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REDUCED ORDER GALERKIN MODELS OF FLOW AROUND NACA-0012 AIRFOIL

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Abstract. The construction of low-dimensional models of the flow, containing only reduced number of degrees of freedom, is the essential prerequisite of closed-loop control of that flow. Presently used models usually base on the Galerkin method, where the flow is approximated by the number of modes and coefficients. The velocities are computed from a system of ordinary differential equations, called Galerkin System, instead of Navier-Stokes equation. In this paper, reduced order models of the flow around NACA-0012 airfoil are presented. The chosen mode sets include POD modes from Karhunen-Loeve decomposition (which require previous direct numerical simulation), as well as different eigenmodes from global stability analysis of the flow.

Key words: Galerkin method, Reduced Order Models (ROM), Proper Orthogonal Decomposition (POD), mode interpolation, stability, control.

1 Introduction

The one of the most challenging areas of fluid dynamics is the control of the flow. Separation in the bluff body wakes, and resulting coherent structures are the source of negative mechanical, environmental and economical effects, especially in aerospace and automotive industry. The examples are increased drag, generated noise, reduced lift force and efficiency [1]. Modification of the state of the flow, e.g. its stabilization, allows the reduction of these effects.

There are two kinds of control techniques. *Passive* control is based on the modification of flow domain, and requires no external energy source and usually utilizes some kind of splitter plates, control cylinders, wake disruptors or riblets to influence the vortex generation [3, 9]. The second possibility is *active flow* control, where there are external energy sources, called actuators. If the actuation is independent on the state of the flow, it is called open-loop control, while closed-loop or feedback flow control requires information about the up-to-date flow state. It is measured by a certain number of sensors and processed by a controller. Usually the controller design requires some kind of mathematical model of the flow. The Navier-Stokes equations after e.g. Finite Element Discretization, can be considered as the high-dimensional flow model – it contains thousands or millions of degrees of freedom.

The flow model, to be suitable for real-time, closed-loop control applications, must contain only a few degrees of freedom, so model reduction is required. One of the most popular ways of Reduced Order Model (ROM) construction is the Galerkin method [4, 7], described in Section 3.

2 Problem Description – NACA-0012

To compare the model reduction techniques, two-dimensional flow around NACA-0012 airfoil at angle of attack $AoA = 30^o$ have been chosen (Fig. 1).

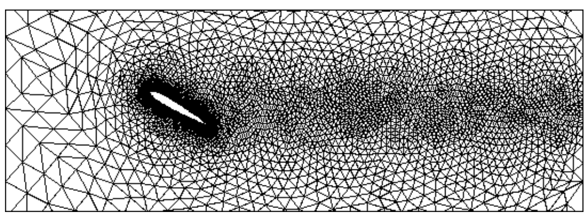

Figure 1. Computational domain with NACA-0012 airfoil, discretized with the finite element method.

In this case, at the Reynolds number $Re = 100$, the separation and vortex-street are visible and the comparison between reduced order models and the Direct Numerical Simulation (DNS) is simple. NACA-0012 have been chosen due to its common usage as the test case in CFD-application development and its application in rotor blades.

For the configuration described above, period of the flow computed by DNS is $T = 68.5$. Streamlines of one of the snapshots, steady solution u_s, time-averaged solution $\bar{u}$ and computed shift-mode for this flow are presented on Fig. 2.

3 Low Dimensional Modelling – Galerkin Method

Flow model reduction is based on the assumption, that the state of the flow u can be approximated by a base flow u_0 (e.g. steady or mean solution) and a sum of products of modes u_j and Fourier coefficients α_j, describing the

