International Journal of Technology (2017) 4: 622-633
ISSN 2086-9614

A STOCHASTIC METHOD BASED ON THE MARKOV MODEL OF UNIT JUMP FOR ANALYZING CRACK JUMP IN A MATERIAL

Karima Selmani Bouayoune[1*], El Mostapha Boudi[1], Aziz Bachir[1]

[1]*Department of Mechanical Engineering, Ecole Mohammadia d'Ingénieurs, University Mohamed V, Avenue Ibn Sina B.P 765, Agdal Rabat, Morocco*

(Received: August 2016 / Revised: June 2017 / Accepted: July 2017)

ABSTRACT

In considering Composite Material Systems, the Markov Model is important for studying the behavior of composite materials. The monitoring of crack growth is suggested as the basis for this study. In fact, crack growth strongly impacts Composite Material Systems. Crack growth may lead to system failure, especially if we cannot prevent the various kinds of risk states and if we do not take necessary actions to maintain this system while in operation. In order to analyze risk states for steel materials, in the Moroccan National Railway Office, the Markov Model of a unit jump is chosen to analyze the crack growth of a composite material. This model is defined by a transition vector and a state vector, with a calculation of the averages and the extensions of the crack. Using these parameters, the jump of each extension of the crack and the number of the crack extensions are considered. A mathematical calculation helps us to find the formula for the transition probability, based on the average. An algorithm allows us to estimate the value of the crack jump. These estimations indicate the level of risk for each system state and values of the crack extension. The obtained results show that more the unit jump approximates to zero, the more the system is maintained in an acceptable operation, despite any disruptions that may influence the results.

Keywords: Crack growth; Maintain in operation; Markov Model of a Unit Jump; Risk

1. INTRODUCTION

In Composite Material Systems, the crack growth is seriously being studied, because it is essential to the security, safety, and operations of these systems. In order to predict the crack growth of materials under certain operational conditions, many stochastic models have been used. Cadini et al. (2009), Corbetta et al. (2014), Mohanty et al. (2009), Nechval et al. (2007), Sankararaman et al. (2011), Xiang et al. (2011), and Zapatero et al. (1990) have studied fatigue-sensitive aircraft structures in order to reduce the risk of failure and to determine the minimum level of reliability. In fact, the prediction of stochastic crack growth accumulation is important for reliability analysis (Yang et al., 1996). Then, when we prevent the probability of the failure, we can estimate reliability and finally decide the level in which we must act, to maintain the continuity of our system operations. However, for studying crack growth in a dynamic system, the Markov Process is the best model. Many Markov Models are applied to predict crack propagation and to study the behavior of a system. Bo-Siou et al. (2009) have proposed a model of Markov chains for predicting the evolution of damage, with a method for constructing a stochastic curve for a number of Composite Material Systems. Beil et al. (2009), Brandejsky et

*Corresponding author's email: karimaselmani@yahoo.fr, Tel. +212-537-687150, Fax. +212-537-778853
Permalink/DOI: https://doi.org/10.14716/ijtech.v8i4.5253

al. (2013), Chiquet et al. (2009), Davis (1984), Davis (1993), Dhondt (1995), Rowatt et al. (1998), de Saporta et al. (2010), de Saporta et al. (2012), Zhang et al. (2014) have presented a Markov Model based on the evolution of damage to characterize fatigue behavior. Based on these behavioral studies, we investigated modeling the crack using a Markov Model with a method of simple analysis. From this perspective, we found that the Markov Model used by Bogdanoff et al. (1985) contains as few parameters as possible even when including all major variables. This model has been used for analyzing the damage to a component under conditions of severity. Bohn (1990) has used the best non-homogeneous Markov model that gives a very good description of the actual behavior of a pitting corrosion system. Also, Xi et al. (1997) found the best methodology, because they used a simple function (a linear function) to calculate the maximum load using a Markov Model of R-curve behavior. Thus, we seek a Markov model that gives us the best possible simulation of crack evolution to enable us to predict risk states, which can be applied to our research.

