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APPROXIMATION PROPERTIES OF SOME DISCRETE FOURIER SUMS FOR PIECEWISE SMOOTH DISCONTINUOUS FUNCTIONS

Abstract. Denote by $L_{n,N}(f,x)$ a trigonometric polynomial of order at most n possessing the least quadratic deviation from f with respect to the system $\left\{t_k = u + \frac{2\pi k}{N}\right\}_{k=0}^{N-1}$, where $u \in \mathbb{R}$ and $n \leqslant N/2$. Let D^1 be the space of 2π-periodic piecewise continuously differentiable functions f with a finite number of jump discontinuity points $-\pi = \xi_1 < \ldots < \xi_m = \pi$ and with absolutely continuous derivatives on each interval (ξ_i, ξ_{i+1}). In the present article, we consider the problem of approximation of functions $f \in D^1$ by the trigonometric polynomials $L_{n,N}(f,x)$. We have found the exact order estimate $|f(x) - L_{n,N}(f,x)| \leqslant c(f,\varepsilon)/n$, $|x-\xi_i| \geqslant \varepsilon$. The proofs of these estimations are based on comparing of approximating properties of discrete and continuous finite Fourier series.

Key words: *function approximation, trigonometric polynomials, Fourier series*

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1. Introduction. Let D^1 be the space of 2π-periodic functions f, each of which has a finite number of jump discontinuity points $\Omega(f) = \{\xi_i\}_{i=0}^m$, where $-\pi = \xi_0 < \xi_1 < \ldots < \xi_m = \pi$, $f(\xi_i) = (f(\xi_i - 0) + f(\xi_i + 0))/2$ and has an absolutely continuous derivative f' on each interval (ξ_i, ξ_{i+1}) $(0 \leqslant i \leqslant m)$ (here we say that a function f is absolutely continuous on an interval (a,b) if the function $\overline{f}$ is absolutely continuous on the segment $[a,b]$, where $\overline{f}(x) = f(x)$ for $x \in (a,b)$, $\overline{f}(a) = f(a+0)$, and $\overline{f}(b) = f(b-0)$). One of such functions is $f(x) = \operatorname{sign}(\sin x)$.

Denote by $L_{n,N}(f,x)$ $(1 \leqslant n \leqslant \lfloor N/2 \rfloor)$ the trigonometric polynomial of order at most n that possesses the least quadratic deviation from the function f with respect to the system $\{t_k\}_{k=0}^{N-1}$, where $t_k = u + 2\pi k/N$ $(u \in \mathbb{R})$. In other words, the minimum of the sums $\sum_{k=0}^{N-1} |f(t_k) - T_n(t_k)|^2$

on the set of trigonometric polynomials T_n of order n is attained when $T_n = L_{n,N}(f)$. In particular, $L_{\lfloor N/2\rfloor,N}(f,t_k) = f(t_k)$. It is easy to show (see [13]) that for $n < N/2$ the polynomial $L_{n,N}(f,x)$ can be represented as follows:

$$L_{n,N}(f,x) = \sum_{\nu=-n}^{n} c_\nu^{(N)}(f) e^{i\nu x}, \quad c_\nu^{(N)}(f) = \frac{1}{N}\sum_{k=0}^{N-1} f(t_k)e^{-i\nu t_k};$$

and for $n = N/2$:

$$L_{N/2,N}(f,x) = L_{N/2-1,N}(f,x) + a_{N/2}^{(N)}(f)\cos\frac{N}{2}(x-u), \tag{1}$$

where

$$a_n^{(2n)}(f) = a_{N/2}^{(N)}(f) = \frac{1}{N}\sum_{k=0}^{N-1} f(t_k)\cos\frac{N}{2}(t_k-u). \tag{2}$$

By $S_n(f,x)$ we denote the partial Fourier sum of order n of f:

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos kx + b_k \sin kx\right),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt dt, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt dt.$$

To read more about approximation of functions by trigonometric polynomials, see [4–7], [9–12], [14].

