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A PIEZOELECTRIC CONTACT PROBLEM WITH SLIP DEPENDENT COEFFICIENT OF FRICTION

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Abstract. We consider a mathematical model which describes the static frictional contact between a piezoelectric body and an obstacle. The constitutive relation of the material is assumed to be electroelastic and involves a nonlinear elasticity operator. The contact is modelled with a version of Coulomb's law of dry friction in which the coefficient of friction depends on the slip. We derive a variational formulation for the model which is in form of a coupled system involving as unknowns the displacement field and the electric potential. Then we provide the existence of a weak solution to the model and, under a smallness assumption, we provide its uniqueness. The proof is based on a result obtained in [14] in the study of elliptic quasi-variational inequalities.

Key words: piezoelectric material, electroelasticity, static frictional contact, Coulomb's law, slip dependent coefficient of friction, quasivariational inequality, weak solution

1. Introduction

The piezoelectric effect was discovered in 1880 by Jacques and Pierre Curie; it consists on the apparition of electric charges on the surfaces of some crystals after their deformation. The reverse effect was outlined in 1881; it consists on the generation of stress and strain in crystals under the action of electric field on the boundary. A deformable material which undergoes piezoelectric effects is called a *piezoelectric* material. An elastic material with piezoelectric effect is called an *electroelastic* material and the discipline dealing with the study of electroelastic materials is the theory of *electroelasticity*. Their bases were underlined by Voigt [24] who provided the first mathematical model of a linear elastic material which takes into account the interaction between mechanical and electrical properties.

General models for elastic materials with piezoelectric effects can be found in [10, 11, 12, 22, 23] and, more recently, in [1, 21]. Currently, there is a considerable interest in frictional contact problems involving piezoelectric materials, see for instance [2, 9] and the references therein. Indeed, situations which involve contact phenomena abound in industry and everyday life. The contact of the braking pads with the wheel, the tire with the road and the piston with skirt are just three simple examples. Because of the importance of contact processes a considerable effort has been made in their modelling and the engineering literature concerning this topic is extensive. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies to models for contact with deformable bodies which include coupling between mechanical and electrical properties.

The aim of this paper is to provide such an extension. Indeed, we consider here a model for the process of frictional contact between an electrolastic body, which is acted upon by forces and electric charges, and a foundation. The process is static, the contact is frictional and it is modeled with a version of Coulomb's law of dry friction in which the coefficient of friction depends on the slip. Such kind of dependence was pointed out in [18] in the study of the stick-slip phenomenon and was considered in various papers, see for instance [16, 19]. Frictional contact boundary value problems with elastic materials and slip dependent friction were considered in [3, 6] in the static case and in [4] in the quasistatic case. Here we extend the frictional model in [3] to the case of nonlinear electroelastic materials. Taking into account the piezoelectric behavior of the body consists the main trait of novelty of the model. We derive a variational formulation of the model then we prove its weak solvability and, under an additional assumption, its unique solvability. As in [3], the proof of these results are based on an abstract theorem on quasivariational inequalities derived in [14]; however, keeping in mind the coupling of the electrical and mechanical effects, we apply this result in a different setting and with a different choice of operators and functionals. An important continuation of this paper consists in the numerical analysis of the model, including numerical simulations, and will be presented in a forthcoming work.

The paper is structured as follows. In Section 2 we state the model of the equilibrium process of the elastic piezoelectric body in frictional contact with a foundation. In Section 3 we introduce some preliminary material, list assumptions on the problem data and state our main existence and uniqueness result, Theorem 1. The proof of the theorem is presented in Section 5; it is based on an abstract existence and uniqueness result that we recall in Section 4.

2. Problem Statement

We consider the following physical setting. An elastic piezoelectric body occupies a bounded domain $\Omega \subset \mathbb{R}^d,\ d = 2, 3$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2, Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$. We assume that the body is clamped on Γ_1 and surfaces tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body is in frictional contact with an obstacle, the so-called foundation. We model the contact with a version of Coulomb's law of dry friction, already used in [3] and [6], in which the normal stress is prescribed and the coeffi cient of friction depends on the slip. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Also, below $\boldsymbol{\nu}$ represents the unit outward normal on Γ while "$\cdot$" and $\|\cdot\|$ denote the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively.

