Testing for Jumps and Cojumps in Financial Markets

by

Cheng Ju

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Author ....
School of Engineering
January 15, 2010

Certified by ....
Scott Joslin
Assistant Professor of Finance
Thesis Supervisor

Accepted by ....
Karen Willcox
Associate Professor of Aeronautics and Astronautics
Codirector, Computation for Design and Optimization Program

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Abstract

In this thesis, we introduce a new testing methodology to detect cojumps in multi-asset returns. We define a cojump as a jump in at least one dimension of the return processes. For a multivariate process that follows a semimartingale, and with no other specific assumptions on the process, we form a test statistic which can easily disentangle jumps from continuous paths of the process. We prove that the test statistics are chi-square distributed in the absence of jumps in any dimensions. We propose a hypothesis testing based on the extreme distribution of the test statistics. If the test statistic observed is beyond the extreme level, then most likely, a cojump occurs.

Monte Carlo simulation is performed to access the effectiveness of the test by examining the size and power of the test. We apply the test to a pair of empirical asset returns data and the findings of jump timing are consistent with existing literature.

Thesis Supervisor: Scott Joslin
Title: Assistant Professor of Finance

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Chapter 1

Introduction

1.1 Motivation

Standard arbitrage-based pricing theories assume that the asset prices follow a continuous path. While empirically, significant discontinuities are often found in the evolution of asset prices. The discontinuities are called jumps, which are empirically hard to identify, because only discrete time data are available in continuous-time models. As a result, finding jumps have a significant impact on asset pricing and portfolio and risk management. Besides, in financial markets, the movements of asset prices are usually correlated, and their processes are collectively modeled. The correlations render it even harder to identify a jump in the multivariate price processes. Our goal in this thesis is to find some statistical methods that can detect jumps in multivariate price processes.

1.2 Literature Review

Identifying jumps in a process has been explored by a number of literature. Among the pioneers, Andersen, Benzoni, and Lund (2002) developed a parametric model by adding Poisson jumps with time-varying intensity to stochastic volatility diffusions. They applied the model to find jumps in daily equity-index returns. At the same time, Aït-Sahalia (2002) imposed a criterion function on the transaction densities of diffusion, and identified jumps from discrete-time sampled data. Barndorff-Nielsen and Shephard (2004) introduced a new measure of variation called bipower variation, which was proved robust to a finite number of jumps and can separate the continuous part from the quadratic variation. Later, Barndorff-Nielsen and Shephard (2006a) (BNS) proposed some nonparametric tests based on an asymptotic distribution theory. Jiang and Oomen (2005) (JO) constructed a test based on the hedging error of a variance swap replication strategy. Lee and Mykland (2008) also introduced a nonparametric test which was accessed by simulation that outperforms the tests by BNS and JO. All these nonparametric tests are for determining jumps in univariate price process.

Research on determining jumps in multivariate price processes, only started recently, when Bollerslev, Law and Tauchen (2008) tested non-diversifiable jumps in a large portfolio of stocks. Jacod and Todorov (2009) considered a bivariate process and tested whether they had at least one jump occurring at the same time. However, to the best of our knowledge, none of the existing literature has so far proposed a method to detect the existence of at least one jump in multivariate price processes (defined as cojump) by taking the correlation into consideration. In the thesis, we concentrate on the problem of determining whether there is a cojump at time t_i in multi-asset returns, which follow multivariate processes.

1.3 Contribution and Organization

To develop a cojump detection technique, we first find a consistent estimator to the local covariation of the multivariate returns. We prove its consistency in estimating the local covariance matrix when the discrete time interval is close to 0. With this estimator, we design a test statistic that standardizes the return vector by the local covariation. We prove that in the absence of jumps at a time t_i, the test statistic $\mathcal{L}(t_i)$ follows a chi-square distribution with degrees of freedom the same as the dimension of the multivariate processes. We study the extreme distribution of the test statistic and propose a nonparametric test to infer jumps. Monte Carlo simulation is designed to examine the effectiveness of our test (we call it multivariate test in the rest of the thesis).

The thesis is organized as follows. Chapter 2 studies the test of detecting jumps in univariate price processes and lays out the background ideas for Chapter 3. Chapter 3 develops a new test to detect cojumps in multivariate processes. Monte Carlo simulation is presented in Chapter 4. Chapter 5 applies the tests to an empirical example and discuses some characteristics of jumps and cojumps. Concluding remarks are followed in Chapter 6.

Chapter 2

Univariate Jump Test

For a single asset, when a jump occurs at time t_i, the change in the prices at that time is expected to be much greater than in regular continuous settings. If we could observe the price movements in continuous times, a jump can be easily detected by identifying the continuous path of the prices. However, in reality, asset prices are observed discretely. In discrete times, a large price change at time t_i might be resulted from two cases: 1. a jump arrives at that time; 2. there is no jump, but the instantaneous volatility is high, pushing up the movements of the prices. Thus, identifying jumps by simply looking at the observed realized returns overestimates the number of jumps. To eliminate the second case, the realized return is standardized by a measure that explains the local variation only from the continuous part of the process.

In this chapter, we present the univariate jump detection technique introduced by Lee and Mykland (2008) to identify jump arrival times and realized jump sizes in single asset returns. Bipower variation is employed to estimate the local variation. A test statistic is then formed based on the idea above. We will derive the asymptotic distribution of the test statistic, and utilize it to detect jumps.

2.1 Notation and Definition

The following notations are used throughout the thesis.

t	The time between 0 and the end of a fixed time horizon T
$(\Omega, \mathcal{F}_t, \mathcal{P})$	A fixed complete probability space
$S(t)$	The price of a single asset at t under $\mathcal{P}$
$d\log S(t)$	The continuously compounded asset return
$W(t)$	A $\mathcal{F}_t$-adapted standard Brownian motion
Δt	The time difference between equally spaced discrete observation times t_i, i.e., $\Delta t = t_i - t_{i-1}$
O_p	Bounded in probability, following Pollard (2001), it means that, for random vectors $\{Z_n\}$ and non-negative random variables $\{b_n\}$, $Z_n = O_p(b_n)$, if for each $\delta > 0$, there exists a finite constant M_δ such that $P(\lvert Z_n\rvert > M_\delta d_n) < \delta$ as $n \to \infty$
$\text{plim}_{n\to\infty}$	The probability limit operator, denoting convergence in probability
$[Y, Y]_t$	The quadratic variation at time t
$\{Y, Y\}_t$	The bipower variation at time t
$[Y_\delta, Y_\delta]_t$	The realized quadratic variation at time t
$\{Y_\delta, Y_\delta\}_t$	The realized bipower variation at time t

2.2 Problem Description

We are interested in finding a jump in single asset returns at time t_i. No assumptions are made about the jump dynamics before or after t_i. The asset price $S(t)$ follows the process below on the probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$, where $\mathcal{F}_t$ is a right-continuous information filtration for market participants, and $\mathcal{P}$ is a data-generating measure.

When there are no jumps in the market, the process is

$$d \log S(t) = \mu(t)dt + \sigma(t)dW(t), \tag{2.1}$$

where $\mu(t)$ and $\sigma(t)$ are $\mathcal{F}_t$-adapted processes, such that the underlying process is an Itô process that has continuous sample paths.

When there are jumps, $S(t)$ is represented as

$$d \log S(t) = \mu(t)dt + \sigma(t)dW(t) + C(t)dJ(t), \tag{2.2}$$

where $J(t)$ is a counting process independent of $W(t)$. $C(t)$ is the jump size. The jump sizes $C(t_1), C(t_2), \ldots, C(t_n)$ are assumed to be independent of each other and identically distributed. They are also independent of other random components $W(t)$ and $J(t)$. The asset prices are observed in discrete time at t_i, for $i = 0, 1, \cdots, n$, with $0 = t_0 < t_1 < \cdots < t_n = T$. We impose the following necessary assumption on single asset price processes.

Assumption 1. *For any $\epsilon > 0$,*

$$\sup_i \sup_{t_i \leq u \leq t_{i+1}} |\mu(u) - \mu(t_i)| = O_p(\Delta t^{\frac{1}{2}-\epsilon}), \tag{2.3}$$

$$\sup_i \sup_{t_i \leq u \leq t_{i+1}} |\sigma(u) - \sigma(t_i)| = O_p(\Delta t^{\frac{1}{2}-\epsilon}). \tag{2.4}$$

The drift term $\mu(t)$ is ignored as it is mathematically negligible for high-frequency data.

From (2.1), in the absence of jumps, the realized instantaneous returns $r(t_i)$ can be approximated by $\log \frac{S(t_i)}{S(t_{i-1})}$. Absolute returns that are too large are plausible to indicate jumps. However, we need to exclude the time when actually there is no jump, but the high return is imposed by a high volatility. This is achieved by standardizing the return using the instantaneous volatility which only counts the continuous part of the process. The standardized return creates the test statistic for detecting jumps. Using this test statistic, we could decide whether the diffusion model (2.1) is rejected.

A jump is found if the model is rejected.

The problem remains: 1. how to consistently estimate the instantaneous volatility; 2. how could the rejection rule be implemented; 3. how powerful the detection method is. We will answer these questions in the following sections.

2.3 Estimation of Volatility

How can we estimate the instantaneous volatility? The realized quadratic variation is commonly used as a nonparametric estimator for the variance. It is defined as the sum of squared returns

$$[Y_\delta]_{t_k} \equiv [Y_\delta, Y_\delta]_{t_k} = \sum_{i=1}^{k} r^2(t_i). \tag{2.5}$$

Barndorff-Nielsen and Shephard (2006a) showed that, for any sequence of partitions $t_0 = 0 < t_1 < \cdots < t_k = t$ as long as $\sup_j \{t_{j+1} - t_j\} \to 0$ for $k \to \infty$, $[Y]_{t_k}$ consistently estimates the total variation comprised of the integrated variance plus the sum of the squared jumps

$$[Y, Y]_{t_k} = \text{plim}_{k\to\infty}[Y_\delta]_{t_k} = \int_{t_0}^{t_k} \sigma^2(s)ds + \sum_i J_i, \tag{2.6}$$

where J_i is the size of the ith jump. Unfortunately, (2.6) reveals that the realized quadratic variation is inconsistent in estimating the continuous part of the local variation in the presence of jumps in a return process.