Also, later we will need the function

$$h_p(x) = \begin{cases} \cos x, & p = 0, \\ \sin x, & p = 1 \end{cases}$$

and the well-known inequalities

$$\left|\sum_{k=1}^{\infty}\frac{\sin kx}{k}\right| \leqslant \frac{\pi}{2}, \tag{3}$$

$$\left|\sum_{k=1}^{n} h_p(kx)\right| \leqslant \frac{1}{|\sin\frac{x}{2}|}, \quad x \neq 2i\pi, \quad i = 0, \pm1, \pm2, \dots, \tag{4}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{5}$$

It is easy to show, that the Fourier series converges pointwise for any function $f \in D^1$ and, therefore, the function can be represented as follows:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) .$$

In the previous works, the author found estimates for the value $|f(x) - L_{n,N}(f,x)|$ for 2π-periodic piecewise-linear and piecewise-smooth continuous functions (see [1], [2]). Also, two particular cases of such functions – 2π-periodic functions $f(x) = |x|$ and $f(x) = \operatorname{sign} x$, $x \in [-\pi, \pi]$ – were considered in [3]. The goal of this work is to estimate $|f(x) - L_{n,N}(f,x)|$ for $f \in D^1$ as $n, N \to \infty$. We obtained the following result:

Theorem 1. *For a function $f \in D^1$, the following estimate holds:*

$$|f(x) - L_{n,N}(f,x)| \leqslant \frac{C(f,\varepsilon)}{n}, \quad 1 \leqslant n \leqslant \lfloor N/2 \rfloor, \quad |x - \xi_i| > \varepsilon, \tag{6}$$

where $i = 0, 1, \ldots, m$. The order of this estimate cannot be improved.

To prove this theorem, we use a lemma from [13]:

Lemma 1. [13] *If the Fourier series of f converges at the points $t_k = u + 2k\pi/N$, then the representation*

$$L_{n,N}(f,x) = S_n(f,x) + R_{n,N}(f,x),$$

where

$$R_{n,N}(f,x) = \frac{2}{\pi} \sum_{\mu=1}^{\infty} \int_{-\pi}^{\pi} f(t) D_n(x-t) \cos \mu N(u-t) dt, \tag{7}$$

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx, \tag{8}$$

holds true when $2n < N$.

From this lemma, we have the following estimate:

$$|f(x) - L_{n,N}(f,x)| \leqslant |f(x) - S_n(f,x)| + |R_{n,N}(f,x)|, \quad n < N/2. \tag{9}$$

In the case $2n = N$, from (1) and (9) we have

$$|f(x) - L_{n,N}(f,x)| \leqslant$$
$$\leqslant |f(x) - S_{n-1}(f,x)| + |R_{n-1,N}(f,x)| + |a_n^{(N)}(f)|, \quad n = N/2. \quad (10)$$

The estimate for $|f(x) - S_n(f,x)|$, where $f \in D^1$, were obtained in [8]:

$$|f(x) - S_n(f,x)| \leqslant \frac{C(f,\varepsilon)}{n}, \quad |x - \xi_i| \geqslant \varepsilon. \quad (11)$$

Now we have to estimate the values $|R_{n,N}(f,x)|$ and $|a_n^{(2n)}(f)|$, which is done in the following sections.

2. The estimate for $|R_{n,N}(f,x)|$. From (7) and (8), we can get the representation

$$R_{n,N}(f,x) = \frac{1}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi} f(t)\cos\mu N(u-t)dt+$$
$$+\frac{2}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi} f(t)\sum_{k=1}^{n}\cos k(x-t)\cos\mu N(u-t)dt =$$
$$= R^1_{n,N}(f,x) + R^2_{n,N}(f,x).$$

Lemma 2. *For $\alpha \in (0, \frac{1}{2}]$, the following inequality holds:*

$$\left|\sum_{k=1}^{\infty}\frac{\sin kx}{k\left(1-\frac{\alpha^2}{k^2}\right)}\right| \leqslant c.$$

