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Dynamic Analysis for Bertrand Competition Model with Exponential Form*

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Abstract. This paper will consider a nonlinear system of difference equations which describes a qualitative study of Bertrand oligopoly games with two boundedly rational players. With nonlinear demand function of exponential form, the local stability of equilibria and the global convergence of positive solutions for the dynamical system are analyzed.

Keywords: difference equations, boundedness, Bertrand model, global asymptotic behavior.

AMS Subject Classification: 39A11; 40A05; 91A25.

1 Introduction

Market economy is fundamentally a dynamic system, which can usually be described mathematically by difference equations. In the dynamic study of economics, a couple of economic models represented by difference equations are investigated, such as the classical cobweb model describing the variation of the supply and demand, the Cournot models of oligopoly, and so on [2], [1], [6], [11]. It is well known that Bertrand duopoly competition game is one of the basic oligopoly games with two players. In this paper we consider a Bertrand duopoly game with an industry where two firms produce heterogeneous products. Price strategic interactions appear because the total demand of the good depends on the price of the industry according to a given demand function. In the classical study of Bertrand game, the demand function is usually linear or quadratic. However the demand function of exponential form can describe the bounded rationality duopoly game more realistic [11] and at the same time the dynamic will be more complicated. We assume in this paper the demand function

$$Q = f(p_1, p_2) = ae^{-(p_1+p_2)}, \quad (1.1)$$

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where a is a parameter of maximum demand in the market and $p_i, i = 1, 2$ denotes the price of the good produced by firm i . Assume the cost function

$$C_i(Q) = c_iQ, \quad i = 1, 2, \tag{1.2}$$

where c_i is the marginal cost of the i th firm. Then the profit resulting from the above Bertrand duopoly game is given by

$$\Pi_i = p_iQ - c_iQ = a(p_i - c_i)e^{-(p_1+p_2)}. \tag{1.3}$$

The classical oligopoly games, and the associated notion of Nash equilibrium, are based on quite demanding notion of rationality. However, since the available information in the oligopoly market is incomplete, the rational players make their price decisions on a local estimate of the expected marginal profit $\frac{\partial \Pi_i}{\partial p_i}$. Hence, the dynamical equation of the bounded rationality player i has the form

$$p_i(t + 1) = p_i(t) + \nu_i p_i(t) \frac{\partial \Pi_i}{\partial p_i}, \quad i = 1, 2,$$

where $\nu_i, i = 1, 2$ is a positive parameter which represents the relative speed of the price adjustment by producer i .

Therefore, using (1.1)–(1.3), the discrete dynamical system becomes an iterated two-dimensional mapping which has the form

$$\begin{cases} p_1(t + 1) = p_1(t) + a\nu_1 e^{-(p_1+p_2)} p_1(t)(1 - p_1(t)) + a\nu_1 c_1 p_1(t) e^{-(p_1+p_2)}, \\ p_2(t + 1) = p_2(t) + a\nu_2 e^{-(p_1+p_2)} p_2(t)(1 - p_2(t)) + a\nu_1 c_1 p_2(t) e^{-(p_1+p_2)}. \end{cases}$$

We can rewrite this system in the new form

$$\begin{cases} x_{n+1} = x_n + (\alpha_1 - \beta_1 x_n) x_n e^{-(x_n+y_n)}, \\ y_{n+1} = y_n + (\alpha_2 - \beta_2 y_n) y_n e^{-(x_n+y_n)}, \end{cases}$$

where $x_n = p_1(t), y_n = p_2(t), \alpha_i = a\nu_i(1 + c_i), \beta_i = a\nu_i, i = 1, 2$.

In this paper we study the boundedness and the global asymptotic behavior of the positive solutions of the system of difference equation

$$\begin{cases} x_{n+1} = x_n + (\alpha_1 - \beta_1 x_n) x_n e^{-(x_n+y_n)}, \\ y_{n+1} = y_n + (\alpha_2 - \beta_2 y_n) y_n e^{-(x_n+y_n)}, \end{cases} \tag{1.4}$$

where $\alpha_i, \beta_i \in (0, \infty)$ with $\alpha_i > \beta_i, i = 1, 2$, and the initial values x_0, y_0 are positive numbers.

