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A Mixed Joint Universality Theorem for Zeta-Functions

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Abstract. In the paper, a joint universality theorem for the Riemann zeta-function and a collection of periodic Hurwitz zeta-functions on approximation of analytic functions is obtained.

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1 Introduction

In 1975, S. M. Voronin discovered [22] a very interesting property of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$. Roughly speaking, he proved that every analytic non-vanishing function on compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ can by uniformly approximated with desired accuracy by shifts $\zeta(s + i\tau)$. Now this property is called the universality of $\zeta(s)$. Later, it was observed that other zeta and L-functions are also universal in the above sense, for results and references, see [1, 3, 4, 12, 15, 19, 20].

The first result on the joint universality also is due to S. M. Voronin. In [21], he obtained that a collection of shifts of Dirichlet L-functions with pairwise non-equivalent characters approximate simultaneously on compact subsets of D with a given accuracy a collection of arbitrary analytic non-vanishing functions.

It is known, see, for example, [12], that the Hurwitz zeta-function $\zeta(s,\alpha)$, $0<\alpha\leq 1$, with transcendental parameter α is also universal, however, in this case an approximated function can be not necessarily non-vanishing.

In [17], the universality of the periodic Hurwitz zeta-function which is a generalization of the function $\zeta(s,\alpha)$ was began to study. Let $\mathfrak{a}=\{a_m : m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ be a periodic sequence of complex numbers with minimal period $k\in\mathbb{N}$. Then the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$, $0<\alpha\leq 1$, is defined, for $\sigma>1$, by

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}.$$

In virtue of the periodicity of the sequence $\mathfrak{a}$, for $\sigma>1$,

$$\zeta(s,\alpha;\mathfrak{a})=\frac{1}{k^s}\sum_{l=0}^{k-1}a_l\zeta\Big(s,\frac{l+\alpha}{k}\Big).$$

Since the Hurwitz zeta-function $\zeta(s,\alpha)$ is meromorphic in the whole complex plane with a single simple pole at $s=1$ with residue 1, the latter equality gives meromorphic continuation for the function $\zeta(s,\alpha;\mathfrak{a})$ with possible simple pole at $s=1$ with residue

$$a\overset{def}{=}\frac{1}{k}\sum_{l=0}^{k-1}a_l.$$

If $a=0$, then the function $\zeta(s,\alpha;\mathfrak{a})$ is entire.

For the statement of results, we use the following notation. Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$, and, for $T>0$, let

$$\nu_T(\dots)=\frac{1}{T}\text{meas}\Big\{\tau\in[0;T]:\ \dots\Big\},$$

where in place of dots a condition satisfied by τ is to be written.

The universality property of the function $\zeta(s,\alpha;\mathfrak{a})$ is contained in the following theorem.

Theorem 1. [18] *Suppose that α is transcendental. Let K be a compact subset of the strip $D=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\epsilon>0$,*

$$\liminf_{T\to\infty}\nu_T\Big(\sup_{s\in K}|\zeta(s+i\tau,\alpha;\mathfrak{a})-f(s)|<\epsilon\Big)>0.$$

A series of works [5, 6, 7, 8, 9, 10] and [11] are devoted to the joint universality of periodic Hurwitz zeta-functions. The most general result is obtained in [10]. For $j=1,\dots,r$, let α_j, $0<\alpha_j\leq 1$, be a fixed parameter, $l_j\in\mathbb{N}$, and, for $j=1,\dots,r$, $l=1,\dots,l_j$, let $\mathfrak{a}_{jl}=\{a_{mjl}:m\in\mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period k_{jl}, and $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$ denote the corresponding periodic Hurwitz zeta-function. Moreover, let

$$\mathrm{L}(\alpha_1,\dots,\alpha_r)=\Big\{\log(m+\alpha_j):m\in\mathbb{N}_0,\ j=1,\dots,r\Big\},$$

and let k_j be the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, $j = 1, \dots, r$. Define

$$B_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_jj1} & a_{k_jj2} & \dots & a_{k_jjl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Theorem 2. [11] *Suppose that the system* $\mathrm{L}(\alpha_1, \dots, \alpha_r)$ *is linearly independent over the field of rational numbers* $\mathbb{Q}$, *and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, \dots, r$. *For every* $j = 1, \dots, r$ *and* $l = 1, \dots, l_j$, *let* K_{jl} *be a compact subset of the strip* D *with connected complement, and let* $f_{jl}(s)$ *be a continuous on* K_{jl} *function which is analytic in the interior of* K_{jl}. *Then, for every* $\epsilon > 0$,

$$\liminf_{T \to \infty} \nu_T \left(\sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \epsilon \right) > 0.$$

The aim of this paper is to consider the joint universality of the Riemann zeta-function $\zeta(s)$ and the functions $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, \dots, r$, $l = 1, \dots, l_j$.

