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TWO-GRID ITERATION METHOD FOR WEAKLY SINGULAR INTEGRAL EQUATIONS

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ABSTRACT

For the solution of weakly singular integral equations by the piecewise polynomial collocation method it is necessary to solve large linear systems. In the present paper a two-grid iteration method for solving such systems is constructed and the convergence of this method is investigated.

1. INTRODUCTION

Consider the linear integral equation

$$u(t) = \int_0^b K(t,s)u(s)ds + f(t), \quad 0 \le t \le b, \tag{1.1}$$

where $b > 0$ and $f\colon [0,b] \to \mathbb{R}$ is a given continuous function. Throughout this paper we shall suppose that the kernel K has the form

$$K(t,s) = a(t,s)\kappa(t-s) \tag{1.2}$$

where

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(A1) the function $\kappa(\tau)$ is $m-1$ times ($m \geq 1$) continuously differentiable with respect to τ for $\tau \in [-b, b] \setminus \{0\}$ and such that the estimates

$$|\kappa^{(k)}(\tau)| \leq b_k |\tau|^{-\alpha-k}, \quad k = 0, 1, \ldots, m-1, \tag{1.3}$$

hold with $0 < \alpha < 1$ and some positive constants $b_0, b_1, \ldots, b_{m-1}$ for all $\tau \in [-b, b] \setminus \{0\}$;

(A2) the function $a(t, s)$ is m times continuously differentiable on $[0, b] \times [0, d]$ and $[0, b] \times [d, b]$, where d is a fixed point in the interval $(0, b)$.

Let $C^k(X)$ denote the space of k times continuously differentiable functions $x: X \to \mathbb{R}$, $X \subset \mathbb{R} \equiv (-\infty; \infty)$, $C(X) = C^0(X)$, and set

$$E^{\alpha,m} \equiv \Big\{ u \in C[0, b] \cap C^m(0, d) \cap C^m(d, b) :$$

$$\sup_{\substack{0<t<b \\ t \neq d}} \frac{|u^{(m)}(t)|}{t^{-(\alpha+m-1)} + |t-d|^{-(\alpha+m-1)} + (b-t)^{-(\alpha+m-1)}} < \infty \Big\};$$

$E^{\alpha,m}$ is a Banach space under the norm

$$||u||_{E^{\alpha,m}} = \max_{0 \leq t \leq b} |u(t)| + \sup_{\substack{0<t<b \\ t \neq d}} \frac{|u^{(m)}(t)|}{t^{-(\alpha+m-1)} + |t-d|^{-(\alpha+m-1)} + (b-t)^{-(\alpha+m-1)}}.$$

Note that $C^m[0, b] \subset E^{\alpha,m}$. It follows from $u \in E^{\alpha,m}$ that $u \in C[0, b] \cap C^m(0, d) \cap C^m(d, b)$ and the estimates

$$|u^{(k)}(t)| \leq c_k \big[t^{-(\alpha+k-1)} + |t-d|^{-(\alpha+k-1)} + (b-t)^{-(\alpha+k-1)} \big], \ k = 1, \ldots, m, \tag{1.4}$$

hold with some positive constants $c_1, \ldots, c_m$ for $0 < t < d$ and $d < t < b$.

The following result (see [11,12,8]) characterizes the regularity properties of solutions of equation (1.1).

Lemma 1.1. *Let the assumptions (A1) and (A2) about the kernel (1.2) hold, and let* $f \in E^{\alpha,m}$*. If integral equation (1.1) has an integrable solution* $u \in L^1(0, b)$ *then* $u \in E^{\alpha,m}$*.*

Remark 1.1. If the conditions of Lemma 1.1 are fulfilled with $a \in C([0, b] \times [0, b])$ then the estimates (1.4) for the derivatives of the solution $u(t)$ of equation (1.1) can be specified (see [11]).

An effective method for the solution of equations (1.1) with kernels (1.2) is the piecewise collocation method on graded grids. By this method the interval of integration is partitioned into suitable small subintervals and the approximate solution is researched in the form of a function which on every subinterval is a polynomial of the same degree. Such a collocation method for equations (1.1) with kernels (1.2) is investigated in [6]. It is shown there how to choose the non-uniform grid so that the method might have the best convergence rate in supremum-norm (see Theorem 2.1 below).