Original works of Meier-Hirmer et al. (2006) show that it is possible, using a Markov process of unit jump, to model the wear of centenary pieces for the National Railway Society in France. Hence, we chose this model to apply to our study of steel materials, which are commonly used for train wheels and rails in a railway system. We chose the Markov Model of a unit jump that was applied by Roh et al. (2000). This model is advantageous because it gives a general formulation for the evolution of the fracture in a system. Roh et al. (2000) used the Markov Model of a unit jump to calculate transition probabilities with a formulation that is characterized by a jump δ. Their work focused on the probability itself more than the unit jump. In their application, Roh et al. (2000) used a constant value for δ. However, we noticed that this unit jump was not constant and it may be the factor that determines whether the system is in a disruptive state or not. In that regard, we will focus on this unit jump and prove its importance.

This article is composed of three sections. In the first section, we present our model, which is defined by a transition probability from one state to another, and which is characterized by an averaging and an extension of the fracture $c = i\delta$, where δ is the unit jump of each extension in the fracture and i is the number of fracture extensions. Our aim is to calculate this unit jump, which is indicated as δ. In the second section, we will use the formulation of δ to estimate the probability of crack growth in a material. We used a numerical application for the steel material in the third section, in order to conclude at what level the system may be in a risk state and in such an instance, what necessary actions should be made to maintain the system in operation, despite this risk.

2. METHODOLOGY OF ESTIMATING CRACK GROWTH

Our methodology aims to calculate crack growth in a material, before making simulations of transition probabilities. It is established in order to estimate the level of risk. It consists of two steps, which are presented in Figure 1.

Step 1. Our methodology begins by defining the Markov Model of a unit jump, with the Transition Probability Matrix P, which is defined by Equation 1 below. From this Matrix, (Roh & Xi, 2000) have demonstrated the Transition Probabilities values $(p_k, k \geq 1)$, which are presented in the Equation 6 below. Consequently, these probabilities are functions of the mean values $(\mu_k, k \geq 1)$ and the jump δ.

Step 2. The crack growth is then calculated by using the formulation of these transition probabilities in Equation 6. Next, the values of a unit jump δ are demonstrated by recurrence, and the results are obtained in Equation 13 as shown below.

