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# CONJUGATION PROBLEM ABOUT JOINTLY SEPARATE FLOW OF VISCOELASTIC AND VISCOUS FLUIDS IN THE PLANE DUCT

V.I. KORZYUK[1], S.V. LEMESHEVSKY[2], P.P. MATUS[3], V.N. SHALIMA[1]

[1] *Belarusian State University*

Skoryna Ave. 4, 220050, Minsk, Belarus

[2,3] *Institute of Mathematics, NAS of Belarus*

Surganov St. 11, 220072, Minsk, Belarus

E-mail: [1] Korzyuk@org.bsu.unibel.by, [2] svl@im.bas-net.by

E-mail: [3] matus@im.bas-net.by

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## ABSTRACT

Conjugation problem about jointly separate flow of viscoelastic and viscous fluids in the plane duct is considered. The results concerning of solvability of this problem are presented. The implicit difference scheme for the conjugation problem is constructed. The consistency conditions are approximated with the second order of approximation with respect to spatial variable. The convergence of the suggested difference scheme is investigated by method of energy inequalities.

## INTRODUCTION

Conjugation problems arise in the study of many phenomena which take place in the media with sharply differing physical characteristics [5; 10; 11]. Various consistency conditions for desired functions are given on the interface of such media. The questions concerning the existence of a unique solution and the numerical methods for such problems are considered in the papers [3; 7; 8; 9].

The study of oil viscoelastic characteristics influence on the technological processes of oil extraction is of great interest [1]. We consider a joint motion of the different non-mixed fluids in the split and porous layers and we into account their viscoelastic characteristics. The motion of viscoelastic and viscous fluids in the plane horizontal split leaving out of the surface is described by the one-dimensional hyperbolic equation and the heat equation supplemented with integral-differential conditions on the interface of moving fluids.

## 1. DIFFERENTIAL PROBLEM

Let the rectangle $Q = \{(x,t) : 0 < x < l,\ 0 < t < T\}$, $0 < T < +\infty$, be separated by the straight line $\gamma = \{(x,t) : x = \xi,\ 0 < \xi < l,\ 0 < t < T\}$ into two sub-domains $Q^{(1)}$ and $Q^{(2)}$.

In the domain $Q^{(1)}$ we consider the hyperbolic (with respect to function $u^{(1)}(t,x)$ of independent variables $t \in (0,T)$ and $x \in (0,\xi)$) equation:

$$\mathcal{L}^{(1)}u^{(1)} \equiv \theta \frac{\partial^2 u^{(1)}}{\partial t^2} + \frac{\partial u^{(1)}}{\partial t} - \frac{\partial^2 u^{(1)}}{\partial x^2} = f^{(1)}(t,x). \tag{1.1}$$

For the function $u^{(2)}(t,x)$ in $Q^{(2)}$ we consider the equation of parabolic type

$$\mathcal{L}^{(2)}u^{(2)} \equiv \frac{1}{\rho}\frac{\partial u^{(2)}}{\partial t} - \frac{1}{\mu}\frac{\partial^2 u^{(2)}}{\partial x^2} = f^{(2)}(t,x). \tag{1.2}$$

Here $\theta$, $\rho$, $\mu$ are positive constants. Equations (1.1), (1.2) are supplemented by the initial conditions

$$\ell u \equiv u(0,x) = u_0(x), \quad x \in (0,l), \tag{1.3}$$

$$\ell_1 u \equiv \frac{\partial u^{(1)}(0,x)}{\partial t} = u_1^{(1)}(x), \quad x \in (0,\xi), \tag{1.4}$$

boundary conditions

$$u(t,0) = u(t,l) = 0, \quad t \in (0,T), \tag{1.5}$$

and the following consistency conditions

$$u^{(1)}\big|_\gamma = u^{(2)}\big|_\gamma, \tag{1.6}$$

$$\frac{1}{\theta}\int\limits_0^t \left(\exp\left(\frac{t'-t}{\theta}\right)\frac{\partial u^{(1)}}{\partial x}\right)\bigg|_\gamma dt' = \frac{1}{\mu}\,\frac{\partial u^{(2)}}{\partial x}\bigg|_\gamma, \tag{1.7}$$

where $u(t,x) = \left(u^{(1)}(t,x),\, (t,x) \in Q^{(1)},\ u^{(2)}(t,x),\, (t,x) \in Q^{(2)}\right)$, $u_0(x) = \left(u_0^{(1)}(x),\, x \in (0,\xi),\ u_0^{(2)}(x),\, x \in (\xi,l)\right)$.

