Published for SISSA by Springer

Received: *January 14, 2010*
Accepted: *February 19, 2010*
Published: *March 15, 2010*

Observations on arithmetic invariants and U-duality orbits in $\mathcal{N} = 8$ supergravity

Massimo Bianchi,[a,b] **Sergio Ferrara**[a,c,d] **and Renata Kallosh**[e]

[a] *Physics Department, Theory Unit, CERN, CH 1211, Geneva 23, Switzerland*

[b] *Dipartimento di Fisica, Università di Roma "Tor Vergata", I.N.F.N. Sezione di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy*

[c] *INFN — Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy*

[d] *Department of Physics and Astronomy, University of California, Los Angeles, CA U.S.A.*

[e] *Department of Physics, Stanford University, Stanford, CA 94305 U.S.A.*

E-mail: massimo.bianchi@roma2.infn.it, Sergio.Ferrara@cern.ch, kallosh@stanford.edu

Abstract: We establish a relation between time-like, light-like and space-like orbits of the non-compact $E_{7(7)}(\mathbb{R})$ symmetry and discrete $E_{7(7)}(\mathbb{Z})$ invariants. We discuss the U-duality invariant formula for the degeneracy of states $d(\mathcal{Q})$ which in the approximation of large occupation numbers reproduces the Bekenstein-Hawking entropy formula for regular black holes with AdS_2 horizon. We explain why the states belonging to light-like orbits, corresponding to classical solutions of "light black holes with null singularity", decouple from the corresponding index. We also present a separate U-duality invariant formula for the class of light-like orbits specified by discrete $E_{7(7)}(\mathbb{Z})$ invariants. We conclude that the present study of the non-perturbative sector of the theory does not reveal any contradiction with the conjectured all-loop perturbative finiteness of D=4 $\mathcal{N}$=8 supergravity.

Keywords: Black Holes in String Theory, Extended Supersymmetry

ArXiv ePrint: 0912.0057

doi:10.1007/JHEP03(2010)081

JHEP03(2010)081

Contents

1 Introduction

We analyze the non-perturbative completion of D=4, $\mathcal{N}$=8 supergravity following the proposal in [1] with the purpose of understanding the possible implications on the conjectured UV finiteness of the perturbative theory [2–6], based on its relation to $\mathcal{N}$=4 SYM theory [7, 8]. The important conclusion from the analysis in [1] was that the string theory states[1] which do not decouple in $D=4$ [9], when viewed as solutions of classical $\mathcal{N}=8$ supergravity, expose null singularity.[2] These states in non-perturbative $D=4$ $\mathcal{N}=8$ supergravity all have no mass gap, they are light and may become massless at the boundary of the moduli space $E_{7(7)}/\mathrm{SU}(8)$, which lies at infinite distance from any interior point and

[1] In [9] the argument of the non-decoupling of string theory states from $D=4$ $\mathcal{N}=8$ supergravity was based mostly on Kaluza-Klein monopoles which are 1/2 BPS states. Other non-perturbative excitations were identified in [9], which preserve less supersymmetry but they are believed to be less crucial for the non-decoupling argument.

[2] It has been suggested in [10] that the tower of states in [9] may be seen in D=4 via the infinite sum of non-planar Feynman graphs which may have an infinite number of Regge-cuts.

where perturbative $D = 4$ $\mathcal{N} = 8$ supergravity fails to be valid. The area of the horizon of such singular solutions with 1/2, 1/4, 1/8 unbroken supersymmetry is zero due to the vanishing of the quartic Cartan invariant of the $E_{7(7)}$ symmetry, i.e. $\mathcal{I}_4(\mathcal{Q}) = 0$.

The studies in [1] raised the following issue: is it possible that the states discussed in [9] may be consistently excluded from the four-dimensional theory and therefore do not affect the UV properties of $\mathcal{N} = 8$ $D = 4$ supergravity or they can be proven to be necessary in $D = 4$ and therefore will affect the perturbative theory? To distinguish between these two possibilities, it is useful to study the $\mathcal{N} = 8$ BPS partition function. One should try to understand if the U-duality symmetry would be broken if the states with null $E_{7(7)}$ quartic invariant and thus vanishing area of the horizon were left out. If this were the case, it would suggest that the conjectured UV finiteness of perturbative theory may be disproved. On the other hand, if one could show that there are independent U-duality invariant partition functions for $\mathcal{I}_4 = 0$ states and for $\mathcal{I}_4 \neq 0$ states, this might be viewed as further evidence for the conjectured finiteness of $D = 4$ $\mathcal{N} = 8$ supergravity. Since such states form separate orbits in the non-compact $E_{7(7)}$ symmetry one would expect that they should not mix with each other. Still, in order to support this expectation, i.e. the absence of U-duality anomalies, one should explicitly find formulae for the degeneracy of states with $\mathcal{I}_4 = 0$ and with $\mathcal{I}_4 \neq 0$ and show that they are separately U-duality invariant.

There is currently no agreement on whether the analysis of the $E_{7(7)}(\mathbb{Z})$ symmetry of the partition function or appropriately modified Witten indices may really serve as an efficient discriminator between finite and non-finite $D = 4$ $\mathcal{N} = 8$ supergravity. However, since higher loop computations are not likely to help to find the difference, it seems that careful derivation of explicit formulae for the degeneracy of $\mathcal{N} = 8$ states and supersymmetric indices is one way to make progress and eventually, reach some definite conclusion.

The purpose of this paper is therefore to establish the properties of $E_{7(7)}(\mathbb{Z})$ invariant partition functions and appropriately modified Witten indices and explore the role of the $\mathcal{I}_4 = 0$ states in the computation of the degeneracy of states. If we knew how to derive the manifestly invariant $E_{7(7)}(\mathbb{Z})$ partition functions, we could realize the full microscopic physics of M-theory. It is not surprising, therefore, that no such explicit formulae are immediately available. However, certain answers, based on the counting of states in string theory [11–14] where U-duality is first broken to some of its subgroups, and afterwards restored, are available, and we will analyze them. Our analysis will be based on the recent studies in [15–17] where the discrete $E_{7(7)}(\mathbb{Z})$ invariants play a prominent role.

The importance of supersymmetric indices is due to their moduli-independence that allows for a robust counting of particular BPS states. In the simplest cases, Witten indices count the number of bosonic states minus the number of fermionic states with assigned charges $\mathcal{Q}$. The index is defined separately for supermultiplets preserveing specific fractions of the original supersymmetry. For instance, the index B_{14} is computed in [17] for 1/8 BPS states of $\mathcal{N} = 8$ supergravity as the difference between the number of bosonic states and fermionic states of the minimal short multiplet with 28 fermionic zero modes. States with 1/4 unbroken supersymmetry would contribute to a different index, B_{12}, since these states have 24 fermionic zero modes and form shorter multiplet. Finally, 1/2 BPS states contribute to a separate index, B_8, as they break 16 superymmetries and form an ultrashort multiplet.

Therefore the properties of the indices suggest that the $\mathcal{I}_4 = 0$ states which are 1/4 or 1/2 BPS are decoupled from the degeneracy formula where 1/8 BPS states contribute. We will also see how this decoupling is realized when the states have particular values of the arithmetic discrete $E_{7(7)}(\mathbb{Z})$ invariants.

The index B_{14} may include 1/8 BPS states with $\mathcal{I}_4 = 0$ as well as states with $\mathcal{I}_4 \neq 0$, as will be clear from the properties of the discrete $E_{7(7)}(\mathbb{Z})$ invariants. We will find out, however, that $\mathcal{I}_4 = 0$ 1/8 BPS states form a separate U-duality invariant orbit whose degeneracy formula will be given explicitly below.

The paper is organized as follows. In section 2 we describe how to derive the $\mathcal{N}$=8 black hole entropy from microscopic state counting. The characterization of orbits in terms of discrete invariants of $E_{7(7)}(\mathbb{Z})$ is addressed in section 3. In section 4 we review Sen's analysis [15–17] for 1/8 BPS states and show that the formulae for $\mathcal{I}_4 \neq 0$ states and for states with $\mathcal{I}_4 = 0$ are separately U-duality invariant. Degeneracy of 1/2 and 1/4 BPS states admitting a perturbative string description are considered in section 5. In section 6, higher dimensional resolution of the null singularity of 'light black hole' is presented based on the decomposition of the quartic invariant of D=4 symmetry, $E_{7(7)}(\mathbb{Z})$, in terms of the cubic invariant of D=5 symmetry, $E_{6(6)}(\mathbb{Z})$, and in terms of the quadratic invariant of the D=6 U-duality symmetry $O(5,5)(\mathbb{Z})$. In section 7 we compare the relation between perturbative and non-perturbative sectors of $\mathcal{N}$=8 supergravity with that on $\mathcal{N}$=4 Yang Mills theory. Section 8 contains our conclusions and a discussion of our results.

In appendix A we provide some details on the elliptic functions, useful for understanding formulae for the degeneracy of states. In appendix B we provide the relation between the 28 electric and 28 magnetic charges of D=4 $\mathcal{N}$=8 supergravity and states of M-theory compactified on T^7 and 27 electric and 27 magnetic charges of the D=5 $\mathcal{N}$=8 supergravity and states of M-theory compactified on T^6.

