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5D/4D U-dualities and $\mathcal{N} = 8$ black holes

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We use the connection between the U-duality groups in $d = 5$ and $d = 4$ to derive properties of the $\mathcal{N} = 8$ black hole potential and its critical points (attractors). This approach allows us to study and compare the supersymmetry features of different solutions.

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I. INTRODUCTION

The $\mathcal{N} = 8$ supergravity theory in $d = 4$ [1] and $d = 5$ [2] dimensions is a remarkable theory which unifies the gravitational fields with other lower spin particles in a rather unique way, due to the high constraints of local $\mathcal{N} = 8$ supersymmetry, the maximal one realized in a four-dimensional (4D) Lagrangian field theory. These theories, particularly in four dimensions, are supposed to enjoy exceptional ultraviolet properties. For this reason, 4D supergravity has been advocated not only as the simplest quantum field theory [3], but also as a potential candidate for a finite theory of quantum gravity, even without its completion into a larger theory [4]. Maximal supergravity in highest dimensions has a large number of classical solutions [5] which may survive at the quantum level. Among them, there are black p branes of several types [6] and interestingly, 4D black holes (BH) of different nature.

On the other hand, theories with lower supersymmetries (such as $\mathcal{N} = 2$) emerging from Calabi-Yau compactifications of M theory or superstring theory admit extremal black hole solutions that have been the subject of intense study because of their wide range of classical and quantum aspects. For asymptotically flat, stationary, and spherically symmetric extremal black holes, the attractor behavior [7,8] has played an important role not only in determining universal features of fields flows toward the horizon, but also to explore dynamical properties such as wall crossing [9] and split attractor flows [10], the connections with string topological partition functions [11], and relations with microstates counting [12]. Therefore, it has become natural to study the properties of extremal black holes not only in the context of $\mathcal{N} = 2$, but also in theories with higher supersymmetries, up to $\mathcal{N} = 8$ [13–22].

In $\mathcal{N} = 8$ supergravity, in the Einsteinian approximation, there is a nice relation between the classification of large black holes which undergo the attractor flow and charge orbits which classify, in a duality invariant manner, the properties of the dyonic vector of electric and magnetic charges $Q = (p^\Lambda, q_\Lambda)$ ($\Lambda = 0, \ldots, 27$ in $d = 4$) [23,24]. The attractor points are given by extrema of the 4D black hole potential, which is given by [16,17]

$$V_{\rm BH} = \tfrac{1}{2} Z_{AB} Z^{*AB} = \langle Q, V_{AB}\rangle\langle Q, \bar{V}^{AB}\rangle, \qquad (1.1)$$

where the central charge is the antisymmetric matrix ($A, B = 1, \ldots, 8$)

$$Z_{AB} = \langle Q, V_{AB}\rangle = Q^T \Omega V_{AB} = f^\Lambda{}_{AB} q_\Lambda - h_{\Lambda AB} p^\Lambda, \qquad (1.2)$$

the symplectic sections are

$$V_{AB} = (f^\Lambda{}_{AB}, h_{\Lambda AB}), \qquad (1.3)$$

and Ω is the symplectic invariant metric.

An important role is played by the Cartan quartic invariant I_4 [1,25] in that it only depends on Q and not on the asymptotic values of the 70 scalar fields φ. This means that if we construct I_4 as a combination of quartic powers of the central charge matrix $Z_{AB}(q, p, \varphi)$ [26], then the φ dependence drops out from the final expression

$$\frac{\partial}{\partial \varphi} I_4(Z_{AB}) = 0. \qquad (1.4)$$

Analogue (cubic) invariants I_3 exist for black holes and/or (black) strings in $d = 5$ [8,23]. These are given by

$$I_3(p^I) = \frac{1}{3!} d_{IJK} p^I p^J p^K, \qquad (1.5)$$

$$I_3(q_I) = \frac{1}{3!} d^{IJK} q_I q_J q_K, \qquad (1.6)$$

where d_{IJK}, d^{IJK} are the $(27)^3$ $E_{6(6)}$ invariants. Consequently, the $d = 4$ $E_{7(7)}$ quartic invariant takes the form

$$I_4(Q) = -(p^0 q_0 + p^I q_I)^2 + 4\left[-p^0 I_3(q) + q_0 I_3(p) + \frac{\partial I_3(q)}{\partial q_I}\frac{\partial I_3(p)}{\partial p^I}\right]. \qquad (1.7)$$

On the other hand, in terms of the central charge matrices $Z_{ab}(\phi, q)$ [in $d = 5$ this is the **27** representation of $USp(8)$] and $Z_{AB}(\phi, p, q)$ [in $d = 4$ this is the **28** of $SU(8)$], their expression is

$$I_3(q) = Z_{ab}\Omega^{bc}Z_{cd}\Omega^{dq}Z_{qp}\Omega^{pa}, \qquad Z_{ab}\Omega^{ab} = 0, \quad (1.8)$$

$$I_4(p,q) = \tfrac{1}{4}[4\,\mathrm{Tr}(ZZ^\dagger ZZ^\dagger) - (\mathrm{Tr}ZZ^\dagger)^2 + 32\,\mathrm{Re}(PfZ_{AB})], \quad (1.9)$$

where $ZZ^\dagger = Z_{AB}\bar{Z}^{CB}$, Ω^{ab} is the 5D symplectic invariant metric, and the Pfaffian of the central charge is [1]

$$Pf(Z_{AB}) = \frac{1}{2^4 4!}\epsilon^{ABCDEFGH}Z_{AB}Z_{CD}Z_{EF}Z_{GH}. \quad (1.10)$$

In fact, these are simply the (totally symmetric) invariants which characterize the 27 dimensional representation of $E_{6(6)}$ and the 56 dimensional representation of $E_{7(7)}$, which are the U-duality [27] symmetries of $\mathcal{N} = 8$ supergravity in $d = 5$ and $d = 4$, respectively.

When charges are chosen such that I_4 and I_3 are not vanishing, one has large black holes, and in the extremal case, the attractor behavior may occur. However, while at $d = 5$ there is a unique ($\frac{1}{8}$-BPS) attractor orbit with $I_3 \neq 0$, associated to the space [24,28]

$$\mathcal{O}_{d=5} = \frac{E_{6(6)}}{F_{4(4)}}, \quad (1.11)$$

at $d = 4$ two orbits emerge, the BPS one

$$\mathcal{O}_{d=4,\mathrm{BPS}} = \frac{E_{7(7)}}{E_{6(2)}}, \quad (1.12)$$

and the non-BPS one with different stabilizer

$$\mathcal{O}_{d=4,\text{non-BPS}} = \frac{E_{7(7)}}{E_{6(6)}}. \quad (1.13)$$

Such orbits have further ramifications in theories with lower supersymmetry, but it is the aim of this paper to confine our attention to the $\mathcal{N} = 8$ theory.

