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Maurer-Cartan equations and black hole superpotential in $\mathcal{N} = 8$ supergravity

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We retrieve the non-BPS extremal black hole superpotential of $\mathcal{N} = 8$, $d = 4$ supergravity by using the Maurer-Cartan equations of the symmetric space $\frac{E_{7(7)}}{SU(8)}$. This superpotential was recently obtained with different 3- and 4-dimensional techniques. The present derivation is independent on the reduction to $d = 3$.

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I. INTRODUCTION

Recently, much progress has been obtained in the description of BPS and non-BPS *extremal* black hole (BH) flows in $\mathcal{N} \geq 2$ supergravities in $d = 4$ space-time dimensions [1–7] (see also Sec. 2 of [8]). In particular, for all theories whose nonlinear scalar sigma model is a symmetric space,[1] *superpotentials* W's exist for all BPS and non-BPS branches, thus yielding that the corresponding radial flow equations are of first order. Namely, the warp factor U of the extremal BH metric and the scalar field trajectories, respectively, read [1]

$$\dot{U} = -e^U W; \tag{1.1}$$

$$\dot{\phi}^i = -2e^U g^{ij} \partial_j W, \tag{1.2}$$

where W is related to the effective BH potential

$$V_{\mathrm{BH}} \equiv \tfrac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I, \tag{1.3}$$

through

$$V_{\mathrm{BH}} = W^2 + 2g^{ij}\partial_i W \partial_j W = W^2 + 2g^{ij}\nabla_i W \nabla_j W. \tag{1.4}$$

Here Z^I denote the matter charges (absent e.g. in $\mathcal{N} = 8$ supergravity), and $Z_{AB} = -Z_{BA}$ is the central charge matrix, entering the supersymmetry algebra as follows:

$$\{\mathcal{Q}_A^\alpha, \mathcal{Q}_B^\beta\} = \epsilon^{\alpha\beta} Z_{AB}(\phi_\infty, Q). \tag{1.5}$$

Moreover, Eq. (1.2) implies that attractor points

$$\dot{\phi}^i = 0 \tag{1.6}$$

correspond to critical points of W itself:

$$\partial_i W = 0. \tag{1.7}$$

For BPS BHs,

$$W(\phi, Q) = |z_I|_{\max}(\phi, Q), \tag{1.8}$$

where Q is the symplectic charge vector, and $|z_I|_{\max}$ is the highest absolute value of the *skew eigenvalues* z_I's of Z_{AB}. Furthermore, the ADM *mass* M_{ADM} [9] is related to W through (r denotes the radial coordinate throughout)

$$M^2_{\mathrm{ADM}} = \lim_{r\to\infty} W^2. \tag{1.9}$$

The Bekenstein-Hawking entropy-area formula [10] exploits as follows:

$$\frac{S_{\mathrm{BH}}(Q)}{\pi} = \frac{A_H}{4\pi} = \lim_{r\to r_H^+} W^2 = W^2|_{\partial W=0} = W^2(\phi_H(Q), Q), \tag{1.10}$$

where r_H and A_H respectively stand for the radius and the area of the event horizon of the considered extremal BH, and $\phi_H(Q)$ denotes the set of scalar fields at the horizon, stabilized in terms of the charges Q.

Explicit ways of constructing W have been given in [5–7] by using different methods, e.g. based on the $\mathcal{N} = 2$ *stu* model [5,7] or on three-dimensional techniques [6]. All these exploit the fact, as generally proven in [4], that

$$W = W(i_n(\phi, Q)), \tag{1.11}$$

where $i_n(\phi, Q)$'s ($n = 1, \ldots, 5$) are duality invariant combinations of the scalars ϕ^i and of charges Q [5,11]. A polynomial in i_n's gives the unique scalar-independent duality invariant $\mathcal{I}(Q)$ [11–13]. In the $\mathcal{N} = 2$ case, it reads [5,11]

$$\mathcal{I} = (i_1 - i_2)^2 + 4i_4 - i_5. \tag{1.12}$$

It is worth remarking that in the considered framework the symplectic vector of charges Q must belong to a *non-*

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[1] Note that this is always the case for $\mathcal{N} \geq 3$, $d = 4$ theories.

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degenerate (i.e. with $\mathcal{I} \neq 0$) orbit of the U-duality group [14–16].

In particular, $\mathcal{I}$ is quartic[2] in charges Q for all rank-three $\mathcal{N} = 2$ symmetric spaces [19], as well as for $\mathcal{N} = 8$ supergravity [see Eqs. (2.8), (2.9), (2.10), (2.11), and (2.12)]. Moreover, since $\mathcal{N} \geq 3$, $d = 4$ supergravities all have symmetric scalar manifolds, they all admit W's for their various scalar flows, i.e. for each different orbit of the charge vector [14–16].

For $\mathcal{N} = 8$ supergravity, it follows that

$$W = W(\rho_0, \rho_1, \rho_2, \rho_3, \varphi), \tag{1.13}$$

where ρ_I's ($I = 0, 1, 2, 3$ throughout) are the absolute values of the skew eigenvalues of Z_{AB}, whose $SU(8)$-invariant phase is φ [see Eq. (2.1)]. In [20] the explicit expressions of ρ_I's and φ were computed in terms of the four roots of a quartic algebraic equation, involving the quantities $(\mathrm{Tr}(ZZ^\dagger))^{m+1}$ ($m = 0, 1, 2, 3$), as well as the quartic invariant $\mathcal{I}_4$ [see e.g. Eqs. (2.10) and (2.11), and also the treatment in [11]].

As shown in [21], two different branches of attractor scalar flows exist, namely, the $\frac{1}{8}$-BPS and the non-BPS branches. Note that W exhibits the same flat directions of V_{BH} at its critical points; such flat directions span the moduli spaces $\frac{E_{6(2)}}{SU(6)\times SU(2)}$ [$\mathcal{I}_4 > 0$, see Eq. (2.21)] and $\frac{E_{6(6)}}{USp(8)}$ [$\mathcal{I}_4 < 0$, see Eq. (2.26)] [22].

This paper is devoted to the derivation of the W's for both these branches. This is done by exploiting the ($d = 4$) Maurer-Cartan equations of the exceptional coset $\frac{E_{7(7)}}{SU(8)}$ (see e.g. [23] and references therein). We will show that, while W_{BPS} is given by the highest absolute value of the skew eigenvalues of Z_{AB} [consistent with Eq. (1.8)], $W_{n\mathrm{BPS}}$ is given by the $USp(8)$ singlet of the **28** of $SU(8)$. These results extend to the whole attractor scalar flow the expression of W which was known for both BPS and non-BPS attractor solutions after [21] (see also e.g. [24]). Our investigation and derivation is complementary to [6], where the expression of $W_{n\mathrm{BPS}}$ was obtained by making use of the nilpotent orbits of the $d = 3$ geodesic flow obtained through a timelike reduction (see e.g. [25–32], and references therein).

The paper is organized as follows.

In Sec. II we recall the $SU(6) \times SU(2)$-covariant normal frame of $\mathcal{N} = 8$ supergravity, which we dub *special* normal frame, and we show that Maurer-Cartan equations yield a partial differential equation (PDE) for W, whose simplest solution is the BPS superpotential W_{BPS}.

