Mathematical Modelling and Analysis
Volume 9 Number 3, 2004, pages 209–222
 ISSN 1392-6292

A REVIEW OF NUMERICAL ASYMPTOTIC AVERAGING FOR WEAKLY NONLINEAR HYPERBOLIC WAVES [1]

A. KRYLOVAS and R. ČIEGIS

Vilnius Gediminas Technical University

Saulėtekio al. 11, LT-10223 Vilnius, Lithuania

E-mail: {akr, rc}@fm.vtu.lt

Received June 8, 2004; revised July 5, 2004

Abstract. We present an overview of averaging method for solving weakly nonlinear hyperbolic systems. An asymptotic solution is constructed, which is uniformly valid in the "large" domain of variables $t + |x| \sim O(\varepsilon^{-1})$. Using this method we obtain the averaged system, which disintegrates into independent equations for the nonresonant systems. A scheme for theoretical justification of such algorithms is given and examples are presented. The averaged systems with periodic solutions are investigated for the following problems of mathematical physics: shallow water waves, gas dynamics and elastic waves. In the resonant case the averaged systems must be solved numerically. They are approximated by the finite difference schemes and the results of numerical experiments are presented.

Key words: small parameter method, perturbations, hyperbolic systems, averaging, resonance, finite difference schemes, numerical solution, gas dynamics, shallow water, elastic waves

1. Introduction

We consider a hyperbolic system of weakly nonlinear differential equations with a small positive parameter ε:

$$U_t + A(U)U_x = \varepsilon B(t, x, \varepsilon t, \varepsilon x, U, U_x, U_{xx}, U_{xxx}) , \tag{1.1}$$

where

$$U(t, x; \varepsilon) = (u_1, u_2, \ldots, u_n)^T , \quad A(U) = \|a_{ij}(U)\|_{n\times n} .$$

Let defi ne a constant solution U_0, which satisfi es the equation

[1] This work was supported by by the Lithuanian State Science and Studies Foundation (V-27) within the framework of the Eureka project OPTPAPER E!-2623, E-2002.02.27

$$B(t, x, \varepsilon t, \varepsilon x, U_0, 0, 0, 0) = 0.$$

We assume that all coeffi cients in (1.1) are suffi ciently smooth functions. Our goal is to fi nd a small amplitude solution

$$U(t, x; \varepsilon) = U_0 + \varepsilon U_1(t, x; \varepsilon).$$

The system (1.1) must be solved in the "large" domain:

$$(t, x) \in \left[0, \frac{\tau_0}{\varepsilon}\right] \times \left[-\frac{\xi_0}{\varepsilon}, \frac{\xi_0}{\varepsilon}\right] \underset{\varepsilon \to 0}{\longrightarrow} [0, +\infty) \times R.$$

For small ε the problem of solving (1.1) numerically is a very diffi cult task. Asymptotic methods are used for the analysis of such problems. Often (but not always) the asymptotic solution satisfi es some simple equations. If a problem for asymptotic solution is still complicated, then a combination of numerical and asymptotic methods can be used (see [1]).

In this article we present an overview of new asymptotic methods for solution of problems with a small parameter. Recent developments in theoretical analysis, as well as numerical algorithms are discussed. We present numerical algorithms for solving the averaged systems which are obtained applying the asymptotic averaging method for the system (1.1) with periodic initial conditions. Three examples of applied problems are investigated, including resonant interaction of shallow water waves, one dimensional waves of gas dynamics and resonances in elastic waves. The theoretical aspects of the asymptotic analysis of these examples where considered in our papers [16, 17, 18, 19]. In this article we focus our attention on the investigation of fi nite difference schemes for solving the averaged systems of equations.

Note, that analogous integro-differential systems were also investigated in [21, 22, 23]. However numerical algorithms were not considered in these papers and no computational examples were given.

2. The Method of Averaging

Let assume that the problem is hyperbolic in neighborhood of U_0, thus we can rewrite the system using the well know Riemann invariants

$$\begin{aligned} \Lambda &= \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = RA(U_0)R^{-1}, \\ V(t, x; \varepsilon) &= RU_1(t, x; \varepsilon) = (v_1, v_2, \ldots, v_n), \end{aligned}$$

then we get:

$$V_t + \Lambda V_x = \varepsilon F(t, x, \varepsilon t, \varepsilon x, V, V_x, V_{xx}, V_{xxx}) + o(\varepsilon). \tag{2.1}$$

If the parameter $\varepsilon = 0$, then system (2.1) disintegrates into independent equations, which describe linear waves $v_j(x - \lambda_j t)$. If $\varepsilon \neq 0$ and $t + |x| \sim \varepsilon^{-1}$, then the exact solution of (2.1) is not close to these simple waves. For example, the initial value problem

$$\begin{cases} v_t + v_x = \varepsilon v v_x \,, \\ v(0,x) = \sin(x) \end{cases}$$

describes a nonlinear wave, which can be obtained from the implicit relation

$$v = \sin(x - t - \varepsilon t v) \,.$$

For $\varepsilon t = O(1)$ the solution $v(t,x;\varepsilon)$ can not be approximated by a simple wave $\sin(x-t)$ and thus it is a nontrivial task to construct an asymptotic approximation, which is uniformly valid in the region $t + |x| = O(\varepsilon^{-1})$.