In this paper, extending a previous result for $\mathcal{N} = 2$ theories [29], we elucidate the connection between these configurations, and we relate the critical points of the $\mathcal{N} = 8$ black hole potential of the five-dimensional (5D) and 4D theories. To achieve this goal, we use a formulation of 4D supergravity in a $E_{6(6)}$ duality covariant basis [30], which is appropriate to discuss a 4D/5D correspondence. This is not the same as the Cremmer-Julia [1] or de Wit-Nicolai [31] manifest $SO(8)$ [and $SL(8,\mathbb{R})$] covariant formulation, but it is rather related to the Sezgin-Van Nieuwenhuizen 5D/4D dimensional reduction [32]. These two formulations are related to one another by dualizing several of the vector fields, and therefore they interchange electric and magnetic charges of some of the 28 vector fields of the final theory. The precise relation between these theories was recently discussed in [33].

The paper is organized as follows. In Sec. II, we rewrite the 4D black hole potential in terms of central charges. This is essential in order to discuss the supersymmetry properties of the solutions. In fact, in the specific solutions we consider in Secs. III and IV, BPS and non-BPS critical points are simply obtained by some charges sign flip. This will manifest in completely different symmetry properties of the central charge matrix, in the normal frame, at the fixed point. These properties reflect the different character of the BPS and non-BPS charge orbits.

The solutions of the critical point equations are particularly simple in the "axion-free" case, discussed in Secs. III and IV, which only occur for some chosen charge configurations. In Sec. III, we derive critical point equations that are completely general and that may be used to study any solution.

The formula for the $\mathcal{N} = 8$ potential given in Sec. II was obtained in an earlier work [33], and it is identical to the $\mathcal{N} = 2$ case [29]. The only difference relies in the kinetic matrix a_{IJ} which, in $\mathcal{N} = 2$ is given by real special geometry, while in $\mathcal{N} = 8$ is given in terms of the $E_{6(6)}$ coset representatives [16,32]. However, in the normal frame, when we suitably restrict to two moduli, this matrix does indeed become an $\mathcal{N} = 2$ matrix, although the interpretation in terms of central charges is completely different.

The supersymmetry properties of the solutions in the $\mathcal{N} = 8$ and $\mathcal{N} = 2$ theories are compared in Sec. IV D. We will see that in the $\mathcal{N} = 2$ interpretation, depending on the sign of the charges, both a BPS and a non-BPS branch exist in $d = 5$, while two non-BPS branches exist in the $d = 4$ theory. In $\mathcal{N} = 8$, the occurrence of one less branch in both dimensions is due to the fact that the central and matter charges of the $\mathcal{N} = 2$ theory are all embedded in the central charge matrix of the $\mathcal{N} = 8$ theory. The higher number of attractive orbits can also be explained by the different form of the relevant noncompact groups and their stabilizers for the moduli space of solutions.

II. 4D/5D RELATIONS FOR THE $\mathcal{N} = 8$ EXTREMAL BLACK HOLE POTENTIAL

In this section, we remind the reader how the $\mathcal{N} = 8$ potential was derived in a basis that illustrates the relation between 4 and 5 dimensions [33].

Using known identities [17,34], the black hole potential can be written as a quadratic form in terms of the charge vector Q and the symplectic 56×56 matrix $\mathcal{M}(\mathcal{N})$, related to the 4D vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$

$$V_{\mathrm{BH}} = -\tfrac{1}{2}Q^T\mathcal{M}(\mathcal{N})Q, \quad (2.1)$$

where $\mathcal{M}$ is

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}(\mathrm{Im}N)^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \quad (2.2)$$

The indices Λ, Σ of $\mathcal{N}_{\Lambda\Sigma}$ are now split as $(0, I)$, according to the decomposition of 4D charges with respect to 5D ones, thus $\mathcal{N}_{\Lambda\Sigma}$ assumes the block form

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \mathcal{N}_{00} & \mathcal{N}_{0J} \\ \mathcal{N}_{I0} & \mathcal{N}_{IJ} \end{pmatrix}. \tag{2.3}$$

The kinetic matrix depends on the 70 scalars of the $\mathcal{N}=8$ theory, which are given, in the 5D/4D Kaluza-Klein reduction, by the 42 scalars of the 5D theory (encoded in the 5D vector kinetic matrix $a_{IJ}=a_{JI}$), by the 27 axions a^I and the dilaton field e^ϕ. In a normalization that is suitable for comparison to $\mathcal{N}=2$, it has the form

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \frac{1}{3}d - i(e^{2\phi}a_{IJ}a^Ia^J + e^{6\phi}) & -\frac{1}{2}d_J + ie^{2\phi}a_{KJ}a^K \\ -\frac{1}{2}d_I + ie^{2\phi}a_{IK}a^K & d_{IJ} - ie^{2\phi}a_{IJ} \end{pmatrix}, \tag{2.4}$$

where

$$d \equiv d_{IJK}a^Ia^Ja^K, \qquad d_I \equiv d_{IJK}a^Ja^K, \qquad d_{IJ} \equiv d_{IJK}a^K. \tag{2.5}$$

The black hole potential of [33], computed from (2.1) using the above formulas, can be rearranged as

$$\begin{aligned} V_{\rm BH} = {}& \frac{1}{2}(p^0e^\phi a^I)a_{IJ}(p^0e^\phi a^J) + \frac{1}{2}(p^0e^{3\phi})^2 + \frac{1}{2}\left(\frac{d}{6}p^0e^{-3\phi}\right)^2 + \frac{1}{2}\left(\frac{1}{2}e^{-\phi}p^0d_I\right)a^{IJ}\left(\frac{1}{2}e^{-\phi}p^0d_J\right) \\ & + \frac{1}{2}\times 2(-p^0e^\phi a_I)a_{IJ}(p^Je^\phi) + \frac{1}{2}\times 2\left(\frac{d}{6}p^0e^{-3\phi}\right)\left(-\frac{1}{2}p^Id_Ie^{-3\phi}\right) - \frac{1}{2}\times 2\left(\frac{1}{2}p^0e^{-\phi}d_I\right)a^{IJ}(p^Kd_{KJ}e^{-\phi}) \\ & + \frac{1}{2}(e^\phi p^I)a_{IJ}(e^\phi p^J) + \frac{1}{2}\left(\frac{1}{2}e^{-3\phi}p^Kd_K\right)^2 + \frac{1}{2}(e^{-\phi}p^Kd_{KI})a^{IJ}(e^{-\phi}p^Ld_{JL}) + \frac{1}{2}\times 2(q_0e^{-3\phi})\left(\frac{d}{6}p^0e^{-3\phi}\right) \\ & + \frac{1}{2}\times 2(q_Ia^Ie^{-3\phi})\left(\frac{d}{6}p^0e^{-3\phi}\right) + \frac{1}{2}\times 2(q_Ie^{-\phi})a^{IJ}\left(\frac{1}{2}p^0d_Je^{-\phi}\right) - \frac{1}{2}\times 2(q_0e^{-3\phi})\left(\frac{1}{2}p^Id_Ie^{-3\phi}\right) \\ & - \frac{1}{2}\times 2(q_Ia^Ie^{-3\phi})\left(\frac{1}{2}p^Jd_Je^{-3\phi}\right) - \frac{1}{2}\times 2(q_Ie^{-\phi})a^{IJ}(p^Kd_{KJ}e^{-\phi}) + \frac{1}{2}(q_0e^{-3\phi})^2 \\ & + \frac{1}{2}\times 2(q_0e^{-3\phi})(q_Ia^Ie^{-3\phi}) + \frac{1}{2}(q_Ia^Ie^{-3\phi})^2 + \frac{1}{2}(q_Ie^{-\phi})a^{IJ}(q_Je^{-\phi}), \end{aligned} \tag{2.6}$$

