Extinction and Non-Extinction of Solutions to a p-Laplace Equation with a Nonlocal Source and an Absorption Term

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Abstract. In this short paper, the authors investigate the extinction and non-extinction of solutions to a fast diffusive p-Laplace equation with a nonlocal source and an absorption term. By applying the super-solution and sub-solution methods, instead of energy estimate methods, they give a classification of the exponents and coefficients for the solutions to vanish in finite time or not, which generalizes some previous results.

Keywords: p-Laplace equation, super-solution and sub-solution, extinction.

AMS Subject Classification: 35K59; 35K67.

1 Introduction

In this paper, we consider a fast diffusive p-Laplace equation with a nonlocal source and an absorption term of the following form

$$
\begin{cases}
u_t = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + a \int_{\Omega} u^q(y,t)\,\mathrm{d}y - bu^r, & x \in \Omega,\ t > 0, \\
u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{1.1}
$$

where $a,b,q,r > 0$, $1 < p < 2$, Ω is a bounded domain in R^N ($N \geq 1$) with smooth boundary $\partial\Omega$, and the initial datum $u_0(x)$ is a nonnegative nontrivial function such that $u_0 \in L^{\infty}(\Omega) \cap W_0^{1,p}(\Omega)$.

The equation in (1.1) is a p-Laplace equation perturbed by both a nonlinear nonlocal source term and an absorption term, which describes the fast

diffusion of the concentration of some non-Newtonian fluids through a porous medium or the temperature of some combustible substance (see [3, 20, 24]). In particular, problem (1.1) is a possible model for the diffusion system of some biological species with human-controlled distribution where $u(x,t)$ represents the density of the species at position x and time t, div$(|\nabla u|^{p-2}\nabla u)$ portrays the mutation (which we view as a spreading of the characteristic), $-bu^r$ is the growth capacity of the species at location x and time t, whereas $a\int_\Omega u^q(y,t)\,\mathrm{d}y$ denotes the human-controlled distribution. Nonlocal term is a way to express that the evolution of the species in a point of space depends not only on nearby density but also on the total amount of species because of the effects of spatial inhomogeneity [2, 5].

Extinction in finite time is a phenomenon whereby the evolution of some nontrivial initial datum $u_0(x)$ produces a nontrivial solution $u(x,t)$ in a time interval $0 < t < T$, but $u(x,t) \equiv 0$ for almost every $(x,t) \in \Omega \times (T, \infty)$. In this case, T is called the extinction time. As one of the most important properties of solutions of evolutionary equations, extinction in finite time of solutions has been intensively studied by several authors (see [6, 7, 8, 9, 10, 13, 14, 17, 18, 21] and references therein). In particular, there are also some works concerning extinction of solutions of problem (1.1) for special cases. For instance, Liu [14] studied (1.1) when $p = 2$ and showed that $q = r$ is the critical extinction exponent of solutions for (1.1). In a recent paper [4], Fang and Xu investigated the case $r = 1$, proved that $q = p - 1$ is the critical extinction exponent of solutions of problem (1.1) and gave the precise decay estimates, by using the methods of energy estimates. Their results show that the linear absorption term does not change the critical extinction exponents. However, when the absorption term is nonlinear, i.e. when $r \neq 1$, the problem is much more challenging and there is no result in this direction. For more works concerning the extinction phenomenon of solutions of fast diffusion equations, readers may refer to [11, 12, 16, 19, 22, 23, 25].

Motivated by the works mentioned above, instead of using the methods of energy estimates, we investigate the extinction and non-extinction properties of solutions to problem (1.1) by constructing suitable super and sub-solutions and give an almost complete classification of the exponents and coefficients for the solutions to vanish in finite time or not.

Throughout this paper, we denote by $\varphi(x)$ the unique positive solution of the following elliptic problem

$$-\mathrm{div}\big(|\nabla\varphi|^{p-2}\nabla\varphi\big) = 1, \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega \qquad (1.2)$$

and set

$$M = \max_{x \in \overline{\Omega}} \varphi(x) \quad \text{and} \quad \mu = \int_\Omega \varphi^{p-1}(x)\,\mathrm{d}x. \qquad (1.3)$$

By the strong maximum principle we know that $M, \mu > 0$. In Section 2, we will give some notations and preliminaries and state our main results. The proofs of the main results will be presented in Section 3.