2 From degeneracy of states to Bekenstein-Hawking entropy

We start by reviewing the derivation of the 1/8 BPS Bekenstein-Hawking black hole entropy starting with the formula for the degeneracy of states in string theory, following [11–15]. In string theory various computations of the degeneracy of 1/8 BPS states corresponding to a set of black holes charges $\mathcal{Q}$ have been performed. Charges are quantized and are often considered to be 'primitive', i.e. such that their "gcd" (greatest common divisor) equals 1. In the past, the goal of such counting formulae was to derive the U-duality invariant 1/8 BPS black hole entropy formula by means of the microscopic counting of states and provide the quantum corrected formula for the degeneracy of states with small charges, for which the classical black hole approximation fails. A formula for the exact degeneracies of 1/8 BPS black holes was originally derived in [11–14], and is given by

$$d_{\mathcal{N}=8}(\mathcal{I}_4(\mathcal{Q})) = \oint d\tau F(\tau) e^{-2\pi i \tau \, \mathcal{I}_4(\mathcal{Q})} \tag{2.1}$$

where the modular form

$$F(\tau) = \frac{\theta_3(2\tau)}{\eta^6(4\tau)} \tag{2.2}$$

is written in terms of the Jacobi theta function

$$\theta_3(q) = \sum_n q^{n^2/2} \tag{2.3}$$

with $q = e^{2\pi i\tau}$ and

$$\eta(q) = q^{1/24} \prod_n (1 - q^n) \tag{2.4}$$

Dedekind's eta function. In order to determine the asymptotic behavior of $F(\tau)$ it is convenient to perform a modular transformation

$$\eta(-1/\tau) = (i\tau)^{1/2}\eta(\tau) \qquad \theta_3(-1/\tau) = (i\tau)^{1/2}\theta_4(\tau) \tag{2.5}$$

where

$$\theta_4(q) = \sum_n (-)^n q^{n^2/2} \tag{2.6}$$

in this way

$$F(\tau) = \frac{\theta(-1/2\tau)(4i\tau)^3}{(2i\tau)^{1/2}\eta^6(-1/4\tau)} \approx 2^{11/2} i^{5/2} \tau^{5/2} e^{i\pi/8\tau}(1 + (6-2)e^{-i\pi/2\tau} + \ldots) \tag{2.7}$$

The dots represent higher powers of $e^{-\pi i/2\tau}$ in the expansion. In [15] this expression was derived for large values of the quartic invariant and large values of the charges. In such a case, one only considers the leading exponential term and performs the integral over τ using a saddle-point approximation. After taking into account the determinant that comes from the integral around the saddle point and evaluating the determinant (together with the integrand) at the saddle point one finds [15]:

$$d_{\mathcal{N}=8}(\mathcal{I}_4(\mathcal{Q})) \sim (-1)^{\mathcal{I}_4(\mathcal{Q})+1}(\mathcal{I}_4(\mathcal{Q}))^{-2} e^{\pi\sqrt{\mathcal{I}_4(\mathcal{Q})}} \tag{2.8}$$

Thus, for large black hole horizon area $\mathcal{I}_4(\mathcal{Q})$ and large charges $\mathcal{Q}$ the quantum mechanical counting of states in string theory agrees with the Bekenstein-Hawking entropy of the black holes, i.e.

$$d_{\mathcal{N}=8}(\mathcal{I}_4(\mathcal{Q})) \sim e^{S_{BH}(\mathcal{Q})+\ldots} \tag{2.9}$$

where $S_{BH} = \frac{1}{4}A_H = \pi\sqrt{|\mathcal{I}_4|}$ [18] and $\mathcal{I}_4 = q^{abcd}\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_c\mathcal{Q}_d$ is the quartic invariant of E_7 and $\mathcal{Q}_a$ is a 56-dimensional vector of electric and magnetic charges. The expansion is valid for large charges and area of the horizon, where the classical approximation is valid. It may be viewed as a limit when the occupation numbers of the quantum states are large and the system is semi-classical. It is important, however, that for small occupation numbers i.e. small charges and vanishing area, the above expansion it not valid.

To understand the properties of the degeneracy formula for states with small charges and vanishing horizon area $\mathcal{I}_4(\mathcal{Q}) = 0$ of the corresponding "black holes", we have to study a recently developed version of eq. (2.8) presented in [15], where the dependence on discrete $E_{7(7)}(\mathbb{Z})$ invariants is introduced.

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3 Orbits and discrete invariants of $E_{7(7)}(\mathbb{Z})$

Classical $\mathcal{N}=8$ supergravity in $D=4$ enjoys a non-compact $E_{7(7)}(\mathbb{R})$ symmetry. This continuous symmetry is broken down to the discrete subgroup $E_{7(7)}(\mathbb{Z})$, when the electric and magnetic charges of black hole solutions are quantized [19, 20].

Meanwhile, the orbits of exceptional non-compact continuous $E_{7(7)}(\mathbb{R})$ group were studied in [21–23]. These orbits are in close parallel to the orbits of time-like, light-like and space-like vectors in Minkowski space. The difference is that no quadratic norm, analogous to $p^2 = p_\mu p_\nu \eta^{\mu\nu}$ in SO(1, 3), is available for $E_{7(7)}$, there is only a quartic invariant $\mathcal{I}_4(\mathcal{Q})$ depending on 56 charges $\mathcal{Q}$.

The "time-like" orbit in $E_{7(7)}(\mathbb{R})$ corresponds to $\mathcal{I}_4 > 0$, the "light-like" to $\mathcal{I}_4 = 0$, and the "space-like" to $\mathcal{I}_4 < 0$. The "time-like" orbit of $E_{7(7)}(\mathbb{R})$ with $\mathcal{I}_4 > 0$ defines a set of regular black holes with 1/8 unbroken supersymmetry. The corresponding minimally short multiplet is generated by 28 fermionic zero modes of the broken supersymmetry. The "space-like" orbit of $E_{7(7)}(\mathbb{R})$ with $\mathcal{I}_4 < 0$ defines a set of regular extremal non-BPS black holes. The corresponding long multiplet is generated by the 32 fermionic zero modes of the broken supersymmetry. Alternatively, it may be viewed as a bound state of a set of two 1/2 BPS black holes with residual 1/4 unbroken supersymmetry, in such a way that the extra 4 fermion zero modes can be used to form the bound state, as explained in [17].

The light-like orbit with $\mathcal{I}_4 = 0$ presents three distinct cases. The distinction between them is $E_{7(7)}(\mathbb{R})$ invariant. Despite the fact that there is only one available invariant, the extra conditions distinguishing among three types of $\mathcal{I}_4 = 0$ states are invariant: they are specified by the vanishing of particular covariant tensors of $E_{7(7)}(\mathbb{R})$. As we will show below, there is no U-duality transformation which will mix the three cases with $\mathcal{I}_4 = 0$.

Here it is instructive to recall that bilinears of two **56** decomposes according

$$(\mathbf{56} \otimes \mathbf{56})_{\text{Symm}} = \mathbf{133} \oplus \mathbf{1463} \tag{3.1}$$

$$(\mathbf{56} \otimes \mathbf{56})_{\text{Anti}} = \mathbf{1} \oplus \mathbf{1539} \tag{3.2}$$

where **133** is the Adjoint.

The first case in the light-like (double critical) orbit has 1/2 unbroken supersymmetry and the corresponding ultra-short multiplet is generated by the 16 fermionic zero modes of the broken supersymmetry. 1/2 BPS-ness requires

$$\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b} = 0 \tag{3.3}$$

which also implies $\mathcal{I}_4 = 0$ and $\frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a} = 0$. Note that the vanishing of the projection into the Adj, $\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_q \partial \mathcal{Q}_b} = 0$, is an $E_{7(7)}(\mathbb{R})$ invariant requirement.

The second case in the light-like (critical) orbit has 1/4 unbroken supersymmetry and the corresponding 'very' short multiplet is generated by the 24 fermionic zero modes of the broken supersymmetry. 1/4 BPS-ness requires

$$\frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a} = 0\,, \qquad \mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_q \partial \mathcal{Q}_b} \neq 0 \tag{3.4}$$

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which also implies $\mathcal{I}_4 = 0$. Here again, the vanishing of $\frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a}$ is an $E_{7(7)}(\mathbb{R})$ invariant requirement.

The third case in the light-like orbit has 1/8 unbroken supersymmetry and the corresponding minimally short multiplet is generated by the 28 fermionic zero modes of the broken supersymmetry. Light-like 1/8 BPS-ness requires that

$$\mathcal{I}_4 = 0\,, \qquad \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a} \neq 0\,, \qquad \mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b} \neq 0 \tag{3.5}$$

Besides $\mathcal{I}_4$, the reduced discrete $E_{7(7)}(\mathbb{Z})$ symmetry allows for more arithmetic invariants which define the physical states in string theory [16]. A general discussion of discrete invariants based on Jordan algebras and Freudenthal duals is presented in [24, 25]. Here we will use the fact that the tensors, derivatives of the quartic invariant, defining distinct orbits, transform covariantly under $E_{7(7)}(\mathbb{R})$.

In particular, for a (non-compact) discrete symmetry one can introduce the notion of gcd (greatest common divisor) of a finite set of not all zero integers, i.e. the greatest integer that divides them all. By definition a gcd is positive. *The discrete $E_{7(7)}(\mathbb{Z})$ invariants are given by the gcd of certain sets of numbers which correspond to covariant tensors of $E_{7(7)}(\mathbb{R})$.* This is the reason why there is a clear relation between the distinct light-like orbits of $E_{7(7)}(\mathbb{R})$ and discrete $E_{7(7)}(\mathbb{Z})$ invariants.