The key idea of all asymptotic methods for solving problem (2.1) (or (1.1)) is to introduce new slow variables, e.g., $\tau = \varepsilon t$, $\xi = \varepsilon x$, and to define explicitly the dependence on fast variables (*principle of multiple scales*, see [8, 24]). For example in [28] the solution of the system (2.1) is obtained in the following form

$$u_j = \psi_j(\eta, \zeta_j), \quad \eta = \varepsilon^{1+a} t, \quad \zeta_j = \varepsilon^a (x - \lambda_j t + \varepsilon^{1-a} \varphi(t,x)) \,.$$

Substituting these expressions into (1.1), using the Taylor expansion with respect to ε, and collecting equal powers of ε^k, we get equations for unknown functions ψ_j. The Burgers and Korteweg – de Vries equations are examples of such problems. We will show, that these asymptotics are not uniformly valid in the region $t + |x| = O(\varepsilon^{-1})$ for the systems with periodic initial conditions in the case of resonant interaction of waves, therefore some modifications of such algorithms should be proposed.

2.1. Formulation of the integro-differential system

Our method of asymptotic integration is based on principles of multiple scales and averaging. We introduce slow variables $\tau = \varepsilon t$, $\xi = \varepsilon x$ and fast characteristic variables $y_j = x - \lambda_j t$, $j = 1, 2, \ldots, n$. Our goal is to construct the asymptotic solution in the following form

$$v_j(t,x;\varepsilon) = V_j(\tau, \xi, y_j) + o(1), \quad j = 1, 2, \ldots, n, \ \varepsilon \to 0 \,.$$

The basic idea of our method is the special averaging along characteristics:

$$M_j[g(\tau,\xi,t,x,y_1,y_2,\ldots,y_n)] \equiv \lim_{T\to+\infty} \frac{1}{T} \int_0^T g\big(\tau,\xi,s, y_j + \lambda_j s, \qquad (2.2)$$
$$y_j + (\lambda_j - \lambda_1)s, \ldots, y_j + (\lambda_j - \lambda_n)s\big)\, ds \,.$$

We also will use the following notation

$$< g >_j (\tau, \xi, y_j) = M_j[g(\tau,\xi,t,x,y_1,y_2,\ldots,y_n)] \,.$$

The asymptotic solution satisfies the averaged system:

$$\frac{\partial V_j}{\partial \tau} + \lambda_j \frac{\partial V_j}{\partial \xi} = M_j[f_j(t,x,\tau,\xi,V_1,V_2,\ldots,V_n,\ldots)] \qquad (2.3)$$

with periodic initial conditions

$$V_j(0,\xi,y_j) = v_{oj}(\xi,y_j) = v_{0j}(\xi,y_j+2\pi),\quad j=1,2,\ldots,n\,.$$

Our method also can be used for the non-periodic solutions, e.g., for almost periodic functions or for functions $v_j(\tau,\xi,y_j)$, which satisfy conditions

$$\lim_{y_j\to\pm\infty} v_0(\tau,\xi,y_j) = 0\,.$$

In the 2π-periodic case with integer coeffi cients λ_1, λ_2, . . ., λ_n the operator (2.2) can be written in the simpler form:

$$< g >_j = \frac{1}{2\pi}\int_0^{2\pi} g(\tau,\xi,s,y_j+\lambda_j s, y_j+(\lambda_j-\lambda_1)s,\ldots,y_j+(\lambda_j-\lambda_n)s)\,ds.$$

After averaging each function depends only on one fast characteristic independent variable y_j. The new feature of this method is that the averaging operator is applied for functions, which themselves are solutions of the obtained averaged equations. Thus we get integro-differential problems (see [12], where our averaging scheme is compared with the other averaging methods). This idea was presented in [27] and developed in papers of the fi rst author of this article [9, 10, 11] (see also [2, 5, 21, 22, 23] and a survey of mathematical results in [7]). The aspects of mathematical substantiation of our method were considered in [13, 14, 15].

