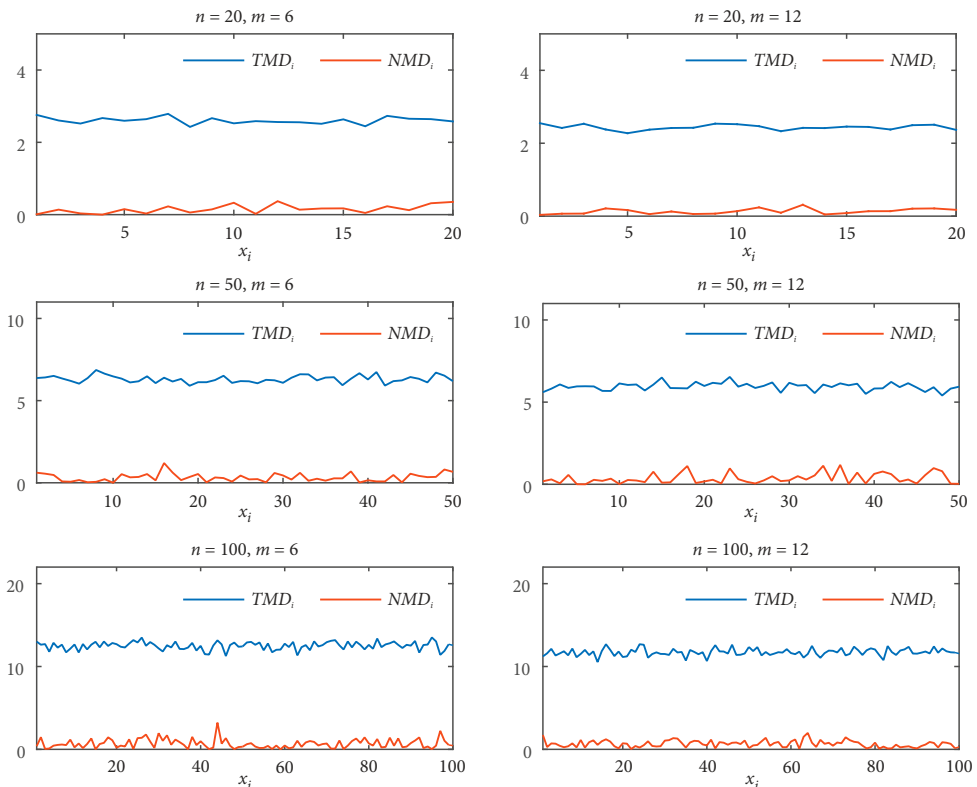

Figure 1. Average values of TMD_i and NMD_i ($i = 1, 2, ..., n$) under the simulation method

To implement above simulation experiment, a simulation method is devised, which is provided in Appendix D.

In the simulation method, we assume that $S = \{(w_1,...,w_m)^T \mid w_j \geq 0, \sum_{j=1}^{m} w_j = 1\}$. Then, we set different values of n and m, and run this simulation method 1000 times to obtain average values of TMD_i and NMD_i, which are shown in Figure 1.

From Figure 1, we find that $TMD_i > NMD_i$ for all alternatives x_i $(i = 1,2,....,n)$, under different parameters n and m. This finding indicates that the proposed ranking range based MADM approach is insensitive to aggregate operators (WA and OWA operators) compared with existing MADM approaches.

Conclusions

This paper investigates the incomplete MADM problem, and develops a ranking range MADM approach to cope with this kind of problems. The main contributions of this paper are summarized as follows.

1) A novel MADM approach, namely, ranking range based MADM approach, is presented to address the incomplete MADM problem. This novel approach combines the minimum, maximum, and average (generated using Monte Carlo simulation method) ranking positions of a specific alternative to generate its comprehensive ranking position.

2) Two ranking range based MADM approaches are further developed by taking the risk attitudes of decision makers into account, which are capable of dealing with venture investment evaluation problems.

3) A case study and a simulation experiment are presented to show the validity of the proposed ranking range based MADM approach.

