ON THE ENERGY LOSS OF A CHARGED RING PASSING A CORRUGATED CYLINDRICAL WAVEGUIDE

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1. Introduction

The energy loss of a charged ring passing a sequence of RF accelerating cavities has recently received much attention in connection with the collective acceleration of ions in electron rings[1]. Various models and techniques were employed to represent the RF structure: modal analysis for closed cylindrical cavities [2], [3], [4] and for corrugated cylindrical waveguides [5], Wiener-Hopf techniques for sets of infinite half-planes [6], [7], and diffraction theory for one infinite half-plane[8]. As in[5] we shall use a periodic corrugated waveguide as a model and obtain the energy loss by an analysis of the resonances of this system. However, we do not restrict the nature of the waveguide modes from the beginning, and continue the calculation to very much smaller wavelengths.

2. The Modes in a Periodic Corrugated Cylindrical Waveguide

If one only considers the axial velocity of the electrons in the rings, and if one considers them perfectly centered in the waveguide, they only interact with TM waveguide modes which are independent of the azimuthal co-ordinate φ. The energy loss in a closed cavity due to the azimuthal motion of the electrons was calculated by Neil[9]. We use the field expansions given by Walkinshaw and Bell [10]. We only write down E_z; E_ρ and H_φ follow from Maxwell's equations.

Expansion of E_z in the slots (I):

$$E_z^{(I)}=\sum_{s=0}^{\infty} B_s[F_0(\Gamma_s\rho)/F_0(\Gamma_s a)]\cos(2\pi sz/g)$$

$$+\sum_{s=1}^{\infty} D_s[F_0(\Gamma'_z\rho)/F_0(\Gamma'_s a)]\sin[(2s-1)\pi z/g] \qquad (1)$$

where

$$\Gamma_s^2=|k^2-(2\pi s/g)^2|, \tag{2}$$

$$\Gamma'^2_s=|k^2-[(2s-1)\pi/g]^2|. \tag{3}$$

If $k>2\pi s/g$ or if $k>(2s-1)\pi/g$ we define:

$$F_j(\Gamma\rho)=Y_0(\Gamma b)J_j(\Gamma\rho)-J_0(\Gamma b)Y_j(\Gamma\rho). \tag{4}$$

Otherwise:

$$F_j(\Gamma\rho)=K_0(\Gamma b)I_j(\Gamma\rho)-(-1)^j I_0(\Gamma b)K_j(\Gamma\rho). \tag{5}$$

In all formulae the time dependence $\exp(i\omega t)$ should be understood. It may be seen that the slot fields satisfy the boundary conditions of a perfectly conducting waveguide at the boundaries drawn in heavy lines in Fig. 1, where also the geometrical parameters a, b, d, g are shown. $k=\omega/c=2\pi$/free-space wavelength.

Expansion of E_z in the axial region (II):

$$E_z^{(II)}=\sum_{m=-\infty}^{\infty} A_m[G_0(x_m\rho)/G_0(x_m a)]\ \exp(-i\beta_m z) \tag{6}$$

where

$$x^2_m=|k^2-\beta_m^2|, \tag{7}$$

$$\beta_m=\beta_0+2\pi m/d, \tag{8}$$

$$\beta_0=k/\beta-2\pi l/d, \tag{9}$$

where l is chosen such that $-\pi/d<\beta_0\leq\pi/d$. $\beta c=v$ is the phase velocity of the travelling wave

$$G_j(x\rho)=J_j(x\rho)\ \text{for}\ k>\beta_m, \tag{10}$$

$$G_j(x\rho)=I_j(x\rho)\ \text{for}\ k<\beta_m. \tag{11}$$

Imposing the condition that E_z and H_φ be equal on both sides of the boundary $\rho=a$ yields the following infinite homogeneous set of linear equations[10]:

$$[\underline{C}^T\ \underline{H}\ \underline{C}\ \underline{M}-\underline{S}^T\ \underline{L}\ \underline{S}\ \underline{M}-\underline{U}]\ \underline{A}=0. \tag{12}$$

