

MATHEMATICAL MODELLING AND ANALYSIS  
Volume 18 Number 1, February 2013, 22–31  
<http://dx.doi.org/10.3846/13926292.2013.756434>  
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Publisher: Taylor&Francis and VGTU  
<http://www.tandfonline.com/TMMA>  
Print ISSN: 1392-6292  
Online ISSN: 1648-3510

# On a Two-Fluid Inclined Film Flow with Evaporation

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Received August 19, 2012; revised November 14, 2012; published online February 1, 2013

**Abstract.** This paper is concerned with a plane steady-state inclined film flow including evaporation effects. The motion is governed by a free boundary value problem for a coupled system of Navier–Stokes and Stefan equations. The flow domain is unbounded in two directions and it contains a geometrical perturbation on the inclined bottom. Existence and uniqueness of a suitable solution in weighted Sobolev spaces can be proved for small data (perturbation, inclination of the bottom) characterizing the problem.

**Keywords:** Navier–Stokes equations, Stefan equations, evaporation, free boundary value problem, perturbed bottom.

**AMS Subject Classification:** 35R35; 35Q30; 76D03; 76D05.

## 1 Introduction

The paper is devoted to a plane stationary nonisothermal two-fluid flow problem with two free boundaries describing an inclined film flow with evaporation. Two heavy viscous incompressible and heat-conducting fluids are flowing down a geometrically perturbed inclined bottom (cf. Figure 1). Both of the a priori unknown free boundaries are noncompact in two directions. The flow domain is unbounded and evaporation is taken into account. This is quite important in several technological and scientific applications; interesting examples may be found in the field of materials science, particularly in coating and solidification processes with evaporation (cf. [2, 4, 5, 8, 10, 17, 21, 25]) or in crystal-growth processes [13, 16, 18].

In this paper we investigate a problem for a 2D steady-state flow with two viscous incompressible heat-conducting fluids (having kinematic viscosities  $\nu_i > 0$ , densities  $\rho_i > 0$  and thermal conductivities  $\lambda_i$ ,  $i = 1, 2$ ) down an inclined bottom  $S_0$  having a slope  $\alpha$  (cf. Figure 1). In fact, the bottom  $S_0$  represents a perturbed plane. Suppose that the bottom is given by the formula  $S_0 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2: x_2 = \varepsilon\varphi_0(x_1), -\infty < x_1 < +\infty\}$  with  $\varphi_0$  having a compact support, i.e.  $\varphi_0(x_1) \equiv 0$  for  $|x_1| \geq R_0 > 0$ , and assume that the direction  $\mathbf{e}_g$

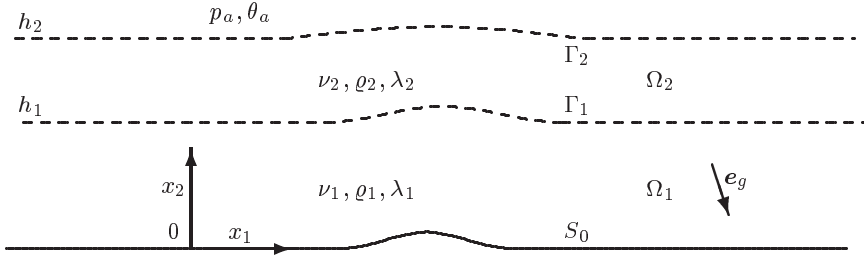


Figure 1. Flow domain of a nonisothermal two-fluid inclined film flow.

of the gravity is the vector  $\mathbf{e}_g = (\sin \alpha, -\cos \alpha)^T$  which makes (with respect to the given co-ordinate system) an angle  $\alpha^* := \pi/2 - \alpha$  ( $0 < \alpha \leq \pi/2$ ) with the  $x_1$ -axis. Note that the corresponding problem will be formulated in dimensionless form. The concrete transition to that formulation can be found in [20].

Now let us explain the problem in more detail. We are concerned with the two-fluid flow down the inclined bottom  $S_0$  caused by gravity  $g \mathbf{e}_g$ , only. This means mathematically that the positive layer thickness at infinity in each liquid layer  $\Omega_i$  ( $i = 1, 2$ ) is *a priori* given. For example, in slide coaters such kind of motions occur on some parts of the coater. The flow fields and layer profiles are essential for associated problems.