Proof. Performing the Abel transformation (summation by parts), we get

$$\sum_{k=1}^{\infty}\frac{\sin kx}{k\left(1-\frac{\alpha^2}{k^2}\right)} = \sum_{k=1}^{\infty}\left(\frac{1}{1-\frac{\alpha^2}{k^2}} - \frac{1}{1-\frac{\alpha^2}{(k+1)^2}}\right)\sum_{j=1}^{k}\frac{\sin jx}{j} =$$
$$= \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\alpha^2\left(2+\frac{1}{k}\right)}{\left(1+\frac{1}{k}\right)^2\left(1-\frac{\alpha^2}{k^2}\right)\left(1-\frac{\alpha^2}{(k+1)^2}\right)}\frac{1}{k}\sum_{j=1}^{k}\frac{\sin jx}{j}.$$

Using (5) and the inequalities

$$\frac{\alpha^2\left(2+\frac{1}{k}\right)}{\left(1+\frac{1}{k}\right)^2\left(1-\frac{\alpha^2}{k^2}\right)\left(1-\frac{\alpha^2}{(k+1)^2}\right)} \leqslant \frac{16}{15}, \quad \left|\frac{1}{k}\sum_{j=1}^{k}\frac{\sin jx}{j}\right| \leqslant 1,$$

we have

$$\left|\sum_{k=1}^{\infty}\frac{\sin kx}{k\left(1-\frac{\alpha^2}{k^2}\right)}\right| \leqslant$$

$$\leqslant \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\alpha^2\left(2+\frac{1}{k}\right)}{\left(1+\frac{1}{k}\right)^2\left(1-\frac{\alpha^2}{k^2}\right)\left(1-\frac{\alpha^2}{(k+1)^2}\right)}\left|\frac{1}{k}\sum_{j=1}^{k}\frac{\sin jx}{j}\right| \leqslant c.$$

This completes the proof. □

Lemma 3. *For $f \in D^1$, the following holds:*

$$\int_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q \mu N}{(\mu N)^2 - k^2}\sum_{i=0}^{m-1}\left(f(\xi_i - 0) - f(\xi_i + 0)\right)h_p(k(\xi_i - x))h_{1-q}(\mu N(\xi_i - u)) -$$

$$- \frac{(-1)^q \mu N}{(\mu N)^2 - k^2}\int_{-\pi}^{\pi} f'(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt +$$

$$+ \frac{(-1)^{1+p} k}{(\mu N)^2 - k^2}\sum_{i=0}^{m-1}\left(f(\xi_i - 0) - f(\xi_i + 0)\right)h_{1-p}(k(\xi_i - x))h_q(\mu N(\xi_i - u)) -$$

$$- \frac{(-1)^{1+p} k}{(\mu N)^2 - k^2}\int_{-\pi}^{\pi} f'(t)h_{1-p}(k(t-x))h_q(\mu N(t-u))dt. \quad (12)$$

Proof. Perform integration by parts:

$$\int_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q}{\mu N}\sum_{i=0}^{m-1}\left(f(\xi_i - 0) - f(\xi_i + 0)\right)h_p(k(\xi_i - x))h_{1-q}(\mu N(\xi_i - u)) -$$

$$- \frac{(-1)^q}{\mu N}\int_{-\pi}^{\pi} f'(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt +$$

$$+\frac{(-1)^{p+q}k}{\mu N}\int\limits_{-\pi}^{\pi} f(t)h_{1-p}(k(t-x))h_{1-q}(\mu N(t-u))dt. \quad (13)$$

Repeat integration by parts for the last integral in (13):

$$\int\limits_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt =$$

$$= \frac{(-1)^q}{\mu N}\sum_{i=0}^{m-1}\left(f(\xi_i-0)-f(\xi_i+0)\right)h_p(k(\xi_i-x))h_{1-q}(\mu N(\xi_i-u)) -$$

$$-\frac{(-1)^q}{\mu N}\int\limits_{-\pi}^{\pi} f'(t)h_p(k(t-x))h_{1-q}(\mu N(t-u))dt +$$

$$+\frac{(-1)^{1+p}k}{(\mu N)^2}\sum_{i=0}^{m-1}\left(f(\xi_i-0)-f(\xi_i+0)\right)h_{1-p}(k(\xi_i-x))h_q(\mu N(\xi_i-u)) -$$

$$-\frac{(-1)^{1+p}k}{(\mu N)^2}\int\limits_{-\pi}^{\pi} f'(t)h_{1-p}(k(t-x))h_q(\mu N(t-u))dt +$$

$$+\frac{k^2}{(\mu N)^2}\int\limits_{-\pi}^{\pi} f(t)h_p(k(t-x))h_q(\mu N(t-u))dt.$$