Meanwhile, two interesting research directions are pointed out:

1) Societal and technological trends demand the management of the GDM problem in complex context (e.g., heterogeneous GDM (Cabrerizo et al., 2013; Pérez, Cabrerizo, Alonso, & Herrera-Viedma, 2014; Zhang, B., Dong, & Herrera-Viedma, 2019), linguistic GDM (Cabrerizo et al., 2017; Liao, Xu, Herrera-Viedma, & Herrera, 2018; Morente-Molinera, Kou, Pang, Cabrerizo, & Herrera-Viedma, 2019; Wu, Li, Chen, & Dong, 2018; Zhang, H., Dong, Palomares, & Zhou, 2019), social network GDM (Dong et al., 2018b; Ureña, Kou, Dong, Chiclana, & Herrera-Viedma, 2019), and large-scale GDM (Dong, Zhao, Zhang, Chiclana, & Herrera-Viedma, 2018c; Li, Dong, & Herrera, 2019; Zhang, Guo, & Martínez, 2017). Moreover, consensus is a key issue in the complex GDM (Dong, Zhang, & Herrera-Viedma, 2016; Gong, Xu, Zhang, & Herrera-Viedma, 2015; Sun & Ma, 2015; B. Zhang, et al., 2018; H. Zhang, Dong, & Chen, 2018; Zhang, Dong, Chiclana, & Yu, 2019). So, it would be interesting in any future research to export the ranking range based approach to deal with complex GDM problems, and design a consensus model to support consensus building.

2) Meanwhile, the real-world MADM problems involve not only mathematical aspects but also psychological behaviors of decision maker(s) (see Dong et al., 2018a; Liu et al., 2018; von Winterfeldt & Edwards, 1986). It would be interesting to study the psychological behaviors of decision maker(s) in the ranking range based MADM approach.

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APPENDIX A

Notations

The mainly notation used in this paper is as follows.

X: The set of alternatives;

A: The set of attributes;

C: The set of risk attributes;

D: The set of income attributes;

$V = [v_{ij}]_{n \times m}$: Decision matrix;

$\overline{V} = [\overline{v}_{ij}]_{n \times m}$: Standardized decision matrix;

$U = [u_{ij}]_{n \times m}$: Decision matrix;

$\overline{U} = [\overline{u}_{ij}]_{n \times m}$: Standardized decision matrix;

$Z = [z_{ij}]_{n \times m}$: Decision matrix;

$\overline{Z} = [\overline{z}_{ij}]_{n \times m}$: Standardized decision matrix;

S: The incomplete information of attribute weight set;

$D_w(x_i)$: The evaluation value of the alternative x_i under attribute weight w;

$\underline{r}^{risk}_{w \in S}(x_i)$: The best risk ranking of the alternative x_i under the incomplete attribute weights set S;

$\overline{r}^{risk}_{w \in S}(x_i)$: The worst risk ranking of the alternative x_i under the incomplete attribute weights set S;

$\overline{\overline{r}}^{risk}_{w \in S}(x_i)$: The average risk ranking of the alternative x_i under the incomplete attribute weights set S;

r^{risk}_i: The risk ranking of the alternative x_i;

$\underline{r}^{income}_{w \in S}(x_i)$: The best income ranking of the alternative x_i under the incomplete attribute weights set S;

$\overline{r}^{income}_{w \in S}(x_i)$: The worst income ranking of the alternative x_i under the incomplete attribute weights set S;

$\overline{\overline{r}}^{income}_{w \in S}(x_i)$: The average income ranking of the alternative x_i under the incomplete attribute weights set S;

r^{income}_i: The income ranking of the alternative x_i.

APPENDIX B

Proofs

Proof of Theorem 1:

We use two steps to prove Theorem 1.

Step 1: According to result (1), i.e., $x_i \succ x_k$ if and only if $y_i = 1$ under the conditions $D(x_i) > D(x_k) - (1 - y_i)M$ and $D(x_i) \leq D(x_k) + y_i M$, when using WA operator as per Eq. (3), we directly have,

$$\underline{r}(x_k) = 1 + \min \left| \{ x_i \mid D(x_i) > D(x_k), \ i = 1, 2, ..., n \} \right|$$

$$= 1 + \min \left\{ \sum_{i=1}^{n} y_i \middle| D(x_i) > D(x_k) - (1 - y_i)M, \ D(x_i) \leq D(x_k) + y_i M, \atop y_i = 1 \ or \ 0, \ (i = 1, 2, ..., n) \right\}$$

$$= 1 + \min \left\{ \sum_{i=1}^{n} y_i \middle| \sum_{j=1}^{m} w_j \overline{v}_{ij} > \sum_{j=1}^{m} w_j \overline{v}_{kj} - (1 - y_i)M, \ D(x_i) \leq D(x_k) + y_i M, \atop y_i = 1 \ or \ 0, \ (i = 1, 2, ..., n) \right\}$$