Assume that the free interface  $\Gamma_1$  separating the two fluids and the upper free surface  $\Gamma_2$  admit the parametrizations  $\Gamma_i = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \psi_i(x_1), -\infty < x_1 < +\infty\}$  ( $i = 1, 2$ ), where the functions  $\psi_i$  ( $i = 1, 2$ ) are a priori unknown and have to be determined. Let  $h_i > 0$  ( $0 < h_1 < h_2$ ) be the prescribed constant limits of  $\psi_i(x_1)$  ( $i = 1, 2$ ), at infinity. The problem under study has the following form: to find a vector of velocity  $\mathbf{v} = (v_1(x_1, x_2), v_2(x_1, x_2))^T$ , a pressure  $p(x_1, x_2)$ , a temperature  $\theta(x_1, x_2)$  and functions  $\psi_i(x_1)$  ( $i = 1, 2$ ) satisfying in the domain  $\Omega = \Omega_1 \cup \Omega_2$  with  $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 : \varepsilon \varphi_0(x_1) < x_2 < \psi_1(x_1), -\infty < x_1 < +\infty\}$  and  $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 : \psi_1(x_1) < x_2 < \psi_2(x_1), -\infty < x_1 < +\infty\}$  the following system of equations for a coupled heat- and mass transfer process

$$\begin{cases} (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \nabla^2 \mathbf{v} + \frac{1}{\rho} \nabla p = g \mathbf{e}_g, \\ \nabla \cdot \mathbf{v} = 0, \\ (\mathbf{v} \cdot \nabla) \theta - \lambda \nabla^2 \theta = 0, \end{cases} \tag{1.1}$$

and the boundary conditions on  $S_0$

$$\mathbf{v}|_{S_0} = \mathbf{0}, \quad \theta|_{S_0} = \theta_0. \tag{1.2}$$

The boundary conditions on the free interface  $\Gamma_1$  are the following ones

$$\begin{cases} [\theta]_{\Gamma_1} = 0, & [\mathbf{v}]_{\Gamma_1} = \mathbf{0}, \\ \mathbf{v} \cdot \mathbf{n}|_{\Gamma_1^-} = \left[ \lambda \frac{\partial \theta}{\partial n} \right]_{\Gamma_1}, & [\boldsymbol{\tau} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]_{\Gamma_1} = 0, \\ \frac{d}{dx_1} \frac{\psi'_1(x_1)}{\sqrt{1 + \psi'_1(x_1)^2}} = \frac{1}{\sigma_1(\theta)} [-p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}]_{\Gamma_1}, \\ \lim_{|x_1| \rightarrow +\infty} \psi_1(x_1) = h_1. \end{cases} \quad (1.3)$$

Finally, at the free surface  $\Gamma_2$  we are given the boundary conditions

$$\begin{cases} \mathbf{v} \cdot \mathbf{n}|_{\Gamma_2} = \left( \theta - \theta_a + \lambda_2 \frac{\partial \theta}{\partial n} \right)_{\Gamma_2}, & \boldsymbol{\tau} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n}|_{\Gamma_2} = 0, \\ \frac{d}{dx_1} \frac{\psi'_2(x_1)}{\sqrt{1 + \psi'_2(x_1)^2}} = \frac{1}{\sigma_2(\theta)} (p_a - p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v}) \mathbf{n})_{\Gamma_2}, \\ \lim_{|x_1| \rightarrow +\infty} \psi_2(x_1) = h_2. \end{cases} \quad (1.4)$$

It is well-known (cf. [3]) that for a large number of fluids the surface tensions  $\sigma_i$  can be represented as linear functions of the temperature  $\theta$  along the free interface  $\Gamma_i$  ( $i = 1, 2$ ) (cf. also [13, 19]) as follows

$$\sigma_i(\theta) = a_i - b_i \theta \quad (a_i, b_i > 0; i = 1, 2). \quad (1.5)$$

Under  $\lambda_m$  we understand the thermal conductivity of the  $m$ -th fluid ( $m = 1, 2$ ) in Problem (1.1)–(1.4). The symbol  $g$  means the acceleration of gravity. The value  $\theta_0$  denotes the (constant) given temperature of the wall  $S_0$ . Without loss of generality one can assume that  $\theta_0 = 0$  and that  $\theta$  is in fact the difference between the physical temperature and  $\theta_0$ . With  $p_a$  and  $\theta_a$  we mention the given (constant) pressure and temperature of the ambient air, respectively.

