NON RELATIVISTIC BROAD BAND WAKE FIELDS AND POTENTIAL-WELL DISTORTION

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Abstract

The study of the interaction between a particle beam and wake fields is usually based on the assumption of ultra relativistic beams. This is not the case, for example, for the Proton Synchrotron Booster (PSB), in which protons cover the energy range 50MeV - 1.4GeV ($\gamma \approx 2.5$).

There are some examples in literature which derive non ultra relativistic formulas for the resistive wall impedance. In this paper we have extended the Broad Band resonator model, allowing the impedance to have poles even in the upper half complex plane, in order to obtain a wake function different from zero for $z > 0$. The Haissinski equation has been numerically solved showing longitudinal bunch shape changes with γ. In addition some longitudinal bunch profile measurements, taken for two different bunch intensities in the PSB, are shown.

INTRODUCTION

The wake functions $W(z)$ have been introduced in order to describe the interaction between the charged particle beams with their surrounding, and the consequent perturbations on the dynamics of the beam itself. The Fourier transform of the wake function is a complex function called impedance $Z(\omega)$.

One of the important contributions to the total machine impedance is the Broad Band impedance modeling the integrated effect of several discontinuities and equipments.

Due to the fact that in the ultra relativistic regime the wake function $W(z)$ has to be zero for any $z > 0$ (causality principle), the broad band wake function has a discontinuity at $z = 0$ being $\lim_{z\to 0^-} W(z) = 2W(0)$ according to the Panofsky-Wensel theorem [1]. On the other hand the wake functions are continuous functions of z for $\beta < 1$ z [2]. This means that the wake function should be non-zero also in front of the source charge and should approach the limit function, with its discontinuity, in the ultra relativistic case when $\beta \to 1$. For proton machines such as the PSB the kinetic energy of the beam ranges from 50 MeV up to 1.4 GeV and consequently β goes from $\beta = 0.314$ up to $\beta = 0.916$.

For this purpose we introduced a sequence of distributions in which the ultra relativistic limit can converge to the wake function in its usual relativistic form. In addition we have numerically solved the Haissinski equation in order to check the differences while trying to simulate charged particle bunches at non ultra relativistic energies.

THE MODEL AND THE IMPEDANCE CALCULATION

In this section we calculate the longitudinal impedance as the inverse transform of the wake function. Starting from the assumption of $\beta \approx 1$ the $m = 0$ longitudinal wake function $W_0'(z)$ reads [1]

$$W_0'(z) = \begin{cases} 0 & \text{if} \quad z > 0, \\ \alpha R_S & \text{if} \quad z = 0, \\ 2\alpha R_S \; e^{\alpha z/c}\big(\cos(\bar{\omega} z/c) + \\ \qquad + \dfrac{\alpha}{\bar{\omega}} \sin(\bar{\omega} z/c)\big) & \text{if} \quad z < 0, \end{cases} \tag{1}$$

where $\alpha = \omega_R/2Q$ and $\bar{\omega} = \sqrt{\omega_R^2 - \alpha^2}$, Q and ω_R being respectively the quality factor and the resonant frequency of the impedance

$$Z_0^{\|}(\omega) = \frac{R_S}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)} \tag{2}$$

with the Eq. (1) being the inverse Fourier transform of Eq. (2): $Z^{\|}(\omega) = 1/c \int_{-\infty}^{\infty} dz \;\; e^{-i\omega z/c} W'(z)$. The transverse Broad Band impedance is simply given by $Z^{\perp} = 2RZ_0^{\|}/b^2 n$, with R being the machine radius, b the beam pipe radius and $n = \omega/\omega_0$.

We easily cast the Eq. (2) in the following form

$$Z^{\|}(\omega) = \frac{R_S \; \omega \; \omega_R}{(\omega - \omega_-)(\omega - \omega_+)} \tag{3}$$

where $\omega_{\pm} = (-i\alpha \pm \bar{\omega})$. The property $W'(z) = 0$ if $z > 0$ follows from the fact that $Z^{\|}(\omega)$ has its poles only in half lower complex plane.

