ON THE UNIPOTENCE OF AUTOEQUIVALENCES OF TORIC COMPLETE INTERSECTION CALABI-YAU CATEGORIES

MANFRED HERBST AND JOHANNES WALCHER

Abstract. We identify a class of autoequivalences of triangulated categories of singularities associated with Calabi-Yau complete intersections in toric varieties. Elements of this class satisfy relations that are directly linked to the toric data.

0. Introduction

Let X be a smooth projective variety over $\mathbb{C}$, and $D_1, \ldots, D_k$ with $k \leq \dim(X)+1$ effective divisor classes satisfying

$$D_1 \cap D_2 \cap \cdots \cap D_k = 0. \tag{1}$$

For each $i = 1, \ldots, k$, we consider the autoequivalence of the bounded derived category of coherent sheaves, $D^b(X)$, given by tensoring with the line bundle $\mathcal{O}(D_i)$, i.e. for $A \in D^b(X)$,

$$\mathcal{M}_i(A) = A \otimes \mathcal{O}(D_i).$$

Choosing for each i a generic section s_i of $\mathcal{O}(D_i)$, we let $\mathcal{N}(s_i)$ be the endofunctor of $D^b(X)$ sending A to

$$\mathcal{N}(s_i)(A) = \mathit{Cone}(s_i : A \to \mathcal{M}_i(A)).$$

Since as a consequence of eq. (1), the complete intersection of $(s_1, \ldots, s_k)$ is empty, the associated Koszul complex $\mathcal{K}(s_1, \ldots, s_k)$ is exact. Therefore,

$$\mathcal{N}(s_1) \circ \mathcal{N}(s_2) \circ \cdots \circ \mathcal{N}(s_k)(A) \cong 0 \tag{2}$$

for any object A in $D^b(X)$. When pushed to K-theory, and denoting the image of $\mathcal{M}_i$ by $M_i : K(D^b(X)) \to K(D^b(X))$, we obtain the relation

$$\prod_{i=1}^{k} (M_i - \mathrm{id}) = 0.$$

In this paper we obtain generalizations of these formulas in a situation coming from toric geometry and of interest in mirror symmetry. We now summarize the ideas involved.

Let $\mathbb{P}_{\hat{\Sigma}}$ be a toric Calabi–Yau variety with fan $\hat{\Sigma}$. Toric varieties birationally equivalent to $\mathbb{P}_{\hat{\Sigma}}$ may be constructed by suitable modifcations of $\hat{\Sigma}$. We here think of modifications $\hat{\Phi}$ obtained by crossing a face in the secondary fan, giving rise to blowdowns, blowups, or flops. Following physics terminology, we shall refer to the various $\hat{\Phi}$ as different *phases* of the secondary fan. For each such modification

Date: November, 2009.

1991 *Mathematics Subject Classification.* Primary 18E30; Secondary 14J32.

CERN-PH-TH/2009-227

of $\hat{\Sigma}$, we have an equivalence of triangulated categories $D^b(\mathbb{P}_{\hat{\Sigma}}) \cong D^b(\mathbb{P}_{\hat{\Phi}})$. The corresponding functors have been described explicitly by van den Bergh [1] and Kawamata [2]. Clearly then, autoequivalences of $D^b(\mathbb{P}_{\hat{\Phi}})$ induce autoequivalences of $D^b(\mathbb{P}_{\hat{\Sigma}})$, and vice-versa. However, certain elements of $\mathrm{Aut}(D^b(\mathbb{P}_{\hat{\Sigma}}))$ and relations among them are easier to see in one description than in the other, which is going to be the main theme of our paper.

The primary interest, however, is in studying compact Calabi–Yau manifolds X that are complete intersection in $\mathbb{P}_\Sigma$ when the latter is a smooth, toric Fano. (This is not the most general case to which our methods apply, but one of much interest.) So let $P_1, P_2, \ldots, P_\ell$ be ℓ anti-effective divisor classes of $\mathbb{P}_\Sigma$, with $\sum_a P_a = K_{\mathbb{P}_{\hat{\Sigma}}}$. Choose generic sections G_a of $\mathcal{O}_{\mathbb{P}_{\hat{\Sigma}}}(-P_a)$ for $a = 1, 2, \ldots, \ell$, and consider

$$X := \{G_1 = 0, G_2 = 0, \ldots, G_\ell = 0\} \subset \mathbb{P}_\Sigma.$$

We recall some preliminaries about the derived category $D^b(X)$ in section 1. In section 2, we give a description of $D^b(X)$ in terms of a singularity category that will be useful for later purposes. We replace $\mathbb{P}_\Sigma$ with a larger combinatorial object $\mathbb{P}_{\hat{\Sigma}}$ that may be thought of as the total space of the bundle $\bigoplus_{a=1}^{l} \mathcal{O}(P_a) \to \mathbb{P}_\Sigma$, with fibers equipped with an additional grading called *R-grading*, a terminology from physics. Let p_a be the fibre coordinate on $\mathcal{O}(P_a)$. Then the function, a.k.a *superpotential* or *Landau–Ginzburg potential*,

$$W = \sum_{a=1}^{l} p_a G_a,$$

is the homological device that reduces $D^b(\mathbb{P}_{\hat{\Sigma}})$ to $D^b(X)$: The singularity category of W (namely, the triangulated category of the singularity $W : \mathbb{P}_{\hat{\Sigma}} \to \mathbb{C}$, in the sense of [3, 4]) is equivalent to $D^b(X)$ (see Theorem 2).

Then, for each modification $\hat{\Phi}$ of $\hat{\Sigma}$ in the above sense, one may construct a triangulated category, $\mathcal{C}_{\hat{\Phi}}$, as a quotient of singularity categories (Definition 4),

$$\mathcal{C}_{\hat{\Phi}} = \frac{D_{\mathrm{sg}}(\mathrm{gr}\hat{S})}{D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}}\hat{S})}.$$

This is done in section 3: The category $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ depends on W, but is independent of $\hat{\Phi}$, whereas the subcategory $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}}\hat{S})$ of torsion modules depends additionally on $\hat{\Phi}$. In fact, however, all $\mathcal{C}_{\hat{\Phi}}$ are equivalent as triangulated categories to the fixed category $\mathcal{C}_{\hat{\Sigma}} \cong D^b(X)$ (Theorem 3).

Working now in a fixed phase $\hat{\Phi}$, we associate in section 4 to any toric divisor class $D \in \mathrm{Pic}(\mathbb{P}_{\hat{\Sigma}})$ an automorphism $\mathcal{M}^D_{\hat{\Phi}}$ of $\mathcal{C}_{\hat{\Phi}}$ that can be thought of as tensoring with a line bundle. We emphasize that the equivalences amongst the $\mathcal{C}_{\hat{\Phi}}$ will not identify the $\mathcal{M}^D_{\hat{\Phi}}$ with each other as $\hat{\Phi}$ varies. (For a familiar example, see comments below.)

To describe the relations analogous to (2) among the $\mathcal{M}^D_{\hat{\Phi}}$, we recall that the toric divisors $D_1, \ldots, D_n$ of $\mathbb{P}_\Sigma$ are in one-to-one correspondence with the set of one-dimensional cones $\Sigma(1) = (v_1, \ldots, v_n)$ of Σ. We denote the canonical sections of $\mathcal{O}(D_i)$ by x_i, $i = 1, \ldots n$. For $\mathbb{P}_{\hat{\Sigma}}$, this list is extended by (the pullbacks of) the divisors $P_a =: D_{n+a}$, for $a = 1, \ldots, \ell$, with canonical section $p_a =: x_{n+a}$. This extended list is in one-to-one correspondence with the set $\hat{\Sigma}(1)$ of one-dimensional cones $(\hat{v}_i)_{1\le i\le n+\ell}$ of $\hat{\Sigma}$, and contains $\hat{\Phi}(1)$ for any $\hat{\Phi}$. We denote the R-grading of x_i by r_i.

$$r_i = \begin{cases} 0\,, & 1 \le i \le n\,, \\ 2\,, & n+1 \le i \le n+\ell\,. \end{cases}$$

Our main result will follow straightforwardly from these definitions in section 4.