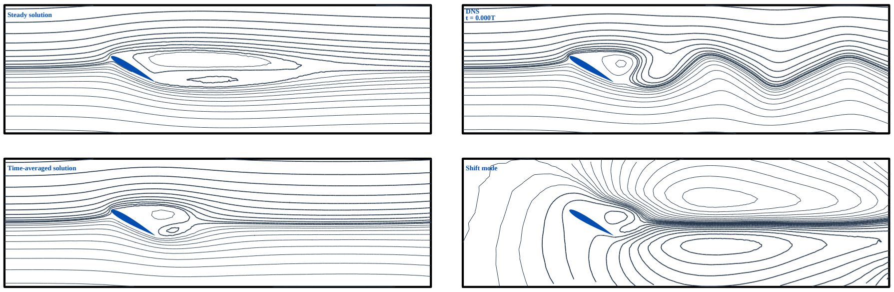

Figure 2. Streamlines of flow around NACA-0012 airfoil at Re=100: steady solution (left-top), time-averaged solution (left-bottom), one of the DNS snapshots (right-top) and shift-mode (right-bottom).

disturbance

$$u^{[N]} = u_0 + \sum_{j=1}^{N} \alpha_j u_j = \sum_{j=0}^{N} \alpha_j u_j, \quad \alpha_0 \equiv 1. \tag{3.1}$$

The use of limited number of modes means, that the right-hand-side of approximated Navier-Stokes equation is equal to residual $R^{[N]}$, depending on the number of used modes N

$$\dot{u}^{[N]} + \nabla \cdot (u^{[N]} \otimes u^{[N]}) + \nabla p^{[N]} - \frac{1}{Re} \triangle u^{[N]} = R^{[N]} \tag{3.2}$$

To obtain the coefficients in every time step, one has to make a projection of approximated Navier-Stokes equation onto space spanned by selected modes (so called Galerkin Projection).

In Hilbert space, it is done by computing the inner products of residual $R^{[N]}$ and each of the modes u_i, and equating them to zero:

$$\left(u_i, R^{[N]}\right)_\Omega = \int_\Omega u_i R^{[N]} d\Omega = 0.$$

This approach leads to the system of ordinary differential equations (the Galerkin system), linking the Fourier coefficients α, their time derivatives $\dot{\alpha}$ and constant parameters l_{ij}, q_{ij} :

$$\dot{\alpha}_i = \frac{1}{Re} \sum_{j=0}^{N} \alpha_j l_{ij} + \sum_{j=0}^{N} \sum_{k=0}^{N} \alpha_j \alpha_k q_{ijk}, \tag{3.3}$$

where:

$$l_{ij} = \left(u_i, \triangle u_j\right)_\Omega, \quad q_{ijk} = -\left(u_i, \nabla \cdot (u_j u_k)\right)_\Omega.$$

Set of modes used for low-dimensional modelling can be obtained mathematically, physically or empirically [8].

4 Reduced Order Models of the Flow

In this paper, empirical and physical modes will be used. In first case, the Karhunen-Loeve or Proper Orthogonal Decomposition (POD) of unsteady flow will be performed. In second case, global stability analysis with various base solutions and continuous mode interpolation will be used.

4.1 POD Galerkin models

The most common set of modes for model reduction results from POD [2]. It is based on the assumption, that snapshots v_i (of size N) of the flow from successive M time-steps are correlated.

The first step in POD is the computation of time-averaged solution $\bar{u}$ and the centering of the flow vectors, by subtraction $\acute{v}_i = v_i - \bar{u}$. Resulting vectors $\acute{v}_i$ describe the fluctuations in the flow. This data is required to compute the autocorrelation matrix C of size $N \times N$:

$$C = \frac{1}{M} S S^T, \quad \text{where} \quad S = [\acute{v}_1, \acute{v}_2, ...\acute{v}_M].$$

Eigenvectors u of standard eigenproblem $Cu = \lambda I u$, related to eigenvalues λ of largest magnitude, are the POD modes used in model reduction (Fig. 3).