2   Preliminaries and Main Results

Because of the singularity of the p-Laplace operator, problem (1.1) does not admit classical solutions in general. Therefore, we consider the nonnegative solutions of (1.1) in some weak sense. To state the definition of the weak solutions, we first introduce some notations which will be used throughout this paper:

$$Q = \Omega \times (0, \infty), \qquad Q_T = \Omega \times (0, T),$$
$$E = \{ u \in L^{2q}(Q_T) \cap L^{2r}(Q_T); \ u \in C([0, T]; L^2(\Omega)), \ \nabla u \in L^p(Q_T) \},$$
$$\widetilde{E} = \left\{ u \in L^2(Q_T); \ \frac{\partial u}{\partial t} \in L^2(Q_T), \ \nabla u \in L^p(Q_T); \ u|_{\partial \Omega} = 0 \right\}.$$

DEFINITION 1. A nonnegative measurable function $u \in E$ is called a weak sub-solution of problem (1.1) provided that for any $T > 0$ and any $0 \le \xi \in \widetilde{E}$ the following conditions hold:

$$u(x, 0) \le u_0(x), \quad x \in \Omega; \qquad u(x, t) \le 0, \quad (x, t) \in \partial \Omega \times (0, T),$$

and for $(x, t) \in Q_T$

$$\int_{\Omega} u(x, T)\xi(x, T) \, \mathrm{d}x - \int_{\Omega} u_0(x)\xi(x, 0) \, \mathrm{d}x$$
$$\le \iint_{Q_T} \left[u\frac{\partial \xi}{\partial t} - |\nabla u|^{p-2}\nabla u \nabla \xi + \left(a \int_{\Omega} u^q(y, t) \, \mathrm{d}y - bu^r \right) \xi(x, t) \right] \mathrm{d}x \, \mathrm{d}t.$$

The definition of a weak super-solution follows by replacing '$\le$' by '$\ge$' in the above inequalities. Furthermore, if u is a weak super-solution as well as a weak sub-solution, then we call it a weak solution of problem (1.1).

Local existence of weak solutions of (1.1) can be obtained by utilizing the methods of standard regularization (see [10]) or by applying the famous Schaefer's fixed point theorem (see [11]). Before proving our main results, we give a comparison principle for the super-solutions and sub-solutions of problem (1.1), which is similar to Proposition 2.3 in [10] and can be proved by a rather standard method as that in [1, 11, 18, 21].

Proposition 1. *Let u and v be a nonnegative subsolution and a nonnegative supersolution of (1.1), respectively. If either $q \ge 1$ and u is bounded from above or $0 < q < 1$ and v has a positive lower bound, then $u(x, t) \le v(x, t)$ in Q_T.*

The main results of this paper read as follows.

Theorem 1. *If $q > p - 1$, then any solution $u(x, t)$ vanishes in finite time for suitably small initial data $u_0(x)$. If $q = p - 1$ and $a\mu < 1$, then $u(x, t)$ vanishes in finite time for any nonnegative bounded initial data.*

Theorem 2. *If $r < \min\{q, 1\}$ or $r = q < 1$ with $a|\Omega| < b$, then the solution $u(x, t)$ of problem (1.1) vanishes in finite time for appropriately small initial data $u_0(x)$.*

Theorems 1 and 2 show that when the source term is suitably weak or the absorption term is in some sense strong, solutions of problem (1.1) vanish in finite time.

Theorem 3. *If $q < \min\{r, p-1\}$ or $r = q < p-1$ with $a\gamma > b$, then problem (1.1) admits at least one nonextinction solution for any nonnegative initial data. Here $\gamma > 0$ will be given in the proof.*

Theorem 4. *If $q = p-1 < r$ with $a\mu > 1$, then problem (1.1) admits at least one non-extinction solution for any nonnegative initial data.*

Theorems 3 and 4 tell us that when the source term is appropriately strong, problem (1.1) admits non-extinction solutions.

