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ON GENERAL REPRESENTATION OF THE MEROMORPHIC SOLUTIONS OF HIGHER ANALOGUES OF THE SECOND PAINLEVE EQUATION

V. I. GROMAK and T. S. STEPANOVA

Belarussian State University
Fr.Skorina Avenue 4,
220050 Minsk, Belarus
E:mail:grom@mmf.bsu.unibel.by

One of the important questions of the nonlinear ordinary differential equations theory is representation of the meromorphic solutions as the ratio of the entire functions similarly to the Weierstrass function $\rho(z)$, which is a solution of an equation ${\rho'}^2 = 4\rho^3 + g_2\rho + g_3$, and it has representation through the entire function $\sigma(z)$

$$\rho(z) = \frac{{\sigma'}^2 - \sigma\sigma''}{\sigma^2} = -\zeta'(z), \quad \zeta = (\ln(\sigma(z))'$$

We shall consider a reduction of the higher - Korteweg-de Vries equations

$$(2m-1)u_t = X_m u, \tag{1}$$

where $X_1 u = Du = D\frac{\partial H_1}{\partial u}$, $D = \frac{\partial}{\partial x}$, $X_m u = (2u + 2DuD^{-1} - D^2)X_{m-1}u = D\frac{\partial H_m}{\partial u}$, $m = 2, 3, ...$
applying the formulas

$$z = xt^{-\frac{1}{2m-1}}, \quad u(x,t) = t^{-\frac{2}{2m-1}}\left(\frac{dw}{dz} + w^2\right) \tag{2}$$

to the ordinary differential equation

$$D^{-1}S_w^{m-1}(w') + zw + \delta = 0, \tag{${}_mP_2$}$$

where $S_w = 4w^2 + 4w'D_w^{-1} - D^2, \quad D = d/dz.$

The order of equation $({}_mP_2)$ is $2m-2$. For $m = 2$ equations (1) and $({}_mP_2)$ become the Korteweg - de Vries equation and the second Painleve equation (P_2) correspondingly. Let us call equation $({}_mP_2)$ the higher analogue of the second Painleve equation similarly to equation (1). For $m = 3$ we have

$$w^{(4)} = 10w^2w'' + 10ww'^2 - 6w^5 - zw - \delta. \qquad ({}_3P_2)$$

It is well known, that solutions of the Painleve equations are, in a general case, meromorphic functions, which we may not state definitely about the higher analogues of equation P_2. In paper [1] representation of the Painleve equations' solutions as the ratio of the entire functions was given and one-to-one correspondence between the solutions of these equations and the systems constructed for them was established.

In the present paper we offer representation of the meromorphic solutions of equations $({}_mP_2)$ as the ratio

$$w(z) = \frac{v(z)}{u(z)} \qquad (3)$$

of entire functions $v(z), u(z)$. We establish one-to-one correspondence between the meromorphic solutions of the equation $({}_mP_2)$ and entire solutions of the constructed system.

For the equation $({}_mP_2)$ let's assume following [2, 3, 1]

$$u(z) = \exp\,(-\int_{z_0}^{z} dz \int_{z_0}^{z} w^2 dz), \qquad (4)$$

where the path of integration does not pass through the singularities of the function $w(z)$. In the neighborhood of the movable pole $z = \alpha$ the meromorphic solution $w(z)$ has expansion [4]

$$w = a_{-1}(z-\alpha)^{-1} + a_1(z-\alpha) + \varphi_1(z,\alpha), \qquad (5)$$

where a_{-1} takes any of the values $\pm 1,\ \pm 2, \ldots\ \pm (m-1)$, and $\varphi_1(z,\alpha)$ is an analytic function in the neighborhood of α. Then from (4) it follows that $u(z)$ is an entire function for any meromorphic solution of the equation $({}_mP_2)$, and at pole α of the solution $w(z)$ the function $u(z)$ has zero of order a_{-1}^2.This fact is easily established by substitution (5) into the right part of (4). Hence, if we will define function $v(z)$ as $v(z) = w(z)u(z)$, then the function $v(z)$ is also an entire one for any meromorphic solution of the equation $({}_mP_2)$. The system for finding of the entire functions $u(z), v(z)$ turns out by differentiation (4) by virtue of (3) and substitution (3) into the equation $({}_mP_2)$. It has the form

$$uu'' - u'^2 = v^2, \quad D^{-1}S_{vu^{-1}}^{m-1}(v'u^{-1} - vu'u^{-2}) + zvu^{-1} + \delta = 0. \qquad (6)$$