Section III is devoted to the analysis of the $USp(8)$-covariant normal frame of $\mathcal{N} = 8$ supergravity (see e.g. the analysis of [33,34], and references therein), which we dub *symplectic* normal frame. We show that in such a normal frame the Maurer-Cartan equations yield a PDE for W, whose simplest solution is the non-BPS superpotential $W_{n\mathrm{BPS}}$. W's are nothing but the singlets in the decomposition of the **28** of $SU(8)$ into the maximal compact subgroup of the stabilizer of the corresponding supporting charge orbit, i.e. respectively into $SU(6) \times SU(2)$ (BPS) and $USp(8)$ (non-BPS).

Derivations of some relevant formulas are given in the Appendix, which concludes the paper.

[2] The quartic invariant $\mathcal{I}_4$ of $\mathcal{N} = 4$ theories was derived in [17,18].

II. SPECIAL NORMAL FRAME

Following [35–37], through a suitable $SU(8)$ transformation the complex skew-symmetric central charge matrix Z_{AB} [$A, B = 1, \ldots, \mathcal{N} = 8$ in the **8** of $\mathcal{R}$-symmetry $SU(8)$] can be *skew diagonalized*, and thus recast in *normal* form [see e.g. Eq. (87) of [2], adopting a different convention on the 2×2 symplectic metric ϵ; $a = 1, 2, 3$ throughout; unwritten matrix components do vanish throughout]:

$$\begin{aligned} Z_{AB} &\overset{SU(8)}{\longrightarrow} \begin{pmatrix} z_0 & & & \\ & z_1 & & \\ & & z_2 & \\ & & & z_3 \end{pmatrix} \otimes \epsilon \\ &= e^{i(\varphi/4)} \begin{pmatrix} \rho_0 & & & \\ & \rho_1 & & \\ & & \rho_2 & \\ & & & \rho_3 \end{pmatrix} \otimes \epsilon, \\ \rho_0, \rho_a &\in \mathbb{R}^+, \qquad \varphi \in [0, 8\pi), \end{aligned} \tag{2.1}$$

where

$$\epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Notice that the second line of Eq. (2.1) can be obtained from the first one by performing a suitable $(U(1))^3$ transformation.

The general definition (1.3) of effective BH potential V_{BH} thus yields

$$V_{\mathrm{BH}} = \rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2. \tag{2.3}$$

Therefore, in the normal frame defined by (2.1) the nonvanishing components of Z_{AB} reads as follows:

$$z_0 \equiv Z_{12} = \rho_0 e^{i(\varphi/4)}; \tag{2.4}$$

$$z_1 \equiv Z_{34} = \rho_1 e^{i(\varphi/4)}; \tag{2.5}$$

$$z_2 \equiv Z_{56} = \rho_2 e^{i(\varphi/4)}; \tag{2.6}$$

$$z_3 \equiv Z_{78} = \rho_3 e^{i(\varphi/4)}. \tag{2.7}$$

Within this parametrization, the unique quartic invariant $\mathcal{I}_4$ of the **56** of the U-duality group $E_{7(7)}$ (see e.g. [11,13], and references therein)

$$\mathcal{I}_4 \equiv \mathrm{Tr}(ZZ^\dagger ZZ^\dagger) - \frac{1}{2^2}\mathrm{Tr}^2(ZZ^\dagger) + 2^3\,\mathrm{Re}[\mathrm{Pfaff}(Z)]; \tag{2.8}$$

$$\mathrm{Pfaff}\,(Z) \equiv \frac{1}{2^4 4!}\epsilon^{ABCDEFGH} Z_{AB}Z_{CD}Z_{EF}Z_{GH}, \tag{2.9}$$

reads as follows (see e.g. [38]):

$$\mathcal{I}_4 = \sum_I \rho_I^4 - 2\sum_{I<J}\rho_I^2\rho_J^2 + 8\rho_0\rho_1\rho_2\rho_3\cos\varphi \tag{2.10}$$

$$\begin{aligned} &= (\rho_0+\rho_1+\rho_2+\rho_3)(\rho_0+\rho_1-\rho_2-\rho_3)\\ &\quad\cdot(\rho_0-\rho_1+\rho_2-\rho_3)(\rho_0-\rho_1-\rho_2+\rho_3)\\ &\quad+8\rho_0\rho_1\rho_2\rho_3(\cos\varphi-1). \end{aligned} \tag{2.11}$$

The Pfaffian of Z_{AB}, defined by Eq. (2.9), simply reads

$$\mathrm{Pfaff}\,(Z) = Z_{12}Z_{34}Z_{56}Z_{78} = e^{i\varphi}\prod_I \rho_I. \tag{2.12}$$

It is worth remarking that the skew-diagonal form of Z_{AB} given by Eq. (2.1) is *democratic*, in the sense that it fixes the phases of the four skew eigenvalues

$$z_I \equiv \rho_I e^{i\varphi_I} \tag{2.13}$$

of Z_{AB} to be *all equal*:

$$\varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 \equiv \frac{\varphi}{4}. \tag{2.14}$$

Actually, this implies some loss of generality, because $SU(8)$ only constrains the phases of z_I's as follows:

$$\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 \equiv \varphi. \tag{2.15}$$

Up to renamings, without loss of generality, the $|z_I|$'s can be ordered as follows:

$$\rho_0 \geq \rho_1 \geq \rho_2 \geq \rho_3. \tag{2.16}$$

Notice that ρ_I's are $U(8)$ invariant, whereas the overall phase φ is invariant under $SU(8)$, but not under $U(8)$.

It turns out that the special skew diagonalization (2.1) is particularly suitable for the treatment of the $\frac{1}{8}$-BPS attractor flow, as shown in the following section.

A. Attractor solutions

In the special normal frame (2.1), the two attractor solutions of $\mathcal{N}=8$, $d=4$ supergravity read as follows (see e.g. [21,33], and references therein; see also the analysis of [34] for further detail):

(i) $\frac{1}{8}$-BPS:

$$\rho_0 \equiv \rho_{\mathrm{BPS}} \in \mathbb{R}_0^+; \tag{2.17}$$

$$\rho_1 = \rho_2 = \rho_3 = 0; \tag{2.18}$$

$$\varphi \text{ undetermined}, \tag{2.19}$$

thus yielding

$$Z_{AB,(1/8)\text{-BPS}} = e^{i(\varphi/4)}\rho_{\mathrm{BPS}}\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \otimes \epsilon; \tag{2.20}$$

$$\mathcal{I}_4(\mathcal{Q}_{\mathrm{BPS}}) = \rho_{\mathrm{BPS}}^4(\mathcal{Q}_{\mathrm{BPS}}) > 0, \tag{2.21}$$

where [39]

$$\mathcal{Q}_{\mathrm{BPS}} \in \mathcal{O}_{(1/8)\text{-BPS,nondeg}} = \frac{E_{7(7)}}{E_{6(2)}}, \tag{2.22}$$

with maximal compact symmetry $SU(6)\times SU(2)$.