Theorem 1. *For each toric divisor D_i, with canonical section x_i of $\mathcal{O}(D_i)$, define an endofunctor $\mathcal{N}_{\hat{\Phi}}$ of $\mathcal{C}_{\hat{\Phi}}$ by*

$$\mathcal{N}_{\hat{\Phi}}(x_i)(-) = Cone(x_i : - \longrightarrow - \otimes \mathcal{O}(D_i)[r_i]).$$

Then for each subset $\mathcal{I} \subset \{1, \dots, n+\ell\}$ such that the corresponding set of edges $(\hat{v}_i)_{i\in I}$ is not contained in any cone of $\hat{\Phi}$, we have the relation

$$\bigcirc_{i\in\mathcal{I}} \mathcal{N}_{\hat{\Phi}}(x_i) \cong 0.$$

A simple consequence is the following

Corollary 1. *Let $M^D_{\hat{\Phi}}$ be the automorphism induced by $\mathcal{M}^D_{\hat{\Phi}}$ on the K-theory $K(\mathcal{C}_{\hat{\Phi}})$. Then for each $\mathcal{I}$ as above,*

$$\prod_{i\in\mathcal{I}} \big(M^{D_i}_{\hat{\Phi}} - \mathrm{id}\big) = 0$$

Our original motivation for this work was to understand the generalization of an old observation of Kontsevich. We let $\mathcal{Y}$ be the family of Calabi–Yau manifolds that is mirror to X according to Batyrev's construction. Let B be the base of the family after removing the singular fibers (and possibly more). Monodromies around loops in B induce symplectic transformations (generalized Dehn twists) that can be lifted to autoequivalences of the symplectic category (the Fukaya category $Fuk(Y)$, where Y is a generic fiber of $\mathcal{Y}$). Via the homological mirror symmetry (HMS) conjecture, $Fuk(Y) \cong D^b(X)$, one is led to expect the existence of a monodromy representation

$$\rho : \pi_1(B) \to \mathrm{Aut}(D^b(X))$$

that has attracted some attention over the years.

When X is the quintic threefold, and $\mathcal{Y}$ its mirror family, we may model B as $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Note that the point at ∞ does not correspond to a singular threefold, but to one with an additional automorphism of order 5. Somewhat surprisingly, this symmetry is only realized projectively at the categorical level. Indeed, if γ_∞ is a path around ∞, the categorical monodromy $\mathcal{M}_\infty = \rho(\gamma_\infty) \in \mathrm{Aut}(D^b(X))$ (modulo HMS conjecture) satisfies the relation

$$(\mathcal{M}_\infty)^5 \cong (-)[2] \tag{3}$$

To fully appreciate this remarkable relation, we rewrite this in more familiar terms using $\gamma_\infty^{-1} = \gamma_0 \circ \gamma_1$. According to HMS for the quintic threefold, the monodromy γ_0 around $0 \in \mathbb{P}^1$ corresponds to the autoequivalence $\mathcal{M}_0$ of tensoring with the line bundle $\mathcal{O}_X(1)$, whilst monodromy γ_1 (around the conifold) is realized as twist $\mathcal{T}_{\mathcal{O}_X}$ by the structure sheaf [5]. These transformations of $D^b(X)$ can be checked to satisfy [6, Chapter 7.1.4]

$$\big(\mathcal{M}_0 \circ \mathcal{T}_{\mathcal{O}_X}\big)^5 \cong (-)[2],$$

which is the way in which (3) is often quoted. (Some hypersurfaces in weighted projective space are treated in [7].)

Our results give a uniform treatment of such relations for the general class of Calabi–Yau complete intersections in toric varieties. We elaborate on this point of view, together with some other applications, in section 5.

Acknowledgements. We thank David Favero, Robert Karp, Ludmil Katzarkov, Maximilian Kreuzer, Marc Nieper-Wißkirchen, and Dmytro Shklyarov for valuable discussions and correspondence. M. H. was supported by the ERC Starting Independent Researcher Grant StG No. 204757-TQFT.

1. Preliminaries

Consider $\mathbb{P}_\Sigma$, a complete, smooth toric variety defined by a fan Σ in the lattice $N \cong \mathbb{Z}^{n-k}$. Let v_i for $i = 1, \dots, n$ be the generators of the one-dimensional cones of Σ. We briefly recall the construction of $\mathbb{P}_\Sigma$ from Σ in terms of homogeneous coordinates $(x_i)_{1\le i\le n}$ associated to the v_i [8]. First, if $M = N^*$ is the dual lattice and $\mathrm{Pic}(\mathbb{P}_\Sigma) \cong \mathbb{Z}^k$ the Picard lattice, the exact sequence

$$M \xrightarrow{v^*} \mathbb{Z}^n \xrightarrow{w} \mathrm{Pic}(\mathbb{P}_\Sigma) \tag{4}$$

induces a $\mathbb{Z}^k$-grading on the homogeneous coordinate ring through the $(\mathbb{C}^\times)^k$-action $(\mathbb{C}^\times)^k \ni \lambda : (x_1, \dots, x_n) \mapsto (\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n)$. Second, if by a (Batyrev) primitive collection [9] we mean a collection of generators v_i that generates none of the cones in Σ, whereas any proper subset of it does, we define the exceptional set as the union

$$Z_\Sigma = \bigcup_p Z_{\mathcal{I}_p},$$

where p indexes all primitive collections, and $Z_{\mathcal{I}_p} = \cap_{i|v_i\in\mathcal{I}_p}\{x_i = 0\}$. Then, the toric variety is the quotient

$$\mathbb{P}_\Sigma = \frac{\mathbb{C}^n - Z_\Sigma}{(\mathbb{C}^\times)^k}.$$

Given $\mathbb{P}_\Sigma$, we consider complete intersections $X \subset \mathbb{P}_\Sigma$ defined by transversal polynomials G_a of $\mathbb{Z}^k$-degree d_a for $a = 1, \dots, \ell$. We require X to be a Calabi–Yau variety, i.e. $\sum_{i=1}^n w_i = \sum_{a=1}^\ell d_a$.

Switching to the algebraic description, let $R = \mathbb{C}[x_1, \dots, x_n]$ be the $\mathbb{Z}^k$-graded coordinate ring associated with the toric data, and $J_\Sigma = \langle \prod_{i|v_i\notin\sigma} x_i | \sigma \in \Sigma\rangle$ the Cox ideal [8], whose vanishing locus is $Z_\Sigma = \mathrm{V}(J_\Sigma)$ [9]. The complete intersection ring is denoted by $S = R/(G_1, \dots, G_\ell)R$, and the image of the Cox ideal along $R \to S$ by the same symbol, J_Σ. The following definitions involving J_Σ can be made over R or over S.

Let $\mathrm{gr}S$ be the abelian category of graded S-modules. The morphisms of $\mathrm{gr}S$ are the module homomorphisms of degree 0. For a given S-module A, and $q \in \mathbb{Z}^k$, a shift in degree is denoted by $A(q)$. We call an S-module J_Σ-torsion if it is annihilated by $(J_\Sigma)^m$ for some positive integer m.

Definition 1. Let $D^b(\mathrm{gr}S)$ be the bounded derived category of graded S-modules and $D^b(\mathrm{tor}_\Sigma S)$ the full triangulated subcategory of graded J_Σ-torsion S-modules, i.e. any object is isomorphic to a complex of J_Σ-torsion modules. The quotient category,

$$D^b(\mathrm{qgr}_\Sigma S) = \frac{D^b(\mathrm{gr}S)}{D^b(\mathrm{tor}_\Sigma S)}, \tag{5}$$

is defined by localization along the multiplicative system of morphisms s that fit into distinguished triangles

$$A \xrightarrow{s} B \longrightarrow C \longrightarrow A[1],$$

where $A, B \in D^b(\mathrm{gr}S)$ and $C \in D^b(\mathrm{tor}_\Sigma S)$.

A generalization of Serre's correspondence identifies the abelian category of coherent sheaves on $X \subset \mathbb{P}_\Sigma$ with the abelian category of graded S-modules modulo J_Σ-torsion modules. This correspondence extends to the derived categories,

$$D^b(X) \cong D^b(\mathrm{qgr}_\Sigma S).$$

When we subsequently consider the derived category of coherent sheaves on X, we will stick to its algebraic description in terms of modules over rings, that is, to $D^b(\mathrm{qgr}_\Sigma S)$.

Since any graded S-module has a projective resolution (though possibly unbounded from below), the derived category $D^b(\mathrm{gr}S)$ can be defined to be the homotopy category of projective complexes $K^-(\mathrm{gr}S)$ with quasi-isomorphisms inverted. Here, the minus index indicates that the complexes may be unbounded to the left, but with bounded cohomology. Namely, in the homotopy category of graded S-modules all quasi-isomorphisms are homotopy equivalences and therefore $D^b(\mathrm{gr}S)$ is in fact $K^-(\mathrm{gr}S)$.