For example, at $m = 3$ for the equation $({}_3P_2)$ we have

$$
\begin{aligned}
&uu'' - {u'}^2 = v^2,\\
&v^{(4)}u^4 - 4v''v'u^3 + 6v''{u'}^2u^2 - 2v''v^2u^2 - 4v'{u'}^3u\\
&\quad +4v^2v'u' + v{u'}^4 - 2v^3{u'}^2 + v^5 + zvu^4 + \delta u^5 = 0.
\end{aligned}
$$

Let us consider system (6).

LEMMA 1. *The system (6) has the solution*

$$v = 0, \quad u = exp(az + b) \tag{7}$$

for any $a, b \in C,\ \delta = 0$.

The choice of the solution of system (6) is defined by the initial conditions

$$u(z_0) = u_0,\ u'(z_0) = u_0',\ v(z_0) = v_0,\ v'(z_0) = v_0', \dots,\ v^{(2m-3)}(z_0) = v_0^{(2m-3)}, \tag{8}$$

where $z_0,\ \ u_0,\ \ u_0',\ \ v_0,\ \ v_0', \dots,\ \ v_0^{(2m-3)} \in C$. Thus, if $u(z_0) = 0$, then the initial conditions (8) is singular.

LEMMA 2. *If* (v, u) *is a solution of system (6), then*

$$(\tilde{v}, \tilde{u}) = (\lambda(z)v, \lambda(z)u), \quad \lambda(z) \not\equiv 0 \tag{9}$$

is a solution of system (6) if and only if $\lambda = e^{az+b},\ \ a, b \in C$.

The validity of lemmas 1 and 2 are confirmed by direct substitution (7) and (9) into system (6).

LEMMA 3. *Any solution of system (6), corresponding to the meromorphic solution of an equation* $({}_mP_2)$ *and satisfying to the initial conditions*

$$u(z_0) = 1,\ \ u'(z_0) = 0,\ \ v(z_0) = v_0,\ \ v'(z_0) = v_0', \dots,\ \ v^{(2m-3)}(z_0) = v_0^{(2m-3)}, \tag{10}$$

is an entire one.

P r o o f. The initial conditions (10) are not singular. Let $w(z)$ be a meromorphic solution of an equation $({}_mP_2)$ with the initial conditions $w(z_0) = w_0, w'(z_0) = w_0', \dots,\ \ w^{(2m-3)}(z_0) = w_0^{(2m-3)}$. We shall consider the functions

$$u_1(z) = \exp\,(-\int_{z_0}^{z} dz \int_{z_0}^{z} w^2 dz), \quad v_1(z) = w(z)u_1(z). \tag{11}$$

From construction functions $u_1(z), v_1(z)$ are the entire ones and satisfy to system (6). By virtue of uniqueness the statement will be proved if we take $w_0 = v_0,\ w_0' = v_0', \dots,\ \ w_0^{(2m-3)} = v_0^{(2m-3)}$.

LEMMA 4. *Any solution of system (6), corresponding to the meromorphic solution of equation* $({}_mP_2)$ *and satisfying to the initial conditions*

$$u(z_0) = u_0 \neq 0, \quad u'(z_0) = u_0', \quad v(z_0) = v_0, \quad v'(z_0) = v_0', \dots, \\ v^{(2m-3)}(z_0) = v_0^{(2m-3)},$$

is an entire one.