with $a^{IJ} = a_{IJ}^{-1}$. This form shows that it can be written in terms of squares of electric and magnetic components as

$$V_{\rm BH} = \tfrac{1}{2}(Z_0^e)^2 + \tfrac{1}{2}(Z_m^0)^2 + \tfrac{1}{2}Z_I^e a^{IJ} Z_J^e + \tfrac{1}{2}Z_m^I a_{IJ} Z_m^J, \tag{2.7}$$

provided one defines

$$\begin{aligned} Z_0^e &= e^{-3\phi}q_0 + e^{-3\phi}q_Ia^I + e^{-3\phi}\frac{d}{6}p^0 - \tfrac{1}{2}e^{-3\phi}p^Id_I, \\ Z_m^0 &= e^{3\phi}p^0, \qquad Z_I^e = \tfrac{1}{2}e^{-\phi}p^0d_I - p^Jd_{IJ}e^{-\phi} + q_Ie^{-\phi}, \\ Z_m^I &= e^\phi p^I - e^\phi p^0a^I. \end{aligned} \tag{2.8}$$

In order to get the symplectic embedding of the four dimensional theory, we still need to complexify the central charges. To this end, we define the two complex vectors

$$Z_0 \equiv \frac{1}{\sqrt{2}}(Z_0^e + iZ_m^0), \qquad Z_a \equiv \frac{1}{\sqrt{2}}(Z_a^e + iZ_m^a), \tag{2.9}$$

where

$$Z_a^e = Z_I^e(a^{-1/2})^I_a, \qquad Z_m^a = Z_m^I(a^{1/2})^a_I, \tag{2.10}$$

such that

$$V_{\rm BH} = |Z_0|^2 + Z_a\bar{Z}_a, \tag{2.11}$$

where now $a = 1, \ldots, 27$ is a flat index, which can be regarded as a $USp(8)$ antisymmetric traceless matrix.

The potential at the critical point gives the black hole entropy corresponding to the given solution, which in $d = 4$ reads

$$\frac{S_{\rm BH}}{\pi} = \sqrt{|I_4|} = V_{\rm BH}^{\rm crit}, \tag{2.12}$$

while in $d = 5$ it is [35]

$$\frac{S_{\rm BH}}{\pi} = 3^{3/2}|I_3|^{1/2} = (3V_5^{\rm crit})^{3/4}, \tag{2.13}$$

where I_4 and I_3 are the invariants of the $\mathcal{N}=8$ theory in $d=4$ and $d=5$, respectively.

Symplectic sections

In virtue of the previous discussion, we can trade the central charge (1.2) for the 28-component vector

$$Z_A = f^\Lambda{}_A q_\Lambda - h_{\Lambda A}p^\Lambda, \tag{2.14}$$

where f and h are symplectic sections satisfying the following properties [36,37]:

(a) $\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda A}(f^{-1})^A{}_\Sigma$,
(b) $i(\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f}) = \mathbf{Id}$,
(c) $\mathbf{f}^T\mathbf{h} - \mathbf{h}^T\mathbf{f} = 0$.

Notice that one still has the freedom of a further transformation

$$h \to hM, \qquad f \to fM, \tag{2.15}$$

as it leaves invariant the vector kinetic matrix $\mathcal{N}$, as well as relations (a)–(c), when M is a unitary matrix

$$MM^\dagger = 1. \tag{2.16}$$

Indeed, when the central charge transforms as

$$Z \to ZM, \qquad ZZ^\dagger \to ZMM^\dagger Z^\dagger = ZZ^\dagger, \tag{2.17}$$

the black hole potential

$$V_{\rm BH} \equiv ZZ^\dagger \tag{2.18}$$

is left invariant. In our case, we rearrange the 28 indices into a single complex vector index, to be identified, for a suitable choice of M, with the two-fold antisymmetric representation of $SU(8)$, according to the decomposition $\mathbf{28} \to \mathbf{27} + \mathbf{1}$ of $SU(8) \to USp(8)$; we thus have

$$\begin{aligned} Z_0 &= f^\Lambda{}_0 q_\Lambda - h_{\Lambda 0} p^\Lambda \\ &= f^0{}_0 q_0 + f^J{}_0 q_J - h_{00} p^0 - h_{J0} p^J, \\ Z_a &= f^\Lambda{}_a q_\Lambda - h_{\Lambda a} p^\Lambda \\ &= f^0{}_a q_0 + f^J{}_a q_J - h_{0a} p^0 - h_{Ja} p^J, \end{aligned} \tag{2.19}$$

which, from the definition in (2.9) yields

$$\begin{aligned} Z_0 &= \frac{1}{\sqrt{2}}\Big[e^{-3\phi} q_0 + e^{-3\phi} a^I q_I + \Big(e^{-3\phi}\frac{d}{6} + i e^{3\phi}\Big) p^0 \\ &\quad - \frac{1}{2}(e^{-3\phi} d_I) p^I \Big], \\ Z_a &= \frac{1}{\sqrt{2}}\Big[e^{-\phi} q_I (a^{-1/2})^I{}_a \\ &\quad + \Big(\frac{1}{2} e^{-\phi} d_I (a^{-1/2})^I{}_a - i e^{\phi} a^J (a^{1/2})_J{}^a\Big) p^0 \\ &\quad - (e^{-\phi} d_{IJ} (a^{-1/2})^I{}_a - i e^{\phi} (a^{1/2})_J{}^a) p^J \Big]. \end{aligned} \tag{2.20}$$

