ENERGY LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

# THE EFFECT OF VIRTUAL MASS ON THE CHARACTERISTICS AND THE NUMERICAL STABILITY IN TWO-PHASE FLOWS

by

H. C. No and M. S. Kazimi

Energy Laboratory Report No. MIT-EL 81-023

April 1981

Energy Laboratory

and

Department of Nuclear Engineering

Massachusetts Institute of Technology
Cambridge, Mass. 02139

# THE EFFECT OF VIRTUAL MASS ON THE CHARACTERISTICS AND THE NUMERICAL STABILITY IN TWO-PHASE FLOWS

by

H. C. No and M. S. Kazimi

April 1981

Sponsored by

U.S. Department of Energy

Agreement No. DE-AC02-78ET37218

MIT Energy Laboratory Report No. MIT-EL 81-023

# REPORTS IN REACTOR THERMAL HYDRAULICS RELATED TO THE MIT ENERGY LABORATORY ELECTRIC POWER PROGRAM

A. Topical Reports (For availability check Energy Laboratory Headquarters, Headquarters, Room E19-439, MIT, Cambridge, Massachusetts 02139)

A.1 General Applications
A.2 PWR Applications
A.3 BWR Applications
A.4 LMFBR Applications

A.1 J.E. Kelly, J. Loomis, L. Wolf, "LWR Core Thermal-Hydraulic Analysis--Assessment and Comparison of the Range of Applicability of the Codes COBRA-IIIC/MIT and COBRA-IV-1," MIT Energy Laboratory Report No. MIT-EL-78-026, September 1978.

M. S. Kazimi and M. Massoud, "A Condensed Review of Nuclear Reactor Thermal-Hydraulic Computer Codes for Two-Phase Flow Analysis," MIT Energy Laboratory Report No. MIT-EL-79-018, February 1979.

J.E. Kelly and M.S. Kazimi, "Development and Testing of the Three Dimensional, Two-Fluid Code THERMIT for LWR Core and Subchannel Applications," MIT Energy Laboratory Report No. MIT-EL-79-046.

J.N. Loomis and W.D. Hinkle, "Reactor Core Thermal-Hydraulic Analysis--Improvement and Application of the Code COBRA-IIIC/MIT," MIT Energy Laboratory Report No. MIT-EL-80-027, September 1980.

D.P. Griggs, A.F. Henry and M.S. Kazimi, "Development of a Three-Dimensional Two-Fluid Code with Transient Neutronic Feedback for LWR Applications," MIT Energy Laboratory No. MIT-EL-81-013, April 1981.

J.E. Kelly, S.P. Kao and M.S. Kazimi, "THERMIT-2: A Two-Fluid Model for Light Water Reactor Subchannel Transient Analysis," MIT Energy Laboratory Report No. MIT-EL-81-014, April 1981.

H.C. No and M.S. Kazimi, "The Effect of Virtual Mass on the Characteristics and the Numerical Stability in Two-Phase Flows," MIT Energy Laboratory Report No. MIT-EL-81-023, April 1981.

J.W. Jackson and N.E. Todreas, "COBRA IIIC/MIT-2: A Digital Computer Program for Steady State and Transient Thermal-Hydraulic Analysis of Rod Bundle Nuclear Fuel Elements," MIT-EL-81-018, June 1981.

J.E. Kelly, S.P. Kao, and M.S. Kazimi, "User's Guide for THERMIT-2: A Version of THERMIT for both Core-Wide and Subchannel Analysis of Light Water Reactors," MIT Energy Laboratory Report No. MIT-EL 81-029, August 1981.

A.2 P. Moreno, C. Chiu, R. Bowring, E. Khan, J. Liu, and N. Todreas, "Methods for Steady-State Thermal/Hydraulic Analysis of PWR Cores," MIT Energy Laboratory Report No. MIT-EL 76-006, Rev. 1, July 1977, (Orig. 3/77).

J. Liu and N. Todreas, "Transient Thermal Analysis of PWR's by a Single Pass Procedure Using a Simplified Model Layout," MIT Energy Laboratory Report No. MIT-EL 77-008, Final, February 1979 (Draft, June 1977).

J. Liu and N. Todreas, "The Comparison of Available Data on PWR Assembly Thermal Behavior with Analytic Predictions," MIT Energy Laboratory Report No. MIT-EL 77-009, Final February 1979, (Draft, June 1977).

A.3 L. Guillebaud, A. Levin, W. Boyd, A. Faya, and L. Wolf, "WOSUB-A Subchannel Code for Steady-State and Transient Thermal-Hydraulic Analysis of Boiling Water Reactor Fuel Bundles," Vol. II, Users Manual, MIT-EL 78-024, July 1977.

L. Wolf, A. Faya, A. Levin, W. Boyd, L. Guillebaud, "WOSUB-A Subchannel Code for Steady State and Transient Thermal-Hydraulic Analysis of Boiling Water Reactor Fuel Pin Bundles," Vol. III, Assessment and Comparison, MIT-EL 78-025, October 1977.

L. Wolf, A. Faya, A. Levin, L. Guillebaud, "WOSUB-A Subchannel Code for Steady-State Reactor Fuel Pin Bundles," Vol. I, Model Description, MIT-EL 78-023, September 1978.

A. Faya L. Wolf and N. Todreas, "Development of a Method for BWR Subchannel Analysis," MIT-EL 79-027, November 1979.

A. Faya, L. Wolf and N. Todreas, "CANAL User's Manual," MIT Energy Laboratory No. MIT-EL 79-028, November 1979.

A.4 W.D. Hinkle, "Water Tests for Determining Post-Voiding Behavior in the LMFBR," MIT Energy Laboratory Report MIT-EL-76-005, June 1976.

W.D. Hinkle, Ed., "LMFBR Safety and Sodium Boiling - A State of the Art Report," Draft DOE Report, June 1978.

M.R. Granziera, P. Griffith, W.D. Hinkle, M.S. Kazimi, A. Levin, M. Manahan, A. Schor, N. Todreas, G. Wilson, "Development of Computer Code for Multi-dimensional Analysis of Sodium Voiding in the LMFBR," Preliminary Draft Report, July 1979.

M. Granziera, P. Griffith, W.D. Hinkle, M.S. Kazimi, A. Levin, M. Manahan, A. Schor, N. Todreas, R. Vilim, G. Wilson, "Development of Computer Code Models for Analysis of Subassembly Voiding in the LMFBR," Interim Report of the MTI Sodium Boiling Project Covering Work Through September 30, 1979, MIT-EL-80-005.