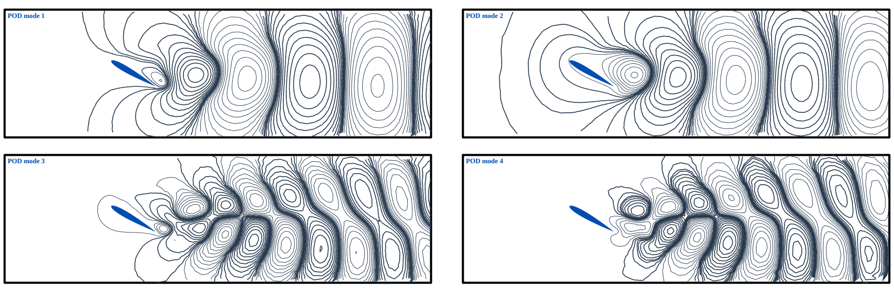

Figure 3. The first four POD modes used in Galerkin modelling.

In this paper two empirical ROMs are presented: one using 2 POD modes and one using 8 POD modes (Fig. 4). In both cases, additional shift-mode (defined as difference between steady and time-averaged solution, Fig. 2) have been used to stabilize the models [8]. Next the flow have been reconstructed and the mean square errors (MSE) for limit cycles of each model and DNS have been computed.

For POD-2 Galerkin model, the percentage time-averaged residual (defined as relation of MSE to integral of original flow) is equal to $res = 5.77\%$, and for POD-8 GM $res = 2.79\%$. As can be seen on the right side of Fig. 4, with POD modes it is possible to reconstruct precisely the attractor of Navier-Stokes equations (limit cycle), but these modes do not suffice to create a model of the flow preserving low dimensionality and expected dynamical properties at the same time. In the transition from steady state to limit cycle oscillations, the kinetic energy of the flow is overestimated – especially in POD-2 Galerkin model. What is more, before the model reduction the DNS computations of the unsteady flow in the given configuration are required.

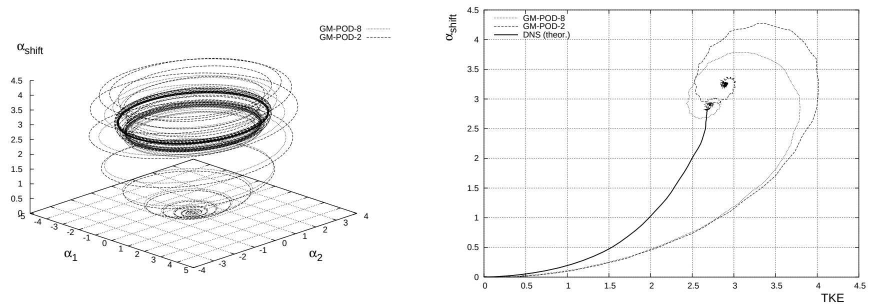

Figure 4. POD-based models characteristics: Fourier coefficients α_1 and α_2 as a function of time (left) and shift-mode coefficient α_{shift} as a function of disturbance kinetic energy (right).

4.2 Stability-mode Galerkin models

To overcome the problem dealing with the accuracy of transients modelling, more POD modes can be used. While low dimensionality of the model is one of the most important of its properties, another mode sets, resulting from stability analysis, have been used.

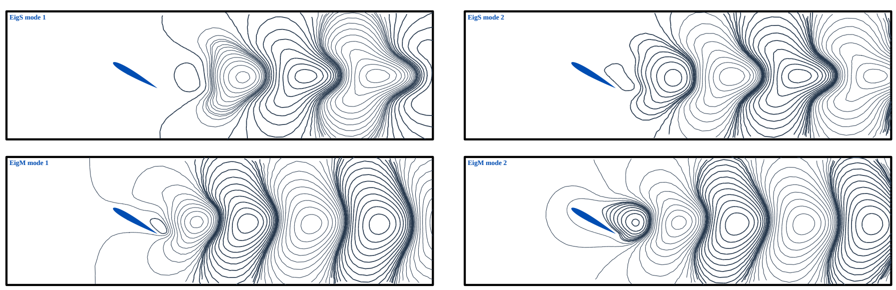

Figure 5. The most dominant eigenmodes based on the steady solution (top, $\lambda = -0.147 \pm 0.720\imath$) and time-averaged solution (bottom, $\lambda = 0.018 \pm 0.915\imath$).

