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Asymptotic Behavior of Solutions to Nonlinear Fractional Differential Equations

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Abstract. It is known that, under certain conditions, solutions of some ordinary differential equations of first, second or even higher order are asymptotic to polynomials as time goes to infinity. We generalize and extend some of the existing results to differential equations of non-integer order. Reasonable conditions and appropriate underlying spaces are determined ensuring that solutions of fractional differential equations with nonlinear right hand sides approach power type functions as time goes to infinity. The case of fractional differential problems with fractional damping is also considered. Our results are obtained by using generalized versions of Gronwall-Bellman inequality and appropriate desingularization techniques.

Keywords: asymptotic behavior, fractional differential equation, Riemann-Liouville fractional integral and fractional derivative.

AMS Subject Classification: 34E10; 34A08; 26A33.

1 Introduction

In this paper, we consider the following fractional differential equation

$$D_0^{1+\alpha} y(t) = f(t, y(t), D_0^{\beta} y(t)) \quad t > 0, \tag{1.1}$$

with initial conditions

$$D_0^{\alpha} y(t)\,|_{t=0} = b_2 \text{ and } I_0^{1-\alpha} y(t)\,|_{t=0} = b_1,\ \ b_1,\ b_2 \in \mathbb{R}, \tag{1.2}$$

where D_0^{σ} is the Riemann-Liouville fractional derivative of order $\sigma > 0$ and $0 \le \beta \le \alpha \le 1$. The existence and uniqueness of solutions in the space $C_{1-\alpha}^{1+\alpha}[0,b]$, $b > 0$, for problem (1.1)–(1.2) has been proven in [14].

The study of asymptotic behavior of solutions of linear and nonlinear differential equations is not only of theoretical importance but it is also extremely useful in applications such as in fluid mechanics, differential geometry (Jacobi fields, [18]), bidimensional gravity and other fields. It has attracted many researchers. For more details, we refer the reader to [1].

In this paper, our main objective is to determine conditions ensuring an asymptotic behavior similar to that of much simpler differential equations (of integer order). In particular, it is well known that, in some situations, solutions of integer order differential equations approach a line or in general a polynomial for large values of time. This is what has motivated the present investigation. We would like to shed some light on this issue.

The asymptotic behavior of solutions of the nonlinear equation

$$y''(t) + f(t, y(t)) = 0 \tag{1.3}$$

has been studied by Cohen [10], Constantin [12], Kusano and Trench [16, 17], Tong [29], Waltman [31] and others. They proved that, under various conditions, every solution of the equation (1.3) is asymptotic to $b + ct$ as $t \to \infty$, for some $b,\ c \in \mathbb{R}$. Some results for the linear case are also known, see, for instance, Trench [30] and Waltman [31].

In the study of asymptotic behavior of solutions to the differential equation

$$y''(t) + f(t, y(t), y'(t)) = 0 \tag{1.4}$$

it is usually assumed that the nonlinearity f in (1.4) satisfies the assumption

$$|f(t, y(t), y'(t))| \leq F(t, |y(t)|, |y'(t)|),$$

where the real-valued function $F(t, u, v)$ is continuous, monotone in the last two arguments, and vanishes at infinity, see, for instance Dannan [13], Constantin [11], Rogovchenko [28], Rogovchenko and Rogovchenko [27], Mustafa and Rogovchenko [24, 25], Lipovan [19]. It is proved that every solution of the equation (1.4) is asymptotic to $b + ct$ as $t \to \infty$, for some $b,\ c \in \mathbb{R}$.

The fractional case of problem (1.3) has been studied by relatively few researchers. In 2009, Mustafa and Băleanu [23] studied the nonlinear fractional differential problem

$${}^{C}D_0^{\alpha} y(t) = f(t, y(t)), \qquad 0 < \alpha < 1,\ t > 0, \tag{1.5}$$

where ${}^{C}D_0^{\alpha}$ is the Caputo derivative of order α. They proved that the solution of (1.5) is asymptotic to

$$o\left(t^{b\alpha}\right) \text{ as } t \to \infty, \text{ for some } b,\ 1 - \alpha < b < 1.$$

Some classes of linear fractional differential equations with Riemann-Liouville fractional derivative have been investigated under various sufficient conditions, for example, in 2010, Băleanu *et al.* [5] considered the linear fractional differential equation

$$D_0^{1+\alpha} y + a(t) y = 0, \quad 0 < \alpha < 1,\ t > 0. \tag{1.6}$$

They proved that the solution of the equation (1.6) is asymptotic to

$$[b+O(1)]\,t^{\alpha-1}+[c+o(1)]\,t^{\alpha}\quad\text{as } t\to\infty,$$

where

$$b=\lim_{t\to 0}\left[t^{1-\alpha}y(t)\right]\ \text{ and } c=\frac{1}{\Gamma(1+\alpha)}\lim_{t\to\infty}D_0^{\alpha}y(t).$$

Also, in 2010, the same authors [4] studied the linear fractional differential equation

$$D_0^{\alpha}(ty'-y)+a(t)\,y=0,\qquad 0<\alpha<1,\ t>0. \tag{1.7}$$

They proved that (1.7) has a solution $y\in C([0,\infty),\mathbb{R})\cap C^1((0,\infty),\mathbb{R})$ satisfying $\lim_{t\to 0}\left[t^{2-\alpha}y'(t)\right]=0$ and

$$y(t)=ct+O\left(t^{b}\right)\ \text{ as } t\to\infty,\ \text{ for some } b,\ b\in(0,1)\ \text{ and some } c\neq 0.$$

In 2011, again the same authors [6] proved that solutions of (1.7), under other conditions, obey the asymptotic property

$$y(t)=ct+[b+O(1)]\,t^{\alpha-1}=ct+O\left(t^{\alpha-1}\right)\ \text{ as } t\to\infty,$$

for some $c\neq 0$ and

$$b=-\frac{1}{(2-\alpha)\,\Gamma(\alpha)}\lim_{t\to\infty}I_0^{1-\alpha}\left[ty'(t)-y(t)\right].$$