A. Levin and P. Griffith, "Development of a Model to Predict Flow Oscillations in Low-Flow Sodium Boiling, "MIT-EL-80-006, April 1980.

M. R. Granziera and M. S. Kazimi, "A Two-Dimensional, Two-Fluid Model for Sodium Boiling in LMFBR Assemblies," MIT-EL-80-011, May 1980.

G. Wilson and M. Kazimi, "Development of Models for the Sodium Version of the Two-Phase Three Dimensional Thermal Hydraulics Code THERMIT," MIT-EL-80-010, May 1980.

R.G. Zielinski and M.S. Kazimi, "Development of Models for the Two-Dimensional, Two-Fluid Code for Sodium Boiling NATOF-2D," MIT Energy Laboratory Report No. MIT-EL-81-030, September 1981.

B. Papers


B.1 General Applications
B.2 PWR Applications
B.3 BWR Applications
B.4 LMFBR Applications



B.1 J.E. Kelly and M.S. Kazimi, "Development of the Two-Fluid Multi-Dimensional Code THERMIT for LWR Analysis," Heat Transfer-Orlando 1980, AIChE Symposium Series 199, Vol. 76, August 1980.

J.E. Kelly and M.S. Kazimi, "THERMIT, A Three-Dimensional, Two-Fluid Code for LWR Transient Analysis," Transactions of American Nuclear Society, 34, p. 893, June 1980.

B.2 P. Moreno, J. Kiu, E. Khan, N. Todreas, "Steady State Thermal Analysis of PWR's by a Single Pass Procedure Using a Simplified Method," American Nuclear Society Transactions, Vol. 26.

P. Moreno, J. Liu, E. Khan, N. Todreas, "Steady-State Thermal Analysis of PWR's by a Single Pass Procedure Using a Simplified Nodal Layout," Nuclear Engineering and Design, Vol. 47, 1978, pp. 35-48.

C. Chiu, P. Moreno, R. Bowring, N. Todreas, "Enthalpy Transfer between PWR Fuel Assemblies in Analysis by the Lumped Subchannel Model," Nuclear Engineering and Design, Vol. 53, 1979, 165-186.

B.3 L. Wolf and A. Faya, "A BWR Subchannel Code with Drift Flux and Vapor Diffusion Transport," American Nuclear Society Transactions, Vol. 28, 1978, p. 553.

S.P. Kao and M.S. Kazimi, "CHF Predictions In Rod Bundles," Trans. ANS, 35, 766 June 1981.

B.4 W.D. Hinkle, (MIT), P.M. Tschamper (GE), M.H. Fontana (ORNL), R.E. Henry (ANL), and A. Padilla (HEDL), for U.S. Department of Energy, "LMFBR Safety & Sodium Boiling," paper presented at the ENS/ANS International Topical Meeting on Nuclear Reactor Safety, October 16-19, 1978, Brussels, Belgium.

M.I. Autruffe, G.J. Wilson, B. Stewart and M. Kazimi, "A Proposed Momentum Exchange Coefficient for Two-Phase Modeling of Sodium Boiling," Proc. Int. Meeting Fast Reactor Safety Technology, Vol. 4, 2512-2521, Seattle, Washington, August 1979.

M.R. Granziera and M.S. Kazimi, "NATOF-2D: A Two Dimensional Two-Fluid Model for Sodium Flow Transient Analysis," Trans. ANS, 33, 515, November 1979.

# THE EFFECT OF VIRTUAL MASS ON THE CHARACTERISTICS AND THE NUMERICAL STABILITY IN TWO-PHASE FLOWS

## ABSTRACT

It is known that the typical six equation two-fluid model of the two-phase flow possesses complex characteristics, exhibits unbounded instabilities in the short-wavelength limit and constitutes an ill-posed initial value problem. Among the suggestions to overcome these difficulties, one model for the virtual mass force terms was studied here, because the virtual mass represents real physical effects to accomplish the dissipation for numerical stability. It was found that the virtual mass has a profound effect upon the mathematical characteristic and numerical stability. Here a quantitative bound on the coefficient of the virtual mass terms was suggested for mathematical hyperbolicity and numerical stability. It was concluded that the finite difference scheme with the virtual mass model is restricted only by the convective stability conditions with the above suggested value.

## INTRODUCTION

Transient two-phase flow analysis is of importance in nuclear reactors under various accident conditions. Among the several models of two-phase flow the two-fluid model offers the most detailed and general description of two-phase flow. But it was reported in early work that the model has inherent instability problems. In 1965 Jarvis [1] tried to solve the two-phase equations using the two-fluid model for modeling the cooldown process in cryogenics and faced instability problems. He attributed the instabilities he encountered to the fact that the system was found to be nonhyperbolic. In 1967 Richtmyer and Morton [2] showed that, if the IVP is ill-posed, then no difference scheme that is consistent with the problem can be stable. In 1971 Siegmann encountered numerical stabilities in his transient sodium boiling code called MOC. In 1973 Boure appears to have encountered severe stability difficulties in the GEVATRAN code. In a round table discussion during the Fifth International Heat Transfer Conference [5], the instability problems in the two-fluid model were shown to be caused by an ill-posed problem. In 1976 Bryce [6] experienced large-scale pressure oscillations with RELAP-UK code. He demonstrated that simple two-fluid model with complex characteristics does not give solutions which converge as the mesh size and time step size are refined. He concluded that, though the solution may be obtained by using numerical technique, the significance of these solutions is not clear. In 1976 Lyczkowski [7] did a sample calculation illustrating error growth caused by complex characteristics. In the 1977 ANS topical meeting of thermal reactor safety, Anderson [8] reported his attempt to acheive stability by using virtual mass terms in his RISQUE code which was proposed by Hilprecht. It was found that the virtual mass has a profound effect on the dynamics of two-phase flow. His code was used to perform numerical calculations on the behavior of interfacial waves. It was observed that for no virtual mass, the wave amplitude grows rapidly, but the computed result with virtual mass shows a stable oscillatory behavior for the wave wave amplitude. In 1979 Rivard and Travis [9] successfully dealt with critical flow with the two-fluid code, K-FIX. But for the results presented, the value of the interfacial drag function was chosed sufficiently large that there is no mean relative motion between the phases. In 1979 investigators at R.P.I. [10] showed that without virtual mass not only was it more costly to run the problem, but they could not even run the complete problem using their code, GEAR. Computer running time without virtual mass was forty times longer than that with virtual mass.