In order to calculate the approximate solution by the piecewise polynomial collocation method it is necessary to solve large linear systems. In the present paper a two-grid iteration scheme is presented for the solution of such systems and fast convergence of this method is shown (see Theorem 3.1 below). Note that similar two-grid iteration methods are considered in [1-5,9,10,12,13].

2. COLLOCATION METHOD

Let $N \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geq 1$. We introduce in the interval $[0, b]$ the following $4N+1$ grid points $\{t_j^{(N)}\}$:

$$\begin{array}{ll} t_j^{(N)} = \left(\frac{j}{N}\right)^r \frac{d}{2}, & j = 0, 1, \ldots, N; \\ t_{N+j}^{(N)} = d - t_{N-j}^{(N)}, & j = 1, \ldots, N-1; \\ t_{2N+j}^{(N)} = d + \left(\frac{j}{N}\right)^r \frac{b-d}{2}, & j = 0, 1, \ldots, N; \\ t_{3N+j}^{(N)} = d + b - t_{3N-j}^{(N)}, & j = 1, \ldots, N. \end{array} \tag{2.1}$$

Here $r \geq 1$ characterizes the degree of the non-uniformity of the grid. If $r = 1$ and $d = b/2$, then the grid points (2.1) are uniformly located in the interval $[0, b]$. If $r > 1$ then the grid points (2.1) are more densely located towards the points 0, d and b.

We determine the collocation points $\{\xi_{j,q}^{(N)}\}$ in the following way. We choose in the interval $[-1, 1]$ m points $\eta_1, \ldots, \eta_m$,

$$-1 \leq \eta_1 < \ldots < \eta_m \leq 1, \tag{2.2}$$

and set

$$\xi_{j,q}^{(N)} = t_{j-1}^{(N)} + \frac{\eta_q + 1}{2}(t_j^{(N)} - t_{j-1}^{(N)}), \quad q = 1, \ldots, m; \; j = 1, \ldots, 4N. \tag{2.3}$$

Note that $\xi_{j,m}^{(N)} = \xi_{j+1,1}^{(N)} = t_j^{(N)}$, if $\eta_1 = -1$, $\eta_m = 1$ $(j = 1, \ldots, 4N-1)$.

For a continuous function $u: [0, b] \to \mathbb{R}$ we construct a piecewise polynomial interpolation function $P_N u: [0, b] \to \mathbb{R}$ as follows: on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ $(j = 1, \ldots, 4N)$ $P_N u$ is a polynomial of degree not exceeding $m-1$, and

$$(P_N u)(\xi_{j,q}^{(N)}) = u(\xi_{j,q}^{(N)}), \quad q = 1, \ldots, m; \; j = 1, \ldots, 4N. \tag{2.4}$$

Thus, the interpolation function $(P_N u)(t)$ is uniquely defined in every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ $(j = 1, \ldots, 4N)$ separately and may have jumps at $t = t_j^{(N)}$,

$j = 1, \ldots, 4N-1$. If $\eta_1 = -1$, $\eta_m = 1$, then $P_N u$ is a continuous function on the interval $[0, b]$. We can define $(P_N u)(t)$ by the formula

$$(P_N u)(t) = \sum_{q=1}^{m} u(\xi_{j,q}^{(N)}) \varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}],\ j = 1, \ldots, 4N, \tag{2.5}$$

where $\varphi_{j,q}^{(N)}(t)$, $t \in [t_{j-1}^{(N)}, t_j^{(N)}]$, $q = 1, \ldots, m$; $j = 1, \ldots, 4N$, are the polynomials of degree $m-1$ such that

$$\varphi_{j,q}^{(N)}(\xi_{j,p}^{(N)}) = \left\{ \begin{array}{ll} 1, & p = q \\ 0, & p \neq q \end{array} \right\}, \quad p, q = 1, \ldots, m. \tag{2.6}$$

Let us denote by E_N the range of the operator $P_N \equiv P_N^{(m)}$, $P_N : C[0, b] \to E_N$; thus, E_N is the space of piecewise polynomial functions u_N which on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1, \ldots, 4N$) are polynomials of the degree not exceeding $m-1$.