Further, the following notations have been used:  $\mathbf{n}$  and  $\boldsymbol{\tau}$  are unit vectors normal and tangential to  $\Gamma_1$  and oriented as  $x_2, x_1$ , respectively. By  $\mathbf{a} \cdot \mathbf{b}$  we denote the inner product of  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ ,  $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^T$  is the gradient operator,  $\nabla p = \text{grad } p$ ,  $\nabla \cdot \mathbf{v} = \text{div } \mathbf{v}$ ,  $\varrho|_{\Omega_m} = \varrho_m$  ( $m = 1, 2$ ) declares the restriction of  $\varrho$  to  $\Omega_m$  (analogously for  $\nu$  and  $\lambda$ ).  $\nabla^2$  denotes the Laplace operator. By  $\mathbf{S}(\mathbf{v})$  we denote the deviatoric stress tensor, i.e. a matrix with elements  $S_{ij}(\mathbf{v}) = 0.5\varrho\nu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$  ( $i, j = 1, 2$ ). The symbol  $[w]_{\Gamma_1}$  is called the jump of  $w$  crossing the free interface  $\Gamma_1$ , that is,

$$[w(\mathbf{x}_0)]_{\Gamma_1} := \lim_{\mathbf{y} \rightarrow \mathbf{x}_0} w(\mathbf{y}) - \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} w(\mathbf{x}) \quad (\mathbf{x}_0 \in \Gamma_1, \mathbf{y} \in \Omega_1, \mathbf{x} \in \Omega_2), \quad (1.6)$$

and the symbol  $w|_{\Gamma_1^-}$  declares the limit from below at the interface  $\Gamma_1$ ; more precisely

$$w(\mathbf{x}_0)|_{\Gamma_1^-} := \lim_{\mathbf{y} \rightarrow \mathbf{x}_0} w(\mathbf{y}) \quad (\mathbf{x}_0 \in \Gamma_1, \mathbf{y} \in \Omega_1). \quad (1.7)$$

Note that the left-hand side of (1.3)<sub>5</sub> (i.e. of the fifth equation in (1.3)) is equal to the curvature  $K_1(x_1)$  of  $\Gamma_1$ . The same is true for  $K_2(x_1)$  in case of  $\Gamma_2$ . Furthermore, Equations (1.3)<sub>3</sub> and (1.4)<sub>1</sub> represent the mathematical expressions of evaporation effects.

## 2 General Solution Scheme

Free boundary value problems for the stationary Navier–Stokes equations or their modifications were the topic of many mathematicians. Numerous references in this field can be found, e.g., in the bibliographies of [9, 12, 23, 24]. In the mathematical papers [6, 7, 10, 13, 15, 19, 21, 22] the dependence on temperature was additionally taken into account. Some of them are devoted to problems with thermocapillary convection whereas others include the effect of evaporation. In [25] a one-dimensional approximation of an evaporation problem was analytically studied. Computational investigations of nonisothermal free boundary problems are described in the studies [3, 14, 20] and many others.

For free boundary value problems where the unknown flow domain is unbounded in two directions as in Problem (1.1)–(1.4) a special linearization procedure was necessary (cf. [9, 12] and others).