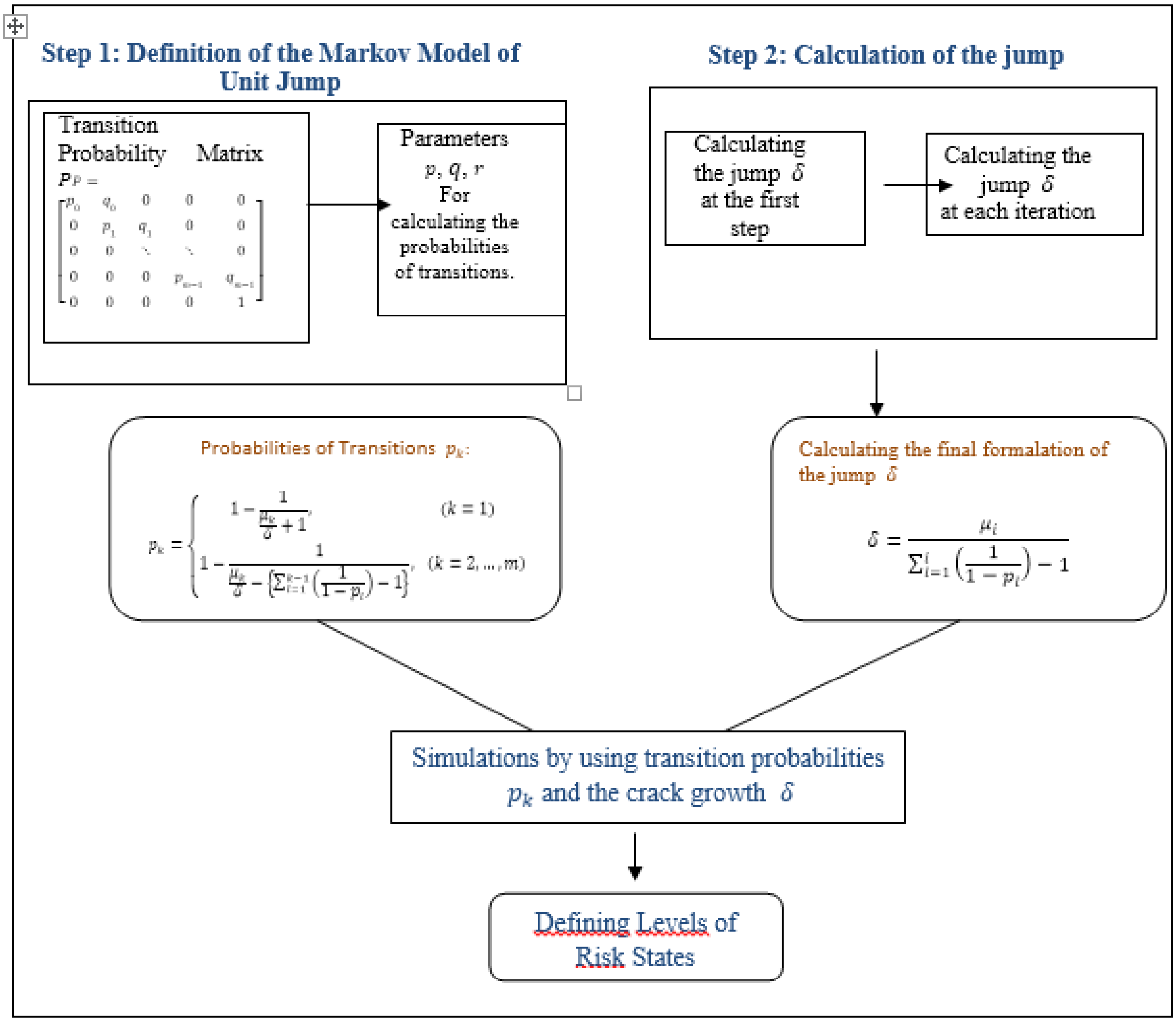

Figure 1 Diagram summarizing our methodology

2.1. Presentation of the Model

The Markov Model of the unit jump is defined by a transition vector and a state vector. The value Γ_i^j is defined, where j is the current state of the system and i is its current transition. To predict the life expectancy of a material, the value Γ_i^j is used to determine the state of the crack, and its evolution.

Our model is defined by the transition probability matrix P in Equation 1, where:

$$P = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{m-1} & q_{m-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

And, for $k = 0, \ldots, m$, p_k is the probability that the crack remains in the state k and q_k is the probability that it will change from state k to state $k + 1$.

The initial transition vector V_0, associated with this Markov model is considered. Generally, it has the value $V_0 = [1, 0, \ldots, 0]^T$, when the transition is $i = 0$. (And, $V_0 = [0, 1, \ldots, 0]^T$ when it is 100% sure that the system maintains the state j = 1 at the initial stage).
Thus, the transition vector at Step 1 is shown in Equation 2, as follows:

$$V_1 = PV_0 \tag{2}$$

The distribution at the Step t is shown in Equation 3:

$$V_t = P^t V_0 \tag{3}$$

Therefore, if it is considered that it is 100% sure that the state $j = m - 1$ is maintained at the initial stage, by the system. The likelihood for maintaining the same transition $n = 1,2,3,\ldots$ is shown in Equation 4:

$$\varphi(n) = p^{n-1}q \tag{4}$$

(See Appendix below). It is also called *the trial of the first failure.*
This is a geometric distribution, hence p and q can be obtained by solving the following system algorithm as shown in Equation 5:

$$\begin{cases} r = \dfrac{VAR}{EX} = \dfrac{p}{q} \\ p + q = 1 \end{cases} \tag{5}$$

Roh and Xi, (2000) have used this model to demonstrate the equation for the transition probability, below as shown in Equation 6:

$$p_k = \begin{cases} 1 - \dfrac{1}{\frac{\mu_k}{\delta} + 1}, & (k = 1) \\ 1 - \dfrac{1}{\frac{\mu_k}{\delta} - \left\{\sum_{i=1}^{k-1}\left(\frac{1}{1 - p_i}\right) - 1\right\}}, & (k = 2, \ldots, m) \end{cases} \tag{6}$$

If the transition variable is selected as the crack extension, then μ_j will be the mean value of the extension under the loading level j.

Then, the unit jump δ of the crack length from one state to another will be calculated.