We look for an approximate solution $u_N \in E_N$ to integral equation (1.1). We require that u_N should satisfy the equation (1.1) at the collocation points (2.3):

$$\left[u_N(t) - \int_0^b K(t, s) u_N(s) ds - f(t) \right]_{t = \xi_{i,p}^{(N)}} = 0, \qquad p = 1, \ldots, m; \quad i = 1, \ldots, 4N. \tag{2.7}$$

By the representation (2.5), we can find $u_N \in E_N$ in the form

$$u_N(t) = \sum_{q=1}^{m} c_{j,q}^{(N)} \varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}],\ j = 1. \ldots, 4N,$$

where, as it follows from (2.6),

$$c_{j,q}^{(N)} = u_N(\xi_{j,q}^{(N)}), \quad q = 1, \ldots, m;\ j = 1, \ldots, 4N.$$

Now the collocation conditions (2.7) will take the form of a system which determines the coefficients $c_{i,p}^{(N)} = u_N(\xi_{i,p}^{(N)})$:

$$c_{i,p}^{(N)} = \sum_{j=1}^{4N} \sum_{q=1}^{m} a_{i,p,j,q}^{(N)} c_{j,q}^{(N)} + f(\xi_{i,p}^{(N)}), \quad p = 1, \ldots, m;\ i = 1, \ldots, 4N, \tag{2.8}$$

where

$$a_{i,p,j,q}^{(N)} = \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} K(\xi_{i,p}^{(N)}, s) \varphi_{j,q}^{(N)}(s) ds, \qquad q = 1, \ldots, m;\ j = 1, \ldots, 4N;\ p = 1, \ldots, m;\ i = 1, \ldots, 4N. \tag{2.9}$$

If $\eta_1 > -1$ or $\eta_m < 1$, then all collocation points $\xi_{j,q}^{(N)}$ ($q = 1, \ldots, m$; $j = 1, \ldots 4N$) are different and there are $4mN$ collocation points. In this case the system (2.8) (system (2.7)) has $4mN = \dim E_N$ equations and the same number of unknowns. If $\eta_1 = -1$, $\eta_m = 1$, then part of the collocation points will coincide. The number of different collocation points is $[4N(m-1)+1] = \dim E_N$ and the system (2.8) (system (2.7)) has the same number of equations and unknowns.

On basis of Lemma 1.1 in [6] the following convergence result is proved.

Theorem 2.1. *Assume that the following conditions are fulfilled:*

1) the kernel (1.2) satisfies the assumptions (A1) and (A2);

2) $f \in E^{\alpha,m}$;

3) the homogeneous integral equation

$$u(t) = \int_0^b K(t,s)u(s)ds \tag{2.10}$$

has only the trivial solution $u = 0$;

4) the collocation points (2.3) with interpolation points (2.2) and grid points (2.1) are used.

Then the equation (1.1) has a unique solution u^* and there exists an integer $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the collocation conditions (2.7) define a unique approximation $u_N^* \in E_N$ to u^*. The following error estimates hold:

$$\sup_{0 \leq t \leq b} |u_N^*(t) - u^*(t)| \leq c \begin{cases} h_N^{r(1-\alpha)} & \text{for } 1 \leq r \leq \frac{m}{1-\alpha} \\ h_N^m & \text{for } \quad r \geq \frac{m}{1-\alpha} \end{cases}, \tag{2.11}$$

where r is the scaling parameter of the grid (2.1), c is a positive constant independent of N (but dependent on r), and

$$h_N = \max\left\{\frac{d}{2N}, \frac{b-d}{2N}\right\} \tag{2.12}$$

Remark 2.1. It is shown in [6] that at special collocation points a more rapid convergence (superconvergence) takes place.