Solving such kind of problems – in [9], and independently in [1], an appropriate procedure was proposed which is based on a linearization of the original problem on a corresponding exact solution in the unperturbed “uniform” flow domain, say  $\Pi = \{\mathbf{x} \in \mathbb{R}^2: 0 < x_2 < h_1 \vee h_1 < x_2 < h_2\}$ . The difference of this technique from previous applied ones is that on each step of iterations the determination of  $\mathbf{v}$ ,  $p$ ,  $\theta$  is not separated from the determination of the free boundaries  $\Gamma_i$  ( $i = 1, 2$ ) (i.e. from the determination of the functions  $\psi_i$  describing  $\Gamma_i$ ). For Problem (1.1)–(1.4) this scheme can be characterized by the diagram

$$(\mathbf{v}^0, p^0, \theta^0, \psi_1^0, \psi_2^0) \rightarrow (\mathbf{v}^1, p^1, \theta^1, \psi_1^1, \psi_2^1) \rightarrow \dots \rightarrow (\mathbf{v}^m, p^m, \theta^m, \psi_1^m, \psi_2^m) \rightarrow \dots$$

where on each step of iterations the linearized problem is solved in the same “strip-like” domain and the functions  $\mathbf{v}$ ,  $p$ ,  $\theta$  and  $\psi_i$  ( $i = 1, 2$ ) are determined simultaneously. Note that the superscripts in the above formula denote the number of the corresponding iteration step.

An important part in the derivation of the correct linearization takes the calculation of exact solutions of the nonlinear problems in a “uniform” (not perturbed) flow domain. These exact (basic) solutions in the uniform domain  $\Pi$  will be determined in Section 4 – Basic solutions. They are also useful for the numerical flow simulation: they can be used as inlet boundary data in more complicated problems. In [12] the analogous isothermal problem (without any inclusion of temperature) to Problem (1.1)–(1.4) was solved by numerical methods. Finally, in [22] a problem was studied which has a lot of common features with the presented one. Instead of evaporation thermocapillary convection of Marangoni type was considered there.

## 3 Function Spaces

When studying Problem (1.1)–(1.4) it is useful to work with weighted Sobolev spaces. Let  $\Pi_m$  ( $m = 1, 2$ ) be the strip-like domains

$$\begin{aligned} \Pi_1 &:= \{\mathbf{x} \in \mathbb{R}^2: 0 < x_2 < h_1, -\infty < x_1 < +\infty\}, \\ \Pi_2 &:= \{\mathbf{x} \in \mathbb{R}^2: h_1 < x_2 < h_2, -\infty < x_1 < +\infty\}, \end{aligned}$$

and  $\Pi = \Pi_1 \cup \Pi_2$  their union. We introduce the space  $W_\beta^{l,2}(\Pi)$  of functions  $\mathbf{u}$  on  $\Pi$  with restrictions  $\mathbf{u}^{(m)} = \mathbf{u}|_{\Pi_m}$  belonging to  $W_\beta^{l,2}(\Pi_m)$  ( $m = 1, 2$ ) and having the finite norms

$$\|\mathbf{u}^{(m)}; W_\beta^{l,2}(\Pi_m)\| = \|\mathbf{u}^{(m)} \exp(\beta\sqrt{1+x_1^2}); W^{l,2}(\Pi_m)\| \quad (m = 1, 2),$$

where  $W^{l,2}(\Pi_m)$  is the usual Sobolev space. The norm in  $W_\beta^{l,2}(\Pi)$  is given by

$$\|\mathbf{u}; W_\beta^{l,2}(\Pi)\| = \sum_{m=1}^2 \|\mathbf{u}^{(m)} \exp(\beta\sqrt{1+x_1^2}); W^{l,2}(\Pi_m)\|.$$

If  $\beta > 0$ , then elements of  $W_\beta^{l,2}(\Pi)$  vanish exponentially as  $|x_1| \rightarrow \infty$  and, if  $\beta < 0$ , then elements  $\mathbf{u} \in W_\beta^{l,2}(\Pi)$  might exponentially increase as  $|x_1| \rightarrow \infty$ .

