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Last scattering, relic gravitons, and the circular polarization of the CMB

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The tensor contribution to the V-mode polarization induced by a magnetized plasma at last scattering vanishes exactly when the magnetic field is parallel to the direction of propagation of the gravitational wave. For propagation perpendicular to the magnetic field orientation the effect depends on the ratio between the Larmor frequency of the electrons and the observational frequency. If the source term of the V-mode equation is averaged over the directions of the magnetic field, the circular polarization evolves independently from all the other Stokes parameters.

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In the current version of the ΛCDM paradigm (where Λ stands for the dark energy component and CDM stands for the cold dark matter component), the tensor modes of the geometry are absent [1]. If included, they could also induce a purported tensor component of the B-mode power spectrum, which has not yet been observed. We cannot *a priori* exclude that the cosmic microwave background (CMB in what follows) is also circularly polarized. It would be highly desirable to have more direct upper limits on the circular polarizations of the CMB with suitable low-frequency instruments [2,3] possibly operating in the GHz range or even below. When photons impinge electrons and ions in the presence of a magnetic field, the circular polarization arises naturally [4] leading to a nonvanishing V mode whose magnitude depends upon the angular frequency of the experiment and upon the intensity of the radiation field. In [4] only the scalar mode contribution to the V-mode polarization has been considered, and it is therefore natural to ask, in a complementary perspective, if an additional source of circular polarization can arise from the tensor fluctuations of the geometry. The magnetized electron-photon scattering induces a scalar V-mode polarization as well as a scalar intensity fluctuation [4]; the tensor modes are then expected to affect, in principle, both the intensity and the circular polarization. The aim of the present paper is an explicit calculation of the tensor contribution to the V-mode polarization in the framework of the magnetized ΛCDM paradigm. Consider the case of a conformally flat Friedmann-Robertson-Walker geometry, i.e. $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric with signature $(+, -, -, -)$, and $a(\tau)$ is the scale factor. The tensor fluctuations are defined as $\delta^{(1)}_{(\mathrm{t})} g_{ij} = -a^2 h_{ij}$ where $h_i^i = \partial_i h_j^i = 0$ (Latin indices run over the three-dimensional spatial submanifold). The tensor fluctuation $h_{ij}(\vec{x}, \tau)$ are

$$h_{ij}(\vec{x}, \tau) = \sum_{\lambda} h_{(\lambda)}(\vec{x}, \tau)\epsilon_{ij}^{(\lambda)}(\hat{k}), \qquad \epsilon_{ij}^{(\lambda)}\epsilon_{ij}^{(\lambda')} = 2\delta^{\lambda\lambda'}, \quad (1)$$

where $\lambda = \oplus, \otimes$ denote the two polarizations; moreover, $\epsilon_{ij}^{\oplus}(\hat{k}) = (\hat{a}_i\hat{a}_j - \hat{b}_i\hat{b}_j)$ and $\epsilon_{ij}^{\otimes}(\hat{k}) = (\hat{a}_i\hat{b}_j + \hat{a}_j\hat{b}_i)$, ($\hat{a}$, $\hat{b}$ and $\hat{k} = \vec{k}/|\vec{k}|$ represent a triplet of mutually orthogonal unit vectors). The two circular polarizations of the graviton can also be defined as

$$\epsilon_{ij}^{(\mathrm{L})}(\hat{k}) = \frac{1}{\sqrt{2}}[\epsilon_{ij}^{\oplus}(\hat{k}) + i\epsilon_{ij}^{\otimes}(\hat{k})],$$
$$\epsilon_{ij}^{(\mathrm{R})}(\hat{k}) = \frac{1}{\sqrt{2}}[\epsilon_{ij}^{\oplus}(\hat{k}) - i\epsilon_{ij}^{\otimes}(\hat{k})]. \quad (2)$$

The problem at hand can be treated, in a magnetized environment, either through the Mueller or through the Jones calculus [5]. While the Mueller approach has been already employed in a related context [4], the Jones method has the advantage of dealing directly with the components of the electric fields which are organized in a two-dimensional column vector. A hybrid approach will be employed hereunder: the Stokes parameters will appear as the components of a 2×2 polarization matrix, i.e. $\mathcal{P}$. The evolution of $\mathcal{P}$ can be formally written as

$$\frac{d\mathcal{P}}{d\tau} + \epsilon'\mathcal{P} = \frac{3\epsilon'}{16\pi}\int M(\Omega, \Omega')\mathcal{P}(\Omega, \Omega')M^{\dagger}(\Omega, \Omega')d\Omega',$$
$$d\Omega' = d\cos\vartheta' d\varphi'; \quad (3)$$

the dagger denotes the transposed and complex conjugate matrix; $\epsilon' = x_{\mathrm{e}}\tilde{n}_{\mathrm{e}}\sigma_{\gamma\mathrm{e}}a(\tau)/a_0$ is the differential optical depth and $\sigma_{\gamma\mathrm{e}} = (8/3)\pi(e^2/m_{\mathrm{e}})^2$. The matrix $\mathcal{P}$ is

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix}$$
$$= \frac{1}{2}(I\mathbf{1} + U\sigma_1 + V\sigma_2 + Q\sigma_3), \quad (4)$$

where $\mathbf{1}$ denotes the identity matrix while σ_1, σ_2 and σ_3 are the three Pauli matrices whose explicit expression allows for a swifter derivation of the collision term of Eq. (3). The primed angles denote conventionally the directions of the incident photons while the unprimed angles describe the scattered radiation field. The matrix