Thus we consider

$$f^\Lambda{}_A = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-3\phi} & 0 \\ e^{-3\phi} a^I & e^{-\phi}(a^{-1/2})^I{}_a \end{pmatrix}, \tag{2.21}$$

$$h_{\Lambda A} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-3\phi}\frac{d}{6} - i e^{3\phi} & -\frac{1}{2} e^{-\phi} d_K (a^{-1/2})^K{}_a + i e^{\phi} a^K (a^{1/2})_K{}^a \\ \frac{1}{2} e^{-3\phi} d_I & e^{-\phi} d_{IJ} (a^{-1/2})^J{}_a - i e^{\phi} (a^{1/2})_I{}^a \end{pmatrix}. \tag{2.22}$$

From $\mathbf{f}^{-1}$

$$(f^{-1})_\Lambda{}^A = \sqrt{2} \begin{pmatrix} e^{3\phi} & 0 \\ -e^{\phi} a^I (a^{1/2})_I{}^a & e^{\phi} (a^{1/2})_I{}^a \end{pmatrix}, \tag{2.23}$$

by matrix multiplication, we find that relations (a), (b), and (c) are fulfilled by $\mathbf{f}$ and $\mathbf{h}$, that we now recognize to be the symplectic sections.

We finally perform the transformation $f' = fM$ (where $M = f^{-1} f' = h^{-1} h'$), with M unitary matrix, in virtue of identities (a), (b), and (c), valid for both (f, h) and (f', h'). A model independent formula for M valid for any $\mathcal{N} = 2$ d geometry (in particular, for any truncation of $\mathcal{N} = 8$ to an $\mathcal{N} = 2$ geometry, such as the models treated in this paper) is given by the matrix [38]

$$M = A^{1/2} \hat{M} G^{-1/2}, \tag{2.24}$$

with

$$A = \begin{pmatrix} 1 & 0\ldots 0 \\ 0 & \\ \cdot & a_{IJ} \\ \cdot & \\ 0 & \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0\ldots 0 \\ 0 & \\ \cdot & g_{IJ} \\ \cdot & \\ 0 & \end{pmatrix}, \quad g_{IJ} = \frac{1}{4} e^{-4\phi} a_{IJ}, \tag{2.25}$$

where $\hat{M}$ is given by

$$\hat{M} = \frac{1}{2} \begin{pmatrix} 1 & \partial_{\bar{J}} K \\ -i\lambda^I e^{-2\phi} & e^{-2\phi}\delta^I_{\bar{J}} + i e^{-2\phi} \lambda^I \partial_{\bar{J}} K \end{pmatrix}, \tag{2.26}$$

where "$-\lambda^I$" are the imaginary parts of the complex moduli $z^I = a^I - i\lambda^I$, and K is the Kähler potential $K = -\ln(8\mathcal{V})$, with $\mathcal{V} = \frac{1}{3!} d_{IJK} \lambda^I \lambda^J \lambda^K$; the matrix $\hat{M}$ satisfies the properties

$$A\hat{M} G^{-1} \hat{M}^\dagger = Id, \qquad G^{-1} \hat{M}^\dagger A \hat{M} = Id. \tag{2.27}$$

For the models considered below, this matrix M does indeed reproduce, for the given special configurations, the formula in Eq. (4.7).

Note that $\hat{M}$ performs the change of basis between the central charges defined as

$$Z_0 = \frac{1}{\sqrt{2}} (Z^e_0 + i Z^0_m), \qquad Z_I = \frac{1}{\sqrt{2}} (Z^e_I + i a_{IJ} Z^J_m), \tag{2.28}$$

and the special geometry charges $(Z, \mathcal{D}_{\bar{I}} \bar{Z})$, that is the charges in "curved" rather than the "flat" indices.

III. ATTRACTORS IN THE FIVE-DIMENSIONAL THEORY

It was shown in [23] that the cubic invariant of the five dimensions can be written as

$$I_3 = Z^5_1 Z^5_2 Z^5_3, \tag{3.1}$$

where Z_a^5's are related to the skew eigenvalues of the $USp(8)$ central charge matrix in the normal frame

$$e_{ab}=\begin{pmatrix} Z_1+Z_2-Z_3 & 0 & 0 & 0\\ 0 & Z_1+Z_3-Z_2 & 0 & 0\\ 0 & 0 & Z_2+Z_3-Z_1 & 0\\ 0 & 0 & 0 & -(Z_1+Z_2+Z_3)\end{pmatrix}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \quad (3.2)$$

We consider a configuration of only three nonvanishing electric charges (q_1, q_2, q_3), that we can take all nonnegative. We further confine to two moduli λ_1, λ_2, describing a geodesic submanifold $SO(1,1)^2 \in E_{6(6)}/USp(8)$ whose special geometry is determined by the constraint

$$\frac{1}{3!}d_{IJK}\hat{\lambda}^I\hat{\lambda}^J\hat{\lambda}^K=\hat{\lambda}^1\hat{\lambda}^2\hat{\lambda}^3=1, \quad (3.3)$$

where $\hat{\lambda}^I=\mathcal{V}^{-1/3}\lambda^I$, defining the *stu* model [29].

The metric a_{IJ}, restricted to this surface, takes the diagonal form

$$a_{IJ}=-\frac{\partial^2}{\partial\hat{\lambda}^I\partial\hat{\lambda}^J}\log\mathcal{V}|_{\mathcal{V}=1}=\begin{pmatrix}\frac{1}{\hat{\lambda}_1^2} & 0 & 0\\ 0 & \frac{1}{\hat{\lambda}_2^2} & 0\\ 0 & 0 & \frac{1}{\hat{\lambda}_3^2}=\hat{\lambda}_1^2\hat{\lambda}_2^2\end{pmatrix}, \quad (3.4)$$

and the five dimensional black hole potential for electric charges is[1]

$$V_5^e=q_Ia^{IJ}q_J=\sum_{a=1}^3 Z_a^5(q)Z_a^5(q), \quad (3.5)$$

with $Z_a=(a^{-1/2})_a^I Z_{\text{el}.I}$ and $Z_{\text{el}.I}=q_I$; the moduli at the attractor point of the five-dimensional solution are (see Eqs. (4.4) and (4.7) of [29])

$$\hat{\lambda}^I_{\text{crit}}=\frac{I_3^{1/3}}{q^I}, \quad (3.6)$$

and

$$V_5^{\text{crit}}=3|q_1q_2q_3|^{2/3}=3I_3^{2/3}, \qquad a^{IJ}_{\text{crit}}=\frac{I_3^{2/3}}{q_I^2}\delta^{IJ} \quad (3.7)$$

with no sum over repeated indices. We find

$$Z_a^{\text{crit}}=I_3^{1/3}, \qquad I_3=Z_1^5Z_2^5Z_3^5. \quad (3.8)$$

These relations also allow us to connect the potential in (3.5)