The method of [21] is very close to the method of averaging along characteristics from [9, 27] but only quadratic nonlinearities were considered in [21]. A mathematic substantiation of the asymptotic method and especially the construction of higher order terms in the asymptotic series leads to the problem of small denominators:

$$\delta_{jl} = l_1(\lambda_j-\lambda_1) + l_2(\lambda_j-\lambda_2) + \cdots + l_n(\lambda_j-\lambda_n)\,,$$

where $l = (l_1, l_2, \ldots, l_n)$ is a vector with integer components. In [27] all combinations $\frac{\lambda_j-\lambda_i}{\lambda_j-\lambda_k}$ were rational numbers and therefore δ_{jl} were equal to zero or $|\delta_{jl}| > const$. In a general case

$$\min_{||l||=L,\,\delta_{jl}\neq 0} |\delta_{jl}| = o(1),\quad L\to\infty\,,$$

therefore numbers λ_1, λ_2, . . ., λ_n, satisfying the following condition

$$\min_{\|l\|=L,\,\delta_{jl}\neq 0} |\delta_{jl}| > \frac{c}{L^{\,r}}\,,$$

where c and r are some positive constants, were considered in [9] (see also [6]). The properties of

$$\min_{\|l\|=L,\quad \delta_{jl}\neq 0} |\delta_{jl}|$$

were studied in [10], here small perturbations of numbers $\lambda_j = \lambda_{j0} + \lambda_{j1}(\varepsilon)$ were investigated. In [21] the conditions of resonant interaction of waves were given only for quadratic nonlinearities, more general relations of resonance $\delta_{jl} = 0$ were proposed in [11].

2.2. Numerical algorithms

The integro-differential system (2.3), must be solved numerically in the compact domain of variables:

$$(\tau, \xi, y_1, y_2, \ldots, y_n) \in [0, \tau_0] \times [-\xi_0, \xi_0] \times [0, 2\pi]^n .$$

It is important to note that the averaged system is solved only once and the obtained solution later can be used for all ε.

We always try to split the operators M_j into a sum of two operators

$$M_j(V_1, \ldots, V_n) = L_j V_j + N_j(V_1, \ldots, V_n)$$

and $L_j V_j$ is included into the differential part of (2.3). For many applications we get systems of nonlinear differential equations such that effi cient numerical methods already exist for solving this type of equations. In some cases even numerical software is available and this fact provides a possibility to tackle real-life problems at small programming cost. The remaining integral part of the integro-differential system is approximated explicitly. A fi xed-point iteration method can be used to improve the stability of the obtained numerical algorithm.

3. Theoretical Justification of the Method

In this section we describe the main steps of the theoretical analysis. *First* we show that the averaging operator $M_j[g(\tau, \xi, t, x)]$ takes out secular terms, which were obtained after integration of the equation

$$\frac{\partial u}{\partial t} + \lambda_j \frac{\partial u}{\partial x} = \varepsilon g(\tau, \xi, t, x)$$

along the characteristic $x - \lambda_j t = const$. Thus the following equality

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^t \big(g - M[g]\big)(t = s,\, x = x - \lambda_j t + \lambda_j s)\, ds = 0 \tag{3.1}$$

should be valid uniformly in $|x| + t \leqslant \dfrac{c_0}{\varepsilon}$.

Let us denote

$$F(\varepsilon) = \max_{\substack{j = 1, 2, \ldots, n \\ \tau + \xi \in [-c_0, c_0] \\ t + |x| \in [0, c_0/\varepsilon]}} \varepsilon \int_0^t (f_j - M[f_j])\, ds. \tag{3.2}$$

Here f_j are functions from (2.1). Let us assume that in (1.1) the right-hand side vector $B = B(U) = \big(b_1(U), \ldots, b_n(U)\big)$ and all functions $b_j(U)$ are continuously differentiable functions. If the averaged system satisfi es condition (3.1), i.e.

$$F(\varepsilon) = o(1), \quad \varepsilon \to 0,$$

then a solution of system (1.1) is approximated uniformly in the region $|x| + t \leqslant \dfrac{c_0}{\varepsilon}$ by a solution of the averaged system, i.e.:

$$u_j(t, x; \varepsilon) = v_j(\tau, \xi, y_j) + O\big(F(\varepsilon)\big)\,.$$

Secondly, we should prove that averaged system (2.1) has a solution, which satisfies condition (3.1). In case of periodical initial conditions it is sufficient to note the following property of operator M_j.

Let us denote by $C^1_{2\pi}([-c_0, c_0] \times [0, 2\pi])$ a class of 2π – periodical functions $u(\tau, y)$, which have a continuous derivative with respect to y. Let assume that

$$f_j(v_1, v_2, \ldots, v_n) \in C^1(R^n), \quad v_j(\tau, y_j) \in C^1_{2\pi}([-c_0, c_0] \times [0, 2\pi])\,.$$

Then we obtain, that

$$M_j[f_j(v_1, \ldots, v_n)] = g_j(\tau, y_j) \in C^1_{2\pi}([-c_0, c_0] \times [0, 2\pi])\,.$$

We see that all averaged functions preserve properties of continuity and periodicity, thus we can prove the existence and uniqueness of the solution of averaged system (2.1) by using the standard Picard method.