$$= 1 + \min \sum_{i=1}^{n} y_i, \ (i = 1, 2, ..., n).$$

Step 2: Similar to step 1, we have,

$$\overline{r}(x_k) = 1 + \max \left| \{ x_i \mid D(x_i) > D(x_k) \} \right|$$

$$= 1 + \max \left\{ \sum_{i=1}^{n} y_i \middle| D(x_i) > D(x_k) - (1 - y_i)M, \ D(x_i) \leq D(x_k) + y_i M, \atop y_i = 1 \ or \ 0, \ (i = 1, 2, ..., n) \right\}$$

$$= 1 + \max \left\{ \sum_{i=1}^{n} y_i \middle| \sum_{j=1}^{m} w_j \overline{v}_{ij} > \sum_{j=1}^{m} w_j \overline{v}_{kj} - (1 - y_i)M, \ D(x_i) \leq D(x_k) + y_i M, \atop y_i = 1 \ or \ 0, \ (i = 1, 2, ..., n) \right\}$$

$$= 1 + \max \sum_{i=1}^{n} y_i, \ (i = 1, 2, ..., n),$$

where M is a large enough number.

This completes the proof of Theorem 1.

Proof of Theorem 2:

In the process of proving Theorem 1, we only replaced the *WA* operator $D(x_i) = \sum_{j=1}^{m} w_j \overline{v}_{ij}$ and $D(x_k) = \sum_{j=1}^{m} w_j \overline{v}_{kj}$ by OWA operator $D(x_i) = \sum_{j=1}^{m} w_j \overline{v}_{i(j)}$ and $D(x_k) = \sum_{j=1}^{m} w_j \overline{v}_{k(j)}$, respectively.

This completes the proof of Theorem 2.

APPENDIX C

Monte Carlo simulation based approach to obtain $\bar{\bar{r}}_{w\in S}^{WA}(x_i)$

Input: $\bar{V} = [\bar{v}_{ij}]_{n\times m}$ and S.

Output: $\bar{\bar{r}}_{w\in S}^{WA}(x_i), \ (i=1,2,...,n)$.

   Step 1: We randomly generate $N \ (N \geq 1)$ attribute weights vectors $w_k = (w_{k,1}, w_{k,2},...,w_{k,m})^T$ from S, where $\sum_{j=1}^{m} w_{k,j} = 1$ and $0 \leq w_{k,j} \leq 1, \ (k=1,2,...,N)$.

   Step 2: Taking $w_k = (w_{k,1}, w_{k,2},...,w_{k,m})^T$ as weight vector of the WA operator (i.e. Eq. (3)), we can obtain the ranking position of alternative x_i from $\bar{V} = [\bar{v}_{ij}]_{n\times m}$, which is denoted as $r_{w_k}^{WA}(x_i), \ (i=1,2,...,m)$.

   Step 3: Let $r_i = \sum_{k=1}^{Num} r_{w_k}^{WA}(x_i) \Big/ N$. Then, we can obtain that $\bar{\bar{r}}_{w\in S}^{WA}(x_i) = g$, if r_i is the gth smallest $\{r_1, r_2,...,r_n\}$. Output $\bar{\bar{r}}_{w\in S}^{WA}(x_i), \ (i=1,2,...,n)$.

APPENDIX D

Simulation method used in the simulation analysis

Input: n, m and S.

Output: TMD_i and $NMD_i, \ (i=1,2,...,n)$

   Step 1: Generate a decision matrix $V = [v_{ij}]_{n\times m}$, where v_{ij} is randomly generated from the interval [0, 1].

   Step 2: We randomly generate a weight vector from S, which is denoted as $w = (w_1, w_2,...,w_n)^T$. Using Eqs (3) and (5) yield $r_i^{TM,WA}$ from V and w. Meanwhile, applying Eqs (4) and (5) produce $r_i^{TM,OWA}$ from V and w. Further, we compute TMD_i by $TMD_i = |r_i^{TM,WA} - r_i^{TM,OWA}|$.

   Step 3: Using the NM with WA and OWA operators presented in section 3, the ranking positions of alternative x_i are generated, respectively, that are $r_i^{NM, WA}$ and $r_i^{NM, OWA}$. Then, we can obtain $NMD_i = |r_i^{NM,WA} - r_i^{NM,OWA}|$.

   Step 4: Output TMD_i and $NMD_i, \ (i=1,2,...,n)$.