$$V_5=(Z_1^5)^2+(Z_2^5)^2+(Z_3^5)^2, \quad (3.9)$$

with the form given in terms of the central charges [35], where it is the trace of the square matrix

$$V_5=\tfrac{1}{2}Z_{ab}^5Z^{5ab}. \quad (3.10)$$

The eigenvalues of Z_{ab}^5 are written in (3.2) in terms of Z_1^5, Z_2^5, Z_3^5. The 5D central charge matrix in the normal frame at the attractor point thus becomes

$$e_{ab}=\begin{pmatrix} I_3^{1/3}\epsilon & 0 & 0 & 0\\ 0 & I_3^{1/3}\epsilon & 0 & 0\\ 0 & 0 & I_3^{1/3}\epsilon & 0\\ 0 & 0 & 0 & -3I_3^{1/3}\epsilon\end{pmatrix}, \quad (3.11)$$

which shows the breaking $USp(8)\to USp(6)\times USp(2)$.

IV. ATTRACTORS IN THE FOUR-DIMENSIONAL THEORY

In this section, we reconsider the attractor solutions found in [29,33], and we reformulate them in terms of the present formalism based on central charges. We separately examine the three axion-free configurations.

A. Electric solution $Q=(p^0, q_i)$

Let us first compute the four-dimensional central charge for the electric charge configuration with vanishing axions; using (2.20), we find

$$Z_0=\frac{i}{\sqrt{2}}e^{3\phi}p^0, \qquad Z_a=\frac{1}{\sqrt{2}}e^{-\phi}q_I(a^{-1/2})^I{}_a. \quad (4.1)$$

The four-dimensional potential is

$$V_{\text{BH}}=\tfrac{1}{2}e^{-2\phi}V_5^e+\tfrac{1}{2}e^{6\phi}(p^0)^2, \quad (4.2)$$

(where ϕ is connected to the volume used in Ref. [29] by the formula $\mathcal{V}=e^{6\phi}$) and has the same critical points of the five-dimensional potential, since

$$\frac{\partial V_{\text{BH}}}{\partial\lambda^I}=0\Leftrightarrow\frac{\partial V_5^e}{\partial\hat{\lambda}^I}=0, \quad \forall\ I=1,2. \quad (4.3)$$

The attractor values of $\hat{\lambda}^I$ are still given by (3.6), while the ϕ field at the critical point is [29]

$$e^{8\phi}|_{\text{crit}}=I_3^{2/3}(p^0)^{-2}. \quad (4.4)$$

This fixes the central charges at the attractor point to be

$$Z_0^{\text{attr}}=\frac{i}{\sqrt{2}}|p^0q_1q_2q_3|^{1/4}\,\text{sign}(p^0)=\frac{i}{2}|I_4|^{1/4}\,\text{sign}(p^0),$$

$$Z_a^{\text{attr}}=\frac{1}{\sqrt{2}}I_3^{-1/12}(p^0)^{1/4}q_I\frac{I_3^{1/3}}{q_I}=\frac{1}{2}|I_4|^{1/4}, \quad (4.5)$$

[1]In an analogous way, the black hole potential for magnetic charges, $V_5^m=\sum_{a=1}^3 Z_a^5(p)Z_a^5(p)$, is obtained by replacing $q_I\to p^I$ and $a^{IJ}\to a_{IJ}$ [29,35] with $Z_a^5(p)=p^I(a^{1/2})_I{}^a$.

where the quartic invariant is $I_4 = -4p^0q_1q_2q_3$. So we find

$$Z_1^{\rm crit} = Z_2^{\rm crit} = Z_3^{\rm crit} = \frac{1}{2}|I_4|^{1/4} \equiv Z, \qquad Z_0^{\rm crit} = \frac{i}{2}|I_4|^{1/4}\,{\rm sign}(p^0) \equiv iZ_0. \tag{4.6}$$

Let us define the 4D central charge matrix as

$$2Z_{AB} = e_{AB} - iZ^0\Omega, \tag{4.7}$$

where e_{AB} is the matrix in (3.2) in which, instead of Z_1^5, Z_2^5, Z_3^5 of the 5D theory, we now write the 4D Z_a's defined in (2.20). It can be readily seen that for axion-free solutions, Eq. (4.7) correctly gives

$$V_{\rm BH} = \sum_i |z_i|^2 = |Z_0|^2 + \sum_a |Z_a|^2, \tag{4.8}$$

where z_i's, for $i = 1, .., 4$, are the (complex skew-diagonal) elements of Z_{AB}. We then have

$$2Z_{AB} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix} + \begin{pmatrix} Z_0\epsilon & 0 & 0 & 0 \\ 0 & Z_0\epsilon & 0 & 0 \\ 0 & 0 & Z_0\epsilon & 0 \\ 0 & 0 & 0 & Z_0\epsilon \end{pmatrix} = \begin{pmatrix} (Z+Z_0)\epsilon & 0 & 0 & 0 \\ 0 & (Z+Z_0)\epsilon & 0 & 0 \\ 0 & 0 & (Z+Z_0)\epsilon & 0 \\ 0 & 0 & 0 & (-3Z+Z_0)\epsilon \end{pmatrix}. \tag{4.9}$$

Since (4.5) and (4.6) yield that $Z = |Z_0|$, depending on the choice $p^0 > 0$ or $p^0 < 0$, two different solutions arise. In fact,

$$Z + Z_0 = 0 \rightarrow Z_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Z_0 \end{pmatrix} \otimes \epsilon, \tag{4.10}$$

gives the $\frac{1}{8}$-BPS solution when $p^0 < 0$ and shows $SU(6) \times SU(2)$ symmetry. Conversely,

$$Z = Z_0 \rightarrow Z_{AB} = \begin{pmatrix} Z_0 & 0 & 0 & 0 \\ 0 & Z_0 & 0 & 0 \\ 0 & 0 & Z_0 & 0 \\ 0 & 0 & 0 & -Z_0 \end{pmatrix} \otimes \epsilon \tag{4.11}$$

is the non-BPS solution that corresponds to the choice $p^0 > 0$, with residual $USp(8)$ symmetry.