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elements M_{ij} will be computed from the electron-photon scattering in the dipole approximation [6], where the outgoing electric field is given as $\vec{E}^{(\mathrm{out})} = -e[\hat{r} \times (\hat{r} \times \vec{A})]/r$, where $\vec{A}_{\mathrm{ei}} = (\vec{a}_{(\mathrm{e})} - \vec{a}_{(\mathrm{i})})$, is the difference between the electron and ion accelerations. By solving for the electron and ion velocities in the guiding center approximation (see, for instance, [7]) $\vec{a}_{(\mathrm{e})}$ and $\vec{a}_{(\mathrm{i})}$ can be derived and the outgoing electric fields computed. Equations (3) and (4) will then give the evolution of the polarization matrix. There are different ways of introducing the guiding center approximation, and the simplest one is to think of a gradient expansion of the background magnetic field, i.e. denoting with $\vec{B}$ the (comoving) magnetic field intensity we can write that

$$B_i(\vec{x}, \tau) \simeq B_i(\vec{x}_0, \tau) + (x^j - x_0^j)\partial_j B_i + \ldots, \tag{5}$$

where the ellipses stand for the higher orders in the gradients leading, both, to curvature and drift corrections, which will be neglected in this investigation as they were neglected in [4]. Higher order in the gradients demand the inclusion of higher multipoles of the field (see, e.g. [8]). The latter effects can be neglected to lowest order in the guiding center approximation. The maximum of the microwave background arises today for typical photon energies of the order of 10^{-3} eV corresponding to a typical wavelength of the mm. At the time of photon decoupling (i.e. $z \simeq z_* \simeq 1090$) the wavelength of the radiation was of the order of μm. Since the magnetic field we are interested in is inhomogeneous on a scale comparable with the Hubble radius (at the corresponding epoch) the guiding center approximation [7] can be safely employed as already discussed in [4]. It should finally be remarked that magnetic fields can also induce relic gravitons on their own and that relic gravitons might have various spectral slopes (see, e.g. [9] and references therein); however, the primary goal of this paper is to discuss the interplay of circular dichroism of the CMB and the tensor modes of the geometry. The components of the vector $\vec{A}_{\mathrm{ei}}$ are computed in a frame characterized by the three orthogonal unit vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, i.e.

$$\begin{aligned} \hat{e}_1 &= (\cos\alpha\cos\beta, \sin\alpha\cos\beta, -\sin\beta), \\ \hat{e}_2 &= (-\sin\alpha, \cos\alpha, 0), \\ \hat{e}_3 &= (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta). \end{aligned} \tag{6}$$

Without loss of generality[1] the direction of the magnetic field can be taken along $\hat{e}_3$ and, in this case, the matrix elements $M_{ij} \equiv M_{ij}(\varphi, \varphi', \mu, \nu, \alpha, \beta)$ (with $i, j = 1, 2$) can be computed explicitly, and they are

$$\begin{aligned} M_{11} &= \frac{\zeta\Lambda_1 - \Lambda_3}{2}[\sqrt{1-\mu^2}\sqrt{1-\nu^2} + \mu\nu\cos(\varphi-\alpha)\cos(\varphi'-\alpha)] + \frac{\zeta\Lambda_1 + \Lambda_3}{2}\cos2\beta[\mu\nu\cos(\varphi-\alpha)\cos(\varphi'-\alpha) \\ &\quad - \sqrt{1-\mu^2}\sqrt{1-\nu^2}] + (\Lambda_3 + \zeta\Lambda_1)\cos\beta\sin\beta[\mu\sqrt{1-\nu^2}\cos(\varphi-\alpha) + \nu\sqrt{1-\mu^2}\cos(\varphi'-\alpha)] \\ &\quad + \zeta\Lambda_1\mu\nu\sin(\varphi-\alpha)\sin(\varphi'-\alpha) + if_{\mathrm{e}}\zeta\Lambda_2\{\sin\beta[\mu\sqrt{1-\nu^2}\sin(\varphi-\alpha) - \nu\sqrt{1-\mu^2}\sin(\varphi'-\alpha)] \\ &\quad + \mu\nu\cos\beta\sin(\varphi-\varphi')\}, \end{aligned} \tag{7}$$

$$\begin{aligned} M_{12} &= \frac{\Lambda_3 - \Lambda_1\zeta}{2}\mu\sin(\varphi'-\alpha)\cos(\varphi-\alpha) - \frac{\Lambda_3 + \Lambda_1\zeta}{2}\mu\sin(\varphi'-\alpha)\cos(\varphi-\alpha)\cos2\beta \\ &\quad - (\Lambda_3 + \zeta\Lambda_1)\sqrt{1-\mu^2}\sin(\varphi'-\alpha)\cos\beta\sin\beta + \zeta\Lambda_1\mu\sin(\varphi-\alpha)\cos(\varphi'-\alpha) - if_{\mathrm{e}}\zeta\Lambda_2[\mu\cos\beta\cos(\varphi'-\varphi) \\ &\quad + \sqrt{1-\mu^2}\sin\beta\cos(\varphi'-\alpha)], \end{aligned} \tag{8}$$