(ii) Non-BPS:

$$\rho_0 = \rho_1 = \rho_2 = \rho_3 \equiv \rho_{n\mathrm{BPS}} \in \mathbb{R}_0^+; \tag{2.23}$$

$$\varphi = \pi, \tag{2.24}$$

thus yielding

$$Z_{AB,n\mathrm{BPS}} = e^{i(\pi/4)}\rho_{n\mathrm{BPS}}\Omega_{AB}; \tag{2.25}$$

$$\mathcal{I}_4(\mathcal{Q}_{n\mathrm{BPS}}) = -2^4\rho_{n\mathrm{BPS}}^4(\mathcal{Q}_{n\mathrm{BPS}}) < 0, \tag{2.26}$$

where

$$\Omega_{AB} \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \otimes \epsilon \tag{2.27}$$

is the 8×8 metric of $USp(8)$, and [39]

$$\mathcal{Q}_{n\mathrm{BPS}} \in \mathcal{O}_{n\mathrm{BPS}} = \frac{E_{7(7)}}{E_{6(6)}}, \tag{2.28}$$

with maximal compact symmetry $USp(8)$.

B. Maurer-Cartan equations and PDE for W

Let us now consider the *Maurer-Cartan* equations of $\mathcal{N}=8$, $d=4$ supergravity (see e.g. [23] and references therein):

$$\nabla_i Z_{AB} = \tfrac{1}{2}P_{ABCD,i}\bar{Z}^{CD}, \tag{2.29}$$

where the *vielbein* 1-form $P_{ABCD} = P_{ABCD,i}d\phi^i$ ($i = 1,\ldots,70$) of the real homogeneous symmetric scalar manifold

$$M_{\mathcal{N}=8,d=4} = \frac{E_{7(7)}}{SU(8)} \tag{2.30}$$

sits in the 4-fold antisymmetric **70** of $SU(8)$, and it satisfies the *self-dual reality* condition (see e.g. [13])

$$P_{ABCD} = P_{[ABCD]} = \frac{1}{4!}\epsilon_{ABCDEFGH}\bar{P}^{EFGH}. \tag{2.31}$$

In order to simplify forthcoming calculations, it is convenient to group $SU(8)$ indices as follows:

$$12 \to 0; \qquad 34 \to 1; \qquad 56 \to 2; \qquad 78 \to 3. \tag{2.32}$$

Thus, for a generic skew-diagonal Z_{AB}, Maurer-Cartan equation (2.29) reads

$$\nabla_i Z_0 = P_{01,i}\bar{Z}^1 + P_{02,i}\bar{Z}^2 + P_{03,i}\bar{Z}^3; \tag{2.33}$$

$$\nabla_i Z_1 = P_{01,i}\bar{Z}^0 + P_{12,i}\bar{Z}^2 + P_{13,i}\bar{Z}^3; \tag{2.34}$$

$$\nabla_i Z_2 = P_{02,i}\bar{Z}^0 + P_{12,i}\bar{Z}^1 + P_{23,i}\bar{Z}^3; \tag{2.35}$$

$$\nabla_i Z_3 = P_{03,i}\bar{Z}^0 + P_{13,i}\bar{Z}^1 + P_{23,i}\bar{Z}^2. \tag{2.36}$$

By disregarding the reality condition (2.31) of the vielbein P_{ABCD}, within the considered special normal frame (2.1) one can determine the PDE for W in an easy way. Indeed, Eq. (2.29) yields

$$\nabla_i \rho_I = \tfrac{1}{2}(e^{i\varphi/4}\nabla_i\bar{Z}^I + e^{-i\varphi/4}\nabla_i Z_J); \tag{2.37}$$

$$\begin{aligned}\nabla_i\varphi &= -2i\nabla_i(\ln Z_I - \ln\bar{Z}^I)\\ &= \frac{2}{\rho_I}(ie^{i\varphi/4}\nabla_i^I\bar{Z} - ie^{-i\varphi/4}\nabla_i Z_I).\end{aligned} \tag{2.38}$$

Consequently, the total covariant differential of W generally reads [the sum is expanded in Eq. (A7)]

$$\begin{aligned}\nabla_i W = \frac{1}{2}\sum_{I<J}\{&e^{i\varphi/2}(W_I\rho_J + W_J\rho_I)\\ &+ e^{-i\varphi/2}\tilde{\epsilon}^{IJKL}(\bar{W}_K\rho_L + \bar{W}_L\rho_K)\}P_{IJ},\end{aligned} \tag{2.39}$$

where the quantity

$$W_I \equiv \frac{\partial W}{\partial\rho_I} + \frac{i}{\rho_I}\frac{\partial W}{\partial\varphi} \tag{2.40}$$

was introduced.

By performing various steps [detailed in the Appendix, see Eqs. (A1)–(A6) therein] and recalling Eqs. (1.4) and (2.3), the final PDE for the fake superpotential W reads

$$\begin{aligned}W^2 + \sum_{I,J\neq I}\Big\{&|(W_I\rho_J + W_J\rho_I)|^2\\ &+ \frac{1}{2}[e^{i\varphi}\tilde{\epsilon}^{IJKL}(W_I\rho_J + W_J\rho_I)(W_K\rho_L + W_L\rho_K)\\ &+ e^{-i\varphi}\tilde{\epsilon}^{IJKL}(\bar{W}_I\rho_J + \bar{W}_J\rho_I)(\bar{W}_K\rho_L + \bar{W}_L\rho_K)]\Big\}\\ &= \rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2,\end{aligned} \tag{2.41}$$

where all terms of the sum can be found in Eq. (A9).

As a consequence of $\mathcal{N} = 8$ supersymmetry, Eq. (2.41) is fully symmetric in $\{\rho_0, \rho_1, \rho_2, \rho_3\}$, and it is straightforward to check that any ρ_I's are a solution. Following [2], a natural *Ansatz* for $\mathcal{N} = 8$ solutions is a linear combination of the skew eigenvalues (with constant coefficients):

$$W = \sum_{I=0}^{3}\alpha_I\rho_I. \tag{2.42}$$

Indeed, by plugging the *Ansatz* (2.42) into Eq. (2.41), the following system is obtained [note it is invariant under permutations of 0, 1, 2, 3; see Eq. (94) of [2]]:

$$\begin{aligned}&\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, && \alpha_0\alpha_1 + \alpha_2\alpha_3\cos\varphi = 0,\\ &\alpha_0\alpha_1\cos\varphi + \alpha_2\alpha_3 = 0, && \alpha_0\alpha_2 + \alpha_1\alpha_3\cos\varphi = 0,\\ &\alpha_0\alpha_2\cos\varphi + \alpha_1\alpha_3 = 0, && \alpha_1\alpha_2 + \alpha_0\alpha_3\cos\varphi = 0,\\ &\alpha_1\alpha_2\cos\varphi + \alpha_0\alpha_3 = 0.\end{aligned} \tag{2.43}$$

Clearly, a solution of this system reads ($a = 1, 2, 3$)

$$\alpha_0 = 1, \qquad \alpha_a = 0. \tag{2.44}$$

Because of the asymptotical meaning of W itself as an ADM mass [see Eq. (1.9)], Eq. (2.44) entails a $\frac{1}{8}$-BPS solution:

$$W_{(1/8)\text{-BPS}} = \rho_0, \tag{2.45}$$

namely the highest of the absolute values of the skew eigenvalues of Z_{AB} as given by ordering (2.16).