Global, linear stability analysis of incompressible flow is based on the assumption, that the unsteady solution V might be written as a sum of base solution $\bar{V}$ and a small disturbance $\tilde{V}$ which allows to linearize the equations and leads to formulation of generalized differential eigenvalue problem

$$\lambda \tilde{V}_i + \tilde{V}_j \bar{V}_{i,j} + \bar{V}_j \tilde{V}_{i,j} + \tilde{P}_{,i} - \frac{1}{Re} \tilde{V}_{i,jj} = 0 \quad \text{or} \quad Ax - \lambda Bx = 0.$$

This eigenproblem has been solved using the algorithms and techniques presented by Morzynski et al. [5]. In the modelling only the most unstable pairs of modes have been used (see Fig. 5).

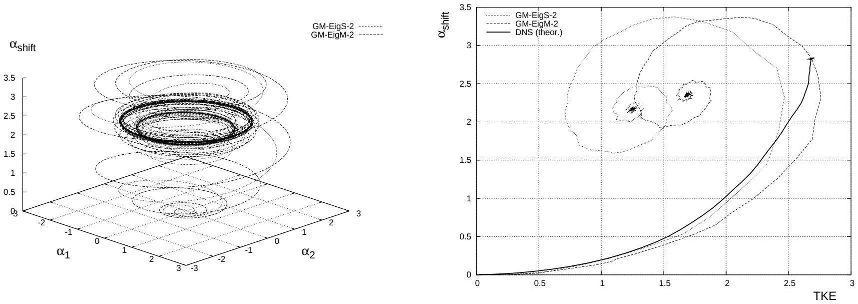

Figure 6. Eigenmode-based models characteristics: Fourier coefficients α_1 and α_2 as a function of time (left) and shift-mode coefficient α_{shift} as a function of disturbance kinetic energy (right).

The models based on the stability eigenmodes reconstruct the transition from the steady solution to the limit cycle better (see, Fig. 6) than POD Galerkin models, but are much worse in attractor reconstruction - the disturbance kinetic energy of periodic flow is underestimated. In case of the model based on two steady-solution-eigenmodes we have that $res = 15.08\%$, and for the model based on two averaged-solution-eigenmodes, $res = 10.10\%$.

4.3 Hybrid Galerkin model

The numerical experiments presented in previous sections lead to conclusion, that POD modes should be used in limit-cycle modelling, and stability eigenmodes - in modelling of flow states close to fixed point of Navier-Stokes equations (steady solution). The idea of using both empirical and physical modes, proposed by Noack et al. for circular cylinder case [7], leads to hybrid model, allowing reconstruction of both limit cycle and dynamical properties of the flow. On the Fig. 7 the characteristics of model of flow around NACA-0012, based on 2 POD modes, 2 eigenmodes and the shift-mode are presented. In this case, the percentage time-averaged residual $res = 8.44\%$, and accuracy of transients modelling is better than in pure POD models.

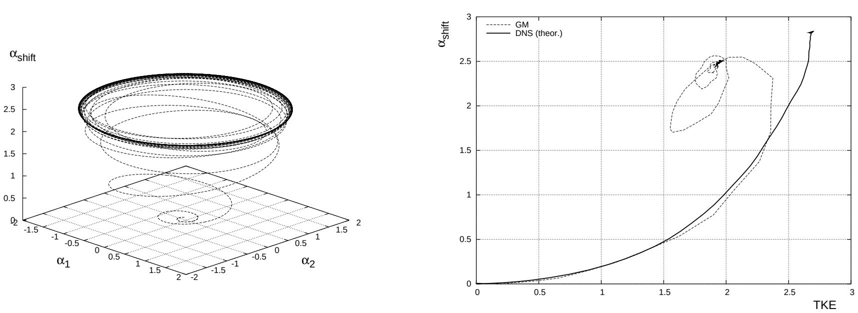

Figure 7. Hybrid model characteristics: Fourier coefficients α_1 and α_2 as a function of time (left) and shift-mode coefficient α_{shift} as a function of disturbance kinetic energy (right).