The spaces  $W_\beta^{l-1/2,2}(\mathbb{R})$  of functions defined on  $\mathbb{R}$  can be introduced analogously. Let  $S = \{x \in \Pi: x_1 \in \mathbb{R}, x_2 = h \in \{0, h_1, h_2\}\}$  be a line. Denote by  $W_\beta^{l-1/2,2}(S)$  the spaces of traces on  $S$  of functions from  $W_\beta^{l,2}(\Pi)$ . Then  $W_\beta^{l-1/2,2}(\mathbb{R})$  coincides with  $W_\beta^{l-1/2,2}(S)$ , i.e. if  $\mathbf{u} \in W_\beta^{l-1/2,2}(\Pi)$ , then  $\mathbf{u}(\cdot, h) \in W_\beta^{l-1/2,2}(\mathbb{R})$ .

In the paper the spaces of scalar and vector-valued functions are not distinguished in notations. The norm for vector-valued functions is then the sum of the norms of the corresponding coordinate functions.

## 4 Basic Flows

Firstly, let us introduce the flow parameter  $r := \nu_1 \varrho_1 / (\nu_2 \varrho_2)$  denoting the ratio of dynamic viscosities. In order to obtain one exact solution to Problem (1.1)–(1.4) in the uniform (not perturbed) domain  $\Pi$  which is in fact an unbounded double-strip (cf. Section 3) we make the subsequent assumptions

$$v_2 \equiv 0, \quad \frac{\partial p}{\partial x_1} \equiv 0, \quad \frac{\partial \theta}{\partial x_1} \equiv 0.$$

The second assumption reflects the circumstance that the fluid flow is caused by gravity, only. The first equation together with the continuity condition imply  $\partial v_1 / \partial x_1 = 0$ . Under these assumptions one gets the following reduced nonisothermal Navier–Stokes system in  $\Pi$

$$-\nu \varrho \nabla^2 v_1 = \varrho g \sin \alpha, \quad \frac{\partial p}{\partial x_2} = -\varrho g \cos \alpha, \quad -\lambda \nabla^2 \theta = 0, \quad (4.1)$$

equipped with the corresponding boundary conditions. The three equations from (4.1) are independent of each other. Finally, we obtain three independent boundary value problems for some ordinary differential equations. The first problem contains the unknown velocity component  $v_1(x_2)$  and takes the form

$$\left\{ \begin{array}{l} \nu_1 \varrho_1 \frac{d^2 v_1^{(1)}}{dx_2^2} = -\varrho_1 g \sin \alpha, \quad \nu_2 \varrho_2 \frac{d^2 v_1^{(2)}}{dx_2^2} = -\varrho_2 g \sin \alpha, \\ v_1^{(1)}(0) = 0, \quad \left. \frac{dv_1^{(2)}}{dx_2} \right|_{x_2=h_2} = 0, \\ v_1^{(1)}|_{x_2=h_1} = v_1^{(2)}|_{x_2=h_1}, \quad \nu_1 \varrho_1 \left. \frac{dv_1^{(1)}}{dx_2} \right|_{x_2=h_1} = \nu_2 \varrho_2 \left. \frac{dv_1^{(2)}}{dx_2} \right|_{x_2=h_1}. \end{array} \right. \quad (4.2)$$

The next boundary value problem relates to the unknown pressure field  $p(x_2)$ :

$$\left\{ \begin{array}{l} \frac{dp^{(1)}}{dx_2} = -\varrho_1 g \cos \alpha, \quad \frac{dp^{(2)}}{dx_2} = -\varrho_2 g \cos \alpha, \\ p^{(1)}|_{x_2=h_1} = p^{(2)}|_{x_2=h_1}, \quad p^{(2)}|_{x_2=h_2} = p_a, \end{array} \right. \quad (4.3)$$

The last problem was established for the unknown temperature field  $\theta(x_2)$ :

$$\left\{ \begin{array}{l} \frac{d^2 \theta^{(1)}}{dx_2^2} = 0, \quad \frac{d^2 \theta^{(2)}}{dx_2^2} = 0, \\ \theta^{(1)}|_{x_2=0} = \theta_0, \quad \left( \theta^{(2)} - \theta_a + \lambda_2 \frac{d\theta^{(2)}}{dx_2} \right) \Big|_{x_2=h_2} = 0, \\ \theta^{(1)}|_{x_2=h_1} = \theta^{(2)}|_{x_2=h_1}, \quad \lambda_1 \left. \frac{d\theta^{(1)}}{dx_2} \right|_{x_2=h_1} = \lambda_2 \left. \frac{d\theta^{(2)}}{dx_2} \right|_{x_2=h_1}, \end{array} \right. \quad (4.4)$$