It may well be argued that equation sets with complex characteristics may still be adequate for a range of phenomena if the numerical method introduces sufficient dissipation to damp the high frequency instabilities. But obviously there are real physical effects to accomplish the dissipation needed for numerical stability. Among several candidates suggested for numerical stability, Cheng and Lahey's model [10] with the virtual mass terms will be studied here.

## A. ILL-POSED PROBLEMS AND MATHEMATICAL STABILITY OF THE INITIAL VALUE PROBLEM (IVP)

Quantitative physical laws are idealizations of reality. As knowledge grows, we observe that a given physical situation can be idealized mathematically in a number of different ways. It is therefore important to characterize those reasonable ideal formulations. Hadamard [11] examined this problem, asserting that a physical formulation is well-posed if its solution exists, is unique, and depends continuously on the external (boundary) conditions. Existence and uniqueness are an affirmation of its principle of determination without which experiments could not be repeated with the expectation of consistent data. The continuous dependence criterion is an expression of the stability of the solution; a small change in any of the problem's data should produce only a correspondingly small change in the solution. The general one-dimensional problem can be represented in matrix form, as follows:

$$A(x) \frac{\partial x}{\partial t} + B(x) \frac{\partial x}{\partial z} + C(x) = 0, \qquad \text{(A.1)}$$

where

x is a column vector of independent variables,

A(x) and B(x) are the coefficient matrices, and

C(x) is a source or sink vector.

The IVP under consideration is to find a solution of Eq. (A.1) in some region

$$a \leq z \leq b,$$

subject to the initial condition

$$x(0,z) = u_0(z), \tag{A.2}$$

and the value of x or its derivatives prescribed on the boundaries,

$$z = a \text{ and } z = b. \tag{A.3}$$

When the definition of a well-posed problem by Hadamard is applied to the above system, the IVP is said to be well-posed if the solution of Eq. (A.1) exists, is unique, and depends continuously on both of the initial condition, Eq. (A.2) and the boundary conditions Eq. (A.3). Also system Eq. (A.1) is defined as hyperbolic if all values of the characteristics of Eq. (A.1) are distinct and real. According to Lax [12] the requirement of a well-posed problem in the linear partial differential equations with the form like Eq. (A.1) is the same as that of hyperbolicity. Along characteristic curves the highest-order derivatives in a PDE are indeterminate; characteristics may separate discontinuities in the solution. Therefore characteristics are trajectories of discontinuities in the solution. Physically it represents the travel of information in a physical system. The typical two-fluid models have four real characteristics (two convective velocities of liquid and vapor, and two relative sound velocities to both convective velocities of liquid and vapor through system) and two imaginary characteristics. The two imaginary characteristics may represent two relative velocities of interfacial wave propagation to both liquid and vapor with complex values which indicate an instability of interfacial waves.

Characteristics and stability are shown to be related only in the limit of large frequency. The characteristics, $\mu$ of Eq. (A.1) are defined by the equation

$$\mathrm{Det}\,(\mu A - B) = 0 \tag{A.4}$$

The local linear stability behavior of Eq. (A.1) is examined. x is replaced by $x_0 + \delta x$, and the result is linearized with respect to $\delta x$, resulting in

$$A(x_0)\frac{\partial(\delta x)}{\partial t} + B(x_0)\frac{\partial(\delta x)}{\partial z} + \left(\delta x\left(\frac{\partial A}{\partial x}\right)_{x_0}\right)\frac{\partial x_0}{\partial t} + \left(\delta x\left(\frac{\partial B}{\partial x}\right)_{x_0}\right)\frac{\partial x_0}{\partial z} + \delta x\left(\frac{\partial C}{\partial x}\right)_{x_0} = 0 \tag{A.5}$$

As Eq. (A.5) describes small perturbations, $\delta x$ about an unperturbed solution, a uniform steady-state unperturbed solution is assumed, which means that $x_0$ is independent of z and t. We take a wave form for the perturbed amount, $\delta x$

$$\delta x = \delta x_0 \exp[i(kz - wt)] \tag{A.6}$$

Then Eq. (A.5) becomes

$$-iw\, A(x_0)\, \delta x_0 + ik\, B(x_0)\, \delta x_0 + \delta x_0 \left(\frac{\partial C}{\partial x}\right)_{x_0} = 0 \tag{A.7}$$

Equation (A.7) is a homogeneous linear system in the components of $\delta x$. For a nontrivial solution the determinant of the coefficient matrix must vanish.

$$\mathrm{Det}(-iwA + ikB + D) = 0, \tag{A.8}$$

where

$$D = \left(\frac{\partial C}{\partial x}\right)^T_{x_0}, \tag{A.9}$$

the superscript T denoting the matrix transpose. For nonzero k Eq. (A.8) can be rewritten as

$$\mathrm{Det}\left(\frac{w}{k} A - B + \frac{i}{k} D\right) = 0 \tag{A.10}$$

Inspection of Eq. (A.6) shows that the condition for stability is that $\mathrm{Im}(w) \leq 0$ for all roots w. Comparison of Eq. (A.4) and Eq. (A.10) shows that, if $D = 0$ or $k \to \infty$, the dispersion relations can be obtained from the characteristics simply by equating $w/k$ to $\mu$. Physically the instability at long wavelengths is the well-known Helmholtz instability. But the system which possesses complex characteristics exhibits unphysical and unbounded instabilities in the short-wavelength limit. One of the reasons why the basic system behaves like the above lies in the simplest choice for the interfacial pressure distribution in which it is constant and equal to the bulk pressure. Physically the above choice is consistent with the assumption that the two fluid velocities are equal. Therefore this leads to an equation set possessing real characteristics only when the two fluid velocities are equal. This is the reason for the present study of the effect of the virtual mass here.