Let us introduce the following notation $\mathcal{J}u^{(1)}(t,x)=\int\limits_0^t \exp\left(\frac{t'-t}{\theta}\right) u^{(1)}(t',x)dt'$. For smooth functions equation (1.1) and the conditions (1.3) are equivalent to the following equation

$$\widetilde{\mathcal{L}}^{(1)}u^{(1)} \equiv \theta\frac{\partial u^{(1)}}{\partial t} - \frac{\partial}{\partial x}\mathcal{J}u^{(1)} = \mathcal{J}f^{(1)}(t,x) + \theta\exp\left(-\frac{t}{\theta}\right)u_1^{(1)}(x). \quad (1.8)$$

Thus, the problem (1.1)–(1.7) is equivalent to the problem (1.2), (1.3), (1.5) — (1.8).

Consider the problem (1.2), (1.3), (1.5)—(1.8) as the operator equation

$$Lu = F, \quad (1.9)$$

where $Lu = (\mathcal{L}u, \ell u, \ell_1 u)$, $\mathcal{L}u = \left(\widetilde{\mathcal{L}}^{(1)}u^{(1)}, \mathcal{L}^{(2)}u^{(2)}\right)$, $F = (f(t,x), u_0(x))$, $f(t,x) = \left(\mathcal{J}f^{(1)}(t,x) + \theta\exp\left(-\frac{t}{\theta}\right)u^{(1)}, f^{(2)}(t,x)\right)$. Here the domain of definition $\mathcal{D}(L)$ of the operator $L$ consists of functions $u(t,x)$, where $u^{(i)}(t,x)$ is a twice continuously differentiable functions in a closure $\overline{Q^{(i)}}$ of the domain $Q^{(i)}$ $(i = 1,2)$, and this functions satisfy the boundary conditions (1.5) and consistency conditions (1.6), (1.7).

We denote $\mathcal{B}$ the Banach space, which is the closure of a subspace $\widetilde{\mathcal{B}}$ with respect to the norm

$$\begin{aligned} \|u\|_{\mathcal{B}} = \sup_{0\leqslant t\leqslant T}\left(\|u(t)\|_{L_2(0,l)} + \left\|\,\left|\operatorname{grad}\mathcal{J}u^{(1)}(t)\right|\,\right\|_{L_2(0,\xi)}\right)(t)+ \\ + \left\|\,\left|\operatorname{grad}\mathcal{J}u^{(1)}\right|\,\right\|_{L_2(Q^{(1)})} + \left\|\,\left|\operatorname{grad}u^{(2)}\right|\,\right\|_{L_2(Q^{(2)})}, \end{aligned} \quad (1.10)$$

where $\widetilde{\mathcal{B}}$ is determined by the set $\mathcal{D}(L)$ and the norm (1.10). Let $\mathcal{H}$ be the Hilbert space $L_2(Q) \times L_2(0,l)$. Consider the operator $L$ as an operator from $\mathcal{B}$ to $\mathcal{H}$ with the domain of definition $\mathcal{D}(L)$.

The proof of solvability of the problem (1.1)—(1.7) for any $F \in \mathcal{H}$ is based on the following energy inequality for operator $L$.

**Theorem 1.1.** *For the operator $L : \mathcal{B} \to \mathcal{H}$ the following energy inequality*

$$\|u\|_{\mathcal{B}} \leqslant c\,\|Lu\|_{\mathcal{H}} \quad (1.11)$$

*holds for any function $u \in \mathcal{D}(L)$ and a positive constant $c$ do not depend on $u$.*

The operator $L$ as an operator from $\mathcal{B}$ to $\mathcal{H}$ admit a closure $\overline{L}$ [4] and the following estimate is correct:

$$\|u\|_{\mathcal{B}} \leqslant c\left\|\overline{L}u\right\|_{\mathcal{H}}, \quad \forall u \in \mathcal{D}(\overline{L}). \quad (1.12)$$

Solution of the operator equation $\overline{L}u = F$ is the *strong solution* of the problem (1.1)—(1.7).