B. Magnetic solution $Q = (p_i, q^0)$

This case is symmetric to the electric solution of Sec. IV A. If we take all positive magnetic charges, then the cubic invariant is $I_3 = p^1p^2p^3$, the quartic invariant is $I_4 = 4q_0p^1p^2p^3$, and the values of the critical 5D moduli are now (see Eq. (5.3) of [29])

$$\hat{\lambda}^I = \frac{p^I}{I_3^{1/3}}. \tag{4.12}$$

The central charges for this configuration are, from (2.20),

$$Z_0 = \frac{1}{\sqrt{2}}e^{-3\phi}q_0, \qquad Z_a = \frac{i}{\sqrt{2}}e^{\phi}p^I(a^{1/2})_I{}^a, \tag{4.13}$$

and the black hole potential is

$$V_{\rm BH} = \tfrac{1}{2}e^{2\phi}V_5^m + \tfrac{1}{2}e^{-6\phi}(q_0)^2. \tag{4.14}$$

This gives the attractor value of the ϕ field as

$$e^{8\phi}|_{\rm crit} = I_3^{-2/3}(q_0)^2. \tag{4.15}$$

At the attractor point $(a^{1/2}_{\rm crit})_{IJ} = (\hat{\lambda}^I)^{-1}\delta_{IJ}$, and the magnetic central charges are

$$Z_a^{\rm crit} = \frac{i}{\sqrt{2}}(I_3)^{1/4}|q_0|^{1/4} = \frac{i}{2}|I_4|^{1/4} \equiv iZ, \qquad a = 1, 2, 3. \tag{4.16}$$

We can then write the central charge matrix corresponding to the **27** representation in the normal frame as

$$e_{AB} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix}. \tag{4.17}$$

To describe the four-dimensional solution, we need the electric central charge, that at the attractor point is

$$Z_0^{\rm crit} = \frac{1}{\sqrt{2}}(I_3)^{1/4}|q_0|^{1/4}\,{\rm sign}(q_0) = \frac{1}{2}|I_4|^{1/4}\,{\rm sign}(q_0) \equiv Z_0.$$

Then, using the definition (4.7), the complete 4D central charge matrix is

$$2Z_{AB} = i\begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & Z\epsilon & 0 & 0 \\ 0 & 0 & Z\epsilon & 0 \\ 0 & 0 & 0 & -3Z\epsilon \end{pmatrix} - i\begin{pmatrix} Z_0\epsilon & 0 & 0 & 0 \\ 0 & Z_0\epsilon & 0 & 0 \\ 0 & 0 & Z_0\epsilon & 0 \\ 0 & 0 & 0 & Z_0\epsilon \end{pmatrix} = e^{i\pi/2}\begin{pmatrix} (Z-Z_0)\epsilon & 0 & 0 & 0 \\ 0 & (Z-Z_0)\epsilon & 0 & 0 \\ 0 & 0 & (Z-Z_0)\epsilon & 0 \\ 0 & 0 & 0 & (-3Z-Z_0)\epsilon \end{pmatrix}. \tag{4.18}$$

The sign(q_0) determines whether the solution is supersymmetric or not. We may have

$$q_0 > 0 \rightarrow Z = Z_0, \qquad Z_{AB} = e^{i\pi/2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2Z_0 \end{pmatrix} \otimes \epsilon, \tag{4.19}$$

which is a magnetic $\frac{1}{8}$-BPS solutions with $SU(6)\times SU(2)$ symmetry, or

$$q_0<0\to Z=-Z_0,$$
$$Z_{AB}=e^{i\pi/2}\begin{pmatrix}-Z_0 & 0 & 0 & 0\\ 0 & -Z_0 & 0 & 0\\ 0 & 0 & -Z_0 & 0\\ 0 & 0 & 0 & Z_0\end{pmatrix}\otimes\epsilon, \tag{4.20}$$

which is the non-BPS solution with $USp(8)$ symmetry. These solutions have the same Z_0 as the electric ones, but now the choice of positive q_0 charge leads to the supersymmetric solution, while the negative q_0 charge gives the nonsupersymmetric one, in contrast with what happened for the choice of p^0 in the electric case in Eq. (4.10) and (4.11).

C. Kaluza-Klein dyonic solution $Q=(p^0,q_0)$

This charge configuration also has vanishing axions, and the only nonzero charges give

$$Z_0^e=e^{-3\phi}q_0,\qquad Z_m^0=e^{3\phi}p^0,$$
$$\Downarrow$$
$$Z_0=\tfrac{1}{\sqrt{2}}(e^{-3\phi}q_0+ie^{3\phi}p^0). \tag{4.21}$$

Since none of the five-dimensional charges are turned on, the four-dimensional black hole potential is

$$V_{\rm BH}=\tfrac{1}{2}[e^{-6\phi}q_0^2+e^{6\phi}(p^0)^2], \tag{4.22}$$

which is extremized at the horizon by the value of the ϕ field

$$e^{6\phi}|_{\rm crit}=\left|\frac{q_0}{p^0}\right|. \tag{4.23}$$

We only focus on the case $p^0>0$ and $q_0>0$, since all of the other choices are related to this by a duality rotation. Evaluating the central charge at the attractor point, we find

$$Z_0^{\rm crit}=\sqrt{|p^0q_0|}\frac{1+i}{\sqrt{2}}=\sqrt{|p^0q_0|}e^{i\pi/4}. \tag{4.24}$$

Following the prescription in (4.7), we find that at the attractor point,

$$2Z_{AB}=-iZ_0\Omega$$
$$=-ie^{i\pi/4}\begin{pmatrix}\sqrt{|p^0q_0|}\epsilon & 0 & 0 & 0\\ 0 & \sqrt{|p^0q_0|}\epsilon & 0 & 0\\ 0 & 0 & \sqrt{|p^0q_0|}\epsilon & 0\\ 0 & 0 & 0 & \sqrt{|p^0q_0|}\epsilon\end{pmatrix} \tag{4.25}$$

that gives a non-BPS four-dimensional black hole with $I_4=-(p^0q_0)^2$.

Note that Eqs. (4.11), (4.20), and (4.25) imply that the sum of the phases of the four complex skew entries is π, as appropriate to a non-BPS $\mathcal{N}=8$ solution [17]. Also, in all cases, $V_{\rm BH}|_{\rm crit}=\sqrt{|I_4|}$.

D. $\mathcal{N}=8$ and $\mathcal{N}=2$ attractive orbits at $d=5$ and $d=4$

We now compare the different interpretations in the $\mathcal{N}=8$ and $\mathcal{N}=2$ theories of the critical points of the very same black hole 4D potential, in terms of the axion-free electric solution (Sec. IV A) as discussed in this paper and in Ref. [29].

Since the "normal frame" solution is common to all symmetric spaces (with rank three), it can be regarded as the generating solution of any model. So we confine our attention to the exceptional $\mathcal{N}=2$ (octonionic) $E_{7(-25)}$ model [39], which has a charge vector in 5D and 4D of the same dimension as in $\mathcal{N}=8$ supergravity. At $d=5$, the duality group is $E_{6(-26)}$, with moduli space of vector multiplets $E_{6(-26)}/F_4$.