Also they proved that the linear fractional equation

$$D_0^{\alpha}y'+a(t)\,y=0,\qquad 0<\alpha<1\ \ t>0 \tag{1.8}$$

has a solution $y\in C([0,\infty),\mathbb{R})$ enjoying the asymptotic property

$$y(t)=b+ct^{\alpha}+O\left(t^{\alpha-1}\right),\qquad \text{as } t\to\infty, \tag{1.9}$$

where

$$b=y(0)\ \text{ and } c=\frac{1}{\Gamma(1+\alpha)}\lim_{t\to\infty}\ {}^{C}D_0^{\alpha}y(t). \tag{1.10}$$

We note that problems (1.6) and (1.8) are different as $D_0^{1+\alpha}y=DD_0^{\alpha}y\neq D_0^{\alpha}Dy=D^{C}D_0^{\alpha}$. In fact

$$D^{C}D_0^{\alpha}y(t)=D_0^{\alpha}Dy(t)=D_0^{1+\alpha}y(t)+\frac{\alpha y(0)}{\Gamma(1-\alpha)}t^{-\alpha-1}.$$

In 2011, Băleanu *et al.* [3] discussed the nonlinear form of (1.8)

$$D_0^{\alpha}y'+f(t,y)=0,\qquad 0<\alpha<1,\ t>0. \tag{1.11}$$

They proved that solutions of (1.11) have the same asymptotic behavior as (1.9) under the same initial conditions (1.10).

In 2012, Medved [20] studied the behavior of solutions of the fractional differential problem with Caputo fractional derivative

$$\begin{cases} {}^{C}D_a^{\alpha+1}y(t) = f(t, y(t)), & t \geq a > 1,\ 0 < \alpha < 1, \\ y(a) = c_1,\ y'(a) = c_2. \end{cases} \tag{1.12}$$

He proved that the solution $y(t)$ of the problem (1.12) is asymptotic to $b + ct$ as $t \to \infty$, for some b, $c \in \mathbb{R}$.

Also, in 2013, the same author [21] considered the following generalization

$${}^{C}D_a^{\alpha+1}y(t) = f(t, y(t), y'(t)),\ t \geq a > 1,\ \alpha \in (0,1) \tag{1.13}$$

and proved that every solution of the equation (1.13) is asymptotic to $b + ct$ as $t \to \infty$, for some $b,\ c \in \mathbb{R}$.

In [9], Brestovanska and Medved considered the fractional initial-value problem

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) + \sum\limits_{i=1}^{m} r_i(t) \int\limits_{0}^{t} (t-s)^{\alpha_i - 1} f_i(s, y(s), y'(s))\, ds = 0, \\ y(1) = b_1,\ y'(1) = b_2,\ 0 < \alpha_i < 1,\ i = 1, 2, ...m. \end{cases} \tag{1.14}$$

They proved that any solution of (1.14) is asymptotic to a straight line.

In 2015, Medved and Pospíšil [22] studied the fractional differential problem

$$\begin{cases} {}^{C}D_a^{\alpha}y(t) = f\left(t, y(t), {}^{C}D_a^{\beta}y(t)\right), & 0 < \beta < \alpha < 1, \\ y(a) = b. \end{cases} \tag{1.15}$$

They proved that any solution $y(t)$ of the problem (1.15) has the asymptotic property $y(t) = ct^{\beta} + o\left(t^{\beta}\right)$ as $t \to \infty$, for some $c \in \mathbb{R}$.

In this paper, we will generalize the results in [3,4,5,6,23] to problem (1.1)–(1.2). Our results will also extend the results obtained for equations (1.3) and (1.4) with $\alpha = \beta = 1$. In particular, the problems treated in [20, 21] become special cases of (1.1) corresponding to $y(0) = y'(0) = 0$. Moreover, by the same occasion, we improve several results found in the literature, concerning the integer order, where the authors have been forced to work away from zero. This difficulty will be solved.

We shall establish some conditions under which all solutions of the fractional differential problem (1.1)–(1.2) have the following property: $\lim_{t\to\infty} y(t)/t^{\alpha} = a$, for some real number a. The proof of this result is based on the Gronwall-Bellman inequality and its generalization due to Bihari [2].

The rest of the paper is divided into four sections. In Section 2, we present some definitions, notations, and lemmas which will be needed later in our proof. In Section 3, we give some properties and inequalities for some classes of functions. In Sections 4 and 5 we present the asymptotic behavior results for non-fractional and fractional source term, respectively.

2 Fractional calculus and preliminaries

In this section we present some definitions, lemmas, properties and notation which will be used in our results later. Also we prove results regarding the asymptotic behavior of fractional integrals. For more details concerning fractional derivatives, we refer the reader to [14].

We denote by $L_1(a,b)$ the space of Lebesgue integrable functions on (a,b). Let $C[a,b]$ and $C^n[a,b]$ denote the spaces of continuous and n times continuously differentiable functions on $[a,b]$, respectively.

Definition 1. We define the weighted spaces of continuous functions

$$\begin{aligned} &C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} \colon (t-a)^\gamma f(t) \in C[a,b]\}, \quad 0 < \gamma < 1, \\ &C_0[a,b] = C[a,b], \\ &C_\gamma^n[a,b] = \left\{ f \in C^{n-1}[a,b] : f^{(n)} \in C_\gamma[a,b] \right\}, \quad n \in \mathbb{N}, \quad C_\gamma^0[a,b] = C_\gamma[a,b]. \end{aligned}$$

The left-sided Riemann–Liouville fractional integral and derivative are defined as follows.

Definition 2. The Riemann-Liouville left-sided fractional integral $I_a^\alpha f$ of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a,$$

provided that the integral exists. Here $\Gamma(\alpha)$ is the Gamma function. When $\alpha = 0$, we define $I_a^0 f = f$.

Definition 3. The Riemann-Liouville left-sided fractional derivative $D_a^\alpha f$ of order $\alpha \geq 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$, is defined by

$$D_a^\alpha f(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \ t > a,$$

provided that the right hand side exists. In particular, when $\alpha = n$ we have $D_a^\alpha f = D^n f$ and when $\alpha = 0$, $D_a^0 f = f$.

From Definition 1 we have the following characterization of the space $C_\gamma^n[a,b]$.

Lemma 1. *[14] The space $C_\gamma^n[a,b], n \in \mathbb{N}$, consists of those and only those functions f which can be represented in the form*

$$f(t) = I_a^n \varphi(t) + \sum\nolimits_{k=0}^{n-1} c_k (t-a)^k,$$

where $\varphi \in C_\gamma[a,b]$ and c_k, $k = 0,1,...,n-1$ are arbitrary constants. Moreover,

$$\varphi(t) = f^{(n)}(t), \qquad c_k = \frac{f^{(k)}(a)}{k!}, \qquad k = 0,1,...,n-1.$$

Remark 1. Note that $C_\gamma^n[a,b] \subset AC^n[a,b]$, $n \geq 1$.