Here $\underline{A}$ is a vector with components A_m; $\underline{H}$, $\underline{L}$ and $\underline{M}$ are diagonal matrices with elements

$$H_{ss}=K_s\Gamma_s F_0(\Gamma_s a)/F_1(\Gamma_s a), \tag{13}$$

$$L_{ss}=2\Gamma'_s F_0(\Gamma'_s a)/F_1(\Gamma'_s a), \tag{14}$$

$$M_{mm}=(g/dx_m)G_1(x_m a)/G_0(x_m a). \tag{15}$$

Here $K_0=1$ and $K_s=2$ for $s>0$.
$\underline{C}$ and $\underline{S}$ are matrices with elements

$$C_{st}=(-1)^s g\frac{\beta_t g}{2}\sin\frac{\beta_t g}{2}\Big/\left[\left(\frac{\beta_t g}{2}\right)^2-(\pi s)^2\right], \tag{16}$$

$$S_{st} = -(-1)^s i g \frac{\beta_t g}{2} \cos \frac{\beta_t g}{2} \Bigg/ \left[\left(\frac{\beta_t g}{2}\right)^2 - \left(\frac{(2s-1)\pi}{2}\right)^2\right]. \tag{17}$$

$\underline{U}$ is the unit matrix and $\underline{X}^T$ is the transpose of $\underline{X}$.

The frequencies ω_λ where (12) has non-trivial solutions are those where waves travelling with the phase velocity βc propagate in the waveguide. They were calculated by computer. Since the infinities of the determinant of (12) can be given analytically, the computer programme uses a function of frequency in which the infinities are eliminated but which has the same zeros as the determinant. This makes it fairly straightforward to find all zeros in a given frequency interval. A further check is the comparison of the average density of resonances with an analytical formula [11].

3. The Excitation of a Single Resonance by the Passage of a Charged Ring

Following the procedure of Akhiezer, Lyubarskij and Fainberg we introduce the vector potential $\vec{A}$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{18}$$

which we normalize such that

$$\int_{V_1} |A|^2 dv = 1/\varepsilon_0 \tag{19}$$

where V_1 is the volume of a single cell. This implies that the coefficients A_m, B_s and D_s are multiplied by the factor

$$N = [1/[\varepsilon_0(I + J + J')]]^{1/2} \tag{20}$$

where the integrals are defined as follows:

$$I = \frac{\pi a^2 d}{c^2} \sum_{m=-\infty}^{+\infty} \frac{A^2_m}{x^2_m} \left[\frac{G^2_1(x_m a)}{G^2_0(x_m a)} \pm 1 \mp \frac{\beta^2_m}{k^2} \frac{2}{x_m a} \frac{G_1(x_m a)}{G_0(x_m a)}\right]. \tag{21}$$

Upper sign for G_j given by (10), lower sign for G_j given by (11).

$$J = \frac{\pi g}{c^2} \sum_{s=0}^{\infty} \frac{B^2_s}{\Gamma^2_s} K_s \left[\frac{b^2}{2} \frac{F^2_1(\Gamma_s b)}{F^2_0(\Gamma_s a)} - \frac{a^2}{2}\left(\frac{F_1^2(\Gamma_s a)}{F^2_0(\Gamma_s a)} \pm 1\right) \pm \frac{a^2}{2}\left(\frac{g\pi s}{gk}\right)^2 \frac{2}{\Gamma_s a} \frac{F_1(\Gamma_s a)}{F_0(\Gamma_s a)}\right), \tag{22}$$

$K_0 = 2$ and $K_s = 1$ for $s > 0$.

$$J' = \frac{\pi g}{c^2} \sum_{s=1}^{\infty} \frac{D^2_s}{\Gamma'^2_s} \left[\frac{b^2}{2} \frac{F_1^2(\Gamma'_s b)}{F^2_0(\Gamma'_s a)} - \frac{a^2}{2}\left(\frac{F_1^2(\Gamma'_s a)}{F^2_0(\Gamma'_s a)} \pm 1\right) \pm \frac{a^2}{2}\left(\frac{(2s-1)\pi}{gk}\right)^2 \frac{2}{\Gamma'_s a} \frac{F_1(\Gamma'_s a)}{F_0(\Gamma'_s a)}\right] \tag{23}$$

Upper sign for F_j given by (4), lawer sign for F_j given by (5).