In [5], by using the inverse demand function of exponential form, MF Elettrey and H El-Metwally studied a Cournot competition model described by difference systems of exponential form and they obtained the local stability of the equilibrium point and the global convergence of positive solutions. In [4], HA El-Metwally and AA Elsadany investigated the chaotic behavior of a duopoly Cournot game model of difference systems of exponential form. As for the studies of the behavior of positive solutions for difference equations of exponential form, we refer the readers to [3], [8], [9], [10] and the references therein. The next theorem will be a useful tool later in Section 3.

Theorem 1. [7] Suppose $T = (f, g)$ be a monotone map on a closed and bounded rectangular region $S \subset \mathbb{R}^2$. If T has a unique fixed point $E = (\bar{x}, \bar{y})$ in S , then E is a global attractor of T on S .

2 Equilibria Points of (1.4)

Observe that the equilibria points of system (1.4) are given by the equations

$$\begin{cases} \bar{x} = \bar{x} + (\alpha_1 - \beta_1 \bar{x})\bar{x}e^{-(\bar{x}+\bar{y})}, \\ \bar{y} = \bar{y} + (\alpha_2 - \beta_2 \bar{y})\bar{y}e^{-(\bar{x}+\bar{y})}, \end{cases} \Rightarrow \begin{cases} (\alpha_1 - \beta_1 \bar{x})\bar{x}e^{-(\bar{x}+\bar{y})} = 0, \\ (\alpha_2 - \beta_2 \bar{y})\bar{y}e^{-(\bar{x}+\bar{y})} = 0. \end{cases}$$

Obviously, system (1.4) has equilibria points $E_0 = (0, 0)$, $E_1 = (0, \frac{\alpha_2}{\beta_2})$, $E_2 = (\frac{\alpha_1}{\beta_1}, 0)$ and $E_3 = (\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2})$.

In the following, we deal with the local stability of the above four equilibrium points of (1.4). Now set $F(x_n, y_n) = x_n + (\alpha_1 - \beta_1 x_n)x_n e^{-(x_n+y_n)}$, $G(x_n, y_n) = y_n + (\alpha_2 - \beta_2 y_n)y_n e^{-(x_n+y_n)}$, where $F(x, y)$ and $G(x, y)$ are continuous functions. Then we obtain

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= 1 + [(\alpha_1 - \beta_1 x) - \beta_1 x - (\alpha_1 - \beta_1 x)x]e^{-(x+y)}, \\ \frac{\partial F(x, y)}{\partial y} &= -(\alpha_1 - \beta_1 x)xe^{-(x+y)}, \quad \frac{\partial G(x, y)}{\partial x} = -(\alpha_2 - \beta_2 y)ye^{-(x+y)}, \\ \frac{\partial G(x, y)}{\partial y} &= 1 + [(\alpha_2 - \beta_2 y) - \beta_2 y - (\alpha_2 - \beta_2 y)y]e^{-(x+y)}. \end{aligned}$$

Proposition 1. The equilibrium point E_0 of system (1.4) is unstable.

Proof. We compute the Jacobian matrix of system (1.4) about the equilibrium point $E_0 = (0, 0)$ and have

$$J(E_0) = \begin{pmatrix} 1 + \alpha_1 & 0 \\ 0 & 1 + \alpha_2 \end{pmatrix}.$$

Therefore the eigenvalues of $J(E_0)$ are given by $\lambda_1 = 1 + \alpha_1$ and $\lambda_2 = 1 + \alpha_2$. Then $|\lambda_i| > 1, i = 1, 2$. So the equilibrium point E_0 of system (1.4) is unstable. The proof is completed. \square

Proposition 2. The equilibria points E_1 and E_2 of system (1.4) are saddle points.