A non-BPS solution to system (2.43) reads [2]

$$W_{n\text{BPS}} = \tfrac{1}{2}(\rho_0 + \rho_1 + \rho_2 + \rho_3), \tag{2.46}$$

with $\varphi = \pi$. Thus, solution (2.46) does not describe the most general non-BPS flow with five parameters, but rather a particular case with a *double-extremal* phase (see Sec. III).

Let us notice that the result (2.45) is an extension to the whole attractor flow (i.e. for all range of the radial coordinate $\tau \in (-\infty, 0]$) of the well-known fact that the solution of the $\frac{1}{8}$-BPS solution to the $\mathcal{N} = 8$ attractor equations is obtained by retaining the singlet in the decomposition of $SU(8)$ with respect to the stabilizer of the $\frac{1}{8}$-BPS nondegenerate charge orbit, namely [13,21,24,40]:

$$\begin{aligned}&E_{7(7)} \to SU(8) \to SU(6)\times SU(2)\times U(1);\\ &\mathbf{56} \to \mathbf{28} + \overline{\mathbf{28}} \to (\mathbf{15},\mathbf{1})_{+1} + (\mathbf{6},\mathbf{2})_{-1} + (\mathbf{1},\mathbf{1})_{-3}\\ &\qquad + (\overline{\mathbf{15}},\mathbf{1})_{-1} + (\overline{\mathbf{6}},\mathbf{2})_{+1} + (\mathbf{1},\mathbf{1})_{+3},\end{aligned} \tag{2.47}$$

where the subscripts denote the charge with respect to $U(1)$. The corresponding extension to the whole $\frac{1}{8}$-BPS attractor flow amounts to stating that the superpotential governing the evolution is given by the singlet sector $(\mathbf{1},\mathbf{1})_{+3} + (\mathbf{1},\mathbf{1})_{+3}$ in the decomposition (2.47). In the normal frame (2.1), by recalling Eqs. (2.4), (2.5), (2.6), and (2.7) and splitting the index of the $\mathbf{8}$ of $SU(8)$ as $A = \hat{a}, \tilde{a}$, with $\hat{a} = 1, 2$ and $\tilde{a} = 3, \ldots, 8$ [consistently with (2.47)], it then follows that

$$W_{(1/8)\text{-BPS}} = |Z_{12}| = \rho_0. \tag{2.48}$$

III. SYMPLECTIC NORMAL FRAME: MAURER-CARTAN EQUATIONS AND PDE FOR W

This section is devoted to the derivation of the non-BPS *fake* superpotential uniquely from Maurer-Cartan equations, with suitable boundary horizon conditions.

We will obtain $W_{n\text{BPS}}$ as a solution of the Maurer-Cartan equations in a suitably defined manifestly $USp(8)$-covariant normal frame [6], in which maximal compact symmetry $USp(8)$ of the non-BPS charge orbit $\frac{E_{7(7)}}{E_{6(6)}}$ [39] is fully manifest (see e.g. also the treatment of [24,33,34]). As will be evident from subsequent treatment, such a normal frame is generally and intrinsically not democratic (in the meaning specified at the start of Sec. II).

In order to derive the non-BPS fake superpotential from the geometric structure encoded in the Maurer-Cartan equations, we extend to the whole attractor flow the well-known fact that the non-BPS solution of the $\mathcal{N} = 8$ attractor equations is obtained by retaining the singlet in the decomposition of $SU(8)$ with respect to the stabilizer of the non-BPS charge orbit, namely [21,24,40]:

$$\begin{aligned} &E_{7(7)} \to SU(8) \to USp(8);\\ &\mathbf{56} \to \mathbf{28} + \overline{\mathbf{28}} \to \mathbf{27} + \mathbf{1} + \mathbf{27}' + \mathbf{1}', \end{aligned} \tag{3.1}$$

where the priming distinguishes the various real irreducible representations of $USp(8)$, namely, the rank-2 antisymmetric skew-traceless $\mathbf{27}^{(\prime)}$ and the related skew-trace $\mathbf{1}^{(\prime)}$. The corresponding extension to the non-BPS attractor flow amounts to stating that the superpotential governing the evolution is given by the $USp(8)$ singlets in the decomposition (3.1) [6,24].

The branching (3.1) corresponds to decomposing the skew-diagonal complex matrix Z_{AB} [within the generic normal frame given by the first line of Eq. (2.1)] into its skew trace and its traceless part. This amounts to introducing the following quantities:

$$\begin{aligned} z_0 &\equiv b + c_1 + c_2 + c_3;\\ z_a &\equiv b - c_a; \end{aligned} \Leftrightarrow \begin{aligned} b &= \tfrac{1}{4}\Big(z_0 + \sum_a z_a\Big);\\ c_a &= \tfrac{1}{4}\Big(z_0 + \sum_a z_a - 4z_a\Big), \end{aligned} \tag{3.2}$$

thus yielding

$$Z_{AB} = b\Omega_{AB} + \mathcal{T}_{0,AB}, \tag{3.3}$$

with b and $\mathcal{T}_0$ respectively being half of the skew-trace and the skew-traceless part of the skew-diagonal complex matrix Z_{AB} [within the generic normal frame given by the first line of Eq. (2.1)]:

$$b \equiv \tfrac{1}{8} Z_{AB}\Omega^{AB}; \tag{3.4}$$

$$\begin{aligned} \mathcal{T}_{0,AB} &\equiv Z_{AB} - \frac{1}{8} Z_{CD}\Omega^{CD}\Omega_{AB}\\ &= \begin{pmatrix} c_1 + c_2 + c_3 & & & \\ & -c_1 & & \\ & & -c_2 & \\ & & & -c_3 \end{pmatrix} \otimes \epsilon, \end{aligned} \tag{3.5}$$

where Ω_{AB} is the 8×8 metric of $USp(8)$ defined in (2.27).

Following the same steps as in Sec. II, with details explained in the Appendix [see Eqs. (A10)–(A14) therein], after some straightforward algebra, one achieves the following result (recall $a = 1, 2, 3$ throughout):