With the assumption above, the problem of equilibrium of the electroelastic body in frictional contact with a foundation is the following.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$*, an electric potential* $\varphi : \Omega \to \mathbb{R}$ *and an electric displacement field* $\boldsymbol{D} : \Omega \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^T\boldsymbol{E}(\varphi) \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\text{div}\,\boldsymbol{D} = q_0 \quad \text{in} \quad \Omega, \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{2.5}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{2.6}$$

$$-\sigma_\nu = S \quad \text{on} \quad \Gamma_3, \tag{2.7}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \le \mu(\|\boldsymbol{u}_\tau\|)|S|, & \text{on} \quad \Gamma_3, \\ \boldsymbol{\sigma}_\tau = -\mu(\|\boldsymbol{u}_\tau\|)|S|\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|}, \ \text{if } \boldsymbol{u}_\tau \neq \boldsymbol{0} & \end{cases} \tag{2.8}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{2.9}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = q_2 \quad \text{on} \quad \Gamma_b. \tag{2.10}$$

In (2.1) – (2.10) and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equations (2.1) and (2.2) represent the electroelastic constitutive law of the material in which $\mathcal{F}$ is a given nonlinear function, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the small strain tensor, $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric fi eld, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^T$ is its transposite and $\boldsymbol{\beta}$ denotes the electric permitivitty tensor. Details of the linear version of the constitutive relations (2.1) and (2.2) can be fi nd in [1, 2]. Equations (2.3) and (2.4) represent the equilibrium equations for the stress and electric-displacement fi elds, respectively, (2.5) and (2.6) are the displacement and traction boundary conditions, respectively, and (2.9), (2.10) represent the electric boundary conditions.

We now provide some comments on the frictional contact conditions (2.7) and (2.8), which are our main interest. Condition (2.7) states that the normal stress σ_ν

is prescribed on Γ_3 since S denotes a given function. Condition (2.8) represents the associated friction law in which $\boldsymbol{\sigma}_\tau$ is the tangential stress, $\boldsymbol{u}_\tau$ denotes the tangential displacement and μ is the coefficient of friction. This law should be seen either as a mathematical model suitable for proportional loadings or as a first approximation of a more realistic model, based on a friction law involving the time derivative of $\boldsymbol{u}_\tau$ (see for instance [4, 13]). Note that in (2.8) the coefficient of friction depends on the slip $\|\boldsymbol{u}_\tau\|$ which leads to a nonstandard frictional contact problem.

3. Variational Formulations and Main Result

In this section we list the assumptions on the data, derive a variational formulation for the contact problem (2.1) – (2.10) and state our main existence and uniqueness result, Theorem 1. To this end we need to introduce notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i\ , \quad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \quad \forall \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\ , \ \|\tau\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d.$$

Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$.

Everywhere below we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$ and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces:

$$L^2(\Omega)^d = \{\ \boldsymbol{v} = (v_i) \mid v_i \in L^2(\Omega)\ \}, \quad H^1(\Omega)^d = \{\ \boldsymbol{v} = (v_i) \mid v_i \in H^1(\Omega)\ \},$$
$$\mathcal{H} = \{\ \boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \boldsymbol{\tau}\in\mathcal{H} \mid \tau_{ij,j}\in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$, are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx, \quad (\boldsymbol{u},\boldsymbol{v})_{H^1(\Omega)^d} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx + \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_{L^2(\Omega)^d}$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here $\boldsymbol{\varepsilon} : H_1 \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, that is

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j}+v_{j,i}) \quad \forall \boldsymbol{v}\in H^1(\Omega)^d,$$

$$\mathrm{Div}\,\boldsymbol{\tau} = (\tau_{ij,j}) \quad \forall \boldsymbol{\tau}\in\mathcal{H}_1.$$

For every element $\boldsymbol{v}\in H^1(\Omega)^d$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$.

Let us now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\ \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K\,\|\boldsymbol{v}\|_{H^1(\Omega)^d} \quad \forall\, \boldsymbol{v} \in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. A proof of Korn's inequality can be found in, for instance, [15] p. 79. Over the space V we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}\|_V \qquad \forall \boldsymbol{v} \in V. \tag{3.3}$$

We also introduce the spaces

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \},$$

$$\mathcal{W} = \{\ \boldsymbol{D} = (D_i) \mid D_i \in L^2(\Omega),\ \operatorname{div} \boldsymbol{D} \in L^2(\Omega)\ \},$$

where div $\boldsymbol{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products

$$(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)}, \quad (\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} = (\boldsymbol{D}, \boldsymbol{E})_{L^2(\Omega)^d} + (\operatorname{div} \boldsymbol{D}, \operatorname{div} \boldsymbol{E})_{L^2(\Omega)}.$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Notice also that, since $meas\,(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_{L^2(\Omega)^d} \geq c_F\,\|\psi\|_W \quad \forall\, \psi \in W, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a.