Instead, a partial generalization of quadratic variation called bipower variation (BV) introduced by Barndorff-Nielsen and Shephard (2004) has shown to be a consistent estimator of the instantaneous volatility even in the presence of jumps (refer to Barndorff-Nielsen and Shephard (2004) and Aït-Sahalia (2004) for details). It is defined as

$$\{Y\}_{t_k} \equiv \{Y, Y\}_{t_k} = \text{plim}_{k\to\infty} \sum_{i=2}^{k} |r(t_i)|\,|r(t_{i-1})|\,. \tag{2.7}$$

Similar to the quadratic variation, the bipower variation can be consistently estimated by the realized bipower variation

$$\{Y_\delta\}_{t_k} \equiv \{Y_\delta, Y_\delta\}_{t_k} = \sum_{i=2}^{k} |r(t_i)|\,|r(t_{i-1})|\,. \tag{2.8}$$

Barndorff-Nielsen and Shephard (2004) showed that $\{Y\}_{t_k} = c^2 \int_{t_0}^{t_k} \sigma^2(s)ds$, with $c = E|U| = \sqrt{2/\pi}$, where $U \sim N(0,1)$. It is obvious that the bipower variation is independent of the presence of jumps over time. Therefore, the realized bipower variation is an consistent estimator of the instantaneous volatility.

2.4 Limit Distribution of Test Statistics

2.4.1 Formulation of Test Statistic

With the consistent estimator of the instantaneous volatility, we can form the test statistic $\mathcal{L}(t_i)$ – the standardized return. We define the test statistic as follows:

Definition 1. The statistic $\mathcal{L}(t_i)$, which tests at time t_i whether there was a jump in the asset return from t_{i-1} to t_i, is defined as

$$\mathcal{L}(t_i) \equiv \frac{r_{t_i}}{\widehat{\sigma(t_{i-1})}}, \tag{2.9}$$

where

$$\widehat{\sigma(t_{i-1})}^2 \equiv \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} |r(t_j)||r(t_{j-1})|. \tag{2.10}$$

Here, we consider a local movement of the process within a window of size K, over which the spot volatility is constant. In this setting, the instantaneous volatility at time t_{i-1} can be approximated by $\widehat{\sigma(t_{i-1})}$, which is the average realized bipower variation over the window. There is a tradeoff in choosing the window size K: K must be large enough to accurately estimate integrated volatility but small enough for the variance to be approximately constant over the window.

2.4.2 Asymptotic Distribution Theory

What would be the distribution of the test statistic when we refine the discretization? Lee and Mykland (2008) shows that under the assumption of no jumps at a particular testing time, the test statistic $\mathcal{L}(t_i)$ follows a normal distribution with mean 0, and variance $\pi/2$. In this subsection, we prove the limiting theory in a clearer and more detailed manner.

Before stating the theorem, we look at two lemmas, which will be helpful to prove the theorem later.

Lemma 1. *For $T > 0, \kappa > 0$, and $\Delta t = T/n$, as $n \to \infty$,*

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n} |x_i + O_p(\Delta t^\kappa)|\,|x_{i-1} + O_p(\Delta t^\kappa)| \\ =&\frac{1}{n}\sum_{i=1}^{n} |x_i|\,|x_{i-1}| + O_p(\Delta t^\kappa).\end{aligned} \tag{2.11}$$

Proof. Since $(|x_i| - \hat{\epsilon}) \leq |x_i + \hat{\epsilon}| \leq (|x_i| + \hat{\epsilon})$ for $\hat{\epsilon} \geq 0$,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} |x_i + \hat{\epsilon}|\,|x_{i-1} + \hat{\epsilon}| &\leq \frac{1}{n}\sum_{i=1}^{n} (|x_i| + \hat{\epsilon})\,(|x_{i-1}| + \hat{\epsilon}) \\ &\leq \frac{1}{n}\sum_{i=1}^{n} |x_i|\,|x_{i-1}| + \hat{\epsilon}\,(|x_i| + |x_{i-1}|) \\ &\leq \frac{1}{n}\sum_{i=1}^{n} |x_i|\,|x_{i-1}| + \frac{2\hat{\epsilon}}{n}\sum_{i=1}^{n} |x_i|\,.\end{aligned}$$

By the weak law of large numbers, $\frac{1}{n}\sum_{i=1}^{n} |x_i|$ converges in probability to its expected value for $n \to \infty$. It implies that $\frac{1}{n}\sum_{i=1}^{n} |x_i| = O_p(1)$. Therefore, as $n \to \infty$, we conclude

$$\frac{1}{n}\sum_{i=1}^{n} |x_i|\,|x_{i-1}| - \hat{\epsilon}O_p(1) \leq \frac{1}{n}\sum_{i=1}^{n} |x_i + \hat{\epsilon}|\,|x_{i-1} + \hat{\epsilon}| \leq \frac{1}{n}\sum_{i=1}^{n} |x_i|\,|x_{i-1}| + \hat{\epsilon}O_p(1).$$

Replacing $\hat{\epsilon}$ by the order in probability, we get the result. □

Lemma 2. *For $U_i \sim$ i.i.d $N(0,1)$,*

$$\frac{1}{n}\sum_{i=1}^{n}|U_i|\,|U_{i-1}| = c^2 + O_p(\Delta t^{\frac{1}{2}}), \tag{2.12}$$

where $c = E|U_i| = \sqrt{2/\pi}$.

Proof. Let $X_{i-1,i} = |U_i||U_{i-1}| - E(|U_i||U_{i-1}|)$. $E(X_{i-1,i}) = 0$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_{i-1,i}$, we have $E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E\left(X_{i-1,i}\right) = 0$, and $\mathrm{var}(\bar{X}_n) = E(\bar{X}_n^2)$.

By Chebyshev's inequality, for any $\alpha > 0$,

$$P(|\bar{X}_n| \geq \alpha) \leq \frac{E(\bar{X}_n^2)}{\alpha^2}. \tag{2.13}$$

The variance of $\bar{X}_n$ is

$$\begin{aligned}
E(\bar{X}_n^2) &= E\left[\frac{1}{n^2}(X_{0,1} + X_{1,2} + \cdots + X_{n-1,n})^2\right] \\
&= \frac{1}{n^2}E\left(\sum_{i=1}^{n} X_{i-1,i}^2 + 2\sum_{i=1}^{n-1} X_{i-1,i}X_{i,i+1}\right) \\
&\quad + \frac{1}{n^2}E\left(2\sum_{i=1}^{n-2} X_{i-1,i}X_{i+1,i+2}\cdots + 2X_{0,1}X_{n-1,n}\right).
\end{aligned}$$

Since $E(X_{i-1,i}X_{j-1,j}) = 0$ for $|i-j| > 1$, and $\sum_{i=1}^{n} E(X_{i-1,i}^2) + \sum_{i=1}^{n-1} E(X_{i-1,i}X_{i,i+1})$ is bounded by $O(n)$,

$$E(\bar{X}_n^2) = \frac{1}{n^2}O(n) \to 0, \qquad \text{as } n \to \infty.$$

It is equivalent to $E[(\sqrt{n}\bar{X}_n)^2] = E(n\bar{X}_n^2) = O(1)$. Apply Chebyshev's inequality again, we have for any finite $M > 0$,

$$P(|\sqrt{n}\bar{X}_n| \geq M) \leq \frac{E[\sqrt{n}\bar{X}_n^2]}{M^2} = \frac{O_p(1)}{M^2} \equiv \gamma. \tag{2.14}$$

By the definition of *bounded in probability*, $\sqrt{n}\bar{X}_n = O_p(1)$, which is equivalent to $\bar{X}_n = O_p(\frac{1}{\sqrt{n}}) = O_p(\Delta t^{\frac{1}{2}})$.

$$\frac{1}{n}\sum_{i=1}^{n} X_{i-1,i} = \bar{X}_n = O_p(\Delta t^{\frac{1}{2}}). \tag{2.15}$$

By the definition of $X_{i-1,i}$, it implies that

$$\frac{1}{n}\sum_{i=1}^{n} |U_i||U_{i-1}| - E(|U_i||U_{i-1}|) = O_p(\Delta t^{\frac{1}{2}}). \tag{2.16}$$

As $U_i \sim$ i.i.d $N(0,1)$, we get

$$\frac{1}{n}\sum_{i=1}^{n} |U_i||U_{i-1}| = E^2(|U_i|) + O_p(\Delta t^{\frac{1}{2}}) = c^2 + O_p(\Delta t^{\frac{1}{2}}). \tag{2.17}$$

□

Theorem 1. *Let $\mathcal{L}(t_i)$ be as in Definition 1 and window size $K = O_p(\Delta t^{\alpha})$, where $-1 < \alpha < -0.5$. Let A_n be the set of $i \in \{1, 2, \ldots, n\}$ so there is no jump at t_i. Then, as $\Delta t \to 0$,*

$$\mathcal{L}(t_i) = \frac{U_i}{c} + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \qquad \text{for } i \in A_n \tag{2.18}$$

where δ satisfies $0 < \delta < \frac{3}{2} + \alpha$ and $U_i = \frac{W(t_i)-W(t_{i-1})}{\sqrt{\Delta t}}$, a standard normal variable and a constant $c = E|U_i| = \sqrt{2/\pi}$.

Proof. For $t \in [t_{i-K}, t_i]$, under Assumption (2.4), similar to Lemma 1 in Mykland and Zhang (2006), we can apply Burkholder's Inequality (Protter (2004)) to get

$$\sup_{i, t \le t_i} \left| \int_{t_{i-K}}^{t} \{\sigma(u) - \sigma(t_{i-K})\}\, dW(u) \right| = O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \tag{2.19}$$

where $0 < \delta < \frac{3}{2} + \alpha$.

Following Lee and Mykland (2008), for $t_{i-K} < t < t_i$, $d\log S(t)$ can be approximated by $d\log S^i(t)$ with

$$d\log S^i(t) = \mu(t_{i-K})dt + \sigma(t_{i-K})dW(t),$$

or

$$d\log S^i(t) = \mu(t_{i-K})dt + \sigma(t_{i-K})dW(t) + Y(t)dJ(t),$$

This is because

$$\begin{aligned}
&\left|\int_{t_{i-K}}^{t} d\log S(t) - \int_{t_{i-K}}^{t} d\log S^i(t)\right| \\
&= \left|(\log S(t) - \log S(t_{i-K})) - \left(\log S^i(t) - \log S^i(t_{i-K})\right)\right| \\
&= \left|\int_{t_{i-K}}^{t} \{\sigma(u) - \sigma(t_{i-K})\}\, dW(u)\right| = O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}). \qquad (2.20)
\end{aligned}$$

For all i, j and $t_j \in [t_{i-K}, t_i]$, the return is

$$\begin{aligned}
r(t_j) &= \log S(t_j) - \log S(t_{j-1}) \\
&= \log S^i(t_j) - \log S^i(t_{j-1}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\
&= \sigma(t_{i-K})(W(t_j) - W(t_{j-1})) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\
&= \sigma(t_{i-K})\sqrt{\Delta t}U_j + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \qquad (2.21)
\end{aligned}$$

where $U_j \sim$ i.i.d. Normal $(0, 1)$.