To apply the collocation method it is necessary to solve the linear system (2.8). We write this system in the form

$$\overline{u}_N = T_N \overline{u}_N + \overline{f}_N, \tag{2.13}$$

where

$$\overline{u}_N = \left(c_{1,1}^{(N)}, \dots, c_{1,m}^{(N)}, c_{2,1}^{(N)}, \dots, c_{2,m}^{(N)}, \dots, c_{4N,1}^{(N)}, \dots, c_{4N,m}^{(N)}\right)^T,$$

$$\overline{f}_N = \left(f(\xi_{1,1}^{(N)}), \dots, f(\xi_{1,m}^{(N)}), f(\xi_{2,1}^{(N)}), \dots, f(\xi_{2,m}^{(N)}), \dots, f(\xi_{4N,1}^{(N)}), \dots, f(\xi_{4N,m}^{(N)})\right)^T$$

are vectors and

$$T_N = (a_{i,p,j,q}^{(N)}) \tag{2.14}$$

is a matrix with elements (2.9) in the following form ($a_{i,p,j,q} = a_{i,p,j,q}^{(N)}$; $n = 4N$):

$$\begin{pmatrix}
a_{1,1,1,1} & \dots & a_{1,1,1,m} & a_{1,1,2,1} & \dots & a_{1,1,2,m} & \dots & a_{1,1,n,1} & \dots & a_{1,1,n,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{1,m,1,1} & \dots & a_{1,m,1,m} & a_{1,m,2,1} & \dots & a_{1,m,2,m} & \dots & a_{1,m,n,1} & \dots & a_{1,m,n,m} \\
a_{2,1,1,1} & \dots & a_{2,1,1,m} & a_{2,1,2,1} & \dots & a_{2,1,2,m} & \dots & a_{2,1,n,1} & \dots & a_{2,1,n,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{2,m,1,1} & \dots & a_{2,m,1,m} & a_{2,m,2,1} & \dots & a_{2,m,2,m} & \dots & a_{2,m,n,1} & \dots & a_{2,m,n,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{n,1,1,1} & \dots & a_{n,1,1,m} & a_{n,1,2,1} & \dots & a_{n,1,2,m} & \dots & a_{n,1,n,1} & \dots & a_{n,1,n,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{n,m,1,1} & \dots & a_{n,m,1,m} & a_{n,m,2,1} & \dots & a_{n,m,2,m} & \dots & a_{n,m,n,1} & \dots & a_{n,m,n,m}
\end{pmatrix}.$$

Usually the number of equations in (2.13) is large and, as a result of this, direct solving of (2.13) is rather complicated. An effective method for solving this system is a two-grid iteration method.

3. TWO-GRID METHOD

In addition to the original grid corresponding to $N \in \mathbb{N}$ (see (2.1)), we define by (2.1) the coarse grid, corresponding to an integer $M \in \mathbb{N}$, $M < N$. More precisely, we choose N and M so that N/M is an integer greater than 1. Then every subinterval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1, \dots, 4N$) is fully contained in some subinterval $[t_{i-1}^{(M)}, t_i^{(M)}]$ ($i = 1, \dots, 4M$) of the coarse grid.

For solving the system (2.13) the following two-grid iteration method is used:

$$\begin{cases} \overline{v}_N^l = \overline{u}_N^l - T_N\overline{u}_N^l - \overline{f}_N, \\ \overline{w}_M^l = (I_M - T_M)^{-1}R_{N,M}T_N\overline{v}_N^l, \\ \overline{u}_N^{l+1} = \overline{u}_N^l - \overline{v}_N^l - Q_{M,N}\overline{w}_M^l, \quad l = 0, 1, \dots, \end{cases} \tag{3.1}$$

where $\overline{u}_N^0$ is the initial guess of $\overline{u}_N$, I_M is the identity matrix, $R_{N,M}\colon \mathbb{R}^{d_N} \to \mathbb{R}^{d_M}$