We expand the vector potential in the presence of the electron rings in terms of the resonances of the empty waveguide.

$$\vec{A}(\vec{r},t)=\sum_{\lambda} q_{\lambda}(t)A_{\lambda}(\vec{r}) \tag{24}$$

and find that the field co-ordinate $q_{\lambda}(t)$ for the λ-th mode obeys the equation:

$$\ddot{q}_{\lambda}+\omega^2_{\lambda}q_{\lambda}=\frac{1}{N_c}\int_{V_N}\vec{j}\vec{A}_{\lambda}dv \tag{25}$$

where N_c is the number of cavities and V_N is their volume. $\vec{j}$ is the current density of the charged ring whose only non-vanishing component is taken to be:

$$j_z(\rho,\varphi,z,t)=\frac{Qv}{2\pi}\frac{2}{R^2_2-R^2_1}\frac{H\left(\frac{h}{2}-|z-vt|\right)}{h}H(\rho-R_1)H(R_2-\rho) \tag{26}$$

$H(x)$ is Heaviside function with $H(x)=1$ for $x\geq0$, and $H(x)=0$ for $x<0$. Q is the ring charge, v its velocity, h its axial dimension, and R_1 and R_2 its smaller and bigger radius, respectively.

The solution of (25) with $q_{\lambda}=\dot{q}_{\lambda}=0$ is:

$$q_{\lambda}=\sum_{m=-\infty}^{+\infty}\frac{A'_m}{\omega^2_{\lambda}-\Omega^2_m}\left[e^{i\Omega_m t}-\frac{1}{2}\left(\frac{\Omega_m}{\omega_{\lambda}}+1\right)e^{i\omega_{\lambda}t}+\frac{1}{2}\left(\frac{\Omega_m}{\omega_{\lambda}}-1\right)e^{-i\omega_{\lambda}t}\right] \tag{27}$$

where

$$A'_m=-\frac{Qvi}{N_c\omega_{\lambda}}A_m\left(\frac{\sin\frac{\beta_m h}{2}}{\frac{\beta_m h}{2}}\right)\left[\frac{2[R_2G_1(x_mR_2)-R_1G_1(x_mR_1)]}{x_m(R^2_2-R_1^2)G_0(x_ma)}\right] \tag{28}$$

and $\Omega_m=\beta_m v$.

The energy radiated per unit time into the λ-th mode becomes:

$$I_{\lambda}=\frac{dQ^2v^2}{4\omega^2_{\lambda}}\left\{\sum_{m=m'}\frac{|A_m|^2}{\left|\frac{d\omega_{\lambda}}{d\beta_0}-v\right|}\left(\frac{\sin\frac{\beta_m h}{2}}{\frac{\beta_m h}{2}}\right)^2\left[\frac{2[R_2G_1(x_mR_2)-R_1G_1(x_mR_1)]}{x_m(R^2_2-R^2_1)G_0(x_ma)}\right]^2\right.$$
$$\left.+\sum_{m=m''}\frac{|A_m|^2}{\left|\frac{d\omega_{\lambda}}{d\beta_0}+v\right|}\left(\frac{\sin\frac{\beta_m h}{2}}{\frac{\beta_m h}{2}}\right)^2\left[\frac{2[R_2G_1(x_mR_2)-R_1G_1(x_mR_1)]}{x_m(R^2_2-R_1^2)G_0(x_ma)}\right]^2\right\} \tag{29}$$

Here, the first sum is to be taken over all m' with $\omega_{\lambda}-\Omega_{m'}=0$, and the second sum over all m'' with $\omega_{\lambda}+\Omega_{m''}=0$. With our choice of travelling waves (9) the first resonance condition takes the form $m'=l$.

In order to fulfil the other resonance condition $\omega_{\lambda}-\Omega_{m''}=0$ we have to replace β_0 by-β_0 and m by-m. It has been verified that changing the sign of β_0 leaves the resonant frequencies unchanged and reverses the order of the A_m. Hence the second sum is just the same as the first one.