Look at the denominator of $\mathcal{L}(t_i)$

$$\begin{aligned}
\widehat{\sigma(t_{i-1})}^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} |r(t_j)||r(t_{j-1})| \\
&= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} \left|r^i(t_j) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon})\right| \left|r^i(t_{j-1}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon})\right|.
\end{aligned}$$

(2.22)

By Lemma 1 and (2.21), (2.22) becomes

$$\begin{aligned}\widehat{\sigma(t_{i-1})}^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} \left|r^i(t_j)\right|\left|r^i(t_{j-1})\right| + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\ &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} \sigma^2(t_{i-K})|\sqrt{\Delta t}U_j||\sqrt{\Delta t}U_{j-1}| + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\ &= \sigma^2(t_{i-K})\Delta t\frac{1}{K-2}\sum_{j=i-K+2}^{i-1} |U_j||U_{j-1}| + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}). \qquad (2.23)\end{aligned}$$

Apply Lemma 2 to (2.23),

$$\begin{aligned}\widehat{\sigma(t_{i-1})}^2 &= c^2\sigma^2(t_{i-K})\Delta t + O_p(\Delta t^{\frac{3}{2}}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\ &= c^2\sigma^2(t_{i-K})\Delta t + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \qquad (2.24)\end{aligned}$$

since $O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon})$ dominates $O_p(\Delta^{\frac{3}{2}})$.

Therefore,

$$\mathcal{L}(t_i) = \frac{r_{t_i}}{\widehat{\sigma(t_{i-1})}} = \frac{U_i}{c} + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}). \qquad (2.25)$$

□

In the presence of jumps at time t_i, $\mathcal{L}(t_i) = \frac{U_i}{c} + \frac{Y(\tau)}{c\sigma\sqrt{\Delta t}}$, where $Y(\tau)$ is the actual jump size at time τ (see Lee and Mykland (2008) for detailed proof). As $\Delta t \to 0, \mathcal{L} \to \infty$.

2.5 Detection Rule

After examining the limiting behavior of the test statistic in the absence of jumps and with the presence of jumps, we can possibly detect a jump if the absolute value of the test statistic is too high. However, how high can $|\mathcal{L}(t_i)|$ be when there is no jump? Lee and Mykland (2008) propose inferring jumps from the distribution of maximums of $|\mathcal{L}(t_i)|$. The idea is that if the observed value of $|\mathcal{L}(t_i)|$ is not even within the usual region of maximums, it is unlikely that the realized return is from the diffusion model without jumps.

By Theorem 2.1.3 of Galambos (1978), the maximum of a standard normal random variable follows a Gumble distribution. Under the null hypothesis of no jumps from t_{i-1} to t_i, the sample maximum of $|\mathcal{L}(t_i)|$ also converges to a Gumble distribution.

We state the hypothesis testing formally (see Galambos (1978) for derivation):

Lemma 3. *Define the null hypothesis as there is no jumps from* $(t_{i-1}, t_i]$, *then as* $\Delta t \to 0$,

$$\max_{i \in A_n} |\mathcal{L}(t_i)| \longrightarrow a_n + b_n x, \tag{2.26}$$

where x *has a cumulative distribution function* $P(x \leq X) = \exp(-e^{-x})$,

$$a_n = \frac{(2\log n)^{1/2}}{c} - \frac{\log \pi + \log(\log n)}{2c(2\log n)^{1/2}} \quad \text{and } b_n = \frac{1}{c(2\log n)^{1/2}}, \tag{2.27}$$

where n *is the number of* $\mathcal{L}(t_i)$.

Suppose we set a significance level of α, the null hypothesis is rejected if $|\mathcal{L}(t_i)| > a_n + b_n\theta^*$, where $P(x \leq \theta^*) = \exp(-e^{-\theta^*}) = 1 - \alpha$.

Lee and Mykland (2008) has proven by simulation that the performance of the test outperforms other univariate nonparametric tests introduced by Barndorff-Nielsen and Shephard (2006a) and Jiang and Oomen (2005).

Chapter 3

Multivariate Jump Test

In this chapter, we introduce a new nonparametric test for detecting cojumps in multivariate price processes. We call it multivariate test in distingushionment from the univariate test in Chapter 2. Our new test utilizes the cross-covariance structure of the asset returns.

When we want to test a cojump in multiple assets at time t_i, we set the null hypothesis: there is no jump in the multiple assets at t_i. If the hypothesis is tested at a significance level of α, one possible approach is to apply the univariate test simultaneously on each asset. Since we want to control the total significance level at α, without taking the correlations of the assets into consideration, we might set the significance level of each test to be α/p, if we have p assets in total. If the assets are uncorrelated, this approach gives a significance level of α. However, if some of the assets are correlated, the total significance level is less than α. Thus, we might lose some power or significance in the inference.

In addition, this approach might have a high Type I error (the probability of rejecting the null hypothesis when it is true). Shaffer (1995) shows that if multiple hypotheses are tested, and each test has a specified Type I error , the probability that at least some Type I errors are committed increases with the number of hypotheses. To solve the problem, we must take into account of the correlations between the multiple assets, and propose a new test to identify the cojumps.

Inspired by the idea of the univariate test of a single asset, in discrete times, a large change of the multiple assets processes can be observed in two cases: 1. an actual jump arises in at least one of the assets; 2. no jump occurrs, but the total return is augmented by high covariations between the assets. To distinguish the two cases, we propose to standardize the returns by the instantaneous covariance matrix which accounts the local covariation of the returns from the continuous part of the process.

We use bipower covariation to estimate the covariance between assets. We will prove that it is a robust estimator. With this estimator, we formulate a test statistic and demonstrate the asymptotic behavior of the statistic. A detection technique is innovated by looking at the limiting distribution of the maximum of the test statistic.

3.1 Problem Formulation

We aim to identify a cojump in a multi-asset returns at t_i. Again, we make no assumption about the jump dynamic before or after t_i. Let the prices of the p assets be written as $X(t) = (S_1(t), S_2(t), ..., S_p(t))'$, for $t \geq 0$. We assume the log price vector $\log X(t)$ is a semimartingale on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. When there are no jumps in the market, the continuously compounded return follows

$$d\log X(t) = \mu(t)dt + \sigma(t)dW(t), \tag{3.1}$$

where $\mu(t)$ is an $p \times 1$ vector, and $\sigma(t)$ is an $p \times p$ matrix. The drift $\mu(t)$ and diffusion rate $\sigma(t)$ are $\mathcal{F}_t$ adapted processes and are càdlàg. $W(t)$ is an p-dimensional vector of independent standard Brownian motions. The $\sigma(t)$ matrix is related to the covariance matrix by $\Sigma(t) = \sigma(t)\sigma(t)^T$.

When there are jumps, $S(t)$ is represented as

$$d\log X(t) = \mu(t)dt + \sigma(t)dW(t) + C(t)dJ(t), \tag{3.2}$$

where $J(t)$ is a vector of counting processes independent of $W(t)$. $C(t)$ is the jump size matrix. The jump sizes at different times $C(t_1), C(t_2), \ldots, C(t_n)$ are assumed to be independent of each other and identically distributed. They are also independent of other random components $W(t)$ and $J(t)$.

Following Chapter 2, we assume the observation times are equally spaced with $\Delta t = t_i - t_{i-1}$. We also assume that the drift and diffusion coefficients do not vary dramatically over a short time interval. The price processes follow:

Assumption 2. *For any* $\epsilon > 0$,

$$\sup_i \sup_{t_i \le u \le t_{i+1}} |\mu_l(u) - \mu_l(t_i)| = O_p(\Delta t^{\frac{1}{2}-\epsilon}), \tag{3.3}$$

$$\sup_i \sup_{t_i \le u \le t_{i+1}} |\sigma_{l,k}(u) - \sigma_{l,k}(t_i)| = O_p(\Delta t^{\frac{1}{2}-\epsilon}), \tag{3.4}$$

where $l, k = 1, 2, \ldots, p$.

The drift term is mathematically negligible for high-frequency data, we can ignore it in developing the test.

We could solve the problem in three steps: 1. standardize the returns by the local covariation; 2. find the asymptotic distribution of the standardized return; 3. formulate a hypothesis testing based on the knowledge of the distribution.

In the following sections, we first look for a consistent estimator of the local covariation. We then prove the asymptotic distribution theory of the standardized return. Finally, we introduce our jump detection rule.

3.2 Estimation of Covariance Matrix

Following Barndorff-Nielsen and Shephard (2006b), we estimate the local variation using the bipower variation for multi-dimensional processes. The bipower variation matrix is defined as

$$\{Y\}_t = \begin{bmatrix} \{Y_1\}_t & \{Y_1, Y_2\}_t & \cdots & \{Y_1, Y_p\}_t \\ \{Y_2, Y_1\}_t & \{Y_2\}_t & \cdots & \{Y_2, Y_p\}_t \\ \vdots & \vdots & \ddots & \vdots \\ \{Y_p, Y_1\}_t & \{Y_p, Y_2\}_t & \cdots & \{Y_p\}_t \end{bmatrix}.$$

Denote the $l, l-th$ elements of the matrix as $\{Y_l\}$. It is exactly the same as the bipower variation for univariate case. For any sequence of partitions $t_0 = 0 < t_1 < \cdots < t_k = t$ as long as $\sup_j \{t_{j+1} - t_j\} \to 0$ for $k \to \infty$,

$$\{Y_l\}_t \equiv \{Y_l, Y_l\}_t = \text{plim}_{k\to\infty} \sum_{i=1}^{k} |r_l(t_i)| \, |r_l(t_{i-1})| \tag{3.5}$$

The $l, q-th$ bipower covariance process is defined as:

$$\{Y_l, Y_q\}_t = \frac{1}{4}(\{Y_l + Y_q\}_t - \{Y_l - Y_q\}_t),$$

where

$$\{Y_l \pm Y_q\}_t \equiv \{Y_l \pm Y_q, Y_l \pm Y_q\}_t = \text{plim}_{k\to\infty} \sum_{i=1}^{k} |r_l(t_i) \pm r_q(t_i)| \, |r_l(t_{i-1}) \pm r_q(t_{i-1})| \,. \tag{3.6}$$

Theorem 1 in Barndorff-Nielsen and Shephard (2006b) states that

$$\{Y\}_t = c^2 \int_0^t \Sigma_u du, \tag{3.7}$$

where Σ_u is the covariance matrix at u. This shows that the bipower variation matrix is independent of the jumps in the process. It is thus a consistent estimator of the local covariation from the continuous part of the process.