Theorem 3. *Suppose that* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$, *and that all hypotheses on* K_{jl} *and* f_{jl} *of Theorem 2 hold. Moreover, let* K *be a compact subset of the strip* D *with connected complement, and let* $f(s)$ *be a continuous non-vanishing on* K *function which is analytic in the interior of* K. *Then, for every* $\epsilon > 0$,

$$\liminf_{T \to \infty} \nu_T \Big(\sup_{s \in K} |\zeta(s + i\tau) - f(s) < \epsilon,$$

$$\sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \epsilon \Big) > 0.$$

2 Limit Theorems

The proof of theorem 3 is based on a joint limit theorem in the space of analytic functions for the functions $\zeta(s)$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, \dots, r$, $l = 1, \dots, l_j$.

Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta, and let

$$H^\kappa(D) = \underbrace{H(D) \times \ldots \times H(D)}_{\kappa}, \quad \text{with} \quad \kappa = \sum_{j=1}^{r} l_j + 1.$$

Moreover, denote by γ the unit circle on the complex plane and define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ and $\gamma_m = \gamma$ for all primes p and all $m \in \mathbb{N}_0$, respectively. By the Tikhonov theorem, with the product topology and pointwise multiplication, the tori $\hat{\Omega}$ and Ω are compact topological Abelian groups. Therefore, on $\left(\hat{\Omega}, \mathcal{B}(\hat{\Omega})\right)$ and $(\Omega, \mathcal{B}(\Omega))$ (where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S) the probability Haar measures $\hat{m}_H$ and m_H, respectively, can by defined. This leads to the probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$ and $(\Omega, \mathcal{B}(\Omega), m_H)$.

Now let

$$\underline{\Omega} = \hat{\Omega} \times \Omega_1 \times \ldots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Then by then Tikhonov theorem again, $\underline{\Omega}$ is a compact topological Abelian group, and we obtain a new probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where $\underline{m}_H$ is the probability Haar measure on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to γ_p, $p \in \mathcal{P}$, $\mathcal{P}$ is the set of all prime numbers, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to γ_m, $m \in \mathbb{N}_0$. For brevity, let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \ldots, \mathfrak{a}_{1l_1}, \ldots, \mathfrak{a}_{r1}, \ldots, \mathfrak{a}_{rl_r})$, and let $\underline{\omega} = (\hat{\omega}, \omega_1, \ldots, \omega_r)$ be an element of $\underline{\Omega}$. On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^\kappa(D)$-valued random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by the formula

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(s, \hat{\omega}), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \ldots,$$
$$\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),$$

where

$$\zeta(s, \hat{\omega}) = \prod_p \left(1 - \frac{\hat{\omega}(p)}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j.$$

Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta}}(A) = \underline{m}_H \left(\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\right), \quad A \in \mathcal{B}(H^\kappa(D)).$$

Let

$$\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(\zeta(s), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \ldots,$$
$$\zeta(s, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big).$$

The main result of this section is the following statement.

Theorem 4. *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measure*

$$P_T(A) \overset{def}{=} \nu_T \left(\underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\right), \quad A \in \mathcal{B}(H^\kappa(D)),$$

converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.

We start the proof of Theorem 4 with a limit theorem on the torus $\underline{\Omega}$. Define

$$Q_T(A) = \nu_T(((p^{-i\tau} : p \in \mathcal{P}), \big((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \ldots,$$
$$\big((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)) \in A), A \in \mathcal{B}(\underline{\Omega}).$$

Lemma 1. *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the measure Q_T converges weakly to $\underline{m}_H$ as $T \to \infty$.*

Proof. The dual group of $\underline{\Omega}$ is isomorphic to

$$\mathcal{D} = \Big(\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p \Big) \bigoplus_{j=1}^{r} \Big(\bigoplus_{m \in \mathbb{N}_0} \mathbb{Z}_{jm} \Big),$$

where $\mathbb{Z}_p = \mathbb{Z}$ and $\mathbb{Z}_{jm} = \mathbb{Z}$ for all $p \in \mathcal{P}$ and $m \in \mathbb{N}_0$, $j = 1, \ldots, r$, respectively. An element $\underline{k} = (\underline{k}_{\mathcal{P}}, \underline{k}_{r\mathbb{N}_0}) \in \mathcal{D}$, $\underline{k}_{\mathcal{P}} = (k_p : p \in \mathcal{P})$, $\underline{k}_{r\mathbb{N}_0} = (k_{jm} : m \in \mathbb{N}_0, j = 1, \ldots, r)$, where only a finite number of integers k_p and k_{jm} are distinct from zero, acts on $\underline{\Omega}$ by

$$\underline{\omega} \to \underline{\omega}^{\underline{k}} = \prod_{p \in \mathcal{P}} \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0} \omega_j^{k_{jm}}(m).$$