The proposed averaging method can be used not only in case of periodical functions. This scheme can be applied to construct almost periodical asymptotical approximations or to consider functions decreasing at infinity

$$\lim_{y_j \to \pm\infty} v_0(\tau, \xi, y_j) = 0\,.$$

The other generalization of the method is obtained for a case when the averaging operator is nonuniform with respect to the arguments of functions:

$$v_j(\tau, \xi, y_j) \approx v^+_{0j}(\tau, \xi, y_j), \quad y_j \to +\infty,$$
$$v_j(\tau, \xi, y_j) \approx v^-_{0j}(\tau, \xi, y_j), \quad y_j \to -\infty.$$

If functions v^+_0, v^-_0 are periodical or almost periodical with respect to y_j or decreasing at infinity, then averaging operator can be applied in each region $(t, x) \in \lambda_i t < x < \lambda_{i+1} t$. Therefore we can consider not only the initial value problem, but also to formulate initial – boundary value problem in $t = 0$, $x > 0$ and $x = 0$, $t > 0$ (see [13, 14, 15]).

4. Shallow Water Equations

4.1. Averaged system

In this section we consider the system of shallow water equations:

$$\begin{pmatrix} Z \\ W \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix}_x = -\varepsilon \begin{pmatrix} \frac{1}{3} W_{xxx} + (H(x)W)_x + (ZW)_x \\ WW_x \end{pmatrix}. \tag{4.1}$$

Here Z denotes the water surface level, W denotes the horizontal velocity of the fluid, L_* is some typical horizontal size, H_* is some typical vertical size, $\varepsilon = \left(\frac{H_*}{L_*}\right)^2 << 1$, $H = 1 + \varepsilon h(x)$ is the bottom equation.

The asymptotic solution of (4.1) satisfies the averaged system of two equations:

$$\begin{cases} \dfrac{\partial V^+}{\partial \tau} + \dfrac{3}{2} V^+ \dfrac{\partial V^+}{\partial y^+} + \dfrac{1}{6} \dfrac{\partial^3 V^+}{\partial {y^+}^3} = -\dfrac{1}{2} \dfrac{\partial}{\partial y^+} \langle H(x) V^- \rangle_+ , \\ \dfrac{\partial V^-}{\partial \tau} - \dfrac{3}{2} V^- \dfrac{\partial V^-}{\partial y^-} - \dfrac{1}{6} \dfrac{\partial^3 V^-}{\partial {y^-}^3} = \dfrac{1}{2} \dfrac{\partial}{\partial y^-} \langle H(x) V^+ \rangle_- . \end{cases} \tag{4.2}$$

Here we use notation

$$W = v^+ - v^-, \;\; Z = v^+ + v^-, \;\; y^\pm = x \mp t \,,$$

and $V^\pm$ are approximations of $v^\pm$:

$$v^\pm(t, x; \varepsilon) = V^\pm(\tau, \xi, y^\pm) + o(1) \,.$$

In the nonresonant case the expressions on the right hand side of (4.2) are equal to zero and we get two independent Korteweg de Vries equations. We consider the resonant interaction of 2π-periodic waves:

$$\begin{cases} V^+_\tau + \dfrac{3}{2} V^+ V^+_{y^+} + \dfrac{1}{6} V^+_{y^+y^+y^+} \\ \qquad\qquad = -\dfrac{1}{4\pi} \dfrac{\partial}{\partial y^+} \int\limits_0^{2\pi} H(y^+ + s) V^-(\tau, y^+ + 2s)\, ds, \\ V^-_\tau - \dfrac{3}{2} V^- V^-_{y^-} - \dfrac{1}{6} V^-_{y^-y^-y^-} \\ \qquad\qquad = \dfrac{1}{4\pi} \dfrac{\partial}{\partial y^-} \int\limits_0^{2\pi} H(y^- - s) V^+(\tau, y^- - 2s)\, ds. \end{cases} \tag{4.3}$$

4.2. Finite difference scheme

We define the space ω_h and time ω_τ meshes and assume that the space mesh size h and time mesh size τ are uniform. We denote by $v^n_j = v(t^n, y_j)$ a discrete function defined on $\omega_h \times \omega_\tau$. The following common notations of difference derivatives are used in our paper (see, e.g., [25]):

$$v_\tau = \frac{v^{n+1} - v^n}{\tau}, \qquad v_{\bar{y}} = \frac{v_j - v_{j-1}}{h},$$
$$v_y = \frac{v_{j+1} - v_j}{h}, \qquad v_{\mathring{y}} = \frac{v_{j+1} - v_{j-1}}{2h}.$$

The finite difference approximation of system (4.3) is defined as follows (see also [4]):

$$P_\tau = -\frac{1}{6}\left(\frac{P^{n+1}+P^n}{2}\right)_{\bar{y}y\mathring{y}} - \frac{3}{4}\left(\frac{\left(P^{n+1}\right)^2 + P^{n+1}P^n + \left(P^n\right)^2}{3}\right)_{\mathring{y}} - \frac{F_+\left(M^{n+1}, M^n, j+1\right) - F_+\left(M^{n+1}, M^n, j-1\right)}{4h},$$