$$\begin{aligned} M_{21} &= \frac{\nu(\zeta\Lambda_1 - \Lambda_3)}{4}[\sin(\varphi'-\varphi) - \cos2\alpha\sin(\varphi'+\varphi)] + \frac{\nu\cos2\beta}{4}(\zeta\Lambda_1 + \nu\Lambda_3)[\sin(\varphi'-\varphi) - \cos2\alpha\sin(\varphi'+\varphi)] \\ &\quad + \frac{\Lambda_3 + \zeta\Lambda_1}{4}[2\sqrt{1-\nu^2}\sin2\beta\cos(\varphi-\alpha) - \cos2\beta\sin2\alpha\sin(\varphi'-\varphi)] + \frac{\zeta\Lambda_1\nu}{2}[\sin(\varphi'-\varphi) + \cos2\alpha\sin(\varphi'+\varphi)] \\ &\quad + \frac{\nu(\Lambda_3 + \zeta\Lambda_1)}{4}\sin2\alpha\sin(\varphi'-\varphi) + if_{\mathrm{e}}\zeta\Lambda_2[\nu\cos\beta\cos(\varphi'-\varphi) + \sqrt{1-\nu^2}\sin\beta\cos(\varphi-\alpha)], \end{aligned} \tag{9}$$

[1]The choice of $\hat{e}_3$ is conventional: what matters is the relative direction between the propagation of the gravitational wave and the magnetic field direction. The direction $\hat{e}_3$ does not coincide necessarily with the third Cartesian direction, as it is clear from Eq. (6).

$$M_{22} = \frac{\zeta\Lambda_1}{2}[\cos(\varphi' - \varphi) + \cos 2\alpha\cos(\varphi' + \varphi)] + \frac{1}{4}\{\zeta\Lambda_1[\cos(\varphi' - \varphi) - \cos 2\alpha\cos(\varphi' + \varphi)] - \Lambda_3[\cos(\varphi' + \varphi) - \cos 2\alpha\cos(\varphi' - \varphi)] + (\zeta\Lambda_1 + \Lambda_3)\sin 2\alpha\sin(\varphi' + \varphi)\} + \frac{\cos 2\beta}{4}\{\zeta\Lambda_1[\cos(\varphi' - \varphi) - \cos 2\alpha\cos(\varphi' + \varphi)] + \Lambda_3[\cos(\varphi + \varphi') - \cos 2\alpha\cos(\varphi' - \varphi)] - (\zeta\Lambda_1 + \Lambda_3)\sin 2\alpha\sin(\varphi + \varphi')\} - if_e\zeta\Lambda_2\cos\beta\sin(\varphi' - \varphi). \tag{10}$$

The functions Λ_i (with $i = 1, 2, 3$) as well as ζ and f_e all depend upon the angular frequency of the photon (i.e. $\omega = 2\pi\nu$) and are defined as

$$\Lambda_1(\omega) = 1 + \left(\frac{\omega_{pi}^2}{\omega_{pe}^2}\right)\left(\frac{\omega^2 - \omega_{Be}^2}{\omega^2 - \omega_{Bi}^2}\right), \qquad \Lambda_2(\omega) = 1 - \left(\frac{\omega_{pi}^2}{\omega_{pe}^2}\right)\left(\frac{\omega_{Bi}}{\omega_{Be}}\right)\left(\frac{\omega^2 - \omega_{Be}^2}{\omega^2 - \omega_{Bi}^2}\right), \tag{11}$$

$$\Lambda_3(\omega) = 1 + \left(\frac{\omega_{pi}^2}{\omega_{pe}^2}\right), \qquad f_e(\omega) = \left(\frac{\omega_{Be}}{\omega}\right), \qquad \zeta(\omega) = \frac{\omega^2}{\omega_{Be}^2 - \omega^2}, \tag{12}$$

where $\omega_{Be,i}$ and $\omega_{pe,i}$ are the Larmor and plasma frequencies for electrons (and ions)

$$\omega_{Be,i} = \frac{e\vec{B}\cdot\hat{e}_3}{m_{ei}a}, \qquad \omega_{pe,i} = \sqrt{\frac{4\pi e^2 n_0}{m_{ei}a}}, \tag{13}$$

where, because of the global neutrality of the plasma, $n_0 = \tilde{n}_0 a^3$ is the common comoving concentration of electrons and ions while m_{ei} denote either the electron or the ion mass, depending upon the relative subscript. The scale factor appears explicitly since the mass breaks the conformal invariance of the system of equations [4]. It is useful to recall [4] that

$$f_e(\omega) = \frac{\omega_{Be}}{\omega} = 2.79\times 10^{-12}\left(\frac{B}{\text{nG}}\right)\left(\frac{\text{GHz}}{\nu}\right)(z_* + 1) \ll 1, \tag{14}$$