Theorem 5. *If $q = p-1 < r < 1$ and $a|\Omega| \leq \lambda_1$, then $u(x,t)$ vanishes in the sense that $\lim_{t \to +\infty} \|u(\cdot, t)\|_2 = 0$. If $q = p-1 < r < 1$ with $a\mu = 1$, then all the solutions $u(x,t)$ of problem (1.1) vanish in finite time for any nonnegative initial data. If $q = p-1 < 1 \leq r$ with $a\mu = 1$, then (1.1) admits at least one non-extinction solution $u(x,t)$ for any positive initial data $u_0(x)$.*

Theorem 5 shows that for the "critical" case $q = p-1$ with $a\mu = 1$, whether the solution of (1.1) vanishes or not depends heavily on the absorption term.

Remark 1. It can be seen from Theorems 1, 3, 4 and 5 that $q = p-1$ is the critical extinction exponent when the absorption is linear($r = 1$), which is consistent with the result obtained in [4] by using the integral estimate methods.

3   Proof of the Main Results

Proof of Theorem 1.   The results of this theorem can be obtained by applying the comparison argument since the solutions of (1.1) is a subsolution of the problem in [11] with $m = 1$. We only sketch the outlines for completeness.

Case (i): $q = p-1$ with $a\mu < 1$. For any bounded smooth domain Ω' such that $\Omega' \supset\supset \Omega$, let $\phi(x)$ be the unique solution of the following elliptic problem

$$\begin{cases} -\text{div}\big(|\nabla \phi|^{p-2} \nabla \phi\big) = 1, & x \in \Omega', \\ \phi(x) = 0, & x \in \partial\Omega'. \end{cases}$$

By the comparison principle for the p-Laplace type elliptic problem we know $\varphi(x) \leq \phi(x)$ in Ω. Set $\mu_1 = \int_\Omega \phi^{p-1}(x)\,dx$, $M_1 = \max_{x \in \overline{\Omega'}} \phi(x)$ and $\delta = \min_{x \in \overline{\Omega}} \phi(x)$. It is known from the strong maximum principle that $\delta > 0$.

We can choose a suitable domain $\Omega' \supset\supset \Omega$ such that $a\mu_1 < 1$ by continuity. Define $v(x,t) = g(t)\phi(x)$, where $g(t)$ satisfies

$$\begin{cases} g'(t)M_1 + (1 - a\mu_1)g^{p-1}(t) = 0, & t > 0, \\ g(0) = A \geq \delta^{-1}\|u_0\|_{L^\infty(\Omega)}. \end{cases}$$

Since $0 < p - 1 < 1$, it follows from the theories in ODEs that $g(t)$ is nonincreasing and $g(t) = 0$ for all

$$t \geq T^* = \frac{M_1}{(1 - a\mu_1)(2 - p)} A^{2-p}.$$

Then it can be verified that for any $0 < T < T^*$, $v(x,t)$ is a super-solution of (1.1) in Q_T. In fact, because $q = p - 1$ and $g'(t) \leq 0$, we know that $v(x,t)$ satisfies the following inequalities (in the weak sense)

$$
\begin{aligned}
\frac{\partial v}{\partial t} &- \operatorname{div}(|\nabla v|^{p-2}\nabla v) - a \int_\Omega v^{p-1}(y,t)\,\mathrm{d}y + bv^r \\
&= g'(t)\phi(x) + g^{p-1}(t) - a\mu_1 g^{p-1}(t) + bg^r(t)\phi^r(x) \\
&\geq g'(t)M_1 + (1 - a\mu_1)g^{p-1}(t) = 0.
\end{aligned}
\tag{3.1}
$$

In addition, $v(x,t) \geq 0$ on $\partial\Omega \times (0,T)$ for any $T > 0$, and $v(x,0) \geq u_0(x)$ by the choice of A. Moreover, there exists a positive constant C_1 such that $v(x,t) \geq C_1$ in Q_T. Therefore, by applying Proposition 1 to (1.1) we see that $u(x,t) \leq v(x,t)$ for $(x,t) \in Q_T$, which implies $u(x,T) \leq v(x,T)$. The arbitrariness of $T < T^*$ and $v(x,T^*) = 0$ ensure that $u(x,T^*) = 0$. Furthermore, let $\widetilde{u}(x,t) = u(x,t + T^*)$, then $\widetilde{u}(x,t)$ satisfies (1.1) with the initial condition $\widetilde{u}(x,0) = 0$. By the aforementioned proof, we see that $\widetilde{u}(x,t) \leq v(x,t)$ with any $A > 0$. From the relation of the extinction time T^* of $v(x,t)$ to A, it follows that $\widetilde{u}(x,t) = 0$ for any $t > 0$, i.e. $u(x,t) = 0$ for any $t \geq T^*$.