## B. CHARACTERISTIC AND STABILITY ANALYSIS

The effect of virtual mass on the characteristics can be illstrated by adding the virtual mass terms to the one-dimensional momentum equations of both liquid and vapor. Here Cheng and Lahey's model for the virtual mass terms is considered. The conservation equations are as follows:

Conservation of vapor mass:

$$\frac{\partial}{\partial t}(\alpha_v \rho_v) + \frac{\partial}{\partial z}(\alpha_v \rho_v V_v) = \Gamma \tag{B.1}$$

Conservation of liquid mass:

$$\frac{\partial}{\partial t}(\alpha_\ell \rho_\ell) + \frac{\partial}{\partial z}(\alpha_\ell \rho_\ell V_\ell) = -\Gamma \tag{B.2}$$

Conservation of vapor momentum:

$$\alpha_v \rho_v \frac{\partial V_v}{\partial t} + \alpha_v \rho_v V_v \frac{\partial V_v}{\partial z} \tag{B.3}$$

$$= \Gamma(V_i - V_v) - \alpha_v \frac{\partial P}{\partial z} - \alpha_v[F_s + F_v] - \alpha_v \rho_v g - F_{wv},$$

where

$V_i$, $F_s$, $F_v$ and $F_{wv}$ are the interfacial velocity, the standard drag force per unit volume, the virtual mass force per unit volume, and the vapor wall friction force per unit volume, respectively.

$$F_s = k(V_v - V_\ell) \tag{B.4}$$

$$F_v = \rho_\ell C_v \left\{ \frac{\partial V_v}{\partial t} - \frac{\partial V_\ell}{\partial t} + V_v \frac{\partial}{\partial z} (V_v - V_\ell) + (V_v - V_\ell) \left[ (\lambda - 2) \frac{\partial V_v}{\partial z} + (1 - \lambda) \frac{\partial V_\ell}{\partial z} \right] \right\} \quad (B.5)$$

Conservation of liquid momentum

$$\alpha_\ell \rho_\ell \frac{\partial V_\ell}{\partial t} + \alpha_\ell \rho_\ell V_\ell \frac{\partial V_\ell}{\partial z} = \Gamma(V_\ell - V_i) - \alpha_\ell \frac{\partial P}{\partial z} + \alpha_v [F_s + F_v] - \alpha_\ell \rho_\ell g - F_{w\ell}, \quad (B.6)$$

where $F_{w\ell}$ is the liquid wall friction force per unit volume. For convenience by summing up Eq. (B.3) and Eq. (B.6), we can get the momentum equation

$$\alpha_v \rho_v \frac{\partial V_v}{\partial t} + \alpha_\ell \rho_\ell \frac{\partial V_\ell}{\partial t} + \alpha_v \rho_v V_v \frac{\partial V_v}{\partial z} + \alpha_\ell \rho_\ell V_\ell \frac{\partial V_\ell}{\partial z} = \Gamma(V_\ell - V_v) - \frac{\partial P}{\partial z} - (\alpha_v \rho_v + \alpha_\ell \rho_\ell) g - (F_{wv} + F_{w\ell}) \quad (B.7)$$

By multiplying Eq. (B.3) by $\alpha_\ell$ and Eq. (B.6) by $\alpha_v$ and then subtracting each other, we obtain

$$\alpha_v (\alpha_\ell \rho_v + \rho_\ell C_v) \frac{\partial V_v}{\partial t} - \alpha_v (\alpha_\ell \rho_\ell + \rho_\ell C_v) \frac{\partial V_\ell}{\partial t} + \{[\alpha_\ell \alpha_v \rho_v + \alpha_v \rho_\ell C_v (\lambda - 1)] V_v - \alpha_v \rho_\ell C_v (\lambda - 2) V_\ell\} \frac{\partial V_v}{\partial z} - \{[\alpha_\ell \alpha_v \rho_\ell + \alpha_v \rho_\ell C_v (1 - \lambda)] V_\ell + \alpha_v \rho_\ell C_v \lambda V_v\} \frac{\partial V_\ell}{\partial z} = \Gamma(V_i - \alpha_v V_\ell - \alpha_\ell V_v) - \alpha_v F_s + \alpha_\ell \alpha_v (\rho_\ell - \rho_v) g - \alpha_\ell F_{wv} + \alpha_v F_{w\ell} \quad (B.8)$$

Conservation of vapor energy:

$$\frac{\partial}{\partial t}(\alpha_v \rho_v e_v) + \frac{\partial}{\partial z}(\alpha_v \rho_v e_v V_v) + P\frac{\partial}{\partial z}(\alpha_v V_v) + P\frac{\partial \alpha_v}{\partial t} = Q_{wv} + Q_i , \quad (B.9)$$

where $Q_{wv}$ and $Q_i$ are the heat transfer per unit volume from wall to vapor and from interface, respectively.

Conservation of liquid energy:

$$\frac{\partial}{\partial t}(\alpha_\ell \rho_\ell e_\ell) + \frac{\partial}{\partial z}(\alpha_\ell \rho_\ell e_\ell V_\ell) + P\frac{\partial}{\partial z}(\alpha_\ell V_\ell) - P\frac{\partial \alpha_v}{\partial t} = Q_{w\ell} - Q_i , \quad (B.10)$$

where $Q_{w\ell}$ is the heat transfer per unit volume from wall to liquid.

Let us construct the form of

$$A(x)\frac{\partial x}{\partial t} + B(x)\frac{\partial x}{\partial z} + C(x) = 0, \quad (B.11)$$

where x is a column vector of independent variables,

$$x = (\alpha, P, V_v, V_\ell, e_v, e_\ell)^T$$

It is assumed that the density of both liquid and vapor is the function of pressure. Here $a_v$ and $a_\ell$ denote the sound velocity of vapor and liquid, respectively.

$$A = \begin{bmatrix} \rho_v & \alpha_v a_v^{-2} & & & & \\ -\rho_\ell & \alpha_\ell a_\ell^{-2} & & & & \\ & & \alpha_v \rho_v & \alpha_\ell \rho_\ell & & \\ & & \alpha_\ell \rho_v + \rho_\ell C_v & -\alpha_\ell \rho_\ell - \rho_\ell C_v & & \\ \rho_v e_v + P & \alpha_v e_v a_v^{-2} & & & \alpha_v \rho_v & \\ -\rho_\ell e_\ell - P & \alpha_\ell \rho_\ell a_\ell^{-2} & & & & \alpha_\ell \rho_\ell \end{bmatrix} \qquad (B.12)$$