Therefore, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is

$$\begin{aligned} g_T(\underline{k}) &= \int_{\underline{\Omega}} \prod_{p \in \mathcal{P}} \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0} \omega_{jm}^{k_{jm}}(m) \mathrm{d}Q_T \\ &= \frac{1}{T} \int_0^T \prod_{p \in \mathcal{P}} p^{-ik_p\tau} \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0} (m + \alpha_j)^{-ik_{jm}\tau} \mathrm{d}\tau, \end{aligned} \tag{2.1}$$

where, as above, only a finite number of integers k_p and k_{jm} are distinct from zero. It is well known that the set $\{\log p : p \in \mathcal{P}\}$ is linearly independent over $\mathbb{Q}$. Since the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, hence it follows that the set

$$\mathrm{L} \overset{def}{=} \Big\{ (\log p : p \in \mathcal{P}), \log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r \Big\}$$

is linearly independent over $\mathbb{Q}$. Really, if there exists integers k_p and k_{jm} not all zeros such that

$$\begin{aligned} k_1 \log p_1 + \ldots + k_n \log p_n + k_{1m_1} \log(m_1 + \alpha_1) + \ldots + k_{n_1 m_n}(m_{n_1} + \alpha_1) + \ldots \\ + k_{rm_r} \log(m_r + \alpha_r) + \ldots + k_{n_r m_{n_r}} \log(m_{n_r} + \alpha_r) = 0, \end{aligned}$$

we obtain that

$$\begin{aligned} &p_1^{k_1} \cdots p_n^{k_n} (p_1 + \alpha_1)^{k_{1m_1}} \cdots (m_{n_1} + \alpha_1)^{k_{n_1 m_{n_1}}} \cdots \\ &(m_r + \alpha_r)^{k_{rm_r}} \cdots (m_{n_r} + \alpha_r)^{k_{n_r m_{n_r}}} = 1, \end{aligned}$$

and this contradicts the algebraic independence of $\alpha_1, \ldots, \alpha_r$.

We find by (2.1) that

$$g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ \dfrac{1 - \exp\Big\{ -iT\Big(\sum_{p \in \mathcal{P}} k_p \log p + \sum_{j=1}^{r} \sum_{m \in \mathbb{N}_0} k_{jm} \log(m + \alpha_j) \Big) \Big\}}{T\Big(\sum_{p \in \mathcal{P}} k_p \log p + \sum_{j=1}^{r} \sum_{m \in \mathbb{N}_0} k_{jm} \log(m + \alpha_j) \Big)} & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Thus,

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

This and a continuity theorem for probability measures on compact topological groups, see, for example, [16], Theorem 1.4.2, prove the lemma. □

Let $\sigma > 1/2$ be a fixed number, and

$$u_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\sigma_1} \right\}, \quad m, n \in \mathbb{N},$$
$$u_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1} \right\}, \quad m, n \in \mathbb{N}_0.$$

From the periodicity it follows that the numbers a_{mjl} are bounded. Therefore, a standard application of the Mellin formula and contour integration shows that the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{u_n(m)}{m^s}$$

and

$$\zeta_n(s, \alpha_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl} u_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,$$

both are absolutely convergent for $\sigma > 1/2$. For $m \in \mathbb{N}$, define

$$\hat{\omega}(m) = \prod_{p^l \| m} \hat{\omega}^l(p),$$

where $p^l \| m$ means that $p^l \mid m$ but $p^{l+1} \nmid m$, and let

$$\zeta_n(s, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{u_n(m)\hat{\omega}(m)}{m^s},$$

and

$$\zeta_n(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m) u_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r.$$

Since $|\hat{\omega}(m)| = |\omega_j(m)| = 1$, the latter series are also absolutely convergent for $\sigma > 1/2$. For brevity, let

$$\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(\zeta_n(s), \zeta_n(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots,$$
$$\zeta_n(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{rl_r})\big)$$

and

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta_n(s, \hat{\omega}), \zeta_n(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots,$$
$$\zeta_n(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big).$$

On $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$, define the probability measures

$$P_{T,n}(A) = \nu_T\Big(\underline{\zeta}_n(s+i\tau), \underline{\alpha}; \underline{\mathfrak{a}}) \in A\Big)$$

and, for fixed $\underline{\omega}_0 = (\hat{\omega}_0, \omega_{10}, \dots, \omega_{r0})$,

$$P_{T,n,\underline{\omega}_0}(A) = \nu_T\Big(\underline{\zeta}_n(s+i\tau), \underline{\alpha}, \underline{\omega}_0; \underline{\mathfrak{a}}) \in A\Big).$$

Lemma 2. *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures $P_{T,n}$ and $P_{T,n,\underline{\omega}_0}$ both converge weakly to the same probability measure P_n on $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$ as $T \to \infty$.*