Lemma 2. *[15] Let $\alpha > 0$ and $0 \leq \gamma < 1$. Then I_a^α is bounded from $C_\gamma[a,b]$ into $C_\gamma[a,b]$.*

Lemma 3. *Let g be a continuous function on $(a,b]$. Then $g^{(n)} \in C_\gamma[a,b]$ if and only if $g \in C_\gamma^n[a,b]$, $0 \le \gamma < 1$.*

Proof. Given $\varepsilon > 0$, then $g^{(n)} \in C[a+\varepsilon,b]$ and thus by the Fundamental Theorem of Calculus, we get

$$g(t) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a+\varepsilon)}{k!}(t-a-\varepsilon)^k + I_{a+\varepsilon}^n D^n g(t),\ t \in [a+\varepsilon,b]. \tag{2.1}$$

Since $g^{(n)} \in C_\gamma[a,b] \subset L_1(a,b)$, then $I_a^1 g^{(n)}(t)$ is bounded on $[a,b]$ and

$$\begin{aligned}\left|I_{a+\varepsilon}^1 g^{(n)}(t) - I_a^1 g^{(n)}(t)\right| &\le \int_a^{a+\varepsilon}(s-a)^{-\gamma}\left|(s-a)^\gamma g^{(n)}(s)\right| ds \\ &\le M\int_a^{a+\varepsilon}(s-a)^{-\gamma} ds = \frac{M}{1-\gamma}\varepsilon^{1-\gamma} \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

Therefore

$$\lim_{\varepsilon\to 0} I_{a+\varepsilon}^1 g^{(n)}(t) = I_a^1 g^{(n)}(t),\ t \in [a,b].$$

Thus by taking the limit of (2.1) we obtain

$$g(t) = \sum_{k=0}^{n-1}\frac{g^{(k)}(a^+)}{k!}(t-a)^k + I_a^n D^n g(t).$$

Now clearly $g^{(k)}(a^+)$, $k = 0,...,n-1$ are finite and the result follows from Lemma 1. The other direction follows directly from the definition of $C_\gamma^n[a,b]$. □

For power function we have the following property.

Property 1. [14] If $\alpha \ge 0$ and $\beta > 0$, then

$$\begin{aligned}\left(I_a^\alpha(s-a)^{\beta-1}\right)(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}, \\ \left(D_a^\alpha(s-a)^{\beta-1}\right)(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}.\end{aligned}$$

Next, we have the semigroup property of the fractional integration operator I_a^α.

Lemma 4. *[14] Let $\alpha > 0$, $\beta > 0$ and $0 \le \gamma < 1$. If $f \in C_\gamma[a,b]$ then the equation*

$$I_a^\alpha I_a^\beta f = I_a^{\alpha+\beta} f$$

holds at any point $t \in (a,b]$. When $f \in C[a,b]$ this relation is valid at any point $t \in [a,b]$.

Another composition property between the fractional differentiation operator and the fractional integration operator is given next.

Property 2. [14] Let $0 < \beta < \alpha$ and $0 \leq \gamma < 1$. If $f \in C_\gamma[a,b]$, then the relation$D_a^\beta I_a^\alpha f = I_a^{\alpha-\beta} f$ holds at any point $t \in (a,b]$. When $f \in C[a,b]$ this relation is valid at any point $t \in [a,b]$. In particular, when $\beta = k \in \mathbb{N}$ and $\alpha > k$, then $D_a^k I_a^\alpha f = I_a^{\alpha-k} f$.

The following result provides another composition of the fractional integration operator I_a^α with the fractional differentiation operator D_a^α.

Lemma 5. *[14] Let $\alpha > 0$, $0 \leq \gamma < 1$, $n = -[-\alpha]$. If $f \in C_\gamma[a,b]$ and $I_a^{n-\alpha} f \in C_\gamma^n[a,b]$, then the equality*

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \sum_{i=1}^{n} \frac{\left(D^{n-i} I_a^{n-\alpha} f\right)(a)}{\Gamma(\alpha - i + 1)} (t-a)^{\alpha-i} \tag{2.2}$$

holds at any point $t \in (a,b]$. In particular, if $0 < \alpha < 1$ then

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}. \tag{2.3}$$

Lemma 6. *Let $0 < \alpha < 1$ and $0 \leq \gamma < 1$. If $f \in C_\gamma[a,b]$ and $I_a^{1-\alpha} f \in C_\gamma^1[a,b]$, then for $0 \leq \beta \leq \alpha < 1$ we have*

$$D_a^\beta f(t) = I_a^{\alpha-\beta} D_a^\alpha f(t) + \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha-\beta)} (t-a)^{\alpha-\beta-1}, \ t \in (a,b].$$

Proof. Since $I_a^{1-\alpha} f \in C_\gamma^1[a,b]$ then by Definition 1 we have $D_a^\alpha f = D I_a^{1-\alpha} f \in C_\gamma[a,b]$. Applying D_a^β to both sides of (2.3), then using Properties 1 and 2, we obtain

$$D_a^\beta f(t) = I_a^{\alpha-\beta} D_a^\alpha f(t) + \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha-\beta)} (t-a)^{\alpha-\beta-1}, \ t \in (a,b].$$

□

The following lemma describes the asymptotic behavior of the Riemann-Liouville fractional integral of a summable function.