One can define a number of discrete U-duality invariants besides the quartic invariant $\mathcal{I}_4(\mathcal{Q})$. Introducing $\tilde{\mathcal{Q}}_a = \partial \mathcal{I}_4(\mathcal{Q})/\partial \mathcal{Q}_a$, that are cubic in $\mathcal{Q}_a$, and following [16, 24–26], one has

$$a_1(\mathcal{Q}) = \gcd\{\mathcal{Q}_a\} \tag{3.6}$$

$$a_2(\mathcal{Q}) = \gcd\left\{\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial^2 \mathcal{I}_4(\mathcal{Q})}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b}\right\} \equiv \psi(\mathcal{Q}) \tag{3.7}$$

$$a_3(\mathcal{Q}) = \gcd\{\tilde{\mathcal{Q}}_a\} \tag{3.8}$$

$$a_4(\mathcal{Q}) = \mathcal{I}_4(\mathcal{Q}) \tag{3.9}$$

$$a'_4(\mathcal{Q}) = \gcd\{\mathcal{P}^{[ab]}_{\mathbf{1539}} \mathcal{Q}_a \tilde{\mathcal{Q}}_b\} \equiv \chi(\mathcal{Q}) \tag{3.10}$$

where the subscript denote the order in $\mathcal{Q}_a$, for example a_1 is linear in $\mathcal{Q}_a$, a_3 is cubic in $\mathcal{Q}_a$ etc. The set of discrete invariants presented above contains all the ones which are relevant for our analysis, namely we have a gcd of a set of integer charges, $\gcd\{\mathcal{Q}_a\}$, a gcd of a set of bilinears of these charges, $\psi(\mathcal{Q})$. Next we have a gcd of a set of charges cubic in the original ones, $\gcd\{\tilde{\mathcal{Q}}_a\}$ and a quartic invariant of $E_{7(7)}(\mathbb{R})$, $\mathcal{I}_4(\mathcal{Q})$, which is also a quartic invariant of $E_{7(7)}(\mathbb{Z})$. Finally we have a gcd of a set of charges quartic in $\mathcal{Q}_a$, $\chi(\mathcal{Q})$.

The list of discrete invariants above may be incomplete, for example other terms like $a'_2(\mathcal{Q}) = \gcd\{\mathcal{P}^{(ab)}_{\mathbf{1463}} \frac{\partial^2 \mathcal{I}_4(\mathcal{Q})}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b}\}$ may be studied in this respect. This is beyond the scope of our current work.

For the discrete $E_{7(7)}(\mathbb{Z})$ the quartic invariant is quantized and takes the values [24–26]

$$\mathcal{I}_4(\mathcal{Q}) \in \{0, 1\} \qquad \mod 4 \tag{3.11}$$

$a_1(\mathcal{Q}) = \gcd\{\mathcal{Q}_a\}$ is obviously linear in the charges Q_a, describing the state, while $a_3(\mathcal{Q}) = \gcd\{\tilde{\mathcal{Q}}_a\}$ is cubic in the charges Q_a.

The states with non-vanishing quartic invariants $\mathcal{I}_4(\mathcal{Q}) \neq 0$ which are 1/8 BPS or non-BPS may have all other discrete invariants non-vanishing. This means that they can be classified according to the values of these invariants. For example, one finds that the states in [15] (corresponding to a particular choice of charges $\mathcal{Q}$) are described by $\psi(\mathcal{Q}) = 1$ and generic integer, in fact even, values of $\chi(\mathcal{Q})$.

A more interesting situation occurs if one tries to classify the states with $\mathcal{I}_4(\mathcal{Q}) = 0$ in terms of discrete invariants of $E_{7(7)}(\mathbb{Z})$. Differently from $a_4 = \mathcal{I}_4(\mathcal{Q})$, which is invariant even under $E_{7(7)}(\mathbb{R})$ transformations, all other invariants are gcd's of certain sets of numbers which correspond to some covariant tensors of $E_{7(7)}(\mathbb{R})$. Prior to taking their gcd, these numbers would not even be invariant under $E_{7(7)}(\mathbb{Z})$. *But all* gcd *are positive by definition*, they never vanish (it does not make sense to divide a set of numbers by zero). Note that the only $E_{7(7)}(\mathbb{Z})$ invariant which can take zero value is $\mathcal{I}_4(\mathcal{Q})$ since it is not a gcd but a specific quartic function of integer charges $\mathcal{Q}$. It follows that:

All *1/2 BPS states with* $\mathcal{I}_4(\mathcal{Q}) = 0$ *and* $\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial^2 \mathcal{I}_4(\mathcal{Q})}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b} = 0$ *are excluded from the classification in terms of discrete* gcd *invariants (3.7) and (3.10)*. The condition $\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial^2 \mathcal{I}_4(\mathcal{Q})}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b} = 0$ contradicts the positivity of $\psi(\mathcal{Q})$. For 1/2 BPS states only $a_2(\mathcal{Q}) \neq 0$.

All *1/4 BPS states with* $\mathcal{I}_4(\mathcal{Q}) = 0$ *and* $\tilde{\mathcal{Q}}_a = \frac{\partial \mathcal{I}_4}{\partial \mathcal{Q}_a} = 0$, $\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial^2 \mathcal{I}_4(\mathcal{Q})}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b} \neq 0$ *are excluded from the classification in terms of discrete* gcd *invariants (3.10)*. The condition $\tilde{\mathcal{Q}}_a = 0$ contradicts the positivity of $\chi(\mathcal{Q})$. For 1/4 BPS states only $a_2(\mathcal{Q}) \neq 0$ and $a_2'(\mathcal{Q}) \neq 0$.

Not all *1/8 BPS states with* $\mathcal{I}_4(\mathcal{Q}) = 0$ *and* $\tilde{\mathcal{Q}}_a \neq 0$, $\mathcal{P}^{(ab)}_{\mathbf{133}} \frac{\partial^2 \mathcal{I}_4(\mathcal{Q})}{\partial \mathcal{Q}_a \partial \mathcal{Q}_b} \neq 0$ *are excluded from the classification in terms of discrete* gcd *invariants* since all positive discrete invariants are still compatible with $\mathcal{I}_4(\mathcal{Q}) = 0$. These states form a class of degenerate orbits, consistent with the arithmetic classification of states. They are described in details in section 4.5 of [24] where the integral version of Freudenthal's construction is developed.

Let us stress here that our conclusion on the decoupling of 1/2 and 1/4 BPS states from the classification using discrete $E_{7(7)}(\mathbb{Z})$ invariants is in a complete agreement with the definition of the supersymmetric index B_{14} which is computed in [15]. The corresponding index is U-duality invariant, according to [15]. The states with 1/4 unbroken supersymmetry would contribute to a different index, B_{12}, and the 1/2 BPS states would contribute to a separate index B_8.

4 Sen's formulae for the degeneracy of $\mathcal{N} = 8$ BPS states

An exact formula for the degeneracy of $\mathcal{N} = 8$ states in string theory was derived in [15–17]. The leading contribution for large areas and charges reproduces the black hole entropy formula (2.9). The exponentially subleading contribution depends on discrete $E_{7(7)}(Z)$ invariants specifying the state.

The formula for the degeneracy of states has been derived in [15–17] in the framework of type IIB string theory compactified on a $T^6 = T^4 \times S^1 \times \tilde{S}^1$. The states are described by a system of D5/D3/D1 branes wrapping 4/2/0 cycles of a T^4 times S^1 or $\tilde{S}^1$. Alternatively, they can be described in type IIA theory as states in the NS-NS sector.

In the context of string theory, it is useful to consider the split of U-duality into S-duality and T-duality subgroups $E_{7(7)}(\mathbb{Z}) \supset \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SO}(6,6;\mathbb{Z})$. In the decomposition $E_{7(7)} \to \mathrm{SO}(6,6) \times \mathrm{SL}(2)$ one has $\mathbf{56} \to (\mathbf{12},\mathbf{2}) + (\mathbf{32},\mathbf{1})$ i.e. $\mathcal{Q}_a \to (P_i, Q^i; S^\alpha)$.

In [15–17] only $(\mathbf{8},\mathbf{2})$ out of the $(\mathbf{12},\mathbf{2})$ charges Q_i, P^i are taken into account, whereas the 32 S_α (corresponding to the R-R sector in the 'standard' T-duality basis but not in the basis chosen in [15–17]) are set to zero. As a result, the string theory method of state counting [15–17] has a manifest $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(4,4;\mathbb{Z})$ subgroup of the U-duality symmetry.