Proof. Since the series $\zeta_n(s)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ converge absolutely for $\sigma > 1/2$, the function $h_n : \underline{\Omega} \to H^\kappa(D)$ given by the formula

$$h_n(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$$

is continuous. Moreover,

$$h_n\big((p^{-i\tau} : p \in \mathcal{P}), \big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \dots, \\ \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)\big) = \underline{\zeta}_n(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Therefore, we have that $P_{T,n} = Q_T h_n^{-1}$. This, the continuity of h_n, Lemma 1 and Theorem 5.1 from [2] show that $P_{T,n}$ converges weakly to $P_n = \underline{m}_H h_n^{-1}$ as $T \to \infty$.

Similarly, we find that $P_{T,n,\underline{\omega}_0}$ converges weakly to $\underline{m}_H g_n^{-1}$ as $T \to \infty$, where $g_n : \underline{\Omega} \to H^\kappa(D)$ is related to h_n by $g_n(\underline{\omega}) = h_n(\underline{\omega}\, \underline{\omega}_0)$. Since the Haar measure $\underline{m}_H$ is invariant, this implies the equality $\underline{m}_H g_n^{-1} = \underline{m}_H h_n^{-1}$, and the lemma is proved. □

Furthermore, we need a metric on $H^\kappa(D)$ which induces its topology of uniform convergence on compacta. It is known, see, for example [13], that there exists a sequence $\{K_k : k \in \mathbb{N}\}$ of compact subsets of D such that

$$D = \bigcup_{k=1}^{\infty} K_k,$$

$K_k \subset K_{k+1}$ for all $k \in \mathbb{N}$, and, for every compact $K \subset D$, there exists k such that $K \subset K_k$. For $f, g \in H(D)$, let

$$\rho(f,g) = \sum_{k=1}^{\infty} 2^{-k} \frac{\sup_{s \in K_k} |f(s) - g(s)|}{1 + \sup_{s \in K_k} |f(s) - g(s)|}.$$

Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta. If, for

$$\underline{f} = (f_0, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r}), \\ \underline{g} = (g_0, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^\kappa(D),$$

$$\rho_\kappa(\underline{f},\underline{g}) = \max\Big(\rho(f_0,g_0), \max_{1\le j\le r}\max_{1\le l\le l_j}\rho(f_{jl},g_{jl})\Big), \tag{2.2}$$

then ρ_κ is a metric on $H^\kappa(D)$ inducing its topology.

Now we will approximate the vectors $\underline{\zeta}(s,\underline{\alpha};\underline{\mathfrak{a}})$ and $\underline{\zeta}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s,\underline{\alpha};\underline{\mathfrak{a}})$ and $\underline{\zeta}_n(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$, respectively.

Lemma 3. *We have*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_\kappa\left(\underline{\zeta}(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}}),\underline{\zeta}_n(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}})\right)\mathrm{d}\tau = 0.$$

Proof. It is known [3] that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\left(\zeta(s+i\tau),\zeta_n(s+i\tau)\right)\mathrm{d}\tau = 0. \tag{2.3}$$

Moreover, from [11] we have that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \max_{1\le j\le r}\max_{1\le l\le l_j}\rho\Big(\hat{\underline{\zeta}}(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}}),\hat{\underline{\zeta}}_n(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}})\Big)\mathrm{d}\tau=0, \tag{2.4}$$

where $\hat{\underline{\zeta}}(s,\underline{\alpha};\underline{\mathfrak{a}})$ and $\hat{\underline{\zeta}}_n(s,\underline{\alpha};\underline{\mathfrak{a}})$ are obtained from $\zeta(s,\underline{\alpha};\underline{\mathfrak{a}})$ and $\zeta_n(s,\underline{\alpha};\underline{\mathfrak{a}})$ by removing $\zeta(s)$ and $\zeta_n(s)$, respectively. Therefore, the equality of the lemma is a result of (2.2)–(2.4). □

Lemma 4. *Suppose that $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega}\in\underline{\Omega}$,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_\kappa\left(\underline{\zeta}(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}),\underline{\zeta}_n(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\right)\mathrm{d}\tau = 0.$$

Proof. In [3], it is obtained that, for almost all $\hat{\omega}\in\Omega$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\Big(\zeta(s+i\tau,\hat{\omega}),\zeta_n(s+i\tau,\hat{\omega})\Big)\mathrm{d}\tau = 0. \tag{2.5}$$