where z_* is the redshift to last scattering, i.e. $z_* = 1090.79^{+0.94}_{-0.92}$ according to the WMAP-7 yr data. In Eq. (14) $B = |\hat{e}_3\cdot\vec{B}|$; grossly speaking, the typical values of ν and B appearing in Eq. (14) do correspond, respectively, to the (very minimal) value of the frequency channel of CMB experiments and to the maximal value of the comoving magnetic field allowed by the distortions of the temperature autocorrelations and of the cross correlations between temperature and polarization (see [4] and references therein). Typical CMB experiments are in the region between 30 and 300 GHz. Since $m_i \gg m_e$, up to corrections $\mathcal{O}(m_e/m_i)$, the Λ_i are all equal to 1. Furthermore, in the limit $f_e(\omega)\to 0$, $\zeta(\omega)\to -1 - f_e^2$: Eqs. (7)–(10) reproduce the standard results for the scattering matrix (see, e.g. [6]). In Eqs. (7)–(10) (ϑ, φ) and (ϑ', φ') denote, respectively, the angular dependence of the outgoing and of the incident polarizations (with $\mu = \cos\vartheta$ and $\nu = \cos\vartheta'$). To be more specific $\hat{\theta} = (\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta)$, $\hat{\varphi} = (-\sin\varphi, \cos\varphi, 0)$ and similarly for the primed quantities. The orientation of the coordinate system implies that $\hat{\vartheta}\times\hat{\varphi} = \hat{r}$. In other classic references such as [6] the orientation is such that $\hat{\vartheta}\times\hat{\varphi} = -\hat{r}$; this different choice entails a sign flip, i.e. $\hat{\varphi} = (\sin\varphi, -\cos\varphi, 0)$ in the notations of [6].

Consider now, without loss of generality, that the propagation of the gravitational wave occurs in the $\hat{z}$ direction. If $\alpha = 0$ and $\beta = 0$, from Eq. (6) the magnetic field will be oriented along the same direction of propagation of the gravitational wave. If $\alpha = -\pi/2$ and $\beta = -\pi/2$, the magnetic field will be orthogonal to the direction of propagation of the gravitational wave and will be oriented, in particular, along the $\hat{y}$ Cartesian direction. In the case $\alpha = \beta = 0$ Eqs. (7)–(10) will lead, respectively, to the following matrix elements:

$$\begin{aligned} M_{11} &= \zeta\mu\nu\Lambda_1\cos\Delta\varphi - \sqrt{1-\mu^2}\sqrt{1-\nu^2}\Lambda_3 - i\Lambda_2 f_e\zeta\mu\nu\sin\Delta\varphi, \\ M_{12} &= -\zeta\mu\Lambda_1\sin\Delta\varphi - i\Lambda_2 f_e\zeta\mu\cos\Delta\varphi, \\ M_{21} &= \zeta\nu\Lambda_1\sin\Delta\varphi + if_e\Lambda_2\zeta\nu\cos\Delta\varphi, \\ M_{22} &= \zeta\Lambda_1\cos\Delta\varphi - if_e\Lambda_2\zeta\sin\Delta\varphi. \end{aligned} \tag{15}$$

where $\mu = \cos\vartheta$, $\nu = \cos\vartheta'$, and $\Delta\varphi = (\varphi' - \varphi)$. In the case $\alpha = \beta = -\pi/2$ Eqs. (7)–(10) will lead instead to the following matrix elements:

$$\begin{aligned} M_{11} &= \zeta\Lambda_1\sqrt{1-\mu^2}\sqrt{1-\nu^2} + \zeta\Lambda_1\mu\nu\cos\varphi'\cos\varphi - \Lambda_3\mu\nu\sin\varphi\sin\varphi' + if_e\Lambda_2\zeta(\nu\sqrt{1-\mu^2}\cos\varphi' - \mu\sqrt{1-\nu^2}\cos\varphi) \\ M_{12} &= -\zeta\mu\Lambda_1\cos\varphi\sin\varphi' - \Lambda_3\mu\cos\varphi'\sin\varphi, \qquad M_{21} = -\Lambda_3\nu\cos\varphi\sin\varphi' - \zeta\Lambda_1\nu\cos\varphi'\sin\varphi + if_e\zeta\Lambda_2\sqrt{1-\nu^2}, \\ M_{22} &= \zeta\Lambda_1\sin\varphi'\sin\varphi - \Lambda_3\cos\varphi'\cos\varphi. \end{aligned} \tag{16}$$

Equation (3) can be written, in components, as

$$\frac{d\mathcal{P}_{ij}}{d\tau} + \epsilon'\mathcal{P}_{ij} = \frac{3\epsilon'}{16\pi}\int M_{ik}(\Omega,\Omega')\mathcal{P}_{km}(\Omega,\Omega')M^*_{jm}(\Omega,\Omega')d\Omega', \tag{17}$$

$$\frac{d\mathcal{P}_{ij}}{d\tau} = \frac{\partial \mathcal{P}_{ij}}{\partial \tau} + n^k \frac{\partial \mathcal{P}_{ij}}{\partial x^k} - \frac{1}{2}\frac{\partial h_{km}}{\partial \tau} n^k n^m \frac{\partial \mathcal{P}_{ij}}{\partial \ln q}, \quad (18)$$

where q is the modulus of the (comoving) three-momentum. The matrix elements $\mathcal{P}_{ij}$ can then be split as $\mathcal{P}_{ij} = f_0(q)[\delta_{ij} + \mathcal{P}^{(1)}_{ij}]$ where $f_0(q)$ is the Bose-Einstein distribution as a function of the modulus of the comoving three-momentum q. The matrix $\mathcal{P}^{(1)}_{ij}$ contains the fluctuations of the brightness perturbations of the Stokes parameters whose explicit evolution is governed by the following set of equations:

$$\frac{\partial \Delta_{\rm I}^{(\rm t)}}{\partial \tau} + n^k \partial_k \Delta_{\rm I}^{(\rm t)} + \epsilon' \Delta_{\rm I}^{(\rm t)} - \frac{\partial h_{km}}{\partial \tau} n^k n^m = \frac{3\epsilon'}{16\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \mathcal{C}_{\rm I}(\mu,\nu,\varphi,\varphi'), \quad (19)$$

$$\frac{\partial \Delta_{\rm Q}^{(\rm t)}}{\partial \tau} + n^k \partial_k \Delta_{\rm Q}^{(\rm t)} + \epsilon' \Delta_{\rm Q}^{(\rm t)} = \frac{3\epsilon'}{16\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \times \mathcal{C}_{\rm Q}(\mu,\nu,\varphi,\varphi'), \quad (20)$$

$$\frac{\partial \Delta_{\rm U}^{(\rm t)}}{\partial \tau} + n^k \partial_k \Delta_{\rm U}^{(\rm t)} + \epsilon' \Delta_{\rm U}^{(\rm t)} = \frac{3\epsilon'}{16\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \times \mathcal{C}_{\rm U}(\mu,\nu,\varphi,\varphi'), \quad (21)$$

$$\frac{\partial \Delta_{\rm V}^{(\rm t)}}{\partial \tau} + n^k \partial_k \Delta_{\rm V}^{(\rm t)} + \epsilon' \Delta_{\rm V}^{(\rm t)} = \frac{3\epsilon'}{16\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \times \mathcal{C}_{\rm V}(\mu,\nu,\varphi,\varphi'), \quad (22)$$

where the integrand appearing in the source terms

$$\mathcal{C}_{\rm I}(\mu,\nu,\varphi,\varphi') = C_{\rm I}^{\rm I}\Delta_{\rm I}^{(\rm t)} + C_{\rm I}^{\rm Q}\Delta_{\rm Q}^{(\rm t)} + C_{\rm I}^{\rm U}\Delta_{\rm U}^{(\rm t)} + C_{\rm I}^{\rm V}\Delta_{\rm V}^{(\rm t)}, \quad (23)$$

$$\mathcal{C}_{\rm Q}(\mu,\nu,\varphi,\varphi') = C_{\rm Q}^{\rm I}\Delta_{\rm I}^{(\rm t)} + C_{\rm Q}^{\rm Q}\Delta_{\rm Q}^{(\rm t)} + C_{\rm Q}^{\rm U}\Delta_{\rm U}^{(\rm t)} + C_{\rm Q}^{\rm V}\Delta_{\rm V}^{(\rm t)}, \quad (24)$$

$$\mathcal{C}_{\rm U}(\mu,\nu,\varphi,\varphi') = C_{\rm U}^{\rm I}(\Delta_{\rm I}^{(\rm t)} + C_{\rm U}^{\rm Q}\Delta_{\rm Q}^{(\rm t)} + C_{\rm U}^{\rm U}\Delta_{\rm U}^{(\rm t)} + C_{\rm U}^{\rm V}\Delta_{\rm V}^{(\rm t)}, \quad (25)$$

$$\mathcal{C}_{\rm V}(\mu,\nu,\varphi,\varphi') = C_{\rm V}^{\rm I}\Delta_{\rm I}^{(\rm t)} + C_{\rm V}^{\rm Q}\Delta_{\rm Q}^{(\rm t)} + C_{\rm V}^{\rm U}\Delta_{\rm U}^{(\rm t)} + C_{\rm V}^{\rm V}\Delta_{\rm V}^{(\rm t)}. \quad (26)$$

If $\alpha = \beta = 0$ (propagation parallel to the magnetic field intensity) the coefficients appearing in Eqs. (23)–(26) are given by