Using averaging operators with variable step [2; 4] we can prove the following statement.

**Theorem 1.2.** *Suppose that* $f^{(2)} \in L_2(Q^{(2)})$, $u_0 \in L_2(0,l)$, $u_1^{(1)} \in L_2(0,\xi)$ *and* $\mathcal{J} f^{(1)} \in L_2(Q^{(1)})$; *then there exists a unique strong solution* $u \in \mathcal{B}$ *of the problem (1.1) — (1.7) and the following estimate*

$$\|u\|_{\mathcal{B}} \leqslant c\left(\left\|\mathcal{J} f^{(1)} + \theta \exp\left(-\frac{t}{\theta}\right) u_1^{(1)}\right\|_{L_2(Q^{(1)})} + \left\|f^{(2)}\right\|_{L_2(Q^{(2)})} + \|u_0\|_{L_2(0,l)}\right) \quad (1.13)$$

*holds, where c is a positive constant that do not depend on u.*

## 2. DIFFERENCE SCHEME

We assume that the following conditions hold

$$f^{(i)}(t,x) \in C\left(Q^{(i)}\right),\ u^{(i)}(t,x) \in C^4\left(Q^{(i)}\right),\ i=1,2,\ u_0(x) \in C^2(0,l). \quad (2.1)$$

On the interval $[0,T]$ let us introduce the uniform grid $\omega_\tau = \{t_j = j\tau,\ j = 1,2,\dots N_t - 1,\ N_t\tau = T\}$. In the domains $Q^{(1)}$ and $Q^{(2)}$ we shall consider the uniform grids $\omega_1 = \omega_{1h_1} \times \omega_\tau$ and $\omega_2 = \omega_{2h_2} \times \omega_\tau$, respectively. Here $\overline{\omega}_{1h_1} = \{x_i = ih_1,\ i = 0,1,2,\dots N_1,\ N_1 h_1 = \xi\}$, $\overline{\omega}_{2h_2} = \{x_{p+i} = \xi + ih_1,\ i = 0,1,2,\dots N_2,\ N_2 h_2 = l - \xi\}$. Let $\overline{\omega}_h = \overline{\omega}_{1h_1} \cup \overline{\omega}_{2h_2}$, $\overline{\omega} = \overline{\omega}_h \times \omega_\tau$.

We approximate the problem (1.1) — (1.7) on the grid $\omega$ by the following implicit difference scheme

$$\theta y_{1\bar{t}t} + \frac{\theta}{\tau}\left(\exp\left(\frac{\tau}{\theta}\right) - 1\right) y_{1t} = \widehat{y}_{1\bar{x}x} + \varphi_1, \quad (t,x) \in \omega_1, \quad (2.2)$$

$$\frac{1}{\rho} y_{2t} = \frac{1}{\mu}\widehat{y}_{2\bar{x}x} + \varphi_2, \quad (t,x) \in \omega_2, \quad (2.3)$$

$$y_1(t,0) = y_2(t,l) = 0, \quad t \in \omega_\tau,, \quad (2.4)$$

$$y(0,x) = u_0(x), \qquad y_t(0,x) = \overline{u}_1(x), \quad x \in \omega_h, \quad (2.5)$$

$$\begin{gathered} y_1(t,\xi) = y_2(t,\xi), \quad t \in \omega_\tau \\ \frac{1}{\theta}\widehat{\mathcal{J}} y_{1\bar{x}} + 0{,}5h_1\left(y_{1t} - \exp\left(-\frac{\tau}{\theta}\right) y_{1t}(0) - \varphi_1\right) \\ = \frac{1}{\mu}\widehat{y}_{2x} - 0{,}5h_2\left(y_{2t} - \varphi_2\right), \quad x = \xi, \quad t \in \omega_\tau, \end{gathered} \quad (2.6)$$