$$\begin{aligned} \nabla W \nabla W = \frac{1}{8}\Bigg\{ \Bigg| & 4\,\mathrm{Re}\Bigg[\Bigg(b\frac{\partial W}{\partial \bar{b}} - \sum_a c_a \frac{\partial W}{\partial \bar{c}_a}\Bigg) + (c_2 + c_3)\Bigg(-\frac{\partial W}{\partial \bar{c}_1} + \frac{\partial W}{\partial \bar{c}_2} + \frac{\partial W}{\partial \bar{c}_3}\Bigg)\Bigg] + -2i\,\mathrm{Im}\Bigg[(c_2 + c_3)\Bigg(\frac{\partial W}{\partial \bar{b}} - \sum_a \frac{\partial W}{\partial \bar{c}_a}\Bigg)\\ & + 2b\Bigg(-\frac{\partial W}{\partial \bar{c}_1} + \frac{\partial W}{\partial \bar{c}_2} + \frac{\partial W}{\partial \bar{c}_3}\Bigg) + 2\Bigg(-c_1\frac{\partial W}{\partial \bar{c}_1} + c_2\frac{\partial W}{\partial \bar{c}_2} + c_3\frac{\partial W}{\partial \bar{c}_3}\Bigg)\Bigg]\Bigg|^2 + \Bigg| 4\,\mathrm{Re}\Bigg[\Bigg(b\frac{\partial W}{\partial \bar{b}} - \sum_a c_a\frac{\partial W}{\partial \bar{c}_a}\Bigg)\\ & + (c_1 + c_3)\Bigg(\frac{\partial W}{\partial \bar{c}_1} - \frac{\partial W}{\partial \bar{c}_2} + \frac{\partial W}{\partial \bar{c}_3}\Bigg)\Bigg] + -2i\,\mathrm{Im}\Bigg[(c_1 + c_3)\Bigg(\frac{\partial W}{\partial \bar{b}} - \sum_a \frac{\partial W}{\partial \bar{c}_a}\Bigg) + 2b\Bigg(\frac{\partial W}{\partial \bar{c}_1} - \frac{\partial W}{\partial \bar{c}_2} + \frac{\partial W}{\partial \bar{c}_3}\Bigg)\\ & + 2\Bigg(c_1\frac{\partial W}{\partial \bar{c}_1} - c_2\frac{\partial W}{\partial \bar{c}_2} + c_3\frac{\partial W}{\partial \bar{c}_3}\Bigg)\Bigg]\Bigg|^2 + \Bigg| 4\,\mathrm{Re}\Bigg[\Bigg(b\frac{\partial W}{\partial \bar{b}} - \sum_a c_a\frac{\partial W}{\partial \bar{c}_a}\Bigg) + (c_1 + c_2)\Bigg(\frac{\partial W}{\partial \bar{c}_1} + \frac{\partial W}{\partial \bar{c}_2} - \frac{\partial W}{\partial \bar{c}_3}\Bigg)\Bigg]\\ & + -2i\,\mathrm{Im}\Bigg[(c_1 + c_2)\Bigg(\frac{\partial W}{\partial \bar{b}} - \sum_a \frac{\partial W}{\partial \bar{c}_a}\Bigg) + 2b\Bigg(\frac{\partial W}{\partial \bar{c}_1} + \frac{\partial W}{\partial \bar{c}_2} - \frac{\partial W}{\partial \bar{c}_3}\Bigg) + 2\Bigg(c_1\frac{\partial W}{\partial \bar{c}_1} + c_2\frac{\partial W}{\partial \bar{c}_2} - c_3\frac{\partial W}{\partial \bar{c}_3}\Bigg)\Bigg]\Bigg|^2\Bigg\}. \end{aligned} \tag{3.6}$$

In order to proceed further, group theoretical arguments based on the reality of the $\mathbf{27}$ and $\mathbf{27}'$ of $USp(8)$ [see Eq. (3.1)] allow for the following polar parametrization of the traceless part $\mathcal{T}_{0,AB}$ [$\varrho_{\mathbf{27},a} \in \mathbb{R}^+$; see Eq. (3.3)]

$$c_a \equiv \varrho_{\mathbf{27},a}\exp(-i\beta) \Rightarrow \begin{pmatrix} \frac{\partial}{\partial c_a} \\ \frac{\partial}{\partial \bar{c}_a} \end{pmatrix} = \begin{pmatrix} e^{i\beta} & \frac{i}{\xi_a}e^{i\beta} \\ e^{-i\beta} & -\frac{i}{\xi_a}e^{-i\beta} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial \varrho_{\mathbf{27},a}} \\ \frac{\partial}{\partial \beta} \end{pmatrix}, \tag{3.7}$$

where, with a slight abuse of language, $\varrho_{\mathbf{27}}$'s generally denote the degrees of freedom pertaining to the traceless part $\mathcal{T}_{0,AB}$ of Z_{AB} [see Eq. (3.3), and the reasoning made above]. Moreover we split the skew trace into its real and imaginary parts

$$b \equiv x + iy, \qquad x, y \in \mathbb{R}. \tag{3.8}$$

The reasoning made at the start of the present section [see Eqs. (3.1) and (3.3)] implies the non-BPS fake superpotential $W_{n\mathrm{BPS}}$ to be related to the skew-trace b.

We now proceed by formulating the *Ansatz* that b is independent on all $\varrho_{\mathbf{27}}$'s introduced in Eq. (3.7). As we will see below, this corresponds to a natural *decoupling Ansatz*[3] for the PDE (3.10) satisfied by W, which will yield to the simplest solution. This yields the vanishing of all the derivatives of W with respect to c_a's. Thus, Eq. (3.6) reduces to

$$\nabla W \nabla W = \frac{1}{8}\left\{12\left(x\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\right) + [(\varrho_{\mathbf{27},1} + \varrho_{\mathbf{27},2})^2 + (\varrho_{\mathbf{27},1} + \varrho_{\mathbf{27},3})^2 + (\varrho_{\mathbf{27},2} + \varrho_{\mathbf{27},3})^2] \times\left(\cos\beta\frac{\partial W}{\partial y} - \sin\beta\frac{\partial W}{\partial x}\right)^2\right\}, \tag{3.9}$$

so that the whole PDE for the W reads

$$W^2 + \frac{1}{4}\left\{12\left(x\frac{\partial W}{\partial x} - y\frac{\partial W}{\partial y}\right)^2 + \Delta_{\mathbf{27}}\left(\cos\beta\frac{\partial W}{\partial y} - \sin\beta\frac{\partial W}{\partial x}\right)^2\right\} = 4(x^2 + y^2) + \Delta_{\mathbf{27}}, \tag{3.10}$$

where the quantity (symmetric in $\{\varrho_{\mathbf{27},1}, \varrho_{\mathbf{27},2}, \varrho_{\mathbf{27},3}\}$)

$$\Delta_{\mathbf{27}} \equiv (\varrho_{\mathbf{27},1} + \varrho_{\mathbf{27},2})^2 + (\varrho_{\mathbf{27},1} + \varrho_{\mathbf{27},3})^2 + (\varrho_{\mathbf{27},2} + \varrho_{\mathbf{27},3})^2 \tag{3.11}$$

was introduced.