$$M_\tau = \frac{1}{6}\left(\frac{M^{n+1}+M^n}{2}\right)_{\bar{y}y\mathring{y}} + \frac{3}{4}\left(\frac{\left(M^{n+1}\right)^2 + M^{n+1}M^n + \left(M^n\right)^2}{3}\right)_{\mathring{y}} + \frac{F_-\left(P^{n+1}, P^n, j+1\right) - F_-\left(P^{n+1}, P^n, j-1\right)}{4h},$$

where the integrals are approximated as follows:

$$F_+\left(M^{n+1}, M^n, j\right) = \frac{1}{2\pi}\sum_{i=1}^{N} H(y_j - ih)\frac{M^{n+1}_{j-2i} + M^n_{j-2i}}{2}\,h\,,$$

$$F_-\left(P^{n+1}, P^n, j\right) = \frac{1}{2\pi}\sum_{i=1}^{N} H(y_j + ih)\frac{P^{n+1}_{j+2i} + P^n_{j+2i}}{2}\,h\,.$$

Here P and M approximate V^+, V^-, respectively.

The approximation error of this finite finite difference scheme is estimated as $O\left(\tau^2 + h^2\right)$. Numerical methods for solving the Korteweg-de Vries equation are investigated in [3, 26]. A special formula for averaging in time is used in order to satisfy some conservation properties, which are valid for the solution of the differential problem.

4.3. Linear dispersion problem

In this section we consider a linear problem

$$\begin{cases} Z_t + (HW)_x = -\dfrac{\varepsilon}{3}\,W_{xxx}, \\ W_t + Z_x = 0. \end{cases} \tag{4.4}$$

First we will prove that system (4.4) defines an ill-posed problem. Let consider the case $H = 1$. After simple computations we get the equation for W:

$$W_{tt} - W_{xx} = \frac{\varepsilon}{3}\,W_{xxxx}. \tag{4.5}$$

Considering the k-th Fourier mode we get that the solution of (4.5) is unstable for $k^2\varepsilon \geq 3$. In order to define a stable solution we use the following regularized problem

$$\begin{cases} Z_t + (HW)_x = -\dfrac{\varepsilon}{3}\,W_{xxx} - \dfrac{\varepsilon^2}{20}\,W_{xxxxx}, \\ W_t + Z_x = 0. \end{cases} \tag{4.6}$$

We note that the averaged system (2.3) also gives a nontrivial regularization of this ill-posed problem.

Extensive computational results are presented in [16, 18].

5. One Dimensional Gas Dynamics Equations

5.1. Averaged system

Let ρ denotes the gas density, u is the velocity and θ is the temperature. We introduce a vector $U = (\rho, u, \theta)^T$ and matrixes

$$A = \begin{pmatrix} u & \rho & 0 \\ \dfrac{R\theta}{\rho} & u & R \\ 0 & \dfrac{R\theta}{c_v} & \dfrac{u}{c_v\rho} \end{pmatrix}, \; B = \frac{1}{\rho} \begin{pmatrix} 0 \\ \gamma \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\kappa}{c_v} \dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\gamma}{c_v} \left(\dfrac{\partial u}{\partial x} \right)^2 \end{pmatrix}, \tag{5.1}$$

where c_v is the specifi c heat at constant volume, R is the gas constant for a politropic ideal gas:

$$p = R\rho\theta \,,$$

κ and γ are small viscosity and heat conduction coeffi cients (i.e. $\sim O(\varepsilon)$ as $\varepsilon \to 0$). Then the gas dynamics problem can be formulated as system (1.1).

In this case the averaged system (2.3) is described by the linear Burgers equations coupled through integral terms:

$$\begin{cases} \dfrac{\partial V_1}{\partial \tau} - f_{111} V_1 \dfrac{\partial V_1}{\partial y_1} - f_{11} \dfrac{\partial^2 V_1}{\partial {y_1}^2} = \left\langle f_{123} V_2 \dfrac{\partial V_3}{\partial y_3} + f_{132} V_3 \dfrac{\partial V_2}{\partial y_2} \right\rangle_1 , \\ \dfrac{\partial V_2}{\partial \tau} - f_{22} \dfrac{\partial^2 V_2}{\partial {y_2}^2} = 0 \,, \\ \dfrac{\partial V_3}{\partial \tau} - f_{333} V_3 \dfrac{\partial V_3}{\partial y_3} - f_{33} \dfrac{\partial^2 V_3}{\partial {y_3}^2} = \left\langle f_{321} V_2 \dfrac{\partial V_1}{\partial y_1} + + f_{312} V_1 \dfrac{\partial V_2}{\partial y_2} \right\rangle_3 . \end{cases} \tag{5.2}$$

The explicit expressions for coeffi cients in (5.2) are presented in [17].