where

$$\overline{u}_1(x) = \begin{cases} \left(1-\dfrac{\tau}{2\theta}\right) u_1^{(1)}(x) + \dfrac{\tau}{2\theta}\left(\dfrac{\partial^2 u_0(x)}{\partial x^2} + f^{(1)}(0,x)\right), & x \in \omega_{1h_1} \cup \{\xi\}, \\ \rho\left(\dfrac{1}{\mu}\dfrac{\partial^2 u_0(x)}{\partial x^2} + f^{(2)}(t,x)\right), & x \in \omega_{2h_2}, \end{cases}$$

$$Jv = (Jv)(t,x) = \sum_{t'=\tau}^{t} \tau \exp\left(\frac{t'-(t+\tau)}{\theta}\right) v(t',x), \quad \widehat{J}v = (Jv)(t+\tau,x),$$

$$\varphi_1(t,x) = \begin{cases} f^{(1)}(t,x), & (t,x) \in \omega_1, \\ (Jf^{(1)})(t,\xi), & t \in \omega_\tau, \end{cases} \quad \varphi_2(t,x) = f_2(t,x), \quad (t,x) \in \omega_2 \cup \xi.$$

Here we also use non-indexed notation of the difference scheme theory [6]:

$$y = y(t,x) = \begin{cases} y_1(t,x), & x \in \omega_1, \\ y_2(t,x), & x \in \omega_2, \end{cases} \qquad y_{k\bar{x}x} = \frac{y_{kx} - y_{k\bar{x}}}{h_k},$$

$$y_{kx} = \frac{y_k(t,x+h_k) - y_k(t,x)}{h_k}, \quad y_{k\bar{x}} = \frac{y_k(t,x) - y_k(t,x-h_k)}{h_k},$$

$$y_{k\bar{t}t} = \frac{y_{kt} - y_{k\bar{t}}}{\tau}, \quad y_{kt} = \frac{y_k(t+\tau,x) - y_k(t,x)}{\tau},$$

$$y_{k\bar{t}} = \frac{y_k(t,x) - y_k(t-\tau,x)}{\tau}, \quad \widehat{y}_k = y_k(t+\tau,x), \quad k = 1,2.$$

Note that consistency condition (1.7) is approximated taking into account the requirement of the second order of approximation with respect to the spatial variable.

## 3. CONVERGENCE OF THE DIFFERENCE SCHEME

On the grids $\omega_{kh_k}$ $(k=1,2)$ let us introduce the scalar product and the norms

$$(u,v)_{\omega_{kh_k}} = \sum_{x \in \omega_{kh_k}} u(x)v(x)h_k, \quad (u,v]_{\omega_{1h_1}} = \sum_{x \in \omega_{1h_1}} u(x)v(x)h_1 + u(\xi)v(\xi)h_1,$$

$$\|v\|_{\omega_{kh_k}} = \sqrt{(v,v)_{\omega_{kh_k}}}, \quad \|v]|_{\omega_{1h_1}} = \sqrt{(v,v]_{\omega_{1h_1}}}.$$

Now let us study the convergence of the proposed difference scheme (2.2) — (2.6). We write the equations for the error $z = y - u$, where $y$ is the solution of the difference scheme (2.2) — (2.6), and $u(t,x)$ is the solution of the differential problem (1.1) — (1.7),

$$\theta z_{\bar{t}t} + \frac{\theta}{\tau}\left(\exp\left(\frac{\tau}{\theta}\right) - 1\right) z_t = \widehat{z}_{\bar{x}x} + \psi, \quad (t,x) \in \omega_1, \tag{3.1}$$

$$\frac{1}{\rho} z_t = \frac{1}{\mu}\widehat{z}_{\bar{x}x} + \psi, \quad (t,x) \in \omega_2, \tag{3.2}$$

$$z(t,0) = z(t,l) = 0, \quad t \in \omega_\tau, \tag{3.3}$$

$$z(0,x) = 0, \quad z_t(0,x) = \phi(x), \quad x \in \omega_h, \tag{3.4}$$

$$\widehat{J} z_{\bar{x}} + 0{,}5 h_1 \left(z_t - \exp\left(-\frac{\tau}{\theta}\right) z_t(0)\right) = \frac{1}{\mu}\widehat{z}_x - 0{,}5 h_2 z_t + \psi, \ x = \xi, \ t \in \omega_\tau. \tag{3.5}$$