Lemma 7. *Let $f \in L_1(0,\infty)$, then*

$$\lim_{t\to\infty} \frac{1}{t^\alpha} I_0^{\alpha+1} f(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(s)\,ds = \frac{1}{\Gamma(\alpha+1)} I_0^1 f(\infty), \quad \alpha > 0.$$

Proof. Indeed, in view of the Definition 2, we have

$$\begin{aligned}&\left| \frac{1}{t^\alpha} I_0^{\alpha+1} f(t) - \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(s)\,ds \right| \\ &\quad = \left| \frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{(t-s)^\alpha}{t^\alpha} f(s)\,ds - \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(s)\,ds \right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha+1)} \left| \int_0^t \left(1-\frac{s}{t}\right)^\alpha f(s)\,ds - \int_0^\infty f(s)\,ds \right| \\
&= \frac{1}{\Gamma(\alpha+1)} \left| \int_0^\infty \chi_{[0,t]}(s) \left(1-\frac{s}{t}\right)^\alpha f(s)\,ds - \int_0^\infty f(s)\,ds \right| \\
&= \frac{1}{\Gamma(\alpha+1)} \left| \int_0^\infty \left[\chi_{[0,t]}(s) \left(1-\frac{s}{t}\right)^\alpha - 1\right] f(s)\,ds \right| \\
&\le \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \left|\chi_{[0,t]}(s) \left(1-\frac{s}{t}\right)^\alpha - 1\right| |f(s)|\,ds,
\end{aligned}
$$

where

$$
\chi_{[0,t]}(s) = \begin{cases} 1, & s \in [0,t], \\ 0, & s \notin [0,t]. \end{cases}
$$

Since

$$
\lim_{t\to\infty} \chi_{[0,t]}(s) \left(1-\frac{s}{t}\right)^\alpha = 1, \ \ s<t,
$$

then, using the Dominated Convergence Theorem (continuous version) [7], we obtain

$$
\begin{aligned}
&\lim_{t\to\infty} \int_0^\infty \left|\chi_{[0,t]}(s) \left(1-\frac{s}{t}\right)^\alpha - 1\right| |f(s)|\,ds \\
&\qquad = \int_0^\infty \lim_{t\to\infty} \left|\chi_{[0,t]}(s) \left(1-\frac{s}{t}\right)^\alpha - 1\right| |f(s)|\,ds = 0.
\end{aligned}
$$

□

Lemma 8. *Let $0<\alpha<1$ and $0\le\gamma<1$. Assume that $y\in C_\gamma[0,\infty)$ and $I_0^{1-\alpha}y \in C_\gamma^2[0,\infty)$. Then*

$$
\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \lim_{t\to\infty} \frac{D_0^\alpha y(t)}{\Gamma(\alpha+1)}. \tag{2.4}
$$

Proof. Since $y\in C_\gamma[0,\infty)$ *and* $I_0^{1-\alpha}y \in C_\gamma^2[0,\infty)$, then we can apply Lemma 5, with α replaced by $1+\alpha$ and $n=2$, to get

$$
I_0^{1+\alpha} D_0^{1+\alpha} y(t) = y(t) - \frac{\left(I_0^{1-\alpha}y\right)(0)}{\Gamma(\alpha)} t^{\alpha-1} - \frac{D_0^\alpha y(0)}{\Gamma(1+\alpha)} t^\alpha. \tag{2.5}
$$

Dividing both sides of (2.5) by t^α and taking the limit as $t\to\infty$, we obtain

$$
\begin{aligned}
\lim_{t\to\infty} \frac{y(t)}{t^\alpha} &= \frac{D_0^\alpha y(0)}{\Gamma(1+\alpha)} + \lim_{t\to\infty} \frac{1}{t^\alpha} I_0^{1+\alpha} D_0^{1+\alpha} y(t) \\
&= \frac{D_0^\alpha y(0)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha+1)} I_0^1 D_0^{1+\alpha} y(\infty),
\end{aligned} \tag{2.6}
$$

where we have used Lemma 7. On the other hand, we have

$$
I_0^1 D_0^{1+\alpha} y(t) = D_0^\alpha y(t) - D_0^\alpha y(0) \tag{2.7}
$$

and (2.4) follows directly from (2.6) and (2.7). □

3 Inequalities

In this section we establish inequalities that involve special classes of functions. These inequalities will be used to obtain our main results. But first we cite here the Bihari inequality.

Theorem 1. *([2], Bihari inequality) Let u and f be nonnegative continuous functions defined on $\mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$). Let $w(u)$ be a continuous nondecreasing function defined on $\mathbb{R}_+$ and $w(u) > 0$ on $(0,\infty)$. If*

$$u(t) \leq k + \int_0^t f(s)\, w(u(s))\, ds,$$

for $t \in \mathbb{R}_+$, where k is a nonnegative constant, then for $0 \leq t \leq t_1$,

$$u(t) \leq G^{-1}\left(G(k) + \int_0^t f(s)\, ds\right),$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{w(s)},\ \ r > 0,\ \ r_0 > 0$$

and G^{-1} is the inverse function of G and $t_1 \in \mathbb{R}_+$ is chosen so that

$$G(k) + \int_0^t f(s)\, ds \in Dom\left(G^{-1}\right),$$

for $0 \leq t \leq t_1$.

Theorem 2. *([2], Gronwall-Bellman inequality) Let u and f be continuous and nonnegative functions defined on $\mathbb{R}_+$ and let $c(t)$ be a continuous, positive and nondecreasing function defined on $\mathbb{R}_+$. Then for $t \geq 0$*

$$u(t) \leq c(t) + \int_0^t f(s)\, u(s)\, ds$$

implies that

$$u(t) \leq c(t) \exp\left(\int_0^t f(s)\, ds\right).$$

Now we define the following classes of functions:

$$\begin{aligned} \varPhi = \big\{ &\varphi \in C(0,\infty) : \varphi \text{ is positive and nondecreasing on } (0,\infty), \\ &u\varphi(v) \leq \varphi(uv),\ 0 < u \leq 1 \big\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \varPsi = \big\{ &F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } 0 \leq F(t,u) - F(t,v) \leq N(t)(u-v), \\ &t > 0,\ u \geq v \geq 0, \text{ for some continuous function } N \text{ on } \mathbb{R}_+ \big\}. \end{aligned} \tag{3.2}$$

The class of functions $\varPhi$ defined above has been widely employed in the literature, see for instance [8]. Two simple functions belonging to spaces $\varPhi$ and $\varPsi$ are

$$\varphi(s) = \sum_{i=1}^n s^{\alpha_i},\ \ \alpha_i \leq 1,\ \ i = 1,2,...,n, \text{ and } F(t,u) = ue^t,$$

respectively.

Remark 2. If φ in the space Φ, then $\int_{x_0}^{\infty} ds/\varphi(s) = \infty$, $x_0 > 0$. (take $u = 1/v$, then $\frac{1}{v}\varphi(v) \le \varphi(1)$).