4.4 Low-dimensional modelling with mode interpolation

One more idea of models adjustable to different operating conditions has been presented by the authors in [6]. The solution of interpolated eigenproblem $\hat{A}$

$$\hat{A}(\kappa) = (1-\kappa)A_S + \kappa\bar{A},$$

where A_S and $\bar{A}$ are two different eigenproblems, leads to the set of modes continuously changing with operating conditions and interpolation parameter κ. The resulting Galerkin model is described by a system of ordinary differential equations:

$$\dot{\alpha}_i^\kappa = \frac{1}{Re}\sum_{j=0}^{N}\alpha_j^\kappa l_{ij}^\kappa + \sum_{j=0}^{N}\sum_{k=0}^{N}\alpha_j^\kappa\alpha_k^\kappa q_{ijk}^\kappa. \tag{4.1}$$

The value of interpolation parameter κ depends on the state of the flow. In this paper, it is related to the shift mode amplitude (coefficient) α_{shift}:

$$\kappa(t) = \frac{\max(\alpha_{shift}) - \alpha_{shift}(t)}{\max(\alpha_{shift}) - \min(\alpha_{shift})}$$

As can be seen on Fig. 8, model based on continuous interpolation between POD modes and stability eigenmodes, based on steady solution, reconstructs the limit cycle oscillations with accuracy of empirical models. It allows accurate modelling of transition from steady state to limit cycle, and remains low-dimensional – it has only one degree of freedom more (κ parameter) than POD-2 Galerkin model.

Mode interpolation can be also used to create a priori model of the flow, requiring no empirical data. Extrapolation of two mode sets resulting from global stability analysis (based on steady and time-averaged solutions), with $\kappa > 1$, leads to modes that can mimic POD modes [6]. Model using such modes is presented on Fig. 9 and has similar accuracy to hybrid model presented before. Its percentage residual $res = 9.37\%$, is smaller than residuals of pure stability-based models.

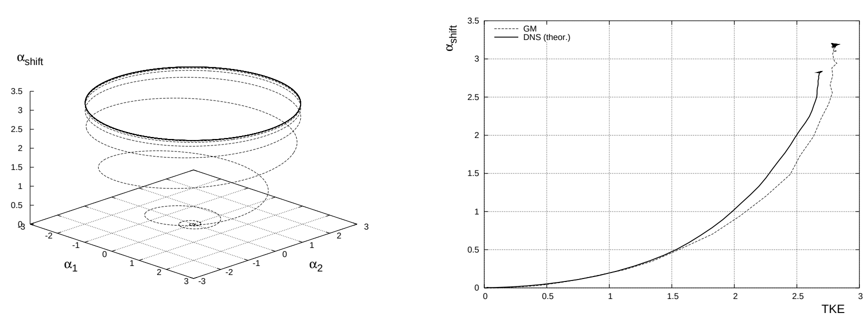

Figure 8. Interpolated model characteristics: Fourier coefficients α_1 and α_2 as a function of time (left) and shift-mode coefficient α_{shift} as a function of disturbance kinetic energy (right).

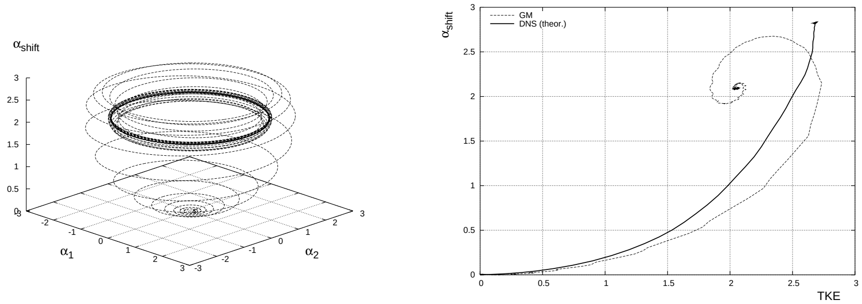

Figure 9. Extrapolated model characteristics: Fourier coefficients α_1 and α_2 as a function of time (left) and shift-mode coefficient α_{shift} as a function of disturbance kinetic energy (right).