Here

$$\psi(t,x) = \begin{cases} \widehat{u}^{(1)}_{\bar{x}x} + \varphi_1 - \theta u^{(1)}_{\bar{t}t} - \dfrac{\theta}{\tau}\left(\exp\left(\dfrac{\tau}{\theta}\right) - 1\right) u^{(1)}_t, & (t,x) \in \omega_1, \\ \dfrac{1}{\mu}\widehat{u}^{(2)}_x - \widehat{J} u^{(1)}_{\bar{x}} + 0{,}5\left(h_1\varphi_1 + h_2\varphi_2\right), & \\ -0{,}5\left(h_1\left(u^{(1)}_t - \exp\left(-\dfrac{\tau}{\theta}\right) u^{(1)}_t(0)\right) + h_2 u^{(2)}_t\right), & x = \xi, \ t \in \omega_\tau, \\ \dfrac{1}{\mu}\widehat{u}^{(2)}_{\bar{x}x} + \varphi_2 - \dfrac{1}{\rho} u^{(2)}_t, & (t,x) \in \omega_2, \end{cases}$$

defines the truncation error of the equations (1.1), (1.2) and the consistency conditions (1.7), $\phi(x) = \overline{u}_1(x) - u_t(0,x) = \mathcal{O}(\tau + h^2)$ is the truncation error of the second initial condition. Note that under the conditions (2.1) $\psi(t,x) = \mathcal{O}(\tau + h^2)$ when $x \neq \xi$ and $\psi(t,\xi) = \mathcal{O}(\tau h + h^2)$, where $h = \max\{h_1, h_2\}$.

Let us reduce *a priori* estimate for the error of method. First let us transform the equation (3.1). Multiplying both sides by $\dfrac{\tau}{\theta}\exp\left(\dfrac{t' - (t+\tau)}{\theta}\right)$ and summing the result with respect to nodes of the time grid $t'$ from $\tau$ to $t$, we obtain

$$\sum_{t'=\tau}^{t} \tau \exp\left(\frac{t' - (t+\tau)}{\theta}\right) z_{\bar{t}t}(t')$$

$$+ \sum_{t'=\tau}^{t} \exp\left(\frac{t' - (t+\tau)}{\theta}\right)\left(\exp\left(\frac{\tau}{\theta}\right) - 1\right) z_t(t') = J\widehat{z}_{\bar{x}x} + J\psi.$$

Using the formula of summation by parts and taking into account the equality $\left(\exp\left(\dfrac{t' - (t+\tau)}{\theta}\right)\right)_t = \dfrac{1}{\tau}\exp\left(\dfrac{t' - (t+\tau)}{\theta}\right)\left(\exp\left(\dfrac{\tau}{\theta}\right) - 1\right)$, we get the following relation

$$z_t = J\widehat{z}_{\bar{x}x} + J\psi + \exp\left(-\frac{t}{\theta}\right) z_t(0), \quad (t,x) \in \omega_1. \tag{3.6}$$

Let us make the scalar product in $\omega_{1h_1}$ of the equation (3.6) with $2\tau\widehat{z}$, and the scalar product in $\omega_{2h_2}$ of the equation (3.2) with $2\tau\widehat{z}$. Summing the results and taking into account the identity $\widehat{z} = 0{,}5(\widehat{z}+z) + 0{,}5\tau z_t$, we have

$$\begin{gathered}
\|\widehat{z}\|^2_{\omega_{1h_1}} + \frac{1}{\rho}\|\widehat{z}\|^2_{\omega_{2h_2}} + \tau^2\left(\|z_t\|^2_{\omega_{1h_1}} + \frac{1}{\rho}\|z_t\|^2_{\omega_{1h_1}}\right) \\
= \|z\|^2_{\omega_{1h_1}} + \frac{1}{\rho}\|z\|^2_{\omega_{2h_2}} + 2\tau\left(\widehat{z}, J\widehat{z}_{\bar{x}x}\right)_{\omega_{1h_1}} \\
+2\tau\left(\widehat{z}, \widehat{z}_{\bar{x}x}\right)_{\omega_{2h_2}} + 2\tau\left(\widehat{z}, J\psi\right)_{\omega_{1h_1}} \\
+2\tau\left(\widehat{z}, \psi\right)\omega_{2h_2} + 2\tau\exp\left(-\frac{t}{\theta}\right)\left(\widehat{z}, z_t(0)\right)_{\omega_{1h_1}}.
\end{gathered} \tag{3.7}$$