5.2. Finite difference scheme

The averaged system (5.2) is approximated by the following fi nite difference scheme:

$$\begin{cases} V_{1,\tau} = f_{11} V^{n+1}_{1,\bar{y}y} + \frac{1}{2} f_{111} \, (V_1^{n+1})^2_{\underset{y}{\circ}} + f_{123} \, S_1(V_2, V_3) + f_{132} \, S_1(V_3, V_2) \,, \\ V_{2,\tau} = f_{22} V^{n+1}_{2,\bar{y}y} \,, \\ V_{3,\tau} = f_{33} V^{n+1}_{3,\bar{y}y} + \frac{1}{2} f_{333} \, (V_3^{n+1})^2_{\underset{y}{\circ}} + f_{312} \, S_2(V_1, V_2) + f_{321} \, S_2(V_2, V_1) \,, \end{cases}$$

where the integrals are approximated as follows:

$$S_i(V_j,V_k)=\frac{1}{4\pi}\sum_{l=1}^{N}\Big(V_k\big(y_i+((\lambda_i-\lambda_k)l+1)h\big)$$
$$-V_k\big(y_i+((\lambda_i-\lambda_k)l-1)h\big)\Big)V_j\big(y_i+((\lambda_i-\lambda_j)l-1)h\big)\,.$$

The approximation error is given by $\mathcal{O}(\tau+h^2)$.

5.3. Numerical experiments

In this section we present results of numerical experiments. The following coeffi - cients

$$c_v=1,\ \ R=1,\ \ \nu=1,\ \ \varkappa=1$$

are used in all tests. Initial conditions are selected as

$$v_{01}(x)=\cos x,\ \ v_{02}(x)=\sin 2x,\ \ v_{03}(x)=\cos x.$$

Figure 1 shows the solution of system (5.1) and the asymptotic solution at $t=\dfrac{1}{\varepsilon}$ for two different values of the small parameter ε. We present graphics of the density and velocity functions.

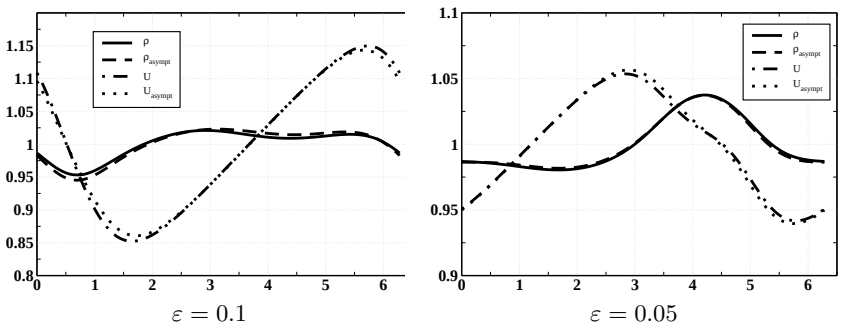

Figure 1. Asymptotic and exact solutions of Euler problem (5.1) for $t=\dfrac{1}{\varepsilon}$.

6. Elastic Waves Equations

6.1. Averaged system

We consider a problem of wave propagation in two dimensional elastic materials and assume that displacements do not depend on the y coordinate. Restricting our attention to the axial displacements along x and y directions, we have equations [20]:

$$\begin{cases} \rho u_{tt} = \sigma_x \,, \\ \rho v_{tt} = \tau_x \,, \end{cases}$$

where ρ is the density of material, u and v denote the displacements along two directions, σ and τ ate longitudinal normal and shear stresses along the x-axes. The linear wave equation can be obtained from this equations if we use assumptions

$$\sigma = (\lambda + 2\mu)u_x, \quad \tau = \mu v_x, \tag{6.1}$$

where λ and μ are the Lamé coeffi cients. The equations (6.1) are obtained by using the simple approximation for the full energy of the system

$$F \approx F_0 + F_{11}u_x^2 + F_{22}v_x^2,$$

then $\sigma = \frac{\partial F}{\partial u_x}$ and $\tau = \frac{\partial F}{\partial v_x}$ (see [20]). In order to get high order approximations we use more terms in a Taylor series of F (see [24]):

$$\begin{cases} \sigma = (\lambda + 2\mu)u_x + 4\rho(a_1u_x^2 + a_2u_xv_x + a_3v_x^2), \\ \tau = \mu v_x + 4\rho(b_1u_x^2 + b_2u_xv_x + a_bv_x^2). \end{cases}$$

Let

$$P = u_x, \;\; Q = v_x, \;\; R = u_t, \;\; S = v_t, \;\; U = (P, Q, R, S)^T \,.$$

Then we get the system (1.1) with

$$A(U) = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\lambda+2\mu}{\rho} & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 \end{pmatrix}, \; B = 4\frac{\partial}{\partial x}\begin{pmatrix} 0 \\ 0 \\ a_1P^2 + a_2PQ + a_3Q^2 \\ b_1P^2 + b_2PQ + b_3Q^2 \end{pmatrix}.$$