Proof. The Jacobian matrix of system (1.4) about the equilibrium point $E_1 = (0, \alpha_2/\beta_2)$ has the form

$$J(E_1) = \begin{pmatrix} 1 + \alpha_1 e^{-\alpha_2/\beta_2} & 0 \\ 0 & 1 - \alpha_2 e^{-\alpha_2/\beta_2} \end{pmatrix}.$$

Therefore the eigenvalues of $J(E_1)$ are given by $\lambda_1 = 1 + \alpha_1 e^{-\alpha_2/\beta_2}$ and $\lambda_2 = 1 - \alpha_2 e^{-\alpha_2/\beta_2}$. Then

$$|\lambda_1| > 1, \quad |\lambda_2| < 1.$$

Thus it follows that the equilibrium point E_1 of system (1.1) is a saddle point. Similarly, one can easily prove that the equilibrium point E_2 of system (1.4) is also a saddle point. This completes the proof. \square

Proposition 3. *The Nash equilibrium point E_3 of system (1.4) is asymptotically stable if either $0 < \alpha_i e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})} < 1$ or $1 < \alpha_i e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})} < 2$, $i = 1, 2$.*

Proof. The Jacobian matrix of system (1.4) about the equilibrium point $E_3 = (\alpha_1/\beta_1, \alpha_2/\beta_2)$ is

$$J(E_1) = \begin{pmatrix} 1 - \alpha_1 e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})} & 0 \\ 0 & 1 - \alpha_2 e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})} \end{pmatrix}.$$

Therefore the eigenvalues of $J(E_3)$ are given by

$$\lambda_1 = 1 - \alpha_1 e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})}, \quad \lambda_2 = 1 - \alpha_2 e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})}.$$

It is well known that the equilibrium point E_3 of system (1.4) is locally asymptotically stable if for $i = 1, 2$, $|\lambda_i| < 1$ are satisfied, which is true when $0 < \alpha_i e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})} < 1$ or $1 < \alpha_i e^{-(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2})} < 2$, $i = 1, 2$. The proof is completed. \square

3 Global Stability Analysis of (1.4)

In this section we first concern with the boundedness properties of the positive solutions for system (1.4). Under appropriate conditions, we give some bounded results related to system (1.4).

Theorem 2. *Assume that $\beta_i e^{-\frac{\alpha_i + \beta_i}{\beta_i}} < 1$. Then every solution $\{(x_n, y_n)\}_{n=0}^\infty$ of system (1.4) with $x_0 > 0$ and $y_0 > 0$, satisfies that $x_n > 0$ and $y_n > 0$ for all $n > 0$.*

Proof. Let $H_i(x, y), i = 1, 2$ be continuous functions defined by

$$H_i(x, y) = 1 + (\alpha_i - \beta_i x)e^{-(x+y)}, \quad i = 1, 2.$$

Then system (1.4) can be rewritten in the form

$$x_{n+1} = x_n H_1(x_n, y_n), \quad y_{n+1} = y_n H_2(x_n, y_n).$$

Now assume that $\{(x_n, y_n)\}_{n=0}^\infty$ is a solution of system (1.4) with positive initial values. Then it suffices to show that $H_i(x, y), i = 1, 2$ are positive for all $x_0 > 0$ and $y_0 > 0$. Observe that

$$\frac{\partial H_i(x, y)}{\partial x} = (-\beta_i - \alpha_i + \beta_i x)e^{-(x+y)}, \quad \frac{\partial H_i(x, y)}{\partial y} = -(\alpha_i - \beta_i x)e^{-(x+y)}, \quad i = 1, 2.$$

Therefore H_i have no positive critical points. Let a and b be arbitrary positive numbers and consider the domain

$$D = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}.$$

Then for $i = 1, 2$, we see that

$$H_i(0, y) = 1 + \alpha_i e^{-y}, \quad H_i(a, y) = 1 + (\alpha_i - a\beta_i) e^{-(a+y)}, \quad 0 \leq y \leq b,$$

$$H_i(x, 0) = 1 + (\alpha_i - \beta_i x) e^{-x}, \quad H_i(x, b) = 1 + (\alpha_i - \beta_i x) e^{-(x+b)}, \quad 0 \leq x \leq a.$$