$$B = \begin{bmatrix} \rho_v V_v & \alpha_v V_v a_v^{-2} & \alpha_v \rho_v & & \\ -\rho_\ell V_\ell & \alpha_\ell V_\ell a_\ell^{-2} & & \alpha_\ell \rho_\ell & \\ & 1 & \alpha_v \rho_v V_v & \alpha_\ell \rho_\ell V_\ell & \\ & & \begin{matrix}[\alpha_\ell \rho_v + \rho_\ell C_v \\ (\lambda - 1)] V_v \\ -\rho_\ell C_v (\lambda - 2) V_\ell\end{matrix} & \begin{matrix}-[(1-\alpha)\rho_\ell + \rho_\ell \\ C_v (1-\lambda)] V_\ell \\ -\rho_\ell C_v V_v \lambda\end{matrix} & \\ \begin{matrix}\rho_v e_v V_v \\ + P V_v\end{matrix} & \alpha_v e_v V_v a_v^{-2} & \alpha_v \rho_v e_v + P\alpha_v & & \alpha_v \rho_v \; V_v \\ \begin{matrix}-\rho_\ell e_\ell V_\ell \\ - P V_\ell\end{matrix} & (\alpha_\ell \;) e_\ell V_\ell a_\ell^{-2} & & \alpha_\ell \rho_\ell e_\ell - \alpha_v P & \alpha_\ell \rho_\ell V_\ell \end{bmatrix} \qquad (B.13)$$

In order to get the characteristics from Eq. (A.2)

$$\text{Det}\,(\mu\, A - B) = 0 \qquad (B.14)$$

Reducing the terms related to the energy equations from Eq. (B.12), we get

$$\alpha_v\rho_v(\mu - V_v) \times \alpha_\ell\rho_\ell(\mu - V_\ell) \times$$

$$\text{Det}\begin{bmatrix} \rho_v(\mu - V_v) & \alpha_v a_v^{-2}(\mu - V_v) & -\alpha_v\rho_v & 0 \\ -\rho_\ell(\mu - V_\ell) & \alpha_\ell a_\ell^{-2}(\mu - V_\ell) & 0 & -\alpha_\ell\rho_\ell \\ 0 & -1 & \alpha_v\rho_v(\mu - V_v) & \alpha_\ell\rho_\ell(\mu - V_\ell) \\ 0 & 0 & \begin{matrix}(\alpha_\ell\rho_v+\rho_\ell C_v)(\mu - V_v) \\ -\rho_\ell C_v(\lambda-2)(V_v - V_\ell)\end{matrix} & \begin{matrix}-(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\mu - V_\ell) \\ +\rho_\ell C_v\lambda(V_v - V_\ell)\end{matrix} \end{bmatrix} = 0 \qquad (B.15)$$

From the terms related to the energy equations we obtained $\mu = V_v, V_\ell$. Physically this means that energy is transferred by only convection. For simplicity with the assumption that $a_v \gg V_v$ and $a_\ell \gg V_\ell$, that is, incompressible flow is used, we neglect

$$\alpha_\ell a_\ell^{-2}(\mu - V_\ell) \text{ and } \alpha_v a_v^{-2}(\mu - V_v)$$

We may predict that these terms are related to the sonic velocities transferred through liquid and vapor.

Now we get the simplified determinant form

$$\text{Det}\begin{bmatrix} \rho_v(\mu - V_v) & -\alpha_v\rho_v & 0 \\ -\rho_\ell(\mu - V_\ell) & 0 & -\alpha_\ell\rho_\ell \\ 0 & \begin{matrix}(\alpha_\ell\rho_v+\rho_\ell C_v)(\mu - V_v) \\ -\rho_\ell C_v(\lambda-2)(V_v - V_\ell)\end{matrix} & \begin{matrix}-(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\mu - V_\ell) \\ +\rho_\ell C_v\lambda\,(V_v - V_\ell)\end{matrix} \end{bmatrix} = 0 \qquad (B.16)$$

From Eq. (B.16) we obtain an algebraic equations of the form

$$aq^2 + bq + c = 0, \tag{B.17}$$

where
$$a = \alpha_\ell^{\,2}\,\rho_v + \alpha_v\alpha_\ell\rho_\ell + \gamma$$

$$b = [\gamma(2-\lambda) + 2\alpha_v\alpha_\ell\rho_\ell]V_r$$

$$c = [\alpha_v\alpha_\ell\rho_\ell + \alpha_v\;\gamma(1-\lambda)]V_r^{\,2}$$

$$\gamma = \rho_\ell C_v$$

$$q = \mu - V_v$$

$$V_r = V_v - V_\ell$$

In order for our system to have real and distinct characteristics

$$b^2 - 4ac > 0$$

$$\frac{b^2-4ac}{V_r^{\,2}} = [\gamma^2(\lambda^2 + 4(1-\lambda)\alpha_\ell) + 4\gamma(1-\lambda)\alpha_v\alpha_\ell^{\,2}(\rho_\ell-\rho_v) -4\alpha_v\alpha_\ell^{\,3}\,\rho_v\rho_\ell] \tag{B.18}$$

i) There is no virtual mass; $C_v = 0$

$$b^2 - 4ac = -4\alpha_v\alpha_\ell^{\,3}\,\rho_v\rho_\ell$$

Therefore the system with no virtual mass always has two complex characteristics except the single phase region ($\alpha_v = 0$ or $\alpha_\ell = 0$)

ii) If $\alpha_v \to 0$, $b^2 - 4ac = \gamma^2(\lambda - 2)^2$

If $C_v \neq 0$ and $\lambda \neq 2$, $\lim_{\alpha_v \to 0} (b^2 - 4ac) > 0$

iii) If $\alpha_\ell \to 0$, $b^2 - 4ac = \gamma^2\lambda^2$

If $C_v \neq 0$ and $\lambda \neq 0$, $\lim_{\alpha_\ell \to 0} (b^2 - 4ac) > 0$

iv) If $\lambda = 1$, $b^2 - 4ac = \gamma^2 - 4\alpha_v\alpha_\ell^3\rho_v\rho_\ell$

$$\text{If } C_v > \sqrt{\frac{4\alpha_v\alpha_\ell^3\rho_v}{\rho_\ell}}\ ,\ b^2 - 4ac > 0 \qquad \text{(B.19)}$$

v) If $0 \leq \lambda < 1$, $C_v^2[\lambda^2 + 4(1-\lambda)\alpha_\ell] + 4\alpha_v\alpha_\ell^2\gamma(1-\lambda)(1 - \frac{\rho_v}{\rho_\ell})$

$$> 4\alpha_v\alpha_\ell^3\frac{\rho_v}{\rho_\ell}$$

As $4(1-\lambda)\alpha_\ell > 0$ and $4\alpha_v\alpha_\ell^2\gamma(1-\lambda)(1 - \frac{\rho_v}{\rho_\ell}) > 0$,

$$C_v^2\lambda^2 > 4\alpha_v\alpha_\ell^3\frac{\rho_v}{\rho_\ell}$$

$$(C_v\lambda) > \sqrt{4\alpha_v\alpha_\ell^3\frac{\rho_v}{\rho_\ell}}$$

vi) If $1 < \lambda < 2$, $C_v^2(\lambda-2)^2 - 4\alpha_v\alpha_\ell^2 C_v > 4\alpha_v\alpha_\ell^3\frac{\rho_v}{\rho_\ell}$

$$C_v^2(\lambda-2)^2 > 4\alpha_v\alpha_\ell^2\left[C_v + \alpha_\ell\frac{\rho_v}{\rho_\ell}\right]$$

Note that as $\lambda$ approaches 2 or 0, the required value of $C_v$ becomes larger and larger.