By taking the quotient (5), all objects of the full subcategory $D^b(\mathrm{tor}_\Sigma S)$ become zero objects in $D^b(\mathrm{qgr}_\Sigma S)$. Over R, an important class of zero objects is given by the Koszul complexes associated with the primitive collections $\mathcal{I}_p$,

$$\mathcal{K}_p(R) := \mathcal{K}(\{x_i\}_{i|v_i\in\mathcal{I}_p}; R) = \bigotimes_{i|v_i\in\mathcal{I}_p} Cone(x_i : R \to R(w_i)) \ .$$

Indeed, the Koszul complex $\mathcal{K}_p(R)$ is nothing but the resolution of a J_Σ-torsion R-module, hence $\mathcal{K}_p(R) \cong 0$ in $D^b(\mathrm{qgr}_\Sigma R)$. Over the complete intersection ring S an even shorter regular sequence $\mathcal{S}_{\mathcal{I}_p} \subset \{x_i\}_{i|v_i\in\mathcal{I}_p}$ may be the resolution of a J_Σ-torsion S-module. For $G_1, \dots, G_\ell$ generic, $\mathcal{S}_{\mathcal{I}_p}$ is just a subset of $\{x_i\}_{i|v_i\in\mathcal{I}_p}$, with corresponding set of divisor clases $\{D_j\}_j$. In general, we can choose representatives s_j of that shorter list of divisor classes, such that $\big(\cap_j\{s_j = 0\}\big) \cap \big(\cap_a\{G_a = 0\}\big)$ has support on $Z_{\mathcal{I}_p}$. Thus, we have

Lemma 1. *Let S be the complete intersection ring associated with the regular sequence $(G_1, \dots, G_\ell)$. For a primitive collection $\mathcal{I}_p$ of the smooth toric fan Σ, we denote by $\mathcal{S}_{\mathcal{I}_p}$ a regular sequence as described above. We let w_j be the degree of s_j. Then the Koszul complex*

$$\mathcal{K}(\mathcal{S}_{\mathcal{I}_p}; S) = \bigotimes_j Cone(s_j : S \to S(w_j)) \ ,$$

is a zero object in $D^b(\mathrm{qgr}_\Sigma S)$. □

The goal of this work is to understand autoequivalences of $D^b(\mathrm{qgr}_\Sigma S) \cong D^b(X)$ and relations among them that can be directly traced back to the toric data, and more specifically to the secondary fan of $X \subset \mathbb{P}_\Sigma$. To this end, we need a realization of $D^b(\mathrm{qgr}_\Sigma S)$ that has a natural generalization to all other maximal cones in the secondary fan. This alternative construction begins with realizing X as the critical locus $\{\mathrm{d}W = 0\}$ of a holomorphic function W on the total space of a certain holomorphic vector bundle over $\mathbb{P}_\Sigma$. This then leads us to consider the singularity category associated with W [3, 4].

2. The singularity category for complete intersections

We append the data of the ℓ polynomials G_a to the toric data of $\mathbb{P}_\Sigma$ to obtain an enhanced fan $\hat\Sigma$ in $\hat N_\mathbb{R}$, where $\hat N = N \oplus \mathbb{Z}^\ell$, cf. [10, Chapter 5]. Explicitly, if v_i has coordinates $(v_i^1,\dots,v_i^{n-k})$ with respect to some basis of N, the generators $\hat v_i$ of the one-dimensional cones of $\hat\Sigma$ are as follows. For $i=1,\dots,n$ the coordinates of $\hat v_i$ are $(v_i^1,\dots,v_i^{n-k},u_i^1,\dots,u_i^\ell)$, where the u_i^a's are chosen to satisfy $\sum_{i=1}^n w_i u_i^a = d_a$. For $a=1,\dots,\ell$, $\hat v_{n+a}$ is given by a vector with 1 at the $(n+a)$-th position and 0 else. Let $\hat w$ subsume the vectors $(w_1,\dots,w_n,-d_1,\dots,-d_\ell)$. We will sometimes use $w_{n+a} := -d_a$. Then the exact sequence (4) is extended to

$$\hat M \xrightarrow{\hat v^*} \mathbb{Z}^{n+\ell} \xrightarrow{\hat w} \mathrm{Pic}(\mathbb{P}_{\hat\Sigma}) = \mathrm{Pic}(\mathbb{P}_\Sigma)\ .$$

The (Calabi–Yau) toric variety of the enhanced fan $\hat\Sigma$ is given by

$$\mathbb{P}_{\hat\Sigma} = \frac{\mathbb{C}^{n+\ell} - Z_{\hat\Sigma}}{(\mathbb{C}^\times)^k} \cong \mathrm{Tot}\left(\oplus_{a=1}^\ell \mathcal{O}(-d_a) \to \mathbb{P}_\Sigma\right)\ .$$

Note that the exceptional set $Z_{\hat\Sigma}$ is just the pull-back of Z_Σ along $\mathbb{C}^{n+\ell}\to\mathbb{C}^n$.

We denote by p_a the homogeneous coordinate associated with $\hat v_{n+a}$ in $\hat\Sigma$. Let $\hat R = \mathbb{C}[x_1,\dots,x_n,p_1,\dots,p_\ell]$ be the $\mathbb{Z}^k$-graded homogeneous coordinate ring associated with the fan $\hat\Sigma$. Similarly, $J_{\hat\Sigma}$ be the Cox ideal for $\hat\Sigma$. For what follows it is necessary to introduce an additional $2\mathbb{Z}$-grading, called R-grading, on $\hat R$. The R-grading of the x_i is 0, that of the p_a is 2. For a given $\hat R$-module A, a shift in R-grading by $2r$ is denoted by $A\{2r\}$.

The holomorphic function W on $\mathbb{P}_{\hat\Sigma}$ is built from the regular sequence $(G_1,\dots,G_\ell)$ and the auxiliary coordinates p_a as

$$W = \sum_{a=1}^\ell p_a G_a\ .$$

W is a polynomial of degree 0 and R-grading 2. Letting $\hat S = \hat R/(W)\hat R$, we have the isomorphism of graded rings, $S \cong \hat S/(G_1,p_1,\dots,G_\ell,p_\ell)\hat S$, which geometrically corresponds to the embedding of the complete intersection X in the toric Calabi–Yau variety $\mathbb{P}_{\hat\Sigma}$ as the critical locus of W,

$$X = \{\mathrm{d}W = 0\} \subset \mathbb{P}_{\hat\Sigma}.$$

This follows on account of the transversality of the polynomials $G_1,\dots,G_\ell$, because $\mathrm{d}W$ vanishes iff $p_a = 0$ and $G_a = 0$ for $a=1,\dots,\ell$.

Following [3, 4], we now introduce the singularity category associated with the polynomial W.

Definition 2. Let $D^b(\mathrm{gr}\hat S)$ be the derived category of graded $\hat S$-modules, and $\mathrm{Perf}(\mathrm{gr}\hat S)$ the full triangulated subcategory of perfect complexes, i.e. bounded complexes of free modules. The singularity category is the quotient

$$D_{\mathrm{sg}}(\mathrm{gr}\hat S) := \frac{D^b(\mathrm{gr}\hat S)}{\mathrm{Perf}(\mathrm{gr}\hat S)}.$$

By a result due to Eisenbud [11], a (minimal) free resolution of any $\hat{S}$-module becomes two-periodic (up to a shift of R-grading) after a finite number of steps. (The same is true for complexes of $\hat{S}$-modules by induction on the length of the complex.) Modding out by perfect complexes means that we can cut off and add finite pieces from the infinite resolution. Therefore, any non-zero object in $D_{\rm sg}({\rm gr}\hat{S})$ can be represented by a half-infinite free complex,

$$A^{\cdot} = \dots \xrightarrow{f_A} A_0\{-2\} \xrightarrow{g_A} A_1\{-2\} \xrightarrow{f_A} A_0 \xrightarrow{g_A} \underline{A_1}, \tag{6}$$

with ${\rm rk}A_0 = {\rm rk}A_1$. The underlined module is in homological degree 0.

Notice that

$$A^{\cdot}\{2\} = A^{\cdot}[2], \tag{7}$$

for any $A^{\cdot} \in D_{\rm sg}({\rm gr}\hat{S})$. The shift in R-grading by -2 in (6) is due to the R-grading of W. In fact, the complex of free $\hat{S}$-modules (6) can be lifted to a sequence of free $\hat{R}$-modules so that the homomorphisms compose as $f_{\tilde{A}}\, g_{\tilde{A}} = W \cdot {\rm id}_{\tilde{A}_0}$ and $g_{\tilde{A}}\, f_{\tilde{A}} = W \cdot {\rm id}_{\tilde{A}_1}$, where $A_j = \tilde{A}_j \otimes_{\hat{R}} \hat{S}$ for $j = 0, 1$. This also leads to Orlovs result in [4] that $D_{\rm sg}({\rm gr}\hat{S})$ can equivalently be described by the triangulated category of matrix factorizations of W over the ring $\hat{R}$.