By moving the last integral to the left-hand side and dividing both sides by $\frac{(\mu N)^2-k^2}{(\mu N)^2}$, we get (12). □

Lemma 4. *The value* $\left|R^1_{n,N}(f,x)\right|$, *where* $f \in D^1$, *can be estimated as follows:*

$$\left|R^1_{n,N}(f,x)\right| \leqslant \frac{c(f)}{N}.$$

Proof. Performing integration by parts twice, we get

$$R^1_{n,N}(f,x) =$$

$$= \frac{1}{\pi}\sum_{\mu=1}^{\infty}\int\limits_{-\pi}^{\pi} f(t)\cos\mu N(t-u)dt = \frac{1}{\pi}\sum_{\mu=1}^{\infty}\sum_{i=0}^{m-1}\int\limits_{\xi_i}^{\xi_{i+1}} f(t)\cos\mu N(t-u)dt =$$

$$= \frac{1}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu}\sum_{i=0}^{m-1}\left(f(\xi_i-0)-f(\xi_i+0)\right)\sin\mu N(\xi_i-u)+$$
$$+\frac{1}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\left[\sum_{i=0}^{m-1}\left(f^{'}(\xi_i-0)-f^{'}(\xi_i+0)\right)\cos\mu N(\xi_i-u)-\right.$$
$$\left.-\int_{-\pi}^{\pi}f^{''}(t)\cos\mu N(t-u)dt\right].$$

Applying some simple transformations and using (3), we have

$$\left|R^1_{n,\,N}(f,x)\right|\leqslant\frac{1}{\pi N}\sum_{i=0}^{m-1}\left|f(\xi_i-0)-f(\xi_i+0)\right|\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N(\xi_i-u)}{\mu}\right|+$$
$$+\frac{1}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\left[\sum_{i=0}^{m-1}\left|f^{'}(\xi_i-0)-f^{'}(\xi_i+0)\right|+\int_{-\pi}^{\pi}\left|f^{''}(t)\right|dt\right]\leqslant\frac{c(f)}{N}.$$

This completes the proof. □

Lemma 5. *The value* $\left|R^2_{n,\,N}(f,x)\right|$, *where* $f\in D^1$, *can be estimated as follows:*

$$\left|R^2_{n,\,N}(f,x)\right|\leqslant\frac{c(f,\varepsilon)}{N},\quad |x-\xi_i|\geqslant\varepsilon.$$

Proof. Using Lemma 3, we have

$$R^2_{n,\,N}(f,x)=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\sum_{k=1}^{n}\int_{-\pi}^{\pi}f(t)\cos k(t-x)\cos\mu N(t-u)dt=$$
$$=\frac{2}{\pi N}\sum_{i=0}^{m-1}\left(f(\xi_i-0)-f(\xi_i+0)\right)\sum_{\mu=1}^{\infty}\frac{\sin\mu N(\xi_i-u)}{\mu}\sum_{k=1}^{n}\frac{\cos k(\xi_i-x)}{1-\left(\frac{k}{\mu N}\right)^2}+$$
$$+\frac{-2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu}\sum_{k=1}^{n}\frac{1}{1-\left(\frac{k}{\mu N}\right)^2}\int_{-\pi}^{\pi}f^{'}(t)\cos k(t-x)\sin\mu N(t-u)dt+$$
$$+\frac{-2}{\pi N^2}\sum_{i=0}^{m-1}\left(f(\xi_i-0)-f(\xi_i+0)\right)\sum_{\mu=1}^{\infty}\frac{\cos\mu N(\xi_i-u)}{\mu^2}\sum_{k=1}^{n}\frac{k\sin k(\xi_i-x)}{1-\left(\frac{k}{\mu N}\right)^2}+$$