($d_N = \dim E_N,\ d_M = \dim E_M$) is the restriction operator defined by

$$(R_{N,M}T_N\overline{v}^l_N)(\xi^{(M)}_{i,p}) = \sum_{j=1}^{4N}\sum_{q=1}^{m}\left(\int\limits_{t^{(N)}_{j-1}}^{t^{(N)}_j} K(\xi^{(M)}_{i,p}, s)\varphi^{(N)}_{j,q}(s)ds\right) v^l_N(\xi^{(N)}_{j,q}), \qquad (3.2)$$
$$p = 1,\dots,m; \quad i = 1,\dots,4M,$$

and $Q_{M,N}\colon \mathbb{R}^{d_M} \to \mathbb{R}^{d_N}$ is the prolongation operator defined by

$$(Q_{M,N}\overline{w}^l_M)(\xi^{(N)}_{j,q}) = \sum_{p=1}^{m} w^l_m(\xi^{(M)}_{i,p})\varphi^{(M)}_{i,p}(\xi^{(N)}_{j,q}) \text{ if } \xi^{(N)}_{j,q} \in [t^{(M)}_{i-1}, t^{(M)}_i] \qquad (3.3)$$
$$i = 1,\dots,4M, \quad q = 1,\dots,m; \quad j = 1,\dots 4N.$$

Here $v^l_N(\xi^{(N)}_{j,q})$, $(R_{N,M}T_N\overline{v}^l_N)(\xi^{(M)}_{i,p})$, $w^l_M(\xi^{(M)}_{i,p})$ and $(Q_{M,N}\overline{w}^l_M)(\xi^{(N)}_{j,q})$ are the corresponding components of the vectors $\overline{v}^l_N$, $R_{N,M}T_N\overline{v}^l_N$, $\overline{w}^l_M$ and $Q_{M,N}\overline{w}_M$, respectively. The ordering of the components of these vectors is the same as that one for the vectors $\overline{u}_N$ and $\overline{f}_N$ above (see (2.13)).

Similar two-grid iteration methods are considered in [1-5,9,10,12,13]. Our treatment will follow the approach of [10].

We write the integral equation in the form

$$u = Tu + f \qquad (3.4)$$

where

$$(Tu)(t) = \int\limits_0^b K(t,s)u(s)ds. \qquad (3.5)$$

The collocation conditions (2.7) are equivalent to the operator equation

$$u_N = P_N T u_N + P_N f, \qquad (3.6)$$

with operators $P_N\colon C[0,b] \to E_N$, introduced in Section 2. Using (A1) and (A2) we can establish the following result.

Lemma 3.1. *Let the conditions (A1) and (A2) about the kernel (1.2) hold, and let the collocation points (2.3) with grid points (2.1) be used. Then for every choice of collocation parameters (2.2),*

$$||T - P_N T||_{L^\infty(0,b)\to L^\infty(0,b)} \le c h_N^{1-\alpha}, \qquad (3.7)$$

where h_N is defined in (2.12) and the constant c is independent of N.

In order to give to the method (3.1) a more convenient form for convergence analysis, we introduce the operators $R_{\infty,N}\colon E_N \to \mathbb{R}^{d_N}$ and $Q_{N,\infty}\colon \mathbb{R}^{d_N} \to E_N$ by the following equalities:

$$R_{\infty,N}u = (u(\xi^{(N)}_{1,1}),\dots,u(\xi^{(N)}_{1,m}),u(\xi^{(N)}_{2,1}),\dots,u(\xi^{(N)}_{2,m}),\dots,u(\xi^{(N)}_{4N,1}),\dots,u(\xi^{(N)}_{4N,m})) \qquad (3.8)$$

for $u \in E_N$, and

$$(Q_{N,\infty}\overline{u}_N)(t) = \sum_{q=1}^{m} u_N(\xi_{j,q}^{(N)})\varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}],\ j = 1, \dots, 4N,$$

for

$$\overline{u}_N = (u_N(\xi_{1,1}^{(N)}), \dots, u_N(\xi_{1,m}^{(N)}), \dots, u_N(\xi_{4N,1}^{(N)}), \dots, u_N(\xi_{4N,m}^{(N)})) \in \mathbb{R}^{d_N}.$$