The effect of virtual mass on stability can be simply illustrated by using the characteristics analysis. If the mass exchange, standard drag force, wall friction, and the energy equations are neglected in order to know clearly the effect of virtual mass on stability, D in Eq. (A.10) reduces to zero.

Now as stated before, the dispersion relationship can be obtained from the characteristics simply by equating $\frac{w}{k}$ to $\mu$.

$$w = k\mu \tag{B.20}$$

The stability condition, $\mathrm{Im}(w) \leq 0$ for all roots w is the same as the condition for the real and distinct characteristics.

## C. NUMERICAL STABILITY

Here we shall examine a difference scheme formed in a similar way to THERMIT [13]. For the same reason as in the stability analysis, mass exchange, standard drag force, and wall friction are neglected. Also energy equations are dropped because they only represent convective properties as we say in the characteristic analysis

vapor mass

$$\frac{(\alpha_v\rho_v)_j^{n+1} - (\alpha_v\rho_v)_j^n}{\Delta t} + \frac{1}{\Delta z}\left[(\alpha_v^n\rho_v^n\, V_v^{n+1})_{j+1/2} - (\alpha_v^n\rho_v^n\, V_v^{n+1})_{j-1/2}\right] = 0 \quad (C.1)$$

liquid mass

$$\frac{(\alpha_\ell\rho_\ell)_j^{n+1} - (\alpha_\ell\rho_\ell)_j^n}{\Delta t} + \frac{1}{\Delta z}\left[(\alpha_\ell^n\rho_\ell^n\, V_\ell^{n+1})_{j+1/2} - (\alpha_\ell^n\rho_\ell^n\, V_\ell^{n+1})_{j-1/2}\right] = 0 \quad (C.2)$$

momentum equations

$$(\alpha_v\rho_v)^n \frac{(V_v^{n+1} - V_v^n)_{j+1/2}}{\Delta t} + (\alpha_\ell\rho_\ell)^n \frac{(V_\ell^{n+1} - V_\ell^n)_{j+1/2}}{\Delta t} + \left(\alpha_v\rho_v V_v \frac{\Delta V_v}{\Delta z}\right)_{j+1/2}^n$$

$$+ \left(\alpha_\ell\rho_\ell V_\ell \frac{\Delta V_\ell}{\Delta z}\right)_{j+1/2}^n + \frac{(P_{j+1} - P_j)^{n+1}}{\Delta z} = 0 \quad (C.3)$$

$$\alpha_v^n\,[\alpha_\ell\rho_v + \rho_\ell C_v]^n \frac{(V_v^{n+1} - V_v^n)_{j+1/2}}{\Delta t} - \alpha_v^n\,[\alpha_\ell\rho_\ell + \rho_\ell C_v]^n \frac{(V_\ell^{n+1} - V_\ell^n)_{j+1/2}}{\Delta t}$$

$$+ \{<\alpha_\ell\alpha_v\rho_v + \alpha_v\rho_\ell C_v(\lambda-1)]V_v - \alpha_v\rho_\ell C_v(\lambda-2)V_\ell> \frac{\Delta V_v}{\Delta z}$$

$$- <[\alpha_\ell\alpha_v\rho_\ell + \alpha_v\rho_\ell C_v(1-\lambda)]V_\ell + \alpha_v\rho_\ell C_v\lambda V_v> \frac{\Delta V_\ell}{\Delta z}\}_{j+1/2}^n = 0 \quad (C.4)$$

The above numerical scheme can be modified as follows:

vapor mass

$$\rho_{vj}^{n+1}\,(\alpha_v^{n+1}-\alpha_v^n)_j+\alpha_{vj}^n\,(\rho_v^{n+1}-\rho_v^n)_j+\frac{\Delta t}{\Delta z}\,[\rho_{vj}^n\,V_{vj+1/2}^{n+1}\,(\alpha_{vj}^n-\alpha_{vj-1}^n)+\alpha_{vj-1}^n\,V_{vj+1/2}^{n+1}\,(\rho_{vj}^n-\rho_{vj-1}^n)+\alpha_{vj-1}^n\,\rho_{vj-1}^n\,(V_{vj+1/2}^{n+1}-V_{vj-1/2}^{n+1})]=0 \qquad (C.5)$$

liquid mass

Same as Eq. (C.5) with $\alpha_v$ replaced by $\alpha_\ell$, subscript v by $\ell$. (C.6)

momentum equations

$$(\alpha_v\rho_v)^n\,(V_v^{n+1}-V_v^n)_j+(\alpha_\ell\rho_\ell)^n\,(V_\ell^{n+1}-V_\ell^n)_j+\frac{\Delta t}{\Delta z}\,[(\alpha_v\rho_v V_v)^n\,(V_{v_j}-V_{v_{j-1}})^n+(\alpha_\ell\rho_\ell V_\ell)^n\,(V_{\ell_j}-V_{\ell_{j-1}})^n+(P_{j+1/2}-P_{j-1/2})^{n+1}]=0 \qquad (C.7)$$

$$[\alpha_\ell\rho_v+\rho_\ell C_v]^n\,(V_v^{n+1}-V_v^n)_j-[\alpha_\ell\rho_\ell+\rho_\ell C_v]^n\,(V_\ell^{n+1}-V_\ell^n)_j+\{<[\alpha_\ell\rho_v+\rho_\ell C_v(\lambda-1)]V_v-\rho_\ell C_v(\lambda-2)V_\ell>\frac{\Delta t}{\Delta z}\,(V_{vj}-V_{vj-1})-<[\alpha_\ell\rho_\ell+\rho_\ell C_v(1-\lambda)]V_\ell+\rho_\ell C_v\lambda V_v>\frac{\Delta t}{\Delta z}\,(V_{\ell j}-V_{\ell j-1})>\}^n=0 \qquad (C.8)$$