Using elementary differential calculus, we obtain that the absolute minimum of H_i is $1 - \beta_i e^{-\frac{\alpha_i + \beta_i}{\beta_i}}$. Therefore $H_i(x, y) \geq 1 - \beta_i e^{-\frac{\alpha_i + \beta_i}{\beta_i}} > 0$ for all $(x, y) \in D$. Since a and b are arbitrary positive numbers, we can conclude that $H_i(x, y) > 0$ for $i = 1, 2$ and for all $(x, y) \in (0, \infty)^2$. \square

Theorem 3. Let $\{(x_n, y_n)\}_{n=0}^\infty$ be a solution of system (1.4) with $(x_{n_0}, y_{n_0}) \in (0, \frac{\alpha_1}{\beta_1}] \times (0, \frac{\alpha_2}{\beta_2}]$ for some $n_0 \geq 0$. If for $i = 1, 2$, $\alpha_i < 1$, then $\{(x_n, y_n)\} \in (0, \frac{\alpha_1}{\beta_1}] \times (0, \frac{\alpha_2}{\beta_2}]$ for all $n \geq n_0$.

Proof. Let $n_0 \geq 0$ be such that $x_{n_0} \in (0, \frac{\alpha_1}{\beta_1}]$. It follows from system (1.4) that

$$x_{n_0+1} \leq x_{n_0} + (\alpha_1 - \beta_1 x_{n_0}) x_{n_0}.$$

Set $h(x) = x + (\alpha_1 - \beta_1 x)x$. We obtain that

$$h'(x) = 1 + \alpha_1 - 2\beta_1 x, \quad h''(x) = -2\beta_1 < 0.$$

Hence $h'(x)$ is decreasing on $(0, \frac{\alpha_1}{\beta_1}]$. Then $h'(\frac{\alpha_1}{\beta_1}) = 1 - \alpha_1 > 0$. Therefore $h(x)$ is increasing on $(0, \frac{\alpha_1}{\beta_1}]$. Consequently, $h(x) \leq h(\frac{\alpha_1}{\beta_1}) = \frac{\alpha_1}{\beta_1}$. So $x_n \leq \frac{\alpha_1}{\beta_1}$ for all $n \geq n_0$. Similar method can be applied for y_n . This completes the proof. \square

Theorem 4. Assume that $\alpha_i < 1$. Then every solution $\{(x_n, y_n)\}_{n=0}^\infty$ of system (1.4) satisfies

$$x_n \leq (1 - \alpha_1)^{n-n_0} x_{n_0} + \frac{\alpha_1}{\beta_1} [1 - (1 - \alpha_1)^{n-n_0}],$$

$$y_n \leq (1 - \alpha_1)^{n-n_0} y_{n_0} + \frac{\alpha_1}{\beta_1} [1 - (1 - \alpha_1)^{n-n_0}]$$

for any n_0 satisfying $x_{n_0} \in (0, \frac{\alpha_1}{\beta_1}]$, $y_{n_0} \in (0, \frac{\alpha_2}{\beta_2}]$.

Proof. Let $n \geq 0$ be such that $x_n \in (0, \frac{\alpha_1}{\beta_1}]$, $y_n \in (0, \frac{\alpha_2}{\beta_2}]$. It follows from system (1.4) that

$$x_{n+1} \leq x_n + (\alpha_1 - \beta_1 x_n) x_n \leq x_n + (\alpha_1 - \beta_1 x_n) \frac{\alpha_1}{\beta_1} = (1 - \alpha_1) x_n + \frac{\alpha_1^2}{\beta_1}.$$

Since $\alpha_1 < 1$, we have

$$x_{n+2} \leq (1 - \alpha_1) x_{n+1} + \frac{\alpha_1^2}{\beta_1} = (1 - \alpha_1)^2 x_n + (1 - \alpha_1) \frac{\alpha_1^2}{\beta_1} + \frac{\alpha_1^2}{\beta_1}.$$