$$+\frac{2}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{k=1}^{n}\frac{k}{1-\left(\frac{k}{\mu N}\right)^2}\int\limits_{-\pi}^{\pi} f^{'}(t)\sin k(t-x)\cos\mu N(t-u)dt =$$

$$= R^{2.1}_{n,N}(f,x)+R^{2.2}_{n,N}(f,x)+R^{2.3}_{n,N}(f,x)+R^{2.4}_{n,N}(f,x).$$

Here we estimate only the values $\left|R^{2.1}_{n,N}(f,x)\right|$ and $\left|R^{2.2}_{n,N}(f,x)\right|$, because $\left|R^{2.3}_{n,N}(f,x)\right|$ and $\left|R^{2.4}_{n,N}(f,x)\right|$ can be estimated in the similar way. Begin with $\left|R^{2.1}_{n,N}(f,x)\right|$. Consider the expression

$$A=\sum_{k=1}^{n}\cos k(\xi_i-x)\sum_{\mu=1}^{\infty}\frac{\sin\mu N(\xi_i-u)}{\mu\left(1-\left(\frac{k}{\mu N}\right)^2\right)}.$$

Applying the Abel transformation, we get

$$A=\sum_{\mu=1}^{\infty}\frac{\sin\mu N(\xi_i-u)}{\mu\left(1-\left(\frac{n}{\mu N}\right)^2\right)}\sum_{j=1}^{n}\cos j(\xi_i-x)+$$

$$+\sum_{k=1}^{n-1}\sum_{\mu=1}^{\infty}\frac{\sin\mu N(\xi_i-u)}{\mu}\left(\frac{1}{1-\left(\frac{k}{\mu N}\right)^2}-\frac{1}{1-\left(\frac{k+1}{\mu N}\right)^2}\right)\sum_{j=1}^{k}\cos j(\xi_i-x).$$

Using (4), Lemma 2 and the fact that

$$\frac{1}{1-\left(\frac{k}{\mu N}\right)^2}-\frac{1}{1-\left(\frac{k+1}{\mu N}\right)^2}=-\frac{k}{(\mu N)^2}\frac{2+\frac{1}{k}}{\left(1-\left(\frac{k}{\mu N}\right)^2\right)\left(1-\left(\frac{k+1}{\mu N}\right)^2\right)},$$

we get

$$|A|\leqslant\frac{c}{\left|\sin\frac{\xi_i-x}{2}\right|}.$$

From this, we get the estimate for $\left|R^{2.1}_{n,N}(f,x)\right|$:

$$\left|R^{2.1}_{n,N}(f,x)\right|\leqslant$$

$$\frac{c}{N}\sum_{i=0}^{m-1}\left|\frac{f(\xi_i-0)-f(\xi_i+0)}{\sin\frac{\xi_i-x}{2}}\right|\leqslant\frac{c(f,\varepsilon)}{N},\quad |x-\xi_i|\geqslant\varepsilon. \quad (14)$$

In the similar way, we get the estimate

$$\left|R^{2.3}_{n,\,N}(f,x)\right| \leqslant \frac{c(f,\varepsilon)}{N}, \quad |x-\xi_i| \geqslant \varepsilon. \tag{15}$$

Now we estimate $\left|R^{2.2}_{n,\,N}(f,x)\right|$. Consider the integral

$$B = \int\limits_{-\pi}^{\pi} f^{'}(t)\cos k(t-x)\sin \mu N(t-u)dt.$$

Using Lemma 3, we estimate the value $|B|$ as follows:

$$|B| \leqslant \frac{c}{\mu N}\left[\sum_{i=0}^{m-1}\left|f^{'}(\xi_i-0)-f^{'}(\xi_i+0)\right| + \int\limits_{-\pi}^{\pi}\left|f^{''}(t)\right|dt\right] \leqslant \frac{c(f)}{\mu N}.$$