Case (ii): $q > p - 1$. Let $\phi(x)$ and M_1 be defined as in Case (i) and denote $k_0 = \max\{1, \frac{2M_1^{q-(p-1)}\mu}{\int_\Omega \phi^q(x)\,\mathrm{d}x}\}$. Set $v(x,t) = k\phi(x)$ with $k = (\frac{1}{ak_0 \int_\Omega \phi^q(x)\,\mathrm{d}x})^{\frac{1}{q-(p-1)}}$, then it is easy to verify that $v(x,t)$ is a supersolution of (1.1) when u_0 is small enough such that $u_0(x) \leq k\phi(x)$ in Ω. Applying Proposition 1 to problem (1.1) in Q_T for any $T > 0$ we obtain that $u(x,t) \leq v(x,t)$ in Q_T, which implies that $u(x,t) \leq kM_1$. Therefore, $u(x,t)$ satisfies

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + bu^r \leq a(kM_1)^{q-(p-1)} \int_\Omega u^{p-1}(y,t)\,\mathrm{d}y, \quad x \in \Omega,\ t > 0.$$

By the choice of k and k_0 it is easily seen that $a(kM_1)^{q-(p-1)}\mu \leq \frac{1}{2} < 1$. Therefore, by the results of Case (i), we can conclude that the solution $u(x,t)$ vanishes in finite time. The proof of this theorem is complete. $\square$

Proof of Theorem 2. We first prove the case $r = q$ with $b > a|\Omega|$. Set $v(x,t) = g(t)$ where $g(t)$ satisfies the following ordinary differential equation

$$
\begin{cases}
g'(t) + (b - a|\Omega|)g^q(t) = 0, & t > 0, \\
g(0) = \|u_0\|_{L^\infty(\Omega)}.
\end{cases}
\tag{3.2}
$$

Since $0 < q < 1$ and $b > a|\Omega|$, we know from the theories in ODEs that $g(t)$ vanishes at some finite time T^*. Moreover, as in the proof of Theorem 1, it can be verified that $g(t)$ is a super-solution of (1.1). Thus, by applying

Proposition 1 to $u(x,t)$ and $g(t)$ for any $0 < T < T^*$ we know that $u(x,t)$ also vanishes at T^*.

In the case $r < \min\{q, 1\}$, let $g(t)$ satisfy the following ODE

$$\begin{cases} g'(t) + \big(b - a|\Omega||g^{q-r}(t)\big)g^r(t) = 0, & t > 0, \\ g(0) = g_0, \end{cases} \tag{3.3}$$

where $0 < g_0 < (\frac{b}{a|\Omega|})^{\frac{1}{q-r}}$. Similar to the first case, it is known that $g(t)$ vanishes in finite time and $g(t)$ is a super-solution of problem (1.1) if $u_0(x)$ is so small that $\|u_0\|_{L^\infty(\Omega)} \le g_0$. Applying Proposition 1 to $u(x,t)$ and $g(t)$ guarantees the finite time extinction of $u(x,t)$. $\square$

Remark 2. The hypotheses that u_0 is small in Theorems 1 and 2 can be removed if a is suitably small.