The morphisms in the singularity category are given by chain maps modulo homotopy and modulo chain maps that factor through perfect complexes. On the two-periodic part the morphisms are also two-periodic, and since we mod out by chain maps that factor through perfect complexes, the two-periodic part determines the morphisms uniquely up to homotopy. On the representatives (6) a chain map is a commutative diagram

$$\begin{array}{ccccccccc} \dots & A_0\{-2\} & \xrightarrow{g_A} & A_1\{-2\} & \xrightarrow{f_A} & A_0 & \xrightarrow{g_A} & \underline{A_1} \\ & \downarrow \psi_0 & & \downarrow \psi_1 & & \downarrow \psi_0 & & \downarrow \psi_1 \\ \dots & B_0\{-2\} & \xrightarrow{g_B} & B_1\{-2\} & \xrightarrow{f_B} & B_0 & \xrightarrow{g_B} & \underline{B_1} \end{array} \tag{8}$$

For the following purposes, a useful representative of the object $A^{\cdot} \in D_{\rm sg}({\rm gr}\hat{S})$ in $D^b({\rm gr}\hat{S})$ is constructed by continuing the complex (6) periodically by $2r$ steps to the right, so that the R-degrees of $A_0\{2r\}$ and $A_1\{2r\}$ are all positive, and then cutting off the finite piece with positive R-degrees, in any homological degree. We denote the corresponding functor by

$$\sigma_{\leq 0} : D_{\rm sg}({\rm gr}\hat{S}) \longrightarrow D^b({\rm gr}\hat{S}).$$

Let $\pi_{\rm sg} : D^b({\rm gr}\hat{S}) \longrightarrow D_{\rm sg}({\rm gr}\hat{S})$, then clearly we have $A^{\cdot} \cong \pi_{\rm sg}\sigma_{\leq 0}A^{\cdot}$ for any object $A^{\cdot} \in D_{\rm sg}({\rm gr}\hat{S})$. It follows from two-periodicity that a free resolution in the image of $\sigma_{\leq 0}$, say $B^{\cdot}$, satisfies the relation

$$\sigma_{\leq 0}\pi_{\rm sg}(B^{\cdot}\{2r\}) = B^{\cdot}[2r], \qquad \text{for} \quad r = 0, 1, 2, \dots\ . \tag{9}$$

In fact, the subcategory of such objects is a full triangulated subcategory of $D^b({\rm gr}\hat{S})$.

Lemma 2. *Let $D_{\leq 0}$ be the full triangulated subcategory of objects satisfying (9). Then the adjoint pair of functors,*

$$D_{\rm sg}({\rm gr}\hat{S}) \quad \underset{\pi_{\rm sg}}{\overset{\sigma_{\leq 0}}{\rightleftarrows}} \quad D_{\leq 0} \ \subset\ D^b({\rm gr}\hat{S}),$$

is an equivalence of triangulated categories.

Proof. We know already that $A^{\cdot} \cong \pi_{\rm sg}\sigma_{\leq 0}A^{\cdot}$ for any object $A^{\cdot} \in D_{\rm sg}(\mathrm{gr}\hat{S})$, and by definiton $B^{\cdot} \cong \sigma_{\leq 0}\pi_{\rm sg}B^{\cdot}$ for any $B^{\cdot} \in D_{\leq 0}$. It remains to check isomorphism of morphisms: Because of two-periodicity a chain map between objects in $D_{\leq 0}$ cannot factor through a perfect complex, and hence

$$\mathrm{Hom}_{\,\rm sg}(\pi_{\rm sg}A^{\cdot}, \pi_{\rm sg}B^{\cdot}) \;\cong\; \mathrm{Hom}_{\,D_{\leq 0}}(A^{\cdot}, B^{\cdot}),$$

for $A^{\cdot}, B^{\cdot} \in D_{\leq 0}$. $\square$

Proposition 1. *The triangulated categories $D_{\rm sg}(\mathrm{gr}\hat{S})$ and $D^b(\mathrm{gr}S)$ are equivalent.*

Proof. Using Lemma 2 it remains to construct an adjoint pair of functors,

$$D^b(\mathrm{gr}R) \;\supset\; D^b(\mathrm{gr}S) \;\underset{\omega}{\overset{E}{\rightleftarrows}}\; D_{\leq 0} \;\subset\; D^b(\mathrm{gr}\hat{S})$$

that implements the equivalence.

The functor E is provided by Eisenbud's work [11] and is constructed in two steps. For the first step we include $D^b(\mathrm{gr}S)$ as full triangulated subcategory in $D^b(\mathrm{gr}R)$ by considering S-modules as R-modules which are annihilated by the regular sequence $(G_1, \ldots, G_\ell)$. For an object $A \in D^b(\mathrm{gr}S)$ we take the (minimal) R-free resolution $P^{\cdot}(A)$. Let $P(A)$ be the free R-module in the complex $P^{\cdot}(A)$ and d_0 the differential.

The second step constructs from $P^{\cdot}(A)$ a half-infinite complex in $D_{\leq 0}$. Theorem 7.1 of [11] allows us to introduce auxiliary endomorphisms $d_{\mathbf{n}} : P(A)[2|\mathbf{n}|] \to P(A)$ for $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^\ell$ and $|\mathbf{n}| = \sum_{a=1}^{\ell} n_a$, which have homological degree 1 as well as degree $\sum_a n_a d_a$ and satisfy

$$\begin{array}{rcll} d_{\mathbf{0}}d_{\mathbf{e_a}} + d_{\mathbf{e_a}}d_{\mathbf{0}} &=& G_a \cdot \mathrm{id}_{P(A)}[-2], & \\ \sum_{\mathbf{m},|\mathbf{m}|\leq|\mathbf{n}|} d_{\mathbf{m}}d_{\mathbf{n}-\mathbf{m}} &=& 0, & \text{for} \quad |\mathbf{n}| > 1. \end{array} \tag{10}$$

Here $e_a = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 at the a$^{\rm th}$ position, and we set $d_{\mathbf{0}} = d_0$.

Define the free $\hat{S}$-module $\hat{P}(A) = (\mathbb{C}[p_1, \ldots, p_\ell] \otimes_{\mathbb{C}} P(A))/(W)$. Then $E(A) \in D_{\leq 0}$ is the $\hat{S}$-module

$$\oplus_{r=0}^{\infty}\, \hat{P}(A)\{-2r\}[2r]$$

together with the differential $d = \sum_{\mathbf{n}} p^{\mathbf{n}} \otimes d_{\mathbf{n}}$. Notice that the latter has degree 0 and preserves the R-grading. In view of (10), $d^2 = W \cdot \mathrm{id}[-2]$, which is zero on $\hat{S}$-modules. $E(A)$ satisfies condition (9) and is therefore an object in $D_{\leq 0}$. In fact, the object is determined by d_0 uniquely up to isomorphism, since finding a solution to the recursive relations (10) admits the freedom, $d \to UdU^{-1}$ for $U = \mathrm{id} + \sum_{\mathbf{n}>0} p^{\mathbf{n}} u_{\mathbf{n}}$.

The functor ω is defined by

$$\begin{array}{rcl} \omega : D_{\leq 0} &\to& D^b(\mathrm{gr}S) \subset D^b(\mathrm{gr}R), \\ A^{\cdot} &\mapsto& (\sigma_{\geq 0}A^{\cdot}) \otimes \hat{S}/(p_1, \ldots, p_\ell)\hat{S}. \end{array}$$

$\sigma_{\geq 0}$ cuts off negative R-gradings of $A^{\cdot}$ and therefore picks the R-grading 0 component, whose differential does not contain the auxiliary coordinates p_a. Tensoring with $\hat{S}/(p_a)\hat{S} \cong R$ then removes the auxiliary coordinates from the module as well, i.e. the image of the functor ω is in $D^b(\mathrm{gr}R)$. Indeed, the image of ω is $D^b(\mathrm{gr}S)$: Given an object $A^{\cdot} \in D_{\leq 0}$ we write its differential as $d = \sum_{\mathbf{n}} p^{\mathbf{n}} d_{\mathbf{n}}$. Then Theorem 7.2 of [11] constructs from the endomorphisms $d_{\mathbf{n}}$ an infinite S-free resolution, that is an object in $D^b(\mathrm{gr}S)$, which is isomorphic to $\omega(A^{\cdot})$ in $D^b(\mathrm{gr}R)$.

From the definitions of E and ω we have

$$\omega E(A) = P^{\cdot}(A) \cong A,$$

for any $A \in D^b(\mathrm{gr}S) \subset D^b(\mathrm{gr}R)$. Conversely, for $B^{\cdot} \in D_{\leq 0}$, since $\omega(B^{\cdot})$ is the R-grading 0 component of $B^{\cdot}$ and E reconstructs the auxiliary endomorphisms in $B^{\cdot}$, we also have

$$E\omega(B^{\cdot}) \cong B^{\cdot}.$$

It remains to show that ω and E are adjoint functors, that is

$$\mathrm{Hom}_{D_{\leq 0}}(E(A), B^{\cdot}) = \mathrm{Hom}_{\mathrm{gr}S}(A, \omega(B^{\cdot})),$$

for $A \in D^b(\mathrm{gr}S)$ and $B^{\cdot} \in D_{\leq 0}$. We use that $D^b(\mathrm{gr}S)$ is full in $D^b(\mathrm{gr}R)$ and write the right-hand side as $\mathrm{Hom}_{\mathrm{gr}R}(P^{\cdot}(A), \omega(B^{\cdot}))$. For the left-hand side write a morphism in $\mathrm{Hom}_{D_{\leq 0}}(E(A), B^{\cdot})$ as $\psi = \sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}} p^{\mathbf{n}}\psi_{\mathbf{n}}$. Just as the differential d, the morphism ψ is determined (up to isomorphisms) by its R-grading 0 component ψ_0, hence the left-hand side is also isomorphic to $\mathrm{Hom}_{\mathrm{gr}R}(P^{\cdot}(A), \omega(B^{\cdot}))$. □