Since the price is observed in discrete times, we can only estimate the bipower variation using realized bipower variation which can consistently estimate bipower variation according to Barndorff-Nielsen and Shephard (2006b). From the definition of the bipower variation matrix, we find that each element of this matrix can be viewed as the bipower variation in the univariate process. Therefore, the estimation of the matrix boils down to the estimation of the univariate bipower variation.

3.3 Cojump Detection Technique

3.3.1 Formulation of Test Statistic

Analogous to the univariate case, we consider a local movement of the multivariate process within a window of size K, over which the spot covariance matrix is constant. Over this window size, as the realized bipower variation consistently estimates the local variation which is constant over the window. Taking average of the total local variation gives a local variation over a small time interval. If we squeeze the time interval small enough, we obtain a good estimator of the instantaneous covariance. Following the idea, we define our test statistic as:

Definition 2. The statistic $\mathcal{L}(t_i)$ which tests at time t_i whether there was at least a jump in the p-asset returns from t_{i-1} to t_i is defined as

$$\mathcal{L}(t_i) = r(t_i)' \widehat{\Sigma(t_i)}^{-1} r(t_i), \tag{3.8}$$

where

$$\widehat{\Sigma(t_i)} = \begin{bmatrix} \widehat{\Sigma_{11}(t_i)} & \widehat{\Sigma_{12}(t_i)} & \dots & \widehat{\Sigma_{1q}(t_i)} \\ \widehat{\Sigma_{21}(t_i)} & \widehat{\Sigma_{22}(t_i)} & \dots & \widehat{\Sigma_{2q}(t_i)} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\Sigma_{q1}(t_i)} & \widehat{\Sigma_{q2}(t_i)} & \dots & \widehat{\Sigma_{qq}(t_i)} \end{bmatrix}. \tag{3.9}$$

Each element of the matrix is defined as

$$\widehat{\Sigma_{ll}(t_i)} = \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} |r_l(t_j)|\,|r_l(t_{j-1})|\,; \tag{3.10}$$

$$\widehat{\Sigma_{lq}(t_i)} = \frac{1}{4}(\widehat{\Sigma_{l\oplus q}(t_i)} - \widehat{\Sigma_{l\ominus q}(t_i)}), \quad \text{for } l \neq q, \tag{3.11}$$

where

$$\widehat{\Sigma_{l\oplus q}(t_i)} = \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} |r_l(t_j) + r_q(t_j)|\,|r_l(t_{j-1}) + r_q(t_{j-1})|\,;$$

$$\widehat{\Sigma_{l\ominus q}(t_i)} = \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} |r_l(t_j) - r_q(t_j)|\,|r_l(t_{j-1}) - r_q(t_{j-1})|\,.$$

3.3.2 Asymptotic Distribution

Before we derive the limiting distribution of our test statistic, we show that the covariance estimator in Definition 2 indeed converges to the local covariance. We first state a Lemma.

Lemma 4. *For random variables* $\{(U_1, V_1), (U_2, V_2), \cdots, (U_n, V_n)\}$ *which is independent and identically bivariate normally distributed with mean* 0 *and a covariance matrix* $\Sigma = \begin{pmatrix} \Sigma_{uu} & \Sigma_{uv} \\ \Sigma_{vu} & \Sigma_{vv} \end{pmatrix}$, *we have*

$$\frac{1}{n}\sum_{i=1}^{n} |U_i + W_i|\,|U_{i-1} + W_{i-1}| = c^2(\Sigma_{uu} + \Sigma_{vv} + 2\Sigma_{uv}). \tag{3.12}$$

Proof. For each $i \in \{1, \cdots, n\}$, let

$$X_{i-1,i} = |U_i + V_i|\,|U_{i-1} + V_{i-1}| - E\left(|U_i + V_i|\,|U_{i-1} + V_{i-1}|\right).$$

$E\left(X_{i-1,i}\right)$ then becomes 0.

Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_{i-1,i}$, we have $E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E\left(X_{i-1,i}\right) = 0$, and $\mathrm{var}(\bar{X}_n) = E(\bar{X}_n^2)$. Similar to the proof in Lemma 2, we have

$$\frac{1}{n}\sum_{i=1}^{n} X_{i-1,i} = \bar{X}_n = O_p(\Delta t^{\frac{1}{2}}). \tag{3.13}$$

By the definition of $X_{i-1,i}$, it implies that

$$\frac{1}{n}\sum_{i=1}^{n} |U_i + V_i||U_{i-1} + V_{i-1}| - E(|U_i + V_i||U_{i-1} + V_{i-1}|) = O_p(\Delta t^{\frac{1}{2}}). \tag{3.14}$$

As (U_i, W_i) is i.i.d bivariate normal, random variable $U_i + V_i$ is then a univariate normal with mean 0 and variance $\Sigma_{uu} + \Sigma_{vv} + 2\Sigma_{uv}$. Therefore,

$$
\begin{aligned}
E\left(|U_i + W_i|\,|U_{i-1} + W_{i-1}|\right) &= E\left(|U_i + W_i|\right) E\left(|U_{i-1} + W_{i-1}|\right) \\
&= \left[E\left(|U_i + W_i|\right)\right]^2 \\
&= c^2(\Sigma_{uu} + \Sigma_{ww} + 2\Sigma_{uw}). \qquad (3.15)
\end{aligned}
$$

Substituting (3.15) into (3.14), we conclude that

$$
\frac{1}{n}\sum_{i=1}^{n} |U_i + V_i|\,|U_{i-1} + V_{i-1}| = c^2(\Sigma_{uu} + \Sigma_{vv} + 2\Sigma_{uv}) + O_p(\Delta t^{\frac{1}{2}}). \qquad (3.16)
$$

□

From Chapter 2, we confirm that the diagonal elements of the covariance matrix estimator approximate the local variance part of a multi-dimensional process. How about the off-diagonal elements? We state the theorem below and prove that the off-diagonal elements do approximate the local covariance part of the multi-dimensional process.

Theorem 2. *Let $\Sigma(t_i)$ be as in Definition 1 and window size $K = O_p(\Delta t^{\alpha})$, where $-1 < \alpha < -0.5$. Let A_k be the set of $i \in \{1, 2, \ldots, k\}$ so there is no jump at t_i. Then, as $\Delta t \to 0$,*

$$
\widehat{\Sigma_{lq}(t_i)} = c^2\Sigma_{lq}(t_{i-K})\Delta t + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \qquad (3.17)
$$

where δ satisfies $0 < \delta < \frac{3}{2} + \alpha$.

Proof. Similar to the univariate case, for $t_{i-K} < t < t_i$, $d\log X(t)$ can be approximated by $d\log X^i(t)$

$$
d\log X^i(t) = \mu(t_{i-K})dt + \sigma(t_{i-K})dW(t),
$$

or

$$
d\log S^i(t) = \mu(t_{i-K})dt + \sigma(t_{i-K})dW(t) + Y(t)dJ(t).
$$

For $t_j \in [t_{i-K}, t_i]$, the log returns $r(t_j)$ is

$$\begin{aligned} r(t_j) &= \log X(t_j) - \log X(t_{j-1}) \\ &= \log X^i(t_j) - \log X^i(t_{j-1}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\ &= \sigma(t_{i-K})(W(t_j) - W(t_{j-1})) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \\ &= \sqrt{\Delta t} U_j + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \end{aligned} \tag{3.18}$$

where $U_j \sim$ i.i.d. Multivariate Normal with mean 0 and covariance matrix $\Sigma(t_{i-K})$ with size $p \times p$, and $0 < \delta < \frac{3}{2} + \alpha$.

Replacing $r(t_i)$ with (3.18)

$$\begin{aligned} \widehat{\Sigma_{l\oplus q}(t_i)} &= \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} |r_l(t_j) + r_q(t_j)| \, |r_l(t_{j-1}) + r_q(t_{j-1})| \\ &= \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left| r_l^i(t_j) + r_q^i(t_j) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \right| \\ &\quad \left| r_l^i(t_{j-1}) + r_q^i(t_{j-1}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \right| \\ &= \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left| r_l^i(t_j) + r_q^i(t_j) \right| \left| r_l^i(t_{j-1}) + r_q^i(t_{j-1}) \right| + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}), \end{aligned} \tag{3.19}$$

where the last equality follows from Lemma 1. Apply Lemma 4 to (3.19), as $O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon})$ dominates $O_p(\Delta^{\frac{3}{2}})$, $O_p(\Delta^{\frac{3}{2}})$ can be ignored, we get

$$\widehat{\Sigma_{l\oplus k}(t_i)} = c^2[\Sigma_{ll}(t_{i-K}) + \Sigma_{kk}(t_{i-K}) + 2\Sigma_{lk}(t_{i-K})]\Delta t + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}).$$

Similarly, we have

$$\widehat{\Sigma_{l\ominus k}(t_i)} = c^2[\Sigma_{ll}(t_{i-K}) + \Sigma_{kk}(t_{i-K}) - 2\Sigma_{lk}(t_{i-K})]\Delta t + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}).$$

By (3.11), we get the result. □

We therefore confirm that

$$\widehat{\Sigma(t_i)} = c^2\Delta t\Sigma(t_{i-K}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}) \tag{3.20}$$

Apply (3.18) and (3.20) to the test statistic defined in Definition 2, and let γ stands for $O_p(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon})$. Under the assumption that there is no jump in any of the assets, the test statistic is

$$\begin{aligned}
\mathcal{L}(t_i) &= r(t_i)'\widehat{\Sigma(t_i)}^{-1}r(t_i) \\
&= [\sqrt{\Delta t}U_i + \gamma]'[c^2\Sigma(t_i)\Delta t + \gamma]^{-1}[\sqrt{\Delta t}U_i + \gamma] \\
&= [\sqrt{\Delta t}U_i + \gamma]'[c^2\Sigma(t_i)\Delta t]^{-1}[1 - \gamma][\sqrt{\Delta t}U_i + \gamma] \\
&= [\sqrt{\Delta t}U_i + \gamma]'[c^2\Sigma(t_i)\Delta t]^{-1}[\sqrt{\Delta t}U_i + \gamma] + \gamma \\
&= \sqrt{\Delta t}U_i'[c^2\Sigma(t_i)\Delta t]^{-1}\sqrt{\Delta t}U_i + \gamma \\
&= c^{-2}\Theta(i) + \gamma,
\end{aligned} \tag{3.21}$$

where $\Theta(i)$ follows a Chi-Square distribution with p degrees of freedom .