Consider the expression $2\tau\left(\widehat{z}, J\widehat{z}_{\bar{x}x}\right)$. Using the formula of summation by parts and taking into account equalities

$$(Jv)_t = \frac{1}{\theta}\widehat{v} + \frac{1}{\tau}\left(1-\exp\left(\frac{\tau}{\theta}\right)\right)\widehat{J}v, \qquad J\widehat{v} = \widehat{J}v - \frac{\tau}{\theta}\exp\left(-\frac{t+\tau}{\theta}\right)v(\tau),$$

we get

$$\begin{gathered}
2\tau(\widehat{z}, J\widehat{z}_{\bar{x}x})_{\omega_{1h_1}} = -\theta\left(\|\widehat{J}z_{\bar{x}}]|^2_{\omega_{1h_1}} - \|Jz_{\bar{x}}]|^2_{\omega_{1h_1}}\right) \\
+2\theta\left(1-\exp\left(\frac{\tau}{\theta}\right)\right)\|\widehat{J}z_{\bar{x}}]|^2_{\omega_{1h_1}} + 2\tau\left.\left(\widehat{z}\widehat{J}z_{\bar{x}}\right)\right|_{x=\xi} \\
-\frac{2}{\theta}\exp\left(-\frac{t+\tau}{\theta}\right)\tau\left(\widehat{z}, z_{\bar{x}x}(\tau)\right) - \theta\tau^2\|(Jz_{\bar{x}})_t]|^2_{\omega_{1h_1}}.
\end{gathered}$$

Similarly,

$$\frac{2}{\mu}\tau\left(\widehat{z}, \widehat{z}_{\bar{x}x}\right)_{\omega_{2h_2}} = -\frac{2}{\mu}\tau\|\widehat{z}_{\bar{x}}]|^2_{\omega_{2h_2}} - 2\tau\left.\left(\widehat{z}\widehat{z}_x\right)\right|_{x=\xi}.$$

Thus, we have obtained the following energy identity

$$\begin{gathered}
\|\widehat{z}\|^2_{\omega_{1h_1}} + \frac{1}{\rho}\|\widehat{z}\|^2_{\omega_{2h_2}} + \|\widehat{J}z]|^2_{\omega_{1h_1}} + 2\theta\left(\exp\left(\frac{\tau}{\theta}\right)-1\right)\|\widehat{J}z]|^2_{\omega_{1h_1}} \\
+\tau^2\left(\|z_t\|^2_{\omega_{1h_1}} + \frac{1}{\rho}\|z_t\|^2_{\omega_{2h_2}} + \theta\|(Jz_{\bar{x}})_t]|^2_{\omega_{1h_1}}\right) + \frac{2}{\mu}\tau\|\widehat{z}_{\bar{x}}]|^2_{\omega_{2h_2}} \\
= \|z\|^2_{\omega_{1h_1}} + \frac{1}{\rho}\|z\|^2_{\omega_{2h_2}} + \|Jz]|^2_{\omega_{1h_1}} + 2\tau\widehat{z}\left.\left(\widehat{J}z_{\bar{x}} - \frac{1}{\mu}\widehat{z}_x\right)\right|_{x=\xi} \\
+2\tau\exp\left(-\frac{t}{\theta}\right)\left(\widehat{z}, z_t(0)\right)_{\omega_{1h_1}} - \frac{2}{\theta}\exp\left(-\frac{t+\tau}{\theta}\right)\tau\left(\widehat{z}, z_{\bar{x}x}(\tau)\right)_{\omega_{1h_1}} \\
+2\tau\left(\widehat{z}, J\psi\right)_{\omega_{1h_1}} + 2\tau\left(\widehat{z}, \psi\right)_{\omega_{2h_2}}.
\end{gathered} \tag{3.8}$$