$$\begin{aligned}
C_{\rm I}^{\rm I} &= \Lambda_3^2\mathcal{A}^2(\mu,\nu) - 2\zeta\Lambda_1\Lambda_3\mu\nu\mathcal{A}(\mu,\nu)c(\varphi',\varphi) + \zeta^2[f_{\rm e}^2\Lambda_2^2(\mu^2+\nu^2) + \Lambda_1^2(1+\nu^2\mu^2)]c^2(\varphi',\varphi) \\
&\quad + \zeta^2[\Lambda_1^2(\mu^2+\nu^2) + f_{\rm e}^2\Lambda_2^2(1+\nu^2\mu^2)]s^2(\varphi',\varphi), \\
C_{\rm I}^{\rm Q} &= \Lambda_3^2\mathcal{A}^2(\mu,\nu) - 2\zeta\Lambda_1\Lambda_3\mu\nu\mathcal{A}(\mu,\nu)c(\varphi',\varphi) + \zeta^2[f_{\rm e}^2\Lambda_2^2(-\mu^2+\nu^2) + \Lambda_1^2(-1+\nu^2\mu^2)]c^2(\varphi',\varphi) \\
&\quad + \zeta^2[\Lambda_1^2(-\mu^2+\nu^2) + f_{\rm e}^2\Lambda_2^2(-1+\nu^2\mu^2)]s^2(\varphi',\varphi), \\
C_{\rm I}^{\rm U} &= 2\zeta[\Lambda_1\Lambda_3\mu\mathcal{A}(\mu,\nu) - \zeta(\Lambda_1^2 - f_{\rm e}^2\Lambda_2^2)(\mu^2-1)\nu c(\varphi',\varphi)]s(\varphi',\varphi), \\
C_{\rm I}^{\rm V} &= 2f_{\rm e}\zeta\Lambda_2[\zeta\Lambda_1(1+\mu^2)\nu - \Lambda_3\mu\mathcal{A}(\mu,\nu)c(\varphi',\varphi)] \\
C_{\rm Q}^{\rm I} &= \Lambda_3^2\mathcal{A}^2(\mu,\nu) - 2\zeta\Lambda_1\Lambda_3\mu\nu\mathcal{A}(\mu,\nu)c(\varphi',\varphi) + \zeta^2[f_{\rm e}^2\Lambda_2^2(\mu^2-\nu^2) + \Lambda_1^2(-1+\nu^2\mu^2)]c^2(\varphi',\varphi) \\
&\quad + \zeta^2[\Lambda_1^2(\mu^2-\nu^2) + f_{\rm e}^2\Lambda_2^2(-1+\nu^2\mu^2)]s^2(\varphi',\varphi), \\
C_{\rm Q}^{\rm Q} &= \Lambda_3^2\mathcal{A}^2(\mu,\nu) - 2\zeta\Lambda_1\Lambda_3\mu\nu\mathcal{A}(\mu,\nu)c(\varphi',\varphi) + \zeta^2[-f_{\rm e}^2\Lambda_2^2(\mu^2+\nu^2) + \Lambda_1^2(1+\nu^2\mu^2)]c^2(\varphi',\varphi) \\
&\quad + \zeta^2[-\Lambda_1^2(\mu^2+\nu^2) + f_{\rm e}^2\Lambda_2^2(1+\nu^2\mu^2)]s^2(\varphi',\varphi), \\
C_{\rm Q}^{\rm U} &= 2\zeta[\Lambda_1\Lambda_3\mu\mathcal{A}(\mu,\nu) - \zeta(\Lambda_1^2 - f_{\rm e}^2\Lambda_2^2)(\mu^2+1)\nu c(\varphi',\varphi)]s(\varphi',\varphi), \\
C_{\rm Q}^{\rm V} &= 2f_{\rm e}\zeta\Lambda_2[\zeta\Lambda_1(-1+\mu^2)\nu - \Lambda_3\mu\mathcal{A}(\mu,\nu)c(\varphi',\varphi), \\
C_{\rm U}^{\rm I} &= 2\zeta[-\Lambda_1\Lambda_3\nu\mathcal{A}(\mu,\nu) + \zeta(\Lambda_1^2 - f_{\rm e}^2\Lambda_2^2)\mu(\nu^2-1)c(\varphi',\varphi)]s(\varphi',\varphi), \\
C_{\rm U}^{\rm Q} &= 2\zeta[-\Lambda_1\Lambda_3\nu\mathcal{A}(\mu,\nu) + \zeta(\Lambda_1^2 - f_{\rm e}^2\Lambda_2^2)\mu(\nu^2+1)c(\varphi',\varphi)]s(\varphi',\varphi), \\
C_{\rm U}^{\rm U} &= -2\zeta[\Lambda_1\Lambda_3\mathcal{A}(\mu,\nu)c(\varphi',\varphi) - \zeta(\Lambda_1^2 - f_{\rm e}^2\Lambda_2^2)\mu\nu(c^2(\varphi',\varphi) - s^2(\varphi',\varphi))], \qquad C_{\rm U}^{\rm V} = -2f_{\rm e}\zeta\Lambda_2\Lambda_3\mathcal{A}(\mu,\nu), \\
C_{\rm V}^{\rm I} &= 2f_{\rm e}\zeta\Lambda_2[\zeta\Lambda_1\mu(1+\nu^2) - \Lambda_3\mathcal{A}^2(\mu,\nu)\nu c(\varphi',\varphi)], \qquad C_{\rm V}^{\rm Q} = 2f_{\rm e}\zeta\Lambda_2[\zeta\Lambda_1\mu(-1+\nu^2) - \Lambda_3\mathcal{A}(\mu,\nu)\nu c(\varphi',\varphi)], \\
C_{\rm V}^{\rm U} &= 2f_{\rm e}\zeta\Lambda_2\Lambda_3\mathcal{A}(\mu,\nu)s(\varphi',\varphi) \qquad C_{\rm V}^{\rm V} = 2i\zeta[\zeta(\Lambda_1^2 + f_{\rm e}^2\Lambda_2^2)\mu\nu - \Lambda_1\Lambda_3\mathcal{A}(\mu,\nu)c(\varphi',\varphi)]; \quad (27)
\end{aligned}$$

in Eq. (27) the shorthand notations $\mathcal{A}(\mu,\nu) = \sqrt{1-\mu^2}\sqrt{1-\nu^2}$, $c(\varphi',\varphi) = \cos(\varphi'-\varphi)$ and $s(\varphi',\varphi) = \sin(\varphi'-\varphi)$ have been introduced. The obtained equations generalize the standard system obtained (in the absence of magnetic fields) for the tensor components of the Stokes parameters (see, for instance, [10]). Consider, for sake of concreteness, a linearly polarized graviton (for instance, $\oplus$). Assuming, without loss of generality, that the graviton travels along the $\hat{z}$ axis, the combination $h_{km}n^k n^m$ equals $h_{\oplus}(1-\mu^2)\cos(2\varphi)$. The dependence of the brightness perturbations upon φ can then be deduced from the whole symmetry of the system, and it is