Treating the coefficients as constants and using Von Neumann local stability analysis, $U_j^n=\zeta^n\exp(ikj\Delta z)$, we obtain the determinant form for a nontrivial solution.

$$
\mathrm{Det}\begin{bmatrix}
\rho_v(\zeta-1+\tilde{V}_v) & a_v^{-2}\alpha_v(\zeta-1+\tilde{V}_v) & \alpha_v\rho_v\zeta q & 0 \\
-\rho_\ell(\zeta-1+\tilde{V}_\ell) & a_\ell^{-2}\alpha_\ell(\zeta-1+\tilde{V}_\ell) & 0 & \alpha_\ell\rho_\ell\,\zeta q \\
0 & \zeta q & \alpha_v\rho_v(\zeta-1+\tilde{V}_v) & \alpha_\ell\rho_\ell(\zeta-1+\tilde{V}_\ell) \\
0 & 0 & \begin{array}{l}(\alpha_\ell\rho_v+\rho_\ell C_v)(\zeta-1) \\ +\{[\alpha_\ell\rho_v+\rho_\ell C_v(\zeta-1)]\tilde{V}_v \\ -\rho_\ell C_v(\lambda-2)\,\tilde{V}_\ell\}\end{array} & \begin{array}{l}-(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\zeta-1) \\ -\{(\alpha_\ell\rho_\ell+\rho_\ell C_v(1-\lambda)]\tilde{V}_\ell \\ +\rho_\ell C_v\lambda\tilde{V}_v\}\end{array}
\end{bmatrix}=0 \tag{C.9}
$$

where $\tilde{V}_v = \frac{\Delta t}{\Delta z} V_v(1-\exp(-ik\Delta z))$

$\tilde{V}_\ell = \frac{\Delta t}{\Delta z} V_\ell(1-\exp(-ik\Delta z))$

$q = 2i\,\frac{\Delta t}{\Delta z}\sin\left(\frac{k\Delta z}{2}\right)$

$k = \frac{\pi}{n\Delta z}$ : $n=1\sim J$ :J: the number of axial mesh

For simplicity we assume in the same way as in the previous analysis that $a_v \gg V_v$, $a_\ell \gg V_\ell$

Then the determinant reduces to the 3 x 3 determinant

$$
\mathrm{Det}\begin{bmatrix}
\rho_v(\zeta-1+\tilde{V}_v) & \alpha_v\rho_v\,\zeta q & 0 \\
-\rho_\ell(\zeta-1+\tilde{V}_v) & 0 & \alpha_\ell\rho_\ell\zeta q \\
0 & \begin{array}{l}(\alpha_\ell\rho_v+\rho_\ell C_v)(\zeta-1+\tilde{V}_v) \\ +\rho_\ell C_v(\lambda-2)(\tilde{V}_v-\tilde{V}_\ell)\end{array} & \begin{array}{l}-(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\zeta-1+\tilde{V}_\ell) \\ -\rho_\ell C_v\lambda(\tilde{V}_v-\tilde{V}_\ell)\end{array}
\end{bmatrix} = 0 \tag{C.10}
$$

As we notice that $\zeta q$ appears as a common coefficient, resolving the determinant we obtain the modified determinant

$$\text{Det}\begin{bmatrix} \rho_v(\mu-\tilde{V}_v) & -\alpha_v\rho_v & 0 \\ -\rho_\ell(\mu-\tilde{V}_\ell) & 0 & -\alpha_\ell\rho_\ell \\ 0 & \begin{matrix}(\alpha_\ell\rho_v+\rho_\ell C_v)(\mu-\tilde{V}_v) \\ -\rho_\ell C_v(\lambda-2)(\tilde{V}_v-\tilde{V}_\ell)\end{matrix} & \begin{matrix}-(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\mu-\tilde{V}_\ell) \\ +\rho_\ell C_v\lambda(\tilde{V}_v-\tilde{V}_\ell)\end{matrix} \end{bmatrix} = 0 \qquad \text{(C.11)}$$

where $\mu = -(\zeta-1)$

The above determinant, Eq. (C.11) is in the exact same form as Eq. (B.16). Therefore we can use Eq. (B.17) as follows:

$$aq^2 + b\tilde{V}_r q + c\tilde{V}_r^2 = 0 \quad , \qquad \text{(C.12)}$$

where
$$a = \alpha_\ell^2\,\rho_v + \alpha_v\alpha_\ell\rho_\ell + \rho_\ell C_v$$
$$b = \rho_\ell C_v(2-\lambda) + 2\alpha_v\alpha_\ell\rho_\ell$$
$$c = \alpha_v\alpha_\ell\rho_\ell + \alpha_v\rho_\ell C_v(1-\lambda)$$
$$q = \mu - \tilde{V}_v$$
$$\tilde{V}_r = \tilde{V}_v - \tilde{V}_\ell$$

For numerical stability the absolute value of the growth factor, $|\zeta|$, should be less than 1.

From Eq. (C.12)

$$q = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\,\tilde{V}_r \qquad \text{(C.13)}$$

$$\zeta - 1 = -\mu = -(\tilde{V}_v + q) \tag{C.14}$$

$$\zeta - 1 = -\left[ \frac{(2a - b \pm \sqrt{b^2 - 4ac}}{2a} \tilde{V}_v + \frac{b \mp \sqrt{b^2 - 4ac}}{2a} \tilde{V}_\ell \right] \tag{C.15}$$

If $\lambda = 1$ and $\rho_\ell^2 C_v^2 >> 4\alpha_v \alpha_\ell^3 \rho_\ell \rho_v$, for $+\sqrt{b^2 - 4ac}$

$$\zeta - 1 = -\frac{(\alpha_\ell^2 \rho_v + \rho_\ell C_v)\tilde{V}_v + (\alpha_v \alpha_\ell \rho_\ell)\tilde{V}_\ell}{\alpha_\ell^2 \rho_v + \alpha_v \alpha_\ell \rho_\ell + \rho_\ell C_v}$$

$$= -p + p \exp(-ik\Delta z) , \tag{C.16}$$

where $p = \dfrac{(\alpha_\ell^2 \rho_v + \rho_\ell C_v) \frac{\Delta t}{\Delta z} V_v + (\alpha_v \alpha_\ell \rho_\ell) \frac{\Delta t}{\Delta z} V_\ell}{\alpha_\ell^2 \rho_v + \alpha_v \alpha_\ell \rho_\ell + \rho_\ell C_v}$