3.3.3 Detection Rule

Following the idea in the univariate test, we propose to infer the cojumps by the extreme distribution of $\mathcal{L}(t_i)$. According to Ramanchandran (1975), the asymptotic distribution of maximimums of Chi-Square also follows a standard Gumbel distribution. We state the result without proof.

Lemma 5. *For a random variable X_i i.i.d $\sim \chi^2(p)$, with $F(x) = P(X_i \leq x)$, and $Z_n = \max(X_1, X_2, \cdots, X_n)$, then there are sequences a_n and $b_n > 0$ such that as $n \to \infty$,*

$$limP(Z_n < a_n + b_n x) = \exp(-e^{-x}), \qquad -\infty < x < +\infty. \tag{3.22}$$

the normalizing constant a_n and b_n can be chosen as

$$a_n = inf\left\{x : 1 - F(x) \leq \frac{1}{n}\right\} \tag{3.23}$$

and

$$b_n = (1 - F(t))^{-1} \int_{a_n}^{+\infty} (1 - F(y))dy. \tag{3.24}$$

To infer the presence of jumps, we set the null hypothesis as there is not a jump in any of the asset at time t_i. For a significance level of α, the null hypothesis is rejected if $\mathcal{L}(t_i) > a_n + b_n\theta^*$, where $P(x \leq \theta^*) = \exp(-e^{-\theta^*}) = 1 - \alpha$.

Approximating the Normalizing Parameters

Compared to the normal distribution, Chi-Square distribution has a more complicated c.d.f. In general, it is hard to find an analytical solution of a_n and b_n. However, given that a $\chi^2(2)$ random variable is exponentially distributed and the sum of k independent identically distributed exponential random variables follows an Erlang–k distribution, we can approximate the constants using the analytical cumulative distribution function (c.d.f) of Erlang–k distribution. In this chapter, we demonstrate the approach using examples when the degrees of freedom of the Chi Square distribution p is 2 or 4. Other Chi Square distributions with degrees of freedom being even number follow the same idea.

Case 1: $p = 2$

When $p = 2$, the test statistic $\mathcal{L}(i)$ is exponentially distributed with the rate parameter $\lambda = 1/2$, i.e., $\mathcal{L}(i) \sim \exp(\frac{1}{2})$. The c.d.f is $F(x) = 1 - e^{-\frac{1}{2}x}$, as $x = \mathcal{L}(i) \geq 0$. Since $F(x)$ is continuous, Equation 3.23 reduces to $1 - F(a_n) = 1/n$. Solving the equation, we get $a_n = \frac{\log n}{\lambda} = 2\log n$. Substitute the result into Equation 3.24, $b_n = 1/\lambda = 2$.

Case 2: $p = 4$

When $N = 4$, the test statistic $\mathcal{L}(i)$ can be interpreted as the sum of two i.i.d exponential random variables with $\lambda = 1/2$. It is therefore an Erlang–k distribution with shape $k = 2$ and rate $\lambda = 1/2$, i.e., $\mathcal{L}(i) \sim$ Erlang $(2, \frac{1}{2})$. The c.d.f is $F(x) =$

$1 - e^{-\lambda x} - \lambda x e^{-\lambda x}$. Similar to case 1, we can get a_n by solving the equation

$$e^{-\lambda a_n} + \lambda a_n e^{-\lambda a_n} = \frac{1}{n}. \tag{3.25}$$

Let $y = -\lambda a_n$, Equation 3.25 reduces to $e^y - ye^y = \frac{1}{n}$. Divide each side by $-e$, we have

$$(y-1)e^{y-1} = -\frac{1}{ne}. \tag{3.26}$$

It follows immediately that $y = 1 + W(-\frac{1}{ne})$, where $W(\cdot)$ is the Lambert W function. That is, $z = W(z)e^{W(z)}$ for every complex number z. Although Lambert W function does not have an analytical form, it can be approximated by Taylor series around 0:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n = z - z^2 + \frac{3}{2}z^3 - \frac{8}{3}z^4 + \cdots \tag{3.27}$$

It is suffice to approximate the Lambert W function around 0 to the 3rd order term. Thus, $y = 1 - \frac{1}{ne} - \frac{1}{n^2e^2} - \frac{3}{2n^3e^3}$ is an approximate solution to Equation 3.26. However, as $y > 1$ for all integer n, and $a_n = -2y_n$, this is not a valid solution to (3.25). If we plot the function $f(a_n) = 1 - F(a_n) - 1/n$, for $n = 2000$, the function approaches $-1/n$ asymptotically for $a_n \geq 0$. a_n can be only be estimated approximately, and it is not unique.

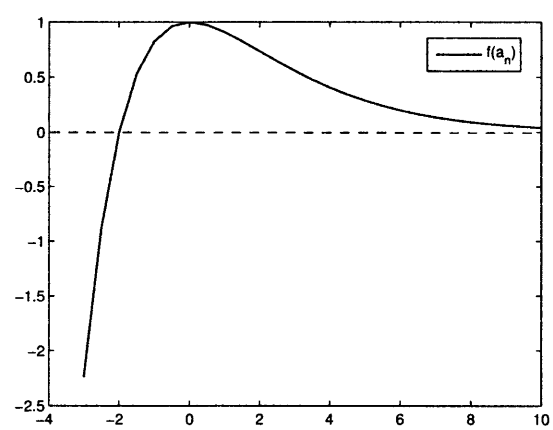

Figure 3-1: Plot of $f(a_n) = 1 - F(a_n) - 1/n$.

Chapter 4

Simulation Study

With the formulation of the multivariate test for a cojump, we access the effectiveness of our test by Monte Carlo simulation that investigates the performance of the test using a finite sample. We first lay out the foundation of the study by choosing a model for the process, and define some terminologies that determine the effectiveness of the test. We then examine the performance of the test under different scenarios. In the end, we compare the effectiveness of the multivariate test to that of the univariate test. We confirm that the univariate test has less power if the significance level is not well designed.

4.1 Simulation Design

Since our test is a nonparametric test, in the sense that there is no specific model assumption of the underlying process, as long as the process is a semi-martingale. We can arbitrarily choose a model to simulate the prices for the test. We choose the Affine Jump-Diffusion (AJD) model in Duffie, Pan and Singleton (2001), as the AJD model allows significant flexibility in setting the drift and volatility processes. Under AJD (ignoring the drift), the p-dimensional log price process $Y = \log X$ follows

$$dY_t = \sigma(Y_t)dW_t + C_t dJ_t, \tag{4.1}$$

where

$$(\sigma(Y_t)\sigma(Y_t)^T)_{ij} = (H_0)_{ij} + (H_1)_{ij}Y_t, \quad \text{for } H = (H_0, H_1) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times p\times p}. \quad (4.2)$$

We further impose the assumption that J follows a Poisson process with intensity λ_t. The jump size C_t has a multivariate normal distribution with mean μ^c and covariance matrix Σ^c.

To facilitate the examination of simulation results, we introduce several concepts which will be used to justify the effectiveness of our test. Analogous to the univariate simulation in Lee and Mykland (2008), for a single testing time t_i, we define the two possible kinds of misclassification. The first is that the test fails to detect the existence of an actual cojump in interval $[t_{i-1}, t_i]$. We call it a failure to detect actual jump (FTD_i) at t_i. The second kind is that the test reports a cojump while there is actually no cojump in $[t_{i-1}, t_i]$. We call it a spurious detection of cojump (SD_i). As the test can be used to detect jumps over time, we introduce the concept of global misclassification. Same as the single testing time, we have two cases of misclassification. The first case is that a number of cojumps present over the time $[0, T]$, but there is at least one actual cojump that the test fails to detect. We call this a global failure to detect actual cojump (GFTD) over $[0, T]$. The second case is that there are some returns not resulting from cojumps, but the test reports at least one of them as a result of a cojump. It is called a global spurious detection of cojump (GSD).

In mathematical terms, suppose there are n observations of returns, we let A_n be a set of times when a cojump presents, and B_n be a set of times at which the test declares a cojump. In addition, we denote each event in A_n and B_n using J_i and D_i respectively. That is $J_i = \{i \in A_n\}$ and $D_i = \{i \in B_n\}$. We then have the following four statements:

$$J_i \bigcap D_i^C = \textit{failure to detect an actual cojump at } t_i \ (FTD_i)$$

$$J_i^C \bigcap D_i = \textit{spurious detection of cojump at } t_i \ (SD_i)$$

$$\bigcup_{i=1}^{n} \left(J_i \bigcap D_i^C \right) = \textit{failure to detect all actual cojumps over } [0,T] \ (GFTD)$$

$$\bigcup_{i=1}^{n} \left(J_i^C \bigcap D_i \right) = \textit{spurious detection of a cojumps over } [0,T] \ (GSD)$$

With this formulation, the power of our test is the probability of global success in detecting actual jumps, which is $1 - P(GFTD)$, and the size of the test is the probability of global spurious detection of jumps, which is $P(GSD)$.

One thousand series of two-dimensional log prices are simulated at different frequencies from 12-hour to 15-minute. The total time horizon is one year, and we assume 252 working days within the year. We compare the performance results under different parameter settings. The significance levels for the study of cojumps are all set at 5%.

4.2 Simulation Result

4.2.1 Null Distribution

Before checking the performance of the test, we first plot the distribution of the test statistic when there is no cojump, that is, neither of the assets has a jump. The test statistic is based on the one thousand simulation at frequency of 15-minute over one year. The plot in Figure 4.2.1 shows that the distribution obtained from the test statistic is close to the exponential distribution with parameter 0.5. The finding is consistent with the theoretical argument that the null distribution is Chi-Square with two degrees of freedom.