Taking into account the consistency condition (3.5), we get

$$2\tau\widehat{z}\left(\frac{1}{\mu}\widehat{z}_x - \widehat{J}z_{\bar{x}}\right)\Big|_{x=\xi} = \Big(0{,}5h_1\widehat{z}^2 - 0{,}5h_1 z^2 + \tau^2 0{,}5(h_1+h_2)z_t^2 - \tau h_1 \exp\left(-\frac{t}{\theta}\right) z_t(0) - \tau\widehat{z}\psi\Big)\Big|_{x=\xi}. \tag{3.9}$$

Using the relation $\exp\left(\frac{\tau}{\theta}\right) - 1 \geqslant 0$ and (3.9), from (3.8) we obtain the inequality

$$\begin{gathered}\|\widehat{z}\|_1^2 + \tau^2\left(\|z_t\|^2_{\omega^{+0,5}_{1h_1}} + \|z_t\|^2_{\omega^{-0,5}_{2h_2}}\right) \leqslant \|z\|_1^2 \\ +2\tau\exp\left(-\frac{t}{\theta}\right)(\widehat{z}, z_t(0))_{\omega^{+0,5}_{1h_1}} - \frac{2}{\theta}\exp\left(-\frac{t+\tau}{\theta}\right)\tau\left(\widehat{z}, z_{\bar{x}x}(\tau)\right)_{\omega_{1h_1}} \\ +2\tau\left(\widehat{z}, J\psi\right)_{\omega_{1h_1}} + 2\tau\left(\widehat{z}, \psi\right)_{\omega_{2h_2}} + \tau(\widehat{z}\psi)|_{x=\xi},\end{gathered} \tag{3.10}$$

where

$$\|z\|_1^2 = \|z\|^2_{\omega^{+0,5}_{1h_1}} + \frac{1}{\rho}\|z\|^2_{\omega^{-0,5}_{2h_2}} + \|Jz_{\bar{x}}]|^2_{\omega_{1h_1}},$$

$$(u,v)_{\omega^{+0,5}_{1h_1}} = \sum_{x\in\omega_{1h_1}} u(x)v(x)h_1 + 0{,}5u(\xi)v(\xi)h_1,$$

$$(u,v)_{\omega^{-0,5}_{2h_2}} = \sum_{x\in\omega_{2h_2}} u(x)v(x)h_1 + 0{,}5u(\xi)v(\xi)h_1, \quad \|v\|^2_{\omega^{+0,5}_{1h_1}} = (v,v)_{\omega^{+0,5}_{1h_1}},$$

$$\|v\|^2_{\omega^{-0,5}_{2h_2}} = (v,v)_{\omega^{-0,5}_{2h_2}}.$$

Taking into account the Cauchy inequality and the relation $ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, $\varepsilon > 0$, we have

$$\begin{gathered}2\tau\exp\left(-\frac{t}{\theta}\right)(\widehat{z}, z_t(0))_{\omega^{+0,5}_{1h_1}} \leqslant 4\varepsilon_1\tau\|z\|^2_{\omega^{+0,5}_{1h_1}} \\ +4\varepsilon_1\tau^3\|z_t\|^2_{\omega^{+0,5}_{1h_1}} + \frac{\tau}{2\varepsilon_1}\|z_t(0)\|^2_{\omega^{+0,5}_{1h_1}},\end{gathered} \tag{3.11}$$

$$\begin{gathered}-\frac{2}{\theta}\exp\left(-\frac{t+\tau}{\theta}\right)\tau\left(\widehat{z}, z_{\bar{x}x}(\tau)\right)_{\omega_{1h_1}} \leqslant \frac{4\varepsilon_2}{\theta}\|z\|^2_{\omega^{+0,5}_{1h_1}}, \\ +\frac{4\varepsilon_2}{\theta}\tau^3\|z_t\|^2_{\omega^{+0,5}_{1h_1}} + \frac{\tau}{2\theta\varepsilon_2}\|z_{\bar{x}x}(\tau)\|^2_{\omega_{1h_1}}\end{gathered} \tag{3.12}$$