By induction, for fixed n_0 satisfying $x_{n_0} \in (0, \alpha_1/\beta_1]$, $y_{n_0} \in (0, \alpha_2/\beta_2]$, we obtain

$$\begin{aligned} x_n &\leq (1-\alpha_1)^{n-n_0}x_{n_0} + \frac{\alpha_1^2}{\beta_1}[(1-\alpha_1)^{n-n_0-1} + (1-\alpha_1)^{n-n_0-2} + \dots + (1-\alpha_1) + 1] \\ &= (1-\alpha_1)^{n-n_0}x_{n_0} + \frac{\alpha_1^2}{\beta_1} \frac{1 - (1-\alpha_1)^{n-n_0}}{\alpha_1} \\ &= (1-\alpha_1)^{n-n_0}x_{n_0} + \frac{\alpha_1}{\beta_1}[1 - (1-\alpha_1)^{n-n_0}]. \end{aligned}$$

Similarly, we could prove

$$y_n \leq (1-\alpha_1)^{n-n_0}y_{n_0} + \frac{\alpha_1}{\beta_1}[1 - (1-\alpha_1)^{n-n_0}]$$

for any n_0 satisfying $x_{n_0} \in (0, \frac{\alpha_1}{\beta_1}]$, $y_{n_0} \in (0, \frac{\alpha_2}{\beta_2}]$. \square

The following corollary is coming immediately from Theorem 4.

Corollary 1. Assume $\alpha_i < 1, i = 1, 2$. Then for any initial point $(x_{n_0}, y_{n_0}) \in (0, \alpha_1/\beta_1] \times (0, \alpha_2/\beta_2]$, every positive solution $\{(x_n, y_n)\}_{n=0}^\infty$ of system (1.4) is bounded. Moreover,

$$\limsup_{n \rightarrow \infty} x_n \leq \alpha_1/\beta_1, \quad \limsup_{n \rightarrow \infty} y_n \leq \alpha_2/\beta_2.$$

Next, in the following theorem, we investigate the global attractivity of the equilibrium point (\bar{x}, \bar{y}) of system (1.4).

Theorem 5. Assume that $\alpha_i < 1$. Then the unique positive equilibrium point (\bar{x}, \bar{y}) of system (1.4) is a global attractor of all positive solutions of system (1.4).

Proof. Let $\{(x_n, y_n)\}_{n=0}^\infty$ be a solution of system (1.4). In the first case, if $x_0 \leq \bar{x} < \frac{\alpha_1}{\beta_1}$,

$$x_1 = x_0 + (\alpha_1 - \beta_1 x_0)x_0 e^{-(x_0+y_0)} > x_0.$$

Then the sequence $\{x_n\}_{n=0}^\infty$ is increasing and since it was shown that it is bounded above by $\frac{\alpha_1}{\beta_1}$, then it converges to the unique positive equilibrium point \bar{x} . The second case is $x_0 \geq \bar{x}$, we will show that there exists a positive integer N such that $x_N \leq \bar{x}$. Notice that in this case

$$x_1 = x_0 + (\alpha_1 - \beta_1 x_0)x_0 e^{-(x_0+y_0)} < x_0.$$

Repeating this step confirms that $\{x_n\}$ is a decreasing sequence. Thus either there exists $N \in \mathbb{N}$ such that $x_N < \bar{x}$ or x_n is a bounded decreasing sequence which has a limit $d_1 > \bar{x}$. In whichever cases, we can assume similarly that y_n has a limit d_2 . Set $f(x, y) = (\beta_1 x - \alpha_1)x e^{-(x+y)}$. By the continuity of f and convergence of (x_n, y_n) , we have there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$, $f(x_n, y_n) \geq \delta$, where $\delta = \frac{f(d_1, d_2)}{2} > 0$. Therefore, for any $n \geq n_1$,

$$x_{n+1} = x_n + (\alpha_1 - \beta_1 x_n)x_n e^{-(x_n+y_n)} = x_n - (\beta_1 x_n - \alpha_1)x_n e^{-(x_n+y_n)} < x_n - \delta.$$

So there exists a positive integer N such that $x_N < \bar{x}$.

Similarly, it can be shown that the sequence $\{y_n\}_{n=0}^\infty$ converges to the unique positive equilibrium point \bar{y} . Thus $\{(x_n, y_n)\}_{n=0}^\infty$ converges to (\bar{x}, \bar{y}) .
 \square

Example 1. Set $\alpha_1 = 0.6, \alpha_2 = 0.5, \beta_1 = 0.3, \beta_2 = 0.1$, Figure 1 shows the stability of equilibrium of (1.4).