If we further assume that $d\omega_\lambda/d\beta_0 \ll v$ we find for the energy radiated per unit time:

$$I_\lambda = \frac{dQ^2v}{2\omega_\lambda^2} |A_1|^2 \left(\frac{\sin \frac{\beta_1 h}{2}}{\frac{\beta_1 h}{2}} \right)^2 \left[\frac{2[R_2G_1(x_1R_2) - R_1G_1(x_1R_1)]}{x_1(R_2^2 - R_1^2)G_0(x_1a)} \right]. \quad (30)$$

The energy radiated into the λ-th mode in a single cell is $U_\lambda = I_\lambda d/v$.

4. Results

Fig. 2 shows a typical example of the frequency spectrum of the radiated energy when electron rings travel along a waveguide with rather shallow corrugations. Tabs. I and II summarise the influence of the waveguide dimensions, and of the ring dimension and velocity expressed in terms of $\gamma = [1-(v/c)^2]^{-1/2}$.

A completely different set of waveguide shapes and the resulting energy loss spectrum are shown in Fig. 3. The waveguide is now loaded with closely spaced infinitely thin irises. In cylindrical geometry this is a close analogy to the infinite half-planes studied in two-dimensional models [6] [7]. The dimensions are chosen such that the approximate formula [5]

$$U = \frac{Q^2}{4\pi\varepsilon_0} \frac{g}{2a^2} \quad (31)$$

should just be valid. The effect of variations in the waveguide dimensions is shown in Tab. III. In the last column the scaling law (31) was applied to the radiated energy. It may be seen that this formula gives an excellent description of the dependence on the waveguide dimensions. However, the total enengy radiated is underestimated by about a factor of three. Again, the ring dimensions and γ have hardly any influence on the radiation loss.

5. Conclusions

Field expansion into Fourier series and matching of all field components at the common boundary surface are shown to be a suitable method for calculating the energy radiated by electrons travelling in a cylindrical corrugated waveguide. In practice, this method is limited to wavelengths longer than a few millimetres for reasons of computer time. The energy losses are calculated for several waveguide dimensions; they are compared with published results [5] for one particular set of waveguide dimensions. In all cases, the electron ring dimensions and their velocity have no significant influence on the energy loss.

Table I

Radiated Energy U for Various Waveguide Dimensions

$N=10^{13}$ electrons $\gamma=20$ Wavelength $\geq 4\pi$ mm
Minor ring dimensions $(R_2-R_1)h=1\times 1$ mm²
Average distance of ring from iris $a-(R_1+R_2)/2=5$ mm

Hole radius a[mm]	Tube radius b[mm]	Cell length d[mm]	Slot width g[mm]	Radiated Energy U $\left[\frac{\text{keV}}{\text{electron cell}}\right]$
53.14	83.14	153.14	73.14	820
53.1[illegible]	166.28	1[illegible]3.14	73.14	860
53.[illegible]4	166.28	153.14	36.57	650
53.14	166.[illegible]8	76.57	36.57	49[illegible]
26.57	[illegible]6[illegible].28	153.14	73.14	2230

Table II

Radiatied Energy U in KeV/electron/cell, for Various Ring Dimensions and γ's

$N=10^{13}$ electrons Wavelength $\geq 3\pi$ mm
Hole radius $a=53.14$ mm Cell length $d=153.14$ mm
Tube radius $b=83.14$ mm Slot width $g=73.14$ mm
Minor ring dimensions $(R_2-R_1)h=1\times 1$ mm²

γ	Average distance of ring from iris $a-\frac{R_1+R_2}{2}$			
	1 mm	2 mm	5 mm	10 mm
5	1080	980	780	610
20	960	940	910	870
50	9[illegible]0	900	890	880

Table III

Radiated Energy U for Waveguide with Infinitely Thin Irises

$N=10^{13}$ electrons $\gamma=50$ Wavelength $\geq 3\pi$ mm
Minor ring dimensions $(R_2-R_1)h=1\times 1$ mm²
Average distance of ring from iris $a-(R_1+R_2)/2=5$ mm

Hole radius a[mm]	Tube radius b[mm]	Cell length d=g[mm]	Radiated Energy U $\left[\frac{\text{keV}}{\text{electron cell}}\right]$	Radiated energy scaled with (31)
50	250	20	190	1.00
40	[illegible]50	20	290	0.98
60	[illegible]50	20	130	0.99
[illegible]0	200	20	180	0.95
50	300	20	190	1.00
50	250	10	100	0.95
50	250	30	280	0.98