In this case the averaged system is given by the system of four equations:

$$\begin{cases} P_\tau^\pm - \dfrac{2a_1}{c_p^2}P^\pm P_{y^\pm}^\pm = -\dfrac{2a_3}{c_s^2}M_{y^\pm}\left[\dfrac{\partial}{\partial x}\left(S^+S^-\right)\right] \\ \qquad\qquad\qquad \mp \dfrac{a_2}{c_sc_p}M_{y^\pm}\left[\dfrac{\partial}{\partial x}\left(P^\mp(S^+ - S^-)\right)\right], \\ S_\tau^\pm - \dfrac{2b_3}{c_s^2}S^\pm S_{z^\pm}^\pm = -\dfrac{2b_1}{c_p^2}M_{z^\pm}\left[\dfrac{\partial}{\partial x}\left(P^+P^-\right)\right] \\ \qquad\qquad\qquad \mp \dfrac{b_2}{c_sc_p}M_{z^\pm}\left[\dfrac{\partial}{\partial x}\left(S^\mp(P^+ - P^-)\right)\right]. \end{cases} \tag{6.2}$$

Here we use notation

$$y^{\pm} = x \mp c_p t, \quad z^{\pm} = x \mp c_s t, \quad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}},$$
$$P^{\pm} = R \pm c_p P, \quad S^{\pm} = S \pm c_s Q.$$

In the nonresonant case we get four independent nonlinear Burgers equations.

6.2. Finite difference scheme

The integrals on the right-hand side of (6.2) are approximated by the trapezoidal rule, the derivatives of functions are computed using the central difference approximations.

The upwind method is used to approximate the Burgers equation

$$U_j^{n+1} = U_j^n + \mu \frac{\tau}{h} \big(F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n) \big),$$

here F is the numerical flux function

$$F(v,w) = \begin{cases} 0.5w^2 & \text{if } \ 0 \leqslant w \leqslant v \ \text{ or } \ (v < w \ \text{ and } \ v + w \geqslant 0), \\ 0.5v^2 & \text{else.} \end{cases}$$

Thus we get the explicit approximation for the system of integro-differential equations. We have used implicit approximations in two previous examples. Now our goal is to show that explicit schemes also can be used to solve averaged equations, if such approximations are effi cient for solving the differential part of the system (i.e. the Burgers equations for this example).

6.3. Numerical experiments

The results of numerical experiments are presented in Fig. 2. Here we present only one wave p^+ and its asymptotic approximation P^+ for $t = \dfrac{1}{\varepsilon}$, $\varepsilon = 0.01$.

We see that the averaged system approximates uniformly the differential problem till time moments $t = \mathcal{O}(\dfrac{1}{\varepsilon})$ and the effect of resonant interaction of waves is also identifi ed correctly.

7. Conclusions

For weakly nonlinear hyperbolic systems with internal resonances the analysis can be done using the combination of numerical and asymptotic methods. The proposed method for constructing asymptotic solution of weakly nonlinear hyperbolic system can be used in nonresonant and in resonant cases. This solution is uniformly valid in large domain $0 \leqslant t \leqslant O(\varepsilon^{-1})$. The averaging of system (2.1) reduces it to the integro-differential system of averaged equations (2.2). The averaged problem gives

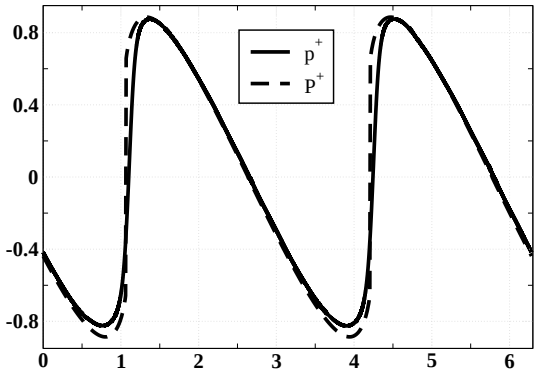

Figure 2. Asymptotic and exact solutions of P^+ wave for $\varepsilon = 0.01$.

a system of integro-differential equations. However, in the nonresonant case the solution can be obtained as independent nonlinear waves. In the resonant case the solution is a superposition of waves, which satisfy the averaged system of nonlinear integro-differential equations. Such systems are solved numerically in the compact domain of variables $(\tau, x) \in [0, \tau_0] \times [0, 2\pi])$.

References

[1] N. S. Bakhvalov, G. P. Panasenko and A. L. Shtaras. The averaging method for partial differential equations (homogenization) and its applications. In: *Partial Differential Equations V*, Springer, New York, 211 – 239, 1999.

[2] S. C. Chikwendu and J. Kevorkian. A perturbation method for hyperbolic equations with small nonlinearities. *SIAM J. Appl. Math.*, **22**, 235 – 258, 1972.

[3] B. Fornberg and T. Driscoll. A fast spectral algorithm for nonlinear wave equations with linear dispersion. *J. of Comp. Phys.*, **155**, 456 – 467, 1999.

[4] D. Furihata. *Finite Difference Schemes that Inherit Energy Conservation or Dissipation Property*. Number 1212. 1998.

[5] J. K. Hunter and J. B. Keller. Weakly nonlinear high frequency waves. *Comm. Pure Appl. Math.*, **36**, 547 – 569, 1983.