To ease this constrain we built a succession of continuous functions $\hat{W}'(z;\beta)$ which have the property

$$\lim_{\beta \to 1} \langle \hat{W}'(z;\beta), \varphi(z) \rangle = \langle W'(z), \varphi(z) \rangle \qquad \forall \varphi \in \mathcal{C}^{\infty} \tag{4}$$

and being the Fourier transform a continuous application we also have

$$\lim_{\beta \to 1} \hat{Z}^{\|}(\omega;\beta) = Z^{\|}(\omega) \tag{5}$$

The used $\hat{W}'(z;\beta)$ functions are Fermi-like distributions multiplied by the wake function $W'(z)$

$$\hat{W}'(z;\beta) = g(z;\beta) W'(z) = \frac{\beta}{1 + e^{\bar{\omega} z \gamma/\beta c}} W'(z). \tag{6}$$

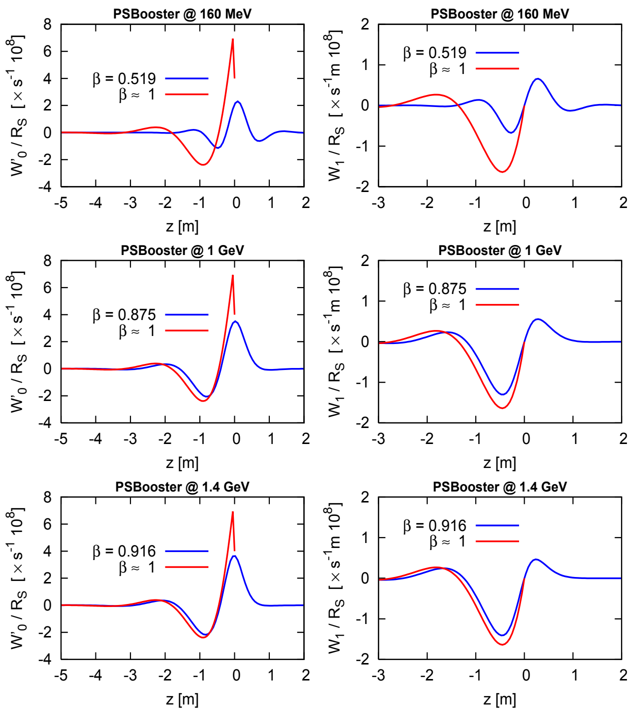

Figure 1: Comparison between longitudinal (left column) and transverse (right column) ultra relativistic wake function Eq. (1) (—) and the the proposed non ultra relativistic one Eq. (6) (—) for three different energies.

When considering the transverse wake function $W_1(z)$ we used the same approach. In Fig. 1 we plot the transverse and the longitudinal broad band wake functions for the three energies of the PSB beam. Performing in Eq. (6) the limit $\beta \to 1$ we have that $g(z;\beta) \to \theta(z)$(Heaviside distribution) and we gain back the causality principle having $W'(z) = 0$ for $z > 0$. The function Eq. (6) has poles in the complex plane for $\hat{\xi} = i\pi/\hat{\omega}\gamma\,(1+2k)$ with $k \in \mathbb{N}$. Introducing the following functions

$$\begin{cases} f_C(z) = \hat{c}e^{Az}\cos(\hat{\omega}z)\,\dfrac{1}{1+e^{\hat{\omega}\gamma z}} \\ f_S(z) = \hat{c}e^{Az}\sin(\hat{\omega}z)\,\dfrac{1}{1+e^{\hat{\omega}\gamma z}} \end{cases} \tag{7}$$

with $A = \alpha/\beta c$, $\hat{c} = 2\alpha^2\beta R_S/\bar{\omega}$ and $\hat{\omega} = \bar{\omega}/\beta c$ we calculate their Fourier transform on the path Fig. 2 and the we solve the linear system

$$\begin{cases} \mathcal{I}_C\left[1 - \mathrm{ch}\left(\dfrac{2\pi}{\gamma}\right)e^{\left(\frac{2\pi}{\gamma\hat{\omega}}(\omega/\beta c+iA)\right)}\right] + i\mathrm{sh}\left(\dfrac{2\pi}{\gamma}\right)\cdot \\ e^{\left(\frac{2\pi}{\gamma\hat{\omega}}(\omega/\beta c+iA)\right)}\mathcal{I}_S = -\dfrac{2\pi i}{\hat{\omega}\gamma}\hat{c}ce^{\left(\frac{2\pi}{\gamma\hat{\omega}}(\omega/\beta c+iA)\right)}\mathrm{ch}\left(\dfrac{\pi}{\gamma}\right) \\ \mathcal{I}_S\left[1 - \mathrm{ch}\left(\dfrac{2\pi}{\gamma}\right)e^{\left(\frac{2\pi}{\gamma\hat{\omega}}(\omega/\beta c+iA)\right)}\right] - i\mathrm{sh}\left(\dfrac{2\pi}{\gamma}\right)\cdot \\ e^{\left(\frac{2\pi}{\gamma\hat{\omega}}(\omega/\beta c+iA)\right)}\mathcal{I}_C = \dfrac{2\pi}{\hat{\omega}\gamma}\hat{c}ce^{\left(\frac{2\pi}{\gamma\hat{\omega}}(\omega/\beta c+iA)\right)}\mathrm{ch}\left(\dfrac{\pi}{\gamma}\right) \end{cases} \tag{8}$$