Now we have

$$\left|R^{2.2}_{n,\,N}(f,x)\right| = \left|\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu}\sum_{k=1}^{n}\frac{B}{1-\left(\frac{k}{\mu N}\right)^2}\right| \leqslant \frac{c(f)}{N}. \tag{16}$$

The value $\left|R^{2.4}_{n,\,N}(f,x)\right|$ can be estimated in the similar way:

$$\left|R^{2.4}_{n,\,N}(f,x)\right| \leqslant \frac{c(f)}{N}. \tag{17}$$

From (14)-(17) we have

$$\left|R^{2}_{n,\,N}(f,x)\right| \leqslant \frac{c(f,\varepsilon)}{N}, \quad |x-\xi_i| \geqslant \varepsilon.$$

Lemma is proved. □

Finally, from Lemmas 4 and 5, we have

$$|R_{n,\,N}(f,x)| \leqslant \frac{c(f,\varepsilon)}{N}, \quad |x-\xi_i| \geqslant \varepsilon. \tag{18}$$

3. The estimate for $|a_n^{(2n)}(f)|$. From (2), using that $t_j = u+2\pi k/N$, we have

$$a_n^{(N)}(f) = \frac{1}{N}\sum_{k=0}^{2n-1}(-1)^k f(t_k) = \frac{1}{N}\sum_{k=0}^{n-1}\left(f(t_{2k})-f(t_{2k+1})\right)$$

and

$$\left|a_n^N(f)\right| \leqslant \frac{1}{N}\sum_{k=0}^{n-1}\left|f(t_{2k}) - f(t_{2k+1})\right|.$$

Denote by G the subset of indexes $\{k\}_{k=0}^{n-1}$, such that for $k \in G$ the segment $[t_{2k}, t_{2k+1}]$ does not contain any point $\xi_i, 0\leqslant i\leqslant m$. Denote $\hat{G}=\{k\}_{k=0}^{n-1}\backslash G$. Now write

$$\left|a_n^N(f)\right| \leqslant \frac{1}{N}\sum_{k\in G}\left|f(t_{2k}) - f(t_{2k+1})\right| + \frac{1}{N}\sum_{k\in\hat{G}}\left|f(t_{2k}) - f(t_{2k+1})\right|. \quad (19)$$

For each $k \in G$, the segment $[t_{2k}, t_{2k+1}]$ lies entirely inside some interval (ξ_i, ξ_{i+1}) and, therefore, the function f is differentiable on it, which allows us to use the mean-value theorem and get the following inequality:

$$|f(t_{2k}) - f(t_{2k+1})| \leqslant c(f)\,|t_{2k} - t_{2k+1}| \leqslant \frac{c(f)}{N}. \quad (20)$$

For a $k \in \hat{G}$, there are $s(k)$ points $\xi_{i_{k,1}} < \xi_{i_{k,2}} < \ldots < \xi_{i_{k,s(k)}}$ inside the segment $[t_{2k}, t_{2k+1}]$. Now we estimate the value $|f(t_{2k}) - f(t_{2k+1})|$ for $k \in \hat{G}$. First, we need the following lemma:

Lemma 6. *For $f \in D^1$ and the segment $[a, b]$, where $[a, b] \subset [-\pi, \pi]$, the following holds:*

$$|f(a) - f(b)| \leqslant c(f)(s + |a - b|),$$

where s is the number of jump discontinuity points $x_1, x_1, \ldots, x_s$ of the function f on the segment $[a, b]$.