It is known [24,40] that in $d=5$, there are two different charge orbits,

$$\mathcal{O}^{\mathcal{N}=2}_{d=5,\rm BPS}=\frac{E_{6(-26)}}{F_4}, \tag{4.26}$$

the BPS one, and the non-BPS one

$$\mathcal{O}^{\mathcal{N}=2}_{d=5,\text{non-BPS}}=\frac{E_{6(-26)}}{F_{4(-20)}}, \tag{4.27}$$

The latter one precisely corresponds to the nonsupersymmetric solution and to $(++-)$, $(--+)$ signs of the q_1, q_2, q_3 charges (implying $\partial Z\neq 0$). For charges of the same sign $(+++)$, $(---)$ one has the $\frac{1}{8}$-BPS solution ($\partial Z=0$), as discussed in [29].

It is easy to see that in the $\mathcal{N}=8$ theory, all of these solutions just interchange Z_1, Z_2, Z_3, and $Z_4=-3Z_3$ but always give a normal frame matrix of the form

$$Z_{ab}=\begin{pmatrix}Z\epsilon & 0 & 0 & 0\\ 0 & Z\epsilon & 0 & 0\\ 0 & 0 & Z\epsilon & 0\\ 0 & 0 & 0 & -3Z\epsilon\end{pmatrix}, \tag{4.28}$$

which has $USp(6)\times USp(2)\in F_{4(4)}$ as maximal symmetry. Another related observation is that while $E_{6(-26)}$ contains both F_4 and $F_{4(-20)}$, so that one expects two orbits and two classes of solution, in the $\mathcal{N}=8$ case $E_{6(6)}$ contains only the noncompact $F_{4(4)}$, thus only one class of solutions is possible.

These orbits and critical points at $d=5$ have a further story when used to study the $d=4$ critical points with axion-free solutions as it is the case for the electric (p^0,q_1,q_2,q_3) configuration. Since in this case $I_4=-4p^0q_1q_2q_3$, in the $\mathcal{N}=8$ case, once one choose q_1, q_2, $q_3>0$, the $I_4>0$, $p^0<0$ solution is BPS, while the $I_4<0$, $p^0>0$ is non-BPS.

Things again change in $\mathcal{N}=2$ [41], when now we consider the solution embedded in the Octonionic model with 4D moduli space $E_{7(-25)}/E_6\times U(1)$. A new non-BPS orbit in $d=4$ is generated, corresponding to $Z=0$ ($\partial Z\neq 0$) solution, so three 4D orbits exist in this case depending whether the $(+++)$ and $(++-)$ solutions are combined with $-p^0 \lessgtr 0$. So

$$(+,+++)\quad \text{is BPS with}\quad I_4>0,\qquad \mathcal{O}=\frac{E_{7(-25)}}{E_6}, \tag{4.29}$$

$$(-,-++)\quad \text{is non-BPS with}\quad I_4>0,\qquad \mathcal{O}=\frac{E_{7(-25)}}{E_{6(-14)}}, \tag{4.30}$$

$$(+,-++)\quad\text{or}\quad(-,+++)\quad \text{is non-BPS with}\quad I_4<0,\qquad \mathcal{O}=\frac{E_{7(-25)}}{E_{6(-26)}}. \tag{4.31}$$

V. MAURER-CARTAN EQUATIONS OF THE FOUR-DIMENSIONAL THEORY

Let us call Maurer-Cartan equations [16] those which give the derivative of the central charges (coset representatives) with respect to the moduli ϕ, a^I, λ^i. Using (2.8), we have

$$\partial_\phi Z_0^e=-3Z_0^e,\qquad \partial_\phi Z_m^0=3Z_m^0,\qquad \partial_\phi Z_I^e=-Z_I^e,\qquad \partial_\phi Z_m^I=Z_m^I, \tag{5.1}$$

and

$$\frac{\partial Z_0^e}{\partial a^I}=e^{-2\phi}Z_I^e,\qquad \frac{\partial Z_m^0}{\partial a^I}=0,\qquad \frac{\partial Z_m^I}{\partial a^J}=-\delta^I_J e^{-2\phi}Z_e^0,\qquad \frac{\partial Z_I^e}{\partial a^J}=-e^{-2\phi}d_{IJK}Z_m^K. \tag{5.2}$$

In our notation, the 5D metric a_{IJ}, $(I,J=1,..,27)$ can also be rewritten with a pair of antisymmetric (traceless) indices

$$a_{\Lambda\Sigma,\Delta\Gamma}=L^{ab}{}_{\Lambda\Sigma}L_{\Delta\Gamma ab}, \tag{5.3}$$

where $L^{ab}{}_{\Lambda\Sigma}$ is the coset representative; in a fixed gauge (where a, b and Λ, Σ indices are identified)

$$L_I{}^a=(a^{1/2})_I{}^a,\qquad (\bar{L}_{Ia}=L^T_{Ia}). \tag{5.4}$$

The object $\mathbb{P}_i\equiv a^{1/2}\partial_i a^{-1/2}$ can be regarded as the Maurer-Cartan connection (see Ref. [32]). In fact, by reminding that $Z_a^e=Z_I^e(a^{-1/2})^I{}_a$, we have $\partial_i Z_a^e=(\partial_i a^{-1/2})^I{}_a Z_I^e$ (since $\partial_i Z_I^e=0$). Since we can also write 0

$$\partial_i Z_a^e=(\partial_i a^{-1/2})^I{}_a(a^{1/2})_I{}^b Z_b^e, \tag{5.5}$$

we find that $\mathbb{P}_{i,a}{}^b$ is such that

$$\partial_i Z_a^e=\mathbb{P}_{i,a}{}^b Z_b^e. \tag{5.6}$$

Notice that using $\mathbb{P}_{i,a}{}^b=Q_{i,a}{}^b+V_{i,a}{}^b$, we identify a connection which satisfies

$$\nabla_i Z_a^e=V_a{}^b Z_b^e, \tag{5.7}$$

with $\nabla_i=\partial_i-Q_i$.