Equation (3.10) is a nonlinear PDE in the real functional variables x and y. The previous statement that b is independent on all $\varrho_{\mathbf{27}}$'s trivially implies that its real and imaginary parts [x, respectively, y, as defined in Eq. (3.8)] do not depend on $\Delta_{\mathbf{27}}$. Thus, PDE (3.10) naturally *decouples* in the following system of PDEs:

$$W^2 + 3\left(x\frac{\partial W}{\partial x} - y\frac{\partial W}{\partial y}\right)^2 = 4(x^2 + y^2); \tag{3.12}$$

$$\left(\cos\beta\frac{\partial W}{\partial y} - \sin\beta\frac{\partial W}{\partial x}\right)^2 = 4. \tag{3.13}$$

PDE (3.12) admits the solution (symmetric in x and y)

$$W(x, y) = (x^{2/3} + y^{2/3})^{3/2}, \tag{3.14}$$

which plugged into PDE (3.13) yields the following algebraic equation for x and y in terms of β:

$$(x^{2/3} + y^{2/3})(x^{1/3}\cos\beta - y^{1/3}\sin\beta)^2 = x^{2/3}y^{2/3}. \tag{3.15}$$

Equation (3.15) is in turn solved by (factor 2 introduced for later convenience)

$$x = -2\varrho\sin^3\beta, \qquad y = 2\varrho\cos^3\beta, \tag{3.16}$$

where ϱ is a real strictly positive number:

$$\varrho \in \mathbb{R}^+. \tag{3.17}$$

In solution (3.16) ϱ is an arbitrary parameter whose introduction is possible as a consequence of the homogeneity of degree 0 of algebraic Eq. (3.15) in x and y. In other words, ϱ can be understood as an integration constant whose meaning has to be clarified by imposing proper boundary conditions. This is the case for the requirement of positivity of ϱ which is an asymptotical boundary condition due to the physical meaning of W that defines the ADM mass M_{ADM} at radial infinity [see Eqs. (1.9) and (3.18)]. Thus, Eqs. (3.14) and (3.16) yield that the final solution for W reads as follows:

$$W(x, y) = 2\varrho. \tag{3.18}$$

By recalling Eqs. (3.3), (3.4), (3.5), and (3.7), in the resulting manifestly $USp(8)$-covariant normal frame the central charge matrix Z_{AB} can thus be written as

$$Z_{AB} = 2(\cos^3\beta + i\sin^3\beta)i\varrho\Omega_{AB} + \exp(-i\beta) \times \begin{pmatrix} \varrho_{\mathbf{27},1}+\varrho_{\mathbf{27},2}+\varrho_{\mathbf{27},3} & & & \\ & -\varrho_{\mathbf{27},1} & & \\ & & -\varrho_{\mathbf{27},2} & \\ & & & -\varrho_{\mathbf{27},3} \end{pmatrix} \otimes \epsilon. \tag{3.19}$$

Equation (3.19) determines a parametrization of the symplectic normal frame (3.3), (3.4), and (3.5) which is *minimal*, because it contains only five parameters (see e.g. [21,38], and references therein) , namely $\{\beta, \varrho, \varrho_{\mathbf{27},1}, \varrho_{\mathbf{27},2}, \varrho_{\mathbf{27},3}\}$.

In order to consistently characterize solution (3.18) as the non-BPS fake superpotential, one can use the boundary condition at the horizon of non-BPS BH. To this end we notice that (see reasoning at the start of the present section) at non-BPS critical points of $V_{\mathrm{BH},\mathcal{N}=8}$ we have

$$\varrho_{\mathbf{27},1} = \varrho_{\mathbf{27},2} = \varrho_{\mathbf{27},3} = 0 \tag{3.20}$$

so that the parametrization (3.19) reduces to

[3]We should also note that this *Ansatz* holds for the particular solution (2.46), with $\beta = -\frac{\pi}{4} + 2k\pi$ ($k \in \mathbb{Z}$) but $\partial W \neq 0$.

$$Z_{AB,n\mathrm{BPS}} = 2(\cos^3\beta_{n\mathrm{BPS}} + i\sin^3\beta_{n\mathrm{BPS}})i\varrho_{n\mathrm{BPS}}\Omega_{AB}. \tag{3.21}$$

This last equation has to be compared with Eq. (2.25), to get

$$2(\cos^3\beta_{n\mathrm{BPS}} + i\sin^3\beta_{n\mathrm{BPS}})i\varrho_{n\mathrm{BPS}} = e^{i(\pi/4)}\rho_{n\mathrm{BPS}}, \tag{3.22}$$

whose splitting in real and imaginary parts, respectively, yields:

$$\sqrt{2}(\sin^3\beta_{n\mathrm{BPS}} - \cos^3\beta_{n\mathrm{BPS}})\varrho_{n\mathrm{BPS}} = \rho_{n\mathrm{BPS}}; \tag{3.23}$$

$$\cos^3\beta_{n\mathrm{BPS}} + \sin^3\beta_{n\mathrm{BPS}} = 0. \tag{3.24}$$

The unique solution of the system (3.23) and (3.24) [consistent with Eq. (3.17)] is found to be

$$\beta_{n\mathrm{BPS}} = -\frac{\pi}{4} + 2k\pi, \qquad k \in \mathbb{Z}; \tag{3.25}$$

$$\varrho_{n\mathrm{BPS}} = \rho_{n\mathrm{BPS}}, \tag{3.26}$$

in agreement with [6].

The non-BPS nature of the solution (3.18) implies the $\mathcal{I}_4$ of the **56** of $E_{7(7)}$ [given by Eqs. (2.10) and (2.11) in the special normal frame (2.1)] to be negative. To show this, we rewrite $\mathcal{I}_4$ in the manifestly $USp(8)$-covariant parametrization (3.19), obtaining [6]

$$\mathcal{I}_4 = -2^4\sin^2 2\beta(\varrho\sin 2\beta - \varrho_{\mathbf{27},1} - \varrho_{\mathbf{27},2} - \varrho_{\mathbf{27},3}) \times \prod_a(\varrho\sin 2\beta + \varrho_{\mathbf{27},a}), \tag{3.27}$$

which evaluated at the horizon of non-BPS BH reads

$$\mathcal{I}_{4,n\mathrm{BPS}} = -2^4\varrho^4_{n\mathrm{BPS}}\sin^6(2\beta_{n\mathrm{BPS}}). \tag{3.28}$$

Using Eqs. (3.25) and (3.26), Eq. (3.28) implies

$$\mathcal{I}_{4,n\mathrm{BPS}} = -2^4\rho^4_{n\mathrm{BPS}} = -W^4_{n\mathrm{BPS}}|_{n\mathrm{BPS}} < 0, \tag{3.29}$$

which confirms the function W given by Eq. (3.18) to be the non-BPS fake superpotential of $\mathcal{N} = 8$, $d = 4$ supergravity:

$$W_{n\mathrm{BPS}} = 2\varrho. \tag{3.30}$$

Thus, $W_{n\mathrm{BPS}}$ given by Eq. (3.30) has been proved to be the simplest solution of the PDE (3.10), determining the non-BPS fake superpotential of $\mathcal{N} = 8$, $d = 4$ supergravity. The proof given in the treatment performed above relies completely on the geometric data encoded into Maurer-Cartan equations (with suitable consistent boundary horizon conditions), and it is alternative with respect to the treatment given in [6].

As the special normal frame (2.1) has been proved in Sec. II to be more suitable to derive $\frac{1}{8}$-BPS attractor flow, so the symplectic normal frame (3.19) has been proved in this section to be more suitable to derive non-BPS attractor flow.

The expression of ϱ in terms of the five parameters $\{\rho_0, \rho_1, \rho_2, \rho_3, \varphi\}$ of the special normal frame (2.1) is not trivial, and it is thoroughly treated in Appendix B of [6]. In general, ϱ^2 turns out to satisfy an algebraic equation of order 6 with coefficients depending on $\{\rho_0, \rho_1, \rho_2, \rho_3, \varphi\}$ and their scalar-independent combination $\mathcal{I}_4$, as given by Eq. (B.14) of [6] (see also the discussion in [7]).