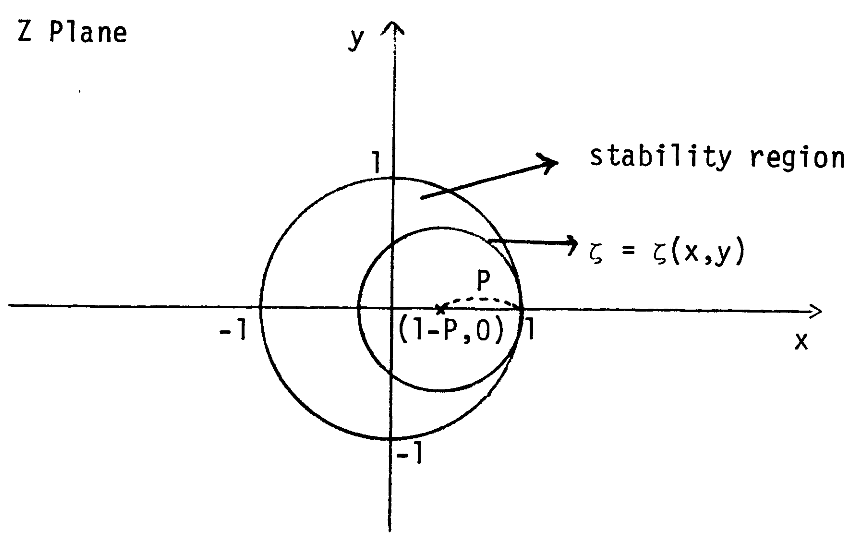


Fig. 1: Stability Diagram of Eq. (C.16)

For numerical stability $0 < P < 1$

$$0 < (\alpha_\ell^2 \rho_v + \rho_\ell C_v) \frac{\Delta t}{\Delta z} V_v + (\alpha_v \alpha_\ell \rho_\ell) \frac{\Delta t}{\Delta z} V_\ell$$

$$< (\alpha_\ell^2 \rho_v + \rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell) \qquad (C.17)$$

If $\frac{\Delta t}{\Delta z} V_v < 1$ and $\frac{\Delta t}{\Delta z} V_\ell < 1$, the above inequality is always met.

For $-\sqrt{b^2 - 4ac}$

$$\zeta - 1 = -\left[\frac{(\alpha_\ell^2 \rho_v) \frac{\Delta t}{\Delta z} \tilde{V}_v + (\rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell) \frac{\Delta t}{\Delta z} \tilde{V}_\ell}{\alpha_\ell^2 \rho_v + \rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell}\right]$$

$$= -r + r \exp(-ik\Delta z) , \qquad (C.18)$$

where $r = \dfrac{(\alpha_\ell^2 \rho_v) \frac{\Delta t}{\Delta z} V_v + (\rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell) \frac{\Delta t}{\Delta z} V_\ell}{\alpha_\ell^2 \rho_v + \rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell}$

For numerical stability $0 < r < 1$

$$0 < (\alpha_\ell^2 \rho_v) \frac{\Delta t}{\Delta z} V_v + (\rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell) \frac{\Delta t}{\Delta z} V_\ell$$

$$< (\alpha_\ell^2 \rho_v + \rho_\ell C_v + \alpha_v \alpha_\ell \rho_\ell) \qquad (C.19)$$

Also if $\frac{\Delta t}{\Delta z} V_v < 1$ and $\frac{\Delta t}{\Delta z} V_\ell < 1$, the above inequality is always met. Therefore, it can be concluded that, if $\lambda = 1$ and $C_v > \sqrt{4\alpha_v \alpha_\ell^3 \rho_v / \rho_\ell}$ , the finite difference scheme is restricted only by the convective stability conditions.

## REFERENCES


1. Stephen Jarvis, Jr., "On the Formulation and Numerical Evaluation of a Set of Two-Phase Flow Equations Modeling the Cooldown Process," Technical Note 301, National Bureau of Standards, Boulder, Colorado (1965).

2. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problem, Interscience Publishers, Inc., New York (1967).

3. E. R. Siegmann, "Theoretical Study of Transient Sodium Boiling in Reactor Coolant Channels Utilizing a Compressible Flow Model," ANL-7842, (1971).

4. J. A. Boure, "GEVATRAN: A Computer Program to Study Two-Phase Flow Dynamics," CONF-720686-3 (June 1972).

5. D. Gidaspow, Proc. 5th Int'l. Heat Transfer Conference, Tokyo, Japan, September 1974, CONF-740925, V. II (1974).

6. W. M. Bryce, "Determination of the Hyperbolic Regime of the Hydrodynamic Equations Modeled in the LOCA Code RELAP-OK," OECD/NEA Specialists' Meeting, Transient Two-Phase Flow, Toronto, Aug. 1976.

7. R. W. Lyczkowski, D. Gidaspow, C. W. Solbrig, and E. D. Hughes, "Characteristics and Stability Analyses of Transient One-Dimensional Two-Phase Flow Equations and Their Finite Difference Approximations," NSE 66, 378-396, 1978.

8. P. S. Andersen, P. Astrup, L. Eget, and O. Rathmann, "Numerical Experience with the Two-Fluid Model RISQUE," Proc. Topical Meeting, Thermal Reactor Safety, Sun Valley, Vol. 2 (1978).

9. W. C. Rivard and J. R. Travis, "A Nonequilibrium Vapor Production Model for Critical Flow," NSE 74,40-78, 1980.

10. R. T. Lahey, Jr., L. Y. Cheng, D. A. Drew, and J. E. Flaherty, "The Effect of Virtual Mass on the Numerical Stability of Accelerating Two-Phase Flows," Int. J. Multiphase Flow, Vol. 6 (1980).

11. J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, New Haven, 1923.

12. P. D. Lax, "Differential Equations, Difference Equations and Matrix Theory," Communications on Pure and Applied Mathematics, Vol. XI, pp. 174-194 (1958).

13. W. H. Reed and H. B. Stewart, "THERMIT - A Computer Program for Three-Dimensional Thermal Hydraulic Analysis of Light Water Reactor Codes,"