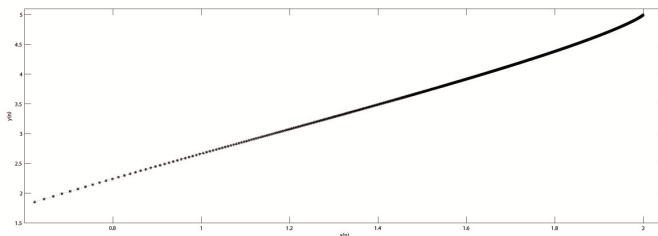


Figure 1. The stability of equilibrium of (1.4) while $\alpha_i > \beta_i$

As a special case at last, when $\alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta$, we contemplate the following system

$$\begin{cases} x_{n+1} = x_n + (\alpha - \beta x_n)x_n e^{-(x_n+y_n)}, \\ y_{n+1} = y_n + (\alpha - \beta y_n)y_n e^{-(x_n+y_n)}, \end{cases} \tag{3.1}$$

we can have

Theorem 6. Assume that $\alpha < 1$. If $\beta > \frac{\alpha^2+4\beta^2}{4}$, then the unique positive equilibrium point (\bar{x}, \bar{y}) of system (1.4) is a global attractor of all positive solutions of system (3.1).

Proof. Rewrite (3.1) as

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = G(x_n, y_n),$$

where $F(x, y) = x + (\alpha - \beta x)xe^{-(x+y)}$ and $G(x, y) = y + (\alpha - \beta y)ye^{-(x+y)}$ are continuous functions. Now consider

$$\begin{cases} m_1 = F(m_1, M_2), & M_1 = F(M_1, m_2), \\ m_2 = G(M_1, m_2), & M_2 = G(m_1, M_2). \end{cases}$$

That is,

$$\begin{cases} m_1 = m_1 + (\alpha - \beta m_1)m_1 e^{-(m_1+M_2)}, & M_1 = M_1 + (\alpha - \beta M_1)M_1 e^{-(M_1+m_2)}, \\ m_2 = m_2 + (\alpha - \beta m_2)m_2 e^{-(M_1+m_2)}, & M_2 = M_2 + (\alpha - \beta M_2)M_2 e^{-(m_1+M_2)}. \end{cases}$$

Obviously, $m_1 = M_1 = m_2 = M_2$ satisfy all the above equations. Solving these equations directly can obtain $m_1 = M_1 = m_2 = M_2 = \alpha/\beta$. So (3.1) has a unique fixed point $(\bar{x}, \bar{y}) = (\alpha/\beta, \alpha/\beta)$.

On the other hand, we can see

$$\frac{\partial F}{\partial x} = 1 + [\alpha - (\alpha + 2\beta)x + \beta x^2]e^{-(x+y)} \geq 1 - \frac{\alpha^2 + 4\beta^2}{4\beta}.$$

Therefore, $\frac{\partial F}{\partial x} > 0$ if $\beta > \frac{\alpha^2 + 4\beta^2}{4}$. We can do the same way to get that $\frac{\partial G}{\partial x} > 0$. Thus, by Theorem 1, it follows that the unique positive equilibrium point (\bar{x}, \bar{y}) is a global attractor of all positive solutions of system (3.1). \square

4 Remark

In this last section, just from the mathematics point of view, we give a remark to discuss the case $\alpha_i \leq \beta_i, i = 1, 2$. We will see the unique positive equilibrium point (\bar{x}, \bar{y}) of system (1.4) is a global attractor of all positive solutions of system (1.4).

Proposition 4. *Let $\{(x_n, y_n)\}_{n=0}^\infty$ be a solution of system (1.4) with $(x_{n_0}, y_{n_0}) \in (0, \frac{\alpha_1}{\beta_1}] \times (0, \frac{\alpha_2}{\beta_2}]$ for some $n_0 \geq 0$. If for $i = 1, 2$, one of the followings is hold:*

(i) $\alpha_i \leq \beta_i, \beta_i e^{-\frac{\alpha_i}{\beta_i}} < 1$; (ii) $\alpha_i < 1$, then $\{(x_n, y_n)\} \in (0, \frac{\alpha_1}{\beta_1}] \times (0, \frac{\alpha_2}{\beta_2}]$ for all $n \geq n_0$.