Similarly [11], for almost all $\underline{\underline{\omega}}=(\omega_1,\dots,\omega_r)\in\Omega_1\times\cdots\times\Omega_r$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \max_{1\le j\le r}\max_{1\le l\le l_j}\rho\Big(\hat{\underline{\zeta}}(s+i\tau,\underline{\alpha},\underline{\underline{\omega}};\underline{\mathfrak{a}}),\hat{\underline{\zeta}}_n(s+i\tau,\underline{\alpha},\underline{\underline{\omega}};\underline{\mathfrak{a}})\Big)\mathrm{d}\tau=0. \tag{2.6}$$

Denote by $\hat{\Omega}_0$ a subset of $\hat{\Omega}$ for which the relation (2.5) holds. Then we have that $\hat{m}_H(\hat{\Omega}_0) = 1$. Similarly, if $\Omega_0^r \subset \Omega_1 \times \cdots \times \Omega_r$ is such that, for $\underline{\underline{\omega}} \in \Omega_0^r$, the relation (2.6) holds, then $\underline{\underline{m}}_H(\Omega_0^r) = 1$, where $\underline{\underline{m}}_H$ is the Haar measure on $\Omega_1 \times \cdots \times \Omega_r$. Now let $\underline{\Omega}_0 = \hat{\Omega}_0 \times \Omega_0^r$. Since the Haar measure $\underline{m}_H$ is the product of $\hat{m}_H$ and $\underline{\underline{m}}_H$, we have that $\underline{m}_H(\underline{\Omega}_0) = 1$. This, (2.5), (2.6) and the definition of ρ_κ prove the lemma. □

Define one more probability measure

$$\hat{P}_T(A) = \nu_T\Big(\underline{\zeta}(s + i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big), \quad A \in \mathcal{B}(H^\kappa(D)).$$

Lemma 5. *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures P_T and $\hat{P}_T$ both converge weakly to the same probability measure P on $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$ as $T \to \infty$.*

Proof. Define on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ a random variable θ uniformly distributed on $[0, 1]$. Let $X_{T,n}$ be an $H^\kappa(D)$-valued random element on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ given by

$$\underline{X}_{T,n}(s) = \big(X_{T,n}(s), X_{T,n,1,1}(s), \ldots, X_{T,n,1,l_1}(s), \ldots, X_{T,n,r,1}(s), \ldots, \\ X_{T,n,r,l_r}(s)\big) = \underline{\zeta}_n(s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then, by Lemma 2,

$$\underline{X}_{T,n}(s) \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n(s), \tag{2.7}$$

where

$$\underline{X}_n(s) = (X_n(s), X_{n,1,1}(s), \ldots, X_{n,1,l_1}(s), \ldots, X_{n,r,1}(s), \ldots, X_{n,r,l_r}(s))$$

is an $H^\kappa(D)$-valued random element with the distribution P_n (P_n is the limit measure in Lemma 2), and $\xrightarrow{\mathcal{D}}$ means convergence in distribution. Since the series for $\zeta_n(s)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ converges absolutely for $\sigma > 1/2$, we have that, for $\sigma > 1/2$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big|\zeta_n(\sigma + it)\Big|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{u_n^2(m)}{m^{2\sigma}} \le \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \tag{2.8}$$

for all $n \in \mathbb{N}$, and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big|\zeta_n(\sigma + it, \alpha_; \mathfrak{a}_{jl})\Big|^2 \mathrm{d}t = \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2 u_n^2(m, \alpha_j)}{(m + \alpha_j)^{2\sigma}} \le \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m + \alpha_j)^{2\sigma}} \tag{2.9}$$

for all $n \in \mathbb{N}_0$.

Using the Caushy integral formula, contour integration, and (2.8), we find that, for $n \in \mathbb{N}$,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_k} \Big|\zeta_n(s+i\tau)\Big| \mathrm{d}\tau \ \le \hat{C}_k \Big(\sum_{m=1}^{\infty} \frac{1}{m^{2\hat{\sigma}_k}} \Big)^{\frac{1}{2}} \tag{2.10}$$

and similarly, by (2.9), for all $n \in \mathbb{N}_0$,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_k} \Big|\zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl})\Big| \mathrm{d}\tau \ \le C_k \Big(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_k}} \Big)^{\frac{1}{2}}, \tag{2.11}$$

with some $\hat{C}_k > 0$, $C_k > 0$ and $\hat{\sigma}_k > \frac{1}{2}$, $\sigma_k > \frac{1}{2}$.