In what follows we give some properties and inequalities that involve these classes of functions. Clearly we have

Lemma 9. *The spaces Φ and Ψ are closed under addition and scalar multiplication.*

Lemma 10. *Let $z(t)$ be a positive solution of the integral equation*

$$z(t) = c_1 + c_2 t + c_3 t \int_0^t h(s)\varphi(z(s))\, ds,\ c_i \in \mathbb{R},\ i = 1,2,3, \quad t \ge 0, \tag{3.3}$$

where φ in the space Φ defined in (3.1) and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $th(t) \in L_1(1,\infty)$. Then

$$z(t) \le \begin{cases} G^{-1}(K), & 0 \le t < 1, \\ tG^{-1}(H), & t \ge 1, \end{cases} \tag{3.4}$$

where G^{-1} is the inverse function of $G(x) = \int_{x_0}^{x} \frac{ds}{\varphi(s)}$,

$$K = G(|c_1| + |c_2|) + |c_3| \int_0^1 h(s)ds, \quad H = G(A) + |c_3| \int_1^{\infty} sh(s)ds,$$

$$A = |c_1| + |c_2| + |c_3|\, \varphi\left(G^{-1}(K)\right) \int_0^1 h(s)ds.$$

Proof. We begin by noting that from the definition of Φ the functions G and G^{-1} are increasing, continuous and defined on $(0,\infty)$ and $(G(0^+),\infty)$, respectively. For $0 \le t < 1$ we have from (3.3)

$$z(t) \le |c_1| + |c_2| + |c_3| \int_0^t h(s)\varphi(z(s))\, ds$$

and the first inequality of (3.4) follows directly from Bihari's inequality (Theorem 1). For $t \ge 1$ we have from (3.3) the estimate

$$\begin{aligned} \frac{z(t)}{t} &\le |c_1| + |c_2| + |c_3| \int_0^t h(s)\varphi(z(s))\, ds \\ &\le |c_1| + |c_2| + |c_3| \int_0^1 h(s)\varphi(z(s))\, ds + |c_3| \int_1^t h(s)\varphi(z(s))\, ds. \end{aligned} \tag{3.5}$$

Therefore, from the first inequality of (3.4) and (3.5) we have

$$\frac{z(t)}{t} \le A + |c_3| \int_1^t h(s)\varphi(z(s))\, ds,\ t \ge 1. \tag{3.6}$$

From (3.1), we can write (3.6) in the form

$$\frac{z(t)}{t} \le A + |c_3| \int_1^t sh(s)\varphi\left(\frac{z(s)}{s}\right) ds,\ t \ge 1.$$

Now the second inequality of (3.4) follows immediately from Bihari's inequality (Theorem 1). $\square$

Lemma 11. *Let $z(t)$ satisfy*

$$z(t) \leq c_2 t + c_3 t \int_0^t \left[F_1(s, c_1 + z(s)) + F_2(s, c_1 + z(s)) + h(s)\right] ds, \ \ t \geq 0, \quad (3.7)$$

where $c_i > 0$, $i = 1, 2, 3$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and F_i, $i = 1, 2$, belong to the space Ψ defined in (3.2). Then

$$z(t) \leq t g(t), \quad t > 0,$$

where

$$g(t) = \left[c_2 + c_3 \int_0^t \left[F_1(s, c_1) + F_2(s, c_1) + h(s)\right] ds\right] \times \exp\left(c_3 \int_0^t s\left[N_1(s) + N_2(s)\right] ds\right)$$

with N_1 and N_2 are the functions in the definition (3.2), corresponding to F_1 and F_2 respectively.

Proof. From (3.7) we have

$$\begin{aligned}
\frac{z(t)}{t} &\leq c_2 + c_3 \int_0^t \left[F_1(s, c_1 + z(s)) + F_2(s, c_1 + z(s)) + h(s)\right] ds \\
&= c_2 + c_3 \int_0^t [F_1(s, c_1 + z(s)) - F_1(s, c_1) + F_1(s, c_1) \\
&\quad + F_2(s, c_1 + z(s)) - F_2(s, c_1) + F_2(s, c_1) + h(s)] ds \\
&= c_2 + c_3 \int_0^t \left[F_1(s, c_1) + F_2(s, c_1) + h(s)\right] ds
\end{aligned}$$

$$+ c_3 \int_0^t \left[F_1(s, c_1 + z(s)) - F_1(s, c_1) + F_2(s, c_1 + z(s)) - F_2(s, c_1)\right] ds, \ \ t > 0.$$

By (3.2) we obtain

$$\begin{aligned}
\frac{z(t)}{t} \leq c_2 &+ c_3 \int_0^t \left[F_1(s, c_1) + F_2(s, c_1) + h(s)\right] ds \\
&+ c_3 \int_0^t \left[N_1(s) + N_2(s)\right] z(s)\, ds, \quad t > 0. \qquad (3.8)
\end{aligned}$$

Put

$$g_1(t) = c_2 + c_3 \int_0^t \left[F_1(s, c_1) + F_2(s, c_1) + h(s)\right] ds. \qquad (3.9)$$

Then from (3) and (3.9) we obtain

$$\frac{z(t)}{t} \leq g_1(t) + c_3 \int_0^t s\left[N_1(s) + N_2(s)\right] \frac{z(s)}{s} ds, \quad t > 0. \qquad (3.10)$$

Clearly g_1 is a continuous, positive and nondecreasing function defined for all $t \geq 0$. Applying Theorem 2 to (3.10), we get

$$\frac{z(t)}{t} \leq g_1(t) \exp\left(c_3 \int_0^t s\left[N_1(s) + N_2(s)\right] ds\right), \quad t > 0.$$

We designate by $g_2(t)$ the expression

$$g_2(t) = \exp\left(c_3 \int_0^t s\left[N_1(s) + N_2(s)\right] ds\right).$$

Therefore

$$z(t) \leq t g(t), \quad g(t) = g_1(t) g_2(t), \ t > 0.$$

□

4 Problems with a non-fractional source

In this section, we study the asymptotic behavior of solutions of (1.1) when $\beta = 0$ and $0 < \alpha < 1$:

$$D_0^{1+\alpha} y(t) = f(t, y(t)), \quad 0 < \alpha \leq 1, \ t > 0 \tag{4.1}$$

with initial conditions

$$D_0^{\alpha} y(t)\,|_{t=0} = b_2 \text{ and } I_0^{1-\alpha} y(t)\,|_{t=0} = b_1, \ b_1, \ b_2 \in \mathbb{R} \tag{4.2}$$

in the space $C_{1-\alpha}^{1+\alpha}[0,\infty)$ defined by

$$C_{1-\alpha}^{1+\alpha}[0,\infty) = \left\{y \in C_{1-\alpha}[0,\infty) : D_0^{1+\alpha} y \in C_{1-\alpha}[0,\infty)\right\}. \tag{4.3}$$

In the sequel, we suppose that the function $f(t,y)$ satisfies the following conditions

(A) $f(t,y) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that $f(., y(.)) \in C_{1-\alpha}[0,\infty)$ for any $y \in C_{1-\alpha}[0,\infty)$.