Actually, we shall use the definition (3.8) for applying $R_{\infty,N}$ to all functions $u(t)$ which are defined at $t = \xi_{j,q}^{(N)}$, $q = 1, \dots, m$; $j = 1, \dots, 4N$. For later use, introduce

$$u_N^l = Q_{N,\infty}\overline{u}_N^l, v_N^l = Q_{N,\infty}\overline{v}_N^l, w_M^l = Q_{M,\infty}\overline{w}_M^l,$$

where $l = 0, 1, \dots$, and $\overline{w}_N^l$ and $\overline{u}_N^{l+1}$ are determined by (3.1) for a initial guess $\overline{u}_N^0 \in \mathbb{R}^{d_N}$. Then

$$\overline{u}_N^l = R_{\infty,N}u_N^l, \quad \overline{v}_N^l = R_{\infty,N}v_N^l, \quad \overline{w}_M^l = R_{\infty,M}w_M^l,$$

$$\begin{array}{llll} R_{\infty,N}Q_{N,\infty} & = I_N, & Q_{N,\infty}R_{\infty,N} = P_N, \\ R_{\infty,N}TQ_{N,\infty} & = T_N, & Q_{N,\infty}Q_{M,N} = Q_{M,\infty}. \end{array}$$

Using these notations and identies we can rewrite formulas (3.1) as follows:

$$\left\{ \begin{array}{l} v_N^l = u_N^l - P_NTu_N^l - P_Nf, \\ w_M^l = P_MTw_M^l + P_MTv_N^l, \\ u_N^{l+1} = u_N^l - v_N^l - w_M^l, \quad l = 0, 1, \dots \ . \end{array} \right. \tag{3.9}$$

Whereas $u_N^0 = Q_{N,\infty}\overline{u}_N^0 \in E_N$ we also have $v_N^l \in E_N$, $w_M^l \in E_M \subset E_N$ and $u_N^{l+1} \in E_N$, $l = 0, 1, \dots$. Therefore the methods (3.1) and (3.9) are equivalent. At the same time the method (3.9) is an iteration method to solve (3.6).

We are now ready to prove the following result about the convergence of the two-grid method (3.1).

Theorem 3.1. *Let the assumptions of Theorem 1.1 hold. Then there exists an integer $M_0 > 0$ such that, for $N \geq M_0$, the system (2.13) has a unique solution $\overline{u}_{N,0}$. The two-grid iteration method (3.1) convergences to this solution for $M \geq M_0$, $N/M = 2, 3, \dots$, and for every choice of the initial guess $\overline{u}_N^0$ to:*

$$||\overline{u}_N^{l+1} - \overline{u}_{N,0}||_\infty \leq \text{const}\ h_M^{1-\alpha}||\overline{u}_N^l - \overline{u}_{N,0}||_\infty, \quad l = 0, 1, \dots, \tag{3.10}$$

where $h_M = \max\{d/2M, (b-d)/2M\}$ and

$$||\overline{u}_N||_\infty = \max_{j=1,\dots,4N; q=1,\dots,m} \left|u_N(\xi_{j,q}^{(N)})\right|. \tag{3.11}$$

Proof. It follows from (A1) and (A2) that the integral operator (3.5) with the kernel (1.2) is a compact operator on $L^\infty(0,b)$ to $L^\infty(0,b)$ [7]. As the homogeneous equation $u = Tu$ has only the trivial solution $u = 0$, the operator $I - T: L^\infty(0,b) \to L^\infty(0,b)$, with the identity operator I, has a bounded inverse $(I-T)^{-1}: L^\infty(0,b) \to L^\infty(0,b)$. By Lemma 3.1, $||T - P_M T||_{L^\infty(0,b)\to L^\infty(0,b)} \to 0$, if $h_M \to 0$. Therefore there occurs $M_0 > 0$ such that for all $M \geq M_0$, $(I - P_M T)^{-1}: L^\infty(0,b) \to L^\infty(0,b)$ exist and their norms are uniformly bounded:

$$||(I - P_M T)^{-1}||_{L^\infty(0,b)\to L^\infty(0,b)} \leq \text{const} \quad (M \geq M_0). \tag{3.12}$$