In the study of the contact problem (2.1) – (2.10) we assume that

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\[4pt]
\text{(b) There exists } M_{\mathcal{F}} > 0 \text{ such that}\\
\qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \leq M_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.\\[4pt]
\text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that}\\
\qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1)) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2\\
\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.\\[4pt]
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \xi) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\xi} \in \mathbb{S}^d.\\[4pt]
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{3.5}$$

$$\begin{cases} \text{(a) } \mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\ \text{(b) } e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{3.6}$$

$$\begin{cases} \text{(a) } \boldsymbol{\beta} = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\ \text{(b) } \beta_{ij} = \beta_{ji} \in L^\infty(\Omega). \\ \text{(c) There exists } m_\beta > 0 \text{ such that } \beta_{ij}(\boldsymbol{x}) E_i E_j \geq m_\beta \|\boldsymbol{E}\|^2 \ \ \forall\, \boldsymbol{E} \in \mathbb{R}^d, \\ \qquad \text{a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{3.7}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_3)^d \tag{3.8}$$

$$q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b), \tag{3.9}$$

$$S \in L^\infty(\Gamma_3) \quad \text{and} \quad \|S\|_{L^\infty(\Gamma_3)} > 0\cdot \tag{3.10}$$

$$\begin{cases} \text{(a) } \mu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exist } c_1^\mu \geq 0 \text{ and } c_2^\mu \geq 0 \text{ such that} \\ \qquad \mu(\boldsymbol{x}, r) \leq c_1^\mu |r| + c_2^\mu \qquad \forall\, r \in \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mu(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_3 \text{ for any } r \in \mathbb{R}. \\ \text{(d) The mapping } r \mapsto \mu(\boldsymbol{x}, r) \text{ is continuous on } \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.11}$$

$$\begin{cases} \text{There exists } L_\mu > 0 \text{ such that} \\ (\mu(\boldsymbol{x}, r_2) - \mu(\boldsymbol{x}, r_1)) \cdot (r_1 - r_2) \leq L_\mu\, |r_1 - r_2|^2 \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.12}$$

We make in what follows some comments on the assumptions (3.5) – (3.12). As stated in Section 2, below we suppress the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$.

First, we note that the condition (3.5) is satisfi ed in the case of the linear elastic constitutive law $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ in which

$$\mathcal{F}\boldsymbol{\xi} = (f_{ijkl}\xi_{kl}), \tag{3.13}$$

provided that $f_{ijkl} \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$f_{ijkl}(\boldsymbol{x})\xi_k\xi_l \geq \alpha\|\boldsymbol{\xi}\|^2 \qquad \forall\, \boldsymbol{\xi} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.$$

To provide examples of nonlinear constitutive laws which satisfy (3.5), for every tensor $\boldsymbol{\xi} \in \mathbb{S}^d$ we denote by $tr\,\boldsymbol{\xi}$ the trace of $\boldsymbol{\xi}$ and let $\boldsymbol{\xi}^D$ be the deviatoric part of $\boldsymbol{\xi}$ given by

$$tr\,\boldsymbol{\xi} = \xi_{ii}, \qquad \boldsymbol{\xi}^D = \boldsymbol{\xi} - \frac{1}{d}\,(tr\,\boldsymbol{\xi})\boldsymbol{I}_d,$$

where $\boldsymbol{I}_d \in \mathbb{S}^d$ represents the identity tensor. Let K denotes a nonempty closed convex set in $\mathbb{S}^d$ and let P_K represents the projection mapping on K. We also consider a forth order symmetric and positively defined tensor $\mathcal{E} : \mathbb{S}^d \to \mathbb{S}^d$ and take

$$\mathcal{F}(\boldsymbol{\xi}) = \mathcal{E}\boldsymbol{\xi} + \frac{1}{\lambda}\,(\boldsymbol{\xi} - P_K\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \mathbb{S}^d, \tag{3.14}$$

where λ is a strictly positive constant. Using the properties of the projection mapping it is straightforward to see that the elasticity operator $\mathcal{F}$ defined by (3.14) satisfies condition (3.5). Constitutive laws of the form $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}))$ with $\mathcal{F}$ given by (3.14) have been considered by many authors, see. e.g. [8], [17] p. 97 and [20] p. 68. Most of them have defined the convex K by the relationship $K = \{\,\boldsymbol{\xi} \in \mathbb{S}^d \mid G(\boldsymbol{\xi}) \leq k\}$ where $G : \mathbb{S}^d \to \mathbb{R}$ is a convex continuous function such that $G(\boldsymbol{0}) = 0$ and $k > 0$.