$$\begin{aligned}
\Delta_{\rm I}^{(t)}(\nu,\varphi') &= \mathcal{Z}(1-\nu^2)\cos 2\varphi',\\
\Delta_{\rm Q}^{(t)}(\nu,\varphi') &= \mathcal{T}(1+\nu^2)\cos 2\varphi',\\
\Delta_{\rm U}^{(t)}(\nu,\varphi') &= -2\nu\mathcal{T}\sin 2\varphi',\\
\Delta_{\rm V}^{(t)}(\nu,\varphi') &= 2\nu\mathcal{S}\cos 2\varphi'.
\end{aligned} \tag{28}$$

Using Eq. (28) into Eqs. (23)–(26) and taking into account the explicit form of the coefficients given in Eq. (27) the evolution of $\mathcal{Z}$, $\mathcal{T}$, and $\mathcal{S}$ become, after the appropriate algebra

$$\frac{\partial\mathcal{Z}}{\partial\tau} + n^k\partial_k\mathcal{Z} + \epsilon'\mathcal{Z} - \frac{\partial h_{\oplus}}{\partial\tau} = \epsilon'\zeta^2(\omega)[\Lambda_1^2(\omega) - f_{\rm e}^2(\omega)\Lambda_2^2(\omega)]\Sigma^{(t)}, \tag{29}$$

$$\frac{\partial\mathcal{T}}{\partial\tau} + n^k\partial_k\mathcal{T} + \epsilon'\mathcal{T} = -\epsilon'\zeta^2(\omega)[\Lambda_1^2(\omega) - f_{\rm e}^2(\omega)\Lambda_2^2(\omega)]\Sigma^{(t)}, \tag{30}$$

$$\frac{\partial\mathcal{S}}{\partial\tau} + n^k\partial_k\mathcal{S} + \epsilon'\mathcal{S} = 0, \tag{31}$$

where the source term $\Sigma^{(t)}$ can also be expressed as

$$\begin{aligned}
\Sigma^{(t)} &= \frac{3}{32}\int_{-1}^{1} d\nu[(1-\nu^2)^2\mathcal{Z}(\nu) - (1+\nu^2)^2\mathcal{T}(\nu) - 4\nu^2\mathcal{T}(\nu)]\\
&= \frac{3}{70}\mathcal{Z}_4 + \frac{\mathcal{Z}_2}{7} - \frac{\mathcal{Z}_0}{10} - \frac{3}{70}\mathcal{T}_4 + \frac{6}{7}\mathcal{T}_2 - \frac{3}{5}\mathcal{T}_0;
\end{aligned} \tag{32}$$

$\mathcal{Z}_\ell$ and $\mathcal{T}_\ell$ define the ℓth multipole of the corresponding quantity, i.e. $\int_{-1}^{1}\mathcal{Z}P_\ell(\nu)d\nu = 2(-i)^\ell\mathcal{Z}_\ell$ and $\int_{-1}^{1}\mathcal{T}P_\ell(\nu)d\nu = 2(-i)^\ell\mathcal{T}_\ell$, where $P_\ell(\nu)$ are the Legendre polynomials. In the case of the orthogonal polarization of the graviton (i.e. $\otimes$) the combination $h_{km}n^k n^m$ equals $h_{\otimes}(1-\mu^2)\sin 2\varphi$ and symmetry considerations imply that

$$\begin{aligned}
\Delta_{\rm I}^{(t)}(\nu,\varphi') &= \mathcal{Z}(1-\nu^2)\sin 2\varphi',\\
\Delta_{\rm Q}^{(t)}(\nu,\varphi') &= \mathcal{T}(1+\nu^2)\sin 2\varphi',\\
\Delta_{\rm U}^{(t)}(\nu,\varphi') &= 2\nu\mathcal{T}\cos 2\varphi',\\
\Delta_{\rm V}^{(t)}(\nu,\varphi') &= 2\nu\mathcal{S}\sin 2\varphi'.
\end{aligned} \tag{33}$$

Inserting Eq. (33) inside Eqs. (23)–(26) and performing the integration over φ' the angular dependence factorizes so that $\mathcal{Z}$ and $\mathcal{T}$ obey, for the two polarization, the same equations, i.e. Eqs. (30)–(33). A similar phenomenon happens also in the absence of magnetic field [10]. The discussion can be carried on when the gravitons are circularly polarized since the left and right polarizations can be expressed as linear combinations of $\oplus$ and $\otimes$ [see Eq. (2)]. We can then conclude that, in the physical limit (i.e. $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$ and $f_{\rm e} \ll 1$) the equations for $\mathcal{Z}$, $\mathcal{T}$, and $\mathcal{S}$ can be written as

$$\mathcal{D}_\tau\mathcal{Z} = \epsilon'\Sigma^{(t)} + \mathcal{O}(f_{\rm e}^2), \qquad \mathcal{D}_\tau\mathcal{T} = -\epsilon'\Sigma^{(t)} + \mathcal{O}(f_{\rm e}^2), \qquad \mathcal{D}_\tau\mathcal{S} = 0, \tag{34}$$

where $\mathcal{D}_\tau\mathcal{Z}$, $\mathcal{D}_\tau\mathcal{T}$, and $\mathcal{D}_\tau\mathcal{S}$ denote, with shorthand notation, the left-hand sides of, respectively, Eqs. (29)–(31).