The discrete invariants in the U-duality subgroup are defined in [15–17] as follows:

$$\ell_1 = gcd\{Q_iP_j - Q_jP_i\}, \ell_2 = gcd\{Q^2/2, P^2/2, Q\cdot P\} \tag{4.1}$$

The degeneracy formula is known [15–17] for the subset of primitive charge vectors with $a_1(Q) = 1$ for which also

$$\mathrm{gcd}(\ell_1, \ell_2) = 1 \tag{4.2}$$

For primitive charge vectors satisfying this condition the degeneracy is given by

$$d^{\mathcal{N}=8}_{1/8BPS}(Q,P) = (-)^{I_4(Q,P)+1} \sum_{s\in\mathbb{Z}, s|\ell_1,\ell_2} s\hat{c}(\mathcal{I}_4(Q,P)/s^2) \tag{4.3}$$

Any generalization would necessarily put the generators of $\mathrm{SO}(6,6)$ (i.e. $\{Q_iP_j - Q_jP_i + S\gamma_{ij}S\}$ in the $(\mathbf{66},\mathbf{1})$) and those of $\mathrm{SL}(2)$ (i.e. $\{Q^2/2, P^2/2, Q\cdot P\}$ in the $(\mathbf{1},\mathbf{3})$) on the same ground as the remaining $(\mathbf{32},\mathbf{2})$ ones $\{P_i\gamma^iS, Q_i\gamma^iS\}$ in $E_{7(7)}/\mathrm{SO}(6,6)\times\mathrm{SL}(2)$, so as to form the full Adj of $E_{7(7)}$ and define

$$\ell_E = \mathrm{gcd}\{P^{ab}_{\mathbf{133}}\mathcal{Q}_a\mathcal{Q}_b\} \tag{4.4}$$

The U-duality invariant generalization of the constraint on charges (4.2) is the requirement that

$$\psi(\mathcal{Q}) = \ell_E = \mathrm{gcd}\{\mathcal{P}^{ab}_{\mathbf{133}}\mathcal{Q}_a\mathcal{Q}_b\} = 1 \tag{4.5}$$

The physical states in the counting formula in [15–17] in addition to satisfying the constraint that $\psi(\mathcal{Q}) = 1$, may differ in the value of the discrete U-duality invariant

$$\chi(\mathcal{Q}) = \tilde{\ell}_E = \mathrm{gcd}\{\mathcal{P}^{[ab]}_{\mathbf{1539}}\mathcal{Q}_a\tilde{\mathcal{Q}}_b\} \tag{4.6}$$

where $\tilde{\mathcal{Q}}_b = \partial\mathcal{I}_4/\partial\mathcal{Q}^b$ are the $\mathbf{56}$ 'dual' charges, cubic in the fundamental ones, introduced before. For the models considered in [15–17] the discrete invariant $\chi(Q,P)$ is even, see for example, eq. (24) in [16]. An additional requirement on the Sen's derivation is that it should be possible to rotate the charges to lie inside the $(\mathbf{12},\mathbf{2})$ subspace.

The formula for the degeneracy of states with $a_1(\mathcal{Q}) = 1$, $a_2(\mathcal{Q}) = \psi(\mathcal{Q}) = 1$ and even $\chi(\mathcal{Q})$ can be written as follows[3]

$$d^{\mathcal{N}=8}_{1/8BPS}(\mathcal{Q}) = (-)^{I_4(\mathcal{Q})+1} \sum_{s\in\mathbb{Z}, 2s|\chi(\mathcal{Q})} s\hat{c}(\mathcal{I}_4(\mathcal{Q})/s^2) \tag{4.7}$$

[3]Note that $(-1)^{Q\cdot P}$ can be written as $(-1)^{I_4(\mathcal{Q})}$ since $I_4(\mathcal{Q})$ is odd (even) when $Q\cdot P$ is odd (even) as explained in [15].

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where $\hat{c}(n) \approx (-)^{n+1} \exp(\pi\sqrt{n})/n^2$ is related to the Fourier coefficients in the expansion

$$\frac{\vartheta_1^2(z|\tau)}{\eta^6(\tau)} = \sum_{k,l} \hat{c}(4k - l^2) e^{2\pi i(k\tau + lz)} \tag{4.8}$$

As a result the degeneracy formula (4.7) is manifestly U-duality invariant.

4.1 Degeneracy formula for 1/8 BPS states with $I_4(\mathcal{Q}) \neq 0$

First we consider the left hand side of the degeneracy formula (4.7), only derived in [15] for states with $\psi(\mathcal{Q}) = 1$ and $I_4(\mathcal{Q}) \neq 0$.

On the right hand side we find an expression of the form

$$\begin{aligned} &d(\mathcal{Q})|^{\mathcal{N}=8}_{I_4(\mathcal{Q})\neq 0} = \\ &(-)^{I_4(\mathcal{Q})+1} \sum_{s\in\mathbb{Z}, 2s|\chi(\mathcal{Q})} s\hat{c}(\mathcal{I}_4(\mathcal{Q})/s^2)|_{I_4(\mathcal{Q})\neq 0} \end{aligned} \tag{4.9}$$

It is manifestly U-duality invariant since it depends on the quartic invariant and a discrete invariant $\chi(\mathcal{Q})$ of $E_{7(7)}(\mathbb{Z})$. For large $\mathcal{I}_4$ the $s = 1$ term dominates and the Bekenstein-Hawking formula for the black hole entropy which depends only on the quartic invariant of $E_{7(7)}(\mathbb{Z})$ is reproduced, as shown in eq. (2.9). When $\mathcal{I}_4(\mathcal{Q})$ is not large, the formula (4.9) is valid and the answer depends on both non-vanishing $\mathcal{I}_4(\mathcal{Q})$ and $\chi(\mathcal{Q})$.

4.2 Degeneracy formula for 1/8 BPS states with $I_4(\mathcal{Q}) = 0$

Here we consider the left hand side of the degeneracy formula (4.7) only for states with $I_4(\mathcal{Q}) = 0$ and $\psi(\mathcal{Q}) = 1$ and some values of the discrete invariant $\chi(\mathcal{Q})$. For 1/8 BPS states with $I_4(\mathcal{Q}) = 0$ we find a simple answer

$$d(\mathcal{Q})|^{\mathcal{N}=8}_{I_4(\mathcal{Q})=0} = \sum_{s\in\mathbb{Z}, 2s|\chi(\mathcal{Q})} 2s = N[\chi(\mathcal{Q})] \tag{4.10}$$

where we used the fact that $\hat{c}(0) = -2$ that can be proven as follows. The elliptic genus

$$\mathcal{E}(z,\tau) = \frac{\theta_1(z|\tau)^2}{\eta^6(\tau)} = \sum_{k,l} \hat{c}(4k - l^2) e^{2\pi i(k\tau + lz)} \tag{4.11}$$

can be written more explicitly using the product expansions

$$\theta_1(z|\tau) = 2iq^{1/8} \sin(\pi z) \prod_n (1 - q^n)(1 - yq^n)(1 - y^{-1}q^n) \tag{4.12}$$

with $y = \exp(2\pi i z)$. Setting $s_z = \sin(\pi z)$ one finds that

$$\mathcal{E} = -4s_z^2 \prod_n \left(1 + \frac{4s_z^2 q^n}{(1 - q^n)^2}\right)^2 \tag{4.13}$$

expanding in powers of q one finds

$$\mathcal{E} = -4s_z^2(1 + 4s_z^2 q + q^2[12s_z^2 + 16s_z^4] + q^3[32s_z^2 + 64s_z^4] + \ldots) \tag{4.14}$$

since $\hat{c}(n)$ is only a function of $n = 4k - \ell^2$ one can read $\hat{c}(0)$ from the term with q^0 $(k = 0)$ and $\ell = 0$. Using $s_z^2 = (1 - \cos(2\pi i z))/2$ one can thus confirm that $\hat{c}(0) = -2$.

The degeneracy formula is U-duality invariant, it depends on the value of the discrete invariant $\chi(\mathcal{Q})$, since the integer $N[\chi(\mathcal{Q})]$ is given by the sum over s which has to be taken over only those integers which are factors of $\chi(\mathcal{Q})/2$. This is indicated by the symbol $2s|\chi(\mathcal{Q})$ in the sum. Thus for example if $\chi(\mathcal{Q}) = 2$, then only the $s = 1$ term contributes. So the sum, $N[\chi(\mathcal{Q})]$, is always finite for a given charge vector $\mathcal{Q}$, but the actual result of the sum depends non-trivially on the arithmetic properties of the charge vector, e.g. divisibility of $\chi(\mathcal{Q})$.

Note that had one first considered the semi-classical approximation of large charges and occupation numbers of particles and large area of the horizon of the regular black holes, the answer would have been $\chi(\mathcal{Q})$ independent as shown in eq. (2.9). Instead, for degenerate orbits, corresponding to singular classical solutions, one should perform a fully quantum mechanical analysis in order to derive the correct degeneracy of such states. One should insert the vanishing value of the quartic invariant directly in the exact formula for the degeneracy of states (4.7). This leads to the simple $E_{7(7)}(\mathbb{Z})$ invariant result shown in eq. (4.10).

5 Counting 1/2 and 1/4 BPS states in perturbative Type II superstrings

The purpose of this section is to illustrate the fact, already discussed above, that the U-duality modular invariant partition function for 1/2 and 1/4 BPS states of $\mathcal{N} = 8$ string theory do not mix with the one for 1/8 BPS states. Moreover in some case, such perturbative BPS states can be interpreted as BH's [27].

After toroidal compactification, the perturbative spectrum of Type II superstrings contains massless, 1/2 BPS, 1/4 BPS, and long multiplets but NO 1/8 BPS ones. The reason is that one needs either R-R charges or K-K monopoles or NS5-branes ('H-monopoles') to be added to perturbative states of type II string theory to get 1/8 BPS states.

5.1 1/2 BPS in $\mathcal{N} = 8$

In $D = 10$, as a result of the GSO projection, the one-loop partition function for Type IIB reads [28]

$$\mathcal{Z} = \frac{|\theta_3^4 - \theta_4^4 - \theta_2^4 - \theta_1^4|^2}{4|\eta^{12}|^2} \tag{5.1}$$

where $\eta(q)$ is a Dedekind's function (2.3) and θ_α with $\alpha = 1, 2, 3, 4$ are Jacobi elliptic functions, see appendix for the details. The partition function vanishes thanks to Jacobi's identity,[4] which accounts for supersymmetry. Modular invariance results after inclusion of the bosonic and (super)ghost zero-modes, producing a factor $V/Im\tau^4$, that nicely combines with the modular invariant measure $d^2\tau/Im\tau^2$. The partition function can be expressed in terms of the characters of the SO(8) current algebra (Little Group for massless states in

[4] *Aequatio identica satis abstrusa.*

$D = 10$) at level $\kappa = 1$ (denoted as V_8, S_8, C_8, O_8 for vector, spinor, co-spinor and singlet conjugacy classes)

$$\mathcal{Z} = \frac{|\mathcal{Q}|^2}{|\eta^8|^2} \tag{5.2}$$

where $\mathcal{Q} = V_8 - S_8 = (\mathbf{8}_v - \mathbf{8}_s)q^{1/3} + massive$ is the super-character introduced in [29–31].