Definition 3. Let $D^b(\mathrm{tor}_{\hat{\Sigma}}\hat{S})$ be the full triangulated subcategory of $D^b(\mathrm{gr}\hat{S})$ consisting of graded $J_{\hat{\Sigma}}$-torsion modules, and consider its image in $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$, that is $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S}) := \pi_{\mathrm{sg}} D^b(\mathrm{tor}_{\hat{\Sigma}}\hat{S})$. This is a full triangulated subcategory of $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$. We define the quotient category

$$D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S}) := \frac{D_{\mathrm{sg}}(\mathrm{gr}\hat{S})}{D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S})}.$$

Theorem 2. *The image of the torsion subcategory $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S})$ under the equivalence $\omega_{\leq 0} = \omega\sigma_{\leq 0} : D_{\mathrm{sg}}(\mathrm{gr}\hat{S}) \to D^b(\mathrm{gr}S)$ is $D^b(\mathrm{tor}_\Sigma S)$, and, consequently,*

$$D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S}) \;\cong\; D^b(\mathrm{qgr}_\Sigma S).$$

Proof. Recalling that $Z_{\hat{\Sigma}}$ is the pull-back of Z_Σ we find from the definition of ω that it maps $J_{\hat{\Sigma}}$-torsion complexes to J_Σ-torsion complexes, hence $\omega_{\leq 0}(D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S})) \subseteq D^b(\mathrm{tor}_\Sigma S)$. Also, from the definition of $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S})$, we know that $\pi_{\mathrm{sg}}(D^b(\mathrm{tor}_\Sigma S)) \subseteq D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S})$. Applying $\omega_{\leq 0}$, we obtain $\omega_{\leq 0} \circ \pi_{\mathrm{sg}}(D^b(\mathrm{tor}_\Sigma S)) = D^b(\mathrm{tor}_\Sigma S) \subseteq \omega_{\leq 0}(D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S}))$, and therefore $D^b(\mathrm{tor}_\Sigma S) \cong \omega_{\leq 0}(D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Sigma}}\hat{S}))$. □

3. Phases of triangulated categories

The realization of the derived category $D^b(X) \cong D^b(\mathrm{qgr}_\Sigma S)$ through the singularity category $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S})$ in Theorem 2 motivates us to define singularity categories for each maximal cone of the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$. We shall label the latter by the toric fan $\hat{\Phi}$.

The corresponding toric variety is

$$\mathbb{P}_{\hat{\Phi}} = \frac{\mathbb{C}^{n+\ell} - Z_{\hat{\Phi}}}{(\mathbb{C}^\times)^k}. \tag{11}$$

Note that in general $\hat{\Phi}$ contains fewer one-dimensional cones than $\hat{\Sigma}$, i.e. fewer than $n+\ell$. This is taken into account in (11) by the exceptional set being the union,

$$Z_{\hat{\Phi}} = Z_{\hat{\Phi}}^{prim} \cup Z_{\hat{\Sigma}(1)\setminus\hat{\Phi}(1)}.$$

Here $Z^{prim}_{\hat{\Phi}}$ is given in terms of Batyrev's primitive collections for the fan $\hat{\Phi}$, and[1]

$$Z_{\hat{\Sigma}(1)\backslash\hat{\Phi}(1)} = \bigcup_{\hat{v}_i\in\hat{\Sigma}(1)\backslash\hat{\Phi}(1)} \{x_i = 0\}.$$

Note that

$$\mathbb{P}_{\hat{\Phi}} = \frac{(\mathbb{C}^{n+\ell-r} - Z^{prim}_{\hat{\Phi}}) \times (\mathbb{C}^\times)^r}{(\mathbb{C}^\times)^k} \cong \frac{(\mathbb{C}^{n+\ell-r} - Z^{prim}_{\hat{\Phi}})}{(\mathbb{C}^\times)^{k-r} \times G_{\hat{\Phi}}}$$

where r is the number of one-dimensional cones in $\hat{\Sigma}(1)\backslash\hat{\Phi}(1)$ and $G_{\hat{\Phi}}$ is a finite group.

To construct quotient singularity categories in every maximal cone $\hat{\Phi}$, it is convenient to start with the graded coordinate rings $\hat{R} = \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_\ell]$ and $\hat{S} = \hat{R}/(W)\hat{R}$, with the degree and R-grading as before. In a maximal cone $\hat{\Phi}$ of the secondary fan consider the ideal $J_{\hat{\Phi}} = \langle \prod_{i|v_i\in\hat{\Sigma}(1), v_i\notin\sigma} x_i | \sigma \in \hat{\Phi}\rangle$. Its vanishing locus is the exceptional set $Z_{\hat{\Phi}}$. We say that a graded $\hat{S}$-module is $J_{\hat{\Phi}}$-torsion if it is annihilated by $J^{\otimes m}_{\hat{\Phi}}$ for some positive integer m.

Definition 4. Let $\hat{S} = \hat{R}/(W)\hat{R}$, and $D^b(\mathrm{gr}\hat{S})$ as well as $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ be as in Definition 2. Take any maximal cone $\hat{\Phi}$ in the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$. Let $D^b(\mathrm{tor}_{\hat{\Phi}}\hat{S})$ be the full triangulated subcategory of $J_{\hat{\Phi}}$-torsion modules in $D^b(\mathrm{gr}\hat{S})$, and $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}}\hat{S})$ its image in $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$. Then we define the quotient

$$\mathcal{C}_{\hat{\Phi}} = D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}}\hat{S}) := \frac{D_{\mathrm{sg}}(\mathrm{gr}\hat{S})}{D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}}\hat{S})}.$$

Remark 1. For every irreducible component $Z_{\mathcal{I}_p}$ of the exceptional set $Z_{\hat{\Phi}}$, the associated Koszul complex

$$\mathcal{K}_p(\hat{S}) := \mathcal{K}(\{x_i\}_{v_i\in\mathcal{I}_p}; \hat{S}) = \bigotimes_{i\in\mathcal{I}_p} Cone(x_i : \hat{S} \to \hat{S}(w_i)\{r_i\}) \ .$$

is an object in $D^b(\mathrm{tor}_{\hat{\Phi}}\hat{S})$. Here, r_i is the R-degree of x_i, i.e. $r_i = 0$ for $i = 1, \ldots, n$ and $r_{n+a} = 2$ for $a = 1, \ldots, \ell$. Furthermore, for any object $A^\cdot$ of $D^b(\mathrm{gr}\hat{S})$, the tensor product $A^\cdot \otimes_{\hat{S}} \mathcal{K}_p$ is in $D^b(\mathrm{tor}_{\hat{\Phi}}\hat{S})$ and via π_{sg} in $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}}\hat{S})$.

We have introduced the quotient singularity categories separately for each maximal cone. The following result relates them.

Theorem 3. *For any pair of neighboring maximal cones, $\hat{\Phi}_1$ and $\hat{\Phi}_2$, in the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$, with Calabi–Yau complete intersecion $X \subset \mathbb{P}_{\hat{\Sigma}}$, there is a family $\{F^{\hat{\Phi}_2\hat{\Phi}_1}_m\}_{m\in\mathbb{Z}}$ of equivalences of the corresponding quotient singularity categories,*

$$F^{\hat{\Phi}_2\hat{\Phi}_1}_m : D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_1}\hat{S}) \xrightarrow{\cong} D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_2}\hat{S}). \tag{12}$$

Proof. Let $\mathbb{P}_{\hat{\Phi}_1}$ and $\mathbb{P}_{\hat{\Phi}_2}$ be the (Calabi–Yau) toric varieties in the neighboring maximal cones. We let $T \in \mathrm{Pic}(\mathbb{P}_{\hat{\Sigma}})^*$ be the primitive dual vector characterizing the face between the two adjacent maximal cones by the condition $T(v) = 0$ for all

[1]Note that the union may include some of the auxiliary coordinates $x_{n+a} = p_a$.

v in the face. The exceptional sets Z_1 and Z_2 in the geometric construction of $\mathbb{P}_{\hat{\Phi}_1}$ and $\mathbb{P}_{\hat{\Phi}_2}$ are related by [12, Chapter 4.5]

$$\begin{aligned} Z_1 &= Z_+ \cup (Z_1 \cap Z_2), \\ Z_2 &= Z_- \cup (Z_1 \cap Z_2), \end{aligned}$$

where $Z_\pm = \cap_{i|T(w_i)\gtrless 0}\{x_i = 0\}$. We let $\mathrm{tor}_{\hat{\Phi}_+}\hat{S}$, $\mathrm{tor}_{\hat{\Phi}_-}\hat{S}$, and $\mathrm{tor}_{\hat{\Phi}_{12}}\hat{S}$ denote the categories of torsion modules associated with Z_+, Z_-, and $Z_1 \cap Z_2$, respectively.