The solution to the systems (4.2)–(4.4) sometimes in the literature is called Nusselt solution of inclined film flow. It can be calculated in a straightforward manner. The solution for the velocity takes the form

$$v_1^0(x_2) = \begin{cases} g \sin \alpha \left[ -\frac{1}{2\nu_1} x_2^2 + \left( \frac{h_2 - h_1}{r\nu_2} + \frac{h_1}{\nu_1} \right) x_2 \right], & 0 \leq x_2 \leq h_1, \\ g \sin \alpha \left[ -\frac{1}{2\nu_2} (h_2 - x_2)^2 + \frac{1}{2\nu_2} (h_2 - h_1)^2 \right. \\ \left. + \frac{1}{2\nu_1} h_1^2 + \frac{1}{r\nu_2} (h_2 - h_1) h_1 \right], & h_1 \leq x_2 \leq h_2. \end{cases} \quad (4.5)$$

The determined pressure is described by

$$p^0(x_2) = \begin{cases} [(h_1 - x_2)\varrho_1 + (h_2 - h_1)\varrho_2] g \cos \alpha + p_a, & 0 \leq x_2 \leq h_1, \\ (h_2 - x_2)\varrho_2 g \cos \alpha + p_a, & h_1 \leq x_2 \leq h_2. \end{cases} \quad (4.6)$$

Finally, the calculated temperature can be written as

$$\theta^0(x_2) = \begin{cases} \frac{\theta_a \lambda_2}{\lambda_2 h_1 + \lambda_1 (h_2 - h_1) + \lambda_1 \lambda_2} x_2, & 0 \leq x_2 \leq h_1, \\ \frac{\theta_a [\lambda_1 x_2 + h_1 (\lambda_2 - \lambda_1)]}{\lambda_2 h_1 + \lambda_1 (h_2 - h_1) + \lambda_1 \lambda_2}, & h_1 \leq x_2 \leq h_2. \end{cases} \quad (4.7)$$

The formula (4.7) was obtained under the assumption  $\theta_0 = 0$  which can be made without loss of generality. The above mentioned solution implies the

subsequent fluxes (i.e. flow rates)

$$F_1 = g \sin \alpha \left[ \frac{1}{3\nu_1} h_1^3 + \frac{1}{2r\nu_2} h_1^2 (h_2 - h_1) \right],$$

$$F_2 = g \sin \alpha \left[ \frac{1}{3\nu_2} (h_2 - h_1)^3 + (h_2 - h_1) \left( \frac{1}{2\nu_1} h_1^2 + \frac{1}{r\nu_2} h_1 (h_2 - h_1) \right) \right],$$

which are small if the inclination angle  $\alpha$  is small.

## 5 Results

**Theorem 1.** *Let  $S_0 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \varepsilon\varphi_0(x_1), -\infty < x_1 < +\infty\}$ ,  $\varphi_0 \in W_\beta^{l+5/2,2}(\mathbb{R})$  with  $l \geq 0$ ,  $\beta = |\beta_0| \sin \alpha > 0$ , where  $\beta_0$  is independent of  $\alpha$  and  $\alpha$  denotes the slope of the inclined bottom  $S_0$ . Assume that  $\alpha$  is sufficiently small. Then there exist real numbers  $\widehat{\varepsilon}, \widehat{r} > 0$  such that for arbitrary  $\varepsilon \in (0, \widehat{\varepsilon})$  the Problem (1.1)–(1.4) has a unique solution  $(\mathbf{v}, p, \theta, \psi_1, \psi_2)^T$  characterized by the structure*