Let $\epsilon > 0$ be an arbitrary number, and

$$\hat{R}_k = \Big(\sum_{m=1}^{\infty} \frac{1}{m^{2\hat{\sigma}_k}} \Big)^{\frac{1}{2}}, \quad R_{jlk} = \Big(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_k}} \Big)^{\frac{1}{2}}.$$

Then, taking $\hat{M}_k = \hat{C}_k \hat{R}_k 2^{l+1} \epsilon^{-1}$ and $M_{jlk} = C_k R_{jlk} 2^{l+1} \epsilon^{-1}$, we deduce from (2.10) and (2.11) that

$$\begin{aligned}
\limsup_{T\to\infty} \mathbb{P}\Big(\Big(& \sup_{s\in K_k} \Big|X_{T,n}(s)\Big| > \hat{M}_k\Big) \\
& \vee \exists j,l : \Big(\sup_{s\in K_k} \Big|X_{T,n,j,l}(s)\Big| > M_{jlk}\Big)\Big) \\
& \le \limsup_{T\to\infty} \mathbb{P}\Big(\sup_{s\in K_k} \Big|X_{T,n}(s)\Big| > \hat{M}_k\Big) \\
& + \sum_{j=1}^{r} \sum_{l=1}^{l_j} \limsup_{T\to\infty} \mathbb{P}\Big(\sup_{s\in K_k} \Big|X_{T,n,j,l}(s)\Big| > M_{jlk}\Big) \\
& \le \frac{1}{\hat{M}_k} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_k} \Big|\zeta_n(s+i\tau)\Big| \mathrm{d}\tau \\
& + \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{1}{M_{jlk}} \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_k} \Big|\zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl})\Big| \mathrm{d}\tau \\
& \le \frac{\hat{C}_k \hat{R}_k}{\hat{M}_k} + \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{C_k R_{jlk}}{M_{jlk}} = \frac{\epsilon}{2^{l+1}} + \frac{\epsilon}{2^{l+1}} = \frac{\epsilon}{2^l}.
\end{aligned}$$

This together with (2.7) leads, for all $n \in \mathbb{N}$, to the inequality

$$\mathbb{P}\Big(\Big(\sup_{s\in K_k} \Big|X_n(s)\Big| > \hat{M}_k\Big) \vee \exists j,l : \Big(\sup_{s\in K_k} \Big|X_{n,j,l}(s)\Big| > M_{jlk}\Big)\Big) \le \frac{\epsilon}{2^l}. \tag{2.12}$$

Define a set

$$\begin{aligned}
H_\epsilon^\kappa = \Big\{ & \Big(g_0, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}\Big) \in H^\kappa(D) : \sup_{s\in K_k} |g_0(s)| \le \hat{M}_k, \\
& \sup_{s\in K_k} |g_{jl}(s)| \le M_{jlk},\ j = 1, \dots, r,\ l = 1, \dots, l_j,\ k \in \mathbb{N} \Big\}.
\end{aligned}$$

Then the set H_ϵ^κ is compact in the space $H^\kappa(D)$, and, in view of (2.12),

$$\mathbb{P}\Big(\underline{X}_n(s) \in H_\epsilon^\kappa\Big) \geq 1 - \epsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \epsilon$$

for all $n \in \mathbb{N}$. This and the definition of $\underline{X}_n(s)$ shows that

$$P_n\Big(H_\epsilon^\kappa\Big) \geq 1 - \epsilon$$

for all $n \in \mathbb{N}$. Thus, we obtained that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. Therefore, by the Prokhorov theorem, it is relatively compact, and thus, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$ as $k \to \infty$. In other words,

$$\underline{X}_{n_k}(s) \underset{k\to\infty}{\xrightarrow{\mathcal{D}}} P. \tag{2.13}$$

Let $X_T(s) = \underline{\zeta}(s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}})$ be one more $H^\kappa(D)$-valued random element on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$. Then, by Lemma 3, we have that, for every $\epsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\Big(\rho_\kappa\Big(\underline{X}_T(s), \underline{X}_{T,n}(s)\Big) \geq \epsilon\Big) \\
&\quad = \lim_{n\to\infty} \limsup_{T\to\infty} \nu_T\Big(\rho_\kappa\Big(\underline{\zeta}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\Big) \geq \epsilon\Big) \\
&\quad \leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\epsilon} \int_0^T \rho_\kappa\Big(\underline{\zeta}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\Big) \mathrm{d}\tau = 0.
\end{aligned}$$

This, (2.13) and (2.7) together with Theorem 4.2 of [1] imply the relation

$$\underline{X}_T(s) \underset{T\to\infty}{\xrightarrow{\mathcal{D}}} P \tag{2.14}$$

which is equivalent to the weak convergence of P_T to P as $T \to \infty$. Moreover, it follows from (2.14) that the measure P is independent of the choice of the sequence $\{P_{n_k}\}$. Thus, we have that

$$\underline{X}_n(s) \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} P. \tag{2.15}$$

Now consider the measure $\hat{P}_T$. For this, define

$$\underline{\hat{X}}_{T,n}(s) = \underline{\zeta}_n(s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$$

and

$$\underline{\hat{X}}_T(s) = \underline{\zeta}(s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Repeating the above arguments for the random elements $\underline{\hat{X}}_{T,n}(s)$ and $\underline{\hat{X}}_T(s)$, and using Lemmas 2 and 4 as well as (2.15), we obtain that the measure $\hat{P}_T$ also converges weakly to P as $T \to \infty$. □