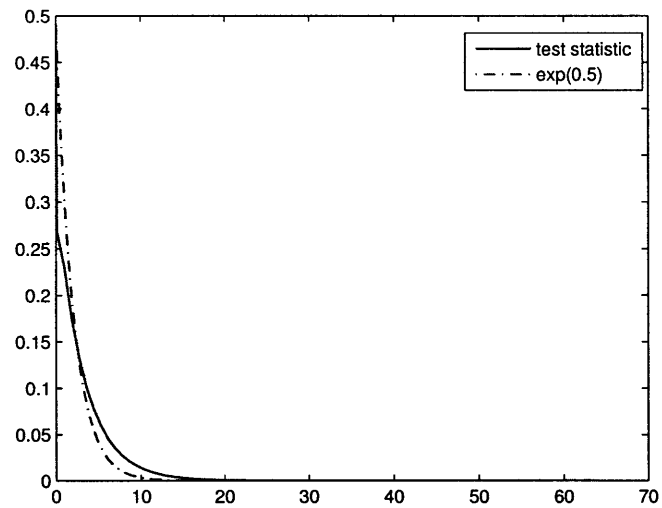

Figure 4-1: Null Distribution of Test Statistics.

4.2.2 Scenario Test

We are going to perform the test under different scenarios of parameters, we set the default parameters first. The default value of the parameters in the model are

$$\Sigma^c = \begin{pmatrix} 0.1 & -0.02 \\ -0.02 & 0.1 \end{pmatrix}, \qquad H_0 = \begin{pmatrix} 0.02 & -0.002 \\ -0.002 & 0.003 \end{pmatrix},$$

$$\mu^c = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}, \qquad H_1^1 = H_1^2 = \begin{pmatrix} 0.01 & 0.005 \\ 0.005 & 0.02 \end{pmatrix}.$$

When the test is performed at different scenarios of the specified parameters, the other non tested parameters are always kept at the default values.

We first consider the effectiveness of the test at three levels of volatilities while keeping the correlations the same.

1. the volatility is constant and at a lower level, i.e., H_0 is lower, and $H_1 = 0$
2. the volatility is constant and at a higher level, i.e., H_0 is higher, and $H_1 = 0$
3. the volatility is not constant

Table 4.1 summarizes the mean and standard errors (in paratheses) of probabilities of spurious detection and detecting actual jumps at a particular time. Looking at a column across different frequencies, it confirms our intuition that as the frequency increases, the probability of spurious detection decreases and the probability of successfully detecting actual jumps increases. At a frequency of 30 minutes or higher, 97% of the jumps can be detected. Across the columns, the test is most effective when the volatility is high, and does not change over time. It is least effective when the volatility stays constantly low. The global performance in Table 4.2.2 indicates the same trends. It is not supprising that changes in volatility over time makes it difficult to detect cojumps. Thus, the test should be less effective when the volatility is not constant. However, the result that the test performs worst under low constant volatility is a bit surprising. It indicates that low level of volatility makes it harder to detect cojumps.

Table 4.1: Local Property for Volatilities

	$P(SD_i)$			$1 - P(FTD_i)$		
freq	1	2	3	1	2	3
12-hour	0.00562	0.00377	0.00424	0.91267	0.93990	0.92986
	(.00192)	(.00144)	(.00111)	(.01258)	(.00984)	(.01023)
6-hour	0.00239	0.00145	0.00165	0.95980	0.96997	0.96010
	(.00028)	(.00014)	(.00018)	(.00982)	(.00489)	(.00694)
1-hour	0.00233	0.00129	0.00163	0.96993	0.97850	0.97299
	(.00064)	(.00055)	(.00048)	(.00883)	(.00465)	(.00670)
30-min	0.00168	0.00084	0.00091	0.97148	0.98020	0.97996
	(.00039)	(.00030)	(.00024)	(.00569)	(.00303)	(.00424)
15-min	0.00166	0.00075	0.00082	0.98883	0.99730	0.99010
	(.00035)	(.00016)	(.00018)	(.00404)	(.00148)	(.00180)

After taking a look at the general trend of the test, we examine the performance of the test at different levels of correlations in returns. To minimize the influence of other factors, the volatility is set to be constant, i.e., $H_1 = 0$. We add another indicator of the performance: Global misclassification (Gmis), which is the combination of GFTD and GSD. This indicator shows how accurately (without making mistakes as GSD or GFTD) the test can locate the jumps over a time horizon. We set the time horizon as one day in the example.

Table 4.2: Global Property for Volatilities

freq	Size 1	Size 2	Size 3	Power 1	Power 2	Power 3
12-hour	0.2497	0.1988	0.2227	0.9958	0.9980	0.9968
	(.0818)	(.0729)	(.0547)	(.0119)	(.0036)	(.0114)
6-hour	0.1116	0.0746	0.0904	0.9963	0.9983	0.9973
	(.0268)	(.0313)	(.0250)	(.0051)	(.0014)	(.0059)
1-hour	0.0597	0.0511	0.0755	0.9975	0.9998	0.9984
	(.0035)	(.0030)	(.0045)	(.0012)	(.0005)	(.0010)
30-min	0.0554	0.0400	0.0411	0.9969	0.9983	0.9978
	(.0079)	(.0091)	(.0066)	(.0031)	(.0019)	(.0015)
15-min	0.0479	0.0352	0.0364	0.9972	0.9993	0.9981
	(.0054)	(.0043)	(.0045)	(.0024)	(.0009)	(.0011)

Table 4.3: Global Property for Levels of Correlation

freq	Power Low	Power high	Size Low	Size high	Gmis Low	Gmis high
1-hour	0.9985	0.9955	0.0747	0.0810	0.0760	0.0847
	(.0007)	(.0011)	(.0042)	(.0046)	(.0042)	(.0047)
30-min	0.9987	0.9964	0.0422	0.0501	0.0433	0.0531
	(.0012)	(.0020)	(.0066)	(.0073)	(.0067)	(.0074)
15-min	0.9989	0.9970	0.0362	0.0436	0.0373	0.0467
	(.0009)	(.0014)	(.0043)	(.0051)	(.0044)	(.0052)

Table 4.2.2 presents the global performance of the test at both low (0.07) and high (0.7) levels of correlation. The performance is measured by power, size and global misclassification, respectively. The results consistently show that the test is more effective when the correlation is low. In fact, the test can detect 0.2% more cojumps when the correlation is lower. This might be because the correlations added more volatility to the process, therefore, making it harder to detect cojumps. The local property of the test in Table 4.4 also confirms that the test performs better when the correlation is lower.

4.2.3 Comparison with Univariate Test

Lastly, we compare the effectiveness of the univariate test and the multivariate test in detecting cojumps in two-asset returns. The idea is that when testing if there is at least a jump in two assets, we can either perform a multivariate test at a significance

Table 4.4: Local Property for Levels of Correlation

	$P(SD_i)$		$1 - P(FTD_i)$	
freq	Low	high	Low	high
1-hour	0.00093	0.00137	0.97361	0.96594
	(.00024)	(.00032)	(.00300)	(.00350)
30-min	0.00082	0.00129	0.98542	0.97845
	(.00017)	(.00027)	(.00230)	(.00260)
15-min	0.00075	0.00090	0.99308	0.98762
	(.00015)	(.00023)	(.00195)	(.00203)

level or an univariate test on each asset to achieve a total significance level the same as the multivariate test.

When performing the univariate test on each asset returns, we set the significance level to be 2.5%, half of the default significance level of the multivariate test. This is a naive way to achieve a total significance level at most as high as that of the multivariate test. In order to minimize the loss of effectiveness in performing the two univariate tests, we do another round of testing by increasing the significance level to 5%, the same level as the multivariate test. The test is performed at both low and high levels of correlation in the returns.

Table 4.5: Probability of Global Success to Detect Cojumps at Low Correlation

		$\alpha = 0.025$		$\alpha = 0.05$	
freq	multivariate	r_1	r_2	r_1	r_2
30-min	0.9953	0.9632	0.9527	0.9645	0.9543
	(.0038)	(.0095)	(.0113)	(.0092)	(.0113)
15-min	0.9986	0.9694	0.9603	0.9703	0.9614
	(.0041)	(.0184)	(.0210)	(.0183)	(.0207)

Table 4.6: Probability of Global Success to Detect Cojumps at High Correlation

		$\alpha = 0.025$		$\alpha = 0.05$	
freq	multivariate	r_1	r_2	r_1	r_2
30-min	0.9872	0.9623	0.9615	0.9635	0.9533
	(.0058)	(.0102)	(.0102)	(.0099)	(.0113)
15-min	0.9989	0.9706	0.9615	0.9717	0.9627
	(.0035)	(.0187)	(.0203)	(.0183)	(.0201)

A common feature from Table 4.5 and Table 4.6 is that the multivariate test outperforms the univariate test in testing the cojumps. As expected, the univariate test has greater power at a higher significance level. However, the powers are still lower than those of the multivariate test. The comparison confirms that our test is more effective in detecting cojumps of multi-asset returns than the univariate test.

Chapter 5

Empirical Example

Both the univariate test and our multivariate test do not assume that the price processes follow any specific parametric models. Therefore, the tests can be applied to detect jumps of different kinds of assets in the financial markets. We apply the tests to two assets that represent the stock market and bond market: the S&P 500 index and 30-year U.S. Treasury bond futures. While our jump detection scheme is based on an infinite small time interval, in financial markets, microstructure complications prevents us from using too frequently traded data. The noise may distort the fundamental semimartingale assumption of the price process. The simulation results for the univariate test in Lee and Mykland (2008) and our simulation results for the multivariate test show that 15-minute is adequate for the tests to gain sufficient power to detect actual jumps and cojumps, we choose 15-minute interval to sample the high frequency data.

We first give an introduction to the two time series used in this empirical example and discuss some interesting findings in the returns. We then apply both the univariate test and multivariate test to detect jumps and cojumps of the two-asset returns.

5.1 Data Description

The data consists of 15-minute returns of two assets: the S&P 500 index and 30-year U.S. Treasury bond futures. The S&P 500 index is a stock market portfolio index which is not traded. An exchange traded fund (ETF) called Standard & Poor's Depositary Receipts (SPDR) is designated to act as a proxy for the S&P 500 index. We downloaded the 15-minute consolidated trades of SPDR in NYSE from the Trade and Quote (TAQ) database. We denote it as the S&P 500 index throughout this chapter. Since the data downloaded from TAQ contain every transactions of trades, we have to construct the 15-minute prices at a time interval of 15 minutes from 9:30 a.m. to 16:00 p.m.. That is, we stamp the prices at every 15-minute time mark. If no trade occurs at the exact interval time mark, previous-tick method (see Dacorogna et al. (2001) for details) is used to construct the interval data, i.e., the last price information (the last tick price) in the previous time interval is put as the price of the interval time mark. We use a subset of the 30-year U.S. Treasury bond futures employed by Wright and Zhou (2007). To construct a single time series, we use the most actively traded contract, which is nearest to its expiration. Following the methodology of Andersen, Bollerslev and Huang (2006), we switch to the next bond futures on the first business day of the delivery month. The trading time for futures is from 8:20 a.m. to 15:00 p.m., which is approximately one hour earlier than the S&P 500 index trading time. We synchronize the data by taking off the prices from 9:30 a.m. to 15:00 p.m., such that we will not miss out the significant jumps at the starting time of a trading day. The full sample of the two-asset price span from February 1, 1993 through August 31, 2006.