P r o o f. We shall take the solution $(\tilde{v}, \tilde{u})$ of system (6) with the initial conditions

$$\tilde{u}(z_0) = 1, \quad \tilde{u}'(z_0) = 0, \quad \tilde{v}(z_0) = \tilde{v}_0, \quad \tilde{v}'(z_0) = \tilde{v}_0', \dots, \quad \tilde{v}^{(2m-3)}(z_0) = \tilde{v}_0^{(2m-3)}.$$

By virtue of lemma3 this solution is an entire one, and by virtue of lemma 2 the functions $(v, u) = (\tilde{v}\exp(az + b),\ \tilde{u}\exp(az + b))$ will also be an entire solution of system (6). The proof of lemma 4 follows from the choice a, b and $\tilde{v}_0, \tilde{v}_0', \dots,\ \tilde{v}_0^{(2m-3)}$ from a condition

$$a = \frac{u_0'}{u_0},\ b = \ln(u_0) - z_0\frac{u_0'}{u_0}, \tilde{v}_0 = \frac{v_0}{u_0}, \dots, \quad \tilde{v}_0^{(2m-3)} = \left(\frac{v}{u}\right)^{(2m-3)}(z_0),$$

where $u''(z_0) = u_0'^2/u_0$.

THEOREM 1. *All solutions of system (6), corresponding to the meromorphic solutions of equation* $({}_mP_2)$*, are entire functions.*

P r o o f. If $u \equiv 0$, then $v \equiv 0$ and this solution is an entire one. Let $u \not\equiv 0$. Then there exists domain D where function $u(z) \neq 0$ and it is an analytic one. Let $z_0 \in D$. Then by virtue of lemma 4 we have the required statement.

THEOREM 2. *Let* (v, u) *be an arbitrary entire non-zero solution of system (6) for some fixed value of parameter* δ*, which is different from the solutions (7). Then a ratio* $v(z)/u(z)$ *represents meromorphic solution of equation* $({}_mP_2)$ *for the same value of parameter* δ*.*

P r o o f. Let (v, u) be an entire non-zero solution of system (6). Then there exists such z_0, that $u(z_0) = u_0 \neq 0$. Let us take the meromorphic solution $w(z)$ of equation $({}_mP_2)$ with the initial conditions $w(z_0) = v_0/u_0$, $w'(z_0) = v_0'/u_0 - v_0u_0'/u_0^2, \dots,$ $w^{(2m-3)}(z_0) = (v/u)^{(2m-3)}(z_0)$, where we assume $u''(z_0) = u_0'^2/u_0$. We shall construct a solution of system (6) applying the formula

$$u_1(z) = u_0\exp((z - z_0)\frac{u_0'}{u_0} - \int_{z_0}^{z} dz \int_{z_0}^{z} w^2(z)dz),\ v_1(z) = u_1(z)w(z).$$

It is not difficult to see that the solution $(v_1(z), u_1(z))$ satisfies to the same initial conditions, as $(v(z), u(z))$. In view of this by virtue of uniqueness

$(v_1(z), u_1(z)) \equiv (v(z), u(z))$ and from the second parity of (11) statement of the theorem follows.

THEOREM 3. *Any meromorphic solution $w(z)$ of equation $({}_mP_2)$ is represented in the form $w(z) = v(z)/u(z)$, where $(v(z), u(z))$ is the corresponding entire solution of system (6), determined up to the factor* $\exp(az + b)$.

P r o o f. If $w(z) \equiv 0$ at $\delta = 0$, we shall take the entire solution of system (6) $(v(z), u(z)) = (0, \exp(az + b))$. Let $w(z) \not\equiv 0$. Then there exists such z_0, that $w(z_0) \neq 0, w(z_0) \neq \infty$. Let us put

$$u(z) = \exp\,(-\int_{z_0}^{z} dz \int_{z_0}^{z} w^2 dz), \quad v(z) = w(z)u(z).$$

These functions are entire solutions of system (6), and $w(z) = v(z)/u(z)$. But by virtue of (9) $w(z)$ is also expressed through the solution

$$(\tilde{v}(z), \tilde{u}(z)) = (v(z)\exp(az + b), u(z)\exp(az + b)).$$

The theorems 2 and 3 establish one-to-one correspondence between the meromorphic solutions of equation $({}_mP_2)$ and the entire solutions of system (6).

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