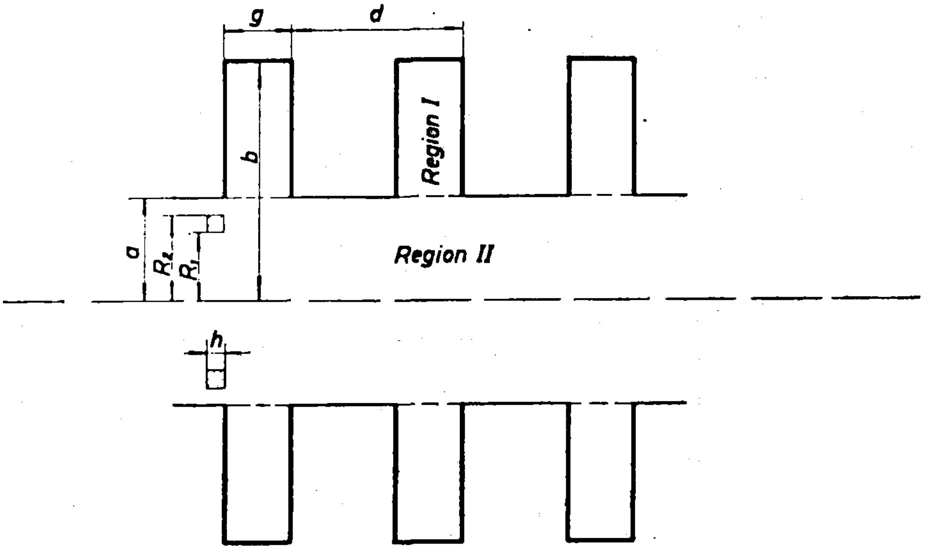

Fig. 1. Geometry of corrugated cylindrical waveguide and electron ring

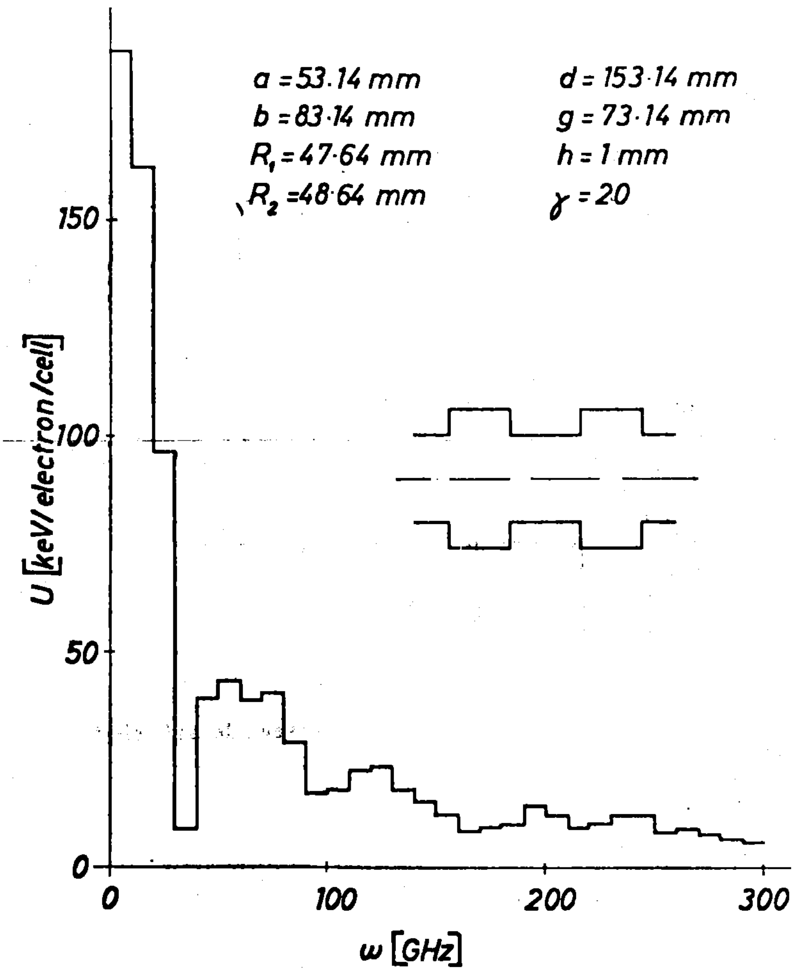

Fig. 2. Energy loss spectrum at $\gamma=20$. Waveguide dimensions a=53 . 14 mm, b=83 . 4 mm, d=.53 . 14 mm, g=73 . 14 mm; ring dimensions $R_1=47.64$ mm, $R_2=48.64$ mm, h=1 mm.