So far we have been concerned with the case when the gravitational wave travels in the *same* direction of the magnetic field. In the case $\alpha = \beta = -\pi/2$ the gravitational wave still travels along $\hat{z}$, while the magnetic field is in the $\hat{y}$ direction, i.e. orthogonal to the direction of propagation. Using Eqs. (7)–(10) in the case $\alpha = \beta = -\pi/2$ the analog of Eq. (27) [obtained in the case $\alpha = \beta = 0$)] can be derived. In what follows, for the sake of simplicity, the formulas will be given already in the case $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$. The final results for the Boltzmann equations for the $\oplus$ polarization are

$$\begin{aligned}
-(\mu^2-1)\cos 2\varphi\,\mathcal{D}_\tau\mathcal{Z} = -\frac{\epsilon'}{2}[&\mu^2(1-2f_{\rm e}^2)\zeta^2 + (1+2f_{\rm e}^2)\zeta^2 - (\mu^2+1)\\
&+ (\zeta^2+1)(\mu^2-1)\cos 2\varphi]\Sigma^{(t)},
\end{aligned} \tag{35}$$

$$-2\mu\sin 2\varphi\,\mathcal{D}_\tau\mathcal{T} = 2\epsilon'\mu\sin 2\varphi(1+\zeta^2)\Sigma^{(t)}, \tag{36}$$

$$2\mu\cos 2\varphi\,\mathcal{D}_\tau\mathcal{S} = -2\epsilon'\sin\varphi\sqrt{1-\mu^2}f_{\rm e}\Sigma^{(t)}. \tag{37}$$

The angular dependence factorizes up to corrections $\mathcal{O}(f_{\rm e}^2)$ for $\mathcal{Z}$ and $\mathcal{T}$ (as in Eq. (34)). For $\mathcal{S}$ the correction is larger, and it is $\mathcal{O}(f_{\rm e})$. The correction can be computed by positing $\Delta_{\rm V}(\mu,\varphi) = 2\mu\cos 2\varphi[\mathcal{S} + f_{\rm e}g(\mu,\varphi)]$, where $g(\mu,\varphi)$ is found to obey the following equation:

$$\frac{\partial g}{\partial \tau} + (n^k \partial^k g + \epsilon' g) = -\epsilon' \frac{\sqrt{1-\mu^2}}{\mu} \frac{\sin\varphi}{\cos 2\varphi} f_e \Sigma^{(t)}, \tag{38}$$

where $\Sigma^{(t)}$ is computed to lowest order in f_e from Eqs. (35) and (36). Up to now it has been shown that the tensor contribution to the V-mode polarization induced by a magnetized plasma at last scattering vanishes exactly when the magnetic field is parallel to the direction of propagation of the gravitational wave. For propagation perpendicular to the magnetic field orientation the effect depends on the ratio between the Larmor frequency of the electrons and the angular frequency of observation (i.e. related to the frequency channel of an hypothetical experiment). The latter statements hold when the magnetic field is oriented along a specific direction. Suppose now that we want to derive the transport equation for Δ_V by averaging over α and β. Using Eqs. (7)–(10), Eq. (22) becomes

$$\frac{\partial \Delta_V^{(t)}}{\partial \tau} + n^k \partial_k \Delta_V^{(t)} + \epsilon' \Delta_V^{(t)} = \frac{3\epsilon'}{64\pi^2} \int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \times \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\alpha \times \mathcal{C}_V(\mu, \nu, \varphi, \varphi', \alpha, \beta). \tag{39}$$

Note that the right-hand side of Eq. (39) has been averaged over the directions of the magnetic field intensity defined by the angles α and β; the integrated expression has been also divided by 4π. We are now interested in performing the integrals over α and β. The result is

$$\begin{aligned}\frac{3\epsilon'}{64\pi^2}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\alpha \mathcal{C}_V(\mu,\nu,\varphi,\varphi',\alpha,\beta) \\ = \frac{i\epsilon'}{32\pi}\int_{-1}^{1} d\nu \int_0^{2\pi} d\varphi' \{[\mu\nu(5+2f_e^2+\cos 2\varphi' \\ +\cos 2\varphi + \cos 2(\varphi'-\varphi)) + 2\sqrt{1-\mu^2}\sqrt{1-\nu^2} \\ \times(\cos(\varphi'-\varphi)+\cos(\varphi'+\varphi))]\Delta_V(\nu,\varphi')\}.\end{aligned} \tag{40}$$

In deriving Eq. (40), it has just been assumed that the brightness perturbations of the incident radiation field only depend, as natural, upon ϑ' and φ' and not upon the magnetic field. The result of Eq. (40) shows that the evolution equation for the V-mode polarization decouples from the others. In spite of the complicated angular dependence of Eq. (40), it is clear that $\Delta_V = 0$ is a solution. The considerations of the last paragraph show that if the source term of the V-mode equation is averaged over the directions of the magnetic field, the circular polarization evolves independently from all the other Stokes parameters.

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