After toroidal compactification, the one-loop partition function reads

$$\mathcal{Z} = \sum_{\mathbf{m},\mathbf{n}} q^{\frac{\alpha'}{4}\mathbf{p}_L^2} \bar{q}^{\frac{\alpha'}{4}\mathbf{p}_R^2} \frac{|\theta_3^4 - \theta_4^4 - \theta_2^4 - \theta_1^4|^2}{4|\eta^{12}|^2} \tag{5.3}$$

where

$$\mathbf{p}_{L/R} = (E^t)^{-1}(\mathbf{m} + B\mathbf{n}) \pm \frac{1}{\alpha'} E\mathbf{n} \tag{5.4}$$

with $E_i^{\hat{i}}$ the 6-bein for the metric $G_{ij} = \delta_{\hat{i}\hat{j}} E_i^{\hat{i}} E_j^{\hat{j}}$ and B_{ij} the anti-symmetric tensor, e.g. $G_{ij} = R^2 \delta_{ij}$ and $B_{ij} = 0$ for a square torus. In this approximation one can account only for the dependence on 36 moduli fields in the NS-NS sector, plus dilaton and axion. Perturbative computations are insensitive to the 32 (pseudo)scalars in the R-R sector. At any rate, degeneracy formulae should be independent of continuous moduli and should be valid anywhere in the interior of moduli space, since no jumping is possible in $\mathcal{N} = 8$.

The 256 massless states correspond to taking the ground states (neither oscillators nor generalized momenta) for both Left and Right movers

$$\mathcal{Z}_{m=0} = (\mathbf{8}_v - \mathbf{8}_s)(\mathbf{8}_v - \mathbf{8}_s) = 128_B - 128_F \tag{5.5}$$

the minus sign accounts for the different statistic of bosons and fermions i.e. $\mathcal{Z}$ is rather a Witten index $\mathcal{I}_W = tr(-)^F (q\bar{q})^H$ than a genuine partition function.

1/2 BPS correspond to excitations of the ground states with only generalized momenta i.e. no oscillators

$$\mathcal{Z}_{1/2BPS} = (\mathbf{8}_v - \mathbf{8}_s)(\mathbf{8}_v - \mathbf{8}_s) \sum_{\mathbf{m},\mathbf{n}} q^{\frac{\alpha'}{4}\mathbf{p}_L^2} \bar{q}^{\frac{\alpha'}{4}\mathbf{p}_R^2} \tag{5.6}$$

and the level matching (i.e. only states with the same power of q and $\bar{q}$ are physical ones) requires $\mathbf{p}_L^2 = \mathbf{p}_R^2$ i.e. $\mathbf{m} \cdot \mathbf{n} = 0$. For each internal direction, only KK momentum or winding but not both are allowed. The degeneracy of such states is thus

$$d_{1/2BPS}^{\mathcal{N}=8}(\mathbf{m}, \mathbf{n}) = 1 \tag{5.7}$$

for given charges such that $\mathbf{m} \cdot \mathbf{n} = 0$. Indeed for any choice of $\mathbf{m}$ and $\mathbf{n} = 0$ such that $\mathbf{m} \cdot \mathbf{n} = 0$ there is only one KK multiplet. Actually the self-conjugate 1/2 BPS multiplet is obtained combining the complex conjugate multiplets associated to $(\mathbf{m}, \mathbf{n})$ and $(-\mathbf{m}, -\mathbf{n})$ (very much as for $W^\pm$ in SYM). By U-duality one expects that the same applies to all 1/2 BPS states, which should include also wrapped branes and KK monopoles.

The structure of 1/2 BPS multiplets is very simple. The spin of the states runs from $S = 0$ to $S = 2$. The multiplicity of the various spins are given by representations of Sp(8) that rotates the 8 real supercharges acting as raising (and as many as lowering) operators in the multiplet. One indeed finds

$$\mathbf{1}(S=2)+\mathbf{8}(S=3/2)+\mathbf{27}(S=1)+\mathbf{48}(S=1/2)+\mathbf{42}(S=0) \tag{5.8}$$

and it is easy to check that the total number of states is 256, i.e. 128 bosons and 128 fermions (up to the doubling mentioned above in order to make the multiplet self-conjugate)

5.2 1/4 BPS in $\mathcal{N}=8$

1/4 BPS states correspond to excitations of the Left (or Right) mover ground states with generalized momenta AND Right (or Left) mover oscillators.

The structure of 1/4 BPS multiplets is less simple than for 1/2 BPS ones. The spin of the states runs from $S=s_{\min}$ to $S=3+s_{\min}$. The multiplicity of the various spins are given by representations of Sp(12) that rotates the 12 real supercharges acting as raising (and as many as lowering) operators in the multiplet. For the simplest case, $s_{\min}=0$, $s_{\max}=3$, one indeed finds

$$\mathbf{1}(S{=}3)+\mathbf{12}(S{=}5/2)+\mathbf{65}(S{=}2)+\mathbf{208}(S{=}3/2)+\mathbf{429}(S{=}1)+\mathbf{572}(S{=}3/2)+\mathbf{429}(S{=}0) \tag{5.9}$$

and it is easy to check that the total number of states is $4096=2^{12}$, i.e. $2048=2^{11}$ bosons and $2048=2^{11}$ fermions (up to the doubling mentioned above in order to make the multiplet self-conjugate)

Let us consider the case with Left movers in the ground state, see also [28]

$$\mathcal{Z}^{L}_{1/4BPS}=(\mathbf{8}_v-\mathbf{8}_s)\frac{\theta_3^4-\theta_4^4-\theta_2^4-\theta_1^4}{2\eta^{12}}\sum_{\mathbf{m},\mathbf{n}}q^{\frac{\alpha'}{4}\mathbf{p}_L^2}\bar{q}^{\frac{\alpha'}{4}\mathbf{p}_R^2} \tag{5.10}$$

level matching (i.e. only states with the same power of q and $\bar{q}$ are physical ones) in this case does not requires $\mathbf{p}^2_{_L}=\mathbf{p}^2_{_R}$ but rather $\alpha'\mathbf{p}^2_{_L}=\alpha'\mathbf{p}^2_{_R}+4\hat{N}_R$ i.e. $\mathbf{m}\cdot\mathbf{n}=\hat{N}_R$, where $\hat{N}_R\geq 1$ is the total oscillator number with respect to the Right mover ground state. A similar expression can be found for $\mathcal{Z}^R_{1/4BPS}$ after exchanging Left and Right movers. Actually the self-conjugate 1/4 BPS multiplets (with R-mover oscillator modes) is obtained combining the complex conjugate multiplets associated to $(\mathbf{m},\mathbf{n})$ and $(-\mathbf{m},-\mathbf{n})$ both with $\mathbf{m}\cdot\mathbf{n}=\hat{N}_R\geq 1$. On the other hand the self-conjugate 1/4 BPS multiplets (with L-mover oscillator modes) is obtained combining the complex conjugate multiplets associated to $(\mathbf{m},-\mathbf{n})$ and $(-\mathbf{m},\mathbf{n})$ both with $-\mathbf{m}\cdot\mathbf{n}=\hat{N}_L\geq 1$. For each choice of $(\mathbf{m},\mathbf{n})$ compatible with the level matching (i.e. only states with the same power of q and $\bar{q}$ are physical ones)condition there is only one state in the lattice sum. The degeneracy comes from the oscillator modes.

At first look $\mathcal{Z}^{L/R}_{1/4BPS}$ vanish faster than the manifest $\mathbf{8}_v-\mathbf{8}_s$ factor would suggest on account of the extra broken supersymmetry (24 out of 32). In order to by-pass the problem and count states instead of computing an index, one can simply 'twist' back (multiplying states by $(-)^F$) the above expression in the Right (or Left) moving sector and simply as well as correctly get

$$\hat{\mathcal{Z}}^{L}_{1/4BPS}=(\mathbf{8}_v+\mathbf{8}_s)\frac{\theta_3^4-\theta_4^4+\theta_2^4+\theta_1^4}{2\eta^{12}}\sum_{\mathbf{m},\mathbf{n}}q^{\frac{\alpha'}{4}\mathbf{p}_L^2}\bar{q}^{\frac{\alpha'}{4}\mathbf{p}_R^2} \tag{5.11}$$

One would have expected to arrive at the same conclusion using the appropriate helicity supertrace formula, but B_{12} vanishes in this case, due to extra fermionic zero-modes, see the discussion [32].