2.2. Calculation of the Unit Jump to Define the State of Risk

Supposing that the length of the crack begins at zero and that μ_1 is the average value of the transition variable $t\delta$. By using the formulation of p_k in Equation 6, δ becomes the result in Equation 7:

$$\delta = \left(\frac{\sum_{i=0}^{\infty} p_1^i}{\sum_{i=0}^{\infty} i p_1^i}\right)\mu_1 \tag{7}$$

By recurrence, the calculation in Equation 8 proves that:

For $t \geq 2$:

$$\delta = \frac{\mu_t}{\sum_{i=1}^{t}\left(\frac{1}{1 - p_i}\right) - 1} \tag{8}$$

Thus, it is relevant to calculate at each stage the value shown in Equation 9:

$$\Delta_t = \sum_{i=1}^{t} \frac{1}{(1 - p_i)} \tag{9}$$

Consequently, using this last formulation, δ is calculated by the following algorithms shown in Equations 10, 11, 12, and 13 as follows:

Initialize: $i = 1$, μ_1, Δ_1, p_1 *:*

Iteration 1:

Calculate δ *by using the following equation:*

$$\delta = (1 - p_1)\mu_1 \tag{10}$$

For $i \geq 2$ *:*

$$\Delta_i = \sum_{i=1}^{i} \frac{1}{(1 - p_i)} \tag{11}$$

And,

$$\mu_i = \left(\frac{\Delta_{i-1} - 1}{\Delta_i - 1}\right)\mu_{i-1} \tag{12}$$

Finally,

$$\delta = \frac{\mu_i}{\sum_{i=1}^{i}\left(\frac{1}{1 - p_i}\right) - 1} \tag{13}$$

3. APPLICATION ON THE CRACK OF A MATERIAL

At each state, the crack in a material is considered as an inverted triangle (Figure 2). The average μ can be calculated by estimating the center of gravity for this triangle. So, parameters of this triangle change from state-to-state. The unit jump is the distance between the center of gravity of a triangle at one state and that of the next state.

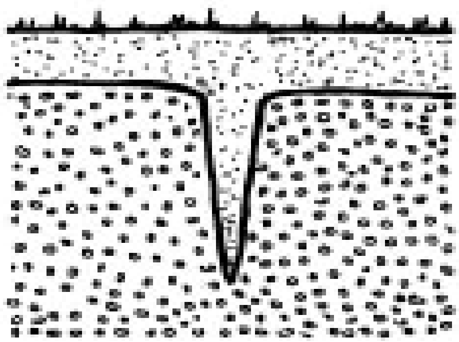

Figure 2 Crack in a material

By applying the previous algorithm on Matlab, the value of δ will be found. In the unit jump model of the Markov chain, the probability mass function (pmf) of the variable at each stage may be assumed to be a geometric distribution. So, the random variables that follow this probability distribution with parameter ρ are generated.

By taking the example of steel materials for the National Railway Office in Morocco, based on the collected data, the parameter ρ will be estimated. As a result, $\rho = 0.3$ and by applying the previous algorithm, the value of δ will be: $\delta = 0.9231$.

Figure 3 represents the graphic of the simulations of crack growth, for detecting the state of risk (the state when its maximum probability exceeds 0.5).