(B) There exist continuous functions $h, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y(t))| \leq h(t) \varphi\left(t^{1-\alpha} |y(t)|\right), \ t \geq 0, \tag{4.4}$$

where φ in the spase Φ defined in (3.1) and $t h(t) \in L_1(1,\infty)$.

The next result provides useful estimates for solutions of Problem (4.1)–(4.2).

Lemma 12. *Assume that $y \in C_{1-\alpha}[0,\infty)$ is a solution of (4.1)–(4.2) and f satisfies (**A**) and (**B**). Then, for all $t > 0$, we have*

$$t^{1-\alpha} |y(t)| \leq z(t),$$

where

$$z(t) = \frac{|b_1|}{\Gamma(\alpha)} + \frac{|b_2| t}{\Gamma(\alpha+1)} + \frac{t}{\Gamma(\alpha+1)} \int_0^t h(s) \varphi\left(s^{1-\alpha} |y(s)|\right) ds, \ t > 0.$$

Proof. Applying $I_0^{1+\alpha}$ to (4.1) we find

$$I_0^{1+\alpha} D_0^{1+\alpha} y\left(t\right) = I_0^{1+\alpha} f\left(t, y\left(t\right)\right), \ t > 0.$$

Since $f \in C_{1-\alpha}[0,\infty)$, (4.1) implies that $D_0^{1+\alpha} y = D^2 I_0^{1-\alpha} y \in C_{1-\alpha}[0,\infty)$, then by Lemma 3, we have $I_0^{1-\alpha} y \in C_{1-\alpha}^2[0,\infty)$. As the hypotheses of Lemma 5 are fulfilled, we infer that

$$y\left(t\right) = \frac{b_1}{\Gamma(\alpha)} t^{\alpha-1} + \frac{b_2}{\Gamma(\alpha+1)} t^{\alpha} + I_0^{1+\alpha} f\left(t, y\left(t\right)\right), \quad t > 0, \tag{4.5}$$

where b_1 and b_2 come from the initial conditions in (4.2). In view of (4.5) we deduce

$$\left|y\left(t\right)\right| \leq \frac{|b_1|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{|b_2|}{\Gamma(\alpha+1)} t^{\alpha} + \frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)} \int_0^t \left|f\left(s, y\left(s\right)\right)\right| ds, \quad t > 0. \tag{4.6}$$

Multiplying both sides of (4.6) by $t^{1-\alpha}$ and using (4.4), we obtain the result. □

Theorem 3. *Let $y \in C_{1-\alpha}[0,\infty)$ be a solution of problem (4.1)–(4.2) and f satisfies (**A**) and (**B**). Then*

$$\lim_{t\to\infty} y\left(t\right)/t^{\alpha} = a \in \mathbb{R}.$$

Proof. It follows from Lemmas 10 and 12 that

$$\left|y\left(t\right)\right|/t^{\alpha} \leq H_0 =: G^{-1}\left(H\right), \quad t \geq 1. \tag{4.7}$$

By (4.4), we see that

$$\begin{aligned}
\left|\int_0^t f\left(s, y\left(s\right)\right) ds\right| &\leq \int_0^t \left|f\left(s, y\left(s\right)\right)\right| ds \leq \int_0^t h(s)\varphi\left(s^{1-\alpha}\left|y(s)\right|\right) ds \\
&\leq \int_0^1 h(s)\varphi\left(s^{1-\alpha}\left|y(s)\right|\right) ds + \int_1^t h(s)\varphi\left(s^{1-\alpha}\left|y(s)\right|\right) ds \\
&\leq \int_0^1 h(s)\varphi\left(z\left(s\right)\right) ds + \int_1^t sh(s)\varphi\left(\frac{|y(s)|}{s^{\alpha}}\right) ds.
\end{aligned} \tag{4.8}$$

Therefore from the first inequality of (3.4), (4.7) and (4.8) we obtain that

$$\begin{aligned}
\left|\int_0^t f\left(s, y\left(s\right)\right) ds\right| &\leq \int_0^1 h(s)\varphi\left(G^{-1}\left(K\right)\right) ds + \int_1^t sh(s)\varphi\left(H_0\right) ds \\
&\leq \varphi\left(G^{-1}\left(K\right)\right) \int_0^1 h(s) ds + \varphi\left(H_0\right) \int_1^t sh(s) ds < \infty.
\end{aligned}$$

Thus the integral $\int_0^t f\left(s, y\left(s\right)\right) ds$ is absolutely convergent and consequently

$$\lim_{t\to\infty} \int_0^t f\left(s, y\left(s\right)\right) ds < \infty. \tag{4.9}$$

Integrating both sides of (4.1), we find

$$D_0^{\alpha} y(t) = b_2 + \int_0^t f(s, y(s))\, ds, \quad t > 0.$$

In virtue of (4.9) we deduce that there exists $c \in \mathbb{R}$ such that

$$\lim_{t\to\infty} D_0^{\alpha} y(t) = c.$$

Further, by Lemma 8, we can write

$$\lim_{t\to\infty} \frac{y(t)}{t^{\alpha}} = \lim_{t\to\infty} \frac{D_0^{\alpha} y(t)}{\Gamma(\alpha+1)} = a$$

and the proof is now complete. □

Example 1. Consider the equation

$$D_0^{1+\alpha} y(t) = t^{\gamma} e^{-t} (y(t))^{r}, \quad t > 0, \tag{4.10}$$

where $0 < \alpha < 1$, $0 < r < 1$, *and* $\gamma + 1 > (1-\alpha) r$. *Then all solutions* $y \in C_{1-\alpha}[0,\infty)$ *of (4.10) enjoy the property* $\lim_{t\to\infty} y(t)/t^{\alpha} = a$ for some real number a.