Using Jacobi's *Aequatio* the 'twisted' index, which is the genuine partition function, can be rewritten as

$$\hat{\mathcal{Z}}^L_{1/4BPS} = (\mathbf{8}_v + \mathbf{8}_s)\frac{\theta_2^4}{2\eta^{12}}\sum_{\mathbf{m},\mathbf{n}} q^{\frac{\alpha'}{4}\mathbf{p}_L^2}\bar{q}^{\frac{\alpha'}{4}\mathbf{p}_R^2} \tag{5.12}$$

that allows to extract the exact degeneracy of perturbative 1/4 BPS states in Type II theories. It is amusing to see that for the first excited level corresponding to $\mathbf{m}\cdot\mathbf{n} = 1$ one has indeed 4096 states as required for a 1/4 BPS multiplet with spin ranging from $s_{\min} = 0$, $s_{\max} = 3$. The precise field content can be obtained by decomposing the product

$$(\mathbf{8}_v - \mathbf{8}_s)^L_0 \times [(\mathbf{8}_v - \mathbf{8}_c)_1 \times (\mathbf{8}_v - \mathbf{8}_s)_0]^R \quad , \tag{5.13}$$

where the subscript 0 stands for the (L- and R-mover) ground states and 1 for the R-mover oscillators, into representations of SO(2) × SO(6) and then lifting SO(2) (Little group of *massless* particles in $D = 4$, i.e. helicity) to SO(3) (Little group of *massive* particles in $D = 4$ i.e. spin) e.g. for the highest spin state with $S = 3$ one has $3^+ + 2^+ + 1^+ + 0 + 1^- + 2^- + 3^- \to (S = 3)$ and similarly for lower spins.

Using the asymptotic growth of the degeneracy of oscillator states at (large) level $\hat{N}$ one finds

$$d_{1/4}(\mathbf{m},\mathbf{n}) \approx \exp(2\pi\sqrt{2\mathbf{m}\cdot\mathbf{n}}) \tag{5.14}$$

that would suggest an entropy/area

$$S^{1/4BPS}_{BH} = 2\pi\sqrt{2\mathbf{m}\cdot\mathbf{n}} \tag{5.15}$$

This formula is not U-duality invariant since the lowest Cartan invariant $\mathcal{I}_4$ is quartic. This is no surprise. In order to derive it we had to twist the partition function (Witten index) by an operator that acts non-trivially on R-R charges. Thus although any 1/4 BPS state with two charges is U-duality equivalent to a perturbative state with momentum and winding such that $\mathbf{m}\cdot\mathbf{n} \neq 0$, one cannot immediately extend to non-perturbative states the above perturbative degeneracy formula in this 1/4 BPS case. See also the discussion [33].

In particular, if we blindly accept the above derivation of the U-duality non-invariant entropy of the 1/4 BPS black holes, we seem to have a contradiction with the fact that corrections, that are quadratic in the charges and stretch the horizon, are absent in D=4 $\mathcal{N} = 8$ supergravity. This is based on the fact that there are no R^2 corrections in D=4 perturbative supergravity. Equivalently, when IIB is reduced on T^6 and not on $K3 \times T^2$ there are no such corrections to the entropy (as different from $\mathcal{N} = 4$ supergravity where such corrections are present and are known [34, 35]).

6 Resolution in $D > 4$ of singularities of BPS states with $I_4(\mathcal{Q}) = 0$

D=4 $\mathcal{N} = 8$ supergravity has 256 massless states which originate from dimensional reduction of D=11 supergravity on T^7. To study the non-perturbative states beyond the massless 256 we considered extremal black holes in D=4 $\mathcal{N}=8$ supergravity in [1] and their masses. They form five distinct orbits of $E_{7(7)}$ defined by the properties of the quartic invariant $\mathcal{I}_4(\mathcal{Q})$ and its derivatives with respect to the 56 charges $\mathcal{Q}$. Namely, there are two orbits with $\mathcal{I}_4>0$ and $\mathcal{I}_4<0$, the corresponding states have a Planck scale mass gap and never become massless. There are three type of orbits with $\mathcal{I}_4(\mathcal{Q})=0$ describing 1/8, 1/4, 1/2 BPS states which may become massless at the boundary[5] of the $E_{7(7)}/\mathrm{SU}(8)$ moduli space of D=4 $\mathcal{N}=8$ supergravity. All such states correspond to classical solutions with null singularities in D=4. We will study these states by uplifting them to higher dimensions.

The uplifting from D=4 to D=5 may be viewed as a decomposition of the U-duality group according to [21–23, 36]

$$E_{7(7)} \to E_{6(6)} \times O(1,1) \tag{6.1}$$

$$\mathbf{56} \to \mathbf{27}_1 + \mathbf{1}_{-3} + \mathbf{27}'_{-1} + \mathbf{1}_3 \tag{6.2}$$

where in D=5 only $E_{6(6)}$ is a symmetry. The $E_{6(6)}$ singlets are p^0, q_0 and the pair of $\mathbf{27}$'s is p^A, q_A with $A=1,\ldots,27$. In order to explore the fate of the singularity of all extremal D=4 black holes, when they are uplifted to D=5, it is useful to exploit the $E_{7(7)}$ decomposition of the D=4 black hole area of the horizon in terms of the D=5 area of the horizon preserving the $E_{6(6)}$ symmetry. This relation is known [14, 22, 23];

$$\mathcal{I}_4(\mathcal{Q}) = -(p^0 q_0 + p^A q_A)^2 + 4\left[p^0 \mathcal{I}_3(q_A) - q_0 \mathcal{I}_3(p^A) + \frac{\partial \mathcal{I}_3(q_A)}{\partial q_A}\frac{\partial \mathcal{I}_3(p^A)}{\partial p^A}\right] \tag{6.3}$$

The extremal electric black holes and magnetic strings in D=5 have horizon area related to the cubic invariant of the U-duality $E_{6(6)}$ symmetry [21, 37]

$$\mathcal{I}_3(q_A) = d^{ABC} q_A q_B q_C \qquad \mathcal{I}_3(p^A) = d_{ABC} p^A p^B p^C \tag{6.4}$$

The three distinct orbits in $E_{6(6)}$ describing the properties of D=5 black holes were classified in [21–23]. All regular horizon extremal black holes in D=5 have a non-vanishing cubic invariant

$$\mathcal{I}_3(q_A) \neq 0 \quad \Rightarrow 1/8\mathrm{BPS} \tag{6.5}$$

The singular solutions are either 1/4 or 1/2 BPS

$$\mathcal{I}_3(q_A) = 0 \quad \frac{\partial \mathcal{I}_3}{\partial q_A} \neq 0 \quad \Rightarrow 1/4\mathrm{BPS} \tag{6.6}$$

$$\frac{\partial \mathcal{I}_3}{\partial q_A} = 0 \quad \Rightarrow 1/2\mathrm{BPS} \tag{6.7}$$

in the second case, $\mathcal{I}_3(q_A)=0$ follows immediately from the 1/2 BPS condition. Regular 1/8 BPS D=4 black holes with $\mathcal{I}_4 \neq 0$ and singular 1/8 BPS D=4 black holes with $\mathcal{I}_4=0$ in D=5 are described by the 1/8 BPS regular black holes with $\mathcal{I}_3(q_A)\neq 0$ or black strings with $\mathcal{I}_3(p^A)\neq 0$.

[5]The boundary itself lies at infinite distance from any interior point of the $E_{7(7)}/\mathrm{SU}(8)$ moduli space.

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Thus the uplifting to D=5 removes the null singularity from all 1/8 BPS extremal black holes in D=4 and makes them regular extremal black holes in D=5. Both regular and singular 1/8 BPS black holes in D=4 become regular 1/8 BPS solutions in D=5. This is in agreement with the fact that the $E_{6(6)}$ orbit with $\mathcal{I}_3(q_A) \neq 0$ in D=5 splits into two distinct $E_{7(7)}$ orbits in D=4, one with $\mathcal{I}_4 \neq 0$ and one with $\mathcal{I}_4 = 0$.

A prototypical example in D=5 is a black hole depending on 27 electric charges with $\mathcal{I}_3 \neq 0$ and $q_0, p^A = 0$. In D=4 the quartic invariant in such case is $\mathcal{I}_4(\mathcal{Q}) = 4p^0\mathcal{I}_3(q_A)$. It may be either zero, for $p^0 = 0$, or non-zero, for $p^0 \neq 0$. However, in D=5 both choices correspond to regular black holes with $AdS_2 \times S^3$ geometry near the horizon, whose area is proportional to $\sqrt{\mathcal{I}_3(q_A)}$.

The higher dimensional resolution of black hole singularities was studied in the past, particularly in [38]. It was shown there that the singularity in the solutions of Einstein theory with scalars in D=4 may be sometimes resolved when the solution is viewed as a solution of Einstein theory in higher dimensions. In particular, some extremal black holes with null singularities become solutions with regular horizon in higher dimensional space-times.

Thus we find that all singular 1/8 solutions, which are massless at the boundary of the moduli space in D=4, become regular black holes with singularity covered by a horizon in D=5. This means that one can cure all singular 1/8 BPS solutions by uplifting them to D=5. In this way, however, one would loose the nice UV properties of the perturbative D=4 $\mathcal{N} = 8$ supergravity. Indeed, D=5 $\mathcal{N} = 8$ supergravity is expected to be UV divergent, whereas D=4 $\mathcal{N} = 8$ supergravity may turn out to be all-loop UV finite.