$$\begin{gathered}2\tau\left(\widehat{z}, J\psi\right)_{\omega_{1h_1}} = \frac{2\tau}{\theta}\sum_{t'=\tau}^{t}\tau\exp\left(\frac{t'-(t+\tau)}{\theta}\right)(z(t+\tau), \psi(t'))_{\omega_{1h_1}} \\ \leqslant 4T\varepsilon_3\tau\|z\|^2_{\omega^{+0,5}_{1h_1}} + 4T\varepsilon_3\tau^3\|z_t\|^2_{\omega^{+0,5}_{1h_1}} + \frac{\tau}{2\varepsilon}\sum_{t'=\tau}^{t}\|\psi(t')\|^2_{\omega_{1h_1}},\end{gathered} \tag{3.13}$$

$$2\tau\,(\widehat{z},\psi)_{\omega_{2h_2}} \leqslant 4\varepsilon_4\tau\|z\|_{\omega_{2h_2}^{-0,5}} + 4\varepsilon_4\tau^3\|z_t\|^2_{\omega_{2h_2}^{-0,5}} + \frac{\tau}{2\varepsilon_4}\|\psi\|^2_{\omega_{2h_2}}, \tag{3.14}$$

$$\tau(\widehat{z}\psi)|_{x=\xi} = \left(0{,}5\varepsilon_5\tau(h_1+h_2)z^2 + \frac{\tau}{2(h_1+h_2)\varepsilon_5}\psi^2\right)\Big|_{x=\xi}, \tag{3.15}$$

where $\varepsilon_k = \text{const} > 0$, $k = \overline{1,5}$.

Substituting the estimates (3.11) — (3.15) in (3.10), we obtain the following recurrence inequality

$$\|z(t+\tau)\|_1^2 \leqslant (1+c\tau)\Bigg(\|z(t)\|_1^2 + \tau\|\phi\|^2_{\omega_{1h_1}^{+0,5}} + \tau\|\phi_{\bar{x}x}\|^2_{\omega_{1h_1}}$$

$$+\tau\sum_{t'=\tau}^{t}\tau\|\psi(t')\|^2_{\omega_{1h_1}} + \tau\|\psi(t)\|^2_{\omega_{2h_2}} + \frac{\tau}{h}\psi^2(t,\xi)\Bigg).$$

Hence we get *a priori* estimate for the error of method

$$\|z(t_n+\tau)\|_1^2 \leqslant M\Bigg(\|\phi\|_1^2 + \|\phi_{\bar{x}x}\|^2_{\omega_{2h_2}}$$
$$+\sum_{t=\tau}^{t_n}\tau\left(\|\psi(t)\|^2_{\omega_{1h_1}} + \|\psi(t)\|^2_{\omega_{2h_2}} + \frac{1}{h}\psi^2(t,\xi)\right)\Bigg), \tag{3.16}$$

where $M$, $c$ are positive constants.

Thus, we have proved the following statement.

**Theorem 3.1.** *Under conditions (2.1) the solution of the difference scheme (2.2) — (2.6) converges to the solution of differential problem (1.1) — (1.7) with rate* $\mathcal{O}(\tau + h^{3/2})$ *in the grid norm* $\|\cdot\|_1$, *i.e. for the error* $z = y - u$ *the following estimate holds*

$$\|z\|_1 \leqslant c_0(\tau + h^{3/2}), \quad c_0 = \text{const} > 0.$$

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## SUJUNGIMO UŽDAVINYS, APRAŠANTIS BENDRĄ KLAMPAUS IR KLAMPIAIELASTINGO SKYSČIŲ TEKĖJIMĄ PLOKŠČIUOSE KANALUOSE

V.I. KORZYUK, S.V. LEMESHEVSKY, P.P. MATUS, V.N. SHALINA

Nagrinėjamos sujungimo sąlygos, kai liečiasi klampus ir klampiaielastingas skysčiai. Ištirtas diferencialinio uždavinio išsprendžiamumas. Uždavinys aproksimuojamas neišreikštine baigtinių skirtumų schema. Suderinamumo sąlygos aproksimuojamos antrosios tikslumo eilės diskrečiuoju analogu. Energetinių nelygybių metodu įrodomas baigtinių skirtumų schemos sprendinio konvergavimas ir įvertinamas konvergavimo greitis.