Attractor equations from Maurer-Cartan equations

We can now use this formalism to write the attractor equations for the potential

$$V_{\rm BH}=\tfrac{1}{2}(Z_0^e)^2+\tfrac{1}{2}(Z_m^0)^2+\tfrac{1}{2}Z_I^e a^{IJ}Z_J^e+\tfrac{1}{2}Z_m^I a_{IJ}Z_m^J. \tag{5.8}$$

By differentiating with respect to ϕ, a^I, λ^i, we get

$$\partial_\phi V_{\rm BH}=-3(Z_0^e)^2+3(Z_m^0)^2-Z_I^e a^{IJ}Z_J^e+Z_m^I a_{IJ}Z_m^J=0, \tag{5.9}$$

$$\partial_{a^I}V_{\rm BH}=e^{-2\phi}[Z_0^eZ_I^e-Z_J^e a^{JK}d_{IKL}Z_m^L-Z_m^0 a_{IJ}Z_m^J]=0, \tag{5.10}$$

$$\partial_{\lambda^i}V_{\rm BH}\equiv\partial_i V_{\rm BH}=\tfrac{1}{2}Z_I^e\partial_i a^{IJ}Z_J^e+\tfrac{1}{2}Z_m^I\partial_i a_{IJ}Z_m^J=0. \tag{5.11}$$

From (5.10), we see that a solution with $a^I=0$ implies

$$\partial_{a^I}V_{\rm BH}|_{a^I=0}=e^{-2\phi}[e^{-4\phi}q_0q_I-q_Ja^{JK}d_{IKL}p^L-e^{4\phi}p^0a_{IJ}p^J]=0, \tag{5.12}$$

which is trivially satisfied if we set $\neq 0$ (q_0,p^0) or (q_0,p^I) or (p^0,q_I).

From (5.9), we see that for an axion-free solution, if Z_0^e, $Z_m^I=0$, we get

$$3(Z_m^0)^2=Z_I^e a^{IJ}Z_J^e, \tag{5.13}$$

and if a_{IJ} is diagonal, $I=J=1,2,3$, we obtain

$$3(Z_m^0)^2=(Z_1^e)^2a^{11}+(Z_2^e)^2a^{22}+(Z_3^e)^2a^{33}, \tag{5.14}$$

which is compatible with $Z_1^e=Z_2^e=Z_3^e=\pm Z_m^0$.

The derivative with respect to the 5D moduli λ^i, $i=1,..,42$ for $\mathcal{N}=8$ theory, only receives contributions from the matrix a_{IJ}. Indeed, since Z_I^e, Z_m^I do not depend on the λ^i [see Eq. (2.8)], one finds

$$\partial_i V_4=0=Z_I^e\partial_i a^{IJ}Z_J^e+Z_m^I\partial_i a_{IJ}Z_m^J. \tag{5.15}$$

By rewriting the charges multiplied by $(a^{-1/2})^I{}_a$ and $(a^{1/2})_I{}^a$ so that

$$Z_a^e\equiv Z_I^e(a^{-1/2})^I{}_a,\qquad Z_m^a=Z_m^I(a^{1/2})^a{}_I, \tag{5.16}$$

we have

$$\partial_i Z_a^e=\mathbb{P}_{i,a}{}^b Z_b^e,\qquad \mathbb{P}_{i,a}{}^b=\partial_i(a^{-1/2})^I{}_a(a^{1/2})_I{}^b,\qquad \partial_i Z_m^a=\mathbb{P}^a_{i\ b}Z_m^b,\qquad \mathbb{P}^a_{i\ b}=\partial_i(a^{1/2})_I{}^a(a^{-1/2})^I{}_b, \tag{5.17}$$

where $\mathbb{P}^{a}_{i\ b} = -\mathbb{P}_{ib}{}^{a}$ since $\partial_i(Z^e_a Z^a_m) = 0$. Then, we also have

$$\partial_i(Z^e_a Z^e_a) = Z^e_a(\mathbb{P}_{ia}{}^{b})Z^e_b = Z^e_a\mathbb{P}_{i,ab}Z^e_b = Z^e_a\mathbb{P}_{i(ab)}Z^e_b = 0, \tag{5.18}$$

and if we split $\mathbb{P}_{i,ab} = Q_{i[ab]} + V_{i(ab)}$, with

$$\mathbb{P}^{\ a}_{i\ b} = Q^{a}_{i\ b} + V^{a}_{i\ b}, \qquad \mathbb{P}_{i,a}{}^{b} = Q_{i,a}{}^{b} - V_{i,a}{}^{b}, \tag{5.19}$$

then the critical condition implies

$$\partial_i(Z^e Z^e) = Z^e_a V_{i(ab)} Z^e_b = 0, \tag{5.20}$$

and the analogue equation for magnetic charges

$$\partial_i(Z^m Z^m) = Z^a_m V_{i(ab)} Z^b_m = 0, \tag{5.21}$$

so that only the vielbein $V_{i,ab}$ enters in the equations of motion.

The criticality condition on the potential of Eq. (5.15) now gives

$$\partial_i V_{\rm BH} = 0 \rightarrow Z^e_a V_i{}^{ab} Z^e_b + Z^a_m V_{i,ab} Z^b_m = 0, \tag{5.22}$$

thus, for electric configurations ($Z^b_m = 0$) with $a^I = 0$,

$$Z^e_a V_i{}^{ab} Z^e_b = 0. \tag{5.23}$$

Comparing results of [35] with our formulas, we see that V_1, V_2, V_3, with $V_1 + V_2 + V_3 = 0$, in the case where the metric a_{IJ} is diagonal, correspond to

$$(a^{-1/2})^I{}_a \partial_i (a^{1/2})_J{}^a = (a^{-1/2})^I \partial_i (a^{1/2})_I = \mathbb{P}^I_{i\ I} = V^I_{i\ I} \equiv V_i{}^I, \tag{5.24}$$

where $(a^{-1/2})^I{}_I \equiv (a^{-1/2})^I$, $(a^{1/2})_I{}^I \equiv (a^{1/2})_I$, $I = 1, 2, 3$, and using (3.4), we find

$$V^I_1 = \left(\frac{1}{\hat{\lambda}_1}, 0, -\frac{1}{\hat{\lambda}_1}\right), \qquad V^I_2 = \left(0, \frac{1}{\hat{\lambda}_2}, -\frac{1}{\hat{\lambda}_2}\right). \tag{5.25}$$

Indeed,

$$\sum_{i=1,2,3} V^I_i = 0, \tag{5.26}$$

so, by using Eqs. (2.31)–(2.33) of Ref. [35], one gets the desired result. In fact, using the definitions of $\mathbb{P}^I_1$ and $\mathbb{P}^I_2$, we get from the $\hat{\lambda}^i$ equations of motion

$$\sum_I Z^e_I V^I_i Z^e_I = 0, \tag{5.27}$$

which explicitly gives

$$Z^e_1 Z^e_1 - Z^e_3 Z^e_3 = 0, \qquad Z^e_2 Z^e_2 - Z^e_3 Z^e_3 = 0, \tag{5.28}$$

whose solution, combined with Eq. (5.14), gives

$$\begin{gathered}(Z^e_1)^2 = (Z^e_2)^2 = (Z^e_3)^2 = (Z^0_m)^2,\\ \Downarrow \\ Z^e_1 = Z^e_2 = Z^e_3 = \pm Z^0_m,\end{gathered} \tag{5.29}$$

all the other sign choices being equivalent in the 5D theory.

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