Proof. Here we consider only the case $a < x_i < \ldots < x_s < b$. The proof for the cases $a = x_1$ or $b = x_s$ is similar. Consider the following inequality:

$$|f(a) - f(b)| \leqslant |f(a) - f(x_1 - 0)| + \sum_{i=1}^{s}|f(x_i - 0) - f(x_i + 0)| +$$

$$+\sum_{i=1}^{s-1}|f(x_i + 0) - f(x_{i+1} - 0)| + |f(x_s + 0) - f(b)|.$$

Function f is differentiable on each of the intervals (a, x_1), (x_1, x_2), $\ldots$, (x_{s-1}, x_s), (x_s, b). Using the mean-value theorem, we can write

$$|f(a)-f(b)| \leqslant c(f)|a-b| + \sum_{i=1}^{s} |f(x_i-0)-f(x_i+0)| \leqslant$$
$$\leqslant c(f)|a-b| + sM \leqslant c(f)|a-b| + c(f)s,$$

where $M = \max\limits_{1\leqslant i\leqslant s} |f'(x_i-0)-f'(x_i+0)|$. $\square$

From this lemma

$$\sum_{k\in\hat{G}} |f(t_{2k})-f(t_{2k+1})| \leqslant$$
$$\leqslant \sum_{k\in\hat{G}} c(f)\left(s(k)+\frac{2\pi}{N}\right) \leqslant c(f)\sum_{k\in\hat{G}} s(k) + \sum_{k\in\hat{G}} \frac{2\pi}{N}.$$

Each point $\xi_1, \xi_2, \ldots, \xi_{m-1}$ may be included in one or two segments $[t_{2k}, t_{2k+1}]$, $k \in \hat{G}$, therefore, $\sum\limits_{k\in\hat{G}} s(k) < 2m$. Using this and the fact that $|\hat{G}| \leqslant m$, we have

$$\sum_{k\in\hat{G}} |f(t_{2k})-f(t_{2k+1})| \leqslant c(f). \tag{21}$$

From (19), (20), and (21) inequality

$$\left|a_n^N(f)\right| \leqslant \frac{c(f)}{N} \tag{22}$$

follows.

4. The proof of Theorem 1. The proof of estimate (6) from Theorem 1 immediately follows from inequalities (9), (10), (11), (18), (22), and $n \leqslant N/2$. To prove that the order of this estimate cannot be improved, consider the value $\left|f_1(\frac{\pi}{2}) - L_{4n,N}(f_1,\frac{\pi}{2})\right|$, where $4n < N/2$ and $f_1(x) = \operatorname{sign}(\sin x)$. From Lemma 1, get the inequality

$$|f(x)-L_{n,N}(f,x)| \geqslant |f(x)-S_n(f,x)| - |R_{n,N}(f,x)|.$$

It is easy to show that the following representation takes place:

$$f_1(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(1-(-1)^k)\sin kx}{k} = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi}{2k-1}, \tag{23}$$

$$S_{2n}(f_1, x) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1}.$$

Using this, we can estimate the value $\left|f_1\left(\frac{\pi}{2}\right) - S_{4n}\left(f_1, \frac{\pi}{2}\right)\right|$ from below:

$$\left|f_1\left(\frac{\pi}{2}\right) - S_{4n}\left(f_1, \frac{\pi}{2}\right)\right| = \\ = \frac{4}{\pi}\left|\sum_{k=2n+1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\right| = \frac{4}{\pi}\sum_{k=n+1}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right) = \\ = \frac{8}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^2\left(4-\frac{1}{k}\right)\left(4-\frac{3}{k}\right)} > \frac{1/4}{4n}.$$

From this and (23) we have

$$\left|f_1\left(\frac{\pi}{2}\right) - L_{4n,N}\left(f_1, \frac{\pi}{2}\right)\right| \geqslant \frac{1/4}{4n} - \left|R_{4n,N}\left(f_1, \frac{\pi}{2}\right)\right|.$$

In the previous sections we showed that $\left|R_{4n,N}\left(f_1, \frac{\pi}{2}\right)\right| \leqslant c/N$. Denote by $N(n)$ a number such that for each $N \geqslant N(n)$ inequality $\left|R_{4n,N}\left(f_1, \frac{\pi}{2}\right)\right| \leqslant \frac{1/8}{4n}$ holds. Now, we have

$$\left|f_1\left(\frac{\pi}{2}\right) - L_{4n,N(n)}\left(f_1, \frac{\pi}{2}\right)\right| \geqslant \frac{1/8}{4n} = \frac{c}{4n}.$$

From this we see that the order of estimate (6) cannot be improved. Theorem 1 is proved.

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