Proof of Theorem 3. (i). Consider first the case $q < r \le p - 1$. Let λ_1 be the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\big(|\nabla\psi|^{p-2}\nabla\psi\big) = \lambda|\psi|^{p-2}\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

and $\psi_1(x) > 0$ $(x \in \Omega)$ be the corresponding eigenfunction. We may normalize $\psi_1(x)$ such that $\|\psi_1\|_{L^\infty(\Omega)} = 1$. Denote $\gamma = \int_\Omega \psi_1^q(x)\,\mathrm{d}x$ and let $g(t)$ satisfy the ODE problem

$$\begin{cases} g'(t) = -\lambda_1 g^{p-1}(t) + a\gamma g^q(t) - bg^r(t), & t > 0, \\ g(0) = 0, & \\ g(t) > 0, & t > 0. \end{cases} \tag{3.5}$$

It is easy to check that $g(t)$ is nondecreasing and bounded from above by $\min\{(\frac{a\gamma}{\lambda_1})^{\frac{1}{p-1-q}}, (\frac{a\gamma}{b})^{\frac{1}{r-q}}\}$. Set $v(x,t) = \rho g(t)\psi_1(x)$. We shall show that $v(x,t)$ is a sub-solution of (1.1) when $\rho > 0$ is sufficiently small. In fact, simple computations show that

$$\frac{\partial v}{\partial t} = \rho\big(-\lambda_1 g^{p-1}(t) + a\gamma g^q(t) - bg^r(t)\big)\psi_1(x)$$

and

$$\mathrm{div}\big(|\nabla v|^{p-2}\nabla v\big) + a\int_\Omega v^q(y,t)\,\mathrm{d}y - bv^r$$
$$= -\lambda_1 \rho^{p-1}g^{p-1}\psi_1 + a\gamma\rho^q g^q - b\rho^r g^r \psi_1^r.$$

Denote $C = \max_{t>0}\{\lambda_1 g^{p-1-q}(t), bg^{r-q}(t)\}$. Then for $v(x,t)$ to be a sub-solution of (1.1), it suffices to show that

$$\lambda_1 g^{p-1}(t)\rho^{p-1} + bg^r(t)\rho^r \le a\gamma g^q(t)\big(\rho^q - \rho\big),$$

which follows from

$$C\big(\rho^{p-1} + \rho^r\big) \le a\gamma\big(\rho^q - \rho\big)$$

or equivalently,

$$C\big(\rho^{p-1-q} + \rho^{r-q}\big) \le a\gamma\big(1 - \rho^{1-q}\big). \tag{3.6}$$

Since $q < r \le p - 1 < 1$, (3.6) holds for sufficiently small $\rho > 0$.

Next, we turn our attention to construct a super-solution of (1.1). Set $w(x,t) = L$, where $L = \max\{\|u_0\|_{L^\infty(\Omega)}, (\frac{a|\Omega|}{b})^{\frac{1}{r-q}}, \rho\max_{t\ge 0} g(t)\}$. Then it is not hard to see that $w(x,t)$ is a super-solution and $v(x,t) \le w(x,t)$. Therefore, by an iteration process, one can obtain a solution of problem (1.1), which satisfies $v(x,t) \le u(x,t) \le w(x,t)$. Because $v(x,t)$ does not vanish, neither does $u(x,t)$.

(ii). The case $q < p - 1 < r$ can be treated similarly to Case (i).

(iii). Finally we consider the case $r = q < p-1$ with $a\gamma > b$. Let $g(t)$ satisfy the following ODE

$$\begin{cases} g'(t) = -\lambda_1 g^{p-1}(t) + (a\gamma - b)g^q(t), & t > 0, \\ g(0) = 0, \\ g(t) > 0, & t > 0. \end{cases} \tag{3.7}$$

Then $g(t)$ is nondecreasing and satisfies $g(t) \le (\frac{a\gamma-b}{\lambda_1})^{\frac{1}{p-1-q}}$. As in the proof of Case (i), we can construct a non-extinction sub-solution $v(x,t) = \rho g(t)\psi(x)$ with $\rho > 0$ sufficiently small.

To construct a super-solution, consider the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\big(|\nabla\psi|^{p-2}\nabla\psi\big) = \lambda|\psi|^{p-2}\psi, & x \in \widetilde{\Omega}, \\ \psi(x) = 0, & x \in \partial\widetilde{\Omega}, \end{cases}$$

where $\widetilde{\Omega} \supset\supset \Omega$ is a bounded domain with smooth boundary $\partial\widetilde{\Omega}$. Let $\widetilde{\lambda_1}$ and $\widetilde{\psi_1}(x) > 0$ $(x \in \widetilde{\Omega})$ be its first eigenvalue and the corresponding eigenfunction, respectively. We may normalize $\widetilde{\psi_1}(x)$ such that $\|\widetilde{\psi_1}\|_{L^\infty(\widetilde{\Omega})} = 1$. Denote $\widetilde{\gamma} = \int_\Omega \widetilde{\psi_1}^q(x)\,\mathrm{d}x$ and $\widetilde{\delta} = \min_{x\in\overline{\Omega}} \widetilde{\psi_1}(x) > 0$.