Proof. For condition (i), we can prove as follows. Let $n_0 \geq 0$ be such that $x_{n_0} \in (0, \frac{\alpha_1}{\beta_1}]$. Since $\alpha_i \leq \beta_i$ in (i), it follows from system (1.4) that

$$x_{n_0+1} \leq x_{n_0} + (\alpha_1 - \beta_1 x_{n_0})e^{-\frac{\alpha_1}{\beta_1}}.$$

Set $g(x) = x + (\alpha_1 - \beta_1 x)e^{-\alpha_1/\beta_1}$. Then $g'(x) = 1 - \beta_1 e^{-\frac{\alpha_1}{\beta_1}}$. If (i) is hold, we have $g'(x) > 0$, and then $g(x)$ is increasing on $(0, \frac{\alpha_1}{\beta_1}]$. Consequently, $g(x) \leq g(\frac{\alpha_1}{\beta_1}) = \frac{\alpha_1}{\beta_1}$. Similar method can be applied for y_n .

As for condition (ii), we have the following proof. Let $n_0 \geq 0$ be such that $x_{n_0} \in (0, \frac{\alpha_1}{\beta_1}]$. It follows from system (1.4) that

$$x_{n_0+1} \leq x_{n_0} + (\alpha_1 - \beta_1 x_{n_0})x_{n_0}.$$

Set $h(x) = x + (\alpha_1 - \beta_1 x)x$. Then

$$h'(x) = 1 + \alpha_1 - 2\beta_1 x, \quad h''(x) = -2\beta_1 < 0.$$

So $h'(x)$ is decreasing on $(0, \frac{\alpha_1}{\beta_1}]$. While $h'(\frac{\alpha_1}{\beta_1}) = 1 - \alpha_1 > 0$ since (ii) is hold. Therefore, $h'(x) > 0$ on $(0, \frac{\alpha_1}{\beta_1}]$, that is, $h(x)$ is increasing on $(0, \frac{\alpha_1}{\beta_1}]$. Hence $h(x) \leq h(\alpha_1/\beta_1) = \alpha_1/\beta_1$. Similar method can be applied for y_n . \square

Proposition 5. *Assume for $i = 1, 2, \alpha_i e^{-\frac{\alpha_i}{\beta_i}} < 1$. Then every solution $\{(x_n, y_n)\}_{n=0}^\infty$ of system (1.4) satisfies*

$$\begin{aligned} x_n &\leq (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0} x_{n_0} + \frac{\alpha_1}{\beta_1} [1 - (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0}], \\ y_n &\leq (1 - \alpha_2 e^{-\frac{\alpha_2}{\beta_2}})^{n-n_0} y_{n_0} + \frac{\alpha_2}{\beta_2} [1 - (1 - \alpha_2 e^{-\frac{\alpha_2}{\beta_2}})^{n-n_0}], \end{aligned}$$

for any n_0 satisfying $x_{n_0} \in (0, \alpha_1/\beta_1], y_{n_0} \in (0, \alpha_2/\beta_2]$.

Proof. Let $n \geq 0$ be such that $x_n \in (0, \frac{\alpha_1}{\beta_1}]$. It follows from system (1.4) that

$$x_{n+1} \leq x_n + (\alpha_1 - \beta_1 x_n)x_n e^{-x_n}.$$

Since $\alpha_1/\beta_1 \leq 1$, we have

$$x_{n+1} \leq x_n + (\alpha_1 - \beta_1 x_n) \frac{\alpha_1}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}} = (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})x_n + \frac{\alpha_1^2}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}}.$$

And if $\alpha_i e^{-\alpha_i/\beta_i} < 1$, then

$$x_{n+2} \leq (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^2 x_n + \frac{\alpha_1^2}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}} (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}}) + \frac{\alpha_1^2}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}}.$$