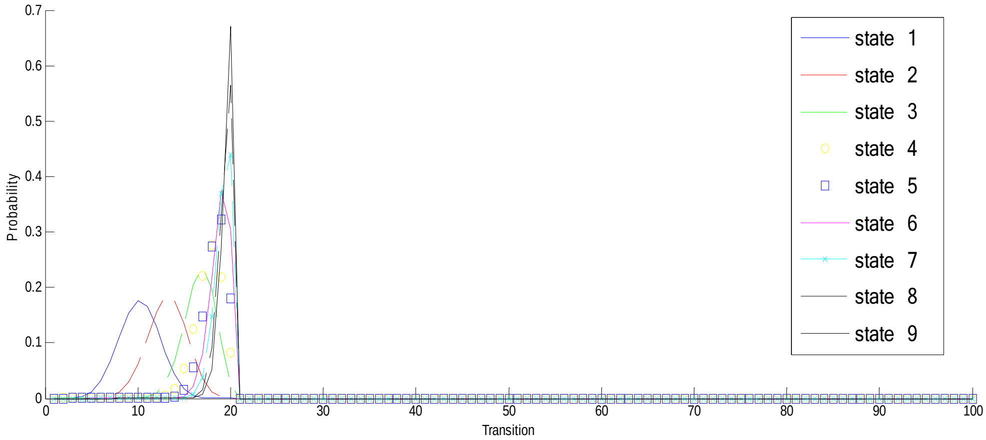

Figure 3 States of crack growth, when $\delta = 0.9231$

Of course, by changing the value of δ, other variations and other states of risk will be displayed. Figure 4 shows variations of crack growth states, when $\rho = 0.35$, so $\delta = 0.5352$.
It should be noted that these transitions and states are chosen because it seems that they are enough to show the low and the high levels of risk. So, the best visualization is obtained and it is easy to deduce levels at which it must act. However, if this choice does not give an acceptable visualization, then these numbers could be increased.

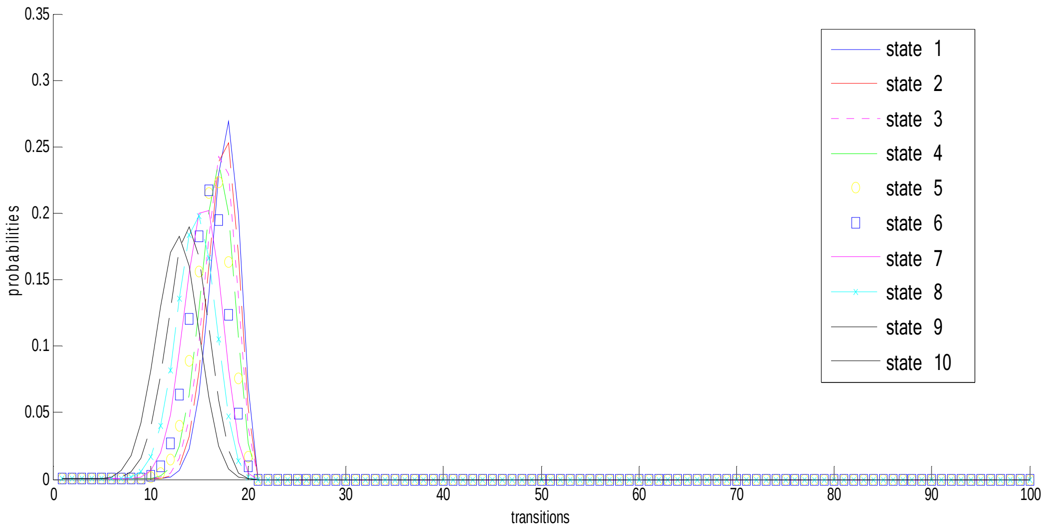

Figure 4 Variations of the crack growth states, when $\delta = 0.5352$

4. INTERPRETATION AND VALIDATION OF RESULTS

4.1. Interpretation of Results

From the Figure 3, the probabilities of crack increase continuously once the number of states increases. And, because the probability exceeds 0.5 in state 8, we can then consider that this state is the step in which the system becomes in a state of risk, so it should make all necessary actions to ensure that the respective system states will be under state 8. In this way, during its period of exploitation, the system in operation is maintained despite disruptions and failures that may influence it.

However, in Figure 4, all the states have low probabilities (the maximum probability is less than 0.3). This can be explained by the presence of good conditions, in which the system operates. Here, the system operation is perfectly maintained, even though there are some disturbances and during the exploitation period of the system only some mild actions are required to remain at this result.

4.2. Validation of Results

In Equation 13, the risk increases when p approximates 1, and then $\delta \to 0$. So, to maintain this probability to a very minimal level, then δ should be moved away from zero to achieve the maximum possible.

To validate this conclusion, our attempt will be to represent graphically δ, and to compare its variations with those of the system states. Thus, Figure 4 is obtained and δ varies during 100 transitions. To compare it with the results obtained, Figure 5 brings together the two graphs obtained previously in Figures 2 and 3, for the two given δ.

To obtain values of δ, random variables X are considered in a manner that follows geometric distribution with a parameter $\rho = 0.3$, and with x as its value. From Equation 13, we can conclude that at every set of state $1, 2, \dots, n$, that $\delta = p_1$. So, during 100 states, if at every time a set of states 1,2,….,n and 2,…,n and 3,…,n and n, n+1,… etc., is considered, then, δ will have respectively p_1 and p_2 and p_3, ….and p_n,…etc.