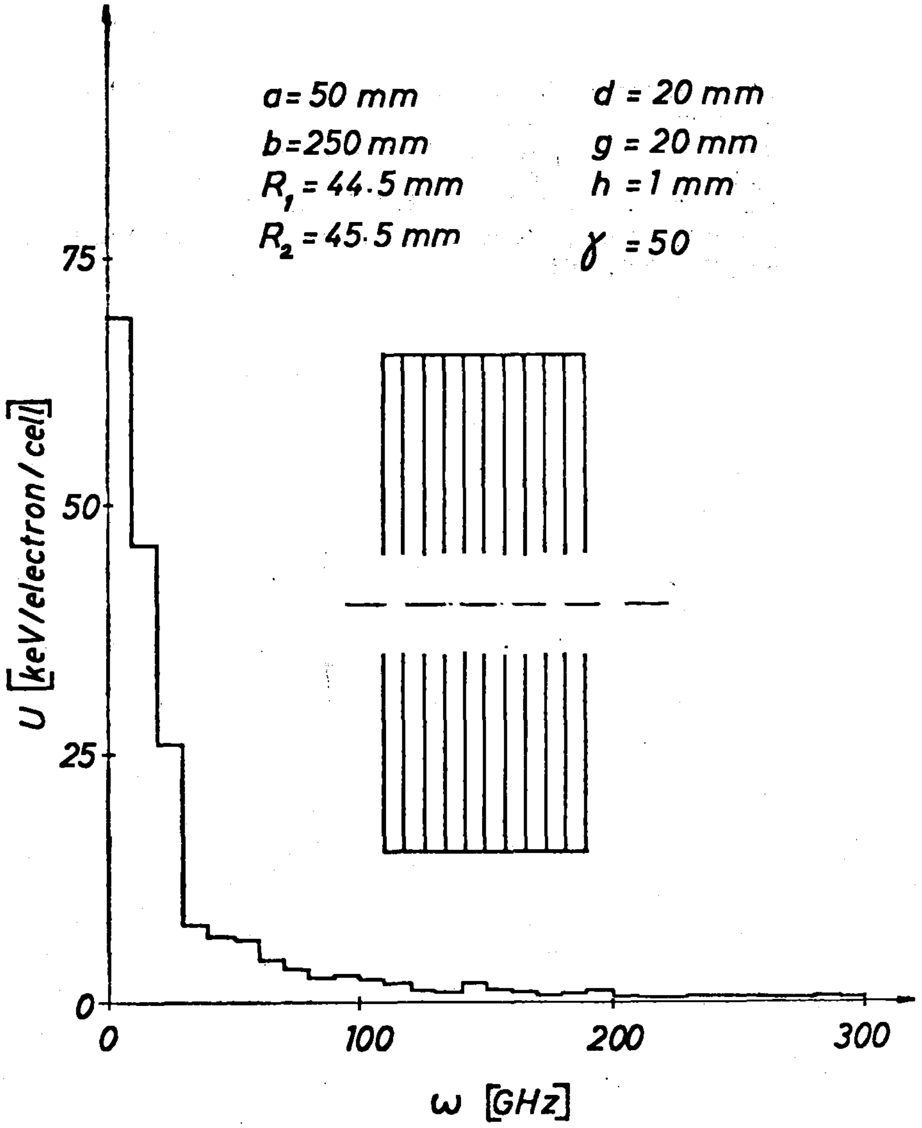

Fig. 3. Energy loss spectrum at $\gamma=50$. Waveguide dimensions $a=50$ mm, $b=250$ mm, $d=g=20$mm; ring dimensions $R_1=44.5$ mm, $R_2=45.5$ mm, $h=1$ mm.

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