Now we consider singular 1/4 BPS solutions in D=4. In D=5 they are still singular: the cubic $E_{6(6)}$ U-duality invariant vanishes, $\mathcal{I}_3(q_A) = 0$. To study the uplifting of $\mathcal{I}_3 = 0$ black holes to D=6 with U-duality group $O(5,5)$ we consider the following decomposition [21, 39] $E_{6(6)}$

$$E_{6(6)} \rightarrow O(5,5) \times O(1,1) \tag{6.8}$$

The **27** charges split as follows $\mathbf{27} \rightarrow \mathbf{1}_4 + \mathbf{10}_{-2} + \mathbf{16}_1$, corresponding to

$$p^A = (p^z, p^r, p^\alpha) \qquad q_A = (q_z, q_r, q_\alpha) \qquad r = 1, \ldots, 10 \quad \alpha = 1, \ldots, 16. \tag{6.9}$$

Using this splitting, the cubic invariant in D=5 can be related to quadratic invariants in D=6 as follows [21, 40]

$$\mathcal{I}_3(q_A) = \frac{1}{2}\Big(q_z\, \mathcal{I}_2(q_r) + q_r\, q_\alpha(\gamma^r)^{\alpha\beta}q_\beta\Big) \tag{6.10}$$

where the quadratic invariant of the U-duality group $O(5,5)$ is

$$\mathcal{I}_2(q_r) = \eta^{rs}q_r q_s \tag{6.11}$$

with η^{rs} the $O(5,5)$ metric. In this setting 1/4 BPS states are defined by the requirement that the non-vanishing **10** vector is not null, or a non-vanishing **16** spinor q_α has a non-vanishing vector bilinear $q_\alpha(\gamma^r)^{\alpha\beta}q_\beta$ (i.e. q_α is not a 'pure' spinor).

$$\mathcal{I}_2(q_r) \neq 0 \quad \text{or} \qquad q_\alpha(\gamma^r)^{\alpha\beta}q_\beta \neq 0 \qquad \Rightarrow \qquad 1/4BPS \tag{6.12}$$

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The 1/4 BPS solutions of D=6 maximal supergravity are dyonic strings. These are regular solutions with $AdS_3 \times S^3$ near horizon geometry. The area of the horizon is proportional to $\sqrt{\mathcal{I}_2(q_r)}$. Thus, we see another example of a large class of singular D=4 and D=5 black holes with $\mathcal{I}_4 = 0$ and $\mathcal{I}_3 = 0$ which in D=6 become regular solutions. This class is described by the vanishing singlet $q_z = 0$, vanishing **16**, $q_\alpha = 0$, and non-vanishing non-null **10** vector. These configurations with non-vanishing quadratic $O(5,5)$ invariant $\sqrt{\mathcal{I}_2(q_r)}$ are regular in D=6.

Thus the uplifting to D=6 removes the null singularity from a class of 1/4 BPS extremal black holes in D=4 and makes them regular dyonic strings in D=6. The invariant $\mathcal{I}_2(q_r)$ is very reminiscent of the 1/4 BPS counting formula, since $4\vec{m} \cdot \vec{n} = \alpha'(p_L^2 - p_R^2)$. In Type II strings in D=6, momenta and windings span an $O(4,4)$ subspace of $O(5,5)$. The missing 8 charges are accounted for either by wrapped branes (D1, D3 or D0, D2, D4) or by uplift to M-theory, where there are 5 p_L and 5 p_R for T^5.

We will not discuss here the intermediate cases of D=7,8,9 and just note that in D=10 the well known 1/2 BPS D3 branes are perfectly regular solutions. In D=11 the 1/2 BPS M2 and M5 branes are regular solutions, too. But once again, we would like to stress that D=10 and D=11 supergravities are not expected to be UV finite and therefore the fact that non-perturbative solutions are regular, may not be of particular significance since once quantum effects are included one should resort to string or M theory for consistent UV completions.

7 On BPS $\mathcal{N}$= 4 Yang-Mills dyons and $\mathcal{N}$= 8 supergravity black holes

Although planar $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity amplitudes seem to secretly share the same structure, so much so that UV divergences are actually absent to the order where computations have been explicitly performed [41], the structure of their moduli space is rather different. While in $\mathcal{N} = 8$ supergravity giving VEV's to scalars does not break any symmetry, the moduli space $E_{7(7)}/\mathrm{SU}(8)$, being a symmetric space, is homogeneous and any point is equivalent to the origin, in $\mathcal{N} = 4$ SYM only at the origin of the moduli space R^{6r}/W_r, with W_r the Weyl group of the rank r gauge group, the theory enjoys unbroken superconformal symmetry.[6] At any other point gauge symmetry and superconformal symmetry are spontaneously broken and an infinite tower of stable BPS dyons appear in the spectrum that play a crucial role in the $\mathrm{SL}(2,Z)$ e-m duality of $\mathcal{N}$=4 SYM theory.

It is interesting to compare the role of non-perturbative states in $\mathcal{N}$=8 supergravity and in $\mathcal{N}$=4 SYM theory at this point. $\mathcal{N}$=4 perturbative YM theory is UV finite, *per se*, when only the 16 massless states of the perturbative theory are included. Non-perturbative monopoles and dyons are solutions of the non-linear classical equations that essentially decouple from perturbation theory. The proof of finiteness is based on the properties of the vertices of the classical action where only 16 elementary massless states enter and there is no infinite tower of light/massless dyons.

[6]Notice that at this point, $\mathcal{N} = 4$ SYM has the same amount of supersymmetries as $\mathcal{N} = 8$ supergravity, the extra (superconformal) charges are precisely the ones needed to balance the supercharges in $\mathcal{N} = 8$ gauged supergravity in $D = 5$ and to make the holographic correspondence kinematically sensible.

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Although at the conformal point, infinitely many 1/2 BPS dyons approach the zero mass limit in $\mathcal{N}$=4 Yang-Mills theory, they don't behave as elementary objects in the following sense. In the conformal limit at weak coupling dyon masses go to zero, but their size L diverges even faster. Indeed, the Compton wavelength of the dyon is related to its mass by $l_{Compt} \sim \frac{1}{M} = \frac{g}{v}$, while the classical size is determined by the Compton wavelength of the massive W-bosons $L \sim \frac{1}{gv}$. One finds that $\frac{l_{Compt}}{L} = g^2$ and, for very small coupling $g^2 \ll 1$, one has $l_{Compt} \ll L$.

As observed in [1], the properties of non-perturbative black hole solutions in $\mathcal{N}$=8 supergravity depend crucially on the value of $\mathcal{I}_4$. Regular BPS and non-BPS extremal black holes $\mathcal{I}_4 \neq 0$ have a mass gap, while solutions with $\mathcal{I}_4 \neq 0$ are singular and can become massless. For regular extremal black holes, assuming that the ADM mass defines the Compton wavelength, $l_{Compt} \sim \frac{1}{M_{ADM}}$ while the classical size is related to the area of the horizon, $L \sim \frac{1}{M_H}$, we may use the attractor properties of the BPS black holes which suggests that $M_H \leq M_{ADM}$ and $l_{Comt} \leq L$. Thus one can see that the classical scale is larger or equal to the Compton scale. For the consistency of the classical interpretation, one should still require that $M_H \gg M_{Pl}$. In string theory the degeneracy of the relevant states with $I_4 \neq 0$ are described by formula (4.9). For light or massless singular solutions with $I_4 = 0$ there is no reasonable classical interpretation in D=4. In M/string theory we have found that the formulae for the degeneracy of states are different from the one for regular black holes.

In summary, UV finiteness of $\mathcal{N}$=4 SYM is well established, while UV finiteness of $\mathcal{N}$=8 is still conjectural. Despite differences between specific properties of the non-perturbative states in the two theories, one may still take a lesson for the UV properties of $\mathcal{N}$=8 supergravity from $\mathcal{N}$=4 perturbative and non-perturbative $\mathcal{N}$= 4 SYM theory. UV finiteness of the perturbative QFT relies only on the 16 massless physical states of the CPT-invariant supermultiplet. Although $\mathcal{N}$=4 SYM theory has an infinite tower of BPS monopoles and dyons with exact mass formula, that become massless in the conformal limit, they anyway decouple from the perturbations theory.[7]

8 Discussion

In this paper we tried to relate M-theory on T^7 and Type II superstring theory on T^6 to D=4 $\mathcal{N}$=8 supergravity. M-theory on T^7 and Type II superstring theory on T^6 have, in addition to $\mathcal{N}$=8 D=4 supergravity multiplet, an infinite number of elementary massless states [9], which are counted in the degeneracy of states formulae. D=4 $\mathcal{N}$=8 supergravity has 256 massless states only and the perturbative QFT includes in the Feynman graphs only these states.

The recent conjecture of all-loop UV finiteness of D=4 $\mathcal{N}$=8 perturbative supergravity is based on the extrapolations of the recent 3- and 4-loop computations to higher loop order [2–5]. Relating these computations to similar ones in N=4 Yang-Mills [7, 8] lends support to the prediction that the critical dimension where the onset of UV divergences is given by $D_c = 4 + \frac{6}{L}$, where L is the number of loops. The all-loop UV finiteness of D=4

[7]We thank M. Porrati for suggesting this analogy.

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$\mathcal{N}$=8 perturbative supergravity is also supported by the light-cone supergraph prediction under condition that the $E_{7(7)}(\mathbb{R})$ be valid at the quantum level [6].

The opinion was expressed that the all-loop UV finiteness of D=4 $\mathcal{N}$=8 perturbative supergravity may contradict U-duality invariance of the degeneracy formulae for states in M/string theory if the light non-perturbative states were to decouple from the theory. Such states were studied in [1] and were shown to correspond to light-like orbits of the U-duality group with vanishing quartic invariant $\mathcal{I}_4(\mathcal{Q}) = 0$ and singular horizon, when viewed as black holes in D=4. They come in 3 types, 1/8, 1/4, 1/2 BPS solutions. *A priori* one would expect that different orbits of $E_{7(7)}(\mathbb{R})$ and $E_{7(7)}(\mathbb{Z})$ should not mix with each other. Still one would like to see that there are no anomalies by explicitly studying the known formulae for the degeneracy of BPS states.