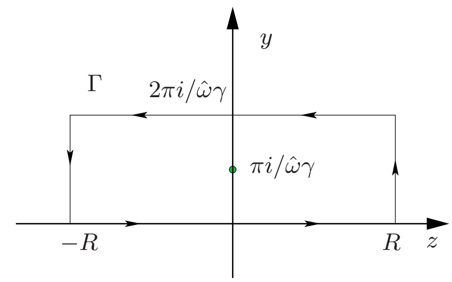

Figure 2: The Γ curve for calculating the Fourier transform.

with $\mathcal{I}_{C/S} = 1/\beta c\int_{-\infty}^{\infty} dz e^{-i\omega z/\beta c} f_{C/S}(z)$. The longitudinal and the transverse Broad Band impedances are hence given by

$$\begin{cases} Z^{\parallel}(\omega) = \mathcal{I}_S(\omega) + \dfrac{\bar{\omega}}{\alpha}\mathcal{I}_C(\omega) \\ Z^{\perp}(\omega) = \mathcal{I}_S(\omega)/\bar{\omega} \end{cases} \tag{9}$$

In Fig. 3 we have the comparison between the impedance functions for two different values of β.

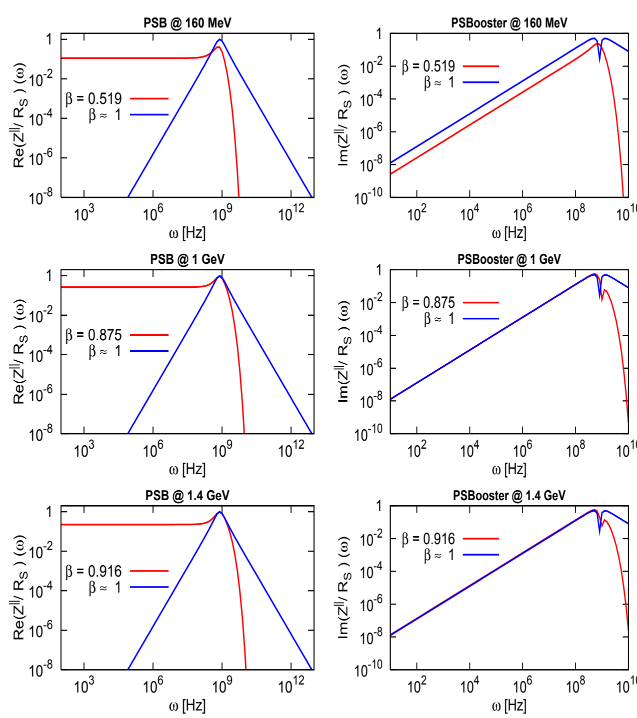

Figure 3: Comparison between the imaginary (left column) and real (right column) part of the longitudinal ultra relativistic Broad Band impedance using Eq. (1) (—) and the proposed non ultra relativistic one Eq. (6) (—) for three different energies.

EFFECTS ON LONGITUDINAL BUNCH DISTRIBUTION

In order to study the effects on the bunch dynamics we numerically solved the longitudinal equilibrium dynamics for a non relativistic bunch. This is done by solving the Vlasov equation for the longitudinal distribution $\hat{\rho}(z,\delta)$, which reduces to the Haissinski equation

$$\begin{cases} \rho(z) = \rho(0)\, exp\left[-\frac{1}{2}\left(\frac{z}{\sigma_z}\right)^2 + \alpha \int dz' \left(\rho * W'\right)(z')\right] \\ \rho(z) = \int d\delta \hat{\rho}(z,\delta), \quad N = \int dz \rho(z), \quad \alpha = \frac{e^2}{\mathcal{C}\eta E_0 c^2 \sigma_\delta^2} \end{cases} \tag{10}$$