Set $w(x) = k\widetilde{\psi_1}(x)$. We shall show that $w(x)$ is a supersolution of problem (1.1) provided that $k > 0$ is suitably large. Indeed, if $k = \max\{(\frac{a\widetilde{\gamma}}{\widetilde{\lambda_1}\widetilde{\delta}^{p-1}})^{\frac{1}{p-1-q}}, \widetilde{\delta}^{-1}\|u_0\|_{L^\infty(\Omega)}, \rho\widetilde{\delta}^{-1}(\frac{a\gamma-b}{\lambda_1})^{\frac{1}{p-1-q}}\}$, we know that $w(x) \ge 0$ on $\partial\Omega \times (0,\infty)$, $w(x) \ge u_0(x)$ in Ω and $w(x)$ satisfies the following inequalities (in the weak sense)

$$\begin{aligned} \frac{\partial w}{\partial t} &- \mathrm{div}\big(|\nabla w|^{p-2}\nabla w\big) - a\int_\Omega w^q(x,t)\,\mathrm{d}x + bw^q \\ &= k^{p-1}\widetilde{\lambda_1}\widetilde{\psi_1}^{p-1}(x) - ak^q\widetilde{\gamma} + bk^q\widetilde{\psi_1}^q(x) \\ &\ge k^q\big(k^{p-1-q}\widetilde{\lambda_1}\widetilde{\delta}^{p-1} - a\widetilde{\gamma}\big) \ge 0. \end{aligned}$$

Moreover, $v(x,t) \le w(x)$ by the choice of k. Therefore, by applying the monotonicity iteration we obtain a non-extinction solution $u(x,t)$ of (1.1) satisfying $v(x,t) \le u(x,t) \le w(x)$. This completes the proof of Theorem 3. $\square$

Proof of Theorem 4.   The proof of this theorem is similar to that of Theorem 3, so we only sketch the outline here. Set $v(x,t) = \rho g(t)\varphi(x)$ where $\varphi(x)$ is defined in (1.2) and $g(t)$ satisfies the following ODE problem

$$\begin{cases} g'(t) = M^{-1}\{(a\mu - 1)g^{p-1}(t) - bM^r g^r(t)\}, & t > 0, \\ g(0) = 0, \\ g(t) > 0, & t > 0. \end{cases} \tag{3.8}$$

Since $a\mu > 1$ and $p-1 < r$, it is known that $g(t)$ is nondecreasing and is bounded from above by $(\frac{a\mu - 1}{bM^r})^{\frac{1}{r-(p-1)}}$. Then $v(x,t)$ is a sub-solution of (1.1) if $\rho > 0$ is sufficiently small. On the other hand, the super-solution $w(x,t)$ can be chosen to be a large positive constant L satisfying $L \geq \max\{\|u_0\|_{L^\infty(\Omega)}, (\frac{a|\Omega|}{b})^{\frac{1}{r-q}}, \rho M \max_{t\geq 0} g(t)\}$. It can be observed that (v, w) is a pair of sub-solution and super-solution of (1.1) satisfying $v(x,t) \leq w(x,t)$. Therefore, by monotonicity iteration, we know that (1.1) admits at least one solution $u(x,t)$ such that $v(x,t) \leq u(x,t) \leq w(x,t)$. Since $v(x,t) > 0$ in $\Omega \times (0,+\infty)$, $u(x,t)$ cannot vanish at any finite time. The proof of Theorem 4 is complete. $\square$

Proof of Theorem 5.   (i). Let $u(x,t)$ be any solution of (1.1). It can be verified that, for the case $p - 1 = q < r < 1$, a suitably large constant L is a supersolution of (1.1). Therefore, we know that $u(x,t) \leq L$ in $\Omega \times (0,+\infty)$. Multiplying Equation (1.1) by u and integrating by parts over Ω yield the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |\nabla u|^p\,dx + b\int_\Omega u^{r+1}\,dx = a\int_\Omega u^{p-1}\,dx \int_\Omega u\,dx. \tag{3.9}$$