Proof. We can rewrite (4.10) as follows

$$D_0^{1+\alpha} y(t) = t^{\gamma+(\alpha-1)r} e^{-t} \left(t^{1-\alpha} y(t)\right)^{r}.$$

Let $h(t) = t^{\gamma+(\alpha-1)r} e^{-t}$ and $\varphi(t) = t^{r}$. Then

$$\int_1^{\infty} s h(s)\, ds < \int_0^{\infty} s h(s)\, ds = \int_0^{\infty} s^{\gamma+(\alpha-1)r+1} e^{-s} ds$$
$$= \Gamma(\gamma + (\alpha-1) r + 2) < \infty.$$

φ is positive, continuous and nondecreasing function such that

$$u\varphi(v) = uv^{r} \le (uv)^{r} = \varphi(uv), \quad v > 0,\ u \le 1$$

and

$$\int_{r_0}^{\infty} \frac{ds}{\varphi(s)} = \int_{r_0}^{\infty} \frac{ds}{s^{r}} = \infty,\ r_0 > 0.$$

Consequently, φ in the space Φ. Clearly, all conditions of Theorem 3 are satisfied and the result follows. □

Remark 3. When $\alpha = 1$ in (4.1), we have

$$y''(t) = f(t, y(t)), \quad t > 0$$

with initial conditions

$$y'(0) = b_2 \text{ and } y(0) = b_1$$

in the space $C^2[0,\infty)$, space of twice continuously differentiable functions, and the function $f(t, y)$ satisfies the following conditions

(A") $f(t, y) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that $f(., y(.)) \in C[0, \infty)$ for any $y \in C[0, \infty)$.

(B") There exist continuous functions h, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y(t))| \leq h(t)\varphi(|y(t)|), \ t > 0.$$

Note that in this case, these assumptions are analogous to those of Philos [26]. On the other hand the function $f(t, y) = t^{-1.6} (y(t))^{1/2}$ satisfies the assumptions of Philos [26] but not ours.

Remark 4. When $b_1 = 0$ in Theorem 3, then the condition (**B**) is replaced by

(B$'$) There exist continuous functions h, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y(t))| \leq h(t)\varphi\left(\frac{|y(t)|}{t^\alpha}\right), \quad t > 0,$$

where φ is positive and nondecreasing and h is such that

$$\int_1^\infty h(s)\, ds < \infty.$$

5 Equations with fractional source term

Now we consider (1.1), with $0 < \beta \leq \alpha < 1$

$$D_0^{1+\alpha} y(t) = f(t, y(t), D_0^\beta y(t)), \quad t > 0 \tag{5.1}$$

with initial conditions

$$D_0^\alpha y(t)\,|_{t=0} = b_2 \text{ and } I_0^{1-\alpha} y(t)\,|_{t=0} = b_1, \ b_1, \ b_2 \in \mathbb{R} \tag{5.2}$$

in the space $C_{1-\alpha}^{1+\alpha}[0, \infty)$ defined in (4.3).
In the sequel, we suppose that the following conditions hold:

(A1) $f(t, u_1, u_2) : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ such that $f(., u_1(.), u_2(.)) \in C_{1-\alpha}[0, \infty)$ for any $u_1, u_2 \in C_{1-\alpha}[0, \infty)$.

(A2) There exist continuous functions $h : \mathbb{R}_+ \to \mathbb{R}_+$, $F_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, such that

$$|f(t, u_1, u_2)| \leq F_1\left(t, t^{1-\alpha} |u_1(t)|\right) + F_2\left(t, t^{1-(\alpha-\beta)} |u_2(t)|\right) + h(t), \tag{5.3}$$

where F_i in the space Ψ defined in (3.2), $i = 1, 2$.

The next result provides useful estimates for solutions of problem (5.1)–(5.2).

Lemma 13. *Assume that $y \in C^{1+\alpha}_{1-\alpha}[0,\infty)$ is a solution of (5.1)–(5.2). Then, for all $t>0$, we have*

$$t^{1-(\alpha-\beta)}\left|D_0^\beta y(t)\right| \le \frac{|b_1|}{\Gamma(\alpha-\beta)} + \frac{t}{\Gamma(1+\alpha-\beta)}\left(|b_2| + I_0^1\left|f(t,y(t),D_0^\beta y(t))\right|\right).$$

Proof. Note that $D_0^{1+\alpha}y = D^2 I_0^{1-\alpha}y \in C_{1-\alpha}[0,\infty)$ implies $I_0^{1-\alpha}y \in C^2_{1-\alpha}[0,\infty)$, see Lemma 3. Since $0\le\beta\le\alpha<1$, by Lemma 6, we see that

$$D_0^\beta y(t) = \frac{I_0^{1-\alpha}y(0)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} + I_0^{\alpha-\beta}D_0^\alpha y(t), \quad t>0. \tag{5.4}$$

Integrating both sides of (5.1), we find

$$D_0^\alpha y(t) = b_2 + I_0^1 f(t,y(t),D_0^\beta y(t)), \quad t>0. \tag{5.5}$$

Let us insert the expression (5.5) into (5.4), use Property 1, we have

$$\begin{aligned} D_0^\beta y(t) &= \frac{b_1}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} + I_0^{\alpha-\beta}\left(b_2 + I_0^1 f\left(s,y(s),D_0^\beta y(s)\right)\right)(t) \\ &= \frac{b_1}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} + \frac{b_2}{\Gamma(1+\alpha-\beta)}t^{\alpha-\beta} + I_0^{\alpha-\beta}I_0^1 f\left(t,y(t),D_0^\beta y(t)\right), t>0. \end{aligned}$$

We deduce the bound

$$\left|D_0^\beta y(t)\right| \le \frac{|b_1|\,t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{|b_2|\,t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + \frac{t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)}I_0^1\left|f\left(t,y(t),D_0^\beta y(t)\right)\right|.$$

Multiplying both sides of this inequality by $t^{1-(\alpha-\beta)}$ and the result follows. □