By comparing Figures 5 and 6, we see that the disruptions in Figure 6 appear mostly between 5 and 20 transitions for the two graphs. And, during this interval [5, 20], δ decreases to its minimum values (Figure 5). Hence, it is simple to conclude that disruptions appear when δ decreases to a minimum value, nearest to zero and disappears when δ increases to a high value of probability.

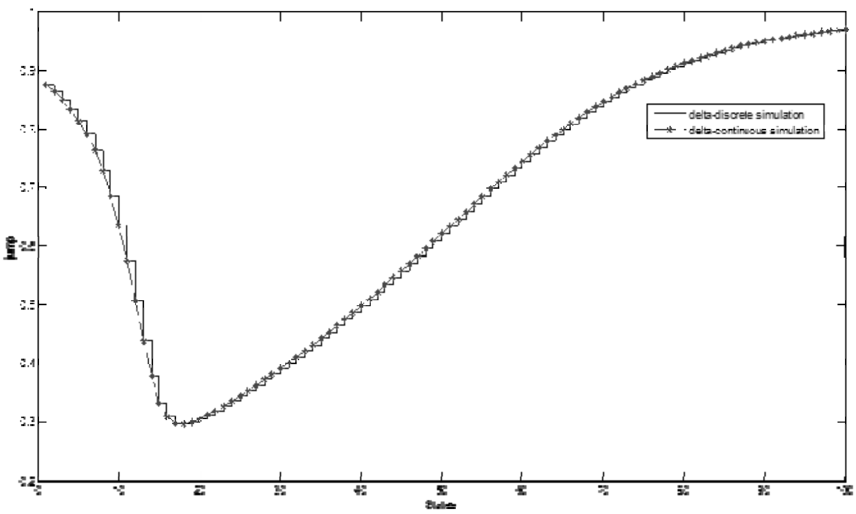

Figure 5 Variations of values of δ, at every interval, during 100 states

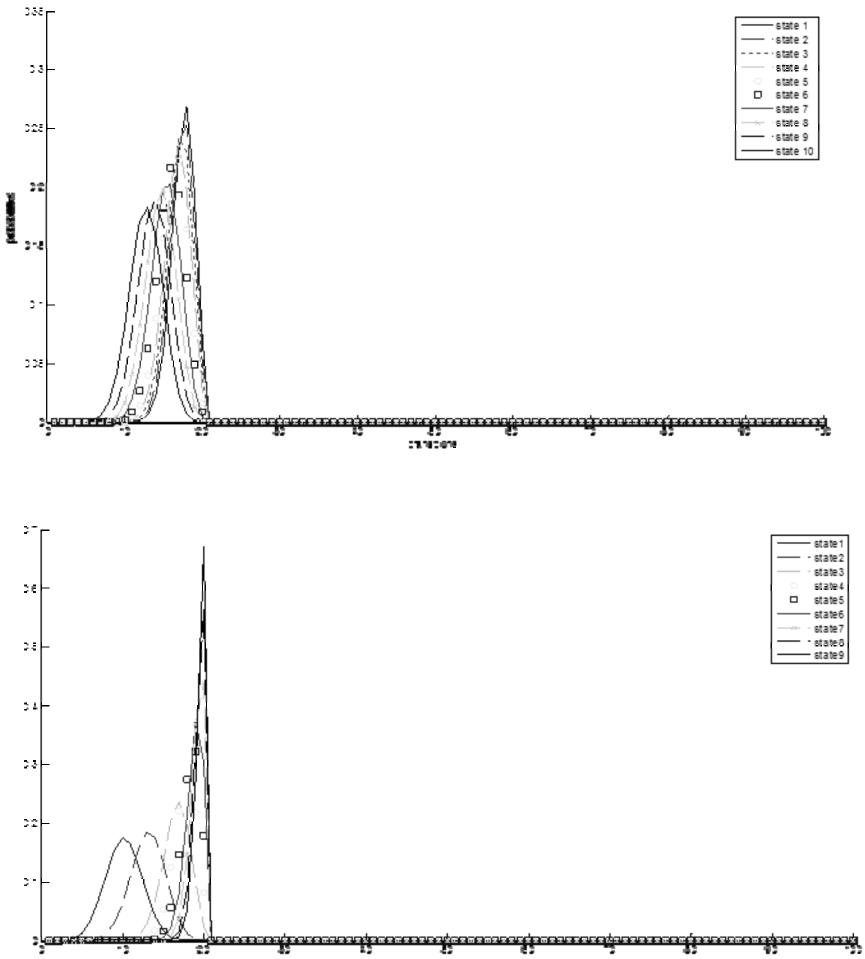

Figure 6 Two examples of variations of states of the system, for 100 transitions

5. CONCLUSION

In this article, our aim is to calculate the unit jump δ, in order to simulate crack variations and to detect states of risk. This is a scientific methodology for maintaining a system operation in security and safety. This calculation gives us an important indicator to analyze risks in a system in operational conditions. By maintaining the unit jump nearest to zero, the security is ensured and the system will operate safely. Our methodology is important for the steps of monitoring and controlling in risk management. But, it is not sufficient to know crack propagation and to predict risk, it must also be known what actions and plans should be established to maintain the system in operation at the same level of safety and security. This is our proposition for future research.

6. REFERENCES

Beil, M., Luck, S., Fleischer, F., Portet, S., Arendt, W., Schmidt, V., 2009. Simulating the Formation of Keratin Filament Networks by a Piecewise-deterministic Markov Process. *J. Theo Biology*, Volume 256(4), pp. 518–532

Bogdanoff, J.L., Kozin, F., 1985. *Probabilistic Models of Cumulative Damage*. Wiley, New York, Volume 1(4), pp. 269–270

Bohn, S.R., 1990. A Markov Model for Fatigue Crack Growth and the Prediction of Component Reliability. *Graduate Thesis*, submitted to the Faculty of Graduate Studies and

Research in partial fulfillment of the requirements for the degree of Master of Engineering. McGill University, Montreal, Canada

Bo-Siou, W., Shane, J., Rami, H.A., 2009. A Stochastic Fatigue Damage Method for Composite Materials based on Markov Chains and Infrared Thermography. *International Journal of Fatigue,* Volume 32(2), pp. 350–360