Thus, in general ϱ^2 seems not to enjoy an analytical expression. However, *at least* one of the solutions of Eq. (B.14) of [6] is a solution of PDE (A9), yielding $W_{n\mathrm{BPS}}$ in the special normal frame (2.1). Analogously, $W_{(1/8)\text{-BPS}}$ given by Eq. (2.45), suitably translated in the notation of the symplectic normal frame (3.19) (see treatment of Appendix B of [6]), is a solution of PDE (3.10), yielding $W_{(1/8)\text{-BPS}}$ in the symplectic normal frame (3.19).

Furthermore, it is here worth mentioning that, through a suitable rewriting in $\mathcal{N} = 2$ language, the results of [5–7] are solutions of PDEs (A9) and/or (3.10), eventually through additional reductions to st^2 or t^3 models [5–7].

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APPENDIX: COMPUTATIONAL DETAILS

In this Appendix we collect details of the computations determining the various formulas of the present paper.

(i) Concerning Sec. II, the details are listed below. Within the index reduction (2.32), the basic multiplication rules for the vielbein

$$P_{ABCD}\bar{P}^{EFGH} = \delta^{EFGH}_{ABCD}; \tag{A1}$$

$$P_{ABCD}P_{EFGH} = \epsilon_{ABCDEFGH} \tag{A2}$$

recast as

$$P_{IJ}\bar{P}^{KL} = \delta^K_I\delta^L_J; \tag{A3}$$

$$P_{IJ}P_{KL} = \tilde{\epsilon}_{IJKL} \equiv |\epsilon_{IJKL}|. \tag{A4}$$

Furthermore such rules and Eq. (2.29) yield

$$\nabla Z_I \nabla Z_J = \tilde{\epsilon}_{IJKL} \bar{Z}^K \bar{Z}^L; \tag{A5}$$

$$\nabla Z_I \nabla \bar{Z}_J = \delta^I_J |Z_I| = \delta^I_J \rho_I. \tag{A6}$$

Using (A4) and the fully explicited form of Eq. (2.39) which reads

$$\begin{aligned}\nabla_i W = \tfrac{1}{2}\{&[e^{i\varphi/2}(W_0\rho_1 + W_1\rho_0) + e^{-i\varphi/2}(\bar{W}_2\rho_3 + \bar{W}_3\rho_2)]P_{23} + [e^{i\varphi/2}(W_0\rho_2 + W_2\rho_0)\\ &+ e^{-i\varphi/2}(\bar{W}_1\rho_3 + \bar{W}_3\rho_1)]P_{13} + [e^{i\varphi/2}(W_0\rho_3 + W_3\rho_0) + e^{-i\varphi/2}(\bar{W}_1\rho_2 + \bar{W}_2\rho_1)]P_{12}\\ &+ [e^{i\varphi/2}(W_1\rho_2 + W_2\rho_1) + e^{-i\varphi/2}(\bar{W}_0\rho_3 + \bar{W}_3\rho_0)]P_{03} + [e^{i\varphi/2}(W_1\rho_3 + W_3\rho_1)\\ &+ e^{-i\varphi/2}(\bar{W}_0\rho_2 + \bar{W}_2\rho_0)]P_{02} + [e^{i\varphi/2}(W_2\rho_3 + W_3\rho_2) + e^{-i\varphi/2}(\bar{W}_0\rho_1 + \bar{W}_1\rho_0)]P_{01}\},\end{aligned} \tag{A7}$$

it can be computed that

$$\begin{aligned}g^{ij}\nabla_i W \nabla_j W = \tfrac{1}{2}\{&|(W_0\rho_1 + W_1\rho_0)|^2 + |(W_0\rho_2 + W_2\rho_0)|^2 + |(W_0\rho_3 + W_3\rho_0)|^2 + |(W_1\rho_2 + W_2\rho_1)|^2\\ &+ |(W_1\rho_3 + W_3\rho_1)|^2 + |(W_2\rho_3 + W_3\rho_2)|^2 + [e^{i\varphi}(W_0\rho_1 + W_1\rho_0)(W_2\rho_3 + W_3\rho_2)\\ &+ e^{-i\varphi}(\bar{W}_0\rho_1 + \bar{W}_1\rho_0)(\bar{W}_2\rho_3 + \bar{W}_3\rho_2)] + [e^{i\varphi}(W_0\rho_2 + W_2\rho_0)(W_1\rho_3 + W_3\rho_1)\\ &+ e^{-i\varphi}(\bar{W}_0\rho_2 + \bar{W}_2\rho_0)(\bar{W}_1\rho_3 + \bar{W}_3\rho_1)] + [e^{i\varphi}(W_0\rho_3 + W_3\rho_0)(W_1\rho_2 + W_2\rho_1)\\ &+ e^{-i\varphi}(\bar{W}_0\rho_3 + \bar{W}_3\rho_0)(\bar{W}_1\rho_2 + \bar{W}_2\rho_1)]\},\end{aligned} \tag{A8}$$

that, in turns, gives the following expanded form of PDE (2.41):

$$\begin{aligned}W^2 + \{&|(W_0\rho_1 + W_1\rho_0)|^2 + |(W_0\rho_2 + W_2\rho_0)|^2 + |(W_0\rho_3 + W_3\rho_0)|^2 + |(W_1\rho_2 + W_2\rho_1)|^2 + |(W_1\rho_3 + W_3\rho_1)|^2\\ &+ |(W_2\rho_3 + W_3\rho_2)|^2 + [e^{i\varphi}(W_0\rho_1 + W_1\rho_0)(W_2\rho_3 + W_3\rho_2) + e^{-i\varphi}(\bar{W}_0\rho_1 + \bar{W}_1\rho_0)(\bar{W}_2\rho_3 + \bar{W}_3\rho_2)]\\ &+ [e^{i\varphi}(W_0\rho_2 + W_2\rho_0)(W_1\rho_3 + W_3\rho_1) + e^{-i\varphi}(\bar{W}_0\rho_2 + \bar{W}_2\rho_0)(\bar{W}_1\rho_3 + \bar{W}_3\rho_1)]\\ &+ [e^{i\varphi}(W_0\rho_3 + W_3\rho_0)(W_1\rho_2 + W_2\rho_1) + e^{-i\varphi}(\bar{W}_0\rho_3 + \bar{W}_3\rho_0)(\bar{W}_1\rho_2 + \bar{W}_2\rho_1)]\} = \rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2.\end{aligned} \tag{A9}$$