A second example of nonlinear elastic equations is provided by nonlinear Hencky materials (see [25] for details). For a Hencky material, the stress-strain relation is given by

$$\boldsymbol{\sigma} = K_0(\operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\boldsymbol{I}_d + \psi(\|\boldsymbol{\varepsilon}^D(\boldsymbol{u})\|^2)\,\boldsymbol{\varepsilon}^D(\boldsymbol{u}),$$

so that the elasticity operator is

$$\mathcal{F}(\boldsymbol{\xi}) = K_0(\operatorname{tr}\boldsymbol{\xi})\,\boldsymbol{I}_d + \psi(\|\boldsymbol{\xi}^D\|^2)\,\boldsymbol{\xi}^D \quad \forall\,\boldsymbol{\xi} \in \mathbb{S}^d. \tag{3.15}$$

Here, $K_0 > 0$ is a material coefficient, the function ψ is assumed to be piecewise continuously differentiable, and there exist positive constants c_1, c_2, d_1 and d_2, such that for $s \geq 0$

$$\psi(s) \leq d_1, \quad -c_1 \leq \psi'(s) \leq 0, \quad c_2 \leq \psi(s) + 2\,\psi'(s)\,s \leq d_2.$$

Under these assumption it can be shown that the elasticity operator $\mathcal{F}$ defined in (3.15) satisfies condition (3.5).

Next, as it is shown in (3.6) and (3.7), we see that the piezoelectric operator $\mathcal{E}$ as well as the electric permitivitty operator $\boldsymbol{\beta}$ are assumed to be linear and, moreover, $\boldsymbol{\beta}$ is symmetric and positive definite. Recall also that the transposite tensor $\mathcal{E}^T$ is given by $\mathcal{E}^T = (e^T_{ijk})$ where $e^T_{ijk} = e_{kij}$, and the following equality holds:

$$\mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\boldsymbol{v} \qquad \forall\boldsymbol{\sigma} \in \mathbb{S}^d,\ \boldsymbol{v} \in \mathbb{R}^d. \tag{3.16}$$

We also remark that (3.8) represent regularity assumptions on the densities of volume forces and surface tractions while (3.9) represent regularity assumptions on the densities of volume and surface electric charges. Condition $\|S\|_{L^\infty(\Gamma_3)} > 0$ in (3.10) is imposed here in order to obtain a genuine frictional contact problem. Indeed, if $S = 0$ a.e. on Γ_3 then by (2.7) and (2.8) it follows that the Cauchy stress vector $\boldsymbol{\sigma\nu}$ vanishes on Γ_3 and therefore problem (2.1) – (2.10) becomes a purely displacement-traction problem for electroelastic materials.

Finally, we observe that the assumptions (3.11) on the coefficient of friction μ are pretty general. Clearly, these assumptions are satisfied if μ is a bounded function which is continuously differentiable with respect to the second variable, as it was considered in [6]. We also remark that assumptions (3.11) and (3.12) are satisfied if μ does not depend on the second argument and is a positive function which belongs to $L^\infty(\Gamma_3)$. This is the case when the coefficient of friction does not depend on the slip. Frictional contact problems involving this last assumption on the coefficient of friction were studied in [5, 17] in the case of purely elastic materials. Notice also that assumption (3.12) is satisfied if $\mu(\boldsymbol{x},\cdot):\mathbb{R}\to\mathbb{R}_+$ is an increasing function, a.e. $\boldsymbol{x}\in\Gamma_3$.

We now turn to the variational formulation of Problem P and, to this end, we introduce further notation. Let $h: V\times V\longrightarrow\mathbb{R}$ be the functional

$$h(\boldsymbol{u},\boldsymbol{v})=\int_{\Gamma_3}\mu(\|\boldsymbol{u}_\tau\|)\,|S|\,\|\boldsymbol{v}_\tau\|da,\quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V \tag{3.17}$$

and, using Riesz's representation theorem, consider the elements $\boldsymbol{f}\in V$ and $q\in W$ given by

$$(\boldsymbol{f},\boldsymbol{v})_V=\int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2}\boldsymbol{f}_2\cdot\boldsymbol{v}\,da+\int_{\Gamma_3}S\,v_\nu\,da\qquad\forall\,\boldsymbol{v}\in V, \tag{3.18}$$

$$(q,\psi)_W=\int_\Omega q_0\psi\,dx+\int_{\Gamma_b}q_2\psi\,da\qquad\forall\,\psi\in W. \tag{3.19}$$

Keeping in mind assumptions (3.8) – (3.11) it follows that the integrals in (3.17) – (3.19) are well-defined.