A technical tool which allowed us to perform an important part of the relevant analysis is the relation which we found between the orbits of $E_{7(7)}(\mathbb{R})$ and discrete invariants of $E_{7(7)}(\mathbb{Z})$. Namely we have established that the states which are described by discrete Sen's invariants $a_2(\mathcal{Q}) = \psi(\mathcal{Q})$ and $a'_4(\mathcal{Q}) = \chi(\mathcal{Q})$ are only 1/8 BPS. The ones which are 1/2 and 1/4 BPS are not included into a set of states classified by discrete invariants. The reason for this is that the set of integers whose greatest common divisor would define the discrete invariant, has to vanish for the state to be in the 1/2 and 1/4 BPS orbits. However, the greatest common divisor is defined for a set of not all vanishing integers and is positive. Thus formulae for the degeneracy of 1/2 and 1/4 BPS states must be separate from the 1/8 BPS ones.

We presented formulae for the degeneracy of 1/2 and 1/4 BPS states with $a_4(\mathcal{Q}) = \mathcal{I}_4(\mathcal{Q}) = 0$. We pointed out a puzzle, already raised by Sen and associated with the degeneracy for 1/4 BPS states, viewed as perturbative string states or as $\mathcal{N}$=4 D=4 supergravity solutions. The 1/4 BPS solutions are singular with vanishing area of the horizon since this is not expected to be stretched by $\mathcal{N}$=8 quantum corrections in D=4. Meanwhile, the string degeneracy formula would suggest a stretching of the horizon quadratic in the charges. The corresponding twisted index, however, is not U-duality invariant.

We have also shown that the degeneracy formula of 1/8 BPS states [15] splits into two separate formulae, a U-duality invariant one for $\mathcal{I}_4(\mathcal{Q}) \neq 0$ related to regular black holes entropy for large values of $\mathcal{I}_4(\mathcal{Q})$. It is given in eq. (4.9). The other one is a relatively simple exact U-duality invariant formula for the degeneracy of 1/ 8 BPS states with $\mathcal{I}_4(\mathcal{Q}) = 0$ that depends on discrete invariants of $E_{7(7)}(\mathbb{Z})$. It is given by eq. (4.10) and suggests that the degeneracy of 1/8 BPS states with $a_4(\mathcal{Q}) = \mathcal{I}_4(\mathcal{Q}) = 0$, $a_1(\mathcal{Q}) = 1$, $a_2(\mathcal{Q}) = \psi(\mathcal{Q}) = 1$ depends on the discrete $E_{7(7)}(\mathbb{Z})$ invariant $a'_4 = \chi(\mathcal{Q})$ as follows:

$$d(\mathcal{Q})|_{\mathcal{I}_4(\mathcal{Q})=0} = \sum_{s\in\mathbb{Z}, 2s|\chi(\mathcal{Q})} 2s \tag{8.1}$$

Thus we have shown explicitly that all singular states with $\mathcal{I}_4(\mathcal{Q}) = 0$ are decoupled from the degeneracy formula for regular black holes states with $\mathcal{I}_4(\mathcal{Q}) \neq 0$ shown in eq. (4.9). The counting formulae for 1/8 BPS regular black holes and 1/8 BPS singular ones are separately U-duality invariant.

We have also explained that all singular $\mathcal{I}_4(\mathcal{Q}) = 0$ solution of $\mathcal{N}$=8 D=4 supergravity resolve the singularities when uplifted to $D > 4$. There is an interesting situation here since in $D > 4$ the maximal supergravity is not expected to be UV finite in perturbation theory and therefore one should resort to string or M theory for consistent UV completions.

We compared $\mathcal{N}$=8 perturbative and non-perturbative supergravity with $\mathcal{N}$=4 perturbative and non-perturbative Yang-Mills theory. The $\mathcal{N}$=4 YM QFT based on 16 massless physical states of the CPT-invariant supermultiplet is UV finite. Meanwhile, the theory also has the infinite tower of the BPS monopoles and dyons, in the conformal limit they become massless, but they do not affect the $\mathcal{N}$=4 perturbation theory. So, there is a precedent of decoupling of the non-perturbative states from the perturbation theory.

Based on the decoupling in the degeneracy of states formulae of the singular non-perturbative $\mathcal{I}_4(\mathcal{Q}) = 0$ light states of D=4 $\mathcal{N}$=8 supergravity, we are lead to conclude that our study of the non-perturbative sector of the theory does not reveal any contradiction with the conjectured all-loop finiteness of D=4 $\mathcal{N}$=8 perturbative supergravity based on 256 massless states of the CPT-invariant supermultiplet.

Acknowledgments

We aknowledge discussions and useful correspondence with T. Banks and A. Strominger and their critical remarks on our analysis in [1] and suggestion to look at formulae for the degeneracy of states with and without including singular states. We are particularly grateful to A. Sen for explaining his work on arithmetic of $\mathcal{N} = 8$ black holes to us. We are grateful to L. Borsten, M. Dine, L. Dixon, M. Duff, D. Harlow, A. Linde, A. Marrani, J. F. Morales, M. Porrati, J. H. Schwarz, D. Simic, S. Shenker, L. Susskind, V. S. Varadarajan and E. Witten for their interest in the problem and for stimulating discussions. This work was partially supported by the ERC Advanced Grant n.226455 *"Superfields"* and by the Italian MIUR-PRIN contract 2007-5ATT78 *"Symmetries of the Universe and of the Fundamental Interactions"*. The work of SF has been supported in part by DOE grant DE-FG03-91ER40662, Task C. The work of RK is supported by the NSF grant 0756174.

A Elliptic functions

Elliptic functions admit both infinite product and series expansions, that we collect in this appendix and read

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2} y^n \tag{A.1}$$

$$\theta_4(z|\tau) = \sum_{n=-\infty}^{+\infty} (-)^n q^{\frac{1}{2}n^2} y^n \tag{A.2}$$

$$\theta_2(z|\tau) = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} y^{n+\frac{1}{2}} \tag{A.3}$$

$$\theta_1(z|\tau) = i \sum_{n=-\infty}^{+\infty} (-)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} y^{n+\frac{1}{2}} \tag{A.4}$$

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where $y = \exp(2\pi i z)$ and

$$\theta_1(z|\tau) = 2iq^{\frac{1}{8}}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \tag{A.5}$$

$$\theta_2(z|\tau) = 2q^{\frac{1}{8}}\cos(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^n) \tag{A.6}$$

$$\theta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n+\frac{1}{2}})(1+y^{-1}q^{n+\frac{1}{2}}) \tag{A.7}$$

$$\theta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n+\frac{1}{2}})(1-y^{-1}q^{n+\frac{1}{2}}) \tag{A.8}$$

Elliptic functions provide a representation of the SL$(2, Z)$ modular group

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau) \qquad \eta(-1/\tau) = \sqrt{i\tau}\eta(\tau) \tag{A.9}$$

$$\theta_3(\tau+1) = \theta_4(\tau) \qquad \theta_3(-1/\tau) = \sqrt{i\tau}\theta_4(\tau) \tag{A.10}$$

$$\theta_4(\tau+1) = \theta_3(\tau) \qquad \theta_4(-1/\tau) = \sqrt{i\tau}\theta_2(\tau) \tag{A.11}$$

$$\theta_2(\tau+1) = -\theta_2(\tau) \qquad \theta_2(-1/\tau) = \sqrt{i\tau}\theta_4(\tau) \tag{A.12}$$

B Small black holes in M-theory

In this appendix, we would like to study the singular states in D=4 from the perspective of M/string theory. For this purpose it may be useful first to identify the black hole charges $\mathcal{Q}$ in M-theory compactified on a seven-torus T^7. Here we follow [14] and identify the 28+28 electric and magnetic charges $\mathcal{Q} = Q + P$ as M2, M5 branes, KK momenta (KKp) and KK monopoles (KKm). In more detail, one can construct two 8x8 antisymmetric matrices, $I, J = 1, \ldots, 7$:

$$Q = +\begin{pmatrix} [M2]^{IJ} & [KKm]^I \\ -[KKm]^I & 0 \end{pmatrix} \qquad P = +\begin{pmatrix} [M5]_{IJ} & [KKp]_I \\ -[KKp]_I & 0 \end{pmatrix} \tag{B.1}$$

The 28 electric charges of the D=4 black holes Q correspond to 21 different M2 branes wrapped on various $[IJ]$ directions and of 7 KKM wrapped on all internal directions but I. The 28 magnetic charges of the D=4 black holes P correspond to 21 different M5 branes wrapped on various directions but $[IJ]$ and of 7 KK momentum along the internal directions I. In terms of these M-theory objects the extremal black hole area of the horizon is $S_{BH,4D} = \pi\sqrt{\mathcal{I}_4(P,Q)}$ where

$$\mathcal{I}_4(\mathcal{Q}) = \mathcal{I}_4(P,Q) = -\mathrm{Tr}(OPQP) + \frac{1}{4}(\mathrm{Tr}QP)^2 - 4[\mathrm{Pf}(P) + \mathrm{Pf}(Q)] \tag{B.2}$$

If one would like to associate the D=5 black holes with the states in M theory compactified on a six torus T^6, one can now split one particular direction out of seven I so that $I = \{1, i\},\ i = 2, \ldots, 7$. Now the the 56 charges $\mathcal{Q}$ are split into 1+27+1+$\overline{27}$ charges

$$q_0 = [KKp]_1, \qquad q_A = \{[M2]^{ij},\ [KKp]_i,\ [M5]_{i1}\} \tag{B.3}$$

$$p^0 = [KKm]^1, \qquad p^A = \{[M2]^{i1},\ [KKm]^i,\ [M5]_{ij}\} \tag{B.4}$$

Here q_A are the 27 electric charges of the D=5 extremal black holes and p^A are the $\overline{27}$ magnetic charges of the D=5 black strings.

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