5 Summary

In this paper, a number of Galerkin models of flow around NACA-0012 have been compared, the results are presented in Fig. 10. In each case, shift mode was used to stabilize the model. Performed numerical simulations have proven, that empirical models (dotted lines on Fig. 10) are very fragile to the change of operating conditions, allowing the accurate reconstruction only for the state, for which POD analysis has been done. Transients modelling requires large number of empirical modes, what makes the dimension of the model to large for feedback control applications.

Eigenmodes resulting from global stability analysis (linearization around steady solution) are not enough to construct accurate Galerkin model of wake/separated flow (see lowest dashed line on Fig. 10), but are very important in construction more advanced models: hybrid, extrapolated and with continuous mode interpolation.

The least stable eigenmodes of Navier-Stokes equation linearized around time-averaged flow are much more similar to POD modes than the ones corresponding to steady solution, and the accuracy of resulting Galerkin model is much higher. Stability eigenmodes extrapolation allows even further increase of accuracy and is a step towards a priori modelling of the flow.

The best results of transition and limit cycle oscillations reconstruction have been obtained for the models using both POD modes and eigenmodes of global stability analysis. First of them is hybrid model proposed by Noack et al. [7]. Another choice is to use continuous mode interpolation, that allows to change the coherent structures with the change of operating condition. It is shown, that interpolated modes allow accurate, least-dimensional Galerkin approximation of Navier-Stokes equation (thin continuous line on Fig. 10), what is important in control design process. Compared to POD models, one additional equation is required to specify the value of interpolation parameter.

Presented methods are instantly applicable to 3D and other geometries.

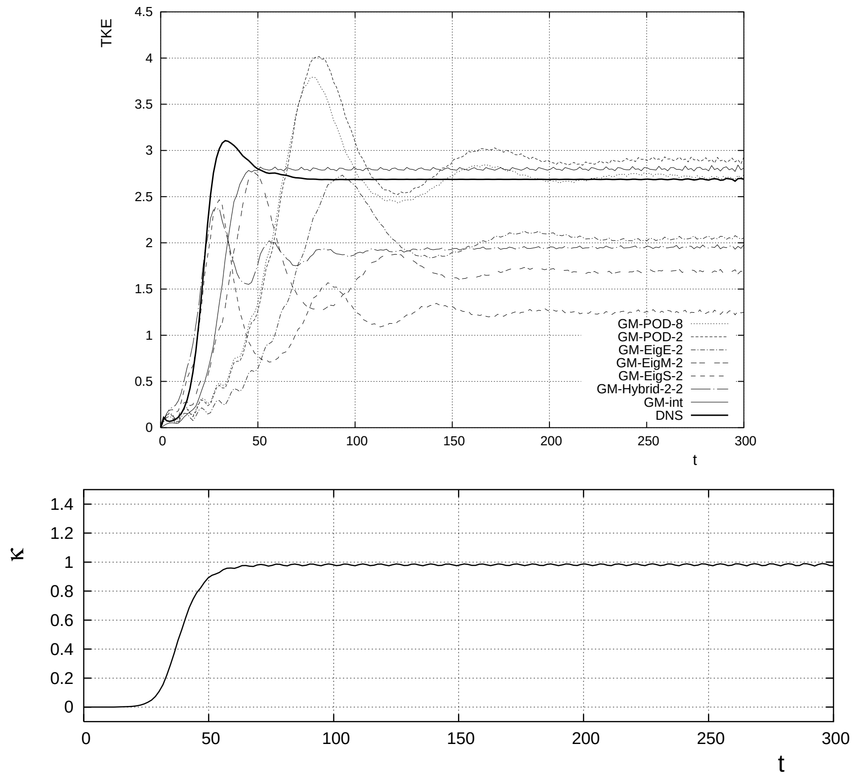

Figure 10. The variation of disturbance kinetic energy (TKE) for different Galerkin models and interpolation parameter κ in function of time, for transition from steady state to the limit cycle oscillation.

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