[6] L. A. Kalyakin. Asymptotic integration of hyperbolic equations with weakly nonlinear perturbation. *Differencialnye uravnenija*, **20**(2), 351 – 353, 1984.

[7] L. A. Kalyakin. Integrability equations as asymptotic limits of nonlinear systems. *Uspechi Matematicheskich Nauk*, **44**(1), 5 – 34, 1989.

[8] J. Kevorkian and J. D. Cole. *Multiple Scale and Singular Pertubation Methods*. Springer – Verlag, Berlin, New-York, 1996.

[9] A. V. Krylov. About the asymptotic interaction of first order hyperbolic systems. *Lith. Math. J.*, **23**(4), 12 –17, 1983.

[10] A. V. Krylov. Asymptotic integration of weakly nonlinear partial differential systems. *Zhurnal Vytchisl. Matem. i Matemat. Fiziki*, **26**(1), 72 – 79, 1986. (in Russian)

[11] A. V. Krylov. The method of research of weakly nonlinear interaction one dimensional waves. *Prikladnaja Matem. i Mech.*, **51**(4), 933 – 940, 1987. (in Russian)

[12] A. V. Krylov. The internal averaging of first order partial differential systems. *Matematitcheskije Zametki*, **46**(6), 112 – 113, 1989. (in Russian)
[13] A. V. Krylov. The substantiation of the method of the internal averaging along characteristics in weakly nonlinear systems. I. *Lith. Math. J.*, **29**(4), 721 – 732, 1989.
[14] A. V. Krylov. The substantiation of the method of the internal averaging along characteristics in weakly nonlinear systems. II. *Lith. Math. J.*, **30**(1), 87 – 100, 1990.
[15] A. V. Krylov. The averaging of weakly nonlinear hyperbolic systems with non-uniform integral average. *Ukrainian Math. J.*, **43**(5), 611 – 618, 1991.
[16] A. Krylovas and R. Čiegis. Asymptotic analysis of weakly nonlinear systems. In: A. A. Samarskii R. Čiegis and M. Sapagovas(Eds.), *Finite Difference Schemes Theory and Applications, Proceedings of the Conference FDS2000, September 1 - 4, Palanga*, Institute of Mathematics and Informatics, Vilnius, 142 – 151, 2000.
[17] A. Krylovas and R. Čiegis. Asymptotical analysis of one dimensional gas dynamics equations. *Mathematical Modelling and Analysis*, **6**(1), 103 – 112, 2001.
[18] A. Krylovas and R. Čiegis. Asymptotical approximation of hyperbolic weakly nonlinear systems. *J. of Nonlinear Math. Phys.*, **8**(4), 458 – 470, 2001.
[19] A. Krylovas and R. Čiegis. On the interaction of elastic waves. *Journal of Civil Engineering and Management*, **9**(3), 218 – 224, 2003.
[20] L. D. Landau and E. M. Lifshitz. *Theory of Elasticity: Course of Theoretical Physics*. Butterworth V Heinemann, Oxford, 1986.
[21] A. Majda and R. Rosales. Resonantly interacting weakly nonlinear hyperbolic waves. 1. a single space variable. *Studies in Appl. Math.*, **71**(2), 149 – 179, 1984.
[22] V. P. Maslov. *Asimptoticheskie metody reshenija psevdodifferencialnych uravnenii*. Nauka, Moskva, 1987. (in Russian)
[23] V. P. Maslov and P. P. Mosolov. *Uravnenija odnomernogo barotropnogo gaza*. Nauka, Moskva, 1990. (in Russian)
[24] A. H. Nayfeh. *Pertubation Methods*. Jon Willay & Sons Inc., New-York, 2000.
[25] A.A. Samarskii. *The theory of difference schemes*. Marcel Dekker, Inc., New-York, Basel, 2001.
[26] J. M. Sanz-Serna. Symplectic integrators for hamiltonian problems. *Acta Numerica*, **1**, 243 – 286, 1992.
[27] A. L. Štaras. The asymptotic integration of weakly nonlinear partial derivatives equations. *Doklady Akademii Nauk SSSR*, **237**(3), 525 – 528, 1977.
[28] T. Taniuti. Reductive perturbation method and far fields of wave equations. *Suppl. Progr. Theor. Phys.*, **55**, 1 – 35, 1974.

Silpnai netiesinių hiperbolinių sistemų skaitinio asimptotinio vidurkinimo apžvalga

A. Krylovas, R. Čiegis

Darbe nagrinėjamas silpnai netiesinių hiperbolinių sistemų ilgųjų bangų asimptotinis sprendinys. Si¯ulomas jo konstravimo metodas, pagrįstas vidurkinimu bei dviejų mastelių principu. Užrašytos skirtuminės schemos suvidurkintų lygčių sistemoms spręsti. Ištirti trys periodinių asimptotinių sprendinių pavyzdžiai: sekliųjų vandenų modelis, dujų dinamikos lygtys bei tampriųjų bangų sąveika.