Lemma 14. *Assume that $y \in C_{1-\alpha}[0,\infty)$ is a solution of (5.1)–(5.2), f satisfies* ***(A1)***, ***(A2)*** *with*

$$\int_0^\infty F_i\left(s,\frac{|b_1|}{\Gamma(\alpha)}\right)ds < \infty, \ \int_0^\infty sN_i(s)\,ds < \infty, \ i=1,2 \tag{5.6}$$

and $h \in L_1(0,\infty)$ where N_i, $i=1,2$, are as in (3.2). Then

$$\lim_{t\to\infty}\int_0^t f\left(s,y(s),D_0^\beta y(s)\right)ds < \infty.$$

Proof. Let, for $t\ge 0$

$$\begin{aligned} z(t) = \frac{t}{\Gamma(1+\alpha-\beta)}\Bigg(|b_2| + \int_0^t \Big[F_1\left(s,s^{1-\alpha}|y(s)|\right) \\ + F_2\left(s,s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right) + h(s)\Big]\Bigg)ds. \end{aligned} \tag{5.7}$$

Then from Lemma 13 and the assumption (5.3) we get

$$t^{1-(\alpha-\beta)}\left|D_0^\beta y(t)\right| \le \frac{|b_1|}{\Gamma(\alpha-\beta)} + z(t), \quad t>0. \tag{5.8}$$

From (5.8) and (3.2), with $\beta = 0$, we have

$$F_1\left(t, t^{1-\alpha}\left|y(t)\right|\right) \leq F_1\left(t, \left|b_1\right|/\Gamma(\alpha) + z(t)\right), \quad t > 0 \tag{5.9}$$

and for $t > 0$

$$F_2\left(t, t^{1-(\alpha-\beta)}\left|D_0^{\beta} y(t)\right|\right) \leq F_2\left(t, \frac{|b_1|}{\Gamma(\alpha-\beta)} + z(t)\right) \leq F_2\left(t, \frac{|b_1|}{\Gamma(\alpha)} + z(t)\right). \tag{5.10}$$

Taking into account (5.7), (5.9) and (5.10) we are lead to

$$\begin{aligned} z(t) \leq \frac{t}{\Gamma(1+\alpha-\beta)} \bigg(|b_2| + \int_0^t \Big[& F_1\left(s, |b_1|/\Gamma(\alpha) + z(s)\right) \\ & + F_2\left(s, |b_1|/\Gamma(\alpha) + z(s)\right) + h(s) \Big] ds \bigg). \end{aligned}$$

Therefore, by Lemma 11, we have

$$z(t) \leq Ct, \quad t > 0, \tag{5.11}$$

where

$$\begin{aligned} C = & \frac{1}{\Gamma(1+\alpha-\beta)} \left(|b_2| + \int_0^{\infty} \left[F_1\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) + F_2\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) + h(s) \right] ds \right) \\ & \times \exp\left(\frac{1}{\Gamma(1+\alpha-\beta)} \int_0^{\infty} s\left[N_1(s) + N_2(s)\right] ds \right) < \infty. \end{aligned}$$

It follows from (5.8) and (5.11) that

$$t^{1-(\alpha-\beta)}\left|D_0^{\beta} y(t)\right| \leq \frac{|b_1|}{\Gamma(\alpha-\beta)} + Ct, \quad t > 0. \tag{5.12}$$

On the other hand, again by our assumption (5.3) we see that

$$\begin{aligned} \left| \int_0^t f\left(s, y(s), D_0^{\beta} y(s)\right) ds \right| & \leq \int_0^t \left| f\left(s, y(s), D_0^{\beta} y(s)\right) \right| ds \\ & \leq \int_0^t \left[F_1\left(s, s^{1-\alpha}|y(s)|\right) + F_2\left(s^{1-(\alpha-\beta)}\left|D_0^{\beta} y(s)\right|\right) + h(s) \right] ds. \end{aligned} \tag{5.13}$$

Therefore from (5.12) and (5.13) we deduce that

$$\begin{aligned} \left| \int_0^t f\left(s, y(s), D_0^{\beta} y(s)\right) ds \right| & \leq \int_0^t \left[F_1\left(s, |b_1|/\Gamma(\alpha) + Cs\right) \right. \\ & \quad \left. + F_2\left(s, |b_1|/\Gamma(\alpha) + Cs\right) + h(s) \right] ds \\ & = \int_0^t \left[F_1\left(s, |b_1|/\Gamma(\alpha) + Cs\right) - F_1\left(s, |b_1|/\Gamma(\alpha)\right) + F_1\left(s, |b_1|/\Gamma(\alpha)\right) \right. \\ & \left. + F_2\left(s, \frac{|b_1|}{\Gamma(\alpha)} + Cs\right) - F_2\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) + F_2\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) + h(s) \right] ds, \ t > 0. \end{aligned}$$

As the functions F_i, $i = 1, 2$, satisfy (3.2), we can write

$$\left|\int_0^t f\left(s, y(s), D_0^\beta y(s)\right) ds\right| \le C \int_0^t s\left[N_1(s) + N_2(s)\right] ds$$
$$+ \int_0^t \left[F_1\left(s, |b_1|/\Gamma(\alpha)\ \right) + F_2\left(s, |b_1|/\Gamma(\alpha)\right) + h(s)\right] ds.$$

That is

$$\left|\int_0^t f\left(s, y(s), D_0^\beta y(s)\right) ds\right| < \infty, \quad t \ge 0.$$

The integral $\int_0^t f\left(s, y(s), D_0^\beta y(s)\right) ds$ is therefore absolutely convergent and the result follows. $\square$

Theorem 4. *Under the same hypotheses as in Lemma 14 any solution* $y \in C_{1-\alpha}[0, \infty)$ *of problem (5.1)–(5.2) has the following property*

$$\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = a, \ \ a \in \mathbb{R}.$$

Proof. It is clear, by virtue of (5.5) and Lemma 14, that there exists $b \in \mathbb{R}$ such that

$$\lim_{t\to\infty} D_0^\alpha y(t) = b.$$

Noting that Lemma 8 remains true for the new problem, we conclude that

$$\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \lim_{t\to\infty} \frac{D_0^\alpha y(t)}{\Gamma(1+\alpha)} = a$$

for some $a \in \mathbb{R}$. $\square$

Remark 5. If $b_1 = 0$ in Theorem 4, then we replace (5.3) and (5.6) by

$$|f(t, u, v)| \le F_1\left(t, \frac{|u|}{t^\alpha}\right) + F_2\left(t, \frac{|v|}{t^{\alpha-\beta}}\right), \quad \int_0^\infty N_i(s)\, ds < \infty,$$

respectively.

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