By the Calabi–Yau condition, $\sum_{i=1}^{n+\ell} w_i = 0$, we may define

$$\sigma := \sum_{i|T(w_i)>0} T(w_i) = -\sum_{i|T(w_i)<0} T(w_i).$$

For $m \in \mathbb{Z}$ let $K^m(\mathrm{gr}\hat{S}) \subset K^-(\mathrm{gr}\hat{S})$ be the homotopy category generated from invertible modules $\hat{S}(q)$ satisfying (cf. [12, 1, 2])

$$m \leq T(q) < m + \sigma. \tag{13}$$

Denote the associated singularity category by $D^m_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ and consider

$$D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_+}\hat{S}) \underset{\pi_+}{\overset{\omega_+}{\rightleftarrows}} D^m_{\mathrm{sg}}(\mathrm{gr}\hat{S}) \underset{\omega_-}{\overset{\pi_-}{\rightleftarrows}} D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_-}\hat{S}), \tag{14}$$

where $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_\pm}\hat{S})$ are the quotients of $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ by $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}_\pm}\hat{S})$. The functors $\pi_\pm$ are the projections of $D_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ to the quotient categories applied to the subcategory $D^m_{\mathrm{sg}}(\mathrm{gr}\hat{S})$. Comparing (13) with $Z_\pm$, we find that the objects in $D^m_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ can not be torsion. Also, the objects of $D^m_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ generate $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_\pm}\hat{S})$, so that $\pi_\pm$ are bijective on the set of objects. $D^m_{\mathrm{sg}}(\mathrm{gr}\hat{S})$ containing no torsion objects also means that there are no non-trivial extensions, and the functors $\pi_\pm$ are fully faithful, hence equivalences. The inverses $\omega_\pm$ take the unique (up to homotopy) representative of an isomorphism class in $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_\pm}\hat{S})$ that satisfies the restriction (13).

Finally, taking at each step in (14) the quotient by $D_{\mathrm{sg}}(\mathrm{tor}_{\hat{\Phi}_{12}}\hat{S})$ we find that the functor in the theorem is given by the compositions,

$$D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_1}\hat{S}) \xrightarrow{\omega_+} D^m_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_{12}}\hat{S}) \xrightarrow{\pi_-} D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}_2}\hat{S}).$$

□

A combination of Theorems 2 and 3 was proved for the hypersurface case by Orlov in [4]. Van den Bergh [1] and Kawamata [2] stated analogous results for the derived categories $D^b(\mathbb{P}_{\hat{\Phi}})$ of Calabi–Yau toric varieties using *noncommutative* crepant resolutions. They build a tilting module, say P, out of a generating set of invertible modules $\hat{S}(q)$ satisfying (13) and use the derived category of the noncommutative endomorphism algebra, $D^b(\mathrm{End}(P))$, to show the equivalence of $D^b(\mathbb{P}_{\hat{\Phi}_1})$ and $D^b(\mathbb{P}_{\hat{\Phi}_2})$.

Remark 2. In string theory the objects of the quotient singularity categories correspond to boundary condition of certain two-dimensional supersymmetric field theories. A physics derivation of the functor (12) was given in [12]. Therein it was found that the choice of the integer m corresponds to a choice of a homotopy class of paths connecting limit points corresponding to $\hat{\Phi}_1$ and $\hat{\Phi}_2$ in the moduli space B of the mirror of X.

4. Autoequivalences: The main formula

In this section we study elements of the group $\mathrm{Aut}_{\hat{\Phi}}$ of autoequivalences of $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}}\hat{S})$. We begin in the phase $\hat{\Phi} = \hat{\Sigma}$.

Immediate elements of $\mathrm{Aut}_{\hat{\Sigma}}$, the group of automorphisms of $D^b(X) \cong D^b(\mathrm{qgr}_\Sigma S)$, are given by the shift functors $[n]$, for $n \in \mathbb{Z}$, and the twists $\mathcal{M}^q$ by the modules $S(q)$ for any $q \in \mathbb{Z}^k \cong \mathrm{Pic}(\mathbb{P}_{\hat{\Sigma}})$. For $A \in D^b(\mathrm{qgr}_\Sigma S)$,

$$\mathcal{M}^q : A \mapsto A \otimes_S S(q).$$

Together with the automorphisms $\mathrm{Aut}(X)$ of X, that is, the graded ring automorphisms of S modulo $(\mathbb{C}^\times)^k$, these generate a subgroup $\mathcal{A}_0(X) \cong \mathrm{Aut}(X) \ltimes \mathrm{Pic}(\mathbb{P}_{\hat{\Sigma}}) \times \mathbb{Z}$ of $\mathrm{Aut}(D^b(X))$.

Because of Theorem 2 the group of automorphisms $\mathrm{Aut}(D^b(X))$, and in particular its subgroup $\mathcal{A}_0(X)$, also acts on the quotient singularity category $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S})$. The latter category appears to have an additional functor of twisting by $\hat{S}\{2r\}$. Recall however from (7) that a shift in R-grading by $\{2r\}$ is equal to the shift functor $[2r]$, hence does not introduce a new autoequivalence.

Let s be an element of S with degree w. Introduce on $D^b(\mathrm{qgr}_\Sigma S)$ the endofunctor

$$\mathcal{N}(s) : A \mapsto \mathit{Cone}\,(s : A \to \mathcal{M}^w(A))\,.$$

Proposition 2. *For every primitive collection $\mathcal{I}_p$ of Σ choose a regular sequence $\mathcal{S}_{\mathcal{I}_p}$ of elements $s_j \in S$, as in Lemma 1. Then,*

$$\bigcirc_{s_j \in \mathcal{S}_{\mathcal{I}_p}} \mathcal{N}(s_j)(A) \cong 0, \tag{15}$$

for any $A \in D^b(\mathrm{qgr}_\Sigma S)$.

Proof. By Lemma 1 the Koszul complex $\mathcal{K}(\mathcal{S}_{\mathcal{I}_p}; S)$ is isomorphic to the zero object in $D^b(\mathrm{qgr}_\Sigma S)$. The same is true for the tensor product $A \otimes_S \mathcal{K}(\mathcal{S}_{\mathcal{I}_p}; S)$ for any object A. This is nothing but the left-hand side of (15) since $\mathcal{N}(s)(A) = A \otimes_S \mathit{Cone}\,(s : S \to S(w))$. □

For any other maximal cone $\hat{\Phi}$ of the secondary fan, there are twist autoequivalences acting on $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}}\hat{S})$ as well. We set

$$\mathcal{M}^q_{\hat{\Phi}} : A^\cdot \mapsto A^\cdot \otimes_{\hat{S}} \hat{S}(q) \qquad \text{for} \quad A^\cdot \in D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}}\hat{S}).$$

For any element $s \in \hat{S}$ with degree w and R-grading $2r$, let

$$\mathcal{N}_{\hat{\Phi}}(s) : A^\cdot \mapsto \mathit{Cone}\left(s : A^\cdot \to \mathcal{M}^w_{\hat{\Phi}}(A^\cdot)[2r]\right).$$

Notice the shift in homological degree and recall that $[2r] \cong \{2r\}$ on $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}}\hat{S})$. Then, we have

Proposition 3. *Let $\hat{\Phi}$ be the fan associated to a maximal cone in the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$. Consider an arbitrary object $A^\cdot$ in the quotient singularity category $D_{\mathrm{sg}}(\mathrm{qgr}_{\hat{\Phi}}\hat{S})$. Then, for every primitive collection $\mathcal{I}_p$ of $\hat{\Phi}$,*

$$\bigcirc_{i|\hat{v}_i \in \mathcal{I}_p} \mathcal{N}_{\hat{\Phi}}(x_i)(A^\cdot) \cong 0,$$

and for every $\hat{v}_i \in \hat{\Sigma}(1) \backslash \hat{\Phi}(1)$ associated with $x_i \in \hat{S}$,

$$\mathcal{N}_{\hat{\Phi}}(x_i)(A^\cdot) \cong 0, \quad \text{or equivalently,} \quad \mathcal{M}^{w_i}_{\hat{\Phi}}(A^\cdot) \cong (A^\cdot)[-2r_i].$$

Proof. Using Remark 1, the proof is similar to the proof of Proposition 2. □

Remark 3. In the maximal cone $\hat{\Sigma}$ we may write $\mathcal{M}^q = \omega\sigma_{\leq 0}\mathcal{M}^q_{\hat{\Sigma}}\pi_{\rm sg}E$. Using this equivalence, the relations of Proposition 2 are in general stronger than those of Proposition 3. One may sometimes obtain similarly strong relations also in the other phases, namely whenever the exceptional set $Z_{\hat{\Phi}}$ admits the solution of $\mathrm{d}W = 0$ to set $p_a = G_a = 0$ for (at least) one $a = 1, \ldots, \ell$. Then, we may work over the ring $\hat{S}/(p_a, G_a)\hat{S}$. In fact, Proposition 2 applies because over $\mathbb{P}_{\hat{\Sigma}}$, $\mathrm{d}W = 0$ admits $p_a = G_a = 0$ for *all* $a = 1, \ldots, \ell$.