Recall the embedding theorem

$$\int_\Omega |\nabla u|^p\,dx \geq \lambda_1 \int_\Omega u^p\,dx.$$

Combining this result with (3.9) and using Hölder's inequality on the right hand side of (3.9) one obtains

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \lambda_1 \int_\Omega u^p\,dx + b\int_\Omega u^{r+1}\,dx \leq a|\Omega| \int_\Omega u^p\,dx. \tag{3.10}$$

Noticing that $a|\Omega| \leq \lambda_1$ and $u(x,t) \leq L$, we see from (3.10) that

$$\frac{d}{dt}\int_\Omega u^2\,dx + 2bL^{r-1} \int_\Omega u^2\,dx \leq 0,$$

which implies

$$\int_\Omega u^2\,dx \leq e^{-2bL^{r-1}t} \int_\Omega u_0^2\,dx.$$

This shows that $\|u(\cdot,t)\|_2$ tends to 0 exponentially as $t \to +\infty$.

(ii). Next, let us prove the extinction results of the case $q = p - 1 < r < 1$. Set $w(x,t) = g(t)\varphi(x)$, where $g(t)$ satisfies

$$\begin{cases} g'(t) + bM^{r-1}g^r(t) = 0, & t > 0, \\ g(0) = A > 0. \end{cases}$$

Since $0 < r < 1$, $g(t)$ is nonincreasing and vanishes at the finite time $T^* = \frac{A^{1-r}}{bM^{r-1}(1-r)}$ for any $A > 0$. Noticing $q = p - 1$ and $a\mu = 1$, one can check that $w(x,t)$ is a super-solution of (1.1) provided that $A > 0$ is appropriately large such that $A\varphi(x) \geq u_0(x)$ in Ω. Now by using the arguments similar to that of the proof of Theorem 1, we know that every solution $u(x,t)$ of (1.1) vanishes at a time no later than T^*.

(iii). Finally we consider the case $q = p - 1 < 1 \leq r$. First we construct a non-extinction sub-solution of (1.1). Set $v(x,t) = h_0 e^{-\alpha t}\varphi(x)$, where h_0, α are two positive constants to be determined. Noticing that $a\mu = 1$, it is easily checked that when $r = 1$, $v(x,t)$ is a sub-solution of (1.1) provided that $\alpha \geq b$ and h_0 is so small such that $h_0\varphi(x) \leq u_0(x)$. When $r > 1$, for $v(x,t)$ to be sub-solution of (1.1) it is reasonable to choose first h_0 so small such that $h_0\varphi(x) \leq u_0(x)$ and then $\alpha \geq bh_0^{r-1}M^{(r-1)}$. Next, since $r > q$ and $v(x,t)$ is bounded, we can choose a sufficiently large constant $L \geq v(x,t)$ to be a super-solution of (1.1). Therefore, by monotonicity iteration, one obtains a solution $u(x,t)$ of (1.1) satisfying $v(x,t) \leq u(x,t) \leq L$. Since $v(x,t)$ does not vanish at any finite time, neither does $u(x,t)$. The proof of Theorem 5 is complete. □

Remark 3. According to Theorems 1, 4 and the first part of Theorem 5 it is easy to see that $q = p - 1$ is the critical extinction exponent for problem (1.1) when $r = 1$, which is consistent with the results obtained in [4]. However, when $r = 1$ and $q = p - 1$, for the nonlocal problem under consideration, the first eigenvalue λ_1 of p-Laplacian in Ω no longer plays the same role as it does in the local case (see Theorem 1 in [15]).

Remark 4. There still exist some gaps of the coefficients in which case we do not know whether problem (1.1) admits non-extinction solutions or not, due to technique reasons. For example, combining Theorem 2 with Theorem 3 one observes that there is no result on whether there exists a non-extinction solution of problem (1.1) or not when $q = r < p - 1$ with $\frac{b}{|\Omega|} \leq a \leq \frac{b}{\gamma}$ (noticing by the definition of γ that $\gamma < |\Omega|$).

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