It follows from this that (2.13) has a unique solution $u_{N,0} \in E_N$ for every $N \geq M_0$. Consequently, for $M \geq M_0$, $N/M = 2, 3, \ldots$, and for a initial guess $u_N^0 \in E_N$ to $u_{N,0}$, the formulas (3.9) define a sequence of elements $u_N^l \in E_N$, $l = 1, 2, \ldots$, and

$$u_N^{l+1} - u_{N,0} = (I - P_M T)^{-1}(P_N - P_M)T(u_N^l - u_{N,0}), \quad l = 0, 1, \ldots. \tag{3.13}$$

Indeed, we have

$$\begin{aligned}
&(I - P_M T)(u_N^{l+1} - u_{N,0}) = (I - P_M T)[u_N^l - v_N^l - w_M^l - u_{N,0}] = \\
&= (I - P_M T)[P_N T u_N^l + P_N f - (I - P_M T)^{-1} P_M T v_N^l - u_{N,0}] = \\
&= P_N T u_N^l + P_N f - P_M T P_N T u_N^l - P_M T P_N f - P_M T v_N^l - u_{N,0} + P_M T u_{N,0} = \\
&= P_N T u_N^l + u_{N,0} - P_N T u_{N,0} - P_M T P_N T u_N^l - P_M T P_N f - P_M T u_N^l + \\
&+ P_M T P_N T u_N^l + P_M T P_N f - u_{N,0} + P_M T u_{N,0} = \\
&= P_N T(u_N^l - u_{N,0}) - P_M T(u_N^l - u_{N,0}) = \\
&= (P_N - P_M)T(u_N^l - u_{N,0}).
\end{aligned}$$

Applying $(I - P_M T)^{-1}$ to the identity $(I - P_M T)(u_N^{l+1} - u_{N,0}) = (P_N - P_M)(T(u_N^l - u_{N,0})$, we obtain (3.13). Further, using Lemma 3.1,

$$\begin{aligned}
&||(P_N - P_M)T||_{L^\infty(0,b)\to L^\infty(0,b)} \leq ||T - P_N T||_{L^\infty(0,b)\to L^\infty(0,b)} + \\
&+ ||T - P_M T||_{L^\infty(0,b)\to L^\infty(0,b)} \leq c(h_N^{1-\alpha} + h_M^{1-\alpha}) \leq c' h_M^{1-\alpha},
\end{aligned}$$

with suitable constants c and c'. Combining this and (3.13) and (3.12) for $M \geq M_0$, $N/M = 2, 3, \ldots$, we have

$$||u_N^{l+1} - u_{N,0}||_{L^\infty(0,b)} \leq c'' h_M^{1-\alpha} ||u_N^l - u_{N,0}||_{L^\infty(0,b)}, \quad l = 0, 1, \ldots, \tag{3.14}$$

with a constant c'' which is independent of M, N and l. The estimate (3.10) now follows from (3.14) because

$$||\overline{u}_N^{l+1} - \overline{u}_{N,0}||_\infty = ||R_{\infty,N}(u_N^{l+1} - u_{N,0})||_\infty \leq ||u_N^{l+1} - u_{N,0}||_\infty$$

and

$$||u_N^l - u_{N,0}||_\infty = ||P_{N,\infty}(\overline{u}_N^l - u_{N,0})||_\infty \leq c''' ||\overline{u}_N^l - \overline{u}_{N,0}||_\infty,$$

with a constant c''' which is independent of N. ■

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SILPNAI SINGULIARIŲ INTEGRALINIŲ LYGČIŲ SPRENDIMAS DVITINKLIU ITERACINIU METODU

K. Hakk, A. Pedas

Sprendžiant silpnai singuliarias integralines lygtis gabalais polinominių kolokacijų metodu tenka spręsti didelės dimensijos tiesinių lygčių sistemas. Šiame darbe tokios sistemos sprendžiamos dvitinkliu iteraciniu metodu. Ištirtas iteracinio metodo konvergavimas, gauti konvergavimo greičio įverčiai.