In virtue of Lemma 5, for the proof of Theorem 4 it suffices to show that the limit measure P in Lemma 5 coincides with $P_{\underline{\zeta}}$. To prove this, we need some results from ergodic theory. Let $\underline{a}_\tau = \{(p^{-i\tau} : p \in \mathcal{P}), ((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)\}$, $\tau \in \mathbb{R}$. Define $\underline{\Phi}_\tau(\underline{\omega}) = \underline{a}_\tau \underline{\omega}$, $\underline{\omega} \in \underline{\Omega}$. Then $\{\underline{\Phi}_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on $\underline{\Omega}$. A set $A \in \mathcal{B}(\underline{\Omega})$ is called invariant with respect to the group $\{\underline{\Phi}_\tau : \tau \in \mathbb{R}\}$ if, for every $\tau \in \mathbb{R}$, the sets A and $\underline{\Phi}_\tau(A)$ may differ one from another only by $\underline{m}_H$-measure zero. The group $\{\underline{\Phi}_\tau : \tau \in \mathbb{R}\}$ is ergodic if its σ-field of invariant sets consists only of the sets of $\underline{m}_H$-measure zero or one.

Lemma 6. *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the one-parameter group $\{\underline{\Phi}_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof of the lemma is given in [9], Lemma 7.

Proof of Theorem 4. We fix a continuity set A of the limit measure P in Lemma 5. Then, by Lemma 5 and Theorem 2.1 of [2],

$$\lim_{T\to\infty} \nu_T\Big(\underline{\zeta}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big) = P(A). \tag{2.16}$$

Consider a random variable ξ defined on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$ by

$$\xi(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, its expectation

$$\mathbb{E}\xi = \underline{m}_H\Big(\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big) = P_{\underline{\zeta}}(A). \tag{2.17}$$

In view of Lemma 6, the process $\xi(\underline{\Phi}_\tau(\underline{\omega}))$ is ergodic. Therefore, the Birkhoff–Khintchine theorem, see, for example, [14], implies that, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi\Big(\underline{\Phi}_\tau(\underline{\omega})\Big) \mathrm{d}\tau = \mathbb{E}\xi. \tag{2.18}$$

On the other hand, the definitions of ξ and $\underline{\Phi}_\tau$ yield

$$\frac{1}{T} \int_0^T \xi\Big(\underline{\Phi}_\tau(\underline{\omega})\Big) \mathrm{d}\tau = \nu_T\Big(\underline{\zeta}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big).$$

Thus, by (2.17) and (2.18), for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \nu_T\Big(\underline{\zeta}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big) = P_{\underline{\zeta}}(A).$$

Combining this with (2.16), we obtain that $P(A) = P_{\underline{\zeta}}(A)$ for all continuity sets A of the measure P. Hence, $P(A) = P_{\underline{\zeta}}(A)$ for all $A \in \mathcal{B}(H^\kappa(D))$ because the continuity sets form a determining class, see [2]. The theorem is proved. □

3 The Support of $P_{\underline{\zeta}}$

In this section, we give explicitly the support of the measure $P_{\underline{\zeta}}$. We recall that the support of $P_{\underline{\zeta}}$ is a minimal closed subset $S_{P_{\underline{\zeta}}}$ of $H^{\kappa}(D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. We also note that $S_{P_{\underline{\zeta}}}$ consists of all points $\underline{g} \in H^{\kappa}(D)$ such that $P_{\underline{\zeta}}(G) > 0$ for every neighbourhood G of $\underline{g}$.

Define $S = \Big\{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\Big\}$.

Theorem 5. *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(B_j) = l_j$, $j = 1 \dots, r$. *Then the support of $P_{\underline{\zeta}}$ is the set $S \times H^r(D)$.*

Proof. We write
$$H^{\kappa}(D) = H(D) \times H^{\kappa_1}(D),$$
where
$$\kappa_1 = \sum_{j=1}^{r} l_j.$$
Since the spaces $H(D)$ and $H^{\kappa_1}(D)$ are separable, it suffices [2] to consider $P_{\underline{\zeta}}(A)$ with $A = A_1 \times A_{\kappa_1}$, $A \in \mathcal{B}(H(D))$, $A_{\kappa_1} \in \mathcal{B}(H^{\kappa_1}(D))$. Let $\Omega^r = \Omega_1 \times \ldots \times \Omega_r$, where $\Omega_j = \Omega$ for all $j = 1, \dots, r$, and let m_H^r by the Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$. Then the Haar measure $\underline{m}_H$ is the product of the Haar measures $\hat{m}_H$ and m_H^r. Hence, we find that

$$\begin{aligned}
P_{\underline{\zeta}}(A) &= \underline{m}_H\Big(\underline{\omega} \in \Omega : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big) \\
&= \underline{m}_H\Big(\underline{\omega} \in \Omega : \zeta(s, \hat{\omega}) \in A_1, \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\
&\quad \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in A_{\kappa_1}\Big) \\
&= \hat{m}_H\Big(\hat{\omega} \in \hat{\Omega} : \zeta(s, \hat{\omega}) \in A_1\Big) \\
&\quad \times m_H^r\Big((\omega_1, \dots, \omega_r) \in \Omega^r : \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\
&\quad \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in A_{\kappa_1}\Big).
\end{aligned} \tag{3.1}$$