The average 15-minute returns and standard deviations (in parentheses) of the S&P 500 index and 30-year U.S. Treasury bond futures are 0.0014%(0.0025) and 0.00004%(0.0014), respectively. As expected, the mean of high frequency returns are extremely close to zero. Figure 5-1 presents the 15-minute returns of the two assets. The full sample correlations of the two-asset returns are small. However, when we only count the returns in the contraction period (March 1,2001 to December 31,

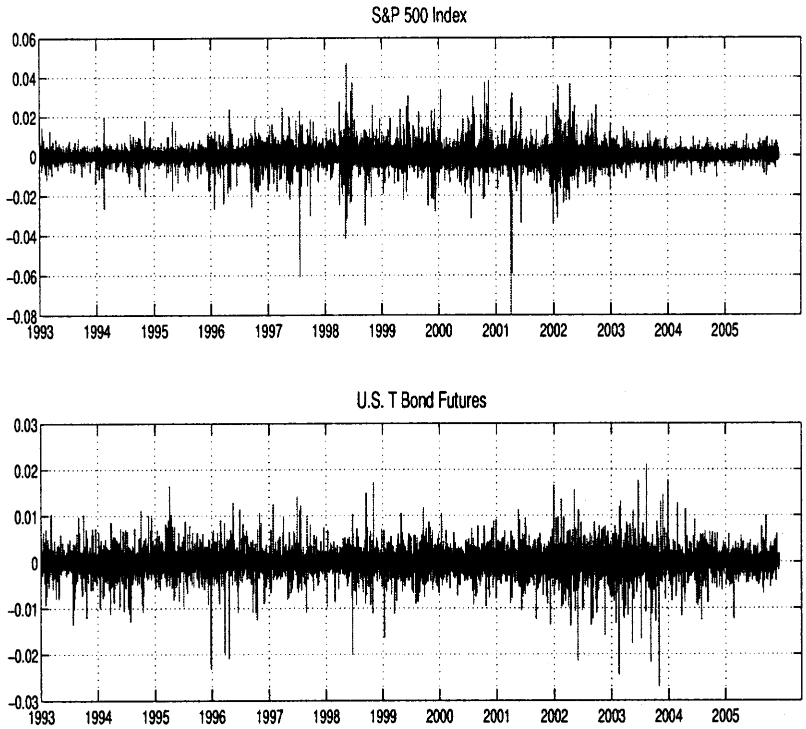

Figure 5-1: 15-minute Returns of the S&P 500 Index and 30-year U.S. Treasury Bond Futures

2002), the correlations are highly negative. This might be explained by the cash flow effect during contraction, as explored by Andersen et al. (2007). Figure 5-2 plots the realized monthly correlations of the two-asset returns, and the realized monthly standard deviations of each asset returns over the full sample period. It confirms that the bond has relatively less volatility than the stock index over time.

5.2 Jump and Cojump in Asset Returns

The jumps are detected in three ways: 1. an univariate test for each asset at a significance level of 2.5%; 2. an univariate test for each asset at a significance level of 5%; 3. a multivariate test for the 2-dimensional process at a significance level of 5%. Table 5.1 summarizes the number of cojumps detected for each method. The

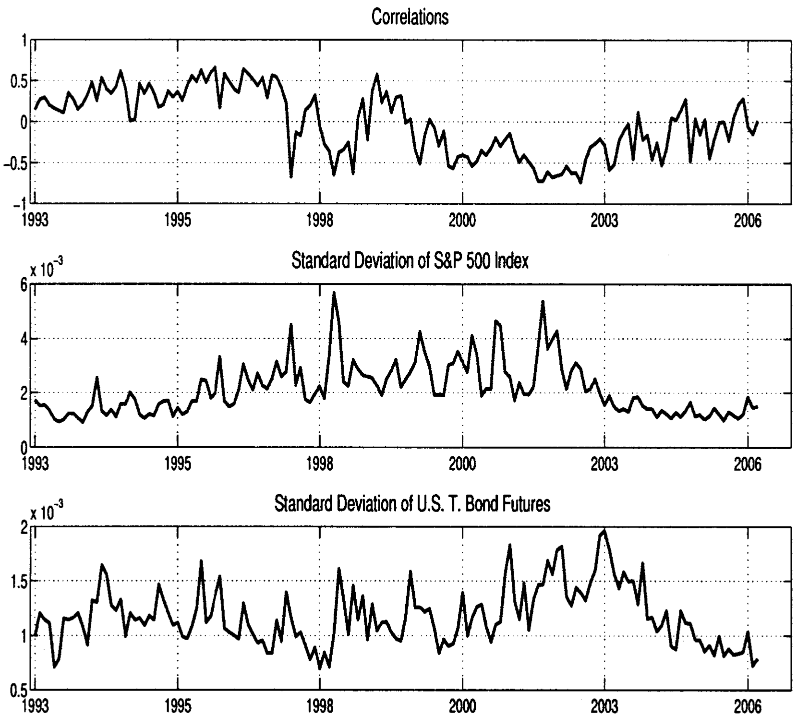

Figure 5-2: Monthly Realized Correlations of the S&P 500 Index and 30-year U.S. Treasury Bond Futures

first row in the table is the number of detected jumps at 15-minute interval for each asset using the univariate test. The second row finds the number of times that both assets have a jump. The last row is obtained by adding the individual jumping times and subtracting the joint jumping times. It represents the number of times that at least one of the asset has a jump (a cojump). It proves that if we use the naive way to choose a significance level (dividing the total significance level by the number of multiple hypothesis testing) to test if there is a jump in any one of the two assets, we lose substantial power in inferring the results. Even when $\alpha = 5\%$, the number of jumps detected is still significantly less than the multivariate test. In the rest of the chapter, the univariate tests are all performed at a significance level of 5%.

Table 5.1: Summary of the Number of Jumps Detected

	$\alpha = 0.025$		$\alpha = 0.05$		$\alpha = 0.05$
	S&P 500	T. Bond	S&P 500	T. Bond	Bivariate
each	600	639	642	686	
joint	175		190		
union	1064		1106		1275

5.2.1 Distribution of Test Statistics

It would be interesting to empirically check the distribution of the test statistic under the condition that there is no jump at t_i. We plot the histograms of the three test statistics at the time in absence of jumps. With some scaling, we find that the two univariate test statistics are approximately normally distributed with mean 0, and the bivariate test is exponentially distributed, which is equivalent to the chi-square distribution with two degrees of freedom. It again, numerically confirmed our theorems in chapter 2 and 3.

5.2.2 Jump Characteristic

We look at the properties of the asset returns in the presence of jumps. Particularly, we plot some variables: the test statistics, the returns (in percentage), and the estimated instantaneous volatility for each of the univariate test. The x-axis of the graphs represents the dates, and y-axis represents the magnitude of these variables. In the plot of the test statistics, the threshold line is drawn as a benchmark.

Figure A-3 plots the jump variables of the S&P 500 index when the jumps are detected using the univariate test. Similarly, Figure A-4 plots the same test on 30-year U.S. Treasury bond futures. Look at the time from mid of 2001 and the end of 2002 in Figure A-3, the changes in returns are relatively high. However, instantaneous volatilities are much higher. Since the univariate test statistics is obtained by standardizing the returns using the volatility, it is thus relatively lower. It empirically explains the rationale of standardizing the returns. Compare Figure A-3 to Figure A-4, it is clear that the bond has less extreme jumps, but the frequency of jumps is higher than the stock index.

One important feature of the multivariate test lies in utilizing the correlations between the two assets. It inspires us to plot the instantaneous correlation as well. Figure A-5 tracks the test statistics and the correlations. The pattern of the instantaneous correlations inherits that of the monthly correlations in Figure 5-2.

When zooming into intra day level, we found that the jumps in the returns of 30-year U.S. Treasury bond futures are most prevalent during the 8:30 a.m. to 8:45 a.m., and the jumps in the S&P 500 index presents mostly during 9:30 a.m. to 9:45 a.m. Since most of the macro news release at 8:30 a.m., it is not surprising that the large movements in returns happen immediately after the news events. The results coincide with the findings among an extensive literature, which explores the relations between jumps and macro news announcements. Lahaye, Laurent and Neely (2009) shows that most large moves in S&P 500 returns are associated with U.S. macroeconomic news announcements.

Chapter 6

Conclusion

To fill up the blank in identifying cojumps in multi-asset returns, we propose a new nonparametric jump test. Monte Carlo simulations prove that our test can accurately identify most of the cojumps under high frequency observations. The nonparametric feature of our test offers great flexibility in investigating jump dynamics in a variety of financial assets. We study an empirical example with returns of the S&P 500 index and 30-year U.S. Treasury bond futures. We find the in intra day level, jumps occur most frequently at the start of the trading day, when the macro news releases. It illustrates the close relation of news events and the reactions of financial markets.

Although our test is designed to detect cojumps on multivariate price processes, so far, it is only been tested for the bivariate case. The reason is twofold: 1. simulation study in higher dimension requires more memory that may be beyond the capability of a personal computer; 2. the normalizing parameters have nice analytical solutions for a bivariate process. As such, future work could be further researched to verify the effectiveness of the test on processes of higher dimension both by simulation and empirically.