where $(f * g)(x) = \int_{\mathbb{R}} dx' f(x') g(x - x')$ is the usual linear convolution between two functions, since $W'(z) \neq 0$ for $z > 0$. Eq. (10) can be numerically solved by the method of successive approximations until convergence is reached. An initial matched Gaussian distribution is taken as first guess

$$\rho_0(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} \text{ with } \begin{cases} \sigma_z = \frac{\sigma_\delta |\eta| \beta c}{\omega_s} \\ \omega_s = \frac{2\pi Q_s}{T_0} = \frac{\beta c}{R}\sqrt{\frac{e|\eta|\hat{V} h}{2\pi \beta^2 E_0}} \end{cases} \tag{11}$$

where $\hat{V}$ is the RF voltage, h is the harmonic number, σ_δ is the rms momentum spread, R the machine radius and T_0 the revolution period, ω_s is the synchrotron frequency and $\mathcal{C}$ is the machine circumference. For a given ω_s the relation between σ_z and σ_δ is fixed by the "matching" condition $\rho = \int d\delta \hat{\rho}\left(-k^2 H(z,\delta)\right)$, $H(z,\delta)$ being the Hamiltonian function for the longitudinal motion and k an arbitrary constant. For bunches much shorter than the RF bucket, the matching condition reads as reported in Eq. (11).

We solved Eq. (10) using the ultra relativistic Broad Band Eq. (1) (U. R. BB) and the non ultra relativistic Broad Band Eq. (9) (N. U. R. BB) wake function for the PSB bunch at a kinetic energy of 1.4 GeV. In Tab. 1 we show the PSB parameters we used to solve the Haissinki equation.

Table 1: Parameters Used to Solve Eq. (10) for the PSB Bunch at 1.4 GeV

relativistic beta	β	0.916
bunch length	σ_z [m]	15.8
RF voltage	$\hat{V}$[KV]	8
Shunt impedance	R_S [KΩ]	1
Resonant frequency	ω_r [GHz]	6
Quality factor	Q	1
Slippage factor	η	-0.1

In Fig. 4 we plot the solution of the Haissinski equation Eq. (10) for three different values of the bunch population N, using the longitudinal wake field Eq. (6) and the machine values written in Tab. 1.

In Fig. 5 we compare the experimental data between the solution of the Haissinski equation for the ultra relativistic and the non ultra relativistic case. We acquired the longitudinal shape of the beam each 75 turn of the PSB and we averaged the shape over 60 acquisitions while the bunch was

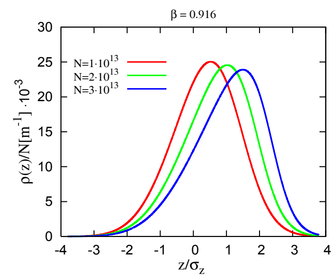

Figure 4: Numerical solution of the Haissinski equation using Eq. (6) as wake function. We used three different bunch populations N: $N = 1 \cdot 10^{13}$ (—), $N = 2 \cdot 10^{13}$ (—) and $N = 3 \cdot 10^{13}$ (—).

stationary. From Fig. 5 we can see that the bunch shift is

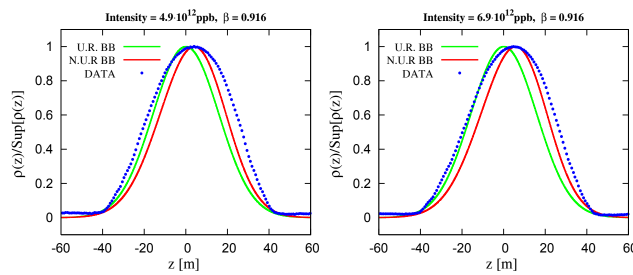

Figure 5: Numerical solution of the Haissinski equation using Eq. (1) — and Eq. (6) — as a wake function. The numerical solution has been compared against the numerical data acquired • at the PSB for a bunch with $N = 4.9 \cdot 10^{12}$(left) and $N = 6.9 \cdot 10^{12}$(right) particles at 1.4 GeV kinetic energy.

enhanced using Eq. (6) instead of the classical Broad Band wake field.

CONCLUSIONS

In this paper we tried to extend the classical Broad Band model normally used to describe the general impedance of the machine. We used a sequence of distributions to smooth the wake functions in the region $z = 0$. We calculated the impedance as the Fourier transform of the wake field.

As a first application of this non ultra relativistic field we solved the Haissinski equation for the longitudinal distribution of a bunch in a linear bucket: the peak of the equilibrium solution shows a bigger shift which better matches with the experimental data taken at PSB.

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