By induction, for fixed n_0 satisfying $x_{n_0} \in (0, \alpha_1/\beta_1]$, we obtain

$$\begin{aligned} x_n &\leq (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0} x_{n_0} + \frac{\alpha_1^2}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}} [(1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0-1} \\ &\quad + (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0-2} + \dots + (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}}) + 1] \\ &= (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0} x_{n_0} + \frac{\alpha_1^2}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}} \frac{1 - (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0}}{\alpha_1 e^{-\frac{\alpha_1}{\beta_1}}} \\ &= (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0} x_{n_0} + \frac{\alpha_1}{\beta_1} [1 - (1 - \alpha_1 e^{-\frac{\alpha_1}{\beta_1}})^{n-n_0}]. \end{aligned}$$

Similarly, we could prove

$$y_n \leq (1 - \alpha_2 e^{-\frac{\alpha_2}{\beta_2}})^{n-n_0} y_{n_0} + \frac{\alpha_2}{\beta_2} [1 - (1 - \alpha_2 e^{-\frac{\alpha_2}{\beta_2}})^{n-n_0}]$$

for any n_0 satisfying $x_{n_0} \in (0, \alpha_1/\beta_1], y_{n_0} \in (0, \alpha_2/\beta_2]$. \square

The following corollary is coming immediately from Proposition 5.

Corollary 2. Assume $\alpha_i e^{-\frac{\alpha_i}{\beta_i}} < 1, i = 1, 2$. Then for any initial point $(x_{n_0}, y_{n_0}) \in (0, \alpha_1/\beta_1] \times (0, \alpha_2/\beta_2]$, every positive solution $\{(x_n, y_n)\}_{n=0}^\infty$ of system (1.4) is bounded. Moreover,

$$\limsup_{n \rightarrow \infty} x_n \leq \alpha_1/\beta_1, \quad \limsup_{n \rightarrow \infty} y_n \leq \alpha_2/\beta_2.$$

Next, in the following theorem, we investigate the global attractivity of the equilibrium point (\bar{x}, \bar{y}) of system (1.4).

Theorem 7. Assume that $\alpha_i \leq \beta_i, \alpha_i e^{-\frac{\alpha_i}{\beta_i}} < 1$. Then the unique positive equilibrium point (\bar{x}, \bar{y}) of system (1.4) is a global attractor of all positive solutions of system (3.1).

Proof. Let $\{(x_n, y_n)\}_{n=0}^\infty$ be a solution of system (1.4) and let $x_n \leq \bar{x} < \frac{\alpha_1}{\beta_1}$ (the case whenever $x_n \geq \bar{x}$ is similar and so it will be omitted). Thus

$$\frac{\alpha_1}{\beta_1} \geq x_{n+1} = x_n + (\alpha_1 - \beta_1 x_n)x_n e^{-(x_n+y_n)} \geq x_n.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is increasing and since it was shown that it is bounded above, then it converges to the unique positive equilibrium point \bar{x} . Similarly, it is easy to show that the sequence $\{y_n\}_{n=0}^{\infty}$ is increasing and since it was shown that it is bounded above, then it converges to the unique positive equilibrium point \bar{y} . Thus $\{(x_n, y_n)\}_{n=0}^{\infty}$ converges to (\bar{x}, \bar{y}) . \square

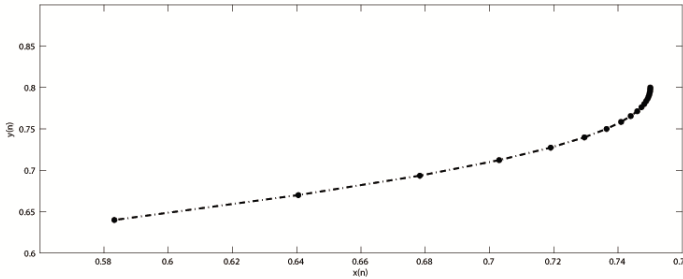


Figure 2. The stability of equilibrium of (1.4) while $\alpha_i \leq \beta_i$

Example 2. Set $\alpha_1 = 1.5, \alpha_2 = 0.8, \beta_1 = 2, \beta_2 = 1$, Figure 2 shows the stability of equilibrium of (1.4).

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