Using integration by parts, it is straightforward to see that if $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{D})$ are sufficiently regular functions which satisfy (2.3) – (2.10) then

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}+h(\boldsymbol{u},\boldsymbol{v})-h(\boldsymbol{u},\boldsymbol{u})\ge(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V\quad\forall\,\boldsymbol{v}\in V, \tag{3.20}$$

$$(\boldsymbol{D},\psi)_{L^2(\Omega)^d}=(q,\psi)_W\quad\forall\psi\in W. \tag{3.21}$$

We plug (2.1) in (3.20), (2.2) in (3.21) and use the notation $\boldsymbol{E}=-\nabla\varphi$ to obtain the following variational formulation of Problem P, in the terms of displacement field and electric potential.

Problem P_V**.** *Find a displacement field $\boldsymbol{u}\in V$ and an electric potential $\varphi\in W$ such that*

$$\begin{aligned}(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}}+(\mathcal{E}^T\nabla\varphi,\boldsymbol{v}-\boldsymbol{u})_{L^2(\Omega)^d}&\\+h(\boldsymbol{u},\boldsymbol{v})-h(\boldsymbol{u},\boldsymbol{u})\ge(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V\quad\forall\boldsymbol{v}\in V,&\end{aligned} \tag{3.22}$$

$$(\boldsymbol{\beta}\nabla\varphi,\nabla\psi)_{L^2(\Omega)^d}-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}),\nabla\psi)_{L^2(\Omega)^d}=(q,\psi)_W\quad\forall\,\psi\in W. \tag{3.23}$$

Our main existence and uniqueness result which we establish in Section 5 is the following.

Theorem 1. *Assume that* (3.5)–(3.10) *hold. Then :*
1) Under the assumption (3.11)*, Problem P_V has at least one solution.*
2) Under the assumptions (3.11) *and* (3.12)*, there exists L_0, which depends only on Ω, Γ_1, Γ_3, Γ_a, $\mathcal{F}$, $\boldsymbol{\beta}$, S, such that if $L_\mu<L_0$ then Problem P_V has unique solution $(\boldsymbol{u},\varphi)$ which depends Lipschitz continuously on $\boldsymbol{f}\in V$ and $q\in W$.*

A “quadriplet" of functions $(\boldsymbol{u},\,\boldsymbol{\sigma},\,\varphi,\,\boldsymbol{D})$ which satisfy (2.1), (2.2), (3.22) and (3.23) is called a *weak solution* of the piezoelectric contact problem P. We conclude by Theorem 1 that, under the assumptions (3.5) – (3.11), the piezoelectric contact problem (2.2) – (2.10) has at least a weak solution $(\boldsymbol{u},\,\boldsymbol{\sigma},\,\varphi,\,\boldsymbol{D})$ such that $\boldsymbol{u} \in V$, $\varphi \in W$. Moreover, it is easy to see that in this case $\boldsymbol{\sigma} \in \mathcal{H}_1$ and $\boldsymbol{D} \in \mathcal{W}$. The solution is unique and depends Lipschitz continuously on the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, q_0 and q_2 if (3.12) holds with a sufficiently small constant L_μ. In particular, this case arise when the coefficient of friction is a given positive bounded function which does not depend on the slip.

4. An Abstract Existence and Uniqueness Result

To prove Theorem 1 we shall use an abstract existence and uniqueness result on elliptic quasivariational inequalities that we recall in what follows, for the convenience of the reader.

Everywhere in this section X will represent a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We denote by “ $\rightharpoonup$" the weak convergence on X. Let $A : X \longrightarrow X$ be a non linear operator, $j : X \times X \longrightarrow \mathbb{R}$ and $f \in X$. With these data we consider the following quasivariational inequality: find $x \in X$ such that

$$(Ax, y - x)_X + j(x, y) - j(x, x) \geq (f, y - x)_X \quad \forall\, y \in X \tag{4.1}$$

In order to solve (4.1) we assume that A is strongly monotone and Lipschitz continuous, i.e.

$$\begin{cases} \text{(a) There exists } m > 0 \text{ such that} \\ \qquad (Ax_1 - Ax_2, x_1 - x_2)_X \geq m\|x_1 - x_2\|_X^2 \quad \forall\, x_1, x_2 \in X. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \qquad \|Ax_1 - Ax_2\|_X \leq M\|x_1 - x_2\|_X \quad \forall\, x_1, x_2 \in X. \end{cases} \tag{4.2}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies

$$j(\eta, \cdot) : X \to \mathbb{R} \quad \text{is a convex functional on } X, \text{ for all } \eta \in X. \tag{4.3}$$