(ii) Concerning Sec. III, the details are as follows:
Within parametrization (3.2), (3.3), (3.4), and (3.5), the Maurer-Cartan equations (2.33), (2.34), (2.35), and (2.36) read as follows:

$$\begin{aligned}\nabla b = \tfrac{1}{4}[&P_{01}(2\bar{b} + \bar{c}_2 + \bar{c}_3) + P_{02}(2\bar{b} + \bar{c}_1 + \bar{c}_3) + P_{03}(2\bar{b} + \bar{c}_1 + \bar{c}_2)\\ &+ P_{12}(2\bar{b} - \bar{c}_1 - \bar{c}_2) + P_{13}(2\bar{b} - \bar{c}_1 - \bar{c}_3) + P_{23}(2\bar{b} - \bar{c}_2 - \bar{c}_3)];\end{aligned} \tag{A10}$$

$$\begin{aligned}\nabla c_1 = \tfrac{1}{4}[&P_{01}(-2\bar{b} - 4\bar{c}_1 - 3\bar{c}_2 - 3\bar{c}_3) + P_{02}(2\bar{b} + \bar{c}_1 + \bar{c}_3) + P_{03}(2\bar{b} + \bar{c}_1 + \bar{c}_2)\\ &+ P_{12}(-2\bar{b} - \bar{c}_1 + 3\bar{c}_2) + P_{13}(-2\bar{b} - \bar{c}_1 + 3\bar{c}_3) + P_{23}(2\bar{b} - \bar{c}_2 - \bar{c}_3)];\end{aligned} \tag{A11}$$

$$\begin{aligned}\nabla c_2 = \tfrac{1}{4}[&P_{01}(2\bar{b} + \bar{c}_2 + \bar{c}_3) + P_{02}(-2\bar{b} - 3\bar{c}_1 - 4\bar{c}_2 - 3\bar{c}_3) + P_{03}(2\bar{b} + \bar{c}_1 + \bar{c}_2)\\ &+ P_{12}(-2\bar{b} + 3\bar{c}_1 - \bar{c}_2) + P_{13}(2\bar{b} - \bar{c}_1 - \bar{c}_3) + P_{23}(-2\bar{b} - \bar{c}_2 + 3\bar{c}_3)];\end{aligned} \tag{A12}$$

$$\begin{aligned}\nabla c_3 = \tfrac{1}{4}[&P_{01}(2\bar{b} + \bar{c}_2 + \bar{c}_3) + P_{02}(2\bar{b} + \bar{c}_1 + \bar{c}_3) + P_{03}(-2\bar{b} - 3\bar{c}_1 - 3\bar{c}_2 - 4\bar{c}_3)\\ &+ P_{12}(2\bar{b} - \bar{c}_1 - \bar{c}_2) + P_{13}(-2\bar{b} + 3\bar{c}_1 - \bar{c}_3) + P_{23}(-2\bar{b} + 3\bar{c}_2 - \bar{c}_3)].\end{aligned} \tag{A13}$$

Then, by following the same steps as in Sec. II, after some algebra, one achieves the following result (recall $a = 1, 2, 3$ throughout):

$$\nabla W = \frac{1}{4}\Big\{P_{01}\Big[(2\bar{b}+\bar{c}_2+\bar{c}_3)\Big(\frac{\partial W}{\partial b}+\sum_a\frac{\partial W}{\partial c_a}\Big)+(2b-c_2-c_3)\Big(\frac{\partial W}{\partial \bar{b}}+\sum_a\frac{\partial W}{\partial \bar{c}_a}\Big)+-4\Big((b-c_2)\frac{\partial W}{\partial \bar{c}_3}+(b-c_3)\frac{\partial W}{\partial \bar{c}_2}\Big)$$
$$+-4(\bar{b}+\bar{c}_1+\bar{c}_2+\bar{c}_3)\frac{\partial W}{\partial c_1}\Big]+P_{02}\Big[(2\bar{b}+\bar{c}_1+\bar{c}_3)\Big(\frac{\partial W}{\partial b}+\sum_a\frac{\partial W}{\partial c_a}\Big)+(2b-c_1-c_3)\Big(\frac{\partial W}{\partial \bar{b}}+\sum_a\frac{\partial W}{\partial \bar{c}_a}\Big)$$
$$+-4\Big((b-c_1)\frac{\partial W}{\partial \bar{c}_3}+(b-c_3)\frac{\partial W}{\partial \bar{c}_1}\Big)+-4(\bar{b}+\bar{c}_1+\bar{c}_2+\bar{c}_3)\frac{\partial W}{\partial c_2}\Big]+P_{03}\Big[(2\bar{b}+\bar{c}_1+\bar{c}_2)\Big(\frac{\partial W}{\partial b}+\sum_a\frac{\partial W}{\partial c_a}\Big)$$
$$+(2b-c_1-c_3)\Big(\frac{\partial W}{\partial \bar{b}}+\sum_a\frac{\partial W}{\partial \bar{c}_a}\Big)+-4\Big((b-c_1)\frac{\partial W}{\partial \bar{c}_2}+(b-c_2)\frac{\partial W}{\partial \bar{c}_1}\Big)+-4(\bar{b}+\bar{c}_1+\bar{c}_2+\bar{c}_3)\frac{\partial W}{\partial c_3}$$
$$+P_{12}\Big[(2\bar{b}-\bar{c}_1-\bar{c}_2)\Big(\frac{\partial W}{\partial b}+\sum_a\frac{\partial W}{\partial c_a}\Big)+(2b+c_1+c_3)\Big(\frac{\partial W}{\partial \bar{b}}+\sum_a\frac{\partial W}{\partial \bar{c}_a}\Big)+-4\Big((\bar{b}-\bar{c}_1)\frac{\partial W}{\partial c_2}+(\bar{b}-\bar{c}_2)\frac{\partial W}{\partial c_1}\Big)$$
$$+-4(b+c_1+c_2+c_3)\frac{\partial W}{\partial \bar{c}_3}\Big]+P_{13}\Big[(2\bar{b}-\bar{c}_1-\bar{c}_3)\Big(\frac{\partial W}{\partial b}+\sum_a\frac{\partial W}{\partial c_a}\Big)+(2b+c_1+c_3)\Big(\frac{\partial W}{\partial \bar{b}}+\sum_a\frac{\partial W}{\partial \bar{c}_a}\Big)$$
$$+-4\Big((\bar{b}-\bar{c}_1)\frac{\partial W}{\partial c_3}+(\bar{b}-\bar{c}_3)\frac{\partial W}{\partial c_1}\Big)+-4(b+c_1+c_2+c_3)\frac{\partial W}{\partial \bar{c}_2}\Big]+P_{23}\Big[(2\bar{b}-\bar{c}_2-\bar{c}_3)\Big(\frac{\partial W}{\partial b}+\sum_a\frac{\partial W}{\partial c_a}\Big)$$
$$+(2b+c_2+c_3)\Big(\frac{\partial W}{\partial \bar{b}}+\sum_a\frac{\partial W}{\partial \bar{c}_a}\Big)+-4\Big((\bar{b}-\bar{c}_2)\frac{\partial W}{\partial c_3}+(\bar{b}-\bar{c}_3)\frac{\partial W}{\partial c_2}\Big)+-4(b+c_1+c_2+c_3)\frac{\partial W}{\partial \bar{c}_1}\Big]\Big\}. \quad \text{(A14)}$$

It is worth noticing that the coefficient of the vielbein P_{IJ} (recall $I = 0, 1, 2, 3$ throughout) is the complex conjugate of the coefficient of P_{KL}, with $K, L \neq I, J$. In other words, in order to compute the term $\nabla W \nabla W$ one has just to sum up the squares of the real and imaginary parts of each coefficient, thus obtaining Eq. (3.6).

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