Theorem 3 relates the singularity categories in neighbouring maximal cones $\hat{\Phi}_1$ and $\hat{\Phi}_2$. The associated twists are only partly independent autoequivalences. In fact, condition (13), which defines the functor $F^{\hat{\Phi}_2\hat{\Phi}_1}_m : \mathcal{C}_{\hat{\Phi}_1} \to \mathcal{C}_{\hat{\Phi}_2}$, implies that

$$\mathcal{M}^q_{\hat{\Phi}_2} \circ F^{\hat{\Phi}_2\hat{\Phi}_1}_m \;\cong\; F^{\hat{\Phi}_2\hat{\Phi}_1}_{m+T(q)} \circ \mathcal{M}^q_{\hat{\Phi}_1}. \tag{16}$$

In particular, the twists with $T(q) = 0$ commute with the functor $F^{\hat{\Phi}_2\hat{\Phi}_1}_m$ (for a fixed integer m), which says that they correspond to the same autoequivalence,

$$\mathcal{M}^q_{\hat{\Phi}_1} \;\cong\; \mathcal{M}^q_{F^{\hat{\Phi}_2\hat{\Phi}_1}_m} := (F^{\hat{\Phi}_2\hat{\Phi}_1}_m)^{-1}\mathcal{M}^q_{\hat{\Phi}_2} \circ F^{\hat{\Phi}_2\hat{\Phi}_1}_m, \quad \text{for } T(q) = 0.$$

On the other hand, if $T(q)$ does not vanish the composition,

$$(F^{\hat{\Phi}_2\hat{\Phi}_1}_m)^{-1}F^{\hat{\Phi}_2\hat{\Phi}_1}_{m+T(q)} \cong \mathcal{M}^q_{F^{\hat{\Phi}_2\hat{\Phi}_1}_m}\mathcal{M}^{-q}_{\hat{\Phi}_1}, \tag{17}$$

is a non-trivial autoequivalence of $\mathcal{C}_{\hat{\Phi}_1}$.

For any $\hat{\Phi}$ let $F_{\hat{\Phi}} : \mathcal{C}_{\hat{\Sigma}} \to \mathcal{C}_{\hat{\Phi}}$ be a composition of functors of Theorem 3. Then the twists in $\hat{\Phi}$ act on $\mathcal{C}_{\hat{\Sigma}} \cong D^b(X)$ via $\mathcal{M}^q_{F_{\hat{\Phi}}} = F^{-1}_{\hat{\Phi}} \circ \mathcal{M}^q_{\hat{\Phi}} \circ F_{\hat{\Phi}}$. By combining Proposition 3 with Theorem 3 we obtain

Theorem 4. *Let X be a smooth Calabi–Yau complete intersection in a toric variety $\mathbb{P}_\Sigma$, and $D_{\rm sg}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S})$ ($\cong D^b(\mathrm{qgr}_\Sigma S)$) its quotient singularity category. Then for every maximal cone $\hat{\Phi}$ in the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$ the autoequivalences $\mathcal{M}^q_{F_{\hat{\Phi}}}$ induce an action of the Picard lattice* $\mathrm{Pic}(\mathbb{P}_{\hat{\Sigma}})$ *on $D_{\rm sg}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S})$, subject to the following relations. For any object $A^\cdot \in D_{\rm sg}(\mathrm{qgr}_{\hat{\Sigma}}\hat{S})$ and every primitive collection $\mathcal{I}_p$ of $\hat{\Phi}$,*

$$\bigcirc_{i|\hat{v}_i\in\mathcal{I}_p}\,\mathcal{N}_{F_{\hat{\Phi}}}(F_{\hat{\Phi}}(x_i))(A^\cdot) \cong 0, \tag{18}$$

and for every $\hat{v}_i \in \hat{\Sigma}(1)\backslash\hat{\Phi}(1)$ and associated element $x_i \in \hat{S}$,

$$\mathcal{N}_{F_{\hat{\Phi}}}(F_{\hat{\Phi}}(x_i))(A^\cdot) \cong 0, \quad \text{or equivalently,} \quad \mathcal{M}^{w_i}_{F_{\hat{\Phi}}}(A^\cdot) \cong (A^\cdot)[-2r_i]. \tag{19}$$

Proof. Writing for any $A^\cdot \in \mathcal{C}_{\hat{\Sigma}}$

$$\begin{aligned}
\mathcal{N}_{F_{\hat{\Phi}}}(F_{\hat{\Phi}}(s))(A^\cdot) \;&:=\; \mathit{Cone}(F_{\hat{\Phi}}(s) : A^\cdot \longrightarrow \mathcal{M}^w_{F_{\hat{\Phi}}}(A^\cdot)[2r]) = \\
&=\; F_{\hat{\Phi}}\,\mathit{Cone}(s : F^{-1}_{\hat{\Phi}}(A^\cdot) \longrightarrow \mathcal{M}^w_{\hat{\Phi}}F^{-1}_{\hat{\Phi}}(A^\cdot)) = \\
&=\; F_{\hat{\Phi}}\mathcal{N}_{\hat{\Phi}}(s)F^{-1}_{\hat{\Phi}}(A^\cdot),
\end{aligned}$$

the claim follows immediately from Proposition 3. □

5. Applications

5.1. **Monodromy representations — more about** $\mathrm{Aut}(D^b(X))$.

According to Theorem 4, we have an action of the toric Picard lattice of the complete intersection X on $D^b(X)$ for every phase $\hat{\Phi}$ in the secondary fan of $\mathbb{P}_{\hat{\Sigma}}$, generalizing the simple twisting by line bundles for $\hat{\Phi} = \hat{\Sigma}$. As is evident from the relations (18) and (19), this action can be very different for each $\hat{\Phi} \neq \hat{\Sigma}$.

In terms of the monodromy representation $\pi_1(B) \to \mathrm{Aut}(D^b(X))$, mentioned in the introduction, the action of $\mathrm{Pic}(\mathbb{P}_{\hat{\Sigma}})$ obtained from $\hat{\Phi}$ is a categorical lift of the monodromies around the boundary divisor of B corresponding to the maximal cone $\hat{\Phi}$ of the secondary fan. This point of view fits in nicely with the interpretation of the equivalence between the various $\mathcal{C}_{\hat{\Phi}}$'s as depending on the homotopy class of a path in B. According to Remark 2 the composition (17) for $T(q) = 1$ is the monodromy along a path around the discriminant locus of B, and therefore corresponds to a Seidel–Thomas twist [5], cf. also [13, 14] and [12, Chapter 10.5].

5.2. **Proof of K-theory formula.**

The projection to K-theory splits the cones in Proposition 3, which implies Corollary 1.

When interpreted in terms of monodromy representations (see previous subsection), the relations in K-theory can also be obtained by studying the analytic continuation of periods of the mirror variety, as in [15]. (See also [16] for a recent study.) Our results provide the precise categorical lift of the monodromies and the relations between them.

5.3. **Some examples.**

Consider the list of 14 Calabi–Yau complete intersections $X(d_1 \dots d_\ell)$ in weighted projective spaces $\mathbb{P}^{3+\ell}(w_0 \dots w_{3+\ell})$ with rank 1 Picard lattice, that is $\mathrm{Pic}(X) \cong \mathbb{Z}$ (cf. [17, 18]):

$X(5) \subset \mathbb{P}^4(11111)$	$X(2,4) \subset \mathbb{P}^5(111111)$	$X(2,12) \subset \mathbb{P}^5(111146)$
$X(6) \subset \mathbb{P}^4(11112)$	$X(3,3) \subset \mathbb{P}^5(111111)$	$X(4,6) \subset \mathbb{P}^5(111223)$
$X(8) \subset \mathbb{P}^4(11114)$	$X(3,4) \subset \mathbb{P}^5(111112)$	$X(6,6) \subset \mathbb{P}^5(112233)$
$X(10) \subset \mathbb{P}^4(11125)$	$X(2,6) \subset \mathbb{P}^5(111113)$	$X(2,2,3) \subset \mathbb{P}^6(1111111)$
	$X(4,4) \subset \mathbb{P}^5(111122)$	$X(2,2,2,2) \subset \mathbb{P}^7(11111111)$

The polynomial ring associated to the toric data is $\hat{R} = \mathbb{C}[x_0, \dots, x_{3+\ell}, p_1, \dots, p_\ell]$. The secondary fan has two maximal cones, say $\hat{\Sigma}$ and $\hat{\Xi}$, where the first shall correspond to the complete intersection itself. The respective (Cox) ideals are

$$J_{\hat{\Sigma}} = \langle \prod_{i=0}^{3+\ell} x_i \rangle, \quad \text{and} \qquad J_{\hat{\Xi}} = \langle \prod_{a=1}^{\ell} p_a \rangle.$$

For the following let $\mathcal{M} = \mathcal{M}^1_{\hat{\Sigma}}$ and $\mathcal{L} := \mathcal{M}^1_{\hat{\Xi}}$ abbreviate the autoequivalences of twisting by $\hat{S}(1)$ in the two associated singularity categories.