Appendix A

Figure

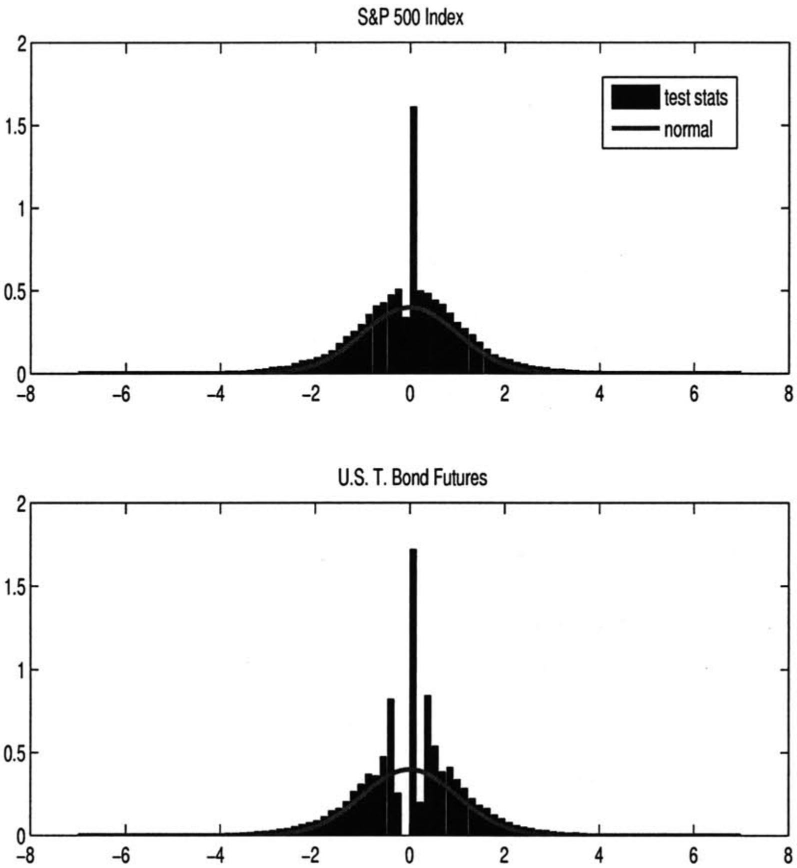

Figure A-1: Distribution of Test Statistics of Univariate Returns in the Absence of Jumps

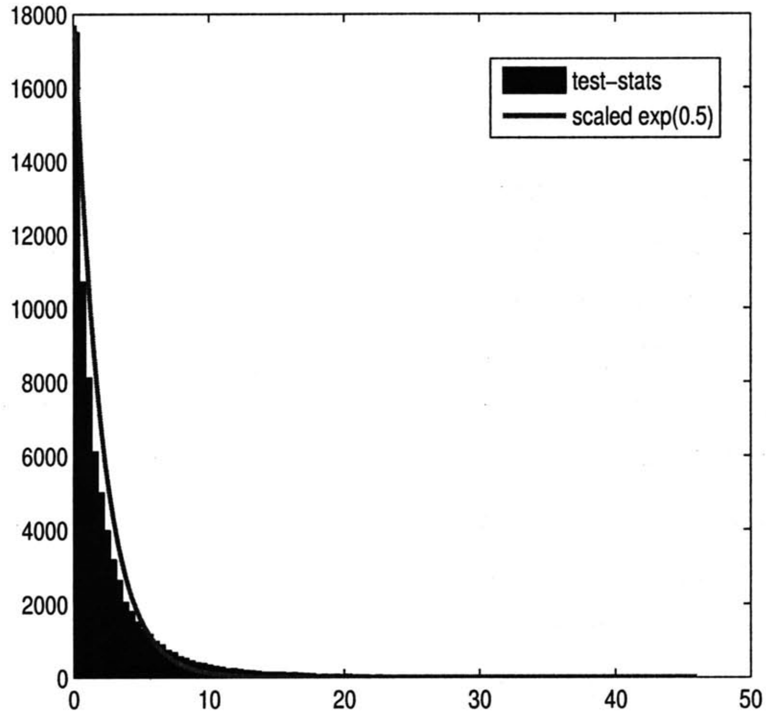

Figure A-2: Distribution of Test Statistics of Bivariate Returns in the Absence of Jumps

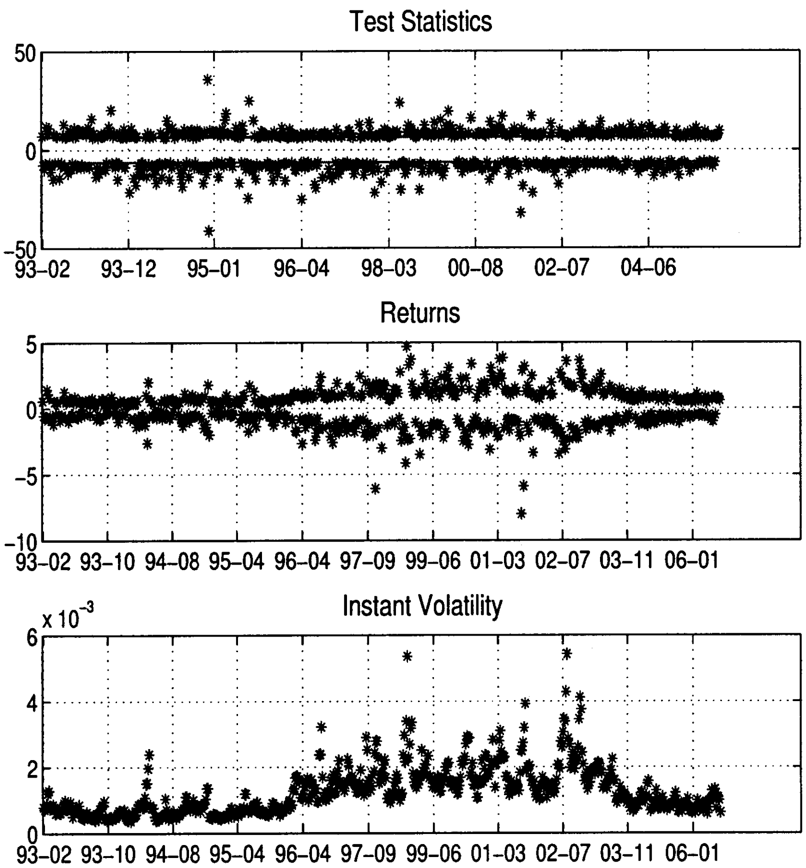

Figure A-3: Jump Characteristics of the S&P 500 Index

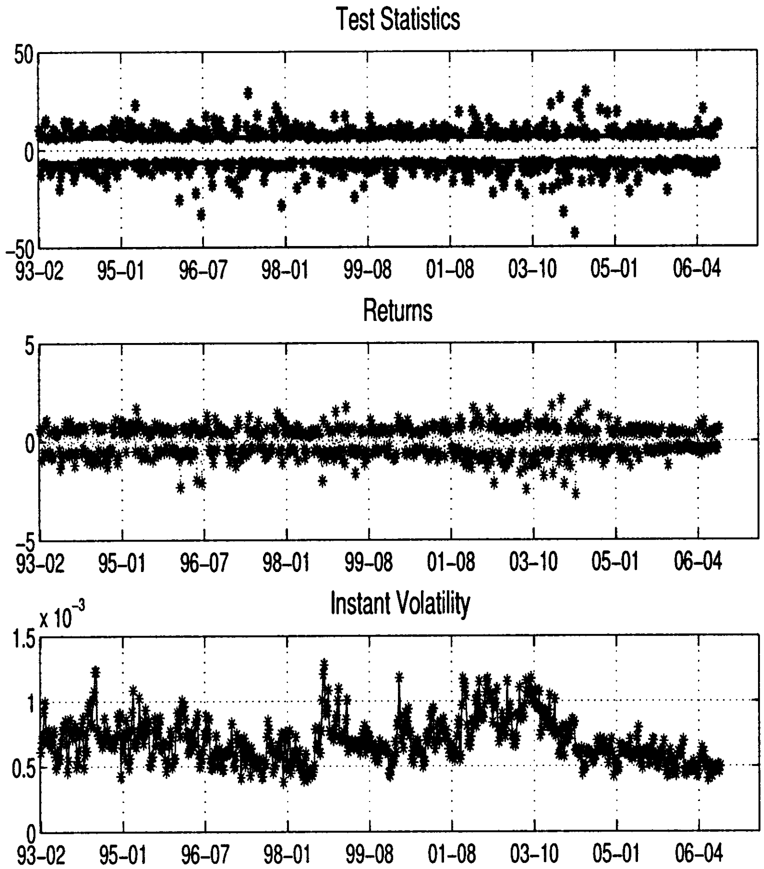

Figure A-4: Jump Characteristics of 30-year U.S. Treasury Bond Futures

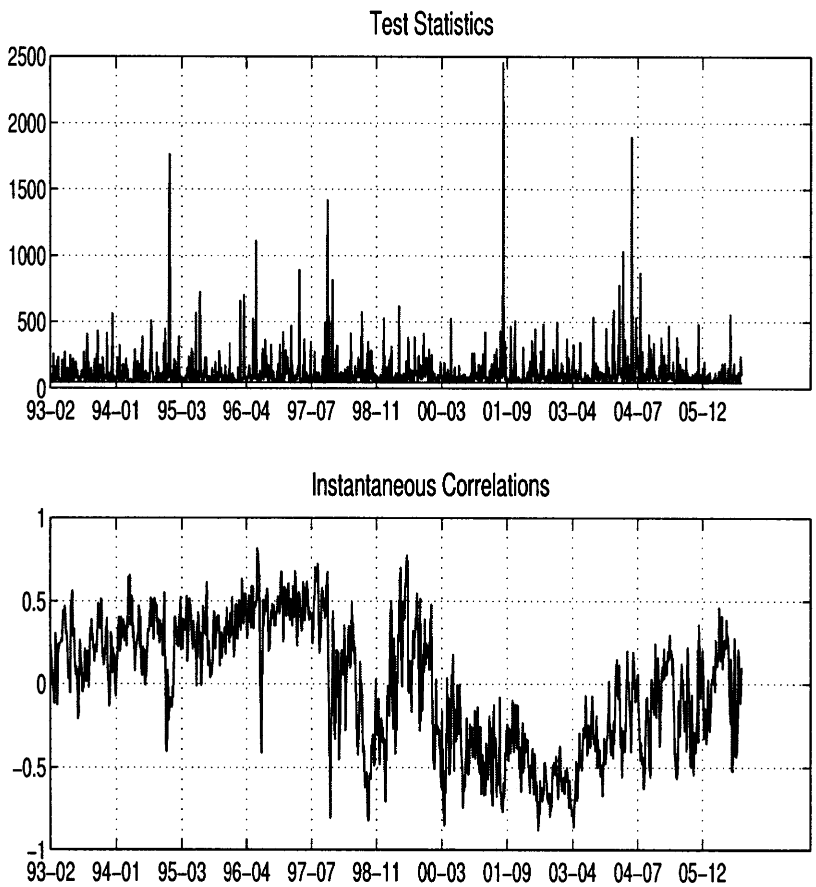

Figure A-5: Cojump Characteristics of the Multivariate Test on the S&P 500 Index and 30-year U.S. Treasury Bond Futures

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