Brandejsky, A., de Saporta, B., Dufour, F., 2013. Optimal Stopping for Partially Observed Piecewise-deterministic Markov Processes. *Stochastic Processes and their Applications*, Volume 123(8), pp. 3201–3238

Cadini, F., Zio, E., Avram, D., 2009. Monte Carlo-based Filtering for Fatigue Crack Growth Estimation. *Probabilistic Engineering Mechanics*, Volume 24(3), pp. 367–373

Chiquet, J., Limnios, N., Eid, M., 2009. Piecewise-deterministic Markov Processes Applied to Fatigue Crack Growth Modelling. *Journal of Statistical Planning and Inference*, Volume 139(5), pp. 1657–1667

Corbetta, M., Sbarufatti, C., Manes, A., Giglio, M., 2014. Sequential Monte Carlo Sampling for Crack Growth Prediction Providing for Several Uncertainties. *In*: Proceedings of the 2nd European Conference of the Prognostics and Health Management Society

Davis, M.H.A., 1984. Piecewise Deterministic Markov Processes: A General Class of Non-Diffusion Stochastic Models. *Journal of the Royal Statistical Society*, *Series B-Methodological*, Volume 46(3), pp. 353–388

Davis, M.H.A., 1993. *Markov Models and* Optimization. Book Series, Monographs on Statistics and Applied Probability. CRC Press, Inc., London

de Saporta, B., Dufour, F., 2012. Numerical Method for Impulse Control of Piecewise Deterministic Markov Processes. *Automatica,* Volume 48(5), pp. 779–793

de Saporta, B., Dufour, F., Gonzalez, K., 2010. Numerical Method for Optimal Stopping of Piecewise Deterministic Markov Processes. *The Annals of Applied Probability*, Volume 20(5), pp. 1607–1637

Dhondt, G., 1995. Application of the Discrete Markov Method to Crack Propagation Problems. *International Journal of Engineering Science*, Volume 33(4), pp. 457–467

Meier-Hirmer, C., Sourget, F., Roussignol, M., 2006. Estimation de la durée de vie des pièces caténaires. *Journal européen des systèmes automatisés,* Volume 40(7), pp. 703–720

Mohanty, J.R., Verma, B.B., Ray, P.K., 2009. Prediction of Fatigue Crack Growth and Residual Life Using an Exponential Model: Part I (Constant Amplitude Loading). *International Journal of Fatigue,* Volume 31(3), pp. 418 –424