In the category $\mathcal{C}_{\hat{\Sigma}}$:

For $(G_1, \dots, G_\ell)$ generic, according to Proposition 2, a regular sequence $\mathcal{S}_{\mathcal{I}}$ of four elements, say $\underline{x} = (x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3})$, gives rise to a complex isomorphic to the zero object,

$$A[4] \xrightarrow{x} \bigoplus_{b=0}^{3} \mathcal{M}^{w_{i_b}}(A)[3] \xrightarrow{x} \ldots \xrightarrow{x} \bigoplus_{b=0}^{3} \mathcal{M}^{w-w_{i_b}}(A)[1] \xrightarrow{x} \mathcal{M}^{w}(A) \cong 0,$$

where $w = \sum_{b=0}^{3} w_{i_b}$. In K-theory for each regular sequence $\mathcal{S}_{\mathcal{I}}$ the relation becomes $\prod_{b=0}^{3}(M^{w_{i_b}} - \mathrm{id}) = 0$.

As an example, for $X(10) \subset \mathbb{P}^4(11125)$ the K-theory relations are

$$\begin{aligned} (M-\mathrm{id})^2(M^2-\mathrm{id})(M^5-\mathrm{id}) &= 0, \\ (M-\mathrm{id})^3(M^2-\mathrm{id}) &= 0, \\ (M-\mathrm{id})^3(M^5-\mathrm{id}) &= 0. \end{aligned}$$

It is easy to verify that for each complete intersection in the above list, the relations imply the well-known result that the autoequivalence M is *maximally* unipotent, that is

$$(M-\mathrm{id})^4 = 0.$$

In the category $\mathcal{C}_{\hat{\Xi}}$:

Proposition 3 tells us that for $\underline{p} = (p_1, \ldots, p_\ell)$ the complex

$$\mathcal{L}^{d}(A^{\cdot}) \xrightarrow{p} \oplus_a \mathcal{L}^{d-d_a}(A^{\cdot})[1] \xrightarrow{p} \ldots \xrightarrow{p} \oplus_a \mathcal{L}^{d_a}(A^{\cdot})[\ell-1] \xrightarrow{p} A^{\cdot}[\ell], \tag{20}$$

is isomorphic to the zero object. Here, $d = \sum_a d_a$. On the level of K-theory this becomes

$$\prod_{a=1}^{\ell}(L^{d_a} - \mathrm{id}) = 0.$$

Back in $\mathcal{C}_{\hat{\Sigma}}$:

Theorem 4 uses the functor $F_{\hat{\Xi}} : \mathcal{C}_{\hat{\Sigma}} \longrightarrow \mathcal{C}_{\hat{\Xi}}$ to map the relation (20) to a relation for the autoequivalence $\mathcal{L}_{F_{\hat{\Xi}}} = F_{\hat{\Xi}}^{-1}\mathcal{L}F_{\hat{\Xi}}$ on $\mathcal{C}_{\hat{\Sigma}}$. Although the relation for $\mathcal{L}_{F_{\hat{\Xi}}}$ is a straight forward consequence of (20), we stress that it is highly non-trivial, even more so, if we "forget" that $\mathcal{L}_{F_{\hat{\Xi}}}$ in fact comes from an action of the Picard lattice in another phase.

We illustrate this for the complete intersection $X = X(3,3)$ of two cubics in $\mathbb{P}^5$, applying the autoequivalence $\mathcal{L}_{F_{\hat{\Xi}}}$ on the structure sheaf $\mathcal{O}$ of X. Let Ω be the pull-back of the cotangent bundle of $\mathbb{P}^5$ to X. We obtain

$$
\begin{array}{rclcccl}
\mathcal{L}_{F_{\hat{\Xi}}}(\mathcal{O}) & = & & & & & \Omega(1)[1]\ , \\
(\mathcal{L}_{F_{\hat{\Xi}}})^2(\mathcal{O}) & = & & & & & \wedge^2\Omega(2)[2]\ , \\
(\mathcal{L}_{F_{\hat{\Xi}}})^3(\mathcal{O}) & = & & & \mathcal{O}[2]^{\oplus 2} & \xrightarrow{\varphi_3} & \wedge^3\Omega(3)[3]\ , \\
(\mathcal{L}_{F_{\hat{\Xi}}})^4(\mathcal{O}) & = & & & \Omega(1)[3]^{\oplus 2} & \xrightarrow{\varphi_4} & \wedge^4\Omega(4)[4]\ , \\
(\mathcal{L}_{F_{\hat{\Xi}}})^5(\mathcal{O}) & = & & & \wedge^2\Omega(2)[4]^{\oplus 2} & \xrightarrow{\varphi_5} & \wedge^5\Omega(5)[5]\ , \\
(\mathcal{L}_{F_{\hat{\Xi}}})^6(\mathcal{O}) & = & \mathcal{O}[4]^{\oplus 3} & \xrightarrow{\varphi_3'} & \wedge^3\Omega(3)[5]^{\oplus 2}, & &
\end{array}
$$

where the arrows are canonical elements in $\mathrm{Ext}^1(-,-)$. In fact, the object in the last line is isomorphic to

$$
\begin{tikzcd}
(\mathcal{L}_{F_{\hat{\Xi}}})^6(\mathcal{O}) \ = \ \mathcal{O}[4]^{\oplus 4} \arrow[r, "\varphi_3"] \arrow[dr, "(1000)"] & \wedge^3\Omega(3)[5]^{\oplus 2}, \\
 & \oplus \\
 & \mathcal{O}[3]
\end{tikzcd}
$$

which confirms the relation for $\mathcal{L}_{F_{\hat{\Xi}}}$, when applied to $\mathcal{O}$,

$$
(\mathcal{L}_{F_{\hat{\Xi}}})^6(\mathcal{O}) \ \cong \ (\mathcal{L}_{F_{\hat{\Xi}}})^3(\mathcal{O})[2]^{\oplus 2} \ \xrightarrow{F_{\hat{\Xi}}(p)} \ \mathcal{O}[3].
$$

References

[1] M. Van den Bergh, "Non-commutative crepant resolutions," arXiv:math/0211064.

[2] Y. Kawamata, "Log crepant birational maps and derived categories," J. Math. Sci. Univ. Tokyo, 12, 211–231 (2005), [arXiv:math.AG/0311139].

[3] D. Orlov, "Triangulated categories of singularities and D-branes in Landau-Ginzburg models," Proc. Steklov Inst. Math. 246, 227–248 (2004), [arXiv:math.AG/0302304].

[4] D. Orlov, "Derived categories of coherent sheaves and triangulated categories of singularities," arXiv:math.AG/0503632.

[5] P. Seidel, R. Thomas, "Braid group actions on derived categories of coherent sheaves," Duke Math. J. 108(1), 37-108 (2001), [arXiv:math.AG/0001043].

[6] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," in "Recent Trends in String Theory", 1-152, World Scientific (2004), arXiv:hep-th/0403166.

[7] A. Canonaco, R. L. Karp, "Derived autoequivalences and a weighted Beilinson resolution," J. Geometry and Physics, 58 (6), 743–760 (2008), arXiv:math/0610848.

[8] D. A. Cox, "The Homogeneous Coordinate Ring of a Toric Variety," J. Algebraic Geometry 4, 17-50 (1995), [arXiv:alg-geom/9210008].

[9] V. V. Batyrev, D. A. Cox, "On the Hodge Structure of Projective Hypersurfaces in Toric Varieties," Duke Math. J. 75, 293-338 (1994), [arXiv:alg-geom/9306011].

[10] D. A. Cox and S. Katz, "Mirror symmetry and algebraic geometry," *Providence, USA: AMS (2000) 469 p*

[11] D. Eisenbud, "Homological algebra on a complete intersection, with an application to group representations," Trans. Amer. Math. Soc. 260, 3564 (1980).

[12] M. Herbst, K. Hori and D. Page, "Phases Of N=2 Theories In 1+1 Dimensions With Boundary," arXiv:0803.2045 [hep-th].

[13] R. Paul Horja, "Derived category automorphisms from mirror symmetry," Duke Mathematical Journal, 127 (1), 1–34 (2005), [arXiv:math.AG/0103231].

[14] P. S. Aspinwall, R. L. Karp and R. P. Horja, "Massless D-branes on Calabi–Yau threefolds and monodromy," Commun. Math. Phys. 259 (1), 45–69 (2005), [arXiv:hep-th/0209161].

[15] L. Borisov and R. P. Horja, "Mellin-Barnes integrals as Fourier-Mukai transforms, Advances in Mathematics, 207, 876–927 (2006), [arXiv:math.AG/0510486].

[16] P. S. Aspinwall and M. R. Plesser, "Decompactifications and Massless D-Branes in Hybrid Models," arXiv:0909.0252 [hep-th].

[17] C. F. Doran and J. W. Morgan, "Mirror symmetry and integral variations of Hodge structure underlying one-parameter families of Calabi–Yau threefolds," Mirror symmetry. V, 517–537, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc., Providence, RI, 2006, [arXiv:math.AG/0505272].

[18] Y. Chen, Y. Yang and N. Yui (Appendix by C. Erdenberger), "Monodromy of Picard–Fuchs differential equations for Calabi–Yau threefolds," J. reine angew. Math. 616, 167–203 (2008), [arXiv:math.AG/0605675].

Institute of Mathematics, University of Augsburg, Germany
E-mail address: `Manfred.Herbst@math.uni-augsburg.de`

PH-TH, CERN, Geneva, Switzerland
E-mail address: `Johannes.Walcher@cern.ch`