Keeping in mind (4.3) it is well known that there exists the directional derivative of j with respect to the second argument given by

$$j_2'(\eta, x; y) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}\Big[j(\eta, x + \lambda y) - j(\eta, x)\Big] \qquad \forall \eta,\, x,\, y \in X. \tag{4.4}$$

We formulate in what follows some conditions on j and we recall that below m represents the positive constant defined in (4.2).

$$\begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ with } \|x_n\|_X \to \infty \\ \text{and every sequence } \{t_n\} \subset [0,1] \text{ one has} \\ \liminf_{n\to\infty} \Big[\dfrac{1}{\|x_n\|_X^2} j'(t_n x_n, x_n; -x_n)\Big] < m. \end{cases} \tag{4.5}$$

$$\begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ with } \|x_n\|_X \to \infty \\ \text{and every bounded sequence } \{\eta_n\} \subset X \text{ one has} \\ \liminf_{n\to\infty} \Big[\dfrac{1}{\|x_n\|_X^2} j'(\eta_n, x_n; -x_n)\Big] < m. \end{cases} \tag{4.6}$$

$$\begin{cases} \text{For every sequences } \{x_n\} \subset X \text{ and } \{\eta_n\} \subset X \text{ such that} \\ x_n \rightharpoonup x \in X,\ \eta_n \rightharpoonup \eta \in X \text{ and for every } y \in X \text{ one has} \\ \limsup_{n\to\infty} \big[j(\eta_n, y) - j(\eta_n, x_n)\big] \le j(\eta, y) - j(\eta, x). \end{cases} \tag{4.7}$$

$$\begin{cases} \text{There exists } \alpha < m \text{ such that} \\ j(x,y) - j(x,x) + j(y,x) - j(y,y) \le \alpha\, \|x-y\|_X^2 \quad \forall x, y \in X. \end{cases} \tag{4.8}$$

In the study of the quasivariational inequality (4.1) we have the following result.

Theorem 2. *Let conditions* (4.2) – (4.3) *hold. Then :*
1) Under the assumptions (4.5) – (4.7) *there exists at least one element* $x \in X$ *which solves* (4.1).
2) Under the assumptions (4.5) – (4.8)*, problem* (4.1) *has unique solution* $x = x_f$ *which depends Lipschitz continuously on* f *with the Lipschitz constant* $(m-\alpha)^{-1}$.

Theorem 2 has been obtained in [14] and therefore we do not provide here the details of the proof. We just specify that the proof was obtained in several steps and it is based on standard arguments of elliptic variational inequalities and topological degree theory.

5. Proof of Theorem 1

The proof of Theorem 1 will be carried out in several steps. To present it we consider the product space $X = V \times W$ together with the inner product

$$(x,y)_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall x = (\boldsymbol{u}, \psi),\ y = (\boldsymbol{v}, \psi) \in X \tag{5.1}$$

and the associated norm $\|\cdot\|_X$. Everywhere below we assume that (3.5) – (3.11) hold.

We introduce the operator $A : X \to X$ defined by

$$\begin{aligned}(Ax, y) = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} + (\mathcal{E}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} \quad \forall x = (\boldsymbol{u}, \psi),\ y = (\boldsymbol{v}, \psi) \in X\end{aligned} \tag{5.2}$$

and we extend the functional h defined by (3.17) to a functional j defined on $X \times X$, that is

$$j(x,y) = h(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall x = (\boldsymbol{u}, \psi),\ y = (\boldsymbol{v}, \psi) \in X. \tag{5.3}$$

Finally, we consider the element $f \in X$ given by

$$f = (\boldsymbol{f}, q) \in X. \tag{5.4}$$

We start with the following equivalence result.

Lemma 1. *The couple $x = (\boldsymbol{u}, \varphi)$ is a solution to Problem P_V if and only if*

$$(Ax, y - x)_X + j(x, y) - j(x, x) \geq (f, y - x)_X \quad \forall\, y \in X. \tag{5.5}$$

Proof. Let $x = (\boldsymbol{u}, \varphi) \in X$ be a solution to Problem P_V and let $y = (\boldsymbol{v}, \psi) \in Y$. We use the test function $\psi - \varphi$ in (3.21), add the corresponding inequality to (3.20) and use (5.1) – (5.4) to obtain (5.5). Conversely, let $x = (\boldsymbol{u}, \varphi) \in X$ be a solution to the quasivariational inequality (5.5). We take $y = (\boldsymbol{v}, \varphi)$ in (5.5) where $\boldsymbol{v}$ is an arbitrary element of V and obtain (3.22); then we take successively $y = (\boldsymbol{v}, \varphi + \psi)$ and $y = (\boldsymbol{v}, \varphi - \psi)$ in (5.5), where ψ is an arbitrary element of W; as a result we obtain (3.23), which concludes the proof. ■

Notice that the quasivariational inequality (5.5) derived in Lemma1 is of the form (4.1). Therefore, in order to apply the abstract result provided by Theorem 2, we start with the study of the the properties of the operator A given by (5.2).