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^0(\mathbf{x}) + \varepsilon \mathbf{u}(\mathbf{x}), \quad p(\mathbf{x}) = p^0(\mathbf{x}) + \varepsilon q(\mathbf{x}), \quad \theta(\mathbf{x}) = \theta^0(\mathbf{x}) + \varepsilon \vartheta(\mathbf{x}),$$

$$\psi_1(x_1) = h_1 + \varepsilon \Psi_1(x_1), \quad \psi_2(x_1) = h_2 + \varepsilon \Psi_2(x_1),$$

where  $\{\mathbf{v}^0, p^0, \theta^0\}$  denotes the basic solution (4.5)–(4.7) and the regular part  $\mathbf{U}$  has the representation

$$\mathbf{U} := (\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2)^T \in [W_\beta^{l+2,2}(\Pi)]^2 \times W_\beta^{l+1,2}(\Pi) \times W_\beta^{l+2,2}(\Pi)$$

$$\times [W_\beta^{l+5/2,2}(\mathbb{R})]^2 \equiv \mathfrak{D}_\beta^{l,2} W(\Pi).$$

Furthermore, the following inequalities hold

$$\|\mathbf{U}; \mathfrak{D}_\beta^{l,2} W(\Pi)\| \leq \widehat{r}, \quad \widehat{\varepsilon} \leq \text{const} \cdot \sin^2 \alpha.$$

Let us shortly describe the main steps and ideas of the proof. We realize the proof by successive approximations. In a first step the original (perturbed) and unknown flow domain  $\Omega$  (cf. Figure 1) is transformed onto the uniform (strip-like) domain  $\Pi$ . Then, by using the transformation mapping, the original flow Problem (1.1)–(1.4) is linearized over the basic solution (4.5)–(4.7) in domain  $\Pi$ . By  $\mathfrak{N}$  we assign the operator on the left-hand side of the corresponding linearized auxiliary problem. Thanks to a related theorem for the linear auxiliary problem (see, e.g., [12]), there exists a bounded inverse operator  $\mathfrak{N}^{-1}$  such that

$$\mathfrak{N}^{-1} : \mathfrak{N}_\beta^{l,2} W(\Pi) \longmapsto \mathfrak{D}_\beta^{l,2} W(\Pi), \quad (5.1)$$

with  $\beta = |\beta_0| \sin \alpha$ . The value  $|\beta_0|$  is independent of  $\alpha$  and depends on eigenvalues of the operator pencils associated with the corresponding linear problem (cf. [12]). Furthermore, the multidimensional space  $\mathfrak{N}_\beta^{l,2} W(\Pi)$  to which the right-hand side of the linearized problem belongs can be described in a similar way as the space  $\mathfrak{D}_\beta^{l,2} W(\Pi)$  (see above). Moreover, one is able to prove that there holds the inequality

$$\|\mathfrak{N}^{-1}\| \leq \frac{C}{\sin \alpha}, \quad (5.2)$$

where the constant  $C$  does not depend on  $\alpha$ . Therefore, Problem (1.1)–(1.4) is equivalent to the subsequent operator equation in the space  $\mathfrak{D}_\beta^{l,2}W(\Pi)$ :

$$U = \mathfrak{N}^{-1}\mathfrak{F}(U) \equiv \mathfrak{K}(U), \quad (5.3)$$

where  $U = (\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2)^T$ , and  $\mathfrak{F}(U) = (f_1(\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2), f_2(\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2), 0, f_3(\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2), \dots)^T$  denotes the long vector of right-hand side after the linearization. The coordinates of the right-hand side vector depend on  $\varepsilon$  via the transformation mapping of the original flow domain. To prove the convergence of the successive approximations  $(\mathbf{u}^n, q^n, \vartheta^n, \Psi_1^n, \Psi_2^n)^T$ , it is sufficient to check that the operator  $\mathfrak{K}$  is a contraction mapping in a ball of the space  $\mathfrak{D}_\beta^{l,2}W(\Pi)$  for small values of  $\varepsilon$  and  $\alpha$ .

Let us remark that the related isothermal problem to Problem (1.1)–(1.4) (i.e. without any inclusion of temperature) was analytically investigated in full detail in the papers [11, 12]. In order to prove Theorem 1 in details, one has to repeat and to modify all the considerations from those papers. Since the temperature equation is a nonlinear elliptic one, there are not essential changes in the proof. Thus we omit the detailed proof here.

## Acknowledgment

The author is grateful for the comments of an anonymous referee.

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