In [11], it is obtained that the support of the $H(D)$-valued random element $\zeta(s, \hat{\omega})$ is the set S, that is, S is a minimal closed set such that
$$\hat{m}_H\Big(\hat{\omega} \in \hat{\Omega} : \zeta(s, \hat{\omega}) \in S\Big) = 1. \tag{3.2}$$

Similarly, in [11], under the hypotheses of the theorem, it was obtained that $H^{\kappa_1}(D)$ is a minimal closed set such that

$$\begin{aligned}
m_H^r\Big((\omega_1, \dots, \omega_r) \in \Omega^r : \big(&\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\
&\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in H^{\kappa_1}(D)\Big) = 1.
\end{aligned}$$

This, (3.1) and (3.2) complete the proof. □

4 Proof of Theorem 3

A proof of Theorem 3 is based on Theorems 4 and 1 as well as on the Mergelyan theorem [23], and is standard.

First suppose that the functions $f(s)$ and $f_{jl}(s)$ have analytic continuations to the whole strip D, and the analytic continuation of $f(s)$ is non-zero. Define

$$G = \Big\{ \Big(g_0, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}\Big) \in H^\kappa(D) :$$
$$\sup_{s\in K} |g_0(s) - f(s)| \le \epsilon, \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |g_{jl}(s) - f_{jl}(s)| < \epsilon \Big\}.$$

The set G is open in $H^\kappa(D)$. Therefore, Theorem 4 together with Theorem 2.1 of [2] (an equivalent of weak convergence in terms of open sets) implies

$$\liminf_{T\to\infty} \nu_T\Big(\underline{\zeta}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in G\Big) \ge P_{\underline{\zeta}}(G). \tag{4.1}$$

However, by Theorem 5, $(f, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r})$ is a point of the support of the measure $P_{\underline{\zeta}}$. Thus, $P_{\underline{\zeta}}(G) > 0$, and the definition of G and (4.1) yield

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \epsilon,$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \epsilon\Big) > 0. \tag{4.2}$$

Now let the functions $f(s)$ and $f_{jl}(s)$ satisfy the hypotheses of the theorem. Then, by the Mergelyan theorem, there exist polynomials $p(s)$, $p(s) \ne 0$ on K, and $p_{jl}(s)$ such that

$$\sup_{s\in K} \Big|f(s) - p(s)\Big| < \frac{\epsilon}{4} \tag{4.3}$$

and

$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} \Big|f_{jl}(s) - p_{jl}(s)\Big| < \frac{\epsilon}{2}. \tag{4.4}$$

Since $p(s) \ne 0$ on K, we can define a continuous branch of the function $\log p(s)$ in K which will be analytic in the interior of K. By the Mergelyan theorem again, we can find a polynomial $q(s)$ such that

$$\sup_{s\in K} \Big|p(s) - e^{q(s)}\Big| < \frac{\epsilon}{4}.$$

This together with (4.3) shows that

$$\sup_{s\in K} \Big|f(s) - e^{q(s)}\Big| < \frac{\epsilon}{2}. \tag{4.5}$$

However, $e^{q(s)} \ne 0$, therefore, the functions $e^{q(s)}$ and $p_{jl}(s)$ satisfy all hypotheses under which (4.2) holds. So, we have that

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K} \Big|\zeta(s+i\tau) - e^{q(s)}\Big| < \frac{\epsilon}{2},$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} \Big|\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - p_{jl}(s)\Big| < \frac{\epsilon}{2}\Big) > 0. \tag{4.6}$$

Clearly, in view of (4.5) and (4.4),

$$\Big\{\tau \in [0,T] : \sup_{s\in K}\Big|\zeta(s+i\tau) - e^{q(s)}\Big| < \frac{\epsilon}{2},$$
$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}\Big|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - p_{jl}(s)\Big| < \frac{\epsilon}{2}\Big\}$$
$$\subseteq \Big\{\tau \in [0,T] : \sup_{s\in K}\Big|\zeta(s+i\tau) - f(s)\Big| < \epsilon,$$
$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}\Big|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)\Big| < \epsilon\Big\}.$$

This and (4.6) prove the theorem.

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