Lemma 2. *The operator $A : X \to X$ is strongly monotone and Lipschitz continuous.*

Proof. Consider two elements $x_1 = (\boldsymbol{u}_1, \varphi_1)$, $x_2 = (\boldsymbol{u}_2, \varphi_2) \in X$. Using (5.2) we have

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} \\ &+ (\boldsymbol{\beta}\nabla\varphi_1 - \boldsymbol{\beta}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d} + (\mathcal{E}^T\nabla\varphi_1 \\ &- \mathcal{E}^T\nabla\varphi_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d}\end{aligned}$$

and, since it follows by (3.16) that $(\mathcal{E}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\varphi)_{L^2(\Omega)^d}$ for all $x = (\boldsymbol{u}, \varphi) \in X$, we fi nd

$$\begin{aligned}&(Ax_1 - Ax_2, x_1 - x_2)_X = \\ &\quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi_1 - \boldsymbol{\beta}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d}.\end{aligned}$$

We use now (3.5), (3.7) and Friedrichs-Poincaré inequality (3.4) to see that there exists $c_1 > 0$ which depends only on $\mathcal{F}$, $\boldsymbol{\beta}$, Ω and Γ_a such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq c_1(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2)$$

and, keeping in mind (5.1), we obtain

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq c_1 \,\|x_1 - x_2\|_X^2. \tag{5.6}$$

In the same way, using (3.5) – (3.7), after some algebra it follows that there exists $c_2 > 0$ which depends only on $\mathcal{F}$, $\boldsymbol{\beta}$ and $\mathcal{E}$ such that

$$\begin{aligned}(Ax_1 - Ax_2, y)_X \geq\, &c_2(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V + \|\varphi_1 - \varphi_2\|_W\|\boldsymbol{v}\|_V \\ &+ \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\psi\|_W + \|\varphi_1 - \varphi_2\|_W\|\psi\|_W)\end{aligned}$$

for all $y = (\boldsymbol{v}, \psi) \in V$. We use (5.1) and the previous inequality to obtain

$$(Ax_1 - Ax_2, y)_X \le 4c_2 \, \|x_1 - x_2\|_V \|y\|_V \qquad \forall y \in X$$

and, taking $y = Ax_1 - Ax_2 \in X$, we find

$$\|Ax_1 - Ax_2\|_X \le 4c_2 \, \|x_1 - x_2\|_V. \tag{5.7}$$

Lemma 2 is now a consequence of inequalities (5.6) and (5.7). ■

Next we investigate the properties of the functional j given by (5.3), (3.17). We first remark that j satisfies condition (4.3). Moreover, we have the following results.

Lemma 3. *The functional j satisfies conditions* (4.5), (4.6) *and* (4.7).

Proof. Let $\eta = (\boldsymbol{w}, \xi)$, $x = (\boldsymbol{u}, \varphi) \in X$ and let $\lambda \in]0, 1]$. Using (5.3) and (3.17) it results that

$$j(\eta, x - \lambda x) - j(\eta, x) = -\lambda \int_{\Gamma_3} \mu(\|\boldsymbol{w}_\tau\|) \, |S| \, \|\boldsymbol{u}_\tau\| \, da$$

and, keeping in mind (4.4), we deduce that

$$j_2'(\eta, x; -x) \le 0 \qquad \forall \eta, \, x \in X. \tag{5.8}$$

We conclude by (5.8) that the functional j satisfies conditions (4.5) and (4.6).