Nechval, K.N., Nicholas A.N., Gundars B., Maris P., 2007. Planning Inspections in Service of Fatigue-sensitive Aircraft Structure Components, under Crack Propagation. *EKSPLOATACJA I NIEZAWODNOŚĆ*, Volume 4(36), pp. 3–8

Roh, Y.S., Xi, Y., 2000. A General Formulation for Transition Probabilities of Markov Model and the Application to Fracture of Composite Materials. *Probabilistic Engineering Mechanics*, Volume 15(3), pp. 241–250

Rowatt, J.D., Spanos, P.D., 1998. Markov Chain Models for Life Prediction of Composite Laminates. *Journal Structural Safety*, Volume 20, pp. 117–135

Sankararaman, S., Ling, Y., Mahadevan, S., 2011. Uncertainty Quantification and Model Validation of Fatigue Crack Growth Prediction. *Engineering Fracture Mechanics*, Volume 78(7), pp. 1487–1504

Xi, Y., Bazant, Z.P., 1997. Random Growth of Crack with R-Curve: Markov Process Model. *Engineering Fracture Mechanics*, Volume 57(6), pp. 593–608

Xiang, Y., Liu, Y., 2011. Application of Inverse First-order Reliability Method for Probabilistic Fatigue Life Prediction. *Probabilistic Engineering Mechanics*, Volume 26(2), pp. 148–156

Yang, J.N., Manning, S.D., 1996. A Simple Second Order Approximation for Stochastic Crack Growth Analysis. *Engineering Fracture Mechanics*, Volume 53, pp. 677–686

Zapatero, J., Domínguez, J., 1990. A Statistical Approach to Fatigue Life Predictions under Random Loading. *International Journal of Fatigue*, Volume 12(2), pp. 107–114

Zhang, H., Innal, F., Dufour, F. Dutuit, Y., 2014. Piecewise Deterministic Markov Processes Based Approach Applied to an Offshore Oil Production System. *Reliability Engineering & System Safety*, Volume 126, pp. 126–134

APPENDIX

Equation 4 is derived from the calculation of the matrix P^n:

Then,

$$P^2 = PP = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{m-1} & q_{m-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{m-1} & q_{m-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

thus,

$$P^2 = \begin{bmatrix} {p_0}^2 & q_0p_0 + q_0p_1 & 0 & 0 & 0 \\ 0 & {p_1}^2 & q_1p_1 + q_1p_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & {p_{m-1}}^2 & p_{m-1}q_{m-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We suppose that

$$P^{n-1} = \begin{bmatrix} {p_0}^{n-1} & & & & \\ 0 & {p_1}^{n-1} & & Q_{n-2} & \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & {p_{m-1}}^{n-1} & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$Q_{n-2} = \begin{pmatrix} {p_0}^{n-2}q_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & {p_{m-1}}^{n-2}q_{m-1} \end{pmatrix}$$

We calculate P^n :

$$P^n = P^{n-1}P = \begin{bmatrix} {p_0}^{n-1} & & & & \\ 0 & {p_1}^{n-1} & & Q_{n-2} & \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & {p_{m-1}}^{n-1} & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{m-1} & q_{m-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We then have:

$$P^n = \begin{bmatrix} {p_0}^{n} & & & & \\ 0 & {p_1}^{n} & & Q_{n-1} & \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & {p_{m-1}}^{n} & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where :

$$Q_{n-1} = \begin{pmatrix} {p_0}^{n-1}q_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & {p_{m-1}}^{n-1}q_{m-1} \end{pmatrix}$$

Consequently, when it is 100% sure that the system maintains the state $j = m - 1$ at the initial stage, the vector $V_0 = [0,0,\ldots,0,1]^T$, so that:

$$\varphi(n) = P^n V_0 = \begin{bmatrix} p_0^{\,n} & & & & \\ 0 & p_1^{\,n} & & Q_{n-1} & \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & p_{m-1}^{\,n} & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\varphi(n) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p_{m-1}^{\,n-1} q_{m-1} \\ 1 \end{bmatrix}$$

In this vector the probability for the first state of failure is: $p_{m-1}^{\,n-1} q_{m-1}$