Let now consider two sequences $\{x_n\} = \{(\boldsymbol{u}_n, \varphi_n)\} \subset X$ and $\{\eta_n\} = \{(\boldsymbol{w}_n, \xi_n)\} \subset X$ such that $x_n \rightharpoonup x = (\boldsymbol{u}, \varphi) \in X$, $\eta_n \rightharpoonup \eta = (\boldsymbol{w}, \xi) \in X$. Using the compactness property of the trace map it follows that $\boldsymbol{u}_n \to \boldsymbol{u}$ and $\boldsymbol{w}_n \to \boldsymbol{w}$ in $L^2(\Gamma_3)^d$, which imply that

$$\|\boldsymbol{u}_{n\tau}\| \to \|\boldsymbol{u}_\tau\| \quad \text{in} \quad L^2(\Gamma_3), \tag{5.9}$$

$$\|\boldsymbol{w}_{n\tau}\| \to \|\boldsymbol{w}_\tau\| \quad \text{in} \quad L^2(\Gamma_3). \tag{5.10}$$

Moreover, (3.12), (5.10) and Kranoselski's theorem (see for instance [7]) yield

$$\mu(\|\boldsymbol{w}_{n\tau}\|) \to \mu(\|\boldsymbol{w}_\tau\|) \quad \text{in} \quad L^2(\Gamma_3). \tag{5.11}$$

Therefore, we use the definition of j, (5.9) and (5.11) to deduce that

$$j(\eta_n, y) \to j(\eta, y) \quad \forall y \in X \quad \text{and} \quad j(\eta_n, x_n) \to j(\eta, x), \quad \text{as } n \to \infty.$$

We conclude that the functional j satisfies the condition (4.7). ■

Lemma 4. *If* (3.12) *holds, then the functional j satisfies the inequality*

$$j(x, y) - j(x, x) + j(y, x) - j(y, y) \le c_0^2 L_\mu \|S\|_{L^\infty(\Gamma_3)} \|x - y\|_X^2 \quad \forall x, y \in X. \tag{5.12}$$

Proof. Let $x = (\boldsymbol{u}, \varphi)$, $y = (\boldsymbol{v}, \psi) \in V$. Using (5.3), (3.17) and (3.12) it follows that

$$\begin{aligned} &j(x,y) - j(x,x) + j(y,x) - j(y,y) \\ &\quad = \int_{\Gamma_3} |S| \left(\mu(\|\boldsymbol{u}_\tau\|) - \mu(\|\boldsymbol{v}_\tau\|)\right)(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, da \\ &\quad \leq L_\mu \|S\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \Big| \|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\| \Big|^2 da \leq L_\mu \|S\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \|\boldsymbol{u} - \boldsymbol{v}\|^2\, da. \end{aligned}$$

Using now (3.3) and (5.1) in the previous inequality we deduce (5.12). ■

We have now all the ingredients to prove the Theorem.

Proof. [Proof of Theorem 1.]

1) Assume that (3.5) – (3.11) hold. Then, Lemmas 2 and 3 allow us to use the abstract results provided by the first part of Theorem 2; we obtain that the quasivariational inequality (5.5) has at least a solution $x = (\boldsymbol{u}, \varphi) \in X$ and, using Lemma1, we deduce that $(\boldsymbol{u}, \varphi)$ is a solution to Problem P_V, which satisfies $\boldsymbol{u} \in V$, $\varphi \in W$.

2) Assume that (3.5) – (3.12) hold and let $L_0 = \dfrac{c_1}{c_0^2 \|S\|_{L^\infty(\Gamma_3)}}$ where c_1 and c_0 are defined by (5.6) and (3.3), respectively. Clearly L_0 depends only on Ω, Γ_1, Γ_3, Γ_a, $\mathcal{F}$, $\boldsymbol{\beta}$ and S. Let now assume that $L_\mu < L_0$. Then, there exists $\alpha \in \mathbb{R}$ such that $c_0^2 L_\mu \|S\|_{L^\infty(\Gamma_3)} < \alpha < c_1$. Using (5.12) and (5.6) we obtain that the functional j satisfies condition (4.8). Therefore, by the second part of Theorem 2, Lemma 1 and (5.4), we obtain that problem P_V has a unique solution which depends Lipschitz continuously on $\boldsymbol{f} \in V$ and $q \in W$, which concludes the proof. ■

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Pjezoelektriko sąlyčio su priklausomu nuo slydimo trinties koeficiento uždavinys

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Mes nagrinėjame matematinį modelį, kuris aprašo sąlytį tarp pjezoelektriko ir kli¯uties. Laikoma, kad medžiaga yra elektroelastinė ir nusakoma netiesiniu elastingumo operatoriumi. Sąlytis modeliuojamas remiamtis sausos trinties Coulomb'o dėsniu, kuriame trinties koeficientas priklauso nuo slydimo. Mes gavome variacinį modelio formulavimą lygčių sistemos formoje, kurios nežinomaisiais yra perkeltasis laukas ir elektrinis potencialas. Įrodomas sprendinio silpnąja prasme egzistavimas ir su nedidelėmis prielaidomis vienatis. Įrodymas paremtas rezultatais gautais [14] darbe, kuriame tiriamos elipsinės kvazivariacinės nelygybės.